# Isabelle/HOLCF — Higher-Order Logic of Computable Functions

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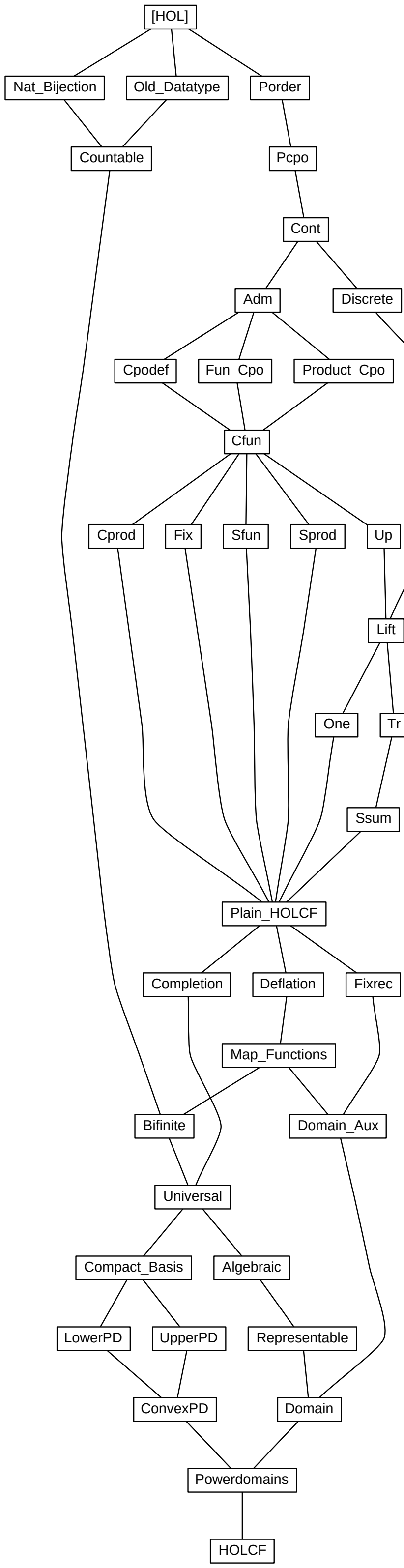
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# 1 Partial orders

**theory** *Porder*
**imports** *Main*
**begin**

**declare** [[*typedef-overloaded*]]

## 1.1 Type class for partial orders

**class** *below* =
  **fixes** *below* :: $'a \Rightarrow 'a \Rightarrow bool$
**begin**

**notation** (*ASCII*)
  *below* (**infix** << *50*)

**notation**
  *below* (**infix** $\sqsubseteq$ *50*)

**abbreviation**
  *not-below* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\not\sqsubseteq$ *50*)
  **where** *not-below* $x\ y \equiv \neg$ *below* $x\ y$

**notation** (*ASCII*)
  *not-below* (**infix** ~<< *50*)

**lemma** *below-eq-trans*: $[\![a \sqsubseteq b;\ b = c]\!] \Longrightarrow a \sqsubseteq c$
  ⟨*proof*⟩

**lemma** *eq-below-trans*: $[\![a = b;\ b \sqsubseteq c]\!] \Longrightarrow a \sqsubseteq c$
  ⟨*proof*⟩

**end**

**class** *po* = *below* +
  **assumes** *below-refl* [*iff*]: $x \sqsubseteq x$
  **assumes** *below-trans*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq z \Longrightarrow x \sqsubseteq z$
  **assumes** *below-antisym*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow x = y$
**begin**

**lemma** *eq-imp-below*: $x = y \Longrightarrow x \sqsubseteq y$
  ⟨*proof*⟩

**lemma** *box-below*: $a \sqsubseteq b \Longrightarrow c \sqsubseteq a \Longrightarrow b \sqsubseteq d \Longrightarrow c \sqsubseteq d$
  ⟨*proof*⟩

**lemma** *po-eq-conv*: $x = y \longleftrightarrow x \sqsubseteq y \wedge y \sqsubseteq x$
  ⟨*proof*⟩

**lemma** *rev-below-trans*: $y \sqsubseteq z \Longrightarrow x \sqsubseteq y \Longrightarrow x \sqsubseteq z$
⟨*proof*⟩

**lemma** *not-below2not-eq*: $x \not\sqsubseteq y \Longrightarrow x \neq y$
⟨*proof*⟩

**end**

**lemmas** *HOLCF-trans-rules* [*trans*] =
*below-trans*
*below-antisym*
*below-eq-trans*
*eq-below-trans*

**context** *po*
**begin**

## 1.2 Upper bounds

**definition** *is-ub* :: *'a set* ⇒ *'a* ⇒ *bool* (**infix** <| *55*) **where**
$S <| x \longleftrightarrow (\forall y \in S.\ y \sqsubseteq x)$

**lemma** *is-ubI*: $(\bigwedge x.\ x \in S \Longrightarrow x \sqsubseteq u) \Longrightarrow S <| u$
⟨*proof*⟩

**lemma** *is-ubD*: $[\![S <| u;\ x \in S]\!] \Longrightarrow x \sqsubseteq u$
⟨*proof*⟩

**lemma** *ub-imageI*: $(\bigwedge x.\ x \in S \Longrightarrow f\ x \sqsubseteq u) \Longrightarrow (\lambda x.\ f\ x)\ `\ S <| u$
⟨*proof*⟩

**lemma** *ub-imageD*: $[\![f\ `\ S <| u;\ x \in S]\!] \Longrightarrow f\ x \sqsubseteq u$
⟨*proof*⟩

**lemma** *ub-rangeI*: $(\bigwedge i.\ S\ i \sqsubseteq x) \Longrightarrow range\ S <| x$
⟨*proof*⟩

**lemma** *ub-rangeD*: $range\ S <| x \Longrightarrow S\ i \sqsubseteq x$
⟨*proof*⟩

**lemma** *is-ub-empty* [*simp*]: $\{\} <| u$
⟨*proof*⟩

**lemma** *is-ub-insert* [*simp*]: $(insert\ x\ A) <| y = (x \sqsubseteq y \land A <| y)$
⟨*proof*⟩

**lemma** *is-ub-upward*: $[\![S <| x;\ x \sqsubseteq y]\!] \Longrightarrow S <| y$
⟨*proof*⟩

## 1.3 Least upper bounds

**definition** *is-lub* :: 'a set ⇒ 'a ⇒ bool (**infix** <<| 55) **where**
S <<| x ⟷ S <| x ∧ (∀ u. S <| u ⟶ x ⊑ u)

**definition** *lub* :: 'a set ⇒ 'a **where**
lub S = (THE x. S <<| x)

**end**

**syntax** (ASCII)
-BLub :: [pttrn, 'a set, 'b] ⇒ 'b ((3LUB -:-./ -) [0,0, 10] 10)

**syntax**
-BLub :: [pttrn, 'a set, 'b] ⇒ 'b ((3⨆-∈-./ -) [0,0, 10] 10)

**translations**
LUB x:A. t == CONST lub ((%x. t) ` A)

**context** *po*
**begin**

**abbreviation**
Lub (**binder** ⨆ 10) **where**
⨆n. t n == lub (range t)

**notation** (ASCII)
Lub (**binder** LUB 10)

access to some definition as inference rule

**lemma** *is-lubD1*: S <<| x ⟹ S <| x
⟨proof⟩

**lemma** *is-lubD2*: ⟦S <<| x; S <| u⟧ ⟹ x ⊑ u
⟨proof⟩

**lemma** *is-lubI*: ⟦S <| x; ⋀u. S <| u ⟹ x ⊑ u⟧ ⟹ S <<| x
⟨proof⟩

**lemma** *is-lub-below-iff*: S <<| x ⟹ x ⊑ u ⟷ S <| u
⟨proof⟩

lubs are unique

**lemma** *is-lub-unique*: ⟦S <<| x; S <<| y⟧ ⟹ x = y
⟨proof⟩

technical lemmas about *lub* and *op* <<|

**lemma** *is-lub-lub*: M <<| x ⟹ M <<| lub M
⟨proof⟩

**lemma** *lub-eqI*: $M <<| l \Longrightarrow lub\ M = l$
⟨*proof*⟩

**lemma** *is-lub-singleton*: $\{x\} <<| x$
⟨*proof*⟩

**lemma** *lub-singleton* [*simp*]: $lub\ \{x\} = x$
⟨*proof*⟩

**lemma** *is-lub-bin*: $x \sqsubseteq y \Longrightarrow \{x, y\} <<| y$
⟨*proof*⟩

**lemma** *lub-bin*: $x \sqsubseteq y \Longrightarrow lub\ \{x, y\} = y$
⟨*proof*⟩

**lemma** *is-lub-maximal*: $[\![S <| x; x \in S]\!] \Longrightarrow S <<| x$
⟨*proof*⟩

**lemma** *lub-maximal*: $[\![S <| x; x \in S]\!] \Longrightarrow lub\ S = x$
⟨*proof*⟩

## 1.4 Countable chains

**definition** *chain* :: $(nat \Rightarrow 'a) \Rightarrow bool$ **where**
— Here we use countable chains and I prefer to code them as functions!
$chain\ Y = (\forall i.\ Y\ i \sqsubseteq Y\ (Suc\ i))$

**lemma** *chainI*: $(\bigwedge i.\ Y\ i \sqsubseteq Y\ (Suc\ i)) \Longrightarrow chain\ Y$
⟨*proof*⟩

**lemma** *chainE*: $chain\ Y \Longrightarrow Y\ i \sqsubseteq Y\ (Suc\ i)$
⟨*proof*⟩

chains are monotone functions

**lemma** *chain-mono-less*: $[\![chain\ Y; i < j]\!] \Longrightarrow Y\ i \sqsubseteq Y\ j$
⟨*proof*⟩

**lemma** *chain-mono*: $[\![chain\ Y; i \leq j]\!] \Longrightarrow Y\ i \sqsubseteq Y\ j$
⟨*proof*⟩

**lemma** *chain-shift*: $chain\ Y \Longrightarrow chain\ (\lambda i.\ Y\ (i + j))$
⟨*proof*⟩

technical lemmas about (least) upper bounds of chains

**lemma** *is-lub-rangeD1*: $range\ S <<| x \Longrightarrow S\ i \sqsubseteq x$
⟨*proof*⟩

**lemma** *is-ub-range-shift*:

$chain\ S \Longrightarrow range\ (\lambda i.\ S\ (i + j)) <|\ x = range\ S <|\ x$
⟨*proof*⟩

**lemma** *is-lub-range-shift*:
$chain\ S \Longrightarrow range\ (\lambda i.\ S\ (i + j)) <<|\ x = range\ S <<|\ x$
⟨*proof*⟩

the lub of a constant chain is the constant

**lemma** *chain-const* [*simp*]: $chain\ (\lambda i.\ c)$
⟨*proof*⟩

**lemma** *is-lub-const*: $range\ (\lambda x.\ c) <<|\ c$
⟨*proof*⟩

**lemma** *lub-const* [*simp*]: $(\bigsqcup i.\ c) = c$
⟨*proof*⟩

## 1.5 Finite chains

**definition** *max-in-chain* :: $nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow bool$ **where**
— finite chains, needed for monotony of continuous functions
$max\text{-}in\text{-}chain\ i\ C \longleftrightarrow (\forall j.\ i \le j \longrightarrow C\ i = C\ j)$

**definition** *finite-chain* :: $(nat \Rightarrow 'a) \Rightarrow bool$ **where**
$finite\text{-}chain\ C = (chain\ C \wedge (\exists i.\ max\text{-}in\text{-}chain\ i\ C))$

results about finite chains

**lemma** *max-in-chainI*: $(\bigwedge j.\ i \le j \Longrightarrow Y\ i = Y\ j) \Longrightarrow max\text{-}in\text{-}chain\ i\ Y$
⟨*proof*⟩

**lemma** *max-in-chainD*: $[\![max\text{-}in\text{-}chain\ i\ Y;\ i \le j]\!] \Longrightarrow Y\ i = Y\ j$
⟨*proof*⟩

**lemma** *finite-chainI*:
$[\![chain\ C;\ max\text{-}in\text{-}chain\ i\ C]\!] \Longrightarrow finite\text{-}chain\ C$
⟨*proof*⟩

**lemma** *finite-chainE*:
$[\![finite\text{-}chain\ C;\ \bigwedge i.\ [\![chain\ C;\ max\text{-}in\text{-}chain\ i\ C]\!] \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

**lemma** *lub-finch1*: $[\![chain\ C;\ max\text{-}in\text{-}chain\ i\ C]\!] \Longrightarrow range\ C <<|\ C\ i$
⟨*proof*⟩

**lemma** *lub-finch2*:
$finite\text{-}chain\ C \Longrightarrow range\ C <<|\ C\ (LEAST\ i.\ max\text{-}in\text{-}chain\ i\ C)$
⟨*proof*⟩

**lemma** *finch-imp-finite-range*: $finite\text{-}chain\ Y \Longrightarrow finite\ (range\ Y)$

⟨*proof*⟩

**lemma** *finite-range-has-max*:
**fixes** $f :: nat \Rightarrow 'a$ **and** $r :: 'a \Rightarrow 'a \Rightarrow bool$
**assumes** *mono*: $\bigwedge i\, j.\ i \le j \Longrightarrow r\ (f\ i)\ (f\ j)$
**assumes** *finite-range*: *finite* (*range f*)
**shows** $\exists k.\ \forall i.\ r\ (f\ i)\ (f\ k)$
⟨*proof*⟩

**lemma** *finite-range-imp-finch*:
⟦*chain Y*; *finite* (*range Y*)⟧ ⟹ *finite-chain Y*
⟨*proof*⟩

**lemma** *bin-chain*: $x \sqsubseteq y \Longrightarrow$ *chain* (λ*i. if i=0 then x else y*)
⟨*proof*⟩

**lemma** *bin-chainmax*:
$x \sqsubseteq y \Longrightarrow$ *max-in-chain* (*Suc 0*) (λ*i. if i=0 then x else y*)
⟨*proof*⟩

**lemma** *is-lub-bin-chain*:
$x \sqsubseteq y \Longrightarrow$ *range* (λ*i::nat. if i=0 then x else y*) <<| *y*
⟨*proof*⟩

the maximal element in a chain is its lub

**lemma** *lub-chain-maxelem*: ⟦$Y\ i = c$; $\forall i.\ Y\ i \sqsubseteq c$⟧ ⟹ *lub* (*range Y*) = *c*
⟨*proof*⟩

**end**

**end**

# 2 Classes cpo and pcpo

**theory** *Pcpo*
**imports** *Porder*
**begin**

## 2.1 Complete partial orders

The class cpo of chain complete partial orders

**class** *cpo* = *po* +
**assumes** *cpo*: *chain S* ⟹ ∃ *x*. *range S* <<| *x*
**begin**

in cpo's everthing equal to THE lub has lub properties for every chain

**lemma** *cpo-lubI*: *chain S* ⟹ *range S* <<| ($\bigsqcup i.\ S\ i$)
⟨*proof*⟩

**lemma** *thelubE*: $[\![chain\ S;\ (\bigsqcup i.\ S\ i) = l]\!] \Longrightarrow range\ S <<|\ l$
⟨*proof*⟩

Properties of the lub

**lemma** *is-ub-thelub*: $chain\ S \Longrightarrow S\ x \sqsubseteq (\bigsqcup i.\ S\ i)$
⟨*proof*⟩

**lemma** *is-lub-thelub*:
$[\![chain\ S;\ range\ S <|\ x]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
⟨*proof*⟩

**lemma** *lub-below-iff*: $chain\ S \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x \longleftrightarrow (\forall i.\ S\ i \sqsubseteq x)$
⟨*proof*⟩

**lemma** *lub-below*: $[\![chain\ S;\ \bigwedge i.\ S\ i \sqsubseteq x]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
⟨*proof*⟩

**lemma** *below-lub*: $[\![chain\ S;\ x \sqsubseteq S\ i]\!] \Longrightarrow x \sqsubseteq (\bigsqcup i.\ S\ i)$
⟨*proof*⟩

**lemma** *lub-range-mono*:
$[\![range\ X \subseteq range\ Y;\ chain\ Y;\ chain\ X]\!]$
$\Longrightarrow (\bigsqcup i.\ X\ i) \sqsubseteq (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

**lemma** *lub-range-shift*:
$chain\ Y \Longrightarrow (\bigsqcup i.\ Y\ (i + j)) = (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

**lemma** *maxinch-is-thelub*:
$chain\ Y \Longrightarrow max\text{-}in\text{-}chain\ i\ Y = ((\bigsqcup i.\ Y\ i) = Y\ i)$
⟨*proof*⟩

the $\sqsubseteq$ relation between two chains is preserved by their lubs

**lemma** *lub-mono*:
$[\![chain\ X;\ chain\ Y;\ \bigwedge i.\ X\ i \sqsubseteq Y\ i]\!]$
$\Longrightarrow (\bigsqcup i.\ X\ i) \sqsubseteq (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

the = relation between two chains is preserved by their lubs

**lemma** *lub-eq*:
$(\bigwedge i.\ X\ i = Y\ i) \Longrightarrow (\bigsqcup i.\ X\ i) = (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

**lemma** *ch2ch-lub*:
**assumes** *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
**assumes** *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
**shows** $chain\ (\lambda i.\ \bigsqcup j.\ Y\ i\ j)$

⟨*proof*⟩

**lemma** *diag-lub*:
  **assumes** *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
  **assumes** *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
  **shows** $(\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j) = (\bigsqcup i.\ Y\ i\ i)$
⟨*proof*⟩

**lemma** *ex-lub*:
  **assumes** *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
  **assumes** *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
  **shows** $(\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j) = (\bigsqcup j.\ \bigsqcup i.\ Y\ i\ j)$
  ⟨*proof*⟩

**end**

## 2.2 Pointed cpos

The class pcpo of pointed cpos

**class** *pcpo* = *cpo* +
  **assumes** *least*: $\exists\, x.\ \forall\, y.\ x \sqsubseteq y$
**begin**

**definition** *bottom* :: *'a* (⊥)
  **where** $bottom = (THE\ x.\ \forall\, y.\ x \sqsubseteq y)$

**lemma** *minimal* [*iff*]: $\bot \sqsubseteq x$
⟨*proof*⟩

**end**

Old "UU" syntax:

**syntax** *UU* :: *logic*

**translations** *UU* => *CONST bottom*

Simproc to rewrite $\bot = x$ to $x = \bot$.

⟨*ML*⟩

useful lemmas about ⊥

**lemma** *below-bottom-iff* [*simp*]: $(x \sqsubseteq \bot) = (x = \bot)$
⟨*proof*⟩

**lemma** *eq-bottom-iff*: $(x = \bot) = (x \sqsubseteq \bot)$
⟨*proof*⟩

**lemma** *bottomI*: $x \sqsubseteq \bot \Longrightarrow x = \bot$
⟨*proof*⟩

**lemma** *lub-eq-bottom-iff*: *chain* $Y \Longrightarrow (\bigsqcup i.\ Y\ i) = \bot \longleftrightarrow (\forall i.\ Y\ i = \bot)$
⟨*proof*⟩

## 2.3 Chain-finite and flat cpos

further useful classes for HOLCF domains

**class** *chfin* = *po* +
  **assumes** *chfin*: *chain* $Y \Longrightarrow \exists n.$ *max-in-chain* $n\ Y$
**begin**

**subclass** *cpo*
⟨*proof*⟩

**lemma** *chfin2finch*: *chain* $Y \Longrightarrow$ *finite-chain* $Y$
  ⟨*proof*⟩

**end**

**class** *flat* = *pcpo* +
  **assumes** *ax-flat*: $x \sqsubseteq y \Longrightarrow x = \bot \vee x = y$
**begin**

**subclass** *chfin*
⟨*proof*⟩

**lemma** *flat-below-iff*:
  **shows** $(x \sqsubseteq y) = (x = \bot \vee x = y)$
  ⟨*proof*⟩

**lemma** *flat-eq*: $a \neq \bot \Longrightarrow a \sqsubseteq b = (a = b)$
  ⟨*proof*⟩

**end**

## 2.4 Discrete cpos

**class** *discrete-cpo* = *below* +
  **assumes** *discrete-cpo* [*simp*]: $x \sqsubseteq y \longleftrightarrow x = y$
**begin**

**subclass** *po*
⟨*proof*⟩

In a discrete cpo, every chain is constant

**lemma** *discrete-chain-const*:
  **assumes** *S*: *chain* *S*
  **shows** $\exists x.\ S = (\lambda i.\ x)$
⟨*proof*⟩

**subclass** *chfin*
⟨*proof*⟩

**end**

**end**

# 3 Continuity and monotonicity

**theory** *Cont*
**imports** *Pcpo*
**begin**

Now we change the default class! Form now on all untyped type variables are of default class po

**default-sort** *po*

## 3.1 Definitions

**definition**
$monofun :: ('a \Rightarrow 'b) \Rightarrow bool$ — monotonicity **where**
$monofun\ f = (\forall x\ y.\ x \sqsubseteq y \longrightarrow f\ x \sqsubseteq f\ y)$

**definition**
$cont :: ('a::cpo \Rightarrow 'b::cpo) \Rightarrow bool$
**where**
$cont\ f = (\forall Y.\ chain\ Y \longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i))$

**lemma** *contI*:
$[\![\bigwedge Y.\ chain\ Y \Longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i)]\!] \Longrightarrow cont\ f$
⟨*proof*⟩

**lemma** *contE*:
$[\![cont\ f;\ chain\ Y]\!] \Longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

**lemma** *monofunI*:
$[\![\bigwedge x\ y.\ x \sqsubseteq y \Longrightarrow f\ x \sqsubseteq f\ y]\!] \Longrightarrow monofun\ f$
⟨*proof*⟩

**lemma** *monofunE*:
$[\![monofun\ f;\ x \sqsubseteq y]\!] \Longrightarrow f\ x \sqsubseteq f\ y$
⟨*proof*⟩

## 3.2 Equivalence of alternate definition

monotone functions map chains to chains

**lemma** *ch2ch-monofun*: $[\![$*monofun f*; *chain Y*$]\!] \Longrightarrow$ *chain* $(\lambda i.\ f\ (Y\ i))$
⟨*proof*⟩

monotone functions map upper bound to upper bounds

**lemma** *ub2ub-monofun*:
$[\![$*monofun f*; *range Y* <| *u*$]\!] \Longrightarrow$ *range* $(\lambda i.\ f\ (Y\ i))$ <| *f u*
⟨*proof*⟩

a lemma about binary chains

**lemma** *binchain-cont*:
$[\![$*cont f*; $x \sqsubseteq y]\!] \Longrightarrow$ *range* $(\lambda i{::}nat.\ f\ (\textit{if } i = 0 \textit{ then } x \textit{ else } y))$ <<| *f y*
⟨*proof*⟩

continuity implies monotonicity

**lemma** *cont2mono*: *cont f* $\Longrightarrow$ *monofun f*
⟨*proof*⟩

**lemmas** *cont2monofunE* = *cont2mono* [*THEN monofunE*]

**lemmas** *ch2ch-cont* = *cont2mono* [*THEN ch2ch-monofun*]

continuity implies preservation of lubs

**lemma** *cont2contlubE*:
$[\![$*cont f*; *chain Y*$]\!] \Longrightarrow f\ (\bigsqcup i.\ Y\ i) = (\bigsqcup i.\ f\ (Y\ i))$
⟨*proof*⟩

**lemma** *contI2*:
**fixes** $f :: {}'a{::}cpo \Rightarrow {}'b{::}cpo$
**assumes** *mono*: *monofun f*
**assumes** *below*: $\bigwedge Y.\ [\![$*chain Y*; *chain* $(\lambda i.\ f\ (Y\ i))]\!]$
$\Longrightarrow f\ (\bigsqcup i.\ Y\ i) \sqsubseteq (\bigsqcup i.\ f\ (Y\ i))$
**shows** *cont f*
⟨*proof*⟩

## 3.3 Collection of continuity rules

**named-theorems** *cont2cont continuity intro rule*

## 3.4 Continuity of basic functions

The identity function is continuous

**lemma** *cont-id* [*simp*, *cont2cont*]: *cont* $(\lambda x.\ x)$
⟨*proof*⟩

constant functions are continuous

**lemma** *cont-const* [*simp*, *cont2cont*]: *cont* $(\lambda x.\ c)$
⟨*proof*⟩

application of functions is continuous

**lemma** *cont-apply*:
**fixes** $f$ :: $'a::cpo \Rightarrow 'b::cpo \Rightarrow 'c::cpo$ **and** $t$ :: $'a \Rightarrow 'b$
**assumes** *1*: $cont\ (\lambda x.\ t\ x)$
**assumes** *2*: $\bigwedge x.\ cont\ (\lambda y.\ f\ x\ y)$
**assumes** *3*: $\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y)$
**shows** $cont\ (\lambda x.\ (f\ x)\ (t\ x))$
$\langle proof \rangle$

**lemma** *cont-compose*:
$[\![cont\ c;\ cont\ (\lambda x.\ f\ x)]\!] \Longrightarrow cont\ (\lambda x.\ c\ (f\ x))$
$\langle proof \rangle$

Least upper bounds preserve continuity

**lemma** *cont2cont-lub* [*simp*]:
**assumes** *chain*: $\bigwedge x.\ chain\ (\lambda i.\ F\ i\ x)$ **and** *cont*: $\bigwedge i.\ cont\ (\lambda x.\ F\ i\ x)$
**shows** $cont\ (\lambda x.\ \bigsqcup i.\ F\ i\ x)$
$\langle proof \rangle$

if-then-else is continuous

**lemma** *cont-if* [*simp*, *cont2cont*]:
$[\![cont\ f;\ cont\ g]\!] \Longrightarrow cont\ (\lambda x.\ if\ b\ then\ f\ x\ else\ g\ x)$
$\langle proof \rangle$

## 3.5 Finite chains and flat pcpos

Monotone functions map finite chains to finite chains.

**lemma** *monofun-finch2finch*:
$[\![monofun\ f;\ finite\text{-}chain\ Y]\!] \Longrightarrow finite\text{-}chain\ (\lambda n.\ f\ (Y\ n))$
$\langle proof \rangle$

The same holds for continuous functions.

**lemma** *cont-finch2finch*:
$[\![cont\ f;\ finite\text{-}chain\ Y]\!] \Longrightarrow finite\text{-}chain\ (\lambda n.\ f\ (Y\ n))$
$\langle proof \rangle$

All monotone functions with chain-finite domain are continuous.

**lemma** *chfindom-monofun2cont*: $monofun\ f \Longrightarrow cont\ (f::'a::chfin \Rightarrow 'b::cpo)$
$\langle proof \rangle$

All strict functions with flat domain are continuous.

**lemma** *flatdom-strict2mono*: $f\ \bot = \bot \Longrightarrow monofun\ (f::'a::flat \Rightarrow 'b::pcpo)$
$\langle proof \rangle$

**lemma** *flatdom-strict2cont*: $f\ \bot = \bot \Longrightarrow cont\ (f::'a::flat \Rightarrow 'b::pcpo)$
$\langle proof \rangle$

All functions with discrete domain are continuous.

**lemma** *cont-discrete-cpo* [*simp*, *cont2cont*]: *cont* ($f$::$'a$::*discrete-cpo* ⇒ $'b$::*cpo*)
⟨*proof*⟩

**end**

# 4 Admissibility and compactness

**theory** *Adm*
**imports** *Cont*
**begin**

**default-sort** *cpo*

## 4.1 Definitions

**definition**
*adm* :: ($'a$::*cpo* ⇒ *bool*) ⇒ *bool* **where**
*adm* $P$ = (∀ $Y$. *chain* $Y$ ⟶ (∀ $i$. $P$ ($Y$ $i$)) ⟶ $P$ (⨆ $i$. $Y$ $i$))

**lemma** *admI*:
(⋀ $Y$. ⟦*chain* $Y$; ∀ $i$. $P$ ($Y$ $i$)⟧ ⟹ $P$ (⨆ $i$. $Y$ $i$)) ⟹ *adm* $P$
⟨*proof*⟩

**lemma** *admD*: ⟦*adm* $P$; *chain* $Y$; ⋀$i$. $P$ ($Y$ $i$)⟧ ⟹ $P$ (⨆ $i$. $Y$ $i$)
⟨*proof*⟩

**lemma** *admD2*: ⟦*adm* (λ$x$. ¬ $P$ $x$); *chain* $Y$; $P$ (⨆ $i$. $Y$ $i$)⟧ ⟹ ∃ $i$. $P$ ($Y$ $i$)
⟨*proof*⟩

**lemma** *triv-admI*: ∀ $x$. $P$ $x$ ⟹ *adm* $P$
⟨*proof*⟩

## 4.2 Admissibility on chain-finite types

For chain-finite (easy) types every formula is admissible.

**lemma** *adm-chfin* [*simp*]: *adm* ($P$::$'a$::*chfin* ⇒ *bool*)
⟨*proof*⟩

## 4.3 Admissibility of special formulae and propagation

**lemma** *adm-const* [*simp*]: *adm* (λ$x$. $t$)
⟨*proof*⟩

**lemma** *adm-conj* [*simp*]:
⟦*adm* (λ$x$. $P$ $x$); *adm* (λ$x$. $Q$ $x$)⟧ ⟹ *adm* (λ$x$. $P$ $x$ ∧ $Q$ $x$)
⟨*proof*⟩

**lemma** *adm-all* [*simp*]:
$(\bigwedge y.\ adm\ (\lambda x.\ P\ x\ y)) \Longrightarrow adm\ (\lambda x.\ \forall\, y.\ P\ x\ y)$
⟨*proof*⟩

**lemma** *adm-ball* [*simp*]:
$(\bigwedge y.\ y \in A \Longrightarrow adm\ (\lambda x.\ P\ x\ y)) \Longrightarrow adm\ (\lambda x.\ \forall\, y{\in}A.\ P\ x\ y)$
⟨*proof*⟩

Admissibility for disjunction is hard to prove. It requires 2 lemmas.

**lemma** *adm-disj-lemma1*:
**assumes** *adm*: *adm P*
**assumes** *chain*: *chain Y*
**assumes** $P$: $\forall\, i.\ \exists\, j{\geq}i.\ P\ (Y\ j)$
**shows** $P\ (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

**lemma** *adm-disj-lemma2*:
$\forall\, n{::}nat.\ P\ n \vee Q\ n \Longrightarrow (\forall\, i.\ \exists\, j{\geq}i.\ P\ j) \vee (\forall\, i.\ \exists\, j{\geq}i.\ Q\ j)$
⟨*proof*⟩

**lemma** *adm-disj* [*simp*]:
$[\![adm\ (\lambda x.\ P\ x);\ adm\ (\lambda x.\ Q\ x)]\!] \Longrightarrow adm\ (\lambda x.\ P\ x \vee Q\ x)$
⟨*proof*⟩

**lemma** *adm-imp* [*simp*]:
$[\![adm\ (\lambda x.\ \neg\ P\ x);\ adm\ (\lambda x.\ Q\ x)]\!] \Longrightarrow adm\ (\lambda x.\ P\ x \longrightarrow Q\ x)$
⟨*proof*⟩

**lemma** *adm-iff* [*simp*]:
$[\![adm\ (\lambda x.\ P\ x \longrightarrow Q\ x);\ adm\ (\lambda x.\ Q\ x \longrightarrow P\ x)]\!]$
$\Longrightarrow adm\ (\lambda x.\ P\ x = Q\ x)$
⟨*proof*⟩

admissibility and continuity

**lemma** *adm-below* [*simp*]:
$[\![cont\ (\lambda x.\ u\ x);\ cont\ (\lambda x.\ v\ x)]\!] \Longrightarrow adm\ (\lambda x.\ u\ x \sqsubseteq v\ x)$
⟨*proof*⟩

**lemma** *adm-eq* [*simp*]:
$[\![cont\ (\lambda x.\ u\ x);\ cont\ (\lambda x.\ v\ x)]\!] \Longrightarrow adm\ (\lambda x.\ u\ x = v\ x)$
⟨*proof*⟩

**lemma** *adm-subst*: $[\![cont\ (\lambda x.\ t\ x);\ adm\ P]\!] \Longrightarrow adm\ (\lambda x.\ P\ (t\ x))$
⟨*proof*⟩

**lemma** *adm-not-below* [*simp*]: $cont\ (\lambda x.\ t\ x) \Longrightarrow adm\ (\lambda x.\ t\ x \not\sqsubseteq u)$
⟨*proof*⟩

## 4.4 Compactness

**definition**
*compact* :: *'a*::*cpo* ⇒ *bool* **where**
*compact k* = *adm* (λ*x*. *k* ⋢ *x*)

**lemma** *compactI*: *adm* (λ*x*. *k* ⋢ *x*) ⟹ *compact k*
⟨*proof*⟩

**lemma** *compactD*: *compact k* ⟹ *adm* (λ*x*. *k* ⋢ *x*)
⟨*proof*⟩

**lemma** *compactI2*:
(⋀*Y*. ⟦*chain Y*; *x* ⊑ (⨆ *i*. *Y i*)⟧ ⟹ ∃ *i*. *x* ⊑ *Y i*) ⟹ *compact x*
⟨*proof*⟩

**lemma** *compactD2*:
⟦*compact x*; *chain Y*; *x* ⊑ (⨆ *i*. *Y i*)⟧ ⟹ ∃ *i*. *x* ⊑ *Y i*
⟨*proof*⟩

**lemma** *compact-below-lub-iff*:
⟦*compact x*; *chain Y*⟧ ⟹ *x* ⊑ (⨆ *i*. *Y i*) ⟷ (∃ *i*. *x* ⊑ *Y i*)
⟨*proof*⟩

**lemma** *compact-chfin* [*simp*]: *compact* (*x*::*'a*::*chfin*)
⟨*proof*⟩

**lemma** *compact-imp-max-in-chain*:
⟦*chain Y*; *compact* (⨆ *i*. *Y i*)⟧ ⟹ ∃ *i*. *max-in-chain i Y*
⟨*proof*⟩

admissibility and compactness

**lemma** *adm-compact-not-below* [*simp*]:
⟦*compact k*; *cont* (λ*x*. *t x*)⟧ ⟹ *adm* (λ*x*. *k* ⋢ *t x*)
⟨*proof*⟩

**lemma** *adm-neq-compact* [*simp*]:
⟦*compact k*; *cont* (λ*x*. *t x*)⟧ ⟹ *adm* (λ*x*. *t x* ≠ *k*)
⟨*proof*⟩

**lemma** *adm-compact-neq* [*simp*]:
⟦*compact k*; *cont* (λ*x*. *t x*)⟧ ⟹ *adm* (λ*x*. *k* ≠ *t x*)
⟨*proof*⟩

**lemma** *compact-bottom* [*simp*, *intro*]: *compact* ⊥
⟨*proof*⟩

Any upward-closed predicate is admissible.

**lemma** *adm-upward*:
**assumes** *P*: ⋀*x y*. ⟦*P x*; *x* ⊑ *y*⟧ ⟹ *P y*

**shows** *adm P*
⟨*proof*⟩

**lemmas** *adm-lemmas* =
*adm-const adm-conj adm-all adm-ball adm-disj adm-imp adm-iff*
*adm-below adm-eq adm-not-below*
*adm-compact-not-below adm-compact-neq adm-neq-compact*

**end**

# 5 Subtypes of pcpos

**theory** *Cpodef*
**imports** *Adm*
**keywords** *pcpodef cpodef* :: *thy-goal*
**begin**

## 5.1 Proving a subtype is a partial order

A subtype of a partial order is itself a partial order, if the ordering is defined in the standard way.

⟨*ML*⟩

**theorem** *typedef-po*:
**fixes** $Abs :: 'a::po \Rightarrow 'b::type$
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
**shows** *OFCLASS*('*b*, *po-class*)
⟨*proof*⟩

⟨*ML*⟩

## 5.2 Proving a subtype is finite

**lemma** *typedef-finite-UNIV*:
**fixes** $Abs :: 'a::type \Rightarrow 'b::type$
**assumes** *type*: *type-definition Rep Abs A*
**shows** $finite\ A \Longrightarrow finite\ (UNIV :: 'b\ set)$
⟨*proof*⟩

## 5.3 Proving a subtype is chain-finite

**lemma** *ch2ch-Rep*:
**assumes** *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
**shows** $chain\ S \Longrightarrow chain\ (\lambda i.\ Rep\ (S\ i))$
⟨*proof*⟩

**theorem** *typedef-chfin*:

**fixes** *Abs* :: *'a::chfin* ⇒ *'b::po*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**shows** *OFCLASS('b, chfin-class)*
⟨*proof*⟩

## 5.4 Proving a subtype is complete

A subtype of a cpo is itself a cpo if the ordering is defined in the standard way, and the defining subset is closed with respect to limits of chains. A set is closed if and only if membership in the set is an admissible predicate.

**lemma** *typedef-is-lubI*:
**assumes** *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**shows** *range* (λ*i*. *Rep* (*S i*)) <<| *Rep x* ⟹ *range S* <<| *x*
⟨*proof*⟩

**lemma** *Abs-inverse-lub-Rep*:
**fixes** *Abs* :: *'a::cpo* ⇒ *'b::po*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *adm*: *adm* (λ*x*. *x* ∈ *A*)
**shows** *chain S* ⟹ *Rep* (*Abs* (⨆*i*. *Rep* (*S i*))) = (⨆*i*. *Rep* (*S i*))
⟨*proof*⟩

**theorem** *typedef-is-lub*:
**fixes** *Abs* :: *'a::cpo* ⇒ *'b::po*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *adm*: *adm* (λ*x*. *x* ∈ *A*)
**shows** *chain S* ⟹ *range S* <<| *Abs* (⨆*i*. *Rep* (*S i*))
⟨*proof*⟩

**lemmas** *typedef-lub* = *typedef-is-lub* [*THEN lub-eqI*]

**theorem** *typedef-cpo*:
**fixes** *Abs* :: *'a::cpo* ⇒ *'b::po*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *adm*: *adm* (λ*x*. *x* ∈ *A*)
**shows** *OFCLASS('b, cpo-class)*
⟨*proof*⟩

### 5.4.1 Continuity of *Rep* and *Abs*

For any sub-cpo, the *Rep* function is continuous.

**theorem** *typedef-cont-Rep*:
**fixes** *Abs* :: *'a::cpo* ⇒ *'b::cpo*
**assumes** *type*: *type-definition Rep Abs A*

**and** *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
**and** *adm*: $adm\ (\lambda x.\ x \in A)$
**shows** $cont\ (\lambda x.\ f\ x) \Longrightarrow cont\ (\lambda x.\ Rep\ (f\ x))$
⟨*proof*⟩

For a sub-cpo, we can make the *Abs* function continuous only if we restrict its domain to the defining subset by composing it with another continuous function.

**theorem** *typedef-cont-Abs*:
**fixes** $Abs :: 'a::cpo \Rightarrow 'b::cpo$
**fixes** $f :: 'c::cpo \Rightarrow 'a::cpo$
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
**and** *adm*: $adm\ (\lambda x.\ x \in A)$
**and** *f-in-A*: $\bigwedge x.\ f\ x \in A$
**shows** $cont\ f \Longrightarrow cont\ (\lambda x.\ Abs\ (f\ x))$
⟨*proof*⟩

## 5.5 Proving subtype elements are compact

**theorem** *typedef-compact*:
**fixes** $Abs :: 'a::cpo \Rightarrow 'b::cpo$
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
**and** *adm*: $adm\ (\lambda x.\ x \in A)$
**shows** $compact\ (Rep\ k) \Longrightarrow compact\ k$
⟨*proof*⟩

## 5.6 Proving a subtype is pointed

A subtype of a cpo has a least element if and only if the defining subset has a least element.

**theorem** *typedef-pcpo-generic*:
**fixes** $Abs :: 'a::cpo \Rightarrow 'b::cpo$
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
**and** *z-in-A*: $z \in A$
**and** *z-least*: $\bigwedge x.\ x \in A \Longrightarrow z \sqsubseteq x$
**shows** $OFCLASS('b,\ pcpo\text{-}class)$
⟨*proof*⟩

As a special case, a subtype of a pcpo has a least element if the defining subset contains ⊥.

**theorem** *typedef-pcpo*:
**fixes** $Abs :: 'a::pcpo \Rightarrow 'b::cpo$
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: $op \sqsubseteq \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
**and** *bottom-in-A*: $\bot \in A$

**shows** *OFCLASS*('*b*, *pcpo-class*)
⟨*proof*⟩

### 5.6.1 Strictness of *Rep* and *Abs*

For a sub-pcpo where ⊥ is a member of the defining subset, *Rep* and *Abs* are both strict.

**theorem** *typedef-Abs-strict*:
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *bottom-in-A*: ⊥ ∈ *A*
**shows** *Abs* ⊥ = ⊥
⟨*proof*⟩

**theorem** *typedef-Rep-strict*:
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *bottom-in-A*: ⊥ ∈ *A*
**shows** *Rep* ⊥ = ⊥
⟨*proof*⟩

**theorem** *typedef-Abs-bottom-iff*:
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *bottom-in-A*: ⊥ ∈ *A*
**shows** *x* ∈ *A* ⟹ (*Abs x* = ⊥) = (*x* = ⊥)
⟨*proof*⟩

**theorem** *typedef-Rep-bottom-iff*:
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *bottom-in-A*: ⊥ ∈ *A*
**shows** (*Rep x* = ⊥) = (*x* = ⊥)
⟨*proof*⟩

## 5.7 Proving a subtype is flat

**theorem** *typedef-flat*:
**fixes** *Abs* :: '*a*::*flat* ⇒ '*b*::*pcpo*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: *op* ⊑ ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *bottom-in-A*: ⊥ ∈ *A*
**shows** *OFCLASS*('*b*, *flat-class*)
⟨*proof*⟩

## 5.8 HOLCF type definition package

⟨*ML*⟩

**end**

# 6 Class instances for the full function space

**theory** *Fun-Cpo*
**imports** *Adm*
**begin**

## 6.1 Full function space is a partial order

**instantiation** *fun* :: (*type*, *below*) *below*
**begin**

**definition**
*below-fun-def*: $(op \sqsubseteq) \equiv (\lambda f\ g.\ \forall x.\ f\ x \sqsubseteq g\ x)$

**instance** ⟨*proof*⟩
**end**

**instance** *fun* :: (*type*, *po*) *po*
⟨*proof*⟩

**lemma** *fun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall x.\ f\ x \sqsubseteq g\ x)$
⟨*proof*⟩

**lemma** *fun-belowI*: $(\bigwedge x.\ f\ x \sqsubseteq g\ x) \Longrightarrow f \sqsubseteq g$
⟨*proof*⟩

**lemma** *fun-belowD*: $f \sqsubseteq g \Longrightarrow f\ x \sqsubseteq g\ x$
⟨*proof*⟩

## 6.2 Full function space is chain complete

Properties of chains of functions.

**lemma** *fun-chain-iff*: $chain\ S \longleftrightarrow (\forall x.\ chain\ (\lambda i.\ S\ i\ x))$
⟨*proof*⟩

**lemma** *ch2ch-fun*: $chain\ S \Longrightarrow chain\ (\lambda i.\ S\ i\ x)$
⟨*proof*⟩

**lemma** *ch2ch-lambda*: $(\bigwedge x.\ chain\ (\lambda i.\ S\ i\ x)) \Longrightarrow chain\ S$
⟨*proof*⟩

Type $'a \Rightarrow 'b$ is chain complete

**lemma** *is-lub-lambda*:
$(\bigwedge x.\ range\ (\lambda i.\ Y\ i\ x) <<|\ f\ x) \Longrightarrow range\ Y <<|\ f$
⟨*proof*⟩

**lemma** *is-lub-fun*:
*chain* $(S::nat \Rightarrow 'a::type \Rightarrow 'b::cpo)$
$\Longrightarrow$ *range* $S <<| (\lambda x. \bigsqcup i. S\ i\ x)$
⟨*proof*⟩

**lemma** *lub-fun*:
*chain* $(S::nat \Rightarrow 'a::type \Rightarrow 'b::cpo)$
$\Longrightarrow (\bigsqcup i. S\ i) = (\lambda x. \bigsqcup i. S\ i\ x)$
⟨*proof*⟩

**instance** *fun* :: (*type*, *cpo*) *cpo*
⟨*proof*⟩

**instance** *fun* :: (*type*, *discrete-cpo*) *discrete-cpo*
⟨*proof*⟩

## 6.3 Full function space is pointed

**lemma** *minimal-fun*: $(\lambda x. \bot) \sqsubseteq f$
⟨*proof*⟩

**instance** *fun* :: (*type*, *pcpo*) *pcpo*
⟨*proof*⟩

**lemma** *inst-fun-pcpo*: $\bot = (\lambda x. \bot)$
⟨*proof*⟩

**lemma** *app-strict* [*simp*]: $\bot\ x = \bot$
⟨*proof*⟩

**lemma** *lambda-strict*: $(\lambda x. \bot) = \bot$
⟨*proof*⟩

## 6.4 Propagation of monotonicity and continuity

The lub of a chain of monotone functions is monotone.

**lemma** *adm-monofun*: *adm monofun*
⟨*proof*⟩

The lub of a chain of continuous functions is continuous.

**lemma** *adm-cont*: *adm cont*
⟨*proof*⟩

Function application preserves monotonicity and continuity.

**lemma** *mono2mono-fun*: *monofun* $f \Longrightarrow$ *monofun* $(\lambda x. f\ x\ y)$
⟨*proof*⟩

**lemma** *cont2cont-fun*: *cont* $f \Longrightarrow$ *cont* $(\lambda x. f\ x\ y)$

⟨*proof*⟩

**lemma** *cont-fun*: *cont* $(\lambda f.\ f\ x)$
⟨*proof*⟩

Lambda abstraction preserves monotonicity and continuity. (Note $(\lambda x.\ \lambda y.\ f\ x\ y) = f$.)

**lemma** *mono2mono-lambda*:
**assumes** $f$: $\bigwedge y.\ monofun\ (\lambda x.\ f\ x\ y)$ **shows** *monofun f*
⟨*proof*⟩

**lemma** *cont2cont-lambda* [*simp*]:
**assumes** $f$: $\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y)$ **shows** *cont f*
⟨*proof*⟩

What D.A.Schmidt calls continuity of abstraction; never used here

**lemma** *contlub-lambda*:
$(\bigwedge x{::}'a{::}type.\ chain\ (\lambda i.\ S\ i\ x{::}'b{::}cpo))$
$\Longrightarrow (\lambda x.\ \bigsqcup i.\ S\ i\ x) = (\bigsqcup i.\ (\lambda x.\ S\ i\ x))$
⟨*proof*⟩

**end**

# 7 The cpo of cartesian products

**theory** *Product-Cpo*
**imports** *Adm*
**begin**

**default-sort** *cpo*

## 7.1 Unit type is a pcpo

**instantiation** *unit* :: *discrete-cpo*
**begin**

**definition**
*below-unit-def* [*simp*]: $x \sqsubseteq (y{::}unit) \longleftrightarrow True$

**instance** ⟨*proof*⟩

**end**

**instance** *unit* :: *pcpo*
⟨*proof*⟩

## 7.2 Product type is a partial order

**instantiation** *prod* :: (*below*, *below*) *below*

**begin**

**definition**
*below-prod-def*: $(op \sqsubseteq) \equiv \lambda p1\ p2.\ (fst\ p1 \sqsubseteq fst\ p2 \wedge snd\ p1 \sqsubseteq snd\ p2)$

**instance** ⟨*proof*⟩
**end**

**instance** *prod* :: (*po*, *po*) *po*
⟨*proof*⟩

## 7.3 Monotonicity of *Pair*, *fst*, *snd*

**lemma** *prod-belowI*: $[\![fst\ p \sqsubseteq fst\ q;\ snd\ p \sqsubseteq snd\ q]\!] \Longrightarrow p \sqsubseteq q$
⟨*proof*⟩

**lemma** *Pair-below-iff* [*simp*]: $(a,\ b) \sqsubseteq (c,\ d) \longleftrightarrow a \sqsubseteq c \wedge b \sqsubseteq d$
⟨*proof*⟩

Pair (-,-) is monotone in both arguments

**lemma** *monofun-pair1*: $monofun\ (\lambda x.\ (x,\ y))$
⟨*proof*⟩

**lemma** *monofun-pair2*: $monofun\ (\lambda y.\ (x,\ y))$
⟨*proof*⟩

**lemma** *monofun-pair*:
$[\![x1 \sqsubseteq x2;\ y1 \sqsubseteq y2]\!] \Longrightarrow (x1,\ y1) \sqsubseteq (x2,\ y2)$
⟨*proof*⟩

**lemma** *ch2ch-Pair* [*simp*]:
$chain\ X \Longrightarrow chain\ Y \Longrightarrow chain\ (\lambda i.\ (X\ i,\ Y\ i))$
⟨*proof*⟩

*fst* and *snd* are monotone

**lemma** *fst-monofun*: $x \sqsubseteq y \Longrightarrow fst\ x \sqsubseteq fst\ y$
⟨*proof*⟩

**lemma** *snd-monofun*: $x \sqsubseteq y \Longrightarrow snd\ x \sqsubseteq snd\ y$
⟨*proof*⟩

**lemma** *monofun-fst*: *monofun fst*
⟨*proof*⟩

**lemma** *monofun-snd*: *monofun snd*
⟨*proof*⟩

**lemmas** *ch2ch-fst* [*simp*] = *ch2ch-monofun* [*OF monofun-fst*]

**lemmas** *ch2ch-snd* [*simp*] = *ch2ch-monofun* [*OF monofun-snd*]

**lemma** *prod-chain-cases*:
**assumes** *chain Y*
**obtains** *A B*
**where** *chain A* **and** *chain B* **and** $Y = (\lambda i.\ (A\ i,\ B\ i))$
⟨*proof*⟩

## 7.4 Product type is a cpo

**lemma** *is-lub-Pair*:
$[\![range\ A <<|\ x;\ range\ B <<|\ y]\!] \Longrightarrow range\ (\lambda i.\ (A\ i,\ B\ i)) <<|\ (x,\ y)$
⟨*proof*⟩

**lemma** *lub-Pair*:
$[\![chain\ (A::nat \Rightarrow 'a::cpo);\ chain\ (B::nat \Rightarrow 'b::cpo)]\!]$
$\Longrightarrow (\bigsqcup i.\ (A\ i,\ B\ i)) = (\bigsqcup i.\ A\ i,\ \bigsqcup i.\ B\ i)$
⟨*proof*⟩

**lemma** *is-lub-prod*:
**fixes** $S :: nat \Rightarrow ('a::cpo \times 'b::cpo)$
**assumes** *S*: *chain S*
**shows** $range\ S <<|\ (\bigsqcup i.\ fst\ (S\ i),\ \bigsqcup i.\ snd\ (S\ i))$
⟨*proof*⟩

**lemma** *lub-prod*:
$chain\ (S::nat \Rightarrow 'a::cpo \times 'b::cpo)$
$\Longrightarrow (\bigsqcup i.\ S\ i) = (\bigsqcup i.\ fst\ (S\ i),\ \bigsqcup i.\ snd\ (S\ i))$
⟨*proof*⟩

**instance** *prod* :: (*cpo*, *cpo*) *cpo*
⟨*proof*⟩

**instance** *prod* :: (*discrete-cpo*, *discrete-cpo*) *discrete-cpo*
⟨*proof*⟩

## 7.5 Product type is pointed

**lemma** *minimal-prod*: $(\bot,\ \bot) \sqsubseteq p$
⟨*proof*⟩

**instance** *prod* :: (*pcpo*, *pcpo*) *pcpo*
⟨*proof*⟩

**lemma** *inst-prod-pcpo*: $\bot = (\bot,\ \bot)$
⟨*proof*⟩

**lemma** *Pair-bottom-iff* [*simp*]: $(x,\ y) = \bot \longleftrightarrow x = \bot \wedge y = \bot$
⟨*proof*⟩

**lemma** *fst-strict* [*simp*]: *fst* ⊥ = ⊥
⟨*proof*⟩

**lemma** *snd-strict* [*simp*]: *snd* ⊥ = ⊥
⟨*proof*⟩

**lemma** *Pair-strict* [*simp*]: (⊥, ⊥) = ⊥
⟨*proof*⟩

**lemma** *split-strict* [*simp*]: *case-prod f* ⊥ = *f* ⊥ ⊥
⟨*proof*⟩

## 7.6 Continuity of *Pair*, *fst*, *snd*

**lemma** *cont-pair1*: *cont* (λ*x*. (*x*, *y*))
⟨*proof*⟩

**lemma** *cont-pair2*: *cont* (λ*y*. (*x*, *y*))
⟨*proof*⟩

**lemma** *cont-fst*: *cont fst*
⟨*proof*⟩

**lemma** *cont-snd*: *cont snd*
⟨*proof*⟩

**lemma** *cont2cont-Pair* [*simp*, *cont2cont*]:
  **assumes** *f*: *cont* (λ*x*. *f x*)
  **assumes** *g*: *cont* (λ*x*. *g x*)
  **shows** *cont* (λ*x*. (*f x*, *g x*))
⟨*proof*⟩

**lemmas** *cont2cont-fst* [*simp*, *cont2cont*] = *cont-compose* [*OF cont-fst*]

**lemmas** *cont2cont-snd* [*simp*, *cont2cont*] = *cont-compose* [*OF cont-snd*]

**lemma** *cont2cont-case-prod*:
  **assumes** *f1*: ⋀*a b*. *cont* (λ*x*. *f x a b*)
  **assumes** *f2*: ⋀*x b*. *cont* (λ*a*. *f x a b*)
  **assumes** *f3*: ⋀*x a*. *cont* (λ*b*. *f x a b*)
  **assumes** *g*: *cont* (λ*x*. *g x*)
  **shows** *cont* (λ*x*. *case g x of* (*a*, *b*) ⇒ *f x a b*)
⟨*proof*⟩

**lemma** *prod-contI*:
  **assumes** *f1*: ⋀*y*. *cont* (λ*x*. *f* (*x*, *y*))
  **assumes** *f2*: ⋀*x*. *cont* (λ*y*. *f* (*x*, *y*))
  **shows** *cont f*
⟨*proof*⟩

**lemma** *prod-cont-iff*:
$cont\ f \longleftrightarrow (\forall y.\ cont\ (\lambda x.\ f\ (x,\ y))) \wedge (\forall x.\ cont\ (\lambda y.\ f\ (x,\ y)))$
⟨*proof*⟩

**lemma** *cont2cont-case-prod'* [*simp*, *cont2cont*]:
**assumes** *f*: $cont\ (\lambda p.\ f\ (fst\ p)\ (fst\ (snd\ p))\ (snd\ (snd\ p)))$
**assumes** *g*: $cont\ (\lambda x.\ g\ x)$
**shows** $cont\ (\lambda x.\ case\text{-}prod\ (f\ x)\ (g\ x))$
⟨*proof*⟩

The simple version (due to Joachim Breitner) is needed if either element type of the pair is not a cpo.

**lemma** *cont2cont-split-simple* [*simp*, *cont2cont*]:
**assumes** $\bigwedge a\ b.\ cont\ (\lambda x.\ f\ x\ a\ b)$
**shows** $cont\ (\lambda x.\ case\ p\ of\ (a,\ b) \Rightarrow f\ x\ a\ b)$
⟨*proof*⟩

Admissibility of predicates on product types.

**lemma** *adm-case-prod* [*simp*]:
**assumes** $adm\ (\lambda x.\ P\ x\ (fst\ (f\ x))\ (snd\ (f\ x)))$
**shows** $adm\ (\lambda x.\ case\ f\ x\ of\ (a,\ b) \Rightarrow P\ x\ a\ b)$
⟨*proof*⟩

## 7.7 Compactness and chain-finiteness

**lemma** *fst-below-iff*: $fst\ (x::'a \times 'b) \sqsubseteq y \longleftrightarrow x \sqsubseteq (y,\ snd\ x)$
⟨*proof*⟩

**lemma** *snd-below-iff*: $snd\ (x::'a \times 'b) \sqsubseteq y \longleftrightarrow x \sqsubseteq (fst\ x,\ y)$
⟨*proof*⟩

**lemma** *compact-fst*: $compact\ x \Longrightarrow compact\ (fst\ x)$
⟨*proof*⟩

**lemma** *compact-snd*: $compact\ x \Longrightarrow compact\ (snd\ x)$
⟨*proof*⟩

**lemma** *compact-Pair*: $[\![compact\ x;\ compact\ y]\!] \Longrightarrow compact\ (x,\ y)$
⟨*proof*⟩

**lemma** *compact-Pair-iff* [*simp*]: $compact\ (x,\ y) \longleftrightarrow compact\ x \wedge compact\ y$
⟨*proof*⟩

**instance** *prod* :: (*chfin*, *chfin*) *chfin*
⟨*proof*⟩

**end**

# 8 The type of continuous functions

**theory** *Cfun*
**imports** *Cpodef Fun-Cpo Product-Cpo*
**begin**

**default-sort** *cpo*

## 8.1 Definition of continuous function type

**definition** *cfun* = {*f*::'*a* => '*b*. *cont f*}

**cpodef** ('*a*, '*b*) *cfun* ((- →/ -) [*1*, *0*] *0*) = *cfun* :: ('*a* => '*b*) *set*
⟨*proof*⟩

**type-notation** (*ASCII*)
*cfun* (**infixr** –> *0*)

**notation** (*ASCII*)
*Rep-cfun* ((-$/-) [*999*,*1000*] *999*)

**notation**
*Rep-cfun* ((-·/-) [*999*,*1000*] *999*)

## 8.2 Syntax for continuous lambda abstraction

**syntax** *-cabs* :: [*logic*, *logic*] ⇒ *logic*

⟨*ML*⟩

Syntax for nested abstractions

**syntax** (*ASCII*)
*-Lambda* :: [*cargs*, *logic*] ⇒ *logic* ((*3LAM* -./ -) [*1000*, *10*] *10*)

**syntax**
*-Lambda* :: [*cargs*, *logic*] ⇒ *logic* ((*3*Λ -./ -) [*1000*, *10*] *10*)

⟨*ML*⟩

Dummy patterns for continuous abstraction

**translations**
Λ -. *t* => *CONST Abs-cfun* (λ -. *t*)

## 8.3 Continuous function space is pointed

**lemma** *bottom-cfun*: ⊥ ∈ *cfun*
⟨*proof*⟩

**instance** *cfun* :: (*cpo*, *discrete-cpo*) *discrete-cpo*

⟨*proof*⟩

**instance** *cfun* :: (*cpo*, *pcpo*) *pcpo*
⟨*proof*⟩

**lemmas** *Rep-cfun-strict* =
*typedef-Rep-strict* [*OF type-definition-cfun below-cfun-def bottom-cfun*]

**lemmas** *Abs-cfun-strict* =
*typedef-Abs-strict* [*OF type-definition-cfun below-cfun-def bottom-cfun*]

function application is strict in its first argument

**lemma** *Rep-cfun-strict1* [*simp*]: $\bot \cdot x = \bot$
⟨*proof*⟩

**lemma** *LAM-strict* [*simp*]: $(\Lambda\ x.\ \bot) = \bot$
⟨*proof*⟩

for compatibility with old HOLCF-Version

**lemma** *inst-cfun-pcpo*: $\bot = (\Lambda\ x.\ \bot)$
⟨*proof*⟩

## 8.4 Basic properties of continuous functions

Beta-equality for continuous functions

**lemma** *Abs-cfun-inverse2*: *cont f* $\Longrightarrow$ *Rep-cfun* (*Abs-cfun f*) = *f*
⟨*proof*⟩

**lemma** *beta-cfun*: *cont f* $\Longrightarrow (\Lambda\ x.\ f\ x) \cdot u = f\ u$
⟨*proof*⟩

Beta-reduction simproc

Given the term $(\Lambda\ x.\ f\ x) \cdot y$, the procedure tries to construct the theorem $(\Lambda\ x.\ f\ x) \cdot y \equiv f\ y$. If this theorem cannot be completely solved by the cont2cont rules, then the procedure returns the ordinary conditional *beta-cfun* rule.

The simproc does not solve any more goals that would be solved by using *beta-cfun* as a simp rule. The advantage of the simproc is that it can avoid deeply-nested calls to the simplifier that would otherwise be caused by large continuity side conditions.

Update: The simproc now uses rule *Abs-cfun-inverse2* instead of *beta-cfun*, to avoid problems with eta-contraction.

⟨*ML*⟩

Eta-equality for continuous functions

**lemma** *eta-cfun*: $(\Lambda\ x.\ f \cdot x) = f$
⟨*proof*⟩

Extensionality for continuous functions

**lemma** *cfun-eq-iff*: $f = g \longleftrightarrow (\forall x.\ f\cdot x = g\cdot x)$
⟨*proof*⟩

**lemma** *cfun-eqI*: $(\bigwedge x.\ f\cdot x = g\cdot x) \Longrightarrow f = g$
⟨*proof*⟩

Extensionality wrt. ordering for continuous functions

**lemma** *cfun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall x.\ f\cdot x \sqsubseteq g\cdot x)$
⟨*proof*⟩

**lemma** *cfun-belowI*: $(\bigwedge x.\ f\cdot x \sqsubseteq g\cdot x) \Longrightarrow f \sqsubseteq g$
⟨*proof*⟩

Congruence for continuous function application

**lemma** *cfun-cong*: $[\![f = g;\ x = y]\!] \Longrightarrow f\cdot x = g\cdot y$
⟨*proof*⟩

**lemma** *cfun-fun-cong*: $f = g \Longrightarrow f\cdot x = g\cdot x$
⟨*proof*⟩

**lemma** *cfun-arg-cong*: $x = y \Longrightarrow f\cdot x = f\cdot y$
⟨*proof*⟩

## 8.5 Continuity of application

**lemma** *cont-Rep-cfun1*: *cont* $(\lambda f.\ f\cdot x)$
⟨*proof*⟩

**lemma** *cont-Rep-cfun2*: *cont* $(\lambda x.\ f\cdot x)$
⟨*proof*⟩

**lemmas** *monofun-Rep-cfun* = *cont-Rep-cfun* [*THEN cont2mono*]

**lemmas** *monofun-Rep-cfun1* = *cont-Rep-cfun1* [*THEN cont2mono*]
**lemmas** *monofun-Rep-cfun2* = *cont-Rep-cfun2* [*THEN cont2mono*]

contlub, cont properties of *Rep-cfun* in each argument

**lemma** *contlub-cfun-arg*: *chain* $Y \Longrightarrow f\cdot(\bigsqcup i.\ Y\ i) = (\bigsqcup i.\ f\cdot(Y\ i))$
⟨*proof*⟩

**lemma** *contlub-cfun-fun*: *chain* $F \Longrightarrow (\bigsqcup i.\ F\ i)\cdot x = (\bigsqcup i.\ F\ i\cdot x)$
⟨*proof*⟩

monotonicity of application

**lemma** *monofun-cfun-fun*: $f \sqsubseteq g \Longrightarrow f\cdot x \sqsubseteq g\cdot x$
⟨*proof*⟩

**lemma** *monofun-cfun-arg*: $x \sqsubseteq y \Longrightarrow f{\cdot}x \sqsubseteq f{\cdot}y$
⟨*proof*⟩

**lemma** *monofun-cfun*: $[\![f \sqsubseteq g;\ x \sqsubseteq y]\!] \Longrightarrow f{\cdot}x \sqsubseteq g{\cdot}y$
⟨*proof*⟩

ch2ch - rules for the type $'a \rightarrow 'b$

**lemma** *chain-monofun*: $chain\ Y \Longrightarrow chain\ (\lambda i.\ f{\cdot}(Y\ i))$
⟨*proof*⟩

**lemma** *ch2ch-Rep-cfunR*: $chain\ Y \Longrightarrow chain\ (\lambda i.\ f{\cdot}(Y\ i))$
⟨*proof*⟩

**lemma** *ch2ch-Rep-cfunL*: $chain\ F \Longrightarrow chain\ (\lambda i.\ (F\ i){\cdot}x)$
⟨*proof*⟩

**lemma** *ch2ch-Rep-cfun* [*simp*]:
$[\![chain\ F;\ chain\ Y]\!] \Longrightarrow chain\ (\lambda i.\ (F\ i){\cdot}(Y\ i))$
⟨*proof*⟩

**lemma** *ch2ch-LAM* [*simp*]:
$[\![\bigwedge x.\ chain\ (\lambda i.\ S\ i\ x);\ \bigwedge i.\ cont\ (\lambda x.\ S\ i\ x)]\!] \Longrightarrow chain\ (\lambda i.\ \Lambda\ x.\ S\ i\ x)$
⟨*proof*⟩

contlub, cont properties of *Rep-cfun* in both arguments

**lemma** *lub-APP*:
$[\![chain\ F;\ chain\ Y]\!] \Longrightarrow (\bigsqcup i.\ F\ i{\cdot}(Y\ i)) = (\bigsqcup i.\ F\ i){\cdot}(\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

**lemma** *lub-LAM*:
$[\![\bigwedge x.\ chain\ (\lambda i.\ F\ i\ x);\ \bigwedge i.\ cont\ (\lambda x.\ F\ i\ x)]\!]$
$\Longrightarrow (\bigsqcup i.\ \Lambda\ x.\ F\ i\ x) = (\Lambda\ x.\ \bigsqcup i.\ F\ i\ x)$
⟨*proof*⟩

**lemmas** *lub-distribs* = *lub-APP lub-LAM*

strictness

**lemma** *strictI*: $f{\cdot}x = \bot \Longrightarrow f{\cdot}\bot = \bot$
⟨*proof*⟩

type $'a \rightarrow 'b$ is chain complete

**lemma** *lub-cfun*: $chain\ F \Longrightarrow (\bigsqcup i.\ F\ i) = (\Lambda\ x.\ \bigsqcup i.\ F\ i{\cdot}x)$
⟨*proof*⟩

## 8.6 Continuity simplification procedure

cont2cont lemma for *Rep-cfun*

**lemma** *cont2cont-APP* [*simp*, *cont2cont*]:

**assumes** *f*: *cont* $(\lambda x.\ f\ x)$
**assumes** *t*: *cont* $(\lambda x.\ t\ x)$
**shows** *cont* $(\lambda x.\ (f\ x)\cdot(t\ x))$
⟨*proof*⟩

Two specific lemmas for the combination of LCF and HOL terms. These lemmas are needed in theories that use types like $'a \to 'b \Rightarrow 'c$.

**lemma** *cont-APP-app* [*simp*]: $[\![cont\ f;\ cont\ g]\!] \Longrightarrow cont\ (\lambda x.\ ((f\ x)\cdot(g\ x))\ s)$
⟨*proof*⟩

**lemma** *cont-APP-app-app* [*simp*]: $[\![cont\ f;\ cont\ g]\!] \Longrightarrow cont\ (\lambda x.\ ((f\ x)\cdot(g\ x))\ s\ t)$
⟨*proof*⟩

cont2mono Lemma for $\lambda x.\ \Lambda\ y.\ c1\ x\ y$

**lemma** *cont2mono-LAM*:
$[\![\bigwedge x.\ cont\ (\lambda y.\ f\ x\ y);\ \bigwedge y.\ monofun\ (\lambda x.\ f\ x\ y)]\!]$
$\Longrightarrow monofun\ (\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

cont2cont Lemma for $\lambda x.\ \Lambda\ y.\ f\ x\ y$

Not suitable as a cont2cont rule, because on nested lambdas it causes exponential blow-up in the number of subgoals.

**lemma** *cont2cont-LAM*:
**assumes** *f1*: $\bigwedge x.\ cont\ (\lambda y.\ f\ x\ y)$
**assumes** *f2*: $\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y)$
**shows** *cont* $(\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

This version does work as a cont2cont rule, since it has only a single subgoal.

**lemma** *cont2cont-LAM'* [*simp*, *cont2cont*]:
**fixes** $f :: 'a::cpo \Rightarrow 'b::cpo \Rightarrow 'c::cpo$
**assumes** *f*: *cont* $(\lambda p.\ f\ (fst\ p)\ (snd\ p))$
**shows** *cont* $(\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

**lemma** *cont2cont-LAM-discrete* [*simp*, *cont2cont*]:
$(\bigwedge y::'a::discrete\text{-}cpo.\ cont\ (\lambda x.\ f\ x\ y)) \Longrightarrow cont\ (\lambda x.\ \Lambda\ y.\ f\ x\ y)$
⟨*proof*⟩

## 8.7 Miscellaneous

Monotonicity of *Abs-cfun*

**lemma** *monofun-LAM*:
$[\![cont\ f;\ cont\ g;\ \bigwedge x.\ f\ x \sqsubseteq g\ x]\!] \Longrightarrow (\Lambda\ x.\ f\ x) \sqsubseteq (\Lambda\ x.\ g\ x)$
⟨*proof*⟩

some lemmata for functions with flat/chfin domain/range types

**lemma** *chfin-Rep-cfunR*: *chain* $(Y::nat => 'a::cpo->'b::chfin)$
$==> !s.\ ?\ n.\ (LUB\ i.\ Y\ i)\$s = Y\ n\$s$
⟨*proof*⟩

**lemma** *adm-chfindom*: *adm* $(\lambda(u::'a::cpo \to 'b::chfin).\ P(u\cdot s))$
⟨*proof*⟩

## 8.8 Continuous injection-retraction pairs

Continuous retractions are strict.

**lemma** *retraction-strict*:
$\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow f\cdot\bot = \bot$
⟨*proof*⟩

**lemma** *injection-eq*:
$\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow (g\cdot x = g\cdot y) = (x = y)$
⟨*proof*⟩

**lemma** *injection-below*:
$\forall x.\ f\cdot(g\cdot x) = x \Longrightarrow (g\cdot x \sqsubseteq g\cdot y) = (x \sqsubseteq y)$
⟨*proof*⟩

**lemma** *injection-defined-rev*:
$[\![\forall x.\ f\cdot(g\cdot x) = x;\ g\cdot z = \bot]\!] \Longrightarrow z = \bot$
⟨*proof*⟩

**lemma** *injection-defined*:
$[\![\forall x.\ f\cdot(g\cdot x) = x;\ z \neq \bot]\!] \Longrightarrow g\cdot z \neq \bot$
⟨*proof*⟩

a result about functions with flat codomain

**lemma** *flat-eqI*: $[\![(x::'a::flat) \sqsubseteq y;\ x \neq \bot]\!] \Longrightarrow x = y$
⟨*proof*⟩

**lemma** *flat-codom*:
$f\cdot x = (c::'b::flat) \Longrightarrow f\cdot\bot = \bot \lor (\forall z.\ f\cdot z = c)$
⟨*proof*⟩

## 8.9 Identity and composition

**definition**
*ID* :: $'a \to 'a$ **where**
*ID* = $(\Lambda\ x.\ x)$

**definition**
*cfcomp* :: $('b \to 'c) \to ('a \to 'b) \to 'a \to 'c$ **where**
*oo-def*: *cfcomp* = $(\Lambda\ f\ g\ x.\ f\cdot(g\cdot x))$

**abbreviation**

*cfcomp-syn* :: $['b \to 'c,\ 'a \to 'b] \Rightarrow 'a \to 'c$ (**infixr** *oo 100*) **where**
*f oo g* == *cfcomp·f·g*

**lemma** *ID1* [*simp*]: *ID·x* = *x*
⟨*proof*⟩

**lemma** *cfcomp1*: (*f oo g*) = (Λ *x*. *f*·(*g*·*x*))
⟨*proof*⟩

**lemma** *cfcomp2* [*simp*]: (*f oo g*)·*x* = *f*·(*g*·*x*)
⟨*proof*⟩

**lemma** *cfcomp-LAM*: *cont g* ⟹ *f oo* (Λ *x*. *g x*) = (Λ *x*. *f*·(*g x*))
⟨*proof*⟩

**lemma** *cfcomp-strict* [*simp*]: ⊥ *oo f* = ⊥
⟨*proof*⟩

Show that interpretation of (pcpo,-->-) is a category. The class of objects is interpretation of syntactical class pcpo. The class of arrows between objects $'a$ and $'b$ is interpret. of $'a \to 'b$. The identity arrow is interpretation of *ID*. The composition of f and g is interpretation of *oo*.

**lemma** *ID2* [*simp*]: *f oo ID* = *f*
⟨*proof*⟩

**lemma** *ID3* [*simp*]: *ID oo f* = *f*
⟨*proof*⟩

**lemma** *assoc-oo*: *f oo* (*g oo h*) = (*f oo g*) *oo h*
⟨*proof*⟩

## 8.10 Strictified functions

**default-sort** *pcpo*

**definition**
*seq* :: $'a \to 'b \to 'b$ **where**
*seq* = (Λ *x*. *if x* = ⊥ *then* ⊥ *else ID*)

**lemma** *cont2cont-if-bottom* [*cont2cont*, *simp*]:
**assumes** *f*: *cont* (λ*x*. *f x*) **and** *g*: *cont* (λ*x*. *g x*)
**shows** *cont* (λ*x*. *if f x* = ⊥ *then* ⊥ *else g x*)
⟨*proof*⟩

**lemma** *seq-conv-if*: *seq·x* = (*if x* = ⊥ *then* ⊥ *else ID*)
⟨*proof*⟩

**lemma** *seq-simps* [*simp*]:
*seq*·⊥ = ⊥

$seq \cdot x \cdot \bot = \bot$
$x \neq \bot \Longrightarrow seq \cdot x = ID$
⟨*proof*⟩

**definition**
*strictify* :: $('a \rightarrow 'b) \rightarrow 'a \rightarrow 'b$ **where**
$strictify = (\Lambda\ f\ x.\ seq \cdot x \cdot (f \cdot x))$

**lemma** *strictify-conv-if*: $strictify \cdot f \cdot x = (if\ x = \bot\ then\ \bot\ else\ f \cdot x)$
⟨*proof*⟩

**lemma** *strictify1* [*simp*]: $strictify \cdot f \cdot \bot = \bot$
⟨*proof*⟩

**lemma** *strictify2* [*simp*]: $x \neq \bot \Longrightarrow strictify \cdot f \cdot x = f \cdot x$
⟨*proof*⟩

### 8.11 Continuity of let-bindings

**lemma** *cont2cont-Let*:
**assumes** *f*: $cont\ (\lambda x.\ f\ x)$
**assumes** *g1*: $\bigwedge y.\ cont\ (\lambda x.\ g\ x\ y)$
**assumes** *g2*: $\bigwedge x.\ cont\ (\lambda y.\ g\ x\ y)$
**shows** $cont\ (\lambda x.\ let\ y = f\ x\ in\ g\ x\ y)$
⟨*proof*⟩

**lemma** *cont2cont-Let'* [*simp*, *cont2cont*]:
**assumes** *f*: $cont\ (\lambda x.\ f\ x)$
**assumes** *g*: $cont\ (\lambda p.\ g\ (fst\ p)\ (snd\ p))$
**shows** $cont\ (\lambda x.\ let\ y = f\ x\ in\ g\ x\ y)$
⟨*proof*⟩

The simple version (suggested by Joachim Breitner) is needed if the type of the defined term is not a cpo.

**lemma** *cont2cont-Let-simple* [*simp*, *cont2cont*]:
**assumes** $\bigwedge y.\ cont\ (\lambda x.\ g\ x\ y)$
**shows** $cont\ (\lambda x.\ let\ y = t\ in\ g\ x\ y)$
⟨*proof*⟩

**end**

# 9 The Strict Function Type

**theory** *Sfun*
**imports** *Cfun*
**begin**

**pcpodef** (*'a*, *'b*) *sfun* (**infixr** →! *0*)
$= \{f :: 'a \rightarrow 'b.\ f \cdot \bot = \bot\}$

⟨*proof*⟩

**type-notation** (*ASCII*)
  *sfun* (**infixr** −>! *0*)

TODO: Define nice syntax for abstraction, application.

**definition**
  *sfun-abs* :: (′*a* → ′*b*) → (′*a* →! ′*b*)
**where**
  *sfun-abs* = (Λ *f*. *Abs-sfun* (*strictify*·*f*))

**definition**
  *sfun-rep* :: (′*a* →! ′*b*) → ′*a* → ′*b*
**where**
  *sfun-rep* = (Λ *f*. *Rep-sfun f*)

**lemma** *sfun-rep-beta*: *sfun-rep*·*f* = *Rep-sfun f*
  ⟨*proof*⟩

**lemma** *sfun-rep-strict1* [*simp*]: *sfun-rep*·⊥ = ⊥
  ⟨*proof*⟩

**lemma** *sfun-rep-strict2* [*simp*]: *sfun-rep*·*f*·⊥ = ⊥
  ⟨*proof*⟩

**lemma** *strictify-cancel*: *f*·⊥ = ⊥ ⟹ *strictify*·*f* = *f*
  ⟨*proof*⟩

**lemma** *sfun-abs-sfun-rep* [*simp*]: *sfun-abs*·(*sfun-rep*·*f*) = *f*
  ⟨*proof*⟩

**lemma** *sfun-rep-sfun-abs* [*simp*]: *sfun-rep*·(*sfun-abs*·*f*) = *strictify*·*f*
  ⟨*proof*⟩

**lemma** *sfun-eq-iff*: *f* = *g* ⟷ *sfun-rep*·*f* = *sfun-rep*·*g*
⟨*proof*⟩

**lemma** *sfun-below-iff*: *f* ⊑ *g* ⟷ *sfun-rep*·*f* ⊑ *sfun-rep*·*g*
⟨*proof*⟩

**end**

# 10 The cpo of cartesian products

**theory** *Cprod*
**imports** *Cfun*
**begin**

**default-sort** *cpo*

### 10.1 Continuous case function for unit type

**definition**
*unit-when* :: $'a \rightarrow unit \rightarrow 'a$ **where**
*unit-when* = $(\Lambda\ a\ \text{-}.\ a)$

**translations**
$\Lambda().\ t == CONST\ unit\text{-}when{\cdot}t$

**lemma** *unit-when* [*simp*]: $unit\text{-}when{\cdot}a{\cdot}u = a$
⟨*proof*⟩

### 10.2 Continuous version of split function

**definition**
*csplit* :: $('a \rightarrow 'b \rightarrow 'c) \rightarrow ('a * 'b) \rightarrow 'c$ **where**
*csplit* = $(\Lambda\ f\ p.\ f{\cdot}(fst\ p){\cdot}(snd\ p))$

**translations**
$\Lambda(CONST\ Pair\ x\ y).\ t == CONST\ csplit{\cdot}(\Lambda\ x\ y.\ t)$

**abbreviation**
*cfst* :: $'a \times 'b \rightarrow 'a$ **where**
$cfst \equiv Abs\text{-}cfun\ fst$

**abbreviation**
*csnd* :: $'a \times 'b \rightarrow 'b$ **where**
$csnd \equiv Abs\text{-}cfun\ snd$

### 10.3 Convert all lemmas to the continuous versions

**lemma** *csplit1* [*simp*]: $csplit{\cdot}f{\cdot}\bot = f{\cdot}\bot{\cdot}\bot$
⟨*proof*⟩

**lemma** *csplit-Pair* [*simp*]: $csplit{\cdot}f{\cdot}(x,\ y) = f{\cdot}x{\cdot}y$
⟨*proof*⟩

**end**

# 11 The type of strict products

**theory** *Sprod*
**imports** *Cfun*
**begin**

**default-sort** *pcpo*

### 11.1 Definition of strict product type

**definition** $sprod = \{p{::}'a \times 'b.\ p = \bot \vee (fst\ p \neq \bot \wedge snd\ p \neq \bot)\}$

**pcpodef** (*'a*, *'b*) *sprod* ((- ⊗/ -) [*21*,*20*] *20*) = *sprod* :: (*'a* × *'b*) *set*
⟨*proof*⟩

**instance** *sprod* :: ({*chfin*,*pcpo*}, {*chfin*,*pcpo*}) *chfin*
⟨*proof*⟩

**type-notation** (*ASCII*)
*sprod* (**infixr** ∗∗ *20*)

## 11.2 Definitions of constants

**definition**
*sfst* :: (*'a* ∗∗ *'b*) → *'a* **where**
*sfst* = (Λ *p*. *fst* (*Rep-sprod p*))

**definition**
*ssnd* :: (*'a* ∗∗ *'b*) → *'b* **where**
*ssnd* = (Λ *p*. *snd* (*Rep-sprod p*))

**definition**
*spair* :: *'a* → *'b* → (*'a* ∗∗ *'b*) **where**
*spair* = (Λ *a b*. *Abs-sprod* (*seq*·*b*·*a*, *seq*·*a*·*b*))

**definition**
*ssplit* :: (*'a* → *'b* → *'c*) → (*'a* ∗∗ *'b*) → *'c* **where**
*ssplit* = (Λ *f p*. *seq*·*p*·(*f*·(*sfst*·*p*)·(*ssnd*·*p*)))

**syntax**
*-stuple* :: [*logic*, *args*] ⇒ *logic* ((*1*'(:-,/ -:')))

**translations**
(:*x*, *y*, *z*:) == (:*x*, (:*y*, *z*:):)
(:*x*, *y*:) == *CONST spair*·*x*·*y*

**translations**
Λ(*CONST spair*·*x*·*y*). *t* == *CONST ssplit*·(Λ *x y*. *t*)

## 11.3 Case analysis

**lemma** *spair-sprod*: (*seq*·*b*·*a*, *seq*·*a*·*b*) ∈ *sprod*
⟨*proof*⟩

**lemma** *Rep-sprod-spair*: *Rep-sprod* (:*a*, *b*:) = (*seq*·*b*·*a*, *seq*·*a*·*b*)
⟨*proof*⟩

**lemmas** *Rep-sprod-simps* =
*Rep-sprod-inject* [*symmetric*] *below-sprod-def*
*prod-eq-iff below-prod-def*
*Rep-sprod-strict Rep-sprod-spair*

**lemma** *sprodE* [*case-names bottom spair*, *cases type*: *sprod*]:
**obtains** $p = \bot \mid x\ y$ **where** $p = (:x,\ y:)$ **and** $x \neq \bot$ **and** $y \neq \bot$
⟨*proof*⟩

**lemma** *sprod-induct* [*case-names bottom spair*, *induct type*: *sprod*]:
$[\![P\ \bot;\ \bigwedge x\ y.\ [\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow P\ (:x,\ y:)]\!] \Longrightarrow P\ x$
⟨*proof*⟩

## 11.4 Properties of *spair*

**lemma** *spair-strict1* [*simp*]: $(:\bot,\ y:) = \bot$
⟨*proof*⟩

**lemma** *spair-strict2* [*simp*]: $(:x,\ \bot:) = \bot$
⟨*proof*⟩

**lemma** *spair-bottom-iff* [*simp*]: $((:x,\ y:) = \bot) = (x = \bot \vee y = \bot)$
⟨*proof*⟩

**lemma** *spair-below-iff*:
$((:a,\ b:) \sqsubseteq (:c,\ d:)) = (a = \bot \vee b = \bot \vee (a \sqsubseteq c \wedge b \sqsubseteq d))$
⟨*proof*⟩

**lemma** *spair-eq-iff*:
$((:a,\ b:) = (:c,\ d:)) =$
$(a = c \wedge b = d \vee (a = \bot \vee b = \bot) \wedge (c = \bot \vee d = \bot))$
⟨*proof*⟩

**lemma** *spair-strict*: $x = \bot \vee y = \bot \Longrightarrow (:x,\ y:) = \bot$
⟨*proof*⟩

**lemma** *spair-strict-rev*: $(:x,\ y:) \neq \bot \Longrightarrow x \neq \bot \wedge y \neq \bot$
⟨*proof*⟩

**lemma** *spair-defined*: $[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow (:x,\ y:) \neq \bot$
⟨*proof*⟩

**lemma** *spair-defined-rev*: $(:x,\ y:) = \bot \Longrightarrow x = \bot \vee y = \bot$
⟨*proof*⟩

**lemma** *spair-below*:
$[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow (:x,\ y:) \sqsubseteq (:a,\ b:) = (x \sqsubseteq a \wedge y \sqsubseteq b)$
⟨*proof*⟩

**lemma** *spair-eq*:
$[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow ((:x,\ y:) = (:a,\ b:)) = (x = a \wedge y = b)$
⟨*proof*⟩

**lemma** *spair-inject*:
$[\![x \neq \bot; y \neq \bot; (:x, y:) = (:a, b:)]\!] \Longrightarrow x = a \wedge y = b$
⟨*proof*⟩

**lemma** *inst-sprod-pcpo2*: $\bot = (:\bot, \bot:)$
⟨*proof*⟩

**lemma** *sprodE2*: $(\bigwedge x\ y.\ p = (:x, y:) \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

## 11.5 Properties of *sfst* and *ssnd*

**lemma** *sfst-strict* [*simp*]: $sfst \cdot \bot = \bot$
⟨*proof*⟩

**lemma** *ssnd-strict* [*simp*]: $ssnd \cdot \bot = \bot$
⟨*proof*⟩

**lemma** *sfst-spair* [*simp*]: $y \neq \bot \Longrightarrow sfst \cdot (:x, y:) = x$
⟨*proof*⟩

**lemma** *ssnd-spair* [*simp*]: $x \neq \bot \Longrightarrow ssnd \cdot (:x, y:) = y$
⟨*proof*⟩

**lemma** *sfst-bottom-iff* [*simp*]: $(sfst \cdot p = \bot) = (p = \bot)$
⟨*proof*⟩

**lemma** *ssnd-bottom-iff* [*simp*]: $(ssnd \cdot p = \bot) = (p = \bot)$
⟨*proof*⟩

**lemma** *sfst-defined*: $p \neq \bot \Longrightarrow sfst \cdot p \neq \bot$
⟨*proof*⟩

**lemma** *ssnd-defined*: $p \neq \bot \Longrightarrow ssnd \cdot p \neq \bot$
⟨*proof*⟩

**lemma** *spair-sfst-ssnd*: $(:sfst \cdot p, ssnd \cdot p:) = p$
⟨*proof*⟩

**lemma** *below-sprod*: $(x \sqsubseteq y) = (sfst \cdot x \sqsubseteq sfst \cdot y \wedge ssnd \cdot x \sqsubseteq ssnd \cdot y)$
⟨*proof*⟩

**lemma** *eq-sprod*: $(x = y) = (sfst \cdot x = sfst \cdot y \wedge ssnd \cdot x = ssnd \cdot y)$
⟨*proof*⟩

**lemma** *sfst-below-iff*: $sfst \cdot x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:y, ssnd \cdot x:)$
⟨*proof*⟩

**lemma** *ssnd-below-iff*: $ssnd \cdot x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:sfst \cdot x, y:)$

⟨*proof*⟩

### 11.6 Compactness

**lemma** *compact-sfst*: *compact x* $\Longrightarrow$ *compact* (*sfst*·*x*)
⟨*proof*⟩

**lemma** *compact-ssnd*: *compact x* $\Longrightarrow$ *compact* (*ssnd*·*x*)
⟨*proof*⟩

**lemma** *compact-spair*: ⟦*compact x*; *compact y*⟧ $\Longrightarrow$ *compact* (:*x*, *y*:)
⟨*proof*⟩

**lemma** *compact-spair-iff*:
*compact* (:*x*, *y*:) = ($x = \bot \lor y = \bot \lor$ (*compact x* $\land$ *compact y*))
⟨*proof*⟩

### 11.7 Properties of *ssplit*

**lemma** *ssplit1* [*simp*]: *ssplit*·*f*·$\bot = \bot$
⟨*proof*⟩

**lemma** *ssplit2* [*simp*]: ⟦$x \neq \bot$; $y \neq \bot$⟧ $\Longrightarrow$ *ssplit*·*f*·(:*x*, *y*:) = *f*·*x*·*y*
⟨*proof*⟩

**lemma** *ssplit3* [*simp*]: *ssplit*·*spair*·*z* = *z*
⟨*proof*⟩

### 11.8 Strict product preserves flatness

**instance** *sprod* :: (*flat*, *flat*) *flat*
⟨*proof*⟩

**end**

## 12 Discrete cpo types

**theory** *Discrete*
**imports** *Cont*
**begin**

**datatype** *'a discr* = *Discr 'a* :: *type*

### 12.1 Discrete cpo class instance

**instantiation** *discr* :: (*type*) *discrete-cpo*
**begin**

**definition**

$(op \sqsubseteq :: 'a\ discr \Rightarrow 'a\ discr \Rightarrow bool) = (op =)$

**instance**
⟨*proof*⟩

**end**

### 12.2 *undiscr*

**definition**
*undiscr* :: (*'a*::*type*)*discr* => *'a* **where**
*undiscr x* = (*case x of Discr y* => *y*)

**lemma** *undiscr-Discr* [*simp*]: *undiscr* (*Discr x*) = *x*
⟨*proof*⟩

**lemma** *Discr-undiscr* [*simp*]: *Discr* (*undiscr y*) = *y*
⟨*proof*⟩

**end**

# 13 The type of lifted values

**theory** *Up*
**imports** *Cfun*
**begin**

**default-sort** *cpo*

## 13.1 Definition of new type for lifting

**datatype** *'a u* ((-$_\perp$) [*1000*] *999*) = *Ibottom* | *Iup 'a*

**primrec** *Ifup* :: $('a \rightarrow 'b::pcpo) \Rightarrow 'a\ u \Rightarrow 'b$ **where**
*Ifup f Ibottom* = $\perp$
| *Ifup f* (*Iup x*) = $f \cdot x$

## 13.2 Ordering on lifted cpo

**instantiation** *u* :: (*cpo*) *below*
**begin**

**definition**
*below-up-def*:
$(op \sqsubseteq) \equiv (\lambda x\ y.\ case\ x\ of\ Ibottom \Rightarrow True \mid Iup\ a \Rightarrow$
$(case\ y\ of\ Ibottom \Rightarrow False \mid Iup\ b \Rightarrow a \sqsubseteq b))$

**instance** ⟨*proof*⟩
**end**

**lemma** *minimal-up* [*iff*]: *Ibottom* $\sqsubseteq$ *z*
⟨*proof*⟩

**lemma** *not-Iup-below* [*iff*]: *Iup x* $\not\sqsubseteq$ *Ibottom*
⟨*proof*⟩

**lemma** *Iup-below* [*iff*]: (*Iup x* $\sqsubseteq$ *Iup y*) = (*x* $\sqsubseteq$ *y*)
⟨*proof*⟩

## 13.3 Lifted cpo is a partial order

**instance** *u* :: (*cpo*) *po*
⟨*proof*⟩

## 13.4 Lifted cpo is a cpo

**lemma** *is-lub-Iup*:
*range S* <<| *x* $\Longrightarrow$ *range* ($\lambda i$. *Iup* (*S i*)) <<| *Iup x*
⟨*proof*⟩

**lemma** *up-chain-lemma*:
**assumes** *Y*: *chain Y* **obtains** $\forall i$. *Y i* = *Ibottom*
| *A k* **where** $\forall i$. *Iup* (*A i*) = *Y* (*i* + *k*) **and** *chain A* **and** *range Y* <<| *Iup* ($\bigsqcup i$. *A i*)
⟨*proof*⟩

**instance** *u* :: (*cpo*) *cpo*
⟨*proof*⟩

## 13.5 Lifted cpo is pointed

**instance** *u* :: (*cpo*) *pcpo*
⟨*proof*⟩

for compatibility with old HOLCF-Version

**lemma** *inst-up-pcpo*: $\bot$ = *Ibottom*
⟨*proof*⟩

## 13.6 Continuity of *Iup* and *Ifup*

continuity for *Iup*

**lemma** *cont-Iup*: *cont Iup*
⟨*proof*⟩

continuity for *Ifup*

**lemma** *cont-Ifup1*: *cont* ($\lambda f$. *Ifup f x*)
⟨*proof*⟩

**lemma** *monofun-Ifup2*: $monofun\ (\lambda x.\ Ifup\ f\ x)$
⟨*proof*⟩

**lemma** *cont-Ifup2*: $cont\ (\lambda x.\ Ifup\ f\ x)$
⟨*proof*⟩

## 13.7 Continuous versions of constants

**definition**
$up\ ::\ 'a \to 'a\ u$ **where**
$up = (\Lambda\ x.\ Iup\ x)$

**definition**
$fup :: ('a \to 'b{::}pcpo) \to 'a\ u \to 'b$ **where**
$fup = (\Lambda\ f\ p.\ Ifup\ f\ p)$

**translations**
$case\ l\ of\ XCONST\ up \cdot x \Rightarrow t == CONST\ fup \cdot (\Lambda\ x.\ t) \cdot l$
$case\ l\ of\ (XCONST\ up :: 'a) \cdot x \Rightarrow t => CONST\ fup \cdot (\Lambda\ x.\ t) \cdot l$
$\Lambda(XCONST\ up \cdot x).\ t == CONST\ fup \cdot (\Lambda\ x.\ t)$

continuous versions of lemmas for $'a_{\bot}$

**lemma** *Exh-Up*: $z = \bot \vee (\exists x.\ z = up \cdot x)$
⟨*proof*⟩

**lemma** *up-eq* [*simp*]: $(up \cdot x = up \cdot y) = (x = y)$
⟨*proof*⟩

**lemma** *up-inject*: $up \cdot x = up \cdot y \Longrightarrow x = y$
⟨*proof*⟩

**lemma** *up-defined* [*simp*]: $up \cdot x \neq \bot$
⟨*proof*⟩

**lemma** *not-up-less-UU*: $up \cdot x \not\sqsubseteq \bot$
⟨*proof*⟩

**lemma** *up-below* [*simp*]: $up \cdot x \sqsubseteq up \cdot y \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

**lemma** *upE* [*case-names bottom up*, *cases type*: *u*]:
$[\![p = \bot \Longrightarrow Q;\ \bigwedge x.\ p = up \cdot x \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

**lemma** *up-induct* [*case-names bottom up*, *induct type*: *u*]:
$[\![P\ \bot;\ \bigwedge x.\ P\ (up \cdot x)]\!] \Longrightarrow P\ x$
⟨*proof*⟩

lifting preserves chain-finiteness

**lemma** *up-chain-cases*:
**assumes** $Y$: *chain* $Y$ **obtains** $\forall\, i.\ Y\ i = \bot$
| $A\ k$ **where** $\forall\, i.\ up \cdot (A\ i) = Y\ (i + k)$ **and** *chain* $A$ **and** $(\bigsqcup i.\ Y\ i) = up \cdot (\bigsqcup i.\ A\ i)$
⟨*proof*⟩

**lemma** *compact-up*: *compact* $x \Longrightarrow$ *compact* $(up \cdot x)$
⟨*proof*⟩

**lemma** *compact-upD*: *compact* $(up \cdot x) \Longrightarrow$ *compact* $x$
⟨*proof*⟩

**lemma** *compact-up-iff* [*simp*]: *compact* $(up \cdot x) =$ *compact* $x$
⟨*proof*⟩

**instance** *u* :: (*chfin*) *chfin*
⟨*proof*⟩

properties of fup

**lemma** *fup1* [*simp*]: $fup \cdot f \cdot \bot = \bot$
⟨*proof*⟩

**lemma** *fup2* [*simp*]: $fup \cdot f \cdot (up \cdot x) = f \cdot x$
⟨*proof*⟩

**lemma** *fup3* [*simp*]: $fup \cdot up \cdot x = x$
⟨*proof*⟩

**end**

# 14 Lifting types of class type to flat pcpo's

**theory** *Lift*
**imports** *Discrete Up*
**begin**

**default-sort** *type*

**pcpodef** *'a lift = UNIV :: 'a discr u set*
⟨*proof*⟩

**lemmas** *inst-lift-pcpo = Abs-lift-strict* [*symmetric*]

**definition**
*Def* :: *'a* ⇒ *'a lift* **where**
*Def x = Abs-lift* $(up \cdot (Discr\ x))$

## 14.1 Lift as a datatype

**lemma** *lift-induct*: $[\![P\ \bot;\ \bigwedge x.\ P\ (Def\ x)]\!] \Longrightarrow P\ y$
⟨*proof*⟩

**old-rep-datatype** $\bot$::$'a$ *lift Def*
⟨*proof*⟩

$\bot$ and *Def*

**lemma** *not-Undef-is-Def*: $(x \neq \bot) = (\exists\, y.\ x = Def\ y)$
⟨*proof*⟩

**lemma** *lift-definedE*: $[\![x \neq \bot;\ \bigwedge a.\ x = Def\ a \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

For $x \neq \bot$ in assumptions *defined* replaces $x$ by *Def a* in conclusion.

⟨*ML*⟩

**lemma** *DefE*: $Def\ x = \bot \Longrightarrow R$
⟨*proof*⟩

**lemma** *DefE2*: $[\![x = Def\ s;\ x = \bot]\!] \Longrightarrow R$
⟨*proof*⟩

**lemma** *Def-below-Def*: $Def\ x \sqsubseteq Def\ y \longleftrightarrow x = y$
⟨*proof*⟩

**lemma** *Def-below-iff* [*simp*]: $Def\ x \sqsubseteq y \longleftrightarrow Def\ x = y$
⟨*proof*⟩

## 14.2 Lift is flat

**instance** *lift* :: (*type*) *flat*
⟨*proof*⟩

## 14.3 Continuity of *case-lift*

**lemma** *case-lift-eq*: $case\text{-}lift\ \bot\ f\ x = fup\cdot(\Lambda\ y.\ f\ (undiscr\ y))\cdot(Rep\text{-}lift\ x)$
⟨*proof*⟩

**lemma** *cont2cont-case-lift* [*simp*]:
$[\![\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y);\ cont\ g]\!] \Longrightarrow cont\ (\lambda x.\ case\text{-}lift\ \bot\ (f\ x)\ (g\ x))$
⟨*proof*⟩

## 14.4 Further operations

**definition**
$flift1 :: ('a \Rightarrow 'b::pcpo) \Rightarrow ('a\ lift \rightarrow 'b)$ (**binder** *FLIFT 10*) **where**
$flift1 = (\lambda f.\ (\Lambda\ x.\ case\text{-}lift\ \bot\ f\ x))$

**translations**
$\Lambda(XCONST\ Def\ x).\ t$ => $CONST\ flift1\ (\lambda x.\ t)$
$\Lambda(CONST\ Def\ x).\ FLIFT\ y.\ t$ <= $FLIFT\ x\ y.\ t$
$\Lambda(CONST\ Def\ x).\ t$ <= $FLIFT\ x.\ t$

**definition**
$flift2 :: ('a \Rightarrow 'b) \Rightarrow ('a\ lift \rightarrow 'b\ lift)$ **where**
$flift2\ f = (FLIFT\ x.\ Def\ (f\ x))$

**lemma** *flift1-Def* [*simp*]: $flift1\ f \cdot (Def\ x) = (f\ x)$
⟨*proof*⟩

**lemma** *flift2-Def* [*simp*]: $flift2\ f \cdot (Def\ x) = Def\ (f\ x)$
⟨*proof*⟩

**lemma** *flift1-strict* [*simp*]: $flift1\ f \cdot \bot = \bot$
⟨*proof*⟩

**lemma** *flift2-strict* [*simp*]: $flift2\ f \cdot \bot = \bot$
⟨*proof*⟩

**lemma** *flift2-defined* [*simp*]: $x \neq \bot \Longrightarrow (flift2\ f) \cdot x \neq \bot$
⟨*proof*⟩

**lemma** *flift2-bottom-iff* [*simp*]: $(flift2\ f \cdot x = \bot) = (x = \bot)$
⟨*proof*⟩

**lemma** *FLIFT-mono*:
$(\bigwedge x.\ f\ x \sqsubseteq g\ x) \Longrightarrow (FLIFT\ x.\ f\ x) \sqsubseteq (FLIFT\ x.\ g\ x)$
⟨*proof*⟩

**lemma** *cont2cont-flift1* [*simp*, *cont2cont*]:
$[\![\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y)]\!] \Longrightarrow cont\ (\lambda x.\ FLIFT\ y.\ f\ x\ y)$
⟨*proof*⟩

**end**

# 15 The type of lifted booleans

**theory** *Tr*
**imports** *Lift*
**begin**

## 15.1 Type definition and constructors

**type-synonym**
$tr = bool\ lift$

**translations**

*(type) tr <= (type) bool lift*

**definition**
$TT$ :: *tr* **where**
$TT = Def\ True$

**definition**
$FF$ :: *tr* **where**
$FF = Def\ False$

Exhaustion and Elimination for type *tr*

**lemma** *Exh-tr*: $t = \bot \vee t = TT \vee t = FF$
⟨*proof*⟩

**lemma** *trE* [*case-names bottom TT FF*, *cases type*: *tr*]:
$[\![p = \bot \Longrightarrow Q;\ p = TT \Longrightarrow Q;\ p = FF \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

**lemma** *tr-induct* [*case-names bottom TT FF*, *induct type*: *tr*]:
$[\![P\ \bot;\ P\ TT;\ P\ FF]\!] \Longrightarrow P\ x$
⟨*proof*⟩

distinctness for type *tr*

**lemma** *dist-below-tr* [*simp*]:
$TT \not\sqsubseteq \bot\ FF \not\sqsubseteq \bot\ TT \not\sqsubseteq FF\ FF \not\sqsubseteq TT$
⟨*proof*⟩

**lemma** *dist-eq-tr* [*simp*]:
$TT \neq \bot\ FF \neq \bot\ TT \neq FF\ \bot \neq TT\ \bot \neq FF\ FF \neq TT$
⟨*proof*⟩

**lemma** *TT-below-iff* [*simp*]: $TT \sqsubseteq x \longleftrightarrow x = TT$
⟨*proof*⟩

**lemma** *FF-below-iff* [*simp*]: $FF \sqsubseteq x \longleftrightarrow x = FF$
⟨*proof*⟩

**lemma** *not-below-TT-iff* [*simp*]: $x \not\sqsubseteq TT \longleftrightarrow x = FF$
⟨*proof*⟩

**lemma** *not-below-FF-iff* [*simp*]: $x \not\sqsubseteq FF \longleftrightarrow x = TT$
⟨*proof*⟩

## 15.2 Case analysis

**default-sort** *pcpo*

**definition** *tr-case* :: $'a \rightarrow 'a \rightarrow tr \rightarrow 'a$ **where**
*tr-case* = ($\Lambda$ *t e* (*Def b*). *if b then t else e*)

**abbreviation**
*cifte-syn* :: [*tr*, ′*c*, ′*c*] ⇒ ′*c* ((*If* (-)/ *then* (-)/ *else* (-)) [*0*, *0*, *60*] *60*)
**where**
*If b then e1 else e2* == *tr-case*·*e1*·*e2*·*b*

**translations**
Λ (*XCONST TT*). *t* == *CONST tr-case*·*t*·⊥
Λ (*XCONST FF*). *t* == *CONST tr-case*·⊥·*t*

**lemma** *ifte-thms* [*simp*]:
*If* ⊥ *then e1 else e2* = ⊥
*If FF then e1 else e2* = *e2*
*If TT then e1 else e2* = *e1*
⟨*proof*⟩

## 15.3 Boolean connectives

**definition**
*trand* :: *tr* → *tr* → *tr* **where**
*andalso-def*: *trand* = (Λ *x y*. *If x then y else FF*)
**abbreviation**
*andalso-syn* :: *tr* ⇒ *tr* ⇒ *tr* (- *andalso* - [*36*,*35*] *35*) **where**
*x andalso y* == *trand*·*x*·*y*

**definition**
*tror* :: *tr* → *tr* → *tr* **where**
*orelse-def*: *tror* = (Λ *x y*. *If x then TT else y*)
**abbreviation**
*orelse-syn* :: *tr* ⇒ *tr* ⇒ *tr* (- *orelse* - [*31*,*30*] *30*) **where**
*x orelse y* == *tror*·*x*·*y*

**definition**
*neg* :: *tr* → *tr* **where**
*neg* = *flift2 Not*

**definition**
*If2* :: [*tr*, ′*c*, ′*c*] ⇒ ′*c* **where**
*If2 Q x y* = (*If Q then x else y*)

tactic for tr-thms with case split

**lemmas** *tr-defs* = *andalso-def orelse-def neg-def tr-case-def TT-def FF-def*

lemmas about andalso, orelse, neg and if

**lemma** *andalso-thms* [*simp*]:
(*TT andalso y*) = *y*
(*FF andalso y*) = *FF*
(⊥ *andalso y*) = ⊥
(*y andalso TT*) = *y*

$(y\ andalso\ y) = y$
⟨*proof*⟩

**lemma** *orelse-thms* [*simp*]:
$(TT\ orelse\ y) = TT$
$(FF\ orelse\ y) = y$
$(\bot\ orelse\ y) = \bot$
$(y\ orelse\ FF) = y$
$(y\ orelse\ y) = y$
⟨*proof*⟩

**lemma** *neg-thms* [*simp*]:
$neg \cdot TT = FF$
$neg \cdot FF = TT$
$neg \cdot \bot = \bot$
⟨*proof*⟩

split-tac for If via If2 because the constant has to be a constant

**lemma** *split-If2*:
$P\ (If2\ Q\ x\ y) = ((Q = \bot \longrightarrow P\ \bot) \wedge (Q = TT \longrightarrow P\ x) \wedge (Q = FF \longrightarrow P\ y))$
⟨*proof*⟩

⟨*ML*⟩

## 15.4 Rewriting of HOLCF operations to HOL functions

**lemma** *andalso-or*:
$t \neq \bot \Longrightarrow ((t\ andalso\ s) = FF) = (t = FF \vee s = FF)$
⟨*proof*⟩

**lemma** *andalso-and*:
$t \neq \bot \Longrightarrow ((t\ andalso\ s) \neq FF) = (t \neq FF \wedge s \neq FF)$
⟨*proof*⟩

**lemma** *Def-bool1* [*simp*]: $(Def\ x \neq FF) = x$
⟨*proof*⟩

**lemma** *Def-bool2* [*simp*]: $(Def\ x = FF) = (\neg\ x)$
⟨*proof*⟩

**lemma** *Def-bool3* [*simp*]: $(Def\ x = TT) = x$
⟨*proof*⟩

**lemma** *Def-bool4* [*simp*]: $(Def\ x \neq TT) = (\neg\ x)$
⟨*proof*⟩

**lemma** *If-and-if*:

$(If\ Def\ P\ then\ A\ else\ B) = (if\ P\ then\ A\ else\ B)$
⟨*proof*⟩

### 15.5 Compactness

**lemma** *compact-TT*: *compact TT*
⟨*proof*⟩

**lemma** *compact-FF*: *compact FF*
⟨*proof*⟩

**end**

# 16 The type of strict sums

**theory** *Ssum*
**imports** *Tr*
**begin**

**default-sort** *pcpo*

## 16.1 Definition of strict sum type

**definition**
*ssum* =
$\{p :: tr \times ('a \times 'b).\ p = \bot \lor$
$(fst\ p = TT \land fst\ (snd\ p) \neq \bot \land snd\ (snd\ p) = \bot) \lor$
$(fst\ p = FF \land fst\ (snd\ p) = \bot \land snd\ (snd\ p) \neq \bot)\}$

**pcpodef** $('a,\ 'b)\ ssum\ \ ((\text{-} \oplus/ \text{-})\ [21,\ 20]\ 20) = ssum :: (tr \times 'a \times 'b)\ set$
⟨*proof*⟩

**instance** *ssum* :: ({*chfin*,*pcpo*}, {*chfin*,*pcpo*}) *chfin*
⟨*proof*⟩

**type-notation** (*ASCII*)
*ssum* (**infixr** ++ *10*)

## 16.2 Definitions of constructors

**definition**
$sinl :: 'a \to ('a\ ++\ 'b)$ **where**
$sinl = (\Lambda\ a.\ Abs\text{-}ssum\ (seq{\cdot}a{\cdot}TT,\ a,\ \bot))$

**definition**
$sinr :: 'b \to ('a\ ++\ 'b)$ **where**
$sinr = (\Lambda\ b.\ Abs\text{-}ssum\ (seq{\cdot}b{\cdot}FF,\ \bot,\ b))$

**lemma** *sinl-ssum*: $(seq{\cdot}a{\cdot}TT,\ a,\ \bot) \in ssum$

⟨*proof*⟩

**lemma** *sinr-ssum*: $(seq{\cdot}b{\cdot}FF,\ \bot,\ b) \in ssum$
⟨*proof*⟩

**lemma** *Rep-ssum-sinl*: $Rep\text{-}ssum\ (sinl{\cdot}a) = (seq{\cdot}a{\cdot}TT,\ a,\ \bot)$
⟨*proof*⟩

**lemma** *Rep-ssum-sinr*: $Rep\text{-}ssum\ (sinr{\cdot}b) = (seq{\cdot}b{\cdot}FF,\ \bot,\ b)$
⟨*proof*⟩

**lemmas** *Rep-ssum-simps* =
*Rep-ssum-inject* [*symmetric*] *below-ssum-def*
*prod-eq-iff below-prod-def*
*Rep-ssum-strict Rep-ssum-sinl Rep-ssum-sinr*

## 16.3 Properties of *sinl* and *sinr*

Ordering

**lemma** *sinl-below* [*simp*]: $(sinl{\cdot}x \sqsubseteq sinl{\cdot}y) = (x \sqsubseteq y)$
⟨*proof*⟩

**lemma** *sinr-below* [*simp*]: $(sinr{\cdot}x \sqsubseteq sinr{\cdot}y) = (x \sqsubseteq y)$
⟨*proof*⟩

**lemma** *sinl-below-sinr* [*simp*]: $(sinl{\cdot}x \sqsubseteq sinr{\cdot}y) = (x = \bot)$
⟨*proof*⟩

**lemma** *sinr-below-sinl* [*simp*]: $(sinr{\cdot}x \sqsubseteq sinl{\cdot}y) = (x = \bot)$
⟨*proof*⟩

Equality

**lemma** *sinl-eq* [*simp*]: $(sinl{\cdot}x = sinl{\cdot}y) = (x = y)$
⟨*proof*⟩

**lemma** *sinr-eq* [*simp*]: $(sinr{\cdot}x = sinr{\cdot}y) = (x = y)$
⟨*proof*⟩

**lemma** *sinl-eq-sinr* [*simp*]: $(sinl{\cdot}x = sinr{\cdot}y) = (x = \bot \wedge y = \bot)$
⟨*proof*⟩

**lemma** *sinr-eq-sinl* [*simp*]: $(sinr{\cdot}x = sinl{\cdot}y) = (x = \bot \wedge y = \bot)$
⟨*proof*⟩

**lemma** *sinl-inject*: $sinl{\cdot}x = sinl{\cdot}y \Longrightarrow x = y$
⟨*proof*⟩

**lemma** *sinr-inject*: $sinr{\cdot}x = sinr{\cdot}y \Longrightarrow x = y$
⟨*proof*⟩

Strictness

**lemma** *sinl-strict* [*simp*]: $sinl \cdot \bot = \bot$
⟨*proof*⟩

**lemma** *sinr-strict* [*simp*]: $sinr \cdot \bot = \bot$
⟨*proof*⟩

**lemma** *sinl-bottom-iff* [*simp*]: $(sinl \cdot x = \bot) = (x = \bot)$
⟨*proof*⟩

**lemma** *sinr-bottom-iff* [*simp*]: $(sinr \cdot x = \bot) = (x = \bot)$
⟨*proof*⟩

**lemma** *sinl-defined*: $x \neq \bot \Longrightarrow sinl \cdot x \neq \bot$
⟨*proof*⟩

**lemma** *sinr-defined*: $x \neq \bot \Longrightarrow sinr \cdot x \neq \bot$
⟨*proof*⟩

Compactness

**lemma** *compact-sinl*: *compact* $x \Longrightarrow$ *compact* $(sinl \cdot x)$
⟨*proof*⟩

**lemma** *compact-sinr*: *compact* $x \Longrightarrow$ *compact* $(sinr \cdot x)$
⟨*proof*⟩

**lemma** *compact-sinlD*: *compact* $(sinl \cdot x) \Longrightarrow$ *compact* $x$
⟨*proof*⟩

**lemma** *compact-sinrD*: *compact* $(sinr \cdot x) \Longrightarrow$ *compact* $x$
⟨*proof*⟩

**lemma** *compact-sinl-iff* [*simp*]: *compact* $(sinl \cdot x) =$ *compact* $x$
⟨*proof*⟩

**lemma** *compact-sinr-iff* [*simp*]: *compact* $(sinr \cdot x) =$ *compact* $x$
⟨*proof*⟩

## 16.4 Case analysis

**lemma** *ssumE* [*case-names bottom sinl sinr*, *cases type*: *ssum*]:
  **obtains** $p = \bot$
  | $x$ **where** $p = sinl \cdot x$ **and** $x \neq \bot$
  | $y$ **where** $p = sinr \cdot y$ **and** $y \neq \bot$
⟨*proof*⟩

**lemma** *ssum-induct* [*case-names bottom sinl sinr*, *induct type*: *ssum*]:
  $[\![P\ \bot;$
  $\bigwedge x.\ x \neq \bot \Longrightarrow P\ (sinl \cdot x);$

$\bigwedge y.\ y \neq \bot \Longrightarrow P\ (sinr{\cdot}y)]\!] \Longrightarrow P\ x$
⟨*proof*⟩

**lemma** *ssumE2* [*case-names sinl sinr*]:
$[\![\bigwedge x.\ p = sinl{\cdot}x \Longrightarrow Q;\ \bigwedge y.\ p = sinr{\cdot}y \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

**lemma** *below-sinlD*: $p \sqsubseteq sinl{\cdot}x \Longrightarrow \exists\, y.\ p = sinl{\cdot}y \land y \sqsubseteq x$
⟨*proof*⟩

**lemma** *below-sinrD*: $p \sqsubseteq sinr{\cdot}x \Longrightarrow \exists\, y.\ p = sinr{\cdot}y \land y \sqsubseteq x$
⟨*proof*⟩

## 16.5 Case analysis combinator

**definition**
$sscase :: ('a \rightarrow 'c) \rightarrow ('b \rightarrow 'c) \rightarrow ('a \mathbin{++} 'b) \rightarrow 'c$ **where**
$sscase = (\Lambda\ f\ g\ s.\ (\lambda(t,\ x,\ y).\ \mathit{If}\ t\ \mathit{then}\ f{\cdot}x\ \mathit{else}\ g{\cdot}y)\ (Rep\text{-}ssum\ s))$

**translations**
$case\ s\ of\ XCONST\ sinl{\cdot}x \Rightarrow t1\ |\ XCONST\ sinr{\cdot}y \Rightarrow t2 == CONST\ sscase{\cdot}(\Lambda\ x.\ t1){\cdot}(\Lambda\ y.\ t2){\cdot}s$
$case\ s\ of\ (XCONST\ sinl :: 'a){\cdot}x \Rightarrow t1\ |\ XCONST\ sinr{\cdot}y \Rightarrow t2 => CONST\ sscase{\cdot}(\Lambda\ x.\ t1){\cdot}(\Lambda\ y.\ t2){\cdot}s$

**translations**
$\Lambda(XCONST\ sinl{\cdot}x).\ t == CONST\ sscase{\cdot}(\Lambda\ x.\ t){\cdot}\bot$
$\Lambda(XCONST\ sinr{\cdot}y).\ t == CONST\ sscase{\cdot}\bot{\cdot}(\Lambda\ y.\ t)$

**lemma** *beta-sscase*:
$sscase{\cdot}f{\cdot}g{\cdot}s = (\lambda(t,\ x,\ y).\ \mathit{If}\ t\ \mathit{then}\ f{\cdot}x\ \mathit{else}\ g{\cdot}y)\ (Rep\text{-}ssum\ s)$
⟨*proof*⟩

**lemma** *sscase1* [*simp*]: $sscase{\cdot}f{\cdot}g{\cdot}\bot = \bot$
⟨*proof*⟩

**lemma** *sscase2* [*simp*]: $x \neq \bot \Longrightarrow sscase{\cdot}f{\cdot}g{\cdot}(sinl{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

**lemma** *sscase3* [*simp*]: $y \neq \bot \Longrightarrow sscase{\cdot}f{\cdot}g{\cdot}(sinr{\cdot}y) = g{\cdot}y$
⟨*proof*⟩

**lemma** *sscase4* [*simp*]: $sscase{\cdot}sinl{\cdot}sinr{\cdot}z = z$
⟨*proof*⟩

## 16.6 Strict sum preserves flatness

**instance** *ssum* :: (*flat*, *flat*) *flat*
⟨*proof*⟩

**end**

# 17 The unit domain

**theory** *One*
**imports** *Lift*
**begin**

**type-synonym**
*one* = *unit lift*

**translations**
(*type*) *one* <= (*type*) *unit lift*

**definition** *ONE* :: *one*
**where** *ONE* == *Def* ()

Exhaustion and Elimination for type *one*

**lemma** *Exh-one*: $t = \bot \lor t = ONE$
⟨*proof*⟩

**lemma** *oneE* [*case-names bottom ONE*]: $⟦p = \bot \Longrightarrow Q; p = ONE \Longrightarrow Q⟧ \Longrightarrow Q$
⟨*proof*⟩

**lemma** *one-induct* [*case-names bottom ONE*]: $⟦P \bot; P\ ONE⟧ \Longrightarrow P\ x$
⟨*proof*⟩

**lemma** *dist-below-one* [*simp*]: $ONE \not\sqsubseteq \bot$
⟨*proof*⟩

**lemma** *below-ONE* [*simp*]: $x \sqsubseteq ONE$
⟨*proof*⟩

**lemma** *ONE-below-iff* [*simp*]: $ONE \sqsubseteq x \longleftrightarrow x = ONE$
⟨*proof*⟩

**lemma** *ONE-defined* [*simp*]: $ONE \neq \bot$
⟨*proof*⟩

**lemma** *one-neq-iffs* [*simp*]:
$x \neq ONE \longleftrightarrow x = \bot$
$ONE \neq x \longleftrightarrow x = \bot$
$x \neq \bot \longleftrightarrow x = ONE$
$\bot \neq x \longleftrightarrow x = ONE$
⟨*proof*⟩

**lemma** *compact-ONE*: *compact ONE*
⟨*proof*⟩

Case analysis function for type *one*

**definition**
*one-case* :: *'a::pcpo* → *one* → *'a* **where**
*one-case* = (Λ *a x*. *seq*·*x*·*a*)

**translations**
*case x of XCONST ONE* ⇒ *t* == *CONST one-case*·*t*·*x*
*case x of XCONST ONE* :: *'a* ⇒ *t* => *CONST one-case*·*t*·*x*
Λ (*XCONST ONE*). *t* == *CONST one-case*·*t*

**lemma** *one-case1* [*simp*]: (*case* ⊥ *of ONE* ⇒ *t*) = ⊥
⟨*proof*⟩

**lemma** *one-case2* [*simp*]: (*case ONE of ONE* ⇒ *t*) = *t*
⟨*proof*⟩

**lemma** *one-case3* [*simp*]: (*case x of ONE* ⇒ *ONE*) = *x*
⟨*proof*⟩

**end**

# 18 Fixed point operator and admissibility

**theory** *Fix*
**imports** *Cfun*
**begin**

**default-sort** *pcpo*

## 18.1 Iteration

**primrec** *iterate* :: *nat* ⇒ (*'a::cpo* → *'a*) → (*'a* → *'a*) **where**
*iterate 0* = (Λ *F x*. *x*)
| *iterate* (*Suc n*) = (Λ *F x*. *F*·(*iterate n*·*F*·*x*))

Derive inductive properties of iterate from primitive recursion

**lemma** *iterate-0* [*simp*]: *iterate 0*·*F*·*x* = *x*
⟨*proof*⟩

**lemma** *iterate-Suc* [*simp*]: *iterate* (*Suc n*)·*F*·*x* = *F*·(*iterate n*·*F*·*x*)
⟨*proof*⟩

**declare** *iterate.simps* [*simp del*]

**lemma** *iterate-Suc2*: *iterate* (*Suc n*)·*F*·*x* = *iterate n*·*F*·(*F*·*x*)
⟨*proof*⟩

**lemma** *iterate-iterate*:

*iterate* $m \cdot F \cdot (iterate\ n \cdot F \cdot x) = iterate\ (m + n) \cdot F \cdot x$
⟨*proof*⟩

The sequence of function iterations is a chain.

**lemma** *chain-iterate* [*simp*]: $chain\ (\lambda i.\ iterate\ i \cdot F \cdot \bot)$
⟨*proof*⟩

## 18.2 Least fixed point operator

**definition**
$fix :: ('a \to 'a) \to 'a$ **where**
$fix = (\Lambda\ F.\ \bigsqcup i.\ iterate\ i \cdot F \cdot \bot)$

Binder syntax for *fix*

**abbreviation**
$fix\text{-}syn :: ('a \Rightarrow 'a) \Rightarrow 'a$ (**binder** $\mu$ *10*) **where**
$fix\text{-}syn\ (\lambda x.\ f\ x) \equiv fix \cdot (\Lambda\ x.\ f\ x)$

**notation** (*ASCII*)
*fix-syn* (**binder** *FIX* *10*)

Properties of *fix*

direct connection between *fix* and iteration

**lemma** *fix-def2*: $fix \cdot F = (\bigsqcup i.\ iterate\ i \cdot F \cdot \bot)$
⟨*proof*⟩

**lemma** *iterate-below-fix*: $iterate\ n \cdot f \cdot \bot \sqsubseteq fix \cdot f$
⟨*proof*⟩

Kleene's fixed point theorems for continuous functions in pointed omega cpo's

**lemma** *fix-eq*: $fix \cdot F = F \cdot (fix \cdot F)$
⟨*proof*⟩

**lemma** *fix-least-below*: $F \cdot x \sqsubseteq x \Longrightarrow fix \cdot F \sqsubseteq x$
⟨*proof*⟩

**lemma** *fix-least*: $F \cdot x = x \Longrightarrow fix \cdot F \sqsubseteq x$
⟨*proof*⟩

**lemma** *fix-eqI*:
**assumes** *fixed*: $F \cdot x = x$ **and** *least*: $\bigwedge z.\ F \cdot z = z \Longrightarrow x \sqsubseteq z$
**shows** $fix \cdot F = x$
⟨*proof*⟩

**lemma** *fix-eq2*: $f \equiv fix \cdot F \Longrightarrow f = F \cdot f$
⟨*proof*⟩

**lemma** *fix-eq3*: $f \equiv fix{\cdot}F \Longrightarrow f{\cdot}x = F{\cdot}f{\cdot}x$
⟨*proof*⟩

**lemma** *fix-eq4*: $f = fix{\cdot}F \Longrightarrow f = F{\cdot}f$
⟨*proof*⟩

**lemma** *fix-eq5*: $f = fix{\cdot}F \Longrightarrow f{\cdot}x = F{\cdot}f{\cdot}x$
⟨*proof*⟩

strictness of *fix*

**lemma** *fix-bottom-iff*: $(fix{\cdot}F = \bot) = (F{\cdot}\bot = \bot)$
⟨*proof*⟩

**lemma** *fix-strict*: $F{\cdot}\bot = \bot \Longrightarrow fix{\cdot}F = \bot$
⟨*proof*⟩

**lemma** *fix-defined*: $F{\cdot}\bot \neq \bot \Longrightarrow fix{\cdot}F \neq \bot$
⟨*proof*⟩

*fix* applied to identity and constant functions

**lemma** *fix-id*: $(\mu\ x.\ x) = \bot$
⟨*proof*⟩

**lemma** *fix-const*: $(\mu\ x.\ c) = c$
⟨*proof*⟩

## 18.3 Fixed point induction

**lemma** *fix-ind*: $[\![adm\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F{\cdot}x)]\!] \Longrightarrow P\ (fix{\cdot}F)$
⟨*proof*⟩

**lemma** *cont-fix-ind*:
$[\![cont\ F;\ adm\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F\ x)]\!] \Longrightarrow P\ (fix{\cdot}(Abs\text{-}cfun\ F))$
⟨*proof*⟩

**lemma** *def-fix-ind*:
$[\![f \equiv fix{\cdot}F;\ adm\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F{\cdot}x)]\!] \Longrightarrow P\ f$
⟨*proof*⟩

**lemma** *fix-ind2*:
**assumes** *adm*: $adm\ P$
**assumes** *0*: $P\ \bot$ **and** *1*: $P\ (F{\cdot}\bot)$
**assumes** *step*: $\bigwedge x.\ [\![P\ x;\ P\ (F{\cdot}x)]\!] \Longrightarrow P\ (F{\cdot}(F{\cdot}x))$
**shows** $P\ (fix{\cdot}F)$
⟨*proof*⟩

**lemma** *parallel-fix-ind*:
**assumes** *adm*: $adm\ (\lambda x.\ P\ (fst\ x)\ (snd\ x))$
**assumes** *base*: $P\ \bot\ \bot$

**assumes** *step*: $\bigwedge x\ y.\ P\ x\ y \Longrightarrow P\ (F{\cdot}x)\ (G{\cdot}y)$
**shows** $P\ (\mathit{fix}{\cdot}F)\ (\mathit{fix}{\cdot}G)$
⟨*proof*⟩

**lemma** *cont-parallel-fix-ind*:
**assumes** *cont F* **and** *cont G*
**assumes** $\mathit{adm}\ (\lambda x.\ P\ (\mathit{fst}\ x)\ (\mathit{snd}\ x))$
**assumes** $P\ \bot\ \bot$
**assumes** $\bigwedge x\ y.\ P\ x\ y \Longrightarrow P\ (F\ x)\ (G\ y)$
**shows** $P\ (\mathit{fix}{\cdot}(\mathit{Abs\text{-}cfun}\ F))\ (\mathit{fix}{\cdot}(\mathit{Abs\text{-}cfun}\ G))$
⟨*proof*⟩

### 18.4 Fixed-points on product types

Bekic's Theorem: Simultaneous fixed points over pairs can be written in terms of separate fixed points.

**lemma** *fix-cprod*:
$\mathit{fix}{\cdot}(F{::}'a \times 'b \to 'a \times 'b) =$
$(\mu\ x.\ \mathit{fst}\ (F{\cdot}(x,\ \mu\ y.\ \mathit{snd}\ (F{\cdot}(x,\ y)))),$
$\mu\ y.\ \mathit{snd}\ (F{\cdot}(\mu\ x.\ \mathit{fst}\ (F{\cdot}(x,\ \mu\ y.\ \mathit{snd}\ (F{\cdot}(x,\ y)))),\ y)))$
(**is** $\mathit{fix}{\cdot}F = (?x,\ ?y)$)
⟨*proof*⟩

**end**

# 19 Plain HOLCF

**theory** *Plain-HOLCF*
**imports** *Cfun Sfun Cprod Sprod Ssum Up Discrete Lift One Tr Fix*
**begin**

Basic HOLCF concepts and types; does not include definition packages.

**hide-const** (**open**) *Filter.principal*

**end**

# 20 Package for defining recursive functions in HOLCF

**theory** *Fixrec*
**imports** *Plain-HOLCF*
**keywords** *fixrec* :: *thy-decl*
**begin**

### 20.1 Pattern-match monad

**default-sort** *cpo*

**pcpodef** *'a match* = *UNIV*::(*one* ++ *'a u*) *set*
⟨*proof*⟩

**definition**
*fail* :: *'a match* **where**
*fail* = *Abs-match* (*sinl*·*ONE*)

**definition**
*succeed* :: *'a* → *'a match* **where**
*succeed* = (Λ *x*. *Abs-match* (*sinr*·(*up*·*x*)))

**lemma** *matchE* [*case-names bottom fail succeed*, *cases type*: *match*]:
$[\![p = \bot \Longrightarrow Q;\ p = fail \Longrightarrow Q;\ \bigwedge x.\ p = succeed\cdot x \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

**lemma** *succeed-defined* [*simp*]: $succeed\cdot x \neq \bot$
⟨*proof*⟩

**lemma** *fail-defined* [*simp*]: $fail \neq \bot$
⟨*proof*⟩

**lemma** *succeed-eq* [*simp*]: $(succeed\cdot x = succeed\cdot y) = (x = y)$
⟨*proof*⟩

**lemma** *succeed-neq-fail* [*simp*]:
$succeed\cdot x \neq fail\ fail \neq succeed\cdot x$
⟨*proof*⟩

### 20.1.1 Run operator

**definition**
*run* :: *'a match* → *'a*::*pcpo* **where**
*run* = (Λ *m*. *sscase*·⊥·(*fup*·*ID*)·(*Rep-match m*))

rewrite rules for run

**lemma** *run-strict* [*simp*]: $run\cdot\bot = \bot$
⟨*proof*⟩

**lemma** *run-fail* [*simp*]: $run\cdot fail = \bot$
⟨*proof*⟩

**lemma** *run-succeed* [*simp*]: $run\cdot(succeed\cdot x) = x$
⟨*proof*⟩

### 20.1.2 Monad plus operator

**definition**
*mplus* :: *'a match* → *'a match* → *'a match* **where**
*mplus* = (Λ *m1 m2*. *sscase*·(Λ -. *m2*)·(Λ -. *m1*)·(*Rep-match m1*))

**abbreviation**
*mplus-syn* :: [*'a match*, *'a match*] ⇒ *'a match* (**infixr** +++ *65*) **where**
*m1* +++ *m2* == *mplus·m1·m2*

rewrite rules for mplus

**lemma** *mplus-strict* [*simp*]: ⊥ +++ *m* = ⊥
⟨*proof*⟩

**lemma** *mplus-fail* [*simp*]: *fail* +++ *m* = *m*
⟨*proof*⟩

**lemma** *mplus-succeed* [*simp*]: *succeed·x* +++ *m* = *succeed·x*
⟨*proof*⟩

**lemma** *mplus-fail2* [*simp*]: *m* +++ *fail* = *m*
⟨*proof*⟩

**lemma** *mplus-assoc*: (*x* +++ *y*) +++ *z* = *x* +++ (*y* +++ *z*)
⟨*proof*⟩

## 20.2 Match functions for built-in types

**default-sort** *pcpo*

**definition**
*match-bottom* :: *'a* → *'c match* → *'c match*
**where**
*match-bottom* = (Λ *x k*. *seq·x·fail*)

**definition**
*match-Pair* :: *'a::cpo* × *'b::cpo* → (*'a* → *'b* → *'c match*) → *'c match*
**where**
*match-Pair* = (Λ *x k*. *csplit·k·x*)

**definition**
*match-spair* :: *'a* ⊗ *'b* → (*'a* → *'b* → *'c match*) → *'c match*
**where**
*match-spair* = (Λ *x k*. *ssplit·k·x*)

**definition**
*match-sinl* :: *'a* ⊕ *'b* → (*'a* → *'c match*) → *'c match*
**where**
*match-sinl* = (Λ *x k*. *sscase·k·*(Λ *b*. *fail*)*·x*)

**definition**
*match-sinr* :: *'a* ⊕ *'b* → (*'b* → *'c match*) → *'c match*
**where**
*match-sinr* = (Λ *x k*. *sscase·*(Λ *a*. *fail*)*·k·x*)

**definition**
*match-up* :: $'a$::*cpo u* → ($'a$ → $'c$ *match*) → $'c$ *match*
**where**
*match-up* = (Λ *x k*. *fup*·*k*·*x*)

**definition**
*match-ONE* :: *one* → $'c$ *match* → $'c$ *match*
**where**
*match-ONE* = (Λ *ONE k*. *k*)

**definition**
*match-TT* :: *tr* → $'c$ *match* → $'c$ *match*
**where**
*match-TT* = (Λ *x k*. *If x then k else fail*)

**definition**
*match-FF* :: *tr* → $'c$ *match* → $'c$ *match*
**where**
*match-FF* = (Λ *x k*. *If x then fail else k*)

**lemma** *match-bottom-simps* [*simp*]:
*match-bottom*·*x*·*k* = (*if* $x = \bot$ *then* ⊥ *else fail*)
⟨*proof*⟩

**lemma** *match-Pair-simps* [*simp*]:
*match-Pair*·(*x*, *y*)·*k* = *k*·*x*·*y*
⟨*proof*⟩

**lemma** *match-spair-simps* [*simp*]:
⟦$x \neq \bot$; $y \neq \bot$⟧ ⟹ *match-spair*·(:*x*, *y*:)·*k* = *k*·*x*·*y*
*match-spair*·⊥·*k* = ⊥
⟨*proof*⟩

**lemma** *match-sinl-simps* [*simp*]:
$x \neq \bot$ ⟹ *match-sinl*·(*sinl*·*x*)·*k* = *k*·*x*
$y \neq \bot$ ⟹ *match-sinl*·(*sinr*·*y*)·*k* = *fail*
*match-sinl*·⊥·*k* = ⊥
⟨*proof*⟩

**lemma** *match-sinr-simps* [*simp*]:
$x \neq \bot$ ⟹ *match-sinr*·(*sinl*·*x*)·*k* = *fail*
$y \neq \bot$ ⟹ *match-sinr*·(*sinr*·*y*)·*k* = *k*·*y*
*match-sinr*·⊥·*k* = ⊥
⟨*proof*⟩

**lemma** *match-up-simps* [*simp*]:
*match-up*·(*up*·*x*)·*k* = *k*·*x*
*match-up*·⊥·*k* = ⊥
⟨*proof*⟩

**lemma** *match-ONE-simps* [*simp*]:
$match\text{-}ONE \cdot ONE \cdot k = k$
$match\text{-}ONE \cdot \bot \cdot k = \bot$
⟨*proof*⟩

**lemma** *match-TT-simps* [*simp*]:
$match\text{-}TT \cdot TT \cdot k = k$
$match\text{-}TT \cdot FF \cdot k = fail$
$match\text{-}TT \cdot \bot \cdot k = \bot$
⟨*proof*⟩

**lemma** *match-FF-simps* [*simp*]:
$match\text{-}FF \cdot FF \cdot k = k$
$match\text{-}FF \cdot TT \cdot k = fail$
$match\text{-}FF \cdot \bot \cdot k = \bot$
⟨*proof*⟩

## 20.3 Mutual recursion

The following rules are used to prove unfolding theorems from fixed-point definitions of mutually recursive functions.

**lemma** *Pair-equalI*: $[\![x \equiv fst\ p;\ y \equiv snd\ p]\!] \Longrightarrow (x,\ y) \equiv p$
⟨*proof*⟩

**lemma** *Pair-eqD1*: $(x,\ y) = (x',\ y') \Longrightarrow x = x'$
⟨*proof*⟩

**lemma** *Pair-eqD2*: $(x,\ y) = (x',\ y') \Longrightarrow y = y'$
⟨*proof*⟩

**lemma** *def-cont-fix-eq*:
$[\![f \equiv fix \cdot (Abs\text{-}cfun\ F);\ cont\ F]\!] \Longrightarrow f = F\ f$
⟨*proof*⟩

**lemma** *def-cont-fix-ind*:
$[\![f \equiv fix \cdot (Abs\text{-}cfun\ F);\ cont\ F;\ adm\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F\ x)]\!] \Longrightarrow P\ f$
⟨*proof*⟩

lemma for proving rewrite rules

**lemma** *ssubst-lhs*: $[\![t = s;\ P\ s = Q]\!] \Longrightarrow P\ t = Q$
⟨*proof*⟩

## 20.4 Initializing the fixrec package

⟨*ML*⟩

**hide-const** (**open**) *succeed fail run*

**end**

# 21 Continuous deflations and ep-pairs

**theory** *Deflation*
**imports** *Plain-HOLCF*
**begin**

**default-sort** *cpo*

## 21.1 Continuous deflations

**locale** *deflation* =
  **fixes** $d :: 'a \to 'a$
  **assumes** *idem*: $\bigwedge x.\ d\cdot(d\cdot x) = d\cdot x$
  **assumes** *below*: $\bigwedge x.\ d\cdot x \sqsubseteq x$
**begin**

**lemma** *below-ID*: $d \sqsubseteq ID$
⟨*proof*⟩

The set of fixed points is the same as the range.

**lemma** *fixes-eq-range*: $\{x.\ d\cdot x = x\} = range\ (\lambda x.\ d\cdot x)$
⟨*proof*⟩

**lemma** *range-eq-fixes*: $range\ (\lambda x.\ d\cdot x) = \{x.\ d\cdot x = x\}$
⟨*proof*⟩

The pointwise ordering on deflation functions coincides with the subset ordering of their sets of fixed-points.

**lemma** *belowI*:
  **assumes** $f$: $\bigwedge x.\ d\cdot x = x \Longrightarrow f\cdot x = x$ **shows** $d \sqsubseteq f$
⟨*proof*⟩

**lemma** *belowD*: $[\![f \sqsubseteq d;\ f\cdot x = x]\!] \Longrightarrow d\cdot x = x$
⟨*proof*⟩

**end**

**lemma** *deflation-strict*: $deflation\ d \Longrightarrow d\cdot\bot = \bot$
⟨*proof*⟩

**lemma** *adm-deflation*: $adm\ (\lambda d.\ deflation\ d)$
⟨*proof*⟩

**lemma** *deflation-ID*: *deflation ID*
⟨*proof*⟩

**lemma** *deflation-bottom*: *deflation* $\bot$
⟨*proof*⟩

**lemma** *deflation-below-iff*:
$[\![$*deflation p*; *deflation q*$]\!] \Longrightarrow p \sqsubseteq q \longleftrightarrow (\forall x.\ p{\cdot}x = x \longrightarrow q{\cdot}x = x)$
⟨*proof*⟩

The composition of two deflations is equal to the lesser of the two (if they are comparable).

**lemma** *deflation-below-comp1*:
**assumes** *deflation f*
**assumes** *deflation g*
**shows** $f \sqsubseteq g \Longrightarrow f{\cdot}(g{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

**lemma** *deflation-below-comp2*:
$[\![$*deflation f*; *deflation g*; $f \sqsubseteq g]\!] \Longrightarrow g{\cdot}(f{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

## 21.2 Deflations with finite range

**lemma** *finite-range-imp-finite-fixes*:
*finite* (*range f*) $\Longrightarrow$ *finite* $\{x.\ f\ x = x\}$
⟨*proof*⟩

**locale** *finite-deflation* = *deflation* +
**assumes** *finite-fixes*: *finite* $\{x.\ d{\cdot}x = x\}$
**begin**

**lemma** *finite-range*: *finite* (*range* $(\lambda x.\ d{\cdot}x)$)
⟨*proof*⟩

**lemma** *finite-image*: *finite* $((\lambda x.\ d{\cdot}x)$ ‘ $A)$
⟨*proof*⟩

**lemma** *compact*: *compact* $(d{\cdot}x)$
⟨*proof*⟩

**end**

**lemma** *finite-deflation-intro*:
*deflation d* $\Longrightarrow$ *finite* $\{x.\ d{\cdot}x = x\} \Longrightarrow$ *finite-deflation d*
⟨*proof*⟩

**lemma** *finite-deflation-imp-deflation*:
*finite-deflation d* $\Longrightarrow$ *deflation d*
⟨*proof*⟩

**lemma** *finite-deflation-bottom*: *finite-deflation* $\bot$

⟨*proof*⟩

## 21.3 Continuous embedding-projection pairs

**locale** *ep-pair* =
  **fixes** $e :: 'a \to 'b$ **and** $p :: 'b \to 'a$
  **assumes** *e-inverse* [*simp*]: $\bigwedge x.\ p\cdot(e\cdot x) = x$
  **and** *e-p-below*: $\bigwedge y.\ e\cdot(p\cdot y) \sqsubseteq y$
**begin**

**lemma** *e-below-iff* [*simp*]: $e\cdot x \sqsubseteq e\cdot y \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

**lemma** *e-eq-iff* [*simp*]: $e\cdot x = e\cdot y \longleftrightarrow x = y$
⟨*proof*⟩

**lemma** *p-eq-iff*:
  $[\![e\cdot(p\cdot x) = x;\ e\cdot(p\cdot y) = y]\!] \Longrightarrow p\cdot x = p\cdot y \longleftrightarrow x = y$
⟨*proof*⟩

**lemma** *p-inverse*: $(\exists\, x.\ y = e\cdot x) = (e\cdot(p\cdot y) = y)$
⟨*proof*⟩

**lemma** *e-below-iff-below-p*: $e\cdot x \sqsubseteq y \longleftrightarrow x \sqsubseteq p\cdot y$
⟨*proof*⟩

**lemma** *compact-e-rev*: *compact* $(e\cdot x) \Longrightarrow$ *compact* $x$
⟨*proof*⟩

**lemma** *compact-e*: *compact* $x \Longrightarrow$ *compact* $(e\cdot x)$
⟨*proof*⟩

**lemma** *compact-e-iff*: *compact* $(e\cdot x) \longleftrightarrow$ *compact* $x$
⟨*proof*⟩

Deflations from ep-pairs

**lemma** *deflation-e-p*: *deflation* (*e oo p*)
⟨*proof*⟩

**lemma** *deflation-e-d-p*:
  **assumes** *deflation d*
  **shows** *deflation* (*e oo d oo p*)
⟨*proof*⟩

**lemma** *finite-deflation-e-d-p*:
  **assumes** *finite-deflation d*
  **shows** *finite-deflation* (*e oo d oo p*)
⟨*proof*⟩

**lemma** *deflation-p-d-e*:
**assumes** *deflation d*
**assumes** *d*: $\bigwedge x.\ d\cdot x \sqsubseteq e\cdot(p\cdot x)$
**shows** *deflation* (*p oo d oo e*)
⟨*proof*⟩

**lemma** *finite-deflation-p-d-e*:
**assumes** *finite-deflation d*
**assumes** *d*: $\bigwedge x.\ d\cdot x \sqsubseteq e\cdot(p\cdot x)$
**shows** *finite-deflation* (*p oo d oo e*)
⟨*proof*⟩

**end**

## 21.4 Uniqueness of ep-pairs

**lemma** *ep-pair-unique-e-lemma*:
**assumes** *1*: *ep-pair e1 p* **and** *2*: *ep-pair e2 p*
**shows** $e1 \sqsubseteq e2$
⟨*proof*⟩

**lemma** *ep-pair-unique-e*:
⟦*ep-pair e1 p*; *ep-pair e2 p*⟧ $\Longrightarrow$ *e1* = *e2*
⟨*proof*⟩

**lemma** *ep-pair-unique-p-lemma*:
**assumes** *1*: *ep-pair e p1* **and** *2*: *ep-pair e p2*
**shows** $p1 \sqsubseteq p2$
⟨*proof*⟩

**lemma** *ep-pair-unique-p*:
⟦*ep-pair e p1*; *ep-pair e p2*⟧ $\Longrightarrow$ *p1* = *p2*
⟨*proof*⟩

## 21.5 Composing ep-pairs

**lemma** *ep-pair-ID-ID*: *ep-pair ID ID*
⟨*proof*⟩

**lemma** *ep-pair-comp*:
**assumes** *ep-pair e1 p1* **and** *ep-pair e2 p2*
**shows** *ep-pair* (*e2 oo e1*) (*p1 oo p2*)
⟨*proof*⟩

**locale** *pcpo-ep-pair* = *ep-pair e p*
**for** *e* :: *'a::pcpo* → *'b::pcpo*
**and** *p* :: *'b::pcpo* → *'a::pcpo*
**begin**

**lemma** *e-strict* [*simp*]: $e\cdot\bot = \bot$

⟨*proof*⟩

**lemma** *e-bottom-iff* [*simp*]: $e \cdot x = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

**lemma** *e-defined*: $x \neq \bot \Longrightarrow e \cdot x \neq \bot$
⟨*proof*⟩

**lemma** *p-strict* [*simp*]: $p \cdot \bot = \bot$
⟨*proof*⟩

**lemmas** *stricts* = *e-strict p-strict*

**end**

**end**

# 22 Map functions for various types

**theory** *Map-Functions*
**imports** *Deflation*
**begin**

## 22.1 Map operator for continuous function space

**default-sort** *cpo*

**definition**
  *cfun-map* :: $('b \to 'a) \to ('c \to 'd) \to ('a \to 'c) \to ('b \to 'd)$
**where**
  *cfun-map* = $(\Lambda\ a\ b\ f\ x.\ b \cdot (f \cdot (a \cdot x)))$

**lemma** *cfun-map-beta* [*simp*]: *cfun-map*$\cdot a \cdot b \cdot f \cdot x = b \cdot (f \cdot (a \cdot x))$
⟨*proof*⟩

**lemma** *cfun-map-ID*: *cfun-map*$\cdot$*ID*$\cdot$*ID* = *ID*
⟨*proof*⟩

**lemma** *cfun-map-map*:
  *cfun-map*$\cdot f1 \cdot g1 \cdot ($*cfun-map*$\cdot f2 \cdot g2 \cdot p)$ =
    *cfun-map*$\cdot(\Lambda\ x.\ f2 \cdot (f1 \cdot x)) \cdot (\Lambda\ x.\ g1 \cdot (g2 \cdot x)) \cdot p$
⟨*proof*⟩

**lemma** *ep-pair-cfun-map*:
  **assumes** *ep-pair e1 p1* **and** *ep-pair e2 p2*
  **shows** *ep-pair* (*cfun-map*$\cdot p1 \cdot e2$) (*cfun-map*$\cdot e1 \cdot p2$)
⟨*proof*⟩

**lemma** *deflation-cfun-map*:

**assumes** *deflation d1* **and** *deflation d2*
**shows** *deflation* (*cfun-map*·*d1*·*d2*)
⟨*proof*⟩

**lemma** *finite-range-cfun-map*:
**assumes** *a*: *finite* (*range* (λ*x*. *a*·*x*))
**assumes** *b*: *finite* (*range* (λ*y*. *b*·*y*))
**shows** *finite* (*range* (λ*f*. *cfun-map*·*a*·*b*·*f*)) (**is** *finite* (*range* ?*h*))
⟨*proof*⟩

**lemma** *finite-deflation-cfun-map*:
**assumes** *finite-deflation d1* **and** *finite-deflation d2*
**shows** *finite-deflation* (*cfun-map*·*d1*·*d2*)
⟨*proof*⟩

Finite deflations are compact elements of the function space

**lemma** *finite-deflation-imp-compact*: *finite-deflation d* ⟹ *compact d*
⟨*proof*⟩

## 22.2 Map operator for product type

**definition**
*prod-map* :: ($'a \rightarrow 'b$) → ($'c \rightarrow 'd$) → $'a \times 'c \rightarrow 'b \times 'd$
**where**
*prod-map* = (Λ *f g p*. (*f*·(*fst p*), *g*·(*snd p*)))

**lemma** *prod-map-Pair* [*simp*]: *prod-map*·*f*·*g*·(*x*, *y*) = (*f*·*x*, *g*·*y*)
⟨*proof*⟩

**lemma** *prod-map-ID*: *prod-map*·*ID*·*ID* = *ID*
⟨*proof*⟩

**lemma** *prod-map-map*:
*prod-map*·*f1*·*g1*·(*prod-map*·*f2*·*g2*·*p*) =
*prod-map*·(Λ *x*. *f1*·(*f2*·*x*))·(Λ *x*. *g1*·(*g2*·*x*))·*p*
⟨*proof*⟩

**lemma** *ep-pair-prod-map*:
**assumes** *ep-pair e1 p1* **and** *ep-pair e2 p2*
**shows** *ep-pair* (*prod-map*·*e1*·*e2*) (*prod-map*·*p1*·*p2*)
⟨*proof*⟩

**lemma** *deflation-prod-map*:
**assumes** *deflation d1* **and** *deflation d2*
**shows** *deflation* (*prod-map*·*d1*·*d2*)
⟨*proof*⟩

**lemma** *finite-deflation-prod-map*:
**assumes** *finite-deflation d1* **and** *finite-deflation d2*

**shows** *finite-deflation* (*prod-map*·*d1*·*d2*)
⟨*proof*⟩

## 22.3 Map function for lifted cpo

**definition**
*u-map* :: ('*a* → '*b*) → '*a u* → '*b u*
**where**
*u-map* = (Λ *f*. *fup*·(*up oo f*))

**lemma** *u-map-strict* [*simp*]: *u-map*·*f*·⊥ = ⊥
⟨*proof*⟩

**lemma** *u-map-up* [*simp*]: *u-map*·*f*·(*up*·*x*) = *up*·(*f*·*x*)
⟨*proof*⟩

**lemma** *u-map-ID*: *u-map*·*ID* = *ID*
⟨*proof*⟩

**lemma** *u-map-map*: *u-map*·*f*·(*u-map*·*g*·*p*) = *u-map*·(Λ *x*. *f*·(*g*·*x*))·*p*
⟨*proof*⟩

**lemma** *u-map-oo*: *u-map*·(*f oo g*) = *u-map*·*f oo u-map*·*g*
⟨*proof*⟩

**lemma** *ep-pair-u-map*: *ep-pair e p* ⟹ *ep-pair* (*u-map*·*e*) (*u-map*·*p*)
⟨*proof*⟩

**lemma** *deflation-u-map*: *deflation d* ⟹ *deflation* (*u-map*·*d*)
⟨*proof*⟩

**lemma** *finite-deflation-u-map*:
**assumes** *finite-deflation d* **shows** *finite-deflation* (*u-map*·*d*)
⟨*proof*⟩

## 22.4 Map function for strict products

**default-sort** *pcpo*

**definition**
*sprod-map* :: ('*a* → '*b*) → ('*c* → '*d*) → '*a* ⊗ '*c* → '*b* ⊗ '*d*
**where**
*sprod-map* = (Λ *f g*. *ssplit*·(Λ *x y*. (:*f*·*x*, *g*·*y*:)))

**lemma** *sprod-map-strict* [*simp*]: *sprod-map*·*a*·*b*·⊥ = ⊥
⟨*proof*⟩

**lemma** *sprod-map-spair* [*simp*]:
*x* ≠ ⊥ ⟹ *y* ≠ ⊥ ⟹ *sprod-map*·*f*·*g*·(:*x*, *y*:) = (:*f*·*x*, *g*·*y*:)
⟨*proof*⟩

**lemma** *sprod-map-spair'*:
$f\cdot\bot = \bot \Longrightarrow g\cdot\bot = \bot \Longrightarrow \mathit{sprod\text{-}map}\cdot f\cdot g\cdot(:x,\ y:) = (:f\cdot x,\ g\cdot y:)$
⟨*proof*⟩

**lemma** *sprod-map-ID*: $\mathit{sprod\text{-}map}\cdot ID\cdot ID = ID$
⟨*proof*⟩

**lemma** *sprod-map-map*:
$[\![f1\cdot\bot = \bot;\ g1\cdot\bot = \bot]\!] \Longrightarrow$
$\mathit{sprod\text{-}map}\cdot f1\cdot g1\cdot(\mathit{sprod\text{-}map}\cdot f2\cdot g2\cdot p) =$
$\mathit{sprod\text{-}map}\cdot(\Lambda\ x.\ f1\cdot(f2\cdot x))\cdot(\Lambda\ x.\ g1\cdot(g2\cdot x))\cdot p$
⟨*proof*⟩

**lemma** *ep-pair-sprod-map*:
**assumes** *ep-pair e1 p1* **and** *ep-pair e2 p2*
**shows** *ep-pair* $(\mathit{sprod\text{-}map}\cdot e1\cdot e2)\ (\mathit{sprod\text{-}map}\cdot p1\cdot p2)$
⟨*proof*⟩

**lemma** *deflation-sprod-map*:
**assumes** *deflation d1* **and** *deflation d2*
**shows** *deflation* $(\mathit{sprod\text{-}map}\cdot d1\cdot d2)$
⟨*proof*⟩

**lemma** *finite-deflation-sprod-map*:
**assumes** *finite-deflation d1* **and** *finite-deflation d2*
**shows** *finite-deflation* $(\mathit{sprod\text{-}map}\cdot d1\cdot d2)$
⟨*proof*⟩

## 22.5 Map function for strict sums

**definition**
$\mathit{ssum\text{-}map} :: ('a \to 'b) \to ('c \to 'd) \to 'a \oplus 'c \to 'b \oplus 'd$
**where**
$\mathit{ssum\text{-}map} = (\Lambda\ f\ g.\ \mathit{sscase}\cdot(\mathit{sinl}\ oo\ f)\cdot(\mathit{sinr}\ oo\ g))$

**lemma** *ssum-map-strict* [*simp*]: $\mathit{ssum\text{-}map}\cdot f\cdot g\cdot\bot = \bot$
⟨*proof*⟩

**lemma** *ssum-map-sinl* [*simp*]: $x \neq \bot \Longrightarrow \mathit{ssum\text{-}map}\cdot f\cdot g\cdot(\mathit{sinl}\cdot x) = \mathit{sinl}\cdot(f\cdot x)$
⟨*proof*⟩

**lemma** *ssum-map-sinr* [*simp*]: $x \neq \bot \Longrightarrow \mathit{ssum\text{-}map}\cdot f\cdot g\cdot(\mathit{sinr}\cdot x) = \mathit{sinr}\cdot(g\cdot x)$
⟨*proof*⟩

**lemma** *ssum-map-sinl'*: $f\cdot\bot = \bot \Longrightarrow \mathit{ssum\text{-}map}\cdot f\cdot g\cdot(\mathit{sinl}\cdot x) = \mathit{sinl}\cdot(f\cdot x)$
⟨*proof*⟩

**lemma** *ssum-map-sinr'*: $g\cdot\bot = \bot \Longrightarrow \mathit{ssum\text{-}map}\cdot f\cdot g\cdot(\mathit{sinr}\cdot x) = \mathit{sinr}\cdot(g\cdot x)$

⟨*proof*⟩

**lemma** *ssum-map-ID*: *ssum-map·ID·ID = ID*
⟨*proof*⟩

**lemma** *ssum-map-map*:
⟦*f1·⊥ = ⊥; g1·⊥ = ⊥*⟧ ⟹
*ssum-map·f1·g1·(ssum-map·f2·g2·p) =*
*ssum-map·(Λ x. f1·(f2·x))·(Λ x. g1·(g2·x))·p*
⟨*proof*⟩

**lemma** *ep-pair-ssum-map*:
**assumes** *ep-pair e1 p1* **and** *ep-pair e2 p2*
**shows** *ep-pair (ssum-map·e1·e2) (ssum-map·p1·p2)*
⟨*proof*⟩

**lemma** *deflation-ssum-map*:
**assumes** *deflation d1* **and** *deflation d2*
**shows** *deflation (ssum-map·d1·d2)*
⟨*proof*⟩

**lemma** *finite-deflation-ssum-map*:
**assumes** *finite-deflation d1* **and** *finite-deflation d2*
**shows** *finite-deflation (ssum-map·d1·d2)*
⟨*proof*⟩

## 22.6 Map operator for strict function space

**definition**
*sfun-map :: ('b → 'a) → ('c → 'd) → ('a →! 'c) → ('b →! 'd)*
**where**
*sfun-map = (Λ a b. sfun-abs oo cfun-map·a·b oo sfun-rep)*

**lemma** *sfun-map-ID*: *sfun-map·ID·ID = ID*
⟨*proof*⟩

**lemma** *sfun-map-map*:
**assumes** *f2·⊥ = ⊥* **and** *g2·⊥ = ⊥* **shows**
*sfun-map·f1·g1·(sfun-map·f2·g2·p) =*
*sfun-map·(Λ x. f2·(f1·x))·(Λ x. g1·(g2·x))·p*
⟨*proof*⟩

**lemma** *ep-pair-sfun-map*:
**assumes** *1*: *ep-pair e1 p1*
**assumes** *2*: *ep-pair e2 p2*
**shows** *ep-pair (sfun-map·p1·e2) (sfun-map·e1·p2)*
⟨*proof*⟩

**lemma** *deflation-sfun-map*:

**assumes** *1*: *deflation d1*
**assumes** *2*: *deflation d2*
**shows** *deflation* (*sfun-map·d1·d2*)
⟨*proof*⟩

**lemma** *finite-deflation-sfun-map*:
**assumes** *1*: *finite-deflation d1*
**assumes** *2*: *finite-deflation d2*
**shows** *finite-deflation* (*sfun-map·d1·d2*)
⟨*proof*⟩

**end**

# 23 Profinite and bifinite cpos

**theory** *Bifinite*
**imports** *Map-Functions ~~/src/HOL/Library/Countable*
**begin**

**default-sort** *cpo*

## 23.1 Chains of finite deflations

**locale** *approx-chain* =
**fixes** *approx* :: *nat* ⇒ *'a* → *'a*
**assumes** *chain-approx* [*simp*]: *chain* (λ*i*. *approx i*)
**assumes** *lub-approx* [*simp*]: (⨆ *i*. *approx i*) = *ID*
**assumes** *finite-deflation-approx* [*simp*]: ⋀*i*. *finite-deflation* (*approx i*)
**begin**

**lemma** *deflation-approx*: *deflation* (*approx i*)
⟨*proof*⟩

**lemma** *approx-idem*: *approx i*·(*approx i*·*x*) = *approx i*·*x*
⟨*proof*⟩

**lemma** *approx-below*: *approx i*·*x* ⊑ *x*
⟨*proof*⟩

**lemma** *finite-range-approx*: *finite* (*range* (λ*x*. *approx i*·*x*))
⟨*proof*⟩

**lemma** *compact-approx* [*simp*]: *compact* (*approx n*·*x*)
⟨*proof*⟩

**lemma** *compact-eq-approx*: *compact x* ⟹ ∃ *i*. *approx i*·*x* = *x*
⟨*proof*⟩

**end**

## 23.2 Omega-profinite and bifinite domains

**class** *bifinite* = *pcpo* +
**assumes** *bifinite*: $\exists (a::nat \Rightarrow 'a \rightarrow 'a).$ *approx-chain a*

**class** *profinite* = *cpo* +
**assumes** *profinite*: $\exists (a::nat \Rightarrow 'a_\bot \rightarrow 'a_\bot).$ *approx-chain a*

## 23.3 Building approx chains

**lemma** *approx-chain-iso*:
**assumes** *a*: *approx-chain a*
**assumes** [*simp*]: $\bigwedge x.\ f\cdot(g\cdot x) = x$
**assumes** [*simp*]: $\bigwedge y.\ g\cdot(f\cdot y) = y$
**shows** *approx-chain* ($\lambda i$. *f oo a i oo g*)
⟨*proof*⟩

**lemma** *approx-chain-u-map*:
**assumes** *approx-chain a*
**shows** *approx-chain* ($\lambda i$. *u-map*$\cdot$(*a i*))
⟨*proof*⟩

**lemma** *approx-chain-sfun-map*:
**assumes** *approx-chain a* **and** *approx-chain b*
**shows** *approx-chain* ($\lambda i$. *sfun-map*$\cdot$(*a i*)$\cdot$(*b i*))
⟨*proof*⟩

**lemma** *approx-chain-sprod-map*:
**assumes** *approx-chain a* **and** *approx-chain b*
**shows** *approx-chain* ($\lambda i$. *sprod-map*$\cdot$(*a i*)$\cdot$(*b i*))
⟨*proof*⟩

**lemma** *approx-chain-ssum-map*:
**assumes** *approx-chain a* **and** *approx-chain b*
**shows** *approx-chain* ($\lambda i$. *ssum-map*$\cdot$(*a i*)$\cdot$(*b i*))
⟨*proof*⟩

**lemma** *approx-chain-cfun-map*:
**assumes** *approx-chain a* **and** *approx-chain b*
**shows** *approx-chain* ($\lambda i$. *cfun-map*$\cdot$(*a i*)$\cdot$(*b i*))
⟨*proof*⟩

**lemma** *approx-chain-prod-map*:
**assumes** *approx-chain a* **and** *approx-chain b*
**shows** *approx-chain* ($\lambda i$. *prod-map*$\cdot$(*a i*)$\cdot$(*b i*))
⟨*proof*⟩

Approx chains for countable discrete types.

**definition** *discr-approx* :: $nat \Rightarrow 'a::countable\ discr\ u \rightarrow 'a\ discr\ u$
**where** *discr-approx* = ($\lambda i$. $\Lambda$(*up*$\cdot$*x*). *if to-nat* (*undiscr x*) < *i then up*$\cdot$*x else* $\bot$)

**lemma** *chain-discr-approx* [*simp*]: *chain discr-approx*
⟨*proof*⟩

**lemma** *lub-discr-approx* [*simp*]: $(\bigsqcup i.\ discr\text{-}approx\ i) = ID$
⟨*proof*⟩

**lemma** *inj-on-undiscr* [*simp*]: *inj-on undiscr A*
⟨*proof*⟩

**lemma** *finite-deflation-discr-approx*: *finite-deflation* (*discr-approx i*)
⟨*proof*⟩

**lemma** *discr-approx*: *approx-chain discr-approx*
⟨*proof*⟩

## 23.4 Class instance proofs

**instance** *bifinite* ⊆ *profinite*
⟨*proof*⟩

**instance** *u* :: (*profinite*) *bifinite*
⟨*proof*⟩

Types $'a \rightarrow\ 'b$ and $'a_\bot \rightarrow!\ 'b$ are isomorphic.

**definition** *encode-cfun* = (Λ *f*. *sfun-abs*·(*fup*·*f*))

**definition** *decode-cfun* = (Λ *g x*. *sfun-rep*·*g*·(*up*·*x*))

**lemma** *decode-encode-cfun* [*simp*]: *decode-cfun*·(*encode-cfun*·*x*) = *x*
⟨*proof*⟩

**lemma** *encode-decode-cfun* [*simp*]: *encode-cfun*·(*decode-cfun*·*y*) = *y*
⟨*proof*⟩

**instance** *cfun* :: (*profinite*, *bifinite*) *bifinite*
⟨*proof*⟩

Types $('a \times\ 'b)_\bot$ and $'a_\bot \otimes\ 'b_\bot$ are isomorphic.

**definition** *encode-prod-u* = (Λ(*up*·(*x*, *y*)). (:*up*·*x*, *up*·*y*:))

**definition** *decode-prod-u* = (Λ(:*up*·*x*, *up*·*y*:). *up*·(*x*, *y*))

**lemma** *decode-encode-prod-u* [*simp*]: *decode-prod-u*·(*encode-prod-u*·*x*) = *x*
⟨*proof*⟩

**lemma** *encode-decode-prod-u* [*simp*]: *encode-prod-u*·(*decode-prod-u*·*y*) = *y*
⟨*proof*⟩

**instance** *prod* :: (*profinite*, *profinite*) *profinite*
⟨*proof*⟩

**instance** *prod* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

**instance** *sfun* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

**instance** *sprod* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

**instance** *ssum* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

**lemma** *approx-chain-unit*: *approx-chain* (⊥ :: *nat* ⇒ *unit* → *unit*)
⟨*proof*⟩

**instance** *unit* :: *bifinite*
⟨*proof*⟩

**instance** *discr* :: (*countable*) *profinite*
⟨*proof*⟩

**instance** *lift* :: (*countable*) *bifinite*
⟨*proof*⟩

**end**

# 24 Defining algebraic domains by ideal completion

**theory** *Completion*
**imports** *Plain-HOLCF*
**begin**

## 24.1 Ideals over a preorder

**locale** *preorder* =
**fixes** $r :: 'a::type \Rightarrow 'a \Rightarrow bool$ (**infix** $\preceq$ *50*)
**assumes** *r-refl*: $x \preceq x$
**assumes** *r-trans*: $\llbracket x \preceq y;\ y \preceq z \rrbracket \Longrightarrow x \preceq z$
**begin**

**definition**
*ideal* :: $'a\ set \Rightarrow bool$ **where**
$ideal\ A = ((\exists x.\ x \in A) \wedge (\forall x{\in}A.\ \forall y{\in}A.\ \exists z{\in}A.\ x \preceq z \wedge y \preceq z) \wedge (\forall x\ y.\ x \preceq y \longrightarrow y \in A \longrightarrow x \in A))$

**lemma** *idealI*:

**assumes** $\exists x.\ x \in A$
**assumes** $\bigwedge x\ y.\ [\![x \in A;\ y \in A]\!] \Longrightarrow \exists z{\in}A.\ x \preceq z \wedge y \preceq z$
**assumes** $\bigwedge x\ y.\ [\![x \preceq y;\ y \in A]\!] \Longrightarrow x \in A$
**shows** *ideal A*
⟨*proof*⟩

**lemma** *idealD1*:
$ideal\ A \Longrightarrow \exists x.\ x \in A$
⟨*proof*⟩

**lemma** *idealD2*:
$[\![ideal\ A;\ x \in A;\ y \in A]\!] \Longrightarrow \exists z{\in}A.\ x \preceq z \wedge y \preceq z$
⟨*proof*⟩

**lemma** *idealD3*:
$[\![ideal\ A;\ x \preceq y;\ y \in A]\!] \Longrightarrow x \in A$
⟨*proof*⟩

**lemma** *ideal-principal*: $ideal\ \{x.\ x \preceq z\}$
⟨*proof*⟩

**lemma** *ex-ideal*: $\exists A.\ A \in \{A.\ ideal\ A\}$
⟨*proof*⟩

The set of ideals is a cpo

**lemma** *ideal-UN*:
**fixes** $A :: nat \Rightarrow {'a}\ set$
**assumes** *ideal-A*: $\bigwedge i.\ ideal\ (A\ i)$
**assumes** *chain-A*: $\bigwedge i\ j.\ i \le j \Longrightarrow A\ i \subseteq A\ j$
**shows** $ideal\ (\bigcup i.\ A\ i)$
⟨*proof*⟩

**lemma** *typedef-ideal-po*:
**fixes** $Abs :: {'a}\ set \Rightarrow {'b}::below$
**assumes** *type*: $type\text{-}definition\ Rep\ Abs\ \{S.\ ideal\ S\}$
**assumes** *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
**shows** $OFCLASS({'b},\ po\text{-}class)$
⟨*proof*⟩

**lemma**
**fixes** $Abs :: {'a}\ set \Rightarrow {'b}::po$
**assumes** *type*: $type\text{-}definition\ Rep\ Abs\ \{S.\ ideal\ S\}$
**assumes** *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
**assumes** *S*: $chain\ S$
**shows** *typedef-ideal-lub*: $range\ S <<|\ Abs\ (\bigcup i.\ Rep\ (S\ i))$
**and** *typedef-ideal-rep-lub*: $Rep\ (\bigsqcup i.\ S\ i) = (\bigcup i.\ Rep\ (S\ i))$
⟨*proof*⟩

**lemma** *typedef-ideal-cpo*:

**fixes** *Abs* :: $'a$ *set* $\Rightarrow$ $'b$::*po*
**assumes** *type*: *type-definition Rep Abs* $\{S.$ *ideal* $S\}$
**assumes** *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow$ *Rep* $x \subseteq$ *Rep* $y$
**shows** *OFCLASS*($'b$, *cpo-class*)
⟨*proof*⟩

**end**

**interpretation** *below*: *preorder below* :: $'a$::*po* $\Rightarrow$ $'a \Rightarrow$ *bool*
⟨*proof*⟩

## 24.2 Lemmas about least upper bounds

**lemma** *is-ub-thelub-ex*: $[\![\exists\, u.\ S <<|\ u;\ x \in S]\!] \Longrightarrow x \sqsubseteq$ *lub* $S$
⟨*proof*⟩

**lemma** *is-lub-thelub-ex*: $[\![\exists\, u.\ S <<|\ u;\ S <|\ x]\!] \Longrightarrow$ *lub* $S \sqsubseteq x$
⟨*proof*⟩

## 24.3 Locale for ideal completion

**locale** *ideal-completion* = *preorder* +
**fixes** *principal* :: $'a$::*type* $\Rightarrow$ $'b$::*cpo*
**fixes** *rep* :: $'b$::*cpo* $\Rightarrow$ $'a$::*type set*
**assumes** *ideal-rep*: $\bigwedge x.$ *ideal* (*rep* $x$)
**assumes** *rep-lub*: $\bigwedge Y.$ *chain* $Y \Longrightarrow$ *rep* $(\bigsqcup i.\ Y\ i) = (\bigcup i.$ *rep* $(Y\ i))$
**assumes** *rep-principal*: $\bigwedge a.$ *rep* (*principal* $a$) $= \{b.\ b \preceq a\}$
**assumes** *belowI*: $\bigwedge x\ y.$ *rep* $x \subseteq$ *rep* $y \Longrightarrow x \sqsubseteq y$
**assumes** *countable*: $\exists f$::$'a \Rightarrow$ *nat*. *inj* $f$
**begin**

**lemma** *rep-mono*: $x \sqsubseteq y \Longrightarrow$ *rep* $x \subseteq$ *rep* $y$
⟨*proof*⟩

**lemma** *below-def*: $x \sqsubseteq y \longleftrightarrow$ *rep* $x \subseteq$ *rep* $y$
⟨*proof*⟩

**lemma** *principal-below-iff-mem-rep*: *principal* $a \sqsubseteq x \longleftrightarrow a \in$ *rep* $x$
⟨*proof*⟩

**lemma** *principal-below-iff* [*simp*]: *principal* $a \sqsubseteq$ *principal* $b \longleftrightarrow a \preceq b$
⟨*proof*⟩

**lemma** *principal-eq-iff*: *principal* $a$ = *principal* $b \longleftrightarrow a \preceq b \wedge b \preceq a$
⟨*proof*⟩

**lemma** *eq-iff*: $x = y \longleftrightarrow$ *rep* $x$ = *rep* $y$
⟨*proof*⟩

**lemma** *principal-mono*: $a \preceq b \Longrightarrow$ *principal* $a \sqsubseteq$ *principal* $b$

⟨*proof*⟩

**lemma** *ch2ch-principal* [*simp*]:
  $\forall\, i.\ Y\ i \preceq Y\ (Suc\ i) \Longrightarrow chain\ (\lambda i.\ principal\ (Y\ i))$
⟨*proof*⟩

### 24.3.1 Principal ideals approximate all elements

**lemma** *compact-principal* [*simp*]: $compact\ (principal\ a)$
⟨*proof*⟩

Construct a chain whose lub is the same as a given ideal

**lemma** *obtain-principal-chain*:
  **obtains** $Y$ **where** $\forall\, i.\ Y\ i \preceq Y\ (Suc\ i)$ **and** $x = (\bigsqcup i.\ principal\ (Y\ i))$
⟨*proof*⟩

**lemma** *principal-induct*:
  **assumes** *adm*: $adm\ P$
  **assumes** $P$: $\bigwedge a.\ P\ (principal\ a)$
  **shows** $P\ x$
⟨*proof*⟩

**lemma** *compact-imp-principal*: $compact\ x \Longrightarrow \exists\, a.\ x = principal\ a$
⟨*proof*⟩

## 24.4 Defining functions in terms of basis elements

**definition**
  $extension :: ('a::type \Rightarrow 'c::cpo) \Rightarrow 'b \rightarrow 'c$ **where**
  $extension = (\lambda f.\ (\Lambda\ x.\ lub\ (f\ `\ rep\ x)))$

**lemma** *extension-lemma*:
  **fixes** $f :: 'a::type \Rightarrow 'c::cpo$
  **assumes** *f-mono*: $\bigwedge a\ b.\ a \preceq b \Longrightarrow f\ a \sqsubseteq f\ b$
  **shows** $\exists\, u.\ f\ `\ rep\ x <<|\ u$
⟨*proof*⟩

**lemma** *extension-beta*:
  **fixes** $f :: 'a::type \Rightarrow 'c::cpo$
  **assumes** *f-mono*: $\bigwedge a\ b.\ a \preceq b \Longrightarrow f\ a \sqsubseteq f\ b$
  **shows** $extension\ f \cdot x = lub\ (f\ `\ rep\ x)$
⟨*proof*⟩

**lemma** *extension-principal*:
  **fixes** $f :: 'a::type \Rightarrow 'c::cpo$
  **assumes** *f-mono*: $\bigwedge a\ b.\ a \preceq b \Longrightarrow f\ a \sqsubseteq f\ b$
  **shows** $extension\ f \cdot (principal\ a) = f\ a$
⟨*proof*⟩

**lemma** *extension-mono*:
**assumes** *f-mono*: $\bigwedge a\ b.\ a \preceq b \Longrightarrow f\ a \sqsubseteq f\ b$
**assumes** *g-mono*: $\bigwedge a\ b.\ a \preceq b \Longrightarrow g\ a \sqsubseteq g\ b$
**assumes** *below*: $\bigwedge a.\ f\ a \sqsubseteq g\ a$
**shows** *extension f* $\sqsubseteq$ *extension g*
⟨*proof*⟩

**lemma** *cont-extension*:
**assumes** *f-mono*: $\bigwedge a\ b\ x.\ a \preceq b \Longrightarrow f\ x\ a \sqsubseteq f\ x\ b$
**assumes** *f-cont*: $\bigwedge a.\ cont\ (\lambda x.\ f\ x\ a)$
**shows** $cont\ (\lambda x.\ extension\ (\lambda a.\ f\ x\ a))$
⟨*proof*⟩

**end**

**lemma** (**in** *preorder*) *typedef-ideal-completion*:
**fixes** *Abs* :: *'a set* ⇒ *'b::cpo*
**assumes** *type*: *type-definition Rep Abs* {*S. ideal S*}
**assumes** *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
**assumes** *principal*: $\bigwedge a.\ principal\ a = Abs\ \{b.\ b \preceq a\}$
**assumes** *countable*: $\exists f{::}'a \Rightarrow nat.\ inj\ f$
**shows** *ideal-completion r principal Rep*
⟨*proof*⟩

**end**

# 25 A universal bifinite domain

**theory** *Universal*
**imports** *Bifinite Completion ~~/src/HOL/Library/Nat-Bijection*
**begin**

## 25.1 Basis for universal domain

### 25.1.1 Basis datatype

**type-synonym** *ubasis = nat*

**definition**
*node* :: *nat* ⇒ *ubasis* ⇒ *ubasis set* ⇒ *ubasis*
**where**
*node i a S = Suc (prod-encode (i, prod-encode (a, set-encode S)))*

**lemma** *node-not-0* [*simp*]: $node\ i\ a\ S \neq 0$
⟨*proof*⟩

**lemma** *node-gt-0* [*simp*]: $0 < node\ i\ a\ S$
⟨*proof*⟩

**lemma** *node-inject* [*simp*]:
$[\![$*finite* $S$; *finite* $T]\!]$
$\Longrightarrow$ *node* $i$ $a$ $S$ = *node* $j$ $b$ $T \longleftrightarrow i = j \wedge a = b \wedge S = T$
⟨*proof*⟩

**lemma** *node-gt0*: $i <$ *node* $i$ $a$ $S$
⟨*proof*⟩

**lemma** *node-gt1*: $a <$ *node* $i$ $a$ $S$
⟨*proof*⟩

**lemma** *nat-less-power2*: $n < 2\hat{}n$
⟨*proof*⟩

**lemma** *node-gt2*: $[\![$*finite* $S$; $b \in S]\!] \Longrightarrow b <$ *node* $i$ $a$ $S$
⟨*proof*⟩

**lemma** *eq-prod-encode-pairI*:
$[\![$*fst* (*prod-decode* $x$) = $a$; *snd* (*prod-decode* $x$) = $b]\!] \Longrightarrow x$ = *prod-encode* $(a, b)$
⟨*proof*⟩

**lemma** *node-cases*:
**assumes** *1*: $x = 0 \Longrightarrow P$
**assumes** *2*: $\bigwedge i\ a\ S$. $[\![$*finite* $S$; $x$ = *node* $i$ $a$ $S]\!] \Longrightarrow P$
**shows** $P$
⟨*proof*⟩

**lemma** *node-induct*:
**assumes** *1*: $P\ 0$
**assumes** *2*: $\bigwedge i\ a\ S$. $[\![P\ a$; *finite* $S$; $\forall\, b{\in}S.\ P\ b]\!] \Longrightarrow P$ (*node* $i$ $a$ $S$)
**shows** $P\ x$
⟨*proof*⟩

### 25.1.2 Basis ordering

**inductive**
*ubasis-le* :: *nat* ⇒ *nat* ⇒ *bool*
**where**
*ubasis-le-refl*: *ubasis-le* $a$ $a$
| *ubasis-le-trans*:
$[\![$*ubasis-le* $a$ $b$; *ubasis-le* $b$ $c]\!] \Longrightarrow$ *ubasis-le* $a$ $c$
| *ubasis-le-lower*:
*finite* $S \Longrightarrow$ *ubasis-le* $a$ (*node* $i$ $a$ $S$)
| *ubasis-le-upper*:
$[\![$*finite* $S$; $b \in S$; *ubasis-le* $a$ $b]\!] \Longrightarrow$ *ubasis-le* (*node* $i$ $a$ $S$) $b$

**lemma** *ubasis-le-minimal*: *ubasis-le* $0$ $x$
⟨*proof*⟩

**interpretation** *udom*: *preorder ubasis-le*
⟨*proof*⟩

### 25.1.3 Generic take function

**function**
*ubasis-until* :: (*ubasis* ⇒ *bool*) ⇒ *ubasis* ⇒ *ubasis*
**where**
*ubasis-until P 0* = *0*
| *finite S* ⟹ *ubasis-until P* (*node i a S*) =
(*if P* (*node i a S*) *then node i a S else ubasis-until P a*)
⟨*proof*⟩

**termination** *ubasis-until*
⟨*proof*⟩

**lemma** *ubasis-until*: *P 0* ⟹ *P* (*ubasis-until P x*)
⟨*proof*⟩

**lemma** *ubasis-until'*: *0* < *ubasis-until P x* ⟹ *P* (*ubasis-until P x*)
⟨*proof*⟩

**lemma** *ubasis-until-same*: *P x* ⟹ *ubasis-until P x* = *x*
⟨*proof*⟩

**lemma** *ubasis-until-idem*:
*P 0* ⟹ *ubasis-until P* (*ubasis-until P x*) = *ubasis-until P x*
⟨*proof*⟩

**lemma** *ubasis-until-0*:
∀ *x*. *x* ≠ *0* ⟶ ¬ *P x* ⟹ *ubasis-until P x* = *0*
⟨*proof*⟩

**lemma** *ubasis-until-less*: *ubasis-le* (*ubasis-until P x*) *x*
⟨*proof*⟩

**lemma** *ubasis-until-chain*:
**assumes** *PQ*: ⋀*x*. *P x* ⟹ *Q x*
**shows** *ubasis-le* (*ubasis-until P x*) (*ubasis-until Q x*)
⟨*proof*⟩

**lemma** *ubasis-until-mono*:
**assumes** ⋀*i a S b*. ⟦*finite S*; *P* (*node i a S*); *b* ∈ *S*; *ubasis-le a b*⟧ ⟹ *P b*
**shows** *ubasis-le a b* ⟹ *ubasis-le* (*ubasis-until P a*) (*ubasis-until P b*)
⟨*proof*⟩

**lemma** *finite-range-ubasis-until*:
*finite* {*x*. *P x*} ⟹ *finite* (*range* (*ubasis-until P*))
⟨*proof*⟩

## 25.2 Defining the universal domain by ideal completion

**typedef** *udom* = {*S*. *udom.ideal S*}
⟨*proof*⟩

**instantiation** *udom* :: *below*
**begin**

**definition**
$x \sqsubseteq y \longleftrightarrow$ *Rep-udom x* ⊆ *Rep-udom y*

**instance** ⟨*proof*⟩
**end**

**instance** *udom* :: *po*
⟨*proof*⟩

**instance** *udom* :: *cpo*
⟨*proof*⟩

**definition**
*udom-principal* :: *nat* ⇒ *udom* **where**
*udom-principal t* = *Abs-udom* {*u*. *ubasis-le u t*}

**lemma** *ubasis-countable*: ∃*f*::*ubasis* ⇒ *nat*. *inj f*
⟨*proof*⟩

**interpretation** *udom*:
*ideal-completion ubasis-le udom-principal Rep-udom*
⟨*proof*⟩

Universal domain is pointed

**lemma** *udom-minimal*: *udom-principal 0* ⊑ *x*
⟨*proof*⟩

**instance** *udom* :: *pcpo*
⟨*proof*⟩

**lemma** *inst-udom-pcpo*: ⊥ = *udom-principal 0*
⟨*proof*⟩

## 25.3 Compact bases of domains

**typedef** *'a compact-basis* = {*x*::*'a*::*pcpo*. *compact x*}
⟨*proof*⟩

**lemma** *Rep-compact-basis'* [*simp*]: *compact* (*Rep-compact-basis a*)
⟨*proof*⟩

**lemma** *Abs-compact-basis-inverse'* [*simp*]:

*compact x* $\Longrightarrow$ *Rep-compact-basis* (*Abs-compact-basis x*) = *x*
⟨*proof*⟩

**instantiation** *compact-basis* :: (*pcpo*) *below*
**begin**

**definition**
*compact-le-def*:
(*op* ⊑) ≡ (λ*x y*. *Rep-compact-basis x* ⊑ *Rep-compact-basis y*)

**instance** ⟨*proof*⟩
**end**

**instance** *compact-basis* :: (*pcpo*) *po*
⟨*proof*⟩

**definition**
*approximants* :: *'a* ⇒ *'a compact-basis set* **where**
*approximants* = (λ*x*. {*a*. *Rep-compact-basis a* ⊑ *x*})

**definition**
*compact-bot* :: *'a::pcpo compact-basis* **where**
*compact-bot* = *Abs-compact-basis* ⊥

**lemma** *Rep-compact-bot* [*simp*]: *Rep-compact-basis compact-bot* = ⊥
⟨*proof*⟩

**lemma** *compact-bot-minimal* [*simp*]: *compact-bot* ⊑ *a*
⟨*proof*⟩

## 25.4 Universality of *udom*

We use a locale to parameterize the construction over a chain of approx functions on the type to be embedded.

**locale** *bifinite-approx-chain* =
*approx-chain approx* **for** *approx* :: *nat* ⇒ *'a::bifinite* → *'a*
**begin**

### 25.4.1 Choosing a maximal element from a finite set

**lemma** *finite-has-maximal*:
**fixes** *A* :: *'a compact-basis set*
**shows** ⟦*finite A*; *A* ≠ {}⟧ $\Longrightarrow$ ∃ *x*∈*A*. ∀ *y*∈*A*. *x* ⊑ *y* ⟶ *x* = *y*
⟨*proof*⟩

**definition**
*choose* :: *'a compact-basis set* ⇒ *'a compact-basis*
**where**
*choose A* = (*SOME x*. *x* ∈ {*x*∈*A*. ∀ *y*∈*A*. *x* ⊑ *y* ⟶ *x* = *y*})

**lemma** *choose-lemma*:
$[\![$*finite A*; $A \neq \{\}]\!] \Longrightarrow$ *choose A* $\in \{x{\in}A.\ \forall\, y{\in}A.\ x \sqsubseteq y \longrightarrow x = y\}$
⟨*proof*⟩

**lemma** *maximal-choose*:
$[\![$*finite A*; $y \in A$; *choose A* $\sqsubseteq y]\!] \Longrightarrow$ *choose A* $= y$
⟨*proof*⟩

**lemma** *choose-in*: $[\![$*finite A*; $A \neq \{\}]\!] \Longrightarrow$ *choose A* $\in A$
⟨*proof*⟩

**function**
*choose-pos* :: *'a compact-basis set* ⇒ *'a compact-basis* ⇒ *nat*
**where**
*choose-pos A x* =
(*if finite A* ∧ *x* ∈ *A* ∧ *x* ≠ *choose A*
*then Suc* (*choose-pos* (*A* − {*choose A*}) *x*) *else 0*)
⟨*proof*⟩

**termination** *choose-pos*
⟨*proof*⟩

**declare** *choose-pos.simps* [*simp del*]

**lemma** *choose-pos-choose*: *finite A* ⟹ *choose-pos A* (*choose A*) = *0*
⟨*proof*⟩

**lemma** *inj-on-choose-pos* [*OF refl*]:
$[\![$*card A* = *n*; *finite A*$]\!] \Longrightarrow$ *inj-on* (*choose-pos A*) *A*
⟨*proof*⟩

**lemma** *choose-pos-bounded* [*OF refl*]:
$[\![$*card A* = *n*; *finite A*; *x* ∈ *A*$]\!] \Longrightarrow$ *choose-pos A x* < *n*
⟨*proof*⟩

**lemma** *choose-pos-lessD*:
$[\![$*choose-pos A x* < *choose-pos A y*; *finite A*; *x* ∈ *A*; *y* ∈ *A*$]\!] \Longrightarrow x \not\sqsubseteq y$
⟨*proof*⟩

### 25.4.2 Compact basis take function

**primrec**
*cb-take* :: *nat* ⇒ *'a compact-basis* ⇒ *'a compact-basis* **where**
*cb-take 0* = (λ*x*. *compact-bot*)
| *cb-take* (*Suc n*) = (λ*a*. *Abs-compact-basis* (*approx n*·(*Rep-compact-basis a*)))

**declare** *cb-take.simps* [*simp del*]

**lemma** *cb-take-zero* [*simp*]: *cb-take 0 a* = *compact-bot*
⟨*proof*⟩

**lemma** *Rep-cb-take*:
*Rep-compact-basis* (*cb-take* (*Suc n*) *a*) = *approx n*·(*Rep-compact-basis a*)
⟨*proof*⟩

**lemmas** *approx-Rep-compact-basis* = *Rep-cb-take* [*symmetric*]

**lemma** *cb-take-covers*: ∃ *n*. *cb-take n x* = *x*
⟨*proof*⟩

**lemma** *cb-take-less*: *cb-take n x* ⊑ *x*
⟨*proof*⟩

**lemma** *cb-take-idem*: *cb-take n* (*cb-take n x*) = *cb-take n x*
⟨*proof*⟩

**lemma** *cb-take-mono*: *x* ⊑ *y* ⟹ *cb-take n x* ⊑ *cb-take n y*
⟨*proof*⟩

**lemma** *cb-take-chain-le*: *m* ≤ *n* ⟹ *cb-take m x* ⊑ *cb-take n x*
⟨*proof*⟩

**lemma** *finite-range-cb-take*: *finite* (*range* (*cb-take n*))
⟨*proof*⟩

### 25.4.3 Rank of basis elements

**definition**
*rank* :: ′*a compact-basis* ⇒ *nat*
**where**
*rank x* = (*LEAST n*. *cb-take n x* = *x*)

**lemma** *compact-approx-rank*: *cb-take* (*rank x*) *x* = *x*
⟨*proof*⟩

**lemma** *rank-leD*: *rank x* ≤ *n* ⟹ *cb-take n x* = *x*
⟨*proof*⟩

**lemma** *rank-leI*: *cb-take n x* = *x* ⟹ *rank x* ≤ *n*
⟨*proof*⟩

**lemma** *rank-le-iff*: *rank x* ≤ *n* ⟷ *cb-take n x* = *x*
⟨*proof*⟩

**lemma** *rank-compact-bot* [*simp*]: *rank compact-bot* = *0*
⟨*proof*⟩

**lemma** *rank-eq-0-iff* [*simp*]: *rank x* = *0* ⟷ *x* = *compact-bot*
⟨*proof*⟩

**definition**
*rank-le* :: ′*a compact-basis* ⇒ ′*a compact-basis set*
**where**
*rank-le x* = {*y*. *rank y* ≤ *rank x*}

**definition**
*rank-lt* :: ′*a compact-basis* ⇒ ′*a compact-basis set*
**where**
*rank-lt x* = {*y*. *rank y* < *rank x*}

**definition**
*rank-eq* :: ′*a compact-basis* ⇒ ′*a compact-basis set*
**where**
*rank-eq x* = {*y*. *rank y* = *rank x*}

**lemma** *rank-eq-cong*: *rank x* = *rank y* ⟹ *rank-eq x* = *rank-eq y*
⟨*proof*⟩

**lemma** *rank-lt-cong*: *rank x* = *rank y* ⟹ *rank-lt x* = *rank-lt y*
⟨*proof*⟩

**lemma** *rank-eq-subset*: *rank-eq x* ⊆ *rank-le x*
⟨*proof*⟩

**lemma** *rank-lt-subset*: *rank-lt x* ⊆ *rank-le x*
⟨*proof*⟩

**lemma** *finite-rank-le*: *finite* (*rank-le x*)
⟨*proof*⟩

**lemma** *finite-rank-eq*: *finite* (*rank-eq x*)
⟨*proof*⟩

**lemma** *finite-rank-lt*: *finite* (*rank-lt x*)
⟨*proof*⟩

**lemma** *rank-lt-Int-rank-eq*: *rank-lt x* ∩ *rank-eq x* = {}
⟨*proof*⟩

**lemma** *rank-lt-Un-rank-eq*: *rank-lt x* ∪ *rank-eq x* = *rank-le x*
⟨*proof*⟩

### 25.4.4 Sequencing basis elements

**definition**
*place* :: ′*a compact-basis* ⇒ *nat*

**where**
*place x = card (rank-lt x) + choose-pos (rank-eq x) x*

**lemma** *place-bounded*: *place x < card (rank-le x)*
⟨*proof*⟩

**lemma** *place-ge*: *card (rank-lt x) ≤ place x*
⟨*proof*⟩

**lemma** *place-rank-mono*:
**fixes** *x y* :: *'a compact-basis*
**shows** *rank x < rank y ⟹ place x < place y*
⟨*proof*⟩

**lemma** *place-eqD*: *place x = place y ⟹ x = y*
⟨*proof*⟩

**lemma** *inj-place*: *inj place*
⟨*proof*⟩

### 25.4.5 Embedding and projection on basis elements

**definition**
*sub* :: *'a compact-basis ⇒ 'a compact-basis*
**where**
*sub x = (case rank x of 0 ⇒ compact-bot | Suc k ⇒ cb-take k x)*

**lemma** *rank-sub-less*: *x ≠ compact-bot ⟹ rank (sub x) < rank x*
⟨*proof*⟩

**lemma** *place-sub-less*: *x ≠ compact-bot ⟹ place (sub x) < place x*
⟨*proof*⟩

**lemma** *sub-below*: *sub x ⊑ x*
⟨*proof*⟩

**lemma** *rank-less-imp-below-sub*: *⟦x ⊑ y; rank x < rank y⟧ ⟹ x ⊑ sub y*
⟨*proof*⟩

**function**
*basis-emb* :: *'a compact-basis ⇒ ubasis*
**where**
*basis-emb x = (if x = compact-bot then 0 else*
*node (place x) (basis-emb (sub x))*
*(basis-emb ‘ {y. place y < place x ∧ x ⊑ y}))*
⟨*proof*⟩

**termination** *basis-emb*
⟨*proof*⟩

**declare** *basis-emb.simps* [*simp del*]

**lemma** *basis-emb-compact-bot* [*simp*]: *basis-emb compact-bot* = *0*
⟨*proof*⟩

**lemma** *fin1*: *finite* {*y*. *place y* < *place x* ∧ *x* ⊑ *y*}
⟨*proof*⟩

**lemma** *fin2*: *finite* (*basis-emb* ‘ {*y*. *place y* < *place x* ∧ *x* ⊑ *y*})
⟨*proof*⟩

**lemma** *rank-place-mono*:
⟦*place x* < *place y*; *x* ⊑ *y*⟧ ⟹ *rank x* < *rank y*
⟨*proof*⟩

**lemma** *basis-emb-mono*:
*x* ⊑ *y* ⟹ *ubasis-le* (*basis-emb x*) (*basis-emb y*)
⟨*proof*⟩

**lemma** *inj-basis-emb*: *inj basis-emb*
⟨*proof*⟩

**definition**
*basis-prj* :: *ubasis* ⇒ ′*a compact-basis*
**where**
*basis-prj x* = *inv basis-emb*
(*ubasis-until* (λ*x*. *x* ∈ *range* (*basis-emb* :: ′*a compact-basis* ⇒ *ubasis*)) *x*)

**lemma** *basis-prj-basis-emb*: ⋀*x*. *basis-prj* (*basis-emb x*) = *x*
⟨*proof*⟩

**lemma** *basis-prj-node*:
⟦*finite S*; *node i a S* ∉ *range* (*basis-emb* :: ′*a compact-basis* ⇒ *nat*)⟧
⟹ *basis-prj* (*node i a S*) = (*basis-prj a* :: ′*a compact-basis*)
⟨*proof*⟩

**lemma** *basis-prj-0*: *basis-prj 0* = *compact-bot*
⟨*proof*⟩

**lemma** *node-eq-basis-emb-iff*:
*finite S* ⟹ *node i a S* = *basis-emb x* ⟷
*x* ≠ *compact-bot* ∧ *i* = *place x* ∧ *a* = *basis-emb* (*sub x*) ∧
*S* = *basis-emb* ‘ {*y*. *place y* < *place x* ∧ *x* ⊑ *y*}
⟨*proof*⟩

**lemma** *basis-prj-mono*: *ubasis-le a b* ⟹ *basis-prj a* ⊑ *basis-prj b*
⟨*proof*⟩

**lemma** *basis-emb-prj-less*: *ubasis-le* (*basis-emb* (*basis-prj x*)) *x*
⟨*proof*⟩

**lemma** *ideal-completion*:
*ideal-completion below Rep-compact-basis* (*approximants* :: *'a* ⇒ -)
⟨*proof*⟩

**end**

**interpretation** *compact-basis*:
*ideal-completion below Rep-compact-basis*
*approximants* :: *'a*::*bifinite* ⇒ *'a compact-basis set*
⟨*proof*⟩

### 25.4.6 EP-pair from any bifinite domain into *udom*

**context** *bifinite-approx-chain* **begin**

**definition**
*udom-emb* :: *'a* → *udom*
**where**
*udom-emb* = *compact-basis.extension* (λ*x*. *udom-principal* (*basis-emb x*))

**definition**
*udom-prj* :: *udom* → *'a*
**where**
*udom-prj* = *udom.extension* (λ*x*. *Rep-compact-basis* (*basis-prj x*))

**lemma** *udom-emb-principal*:
*udom-emb*·(*Rep-compact-basis x*) = *udom-principal* (*basis-emb x*)
⟨*proof*⟩

**lemma** *udom-prj-principal*:
*udom-prj*·(*udom-principal x*) = *Rep-compact-basis* (*basis-prj x*)
⟨*proof*⟩

**lemma** *ep-pair-udom*: *ep-pair udom-emb udom-prj*
⟨*proof*⟩

**end**

**abbreviation** *udom-emb* ≡ *bifinite-approx-chain.udom-emb*
**abbreviation** *udom-prj* ≡ *bifinite-approx-chain.udom-prj*

**lemmas** *ep-pair-udom* =
*bifinite-approx-chain.ep-pair-udom* [*unfolded bifinite-approx-chain-def*]

## 25.5 Chain of approx functions for type *udom*

**definition**

*udom-approx* :: *nat* ⇒ *udom* → *udom*
**where**
*udom-approx i* =
*udom.extension* (λ*x*. *udom-principal* (*ubasis-until* (λ*y*. *y* ≤ *i*) *x*))

**lemma** *udom-approx-mono*:
*ubasis-le a b* ⟹
*udom-principal* (*ubasis-until* (λ*y*. *y* ≤ *i*) *a*) ⊑
*udom-principal* (*ubasis-until* (λ*y*. *y* ≤ *i*) *b*)
⟨*proof*⟩

**lemma** *adm-mem-finite*: ⟦*cont f*; *finite S*⟧ ⟹ *adm* (λ*x*. *f x* ∈ *S*)
⟨*proof*⟩

**lemma** *udom-approx-principal*:
*udom-approx i*·(*udom-principal x*) =
*udom-principal* (*ubasis-until* (λ*y*. *y* ≤ *i*) *x*)
⟨*proof*⟩

**lemma** *finite-deflation-udom-approx*: *finite-deflation* (*udom-approx i*)
⟨*proof*⟩

**interpretation** *udom-approx*: *finite-deflation udom-approx i*
⟨*proof*⟩

**lemma** *chain-udom-approx* [*simp*]: *chain* (λ*i*. *udom-approx i*)
⟨*proof*⟩

**lemma** *lub-udom-approx* [*simp*]: (⨆*i*. *udom-approx i*) = *ID*
⟨*proof*⟩

**lemma** *udom-approx* [*simp*]: *approx-chain udom-approx*
⟨*proof*⟩

**instance** *udom* :: *bifinite*
⟨*proof*⟩

**hide-const** (**open**) *node*

**end**

# 26 Algebraic deflations

**theory** *Algebraic*
**imports** *Universal Map-Functions*
**begin**

**default-sort** *bifinite*

## 26.1 Type constructor for finite deflations

**typedef** *'a fin-defl* = {*d*::*'a* → *'a*. *finite-deflation d*}
⟨*proof*⟩

**instantiation** *fin-defl* :: (*bifinite*) *below*
**begin**

**definition** *below-fin-defl-def*:
*below* ≡ λ*x y*. *Rep-fin-defl x* ⊑ *Rep-fin-defl y*

**instance** ⟨*proof*⟩
**end**

**instance** *fin-defl* :: (*bifinite*) *po*
⟨*proof*⟩

**lemma** *finite-deflation-Rep-fin-defl*: *finite-deflation* (*Rep-fin-defl d*)
⟨*proof*⟩

**lemma** *deflation-Rep-fin-defl*: *deflation* (*Rep-fin-defl d*)
⟨*proof*⟩

**interpretation** *Rep-fin-defl*: *finite-deflation Rep-fin-defl d*
⟨*proof*⟩

**lemma** *fin-defl-belowI*:
(⋀*x*. *Rep-fin-defl a*·*x* = *x* ⟹ *Rep-fin-defl b*·*x* = *x*) ⟹ *a* ⊑ *b*
⟨*proof*⟩

**lemma** *fin-defl-belowD*:
⟦*a* ⊑ *b*; *Rep-fin-defl a*·*x* = *x*⟧ ⟹ *Rep-fin-defl b*·*x* = *x*
⟨*proof*⟩

**lemma** *fin-defl-eqI*:
(⋀*x*. *Rep-fin-defl a*·*x* = *x* ⟷ *Rep-fin-defl b*·*x* = *x*) ⟹ *a* = *b*
⟨*proof*⟩

**lemma** *Rep-fin-defl-mono*: *a* ⊑ *b* ⟹ *Rep-fin-defl a* ⊑ *Rep-fin-defl b*
⟨*proof*⟩

**lemma** *Abs-fin-defl-mono*:
⟦*finite-deflation a*; *finite-deflation b*; *a* ⊑ *b*⟧
⟹ *Abs-fin-defl a* ⊑ *Abs-fin-defl b*
⟨*proof*⟩

**lemma** (**in** *finite-deflation*) *compact-belowI*:
**assumes** ⋀*x*. *compact x* ⟹ *d*·*x* = *x* ⟹ *f*·*x* = *x* **shows** *d* ⊑ *f*
⟨*proof*⟩

**lemma** *compact-Rep-fin-defl* [*simp*]: *compact* (*Rep-fin-defl a*)
⟨*proof*⟩

## 26.2 Defining algebraic deflations by ideal completion

**typedef** *'a defl* = {*S*::*'a fin-defl set. below.ideal S*}
⟨*proof*⟩

**instantiation** *defl* :: (*bifinite*) *below*
**begin**

**definition**
$x \sqsubseteq y \longleftrightarrow$ *Rep-defl x* $\subseteq$ *Rep-defl y*

**instance** ⟨*proof*⟩
**end**

**instance** *defl* :: (*bifinite*) *po*
⟨*proof*⟩

**instance** *defl* :: (*bifinite*) *cpo*
⟨*proof*⟩

**definition**
*defl-principal* :: *'a fin-defl* ⇒ *'a defl* **where**
*defl-principal t* = *Abs-defl* {*u. u* $\sqsubseteq$ *t*}

**lemma** *fin-defl-countable*: ∃*f*::*'a fin-defl* ⇒ *nat. inj f*
⟨*proof*⟩

**interpretation** *defl*: *ideal-completion below defl-principal Rep-defl*
⟨*proof*⟩

Algebraic deflations are pointed

**lemma** *defl-minimal*: *defl-principal* (*Abs-fin-defl* ⊥) $\sqsubseteq$ *x*
⟨*proof*⟩

**instance** *defl* :: (*bifinite*) *pcpo*
⟨*proof*⟩

**lemma** *inst-defl-pcpo*: ⊥ = *defl-principal* (*Abs-fin-defl* ⊥)
⟨*proof*⟩

## 26.3 Applying algebraic deflations

**definition**
*cast* :: *'a defl* → *'a* → *'a*
**where**
*cast* = *defl.extension Rep-fin-defl*

**lemma** *cast-defl-principal*:
*cast*·(*defl-principal a*) = *Rep-fin-defl a*
⟨*proof*⟩

**lemma** *deflation-cast*: *deflation* (*cast*·*d*)
⟨*proof*⟩

**lemma** *finite-deflation-cast*:
*compact d* ⟹ *finite-deflation* (*cast*·*d*)
⟨*proof*⟩

**interpretation** *cast*: *deflation cast*·*d*
⟨*proof*⟩

**declare** *cast.idem* [*simp*]

**lemma** *compact-cast* [*simp*]: *compact d* ⟹ *compact* (*cast*·*d*)
⟨*proof*⟩

**lemma** *cast-below-cast*: *cast*·*A* ⊑ *cast*·*B* ⟷ *A* ⊑ *B*
⟨*proof*⟩

**lemma** *compact-cast-iff*: *compact* (*cast*·*d*) ⟷ *compact d*
⟨*proof*⟩

**lemma** *cast-below-imp-below*: *cast*·*A* ⊑ *cast*·*B* ⟹ *A* ⊑ *B*
⟨*proof*⟩

**lemma** *cast-eq-imp-eq*: *cast*·*A* = *cast*·*B* ⟹ *A* = *B*
⟨*proof*⟩

**lemma** *cast-strict1* [*simp*]: *cast*·⊥ = ⊥
⟨*proof*⟩

**lemma** *cast-strict2* [*simp*]: *cast*·*A*·⊥ = ⊥
⟨*proof*⟩

## 26.4 Deflation combinators

**definition**
*defl-fun1 e p f* =
*defl.extension* (λ*a*.
*defl-principal* (*Abs-fin-defl*
(*e oo f*·(*Rep-fin-defl a*) *oo p*)))

**definition**
*defl-fun2 e p f* =
*defl.extension* (λ*a*.

*defl.extension* (λ*b*.
*defl-principal* (*Abs-fin-defl*
(*e oo f*·(*Rep-fin-defl a*)·(*Rep-fin-defl b*) *oo p*))))

**lemma** *cast-defl-fun1*:
**assumes** *ep*: *ep-pair e p*
**assumes** *f*: ⋀*a*. *finite-deflation a* ⟹ *finite-deflation* (*f*·*a*)
**shows** *cast*·(*defl-fun1 e p f*·*A*) = *e oo f*·(*cast*·*A*) *oo p*
⟨*proof*⟩

**lemma** *cast-defl-fun2*:
**assumes** *ep*: *ep-pair e p*
**assumes** *f*: ⋀*a b*. *finite-deflation a* ⟹ *finite-deflation b* ⟹
*finite-deflation* (*f*·*a*·*b*)
**shows** *cast*·(*defl-fun2 e p f*·*A*·*B*) = *e oo f*·(*cast*·*A*)·(*cast*·*B*) *oo p*
⟨*proof*⟩

**end**

# 27 Representable domains

**theory** *Representable*
**imports** *Algebraic Map-Functions ~~/src/HOL/Library/Countable*
**begin**

**default-sort** *cpo*

## 27.1 Class of representable domains

We define a "domain" as a pcpo that is isomorphic to some algebraic deflation over the universal domain; this is equivalent to being omega-bifinite.

A predomain is a cpo that, when lifted, becomes a domain. Predomains are represented by deflations over a lifted universal domain type.

**class** *predomain-syn* = *cpo* +
**fixes** *liftemb* :: $'a_\bot \rightarrow udom_\bot$
**fixes** *liftprj* :: $udom_\bot \rightarrow 'a_\bot$
**fixes** *liftdefl* :: *'a itself* ⇒ *udom u defl*

**class** *predomain* = *predomain-syn* +
**assumes** *predomain-ep*: *ep-pair liftemb liftprj*
**assumes** *cast-liftdefl*: *cast*·(*liftdefl TYPE*(*'a*)) = *liftemb oo liftprj*

**syntax** *-LIFTDEFL* :: *type* ⇒ *logic* ((*1LIFTDEFL*/(*1'*(-'))))
**translations** *LIFTDEFL*(*'t*) ⇌ *CONST liftdefl TYPE*(*'t*)

**definition** *liftdefl-of* :: *udom defl* → *udom u defl*
**where** *liftdefl-of* = *defl-fun1 ID ID u-map*

**lemma** *cast-liftdefl-of*: *cast*·(*liftdefl-of*·*t*) = *u-map*·(*cast*·*t*)
⟨*proof*⟩

**class** *domain* = *predomain-syn* + *pcpo* +
  **fixes** *emb* :: ′*a* → *udom*
  **fixes** *prj* :: *udom* → ′*a*
  **fixes** *defl* :: ′*a itself* ⇒ *udom defl*
  **assumes** *ep-pair-emb-prj*: *ep-pair emb prj*
  **assumes** *cast-DEFL*: *cast*·(*defl TYPE*(′*a*)) = *emb oo prj*
  **assumes** *liftemb-eq*: *liftemb* = *u-map*·*emb*
  **assumes** *liftprj-eq*: *liftprj* = *u-map*·*prj*
  **assumes** *liftdefl-eq*: *liftdefl TYPE*(′*a*) = *liftdefl-of*·(*defl TYPE*(′*a*))

**syntax** *-DEFL* :: *type* ⇒ *logic* ((*1DEFL*/(*1*′(-′))))
**translations** *DEFL*(′*t*) ⇌ *CONST defl TYPE*(′*t*)

**instance** *domain* ⊆ *predomain*
⟨*proof*⟩

Constants *liftemb* and *liftprj* imply class predomain.

⟨*ML*⟩

**interpretation** *predomain*: *pcpo-ep-pair liftemb liftprj*
  ⟨*proof*⟩

**interpretation** *domain*: *pcpo-ep-pair emb prj*
  ⟨*proof*⟩

**lemmas** *emb-inverse* = *domain.e-inverse*
**lemmas** *emb-prj-below* = *domain.e-p-below*
**lemmas** *emb-eq-iff* = *domain.e-eq-iff*
**lemmas** *emb-strict* = *domain.e-strict*
**lemmas** *prj-strict* = *domain.p-strict*

## 27.2 Domains are bifinite

**lemma** *approx-chain-ep-cast*:
  **assumes** *ep*: *ep-pair* (*e*::′*a*::*pcpo* → ′*b*::*bifinite*) (*p*::′*b* → ′*a*)
  **assumes** *cast-t*: *cast*·*t* = *e oo p*
  **shows** ∃(*a*::*nat* ⇒ ′*a*::*pcpo* → ′*a*). *approx-chain a*
⟨*proof*⟩

**instance** *domain* ⊆ *bifinite*
⟨*proof*⟩

**instance** *predomain* ⊆ *profinite*
⟨*proof*⟩

## 27.3 Universal domain ep-pairs

**definition** *u-emb* = *udom-emb* (λ*i*. *u-map*·(*udom-approx i*))
**definition** *u-prj* = *udom-prj* (λ*i*. *u-map*·(*udom-approx i*))

**definition** *prod-emb* = *udom-emb* (λ*i*. *prod-map*·(*udom-approx i*)·(*udom-approx i*))
**definition** *prod-prj* = *udom-prj* (λ*i*. *prod-map*·(*udom-approx i*)·(*udom-approx i*))

**definition** *sprod-emb* = *udom-emb* (λ*i*. *sprod-map*·(*udom-approx i*)·(*udom-approx i*))
**definition** *sprod-prj* = *udom-prj* (λ*i*. *sprod-map*·(*udom-approx i*)·(*udom-approx i*))

**definition** *ssum-emb* = *udom-emb* (λ*i*. *ssum-map*·(*udom-approx i*)·(*udom-approx i*))
**definition** *ssum-prj* = *udom-prj* (λ*i*. *ssum-map*·(*udom-approx i*)·(*udom-approx i*))

**definition** *sfun-emb* = *udom-emb* (λ*i*. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))
**definition** *sfun-prj* = *udom-prj* (λ*i*. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))

**lemma** *ep-pair-u*: *ep-pair u-emb u-prj*
⟨*proof*⟩

**lemma** *ep-pair-prod*: *ep-pair prod-emb prod-prj*
⟨*proof*⟩

**lemma** *ep-pair-sprod*: *ep-pair sprod-emb sprod-prj*
⟨*proof*⟩

**lemma** *ep-pair-ssum*: *ep-pair ssum-emb ssum-prj*
⟨*proof*⟩

**lemma** *ep-pair-sfun*: *ep-pair sfun-emb sfun-prj*
⟨*proof*⟩

## 27.4 Type combinators

**definition** *u-defl* :: *udom defl* → *udom defl*
**where** *u-defl* = *defl-fun1 u-emb u-prj u-map*

**definition** *prod-defl* :: *udom defl* → *udom defl* → *udom defl*
**where** *prod-defl* = *defl-fun2 prod-emb prod-prj prod-map*

**definition** *sprod-defl* :: *udom defl* → *udom defl* → *udom defl*
**where** *sprod-defl* = *defl-fun2 sprod-emb sprod-prj sprod-map*

**definition** *ssum-defl* :: *udom defl* → *udom defl* → *udom defl*

**where** *ssum-defl* = *defl-fun2 ssum-emb ssum-prj ssum-map*

**definition** *sfun-defl* :: *udom defl* → *udom defl* → *udom defl*
  **where** *sfun-defl* = *defl-fun2 sfun-emb sfun-prj sfun-map*

**lemma** *cast-u-defl*:
  *cast*·(*u-defl*·*A*) = *u-emb oo u-map*·(*cast*·*A*) *oo u-prj*
⟨*proof*⟩

**lemma** *cast-prod-defl*:
  *cast*·(*prod-defl*·*A*·*B*) =
    *prod-emb oo prod-map*·(*cast*·*A*)·(*cast*·*B*) *oo prod-prj*
⟨*proof*⟩

**lemma** *cast-sprod-defl*:
  *cast*·(*sprod-defl*·*A*·*B*) =
    *sprod-emb oo sprod-map*·(*cast*·*A*)·(*cast*·*B*) *oo sprod-prj*
⟨*proof*⟩

**lemma** *cast-ssum-defl*:
  *cast*·(*ssum-defl*·*A*·*B*) =
    *ssum-emb oo ssum-map*·(*cast*·*A*)·(*cast*·*B*) *oo ssum-prj*
⟨*proof*⟩

**lemma** *cast-sfun-defl*:
  *cast*·(*sfun-defl*·*A*·*B*) =
    *sfun-emb oo sfun-map*·(*cast*·*A*)·(*cast*·*B*) *oo sfun-prj*
⟨*proof*⟩

Special deflation combinator for unpointed types.

**definition** *u-liftdefl* :: *udom u defl* → *udom defl*
  **where** *u-liftdefl* = *defl-fun1 u-emb u-prj ID*

**lemma** *cast-u-liftdefl*:
  *cast*·(*u-liftdefl*·*A*) = *u-emb oo cast*·*A oo u-prj*
⟨*proof*⟩

**lemma** *u-liftdefl-liftdefl-of*:
  *u-liftdefl*·(*liftdefl-of*·*A*) = *u-defl*·*A*
⟨*proof*⟩

## 27.5 Class instance proofs

### 27.5.1 Universal domain

**instantiation** *udom* :: *domain*
**begin**

**definition** [*simp*]:
  *emb* = (*ID* :: *udom* → *udom*)

**definition** [*simp*]:
  $prj = (ID :: udom \rightarrow udom)$

**definition**
  $defl\ (t::udom\ itself) = (\bigsqcup i.\ defl\text{-}principal\ (Abs\text{-}fin\text{-}defl\ (udom\text{-}approx\ i)))$

**definition**
  $(liftemb :: udom\ u \rightarrow udom\ u) = u\text{-}map{\cdot}emb$

**definition**
  $(liftprj :: udom\ u \rightarrow udom\ u) = u\text{-}map{\cdot}prj$

**definition**
  $liftdefl\ (t::udom\ itself) = liftdefl\text{-}of{\cdot}DEFL(udom)$

**instance** ⟨*proof*⟩

**end**

### 27.5.2 Lifted cpo

**instantiation** *u* :: (*predomain*) *domain*
**begin**

**definition**
  $emb = u\text{-}emb\ oo\ liftemb$

**definition**
  $prj = liftprj\ oo\ u\text{-}prj$

**definition**
  $defl\ (t::'a\ u\ itself) = u\text{-}liftdefl{\cdot}LIFTDEFL('a)$

**definition**
  $(liftemb :: 'a\ u\ u \rightarrow udom\ u) = u\text{-}map{\cdot}emb$

**definition**
  $(liftprj :: udom\ u \rightarrow 'a\ u\ u) = u\text{-}map{\cdot}prj$

**definition**
  $liftdefl\ (t::'a\ u\ itself) = liftdefl\text{-}of{\cdot}DEFL('a\ u)$

**instance** ⟨*proof*⟩

**end**

**lemma** *DEFL-u*: $DEFL('a::predomain\ u) = u\text{-}liftdefl{\cdot}LIFTDEFL('a)$
⟨*proof*⟩

### 27.5.3 Strict function space

**instantiation** *sfun* :: (*domain*, *domain*) *domain*
**begin**

**definition**
*emb* = *sfun-emb oo sfun-map*·*prj*·*emb*

**definition**
*prj* = *sfun-map*·*emb*·*prj oo sfun-prj*

**definition**
*defl* (*t*::(*'a* →! *'b*) *itself*) = *sfun-defl*·*DEFL*(*'a*)·*DEFL*(*'b*)

**definition**
(*liftemb* :: (*'a* →! *'b*) *u* → *udom u*) = *u-map*·*emb*

**definition**
(*liftprj* :: *udom u* → (*'a* →! *'b*) *u*) = *u-map*·*prj*

**definition**
*liftdefl* (*t*::(*'a* →! *'b*) *itself*) = *liftdefl-of*·*DEFL*(*'a* →! *'b*)

**instance** ⟨*proof*⟩

**end**

**lemma** *DEFL-sfun*:
*DEFL*(*'a*::*domain* →! *'b*::*domain*) = *sfun-defl*·*DEFL*(*'a*)·*DEFL*(*'b*)
⟨*proof*⟩

### 27.5.4 Continuous function space

**instantiation** *cfun* :: (*predomain*, *domain*) *domain*
**begin**

**definition**
*emb* = *emb oo encode-cfun*

**definition**
*prj* = *decode-cfun oo prj*

**definition**
*defl* (*t*::(*'a* → *'b*) *itself*) = *DEFL*(*'a u* →! *'b*)

**definition**
(*liftemb* :: (*'a* → *'b*) *u* → *udom u*) = *u-map*·*emb*

**definition**
(*liftprj* :: *udom u* → (*'a* → *'b*) *u*) = *u-map*·*prj*

**definition**
*liftdefl* $(t::('a \to 'b)\ itself) = liftdefl\text{-}of\cdot DEFL('a \to 'b)$

**instance** ⟨*proof*⟩

**end**

**lemma** *DEFL-cfun*:
$DEFL('a::predomain \to 'b::domain) = DEFL('a\ u \to! 'b)$
⟨*proof*⟩

### 27.5.5 Strict product

**instantiation** *sprod* :: (*domain*, *domain*) *domain*
**begin**

**definition**
$emb = sprod\text{-}emb\ oo\ sprod\text{-}map\cdot emb\cdot emb$

**definition**
$prj = sprod\text{-}map\cdot prj\cdot prj\ oo\ sprod\text{-}prj$

**definition**
*defl* $(t::('a \otimes 'b)\ itself) = sprod\text{-}defl\cdot DEFL('a)\cdot DEFL('b)$

**definition**
$(liftemb :: ('a \otimes 'b)\ u \to udom\ u) = u\text{-}map\cdot emb$

**definition**
$(liftprj :: udom\ u \to ('a \otimes 'b)\ u) = u\text{-}map\cdot prj$

**definition**
*liftdefl* $(t::('a \otimes 'b)\ itself) = liftdefl\text{-}of\cdot DEFL('a \otimes 'b)$

**instance** ⟨*proof*⟩

**end**

**lemma** *DEFL-sprod*:
$DEFL('a::domain \otimes 'b::domain) = sprod\text{-}defl\cdot DEFL('a)\cdot DEFL('b)$
⟨*proof*⟩

### 27.5.6 Cartesian product

**definition** *prod-liftdefl* :: *udom u defl* → *udom u defl* → *udom u defl*
**where** *prod-liftdefl* = *defl-fun2* (*u-map*·*prod-emb oo decode-prod-u*)
(*encode-prod-u oo u-map*·*prod-prj*) *sprod-map*

**lemma** *cast-prod-liftdefl*:

*cast*·(*prod-liftdefl*·*a*·*b*) =
(*u-map*·*prod-emb oo decode-prod-u*) *oo sprod-map*·(*cast*·*a*)·(*cast*·*b*) *oo*
(*encode-prod-u oo u-map*·*prod-prj*)
⟨*proof*⟩

**instantiation** *prod* :: (*predomain*, *predomain*) *predomain*
**begin**

**definition**
*liftemb* = (*u-map*·*prod-emb oo decode-prod-u*) *oo*
(*sprod-map*·*liftemb*·*liftemb oo encode-prod-u*)

**definition**
*liftprj* = (*decode-prod-u oo sprod-map*·*liftprj*·*liftprj*) *oo*
(*encode-prod-u oo u-map*·*prod-prj*)

**definition**
*liftdefl* (*t*::('*a* × '*b*) *itself*) = *prod-liftdefl*·*LIFTDEFL*('*a*)·*LIFTDEFL*('*b*)

**instance** ⟨*proof*⟩

**end**

**instantiation** *prod* :: (*domain*, *domain*) *domain*
**begin**

**definition**
*emb* = *prod-emb oo prod-map*·*emb*·*emb*

**definition**
*prj* = *prod-map*·*prj*·*prj oo prod-prj*

**definition**
*defl* (*t*::('*a* × '*b*) *itself*) = *prod-defl*·*DEFL*('*a*)·*DEFL*('*b*)

**instance** ⟨*proof*⟩

**end**

**lemma** *DEFL-prod*:
*DEFL*('*a*::*domain* × '*b*::*domain*) = *prod-defl*·*DEFL*('*a*)·*DEFL*('*b*)
⟨*proof*⟩

**lemma** *LIFTDEFL-prod*:
*LIFTDEFL*('*a*::*predomain* × '*b*::*predomain*) =
*prod-liftdefl*·*LIFTDEFL*('*a*)·*LIFTDEFL*('*b*)
⟨*proof*⟩

### 27.5.7 Unit type

**instantiation** *unit* :: *domain*
**begin**

**definition**
*emb* = (⊥ :: *unit* → *udom*)

**definition**
*prj* = (⊥ :: *udom* → *unit*)

**definition**
*defl* (*t*::*unit itself*) = ⊥

**definition**
(*liftemb* :: *unit u* → *udom u*) = *u-map*·*emb*

**definition**
(*liftprj* :: *udom u* → *unit u*) = *u-map*·*prj*

**definition**
*liftdefl* (*t*::*unit itself*) = *liftdefl-of*·*DEFL*(*unit*)

**instance** ⟨*proof*⟩

**end**

### 27.5.8 Discrete cpo

**instantiation** *discr* :: (*countable*) *predomain*
**begin**

**definition**
(*liftemb* :: ′*a discr u* → *udom u*) = *strictify*·*up oo udom-emb discr-approx*

**definition**
(*liftprj* :: *udom u* → ′*a discr u*) = *udom-prj discr-approx oo fup*·*ID*

**definition**
*liftdefl* (*t*::′*a discr itself*) =
(⨆ *i*. *defl-principal* (*Abs-fin-defl* (*liftemb oo discr-approx i oo* (*liftprj*::*udom u* → ′*a discr u*))))

**instance** ⟨*proof*⟩

**end**

### 27.5.9 Strict sum

**instantiation** *ssum* :: (*domain*, *domain*) *domain*

**begin**

**definition**
*emb* = *ssum-emb oo ssum-map·emb·emb*

**definition**
*prj* = *ssum-map·prj·prj oo ssum-prj*

**definition**
*defl* (*t*::('*a* ⊕ '*b*) *itself*) = *ssum-defl·DEFL*('*a*)·*DEFL*('*b*)

**definition**
(*liftemb* :: ('*a* ⊕ '*b*) *u* → *udom u*) = *u-map·emb*

**definition**
(*liftprj* :: *udom u* → ('*a* ⊕ '*b*) *u*) = *u-map·prj*

**definition**
*liftdefl* (*t*::('*a* ⊕ '*b*) *itself*) = *liftdefl-of·DEFL*('*a* ⊕ '*b*)

**instance** ⟨*proof*⟩

**end**

**lemma** *DEFL-ssum*:
*DEFL*('*a*::*domain* ⊕ '*b*::*domain*) = *ssum-defl·DEFL*('*a*)·*DEFL*('*b*)
⟨*proof*⟩

### 27.5.10 Lifted HOL type

**instantiation** *lift* :: (*countable*) *domain*
**begin**

**definition**
*emb* = *emb oo* (Λ *x*. *Rep-lift x*)

**definition**
*prj* = (Λ *y*. *Abs-lift y*) *oo prj*

**definition**
*defl* (*t*::'*a lift itself*) = *DEFL*('*a discr u*)

**definition**
(*liftemb* :: '*a lift u* → *udom u*) = *u-map·emb*

**definition**
(*liftprj* :: *udom u* → '*a lift u*) = *u-map·prj*

**definition**

*liftdefl* (*t*::*'a lift itself*) = *liftdefl-of*·*DEFL*(*'a lift*)

**instance** ⟨*proof*⟩

**end**

**end**

# 28 Domain package support

**theory** *Domain-Aux*
**imports** *Map-Functions Fixrec*
**begin**

## 28.1 Continuous isomorphisms

A locale for continuous isomorphisms

**locale** *iso* =
 **fixes** *abs* :: *'a* → *'b*
 **fixes** *rep* :: *'b* → *'a*
 **assumes** *abs-iso* [*simp*]: *rep*·(*abs*·*x*) = *x*
 **assumes** *rep-iso* [*simp*]: *abs*·(*rep*·*y*) = *y*
**begin**

**lemma** *swap*: *iso rep abs*
 ⟨*proof*⟩

**lemma** *abs-below*: (*abs*·*x* ⊑ *abs*·*y*) = (*x* ⊑ *y*)
⟨*proof*⟩

**lemma** *rep-below*: (*rep*·*x* ⊑ *rep*·*y*) = (*x* ⊑ *y*)
 ⟨*proof*⟩

**lemma** *abs-eq*: (*abs*·*x* = *abs*·*y*) = (*x* = *y*)
 ⟨*proof*⟩

**lemma** *rep-eq*: (*rep*·*x* = *rep*·*y*) = (*x* = *y*)
 ⟨*proof*⟩

**lemma** *abs-strict*: *abs*·⊥ = ⊥
⟨*proof*⟩

**lemma** *rep-strict*: *rep*·⊥ = ⊥
 ⟨*proof*⟩

**lemma** *abs-defin'*: *abs*·*x* = ⊥ ⟹ *x* = ⊥
⟨*proof*⟩

**lemma** *rep-defin'*: $rep \cdot z = \bot \Longrightarrow z = \bot$
⟨*proof*⟩

**lemma** *abs-defined*: $z \neq \bot \Longrightarrow abs \cdot z \neq \bot$
⟨*proof*⟩

**lemma** *rep-defined*: $z \neq \bot \Longrightarrow rep \cdot z \neq \bot$
⟨*proof*⟩

**lemma** *abs-bottom-iff*: $(abs \cdot x = \bot) = (x = \bot)$
⟨*proof*⟩

**lemma** *rep-bottom-iff*: $(rep \cdot x = \bot) = (x = \bot)$
⟨*proof*⟩

**lemma** *casedist-rule*: $rep \cdot x = \bot \vee P \Longrightarrow x = \bot \vee P$
⟨*proof*⟩

**lemma** *compact-abs-rev*: $compact\ (abs \cdot x) \Longrightarrow compact\ x$
⟨*proof*⟩

**lemma** *compact-rep-rev*: $compact\ (rep \cdot x) \Longrightarrow compact\ x$
⟨*proof*⟩

**lemma** *compact-abs*: $compact\ x \Longrightarrow compact\ (abs \cdot x)$
⟨*proof*⟩

**lemma** *compact-rep*: $compact\ x \Longrightarrow compact\ (rep \cdot x)$
⟨*proof*⟩

**lemma** *iso-swap*: $(x = abs \cdot y) = (rep \cdot x = y)$
⟨*proof*⟩

**end**

## 28.2 Proofs about take functions

This section contains lemmas that are used in a module that supports the domain isomorphism package; the module contains proofs related to take functions and the finiteness predicate.

**lemma** *deflation-abs-rep*:
**fixes** *abs* **and** *rep* **and** *d*
**assumes** *abs-iso*: $\bigwedge x.\ rep \cdot (abs \cdot x) = x$
**assumes** *rep-iso*: $\bigwedge y.\ abs \cdot (rep \cdot y) = y$
**shows** $deflation\ d \Longrightarrow deflation\ (abs\ oo\ d\ oo\ rep)$
⟨*proof*⟩

**lemma** *deflation-chain-min*:
**assumes** *chain*: *chain d*

**assumes** *defl*: $\bigwedge n.\ deflation\ (d\ n)$
**shows** $d\ m{\cdot}(d\ n{\cdot}x) = d\ (min\ m\ n){\cdot}x$
⟨*proof*⟩

**lemma** *lub-ID-take-lemma*:
**assumes** *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
**assumes** $\bigwedge n.\ t\ n{\cdot}x = t\ n{\cdot}y$ **shows** $x = y$
⟨*proof*⟩

**lemma** *lub-ID-reach*:
**assumes** *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
**shows** $(\bigsqcup n.\ t\ n{\cdot}x) = x$
⟨*proof*⟩

**lemma** *lub-ID-take-induct*:
**assumes** *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
**assumes** *adm P* **and** $\bigwedge n.\ P\ (t\ n{\cdot}x)$ **shows** $P\ x$
⟨*proof*⟩

## 28.3 Finiteness

Let a "decisive" function be a deflation that maps every input to either itself or bottom. Then if a domain's take functions are all decisive, then all values in the domain are finite.

**definition**
*decisive* :: $('a{::}pcpo \rightarrow 'a) \Rightarrow bool$
**where**
$decisive\ d \longleftrightarrow (\forall x.\ d{\cdot}x = x \lor d{\cdot}x = \bot)$

**lemma** *decisiveI*: $(\bigwedge x.\ d{\cdot}x = x \lor d{\cdot}x = \bot) \Longrightarrow decisive\ d$
⟨*proof*⟩

**lemma** *decisive-cases*:
**assumes** *decisive d* **obtains** $d{\cdot}x = x \mid d{\cdot}x = \bot$
⟨*proof*⟩

**lemma** *decisive-bottom*: *decisive* $\bot$
⟨*proof*⟩

**lemma** *decisive-ID*: *decisive ID*
⟨*proof*⟩

**lemma** *decisive-ssum-map*:
**assumes** *f*: *decisive f*
**assumes** *g*: *decisive g*
**shows** *decisive* $(ssum\text{-}map{\cdot}f{\cdot}g)$
⟨*proof*⟩

**lemma** *decisive-sprod-map*:

**assumes** $f$: *decisive* $f$
**assumes** $g$: *decisive* $g$
**shows** *decisive* ($sprod\text{-}map{\cdot}f{\cdot}g$)
⟨*proof*⟩

**lemma** *decisive-abs-rep*:
**fixes** *abs rep*
**assumes** *iso*: *iso abs rep*
**assumes** $d$: *decisive* $d$
**shows** *decisive* (*abs oo d oo rep*)
⟨*proof*⟩

**lemma** *lub-ID-finite*:
**assumes** *chain*: *chain* $d$
**assumes** *lub*: $(\bigsqcup n.\ d\ n) = ID$
**assumes** *decisive*: $\bigwedge n.$ *decisive* $(d\ n)$
**shows** $\exists n.\ d\ n{\cdot}x = x$
⟨*proof*⟩

**lemma** *lub-ID-finite-take-induct*:
**assumes** *chain* $d$ **and** $(\bigsqcup n.\ d\ n) = ID$ **and** $\bigwedge n.$ *decisive* $(d\ n)$
**shows** $(\bigwedge n.\ P\ (d\ n{\cdot}x)) \Longrightarrow P\ x$
⟨*proof*⟩

## 28.4 Proofs about constructor functions

Lemmas for proving nchotomy rule:

**lemma** *ex-one-bottom-iff*:
$(\exists x.\ P\ x \wedge x \neq \bot) = P\ ONE$
⟨*proof*⟩

**lemma** *ex-up-bottom-iff*:
$(\exists x.\ P\ x \wedge x \neq \bot) = (\exists x.\ P\ (up{\cdot}x))$
⟨*proof*⟩

**lemma** *ex-sprod-bottom-iff*:
$(\exists y.\ P\ y \wedge y \neq \bot) =$
$(\exists x\ y.\ (P\ (:x,\ y:) \wedge x \neq \bot) \wedge y \neq \bot)$
⟨*proof*⟩

**lemma** *ex-sprod-up-bottom-iff*:
$(\exists y.\ P\ y \wedge y \neq \bot) =$
$(\exists x\ y.\ P\ (:up{\cdot}x,\ y:) \wedge y \neq \bot)$
⟨*proof*⟩

**lemma** *ex-ssum-bottom-iff*:
$(\exists x.\ P\ x \wedge x \neq \bot) =$
$((\exists x.\ P\ (sinl{\cdot}x) \wedge x \neq \bot) \vee$
$(\exists x.\ P\ (sinr{\cdot}x) \wedge x \neq \bot))$

⟨*proof*⟩

**lemma** *exh-start*: $p = \bot \vee (\exists x.\ p = x \wedge x \neq \bot)$
⟨*proof*⟩

**lemmas** *ex-bottom-iffs* =
*ex-ssum-bottom-iff*
*ex-sprod-up-bottom-iff*
*ex-sprod-bottom-iff*
*ex-up-bottom-iff*
*ex-one-bottom-iff*

Rules for turning nchotomy into exhaust:

**lemma** *exh-casedist0*: $[\![R;\ R \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

**lemma** *exh-casedist1*: $((P \vee Q \Longrightarrow R) \Longrightarrow S) \equiv ([\![P \Longrightarrow R;\ Q \Longrightarrow R]\!] \Longrightarrow S)$
⟨*proof*⟩

**lemma** *exh-casedist2*: $(\exists x.\ P\ x \Longrightarrow Q) \equiv (\bigwedge x.\ P\ x \Longrightarrow Q)$
⟨*proof*⟩

**lemma** *exh-casedist3*: $(P \wedge Q \Longrightarrow R) \equiv (P \Longrightarrow Q \Longrightarrow R)$
⟨*proof*⟩

**lemmas** *exh-casedists* = *exh-casedist1 exh-casedist2 exh-casedist3*

Rules for proving constructor properties

**lemmas** *con-strict-rules* =
*sinl-strict sinr-strict spair-strict1 spair-strict2*

**lemmas** *con-bottom-iff-rules* =
*sinl-bottom-iff sinr-bottom-iff spair-bottom-iff up-defined ONE-defined*

**lemmas** *con-below-iff-rules* =
*sinl-below sinr-below sinl-below-sinr sinr-below-sinl con-bottom-iff-rules*

**lemmas** *con-eq-iff-rules* =
*sinl-eq sinr-eq sinl-eq-sinr sinr-eq-sinl con-bottom-iff-rules*

**lemmas** *sel-strict-rules* =
*cfcomp2 sscase1 sfst-strict ssnd-strict fup1*

**lemma** *sel-app-extra-rules*:
$sscase \cdot ID \cdot \bot \cdot (sinr \cdot x) = \bot$
$sscase \cdot ID \cdot \bot \cdot (sinl \cdot x) = x$
$sscase \cdot \bot \cdot ID \cdot (sinl \cdot x) = \bot$
$sscase \cdot \bot \cdot ID \cdot (sinr \cdot x) = x$
$fup \cdot ID \cdot (up \cdot x) = x$

⟨*proof*⟩

**lemmas** *sel-app-rules* =
*sel-strict-rules sel-app-extra-rules*
*ssnd-spair sfst-spair up-defined spair-defined*

**lemmas** *sel-bottom-iff-rules* =
*cfcomp2 sfst-bottom-iff ssnd-bottom-iff*

**lemmas** *take-con-rules* =
*ssum-map-sinl′ ssum-map-sinr′ sprod-map-spair′ u-map-up*
*deflation-strict deflation-ID ID1 cfcomp2*

### 28.5 ML setup

**named-theorems** *domain-deflation theorems like deflation a ==> deflation (foo-map$a)*
**and** *domain-map-ID theorems like foo-map$ID = ID*

⟨*ML*⟩

**end**

## 29 Domain package

**theory** *Domain*
**imports** *Representable Domain-Aux*
**keywords**
*domaindef* :: *thy-decl* **and** *lazy unsafe* **and**
*domain-isomorphism domain* :: *thy-decl*
**begin**

**default-sort** *domain*

### 29.1 Representations of types

**lemma** *emb-prj*: $emb\cdot((prj\cdot x)::'a) = cast\cdot DEFL('a)\cdot x$
⟨*proof*⟩

**lemma** *emb-prj-emb*:
**fixes** $x :: 'a$
**assumes** $DEFL('a) \sqsubseteq DEFL('b)$
**shows** $emb\cdot(prj\cdot(emb\cdot x) :: 'b) = emb\cdot x$
⟨*proof*⟩

**lemma** *prj-emb-prj*:
**assumes** $DEFL('a) \sqsubseteq DEFL('b)$
**shows** $prj\cdot(emb\cdot(prj\cdot x :: 'b)) = (prj\cdot x :: 'a)$
⟨*proof*⟩

Isomorphism lemmas used internally by the domain package:

**lemma** *domain-abs-iso*:
  **fixes** *abs* **and** *rep*
  **assumes** *DEFL*: $DEFL('b) = DEFL('a)$
  **assumes** *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
  **assumes** *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
  **shows** $rep\cdot(abs\cdot x) = x$
⟨*proof*⟩

**lemma** *domain-rep-iso*:
  **fixes** *abs* **and** *rep*
  **assumes** *DEFL*: $DEFL('b) = DEFL('a)$
  **assumes** *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
  **assumes** *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
  **shows** $abs\cdot(rep\cdot x) = x$
⟨*proof*⟩

## 29.2 Deflations as sets

**definition** *defl-set* :: *'a::bifinite defl* ⇒ *'a set*
**where** *defl-set A* = $\{x.\ cast\cdot A\cdot x = x\}$

**lemma** *adm-defl-set*: *adm* ($\lambda x.\ x \in$ *defl-set A*)
⟨*proof*⟩

**lemma** *defl-set-bottom*: $\bot \in$ *defl-set A*
⟨*proof*⟩

**lemma** *defl-set-cast* [*simp*]: $cast\cdot A\cdot x \in$ *defl-set A*
⟨*proof*⟩

**lemma** *defl-set-subset-iff*: *defl-set A* ⊆ *defl-set B* ⟷ $A \sqsubseteq B$
⟨*proof*⟩

## 29.3 Proving a subtype is representable

Temporarily relax type constraints.

⟨*ML*⟩

**lemma** *typedef-domain-class*:
  **fixes** *Rep* :: *'a::pcpo* ⇒ *udom*
  **fixes** *Abs* :: *udom* ⇒ *'a::pcpo*
  **fixes** *t* :: *udom defl*
  **assumes** *type*: *type-definition Rep Abs* (*defl-set t*)
  **assumes** *below*: $op \sqsubseteq\ \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
  **assumes** *emb*: $emb \equiv (\Lambda\ x.\ Rep\ x)$
  **assumes** *prj*: $prj \equiv (\Lambda\ x.\ Abs\ (cast\cdot t\cdot x))$
  **assumes** *defl*: $defl \equiv (\lambda\ a::'a\ itself.\ t)$

**assumes** *liftemb*: $(liftemb :: 'a\ u \rightarrow udom\ u) \equiv u\text{-}map\cdot emb$
**assumes** *liftprj*: $(liftprj :: udom\ u \rightarrow 'a\ u) \equiv u\text{-}map\cdot prj$
**assumes** *liftdefl*: $(liftdefl :: 'a\ itself \Rightarrow \text{-}) \equiv (\lambda t.\ liftdefl\text{-}of\cdot DEFL('a))$
**shows** $OFCLASS('a,\ domain\text{-}class)$
⟨*proof*⟩

**lemma** *typedef-DEFL*:
**assumes** $defl \equiv (\lambda a::'a::pcpo\ itself.\ t)$
**shows** $DEFL('a::pcpo) = t$
⟨*proof*⟩

Restore original typing constraints.

⟨*ML*⟩

## 29.4 Isomorphic deflations

**definition** *isodefl* :: $('a \rightarrow 'a) \Rightarrow udom\ defl \Rightarrow bool$
**where** $isodefl\ d\ t \longleftrightarrow cast\cdot t = emb\ oo\ d\ oo\ prj$

**definition** *isodefl'* :: $('a::predomain \rightarrow 'a) \Rightarrow udom\ u\ defl \Rightarrow bool$
**where** $isodefl'\ d\ t \longleftrightarrow cast\cdot t = liftemb\ oo\ u\text{-}map\cdot d\ oo\ liftprj$

**lemma** *isodeflI*: $(\bigwedge x.\ cast\cdot t\cdot x = emb\cdot(d\cdot(prj\cdot x))) \Longrightarrow isodefl\ d\ t$
⟨*proof*⟩

**lemma** *cast-isodefl*: $isodefl\ d\ t \Longrightarrow cast\cdot t = (\Lambda\ x.\ emb\cdot(d\cdot(prj\cdot x)))$
⟨*proof*⟩

**lemma** *isodefl-strict*: $isodefl\ d\ t \Longrightarrow d\cdot\bot = \bot$
⟨*proof*⟩

**lemma** *isodefl-imp-deflation*:
**fixes** $d :: 'a \rightarrow 'a$
**assumes** *isodefl d t* **shows** *deflation d*
⟨*proof*⟩

**lemma** *isodefl-ID-DEFL*: $isodefl\ (ID :: 'a \rightarrow 'a)\ DEFL('a)$
⟨*proof*⟩

**lemma** *isodefl-LIFTDEFL*:
$isodefl'\ (ID :: 'a \rightarrow 'a)\ LIFTDEFL('a::predomain)$
⟨*proof*⟩

**lemma** *isodefl-DEFL-imp-ID*: $isodefl\ (d :: 'a \rightarrow 'a)\ DEFL('a) \Longrightarrow d = ID$
⟨*proof*⟩

**lemma** *isodefl-bottom*: $isodefl\ \bot\ \bot$
⟨*proof*⟩

**lemma** *adm-isodefl*:
  *cont f* $\Longrightarrow$ *cont g* $\Longrightarrow$ *adm* ($\lambda x.$ *isodefl* (*f x*) (*g x*))
⟨*proof*⟩

**lemma** *isodefl-lub*:
  **assumes** *chain d* **and** *chain t*
  **assumes** $\bigwedge i.$ *isodefl* (*d i*) (*t i*)
  **shows** *isodefl* ($\bigsqcup i.$ *d i*) ($\bigsqcup i.$ *t i*)
⟨*proof*⟩

**lemma** *isodefl-fix*:
  **assumes** $\bigwedge d\ t.$ *isodefl d t* $\Longrightarrow$ *isodefl* ($f \cdot d$) ($g \cdot t$)
  **shows** *isodefl* ($fix \cdot f$) ($fix \cdot g$)
⟨*proof*⟩

**lemma** *isodefl-abs-rep*:
  **fixes** *abs* **and** *rep* **and** *d*
  **assumes** *DEFL*: *DEFL*('*b*) = *DEFL*('*a*)
  **assumes** *abs-def*: (*abs* :: '*a* → '*b*) ≡ *prj oo emb*
  **assumes** *rep-def*: (*rep* :: '*b* → '*a*) ≡ *prj oo emb*
  **shows** *isodefl d t* $\Longrightarrow$ *isodefl* (*abs oo d oo rep*) *t*
⟨*proof*⟩

**lemma** *isodefl'-liftdefl-of*: *isodefl d t* $\Longrightarrow$ *isodefl' d* (*liftdefl-of*$\cdot$*t*)
⟨*proof*⟩

**lemma** *isodefl-sfun*:
  *isodefl d1 t1* $\Longrightarrow$ *isodefl d2 t2* $\Longrightarrow$
    *isodefl* (*sfun-map*$\cdot$*d1*$\cdot$*d2*) (*sfun-defl*$\cdot$*t1*$\cdot$*t2*)
⟨*proof*⟩

**lemma** *isodefl-ssum*:
  *isodefl d1 t1* $\Longrightarrow$ *isodefl d2 t2* $\Longrightarrow$
    *isodefl* (*ssum-map*$\cdot$*d1*$\cdot$*d2*) (*ssum-defl*$\cdot$*t1*$\cdot$*t2*)
⟨*proof*⟩

**lemma** *isodefl-sprod*:
  *isodefl d1 t1* $\Longrightarrow$ *isodefl d2 t2* $\Longrightarrow$
    *isodefl* (*sprod-map*$\cdot$*d1*$\cdot$*d2*) (*sprod-defl*$\cdot$*t1*$\cdot$*t2*)
⟨*proof*⟩

**lemma** *isodefl-prod*:
  *isodefl d1 t1* $\Longrightarrow$ *isodefl d2 t2* $\Longrightarrow$
    *isodefl* (*prod-map*$\cdot$*d1*$\cdot$*d2*) (*prod-defl*$\cdot$*t1*$\cdot$*t2*)
⟨*proof*⟩

**lemma** *isodefl-u*:
  *isodefl d t* $\Longrightarrow$ *isodefl* (*u-map*$\cdot$*d*) (*u-defl*$\cdot$*t*)
⟨*proof*⟩

**lemma** *isodefl-u-liftdefl*:
*isodefl′ d t* $\Longrightarrow$ *isodefl* (*u-map·d*) (*u-liftdefl·t*)
⟨*proof*⟩

**lemma** *encode-prod-u-map*:
*encode-prod-u·*(*u-map·*(*prod-map·f·g*)·(*decode-prod-u·x*))
= *sprod-map·*(*u-map·f*)·(*u-map·g*)·*x*
⟨*proof*⟩

**lemma** *isodefl-prod-u*:
**assumes** *isodefl′ d1 t1* **and** *isodefl′ d2 t2*
**shows** *isodefl′* (*prod-map·d1·d2*) (*prod-liftdefl·t1·t2*)
⟨*proof*⟩

**lemma** *encode-cfun-map*:
*encode-cfun·*(*cfun-map·f·g·*(*decode-cfun·x*))
= *sfun-map·*(*u-map·f*)·*g·x*
⟨*proof*⟩

**lemma** *isodefl-cfun*:
**assumes** *isodefl* (*u-map·d1*) *t1* **and** *isodefl d2 t2*
**shows** *isodefl* (*cfun-map·d1·d2*) (*sfun-defl·t1·t2*)
⟨*proof*⟩

## 29.5 Setting up the domain package

**named-theorems** *domain-defl-simps theorems like DEFL*(*′a t*) = *t-defl$DEFL*(*′a*)
**and** *domain-isodefl theorems like isodefl d t ==> isodefl* (*foo-map$d*) (*foo-defl$t*)

⟨*ML*⟩

**lemmas** [*domain-defl-simps*] =
*DEFL-cfun DEFL-sfun DEFL-ssum DEFL-sprod DEFL-prod DEFL-u*
*liftdefl-eq LIFTDEFL-prod u-liftdefl-liftdefl-of*

**lemmas** [*domain-map-ID*] =
*cfun-map-ID sfun-map-ID ssum-map-ID sprod-map-ID prod-map-ID u-map-ID*

**lemmas** [*domain-isodefl*] =
*isodefl-u isodefl-sfun isodefl-ssum isodefl-sprod*
*isodefl-cfun isodefl-prod isodefl-prod-u isodefl′-liftdefl-of*
*isodefl-u-liftdefl*

**lemmas** [*domain-deflation*] =
*deflation-cfun-map deflation-sfun-map deflation-ssum-map*
*deflation-sprod-map deflation-prod-map deflation-u-map*

⟨*ML*⟩

**end**

# 30 A compact basis for powerdomains

**theory** *Compact-Basis*
**imports** *Universal*
**begin**

**default-sort** *bifinite*

## 30.1 A compact basis for powerdomains

**definition** *pd-basis* = {*S*::′*a compact-basis set*. *finite S* ∧ *S* ≠ {}}

**typedef** ′*a pd-basis* = *pd-basis* :: ′*a compact-basis set set*
⟨*proof*⟩

**lemma** *finite-Rep-pd-basis* [*simp*]: *finite* (*Rep-pd-basis u*)
⟨*proof*⟩

**lemma** *Rep-pd-basis-nonempty* [*simp*]: *Rep-pd-basis u* ≠ {}
⟨*proof*⟩

The powerdomain basis type is countable.

**lemma** *pd-basis-countable*: ∃*f*::′*a pd-basis* ⇒ *nat*. *inj f*
⟨*proof*⟩

## 30.2 Unit and plus constructors

**definition**
*PDUnit* :: ′*a compact-basis* ⇒ ′*a pd-basis* **where**
*PDUnit* = (λ*x*. *Abs-pd-basis* {*x*})

**definition**
*PDPlus* :: ′*a pd-basis* ⇒ ′*a pd-basis* ⇒ ′*a pd-basis* **where**
*PDPlus t u* = *Abs-pd-basis* (*Rep-pd-basis t* ∪ *Rep-pd-basis u*)

**lemma** *Rep-PDUnit*:
*Rep-pd-basis* (*PDUnit x*) = {*x*}
⟨*proof*⟩

**lemma** *Rep-PDPlus*:
*Rep-pd-basis* (*PDPlus u v*) = *Rep-pd-basis u* ∪ *Rep-pd-basis v*
⟨*proof*⟩

**lemma** *PDUnit-inject* [*simp*]: (*PDUnit a* = *PDUnit b*) = (*a* = *b*)
⟨*proof*⟩

**lemma** *PDPlus-assoc*: *PDPlus* (*PDPlus t u*) *v* = *PDPlus t* (*PDPlus u v*)
⟨*proof*⟩

**lemma** *PDPlus-commute*: *PDPlus t u* = *PDPlus u t*
⟨*proof*⟩

**lemma** *PDPlus-absorb*: *PDPlus t t* = *t*
⟨*proof*⟩

**lemma** *pd-basis-induct1*:
  **assumes** *PDUnit*: $\bigwedge a.\ P\ (PDUnit\ a)$
  **assumes** *PDPlus*: $\bigwedge a\ t.\ P\ t \Longrightarrow P\ (PDPlus\ (PDUnit\ a)\ t)$
  **shows** *P x*
⟨*proof*⟩

**lemma** *pd-basis-induct*:
  **assumes** *PDUnit*: $\bigwedge a.\ P\ (PDUnit\ a)$
  **assumes** *PDPlus*: $\bigwedge t\ u.\ [\![P\ t;\ P\ u]\!] \Longrightarrow P\ (PDPlus\ t\ u)$
  **shows** *P x*
⟨*proof*⟩

### 30.3 Fold operator

**definition**
  *fold-pd* ::
    ('*a compact-basis* ⇒ '*b*::*type*) ⇒ ('*b* ⇒ '*b* ⇒ '*b*) ⇒ '*a pd-basis* ⇒ '*b*
  **where** *fold-pd g f t* = *semilattice-set.F f* (*g* ' *Rep-pd-basis t*)

**lemma** *fold-pd-PDUnit*:
  **assumes** *semilattice f*
  **shows** *fold-pd g f* (*PDUnit x*) = *g x*
⟨*proof*⟩

**lemma** *fold-pd-PDPlus*:
  **assumes** *semilattice f*
  **shows** *fold-pd g f* (*PDPlus t u*) = *f* (*fold-pd g f t*) (*fold-pd g f u*)
⟨*proof*⟩

**end**

## 31 Upper powerdomain

**theory** *UpperPD*
**imports** *Compact-Basis*
**begin**

### 31.1 Basis preorder

**definition**

*upper-le* :: *'a pd-basis* ⇒ *'a pd-basis* ⇒ *bool* (**infix** $\leq\sharp$ *50*) **where**
*upper-le* = ($\lambda u\ v.\ \forall y{\in}Rep\text{-}pd\text{-}basis\ v.\ \exists x{\in}Rep\text{-}pd\text{-}basis\ u.\ x \sqsubseteq y$)

**lemma** *upper-le-refl* [*simp*]: $t \leq\sharp\ t$
⟨*proof*⟩

**lemma** *upper-le-trans*: $[\![t \leq\sharp\ u;\ u \leq\sharp\ v]\!] \Longrightarrow t \leq\sharp\ v$
⟨*proof*⟩

**interpretation** *upper-le*: *preorder upper-le*
⟨*proof*⟩

**lemma** *upper-le-minimal* [*simp*]: *PDUnit compact-bot* $\leq\sharp\ t$
⟨*proof*⟩

**lemma** *PDUnit-upper-mono*: $x \sqsubseteq y \Longrightarrow$ *PDUnit x* $\leq\sharp$ *PDUnit y*
⟨*proof*⟩

**lemma** *PDPlus-upper-mono*: $[\![s \leq\sharp\ t;\ u \leq\sharp\ v]\!] \Longrightarrow$ *PDPlus s u* $\leq\sharp$ *PDPlus t v*
⟨*proof*⟩

**lemma** *PDPlus-upper-le*: *PDPlus t u* $\leq\sharp\ t$
⟨*proof*⟩

**lemma** *upper-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* $\leq\sharp$ *PDUnit b*) = ($a \sqsubseteq b$)
⟨*proof*⟩

**lemma** *upper-le-PDPlus-PDUnit-iff*:
(*PDPlus t u* $\leq\sharp$ *PDUnit a*) = ($t \leq\sharp$ *PDUnit a* ∨ $u \leq\sharp$ *PDUnit a*)
⟨*proof*⟩

**lemma** *upper-le-PDPlus-iff*: ($t \leq\sharp$ *PDPlus u v*) = ($t \leq\sharp\ u \wedge t \leq\sharp\ v$)
⟨*proof*⟩

**lemma** *upper-le-induct* [*induct set*: *upper-le*]:
**assumes** *le*: $t \leq\sharp\ u$
**assumes** *1*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P$ (*PDUnit a*) (*PDUnit b*)
**assumes** *2*: $\bigwedge t\ u\ a.\ P\ t$ (*PDUnit a*) $\Longrightarrow P$ (*PDPlus t u*) (*PDUnit a*)
**assumes** *3*: $\bigwedge t\ u\ v.\ [\![P\ t\ u;\ P\ t\ v]\!] \Longrightarrow P\ t$ (*PDPlus u v*)
**shows** *P t u*
⟨*proof*⟩

## 31.2 Type definition

**typedef** *'a upper-pd* (('(-')♯)) =
{*S*::*'a pd-basis set*. *upper-le.ideal S*}
⟨*proof*⟩

**instantiation** *upper-pd* :: (*bifinite*) *below*
**begin**

**definition**
$x \sqsubseteq y \longleftrightarrow \textit{Rep-upper-pd}\ x \subseteq \textit{Rep-upper-pd}\ y$

**instance** ⟨*proof*⟩
**end**

**instance** *upper-pd* :: (*bifinite*) *po*
⟨*proof*⟩

**instance** *upper-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

**definition**
*upper-principal* :: *'a pd-basis* ⇒ *'a upper-pd* **where**
$\textit{upper-principal}\ t = \textit{Abs-upper-pd}\ \{u.\ u \leq\sharp\ t\}$

**interpretation** *upper-pd*:
*ideal-completion upper-le upper-principal Rep-upper-pd*
⟨*proof*⟩

Upper powerdomain is pointed

**lemma** *upper-pd-minimal*: *upper-principal* (*PDUnit compact-bot*) ⊑ *ys*
⟨*proof*⟩

**instance** *upper-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

**lemma** *inst-upper-pd-pcpo*: ⊥ = *upper-principal* (*PDUnit compact-bot*)
⟨*proof*⟩

## 31.3 Monadic unit and plus

**definition**
*upper-unit* :: *'a* → *'a upper-pd* **where**
*upper-unit* = *compact-basis.extension* (λ*a*. *upper-principal* (*PDUnit a*))

**definition**
*upper-plus* :: *'a upper-pd* → *'a upper-pd* → *'a upper-pd* **where**
*upper-plus* = *upper-pd.extension* (λ*t*. *upper-pd.extension* (λ*u*.
*upper-principal* (*PDPlus t u*)))

**abbreviation**
*upper-add* :: *'a upper-pd* ⇒ *'a upper-pd* ⇒ *'a upper-pd*
(**infixl** ∪♯ *65*) **where**
*xs* ∪♯ *ys* == *upper-plus·xs·ys*

**syntax**
  *-upper-pd* :: *args* ⇒ *logic* ({-}♯)

**translations**
  $\{x, xs\}\sharp == \{x\}\sharp \cup\sharp \{xs\}\sharp$
  $\{x\}\sharp ==$ *CONST upper-unit*·$x$

**lemma** *upper-unit-Rep-compact-basis* [*simp*]:
  {*Rep-compact-basis a*}♯ = *upper-principal* (*PDUnit a*)
⟨*proof*⟩

**lemma** *upper-plus-principal* [*simp*]:
  *upper-principal t* ∪♯ *upper-principal u* = *upper-principal* (*PDPlus t u*)
⟨*proof*⟩

**interpretation** *upper-add*: *semilattice upper-add* ⟨*proof*⟩

**lemmas** *upper-plus-assoc* = *upper-add.assoc*
**lemmas** *upper-plus-commute* = *upper-add.commute*
**lemmas** *upper-plus-absorb* = *upper-add.idem*
**lemmas** *upper-plus-left-commute* = *upper-add.left-commute*
**lemmas** *upper-plus-left-absorb* = *upper-add.left-idem*

Useful for *simp add*: *upper-plus-ac*

**lemmas** *upper-plus-ac* =
  *upper-plus-assoc upper-plus-commute upper-plus-left-commute*

Useful for *simp only*: *upper-plus-aci*

**lemmas** *upper-plus-aci* =
  *upper-plus-ac upper-plus-absorb upper-plus-left-absorb*

**lemma** *upper-plus-below1*: $xs \cup\sharp\ ys \sqsubseteq xs$
⟨*proof*⟩

**lemma** *upper-plus-below2*: $xs \cup\sharp\ ys \sqsubseteq ys$
⟨*proof*⟩

**lemma** *upper-plus-greatest*: $[\![xs \sqsubseteq ys;\ xs \sqsubseteq zs]\!] \Longrightarrow xs \sqsubseteq ys \cup\sharp\ zs$
⟨*proof*⟩

**lemma** *upper-below-plus-iff* [*simp*]:
  $xs \sqsubseteq ys \cup\sharp\ zs \longleftrightarrow xs \sqsubseteq ys \wedge xs \sqsubseteq zs$
⟨*proof*⟩

**lemma** *upper-plus-below-unit-iff* [*simp*]:
  $xs \cup\sharp\ ys \sqsubseteq \{z\}\sharp \longleftrightarrow xs \sqsubseteq \{z\}\sharp \vee ys \sqsubseteq \{z\}\sharp$
⟨*proof*⟩

**lemma** *upper-unit-below-iff* [*simp*]: $\{x\}\sharp \sqsubseteq \{y\}\sharp \longleftrightarrow x \sqsubseteq y$

⟨*proof*⟩

**lemmas** *upper-pd-below-simps* =
*upper-unit-below-iff*
*upper-below-plus-iff*
*upper-plus-below-unit-iff*

**lemma** *upper-unit-eq-iff* [*simp*]: $\{x\}\sharp = \{y\}\sharp \longleftrightarrow x = y$
⟨*proof*⟩

**lemma** *upper-unit-strict* [*simp*]: $\{\bot\}\sharp = \bot$
⟨*proof*⟩

**lemma** *upper-plus-strict1* [*simp*]: $\bot \cup\sharp\ ys = \bot$
⟨*proof*⟩

**lemma** *upper-plus-strict2* [*simp*]: $xs \cup\sharp\ \bot = \bot$
⟨*proof*⟩

**lemma** *upper-unit-bottom-iff* [*simp*]: $\{x\}\sharp = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

**lemma** *upper-plus-bottom-iff* [*simp*]:
$xs \cup\sharp\ ys = \bot \longleftrightarrow xs = \bot \vee ys = \bot$
⟨*proof*⟩

**lemma** *compact-upper-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\sharp$
⟨*proof*⟩

**lemma** *compact-upper-unit-iff* [*simp*]: *compact* $\{x\}\sharp \longleftrightarrow$ *compact* $x$
⟨*proof*⟩

**lemma** *compact-upper-plus* [*simp*]:
$[\![$*compact xs*; *compact ys*$]\!] \Longrightarrow$ *compact* $(xs \cup\sharp\ ys)$
⟨*proof*⟩

## 31.4 Induction rules

**lemma** *upper-pd-induct1*:
**assumes** *P*: *adm P*
**assumes** *unit*: $\bigwedge x.\ P\ \{x\}\sharp$
**assumes** *insert*: $\bigwedge x\ ys.\ [\![P\ \{x\}\sharp;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\sharp \cup\sharp\ ys)$
**shows** *P* (*xs*::'*a upper-pd*)
⟨*proof*⟩

**lemma** *upper-pd-induct*
[*case-names adm upper-unit upper-plus*, *induct type*: *upper-pd*]:
**assumes** *P*: *adm P*
**assumes** *unit*: $\bigwedge x.\ P\ \{x\}\sharp$

**assumes** *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\sharp\ ys)$
**shows** $P\ (xs{::}'a\ upper\text{-}pd)$
⟨*proof*⟩

## 31.5 Monadic bind

**definition**
*upper-bind-basis* ::
$'a$ *pd-basis* ⇒ ($'a$ → $'b$ *upper-pd*) → $'b$ *upper-pd* **where**
*upper-bind-basis* = *fold-pd*
(λ*a*. Λ *f*. *f*·(*Rep-compact-basis a*))
(λ*x y*. Λ *f*. *x*·*f* ∪♯ *y*·*f*)

**lemma** *ACI-upper-bind*:
*semilattice* (λ*x y*. Λ *f*. *x*·*f* ∪♯ *y*·*f*)
⟨*proof*⟩

**lemma** *upper-bind-basis-simps* [*simp*]:
*upper-bind-basis* (*PDUnit a*) =
(Λ *f*. *f*·(*Rep-compact-basis a*))
*upper-bind-basis* (*PDPlus t u*) =
(Λ *f*. *upper-bind-basis t*·*f* ∪♯ *upper-bind-basis u*·*f*)
⟨*proof*⟩

**lemma** *upper-bind-basis-mono*:
$t \le\sharp\ u \Longrightarrow$ *upper-bind-basis t* ⊑ *upper-bind-basis u*
⟨*proof*⟩

**definition**
*upper-bind* :: $'a$ *upper-pd* → ($'a$ → $'b$ *upper-pd*) → $'b$ *upper-pd* **where**
*upper-bind* = *upper-pd.extension upper-bind-basis*

**syntax**
*-upper-bind* :: [*logic*, *logic*, *logic*] ⇒ *logic*
((*3*⋃♯-∈-./ -) [*0*, *0*, *10*] *10*)

**translations**
⋃♯*x*∈*xs*. *e* == *CONST upper-bind*·*xs*·(Λ *x*. *e*)

**lemma** *upper-bind-principal* [*simp*]:
*upper-bind*·(*upper-principal t*) = *upper-bind-basis t*
⟨*proof*⟩

**lemma** *upper-bind-unit* [*simp*]:
*upper-bind*·{*x*}♯·*f* = *f*·*x*
⟨*proof*⟩

**lemma** *upper-bind-plus* [*simp*]:
*upper-bind*·(*xs* ∪♯ *ys*)·*f* = *upper-bind*·*xs*·*f* ∪♯ *upper-bind*·*ys*·*f*

⟨*proof*⟩

**lemma** *upper-bind-strict* [*simp*]: $upper\text{-}bind\cdot\bot\cdot f = f\cdot\bot$
⟨*proof*⟩

**lemma** *upper-bind-bind*:
$upper\text{-}bind\cdot(upper\text{-}bind\cdot xs\cdot f)\cdot g = upper\text{-}bind\cdot xs\cdot(\Lambda\ x.\ upper\text{-}bind\cdot(f\cdot x)\cdot g)$
⟨*proof*⟩

## 31.6 Map

**definition**
$upper\text{-}map :: ('a \rightarrow 'b) \rightarrow 'a\ upper\text{-}pd \rightarrow 'b\ upper\text{-}pd$ **where**
$upper\text{-}map = (\Lambda\ f\ xs.\ upper\text{-}bind\cdot xs\cdot(\Lambda\ x.\ \{f\cdot x\}\sharp))$

**lemma** *upper-map-unit* [*simp*]:
$upper\text{-}map\cdot f\cdot\{x\}\sharp = \{f\cdot x\}\sharp$
⟨*proof*⟩

**lemma** *upper-map-plus* [*simp*]:
$upper\text{-}map\cdot f\cdot(xs \cup\sharp\ ys) = upper\text{-}map\cdot f\cdot xs \cup\sharp\ upper\text{-}map\cdot f\cdot ys$
⟨*proof*⟩

**lemma** *upper-map-bottom* [*simp*]: $upper\text{-}map\cdot f\cdot\bot = \{f\cdot\bot\}\sharp$
⟨*proof*⟩

**lemma** *upper-map-ident*: $upper\text{-}map\cdot(\Lambda\ x.\ x)\cdot xs = xs$
⟨*proof*⟩

**lemma** *upper-map-ID*: $upper\text{-}map\cdot ID = ID$
⟨*proof*⟩

**lemma** *upper-map-map*:
$upper\text{-}map\cdot f\cdot(upper\text{-}map\cdot g\cdot xs) = upper\text{-}map\cdot(\Lambda\ x.\ f\cdot(g\cdot x))\cdot xs$
⟨*proof*⟩

**lemma** *upper-bind-map*:
$upper\text{-}bind\cdot(upper\text{-}map\cdot f\cdot xs)\cdot g = upper\text{-}bind\cdot xs\cdot(\Lambda\ x.\ g\cdot(f\cdot x))$
⟨*proof*⟩

**lemma** *upper-map-bind*:
$upper\text{-}map\cdot f\cdot(upper\text{-}bind\cdot xs\cdot g) = upper\text{-}bind\cdot xs\cdot(\Lambda\ x.\ upper\text{-}map\cdot f\cdot(g\cdot x))$
⟨*proof*⟩

**lemma** *ep-pair-upper-map*: $ep\text{-}pair\ e\ p \Longrightarrow ep\text{-}pair\ (upper\text{-}map\cdot e)\ (upper\text{-}map\cdot p)$
⟨*proof*⟩

**lemma** *deflation-upper-map*: $deflation\ d \Longrightarrow deflation\ (upper\text{-}map\cdot d)$
⟨*proof*⟩

**lemma** *finite-deflation-upper-map*:
  **assumes** *finite-deflation d* **shows** *finite-deflation* (*upper-map*·*d*)
⟨*proof*⟩

## 31.7 Upper powerdomain is bifinite

**lemma** *approx-chain-upper-map*:
  **assumes** *approx-chain a*
  **shows** *approx-chain* (λ*i*. *upper-map*·(*a i*))
  ⟨*proof*⟩

**instance** *upper-pd* :: (*bifinite*) *bifinite*
⟨*proof*⟩

## 31.8 Join

**definition**
  *upper-join* :: ′*a upper-pd upper-pd* → ′*a upper-pd* **where**
  *upper-join* = (Λ *xss*. *upper-bind*·*xss*·(Λ *xs*. *xs*))

**lemma** *upper-join-unit* [*simp*]:
  *upper-join*·{*xs*}♯ = *xs*
⟨*proof*⟩

**lemma** *upper-join-plus* [*simp*]:
  *upper-join*·(*xss* ∪♯ *yss*) = *upper-join*·*xss* ∪♯ *upper-join*·*yss*
⟨*proof*⟩

**lemma** *upper-join-bottom* [*simp*]: *upper-join*·⊥ = ⊥
⟨*proof*⟩

**lemma** *upper-join-map-unit*:
  *upper-join*·(*upper-map*·*upper-unit*·*xs*) = *xs*
⟨*proof*⟩

**lemma** *upper-join-map-join*:
  *upper-join*·(*upper-map*·*upper-join*·*xsss*) = *upper-join*·(*upper-join*·*xsss*)
⟨*proof*⟩

**lemma** *upper-join-map-map*:
  *upper-join*·(*upper-map*·(*upper-map*·*f*)·*xss*) =
   *upper-map*·*f*·(*upper-join*·*xss*)
⟨*proof*⟩

**end**

# 32 Lower powerdomain

**theory** *LowerPD*
**imports** *Compact-Basis*
**begin**

## 32.1 Basis preorder

**definition**
*lower-le* :: $'a$ *pd-basis* ⇒ $'a$ *pd-basis* ⇒ *bool* (**infix** ≤♭ *50*) **where**
*lower-le* = (λ*u v*. ∀ *x*∈*Rep-pd-basis u*. ∃ *y*∈*Rep-pd-basis v*. *x* ⊑ *y*)

**lemma** *lower-le-refl* [*simp*]: *t* ≤♭ *t*
⟨*proof*⟩

**lemma** *lower-le-trans*: ⟦*t* ≤♭ *u*; *u* ≤♭ *v*⟧ ⟹ *t* ≤♭ *v*
⟨*proof*⟩

**interpretation** *lower-le*: *preorder lower-le*
⟨*proof*⟩

**lemma** *lower-le-minimal* [*simp*]: *PDUnit compact-bot* ≤♭ *t*
⟨*proof*⟩

**lemma** *PDUnit-lower-mono*: *x* ⊑ *y* ⟹ *PDUnit x* ≤♭ *PDUnit y*
⟨*proof*⟩

**lemma** *PDPlus-lower-mono*: ⟦*s* ≤♭ *t*; *u* ≤♭ *v*⟧ ⟹ *PDPlus s u* ≤♭ *PDPlus t v*
⟨*proof*⟩

**lemma** *PDPlus-lower-le*: *t* ≤♭ *PDPlus t u*
⟨*proof*⟩

**lemma** *lower-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* ≤♭ *PDUnit b*) = (*a* ⊑ *b*)
⟨*proof*⟩

**lemma** *lower-le-PDUnit-PDPlus-iff*:
(*PDUnit a* ≤♭ *PDPlus t u*) = (*PDUnit a* ≤♭ *t* ∨ *PDUnit a* ≤♭ *u*)
⟨*proof*⟩

**lemma** *lower-le-PDPlus-iff*: (*PDPlus t u* ≤♭ *v*) = (*t* ≤♭ *v* ∧ *u* ≤♭ *v*)
⟨*proof*⟩

**lemma** *lower-le-induct* [*induct set*: *lower-le*]:
**assumes** *le*: *t* ≤♭ *u*
**assumes** *1*: ⋀*a b*. *a* ⊑ *b* ⟹ *P* (*PDUnit a*) (*PDUnit b*)
**assumes** *2*: ⋀*t u a*. *P* (*PDUnit a*) *t* ⟹ *P* (*PDUnit a*) (*PDPlus t u*)
**assumes** *3*: ⋀*t u v*. ⟦*P t v*; *P u v*⟧ ⟹ *P* (*PDPlus t u*) *v*
**shows** *P t u*

⟨*proof*⟩

## 32.2 Type definition

**typedef** *'a lower-pd* (('(-')♭)) =
{*S*::*'a pd-basis set. lower-le.ideal S*}
⟨*proof*⟩

**instantiation** *lower-pd* :: (*bifinite*) *below*
**begin**

**definition**
$x \sqsubseteq y \longleftrightarrow$ *Rep-lower-pd x* $\subseteq$ *Rep-lower-pd y*

**instance** ⟨*proof*⟩
**end**

**instance** *lower-pd* :: (*bifinite*) *po*
⟨*proof*⟩

**instance** *lower-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

**definition**
*lower-principal* :: *'a pd-basis* ⇒ *'a lower-pd* **where**
*lower-principal t* = *Abs-lower-pd* {*u. u* ≤♭ *t*}

**interpretation** *lower-pd*:
*ideal-completion lower-le lower-principal Rep-lower-pd*
⟨*proof*⟩

Lower powerdomain is pointed

**lemma** *lower-pd-minimal*: *lower-principal* (*PDUnit compact-bot*) $\sqsubseteq$ *ys*
⟨*proof*⟩

**instance** *lower-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

**lemma** *inst-lower-pd-pcpo*: ⊥ = *lower-principal* (*PDUnit compact-bot*)
⟨*proof*⟩

## 32.3 Monadic unit and plus

**definition**
*lower-unit* :: *'a* → *'a lower-pd* **where**
*lower-unit* = *compact-basis.extension* (λ*a. lower-principal* (*PDUnit a*))

**definition**
*lower-plus* :: *'a lower-pd* → *'a lower-pd* → *'a lower-pd* **where**

*lower-plus* = *lower-pd.extension* (λ*t*. *lower-pd.extension* (λ*u*.
*lower-principal* (*PDPlus t u*)))

**abbreviation**
*lower-add* :: *'a lower-pd* ⇒ *'a lower-pd* ⇒ *'a lower-pd*
(**infixl** ∪♭ *65*) **where**
*xs* ∪♭ *ys* == *lower-plus·xs·ys*

**syntax**
*-lower-pd* :: *args* ⇒ *logic* ({-}♭)

**translations**
{*x*,*xs*}♭ == {*x*}♭ ∪♭ {*xs*}♭
{*x*}♭ == *CONST lower-unit·x*

**lemma** *lower-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♭ = *lower-principal* (*PDUnit a*)
⟨*proof*⟩

**lemma** *lower-plus-principal* [*simp*]:
*lower-principal t* ∪♭ *lower-principal u* = *lower-principal* (*PDPlus t u*)
⟨*proof*⟩

**interpretation** *lower-add*: *semilattice lower-add* ⟨*proof*⟩

**lemmas** *lower-plus-assoc* = *lower-add.assoc*
**lemmas** *lower-plus-commute* = *lower-add.commute*
**lemmas** *lower-plus-absorb* = *lower-add.idem*
**lemmas** *lower-plus-left-commute* = *lower-add.left-commute*
**lemmas** *lower-plus-left-absorb* = *lower-add.left-idem*

Useful for *simp add*: *lower-plus-ac*

**lemmas** *lower-plus-ac* =
*lower-plus-assoc lower-plus-commute lower-plus-left-commute*

Useful for *simp only*: *lower-plus-aci*

**lemmas** *lower-plus-aci* =
*lower-plus-ac lower-plus-absorb lower-plus-left-absorb*

**lemma** *lower-plus-below1*: *xs* ⊑ *xs* ∪♭ *ys*
⟨*proof*⟩

**lemma** *lower-plus-below2*: *ys* ⊑ *xs* ∪♭ *ys*
⟨*proof*⟩

**lemma** *lower-plus-least*: ⟦*xs* ⊑ *zs*; *ys* ⊑ *zs*⟧ ⟹ *xs* ∪♭ *ys* ⊑ *zs*
⟨*proof*⟩

**lemma** *lower-plus-below-iff* [*simp*]:

$xs \cup\flat\ ys \sqsubseteq zs \longleftrightarrow xs \sqsubseteq zs \wedge ys \sqsubseteq zs$
⟨*proof*⟩

**lemma** *lower-unit-below-plus-iff* [*simp*]:
$\{x\}\flat \sqsubseteq ys \cup\flat\ zs \longleftrightarrow \{x\}\flat \sqsubseteq ys \vee \{x\}\flat \sqsubseteq zs$
⟨*proof*⟩

**lemma** *lower-unit-below-iff* [*simp*]: $\{x\}\flat \sqsubseteq \{y\}\flat \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

**lemmas** *lower-pd-below-simps* =
*lower-unit-below-iff*
*lower-plus-below-iff*
*lower-unit-below-plus-iff*

**lemma** *lower-unit-eq-iff* [*simp*]: $\{x\}\flat = \{y\}\flat \longleftrightarrow x = y$
⟨*proof*⟩

**lemma** *lower-unit-strict* [*simp*]: $\{\bot\}\flat = \bot$
⟨*proof*⟩

**lemma** *lower-unit-bottom-iff* [*simp*]: $\{x\}\flat = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

**lemma** *lower-plus-bottom-iff* [*simp*]:
$xs \cup\flat\ ys = \bot \longleftrightarrow xs = \bot \wedge ys = \bot$
⟨*proof*⟩

**lemma** *lower-plus-strict1* [*simp*]: $\bot \cup\flat\ ys = ys$
⟨*proof*⟩

**lemma** *lower-plus-strict2* [*simp*]: $xs \cup\flat\ \bot = xs$
⟨*proof*⟩

**lemma** *compact-lower-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\flat$
⟨*proof*⟩

**lemma** *compact-lower-unit-iff* [*simp*]: *compact* $\{x\}\flat \longleftrightarrow$ *compact* $x$
⟨*proof*⟩

**lemma** *compact-lower-plus* [*simp*]:
$[\![$*compact xs*; *compact ys*$]\!] \Longrightarrow$ *compact* $(xs \cup\flat\ ys)$
⟨*proof*⟩

## 32.4 Induction rules

**lemma** *lower-pd-induct1*:
**assumes** *P*: *adm P*
**assumes** *unit*: $\bigwedge x.\ P\ \{x\}\flat$

**assumes** *insert*:
$\bigwedge x\ ys.\ [\![P\ \{x\}\flat;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\flat \cup\flat\ ys)$
**shows** $P\ (xs{::}'a\ lower\text{-}pd)$
⟨*proof*⟩

**lemma** *lower-pd-induct*
[*case-names adm lower-unit lower-plus*, *induct type*: *lower-pd*]:
**assumes** *P*: *adm P*
**assumes** *unit*: $\bigwedge x.\ P\ \{x\}\flat$
**assumes** *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\flat\ ys)$
**shows** $P\ (xs{::}'a\ lower\text{-}pd)$
⟨*proof*⟩

## 32.5 Monadic bind

**definition**
*lower-bind-basis* ::
$'a\ pd\text{-}basis \Rightarrow ('a \rightarrow 'b\ lower\text{-}pd) \rightarrow 'b\ lower\text{-}pd$ **where**
*lower-bind-basis* = *fold-pd*
$(\lambda a.\ \Lambda\ f.\ f{\cdot}(Rep\text{-}compact\text{-}basis\ a))$
$(\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\flat\ y{\cdot}f)$

**lemma** *ACI-lower-bind*:
*semilattice* $(\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\flat\ y{\cdot}f)$
⟨*proof*⟩

**lemma** *lower-bind-basis-simps* [*simp*]:
*lower-bind-basis* (*PDUnit a*) =
$(\Lambda\ f.\ f{\cdot}(Rep\text{-}compact\text{-}basis\ a))$
*lower-bind-basis* (*PDPlus t u*) =
$(\Lambda\ f.\ lower\text{-}bind\text{-}basis\ t{\cdot}f \cup\flat\ lower\text{-}bind\text{-}basis\ u{\cdot}f)$
⟨*proof*⟩

**lemma** *lower-bind-basis-mono*:
$t \le\flat\ u \Longrightarrow lower\text{-}bind\text{-}basis\ t \sqsubseteq lower\text{-}bind\text{-}basis\ u$
⟨*proof*⟩

**definition**
*lower-bind* :: $'a\ lower\text{-}pd \rightarrow ('a \rightarrow 'b\ lower\text{-}pd) \rightarrow 'b\ lower\text{-}pd$ **where**
*lower-bind* = *lower-pd.extension lower-bind-basis*

**syntax**
*-lower-bind* :: [*logic*, *logic*, *logic*] ⇒ *logic*
((*3*⋃♭-∈-./ -) [*0*, *0*, *10*] *10*)

**translations**
⋃♭*x*∈*xs*. *e* == *CONST lower-bind*·*xs*·(Λ *x*. *e*)

**lemma** *lower-bind-principal* [*simp*]:

*lower-bind*·(*lower-principal t*) = *lower-bind-basis t*
⟨*proof*⟩

**lemma** *lower-bind-unit* [*simp*]:
*lower-bind*·{*x*}♭·*f* = *f*·*x*
⟨*proof*⟩

**lemma** *lower-bind-plus* [*simp*]:
*lower-bind*·(*xs* ∪♭ *ys*)·*f* = *lower-bind*·*xs*·*f* ∪♭ *lower-bind*·*ys*·*f*
⟨*proof*⟩

**lemma** *lower-bind-strict* [*simp*]: *lower-bind*·⊥·*f* = *f*·⊥
⟨*proof*⟩

**lemma** *lower-bind-bind*:
*lower-bind*·(*lower-bind*·*xs*·*f*)·*g* = *lower-bind*·*xs*·(Λ *x*. *lower-bind*·(*f*·*x*)·*g*)
⟨*proof*⟩

## 32.6 Map

**definition**
*lower-map* :: (′*a* → ′*b*) → ′*a lower-pd* → ′*b lower-pd* **where**
*lower-map* = (Λ *f xs*. *lower-bind*·*xs*·(Λ *x*. {*f*·*x*}♭))

**lemma** *lower-map-unit* [*simp*]:
*lower-map*·*f*·{*x*}♭ = {*f*·*x*}♭
⟨*proof*⟩

**lemma** *lower-map-plus* [*simp*]:
*lower-map*·*f*·(*xs* ∪♭ *ys*) = *lower-map*·*f*·*xs* ∪♭ *lower-map*·*f*·*ys*
⟨*proof*⟩

**lemma** *lower-map-bottom* [*simp*]: *lower-map*·*f*·⊥ = {*f*·⊥}♭
⟨*proof*⟩

**lemma** *lower-map-ident*: *lower-map*·(Λ *x*. *x*)·*xs* = *xs*
⟨*proof*⟩

**lemma** *lower-map-ID*: *lower-map*·*ID* = *ID*
⟨*proof*⟩

**lemma** *lower-map-map*:
*lower-map*·*f*·(*lower-map*·*g*·*xs*) = *lower-map*·(Λ *x*. *f*·(*g*·*x*))·*xs*
⟨*proof*⟩

**lemma** *lower-bind-map*:
*lower-bind*·(*lower-map*·*f*·*xs*)·*g* = *lower-bind*·*xs*·(Λ *x*. *g*·(*f*·*x*))
⟨*proof*⟩

**lemma** *lower-map-bind*:
*lower-map·f·(lower-bind·xs·g) = lower-bind·xs·(Λ x. lower-map·f·(g·x))*
⟨*proof*⟩

**lemma** *ep-pair-lower-map*: *ep-pair e p ⟹ ep-pair (lower-map·e) (lower-map·p)*
⟨*proof*⟩

**lemma** *deflation-lower-map*: *deflation d ⟹ deflation (lower-map·d)*
⟨*proof*⟩

**lemma** *finite-deflation-lower-map*:
**assumes** *finite-deflation d* **shows** *finite-deflation (lower-map·d)*
⟨*proof*⟩

## 32.7 Lower powerdomain is bifinite

**lemma** *approx-chain-lower-map*:
**assumes** *approx-chain a*
**shows** *approx-chain (λi. lower-map·(a i))*
⟨*proof*⟩

**instance** *lower-pd* :: (*bifinite*) *bifinite*
⟨*proof*⟩

## 32.8 Join

**definition**
*lower-join* :: *'a lower-pd lower-pd → 'a lower-pd* **where**
*lower-join = (Λ xss. lower-bind·xss·(Λ xs. xs))*

**lemma** *lower-join-unit* [*simp*]:
*lower-join·{xs}♭ = xs*
⟨*proof*⟩

**lemma** *lower-join-plus* [*simp*]:
*lower-join·(xss ∪♭ yss) = lower-join·xss ∪♭ lower-join·yss*
⟨*proof*⟩

**lemma** *lower-join-bottom* [*simp*]: *lower-join·⊥ = ⊥*
⟨*proof*⟩

**lemma** *lower-join-map-unit*:
*lower-join·(lower-map·lower-unit·xs) = xs*
⟨*proof*⟩

**lemma** *lower-join-map-join*:
*lower-join·(lower-map·lower-join·xsss) = lower-join·(lower-join·xsss)*
⟨*proof*⟩

**lemma** *lower-join-map-map*:
*lower-join*·(*lower-map*·(*lower-map*·*f*)·*xss*) =
*lower-map*·*f*·(*lower-join*·*xss*)
⟨*proof*⟩

**end**

# 33 Convex powerdomain

**theory** *ConvexPD*
**imports** *UpperPD LowerPD*
**begin**

## 33.1 Basis preorder

**definition**
*convex-le* :: *'a pd-basis* ⇒ *'a pd-basis* ⇒ *bool* (**infix** ≤♮ *50*) **where**
*convex-le* = (λ*u v*. *u* ≤♯ *v* ∧ *u* ≤♭ *v*)

**lemma** *convex-le-refl* [*simp*]: *t* ≤♮ *t*
⟨*proof*⟩

**lemma** *convex-le-trans*: ⟦*t* ≤♮ *u*; *u* ≤♮ *v*⟧ ⟹ *t* ≤♮ *v*
⟨*proof*⟩

**interpretation** *convex-le*: *preorder convex-le*
⟨*proof*⟩

**lemma** *upper-le-minimal* [*simp*]: *PDUnit compact-bot* ≤♮ *t*
⟨*proof*⟩

**lemma** *PDUnit-convex-mono*: *x* ⊑ *y* ⟹ *PDUnit x* ≤♮ *PDUnit y*
⟨*proof*⟩

**lemma** *PDPlus-convex-mono*: ⟦*s* ≤♮ *t*; *u* ≤♮ *v*⟧ ⟹ *PDPlus s u* ≤♮ *PDPlus t v*
⟨*proof*⟩

**lemma** *convex-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* ≤♮ *PDUnit b*) = (*a* ⊑ *b*)
⟨*proof*⟩

**lemma** *convex-le-PDUnit-lemma1*:
(*PDUnit a* ≤♮ *t*) = (∀ *b*∈*Rep-pd-basis t*. *a* ⊑ *b*)
⟨*proof*⟩

**lemma** *convex-le-PDUnit-PDPlus-iff* [*simp*]:
(*PDUnit a* ≤♮ *PDPlus t u*) = (*PDUnit a* ≤♮ *t* ∧ *PDUnit a* ≤♮ *u*)
⟨*proof*⟩

**lemma** *convex-le-PDUnit-lemma2*:
 $(t \leq\natural\ PDUnit\ b) = (\forall\ a{\in}Rep\text{-}pd\text{-}basis\ t.\ a \sqsubseteq b)$
⟨*proof*⟩

**lemma** *convex-le-PDPlus-PDUnit-iff* [*simp*]:
 $(PDPlus\ t\ u \leq\natural\ PDUnit\ a) = (t \leq\natural\ PDUnit\ a \wedge u \leq\natural\ PDUnit\ a)$
⟨*proof*⟩

**lemma** *convex-le-PDPlus-lemma*:
 **assumes** *z*: $PDPlus\ t\ u \leq\natural\ z$
 **shows** $\exists\ v\ w.\ z = PDPlus\ v\ w \wedge t \leq\natural\ v \wedge u \leq\natural\ w$
⟨*proof*⟩

**lemma** *convex-le-induct* [*induct set*: *convex-le*]:
 **assumes** *le*: $t \leq\natural\ u$
 **assumes** *2*: $\bigwedge t\ u\ v.\ [\![P\ t\ u;\ P\ u\ v]\!] \Longrightarrow P\ t\ v$
 **assumes** *3*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P\ (PDUnit\ a)\ (PDUnit\ b)$
 **assumes** *4*: $\bigwedge t\ u\ v\ w.\ [\![P\ t\ v;\ P\ u\ w]\!] \Longrightarrow P\ (PDPlus\ t\ u)\ (PDPlus\ v\ w)$
 **shows** $P\ t\ u$
⟨*proof*⟩

## 33.2 Type definition

**typedef** *'a convex-pd* (('(-')♮)) =
 {*S*::*'a pd-basis set*. *convex-le.ideal S*}
⟨*proof*⟩

**instantiation** *convex-pd* :: (*bifinite*) *below*
**begin**

**definition**
 $x \sqsubseteq y \longleftrightarrow Rep\text{-}convex\text{-}pd\ x \subseteq Rep\text{-}convex\text{-}pd\ y$

**instance** ⟨*proof*⟩
**end**

**instance** *convex-pd* :: (*bifinite*) *po*
⟨*proof*⟩

**instance** *convex-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

**definition**
 *convex-principal* :: *'a pd-basis* ⇒ *'a convex-pd* **where**
 $convex\text{-}principal\ t = Abs\text{-}convex\text{-}pd\ \{u.\ u \leq\natural\ t\}$

**interpretation** *convex-pd*:
 *ideal-completion convex-le convex-principal Rep-convex-pd*
⟨*proof*⟩

Convex powerdomain is pointed

**lemma** *convex-pd-minimal*: *convex-principal* (*PDUnit compact-bot*) $\sqsubseteq$ *ys*
⟨*proof*⟩

**instance** *convex-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

**lemma** *inst-convex-pd-pcpo*: $\bot$ = *convex-principal* (*PDUnit compact-bot*)
⟨*proof*⟩

## 33.3 Monadic unit and plus

**definition**
*convex-unit* :: *'a* → *'a convex-pd* **where**
*convex-unit* = *compact-basis.extension* (λ*a*. *convex-principal* (*PDUnit a*))

**definition**
*convex-plus* :: *'a convex-pd* → *'a convex-pd* → *'a convex-pd* **where**
*convex-plus* = *convex-pd.extension* (λ*t*. *convex-pd.extension* (λ*u*.
*convex-principal* (*PDPlus t u*)))

**abbreviation**
*convex-add* :: *'a convex-pd* ⇒ *'a convex-pd* ⇒ *'a convex-pd*
(**infixl** ∪♮ *65*) **where**
*xs* ∪♮ *ys* == *convex-plus·xs·ys*

**syntax**
*-convex-pd* :: *args* ⇒ *logic* ({-}♮)

**translations**
{*x*,*xs*}♮ == {*x*}♮ ∪♮ {*xs*}♮
{*x*}♮ == *CONST convex-unit·x*

**lemma** *convex-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♮ = *convex-principal* (*PDUnit a*)
⟨*proof*⟩

**lemma** *convex-plus-principal* [*simp*]:
*convex-principal t* ∪♮ *convex-principal u* = *convex-principal* (*PDPlus t u*)
⟨*proof*⟩

**interpretation** *convex-add*: *semilattice convex-add* ⟨*proof*⟩

**lemmas** *convex-plus-assoc* = *convex-add.assoc*
**lemmas** *convex-plus-commute* = *convex-add.commute*
**lemmas** *convex-plus-absorb* = *convex-add.idem*
**lemmas** *convex-plus-left-commute* = *convex-add.left-commute*
**lemmas** *convex-plus-left-absorb* = *convex-add.left-idem*

Useful for *simp add*: *convex-plus-ac*

**lemmas** *convex-plus-ac* =
*convex-plus-assoc convex-plus-commute convex-plus-left-commute*

Useful for *simp only*: *convex-plus-aci*

**lemmas** *convex-plus-aci* =
*convex-plus-ac convex-plus-absorb convex-plus-left-absorb*

**lemma** *convex-unit-below-plus-iff* [*simp*]:
$\{x\}\natural \sqsubseteq ys \cup\natural\ zs \longleftrightarrow \{x\}\natural \sqsubseteq ys \wedge \{x\}\natural \sqsubseteq zs$
⟨*proof*⟩

**lemma** *convex-plus-below-unit-iff* [*simp*]:
$xs \cup\natural\ ys \sqsubseteq \{z\}\natural \longleftrightarrow xs \sqsubseteq \{z\}\natural \wedge ys \sqsubseteq \{z\}\natural$
⟨*proof*⟩

**lemma** *convex-unit-below-iff* [*simp*]: $\{x\}\natural \sqsubseteq \{y\}\natural \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

**lemma** *convex-unit-eq-iff* [*simp*]: $\{x\}\natural = \{y\}\natural \longleftrightarrow x = y$
⟨*proof*⟩

**lemma** *convex-unit-strict* [*simp*]: $\{\bot\}\natural = \bot$
⟨*proof*⟩

**lemma** *convex-unit-bottom-iff* [*simp*]: $\{x\}\natural = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

**lemma** *compact-convex-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\natural$
⟨*proof*⟩

**lemma** *compact-convex-unit-iff* [*simp*]: *compact* $\{x\}\natural \longleftrightarrow$ *compact* $x$
⟨*proof*⟩

**lemma** *compact-convex-plus* [*simp*]:
⟦*compact xs*; *compact ys*⟧ $\Longrightarrow$ *compact* ($xs \cup\natural\ ys$)
⟨*proof*⟩

## 33.4 Induction rules

**lemma** *convex-pd-induct1*:
**assumes** *P*: *adm P*
**assumes** *unit*: $\bigwedge x.\ P\ \{x\}\natural$
**assumes** *insert*: $\bigwedge x\ ys.\ [\![P\ \{x\}\natural;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\natural \cup\natural\ ys)$
**shows** *P* (*xs*::'*a convex-pd*)
⟨*proof*⟩

**lemma** *convex-pd-induct*
[*case-names adm convex-unit convex-plus*, *induct type*: *convex-pd*]:

**assumes** *P*: *adm P*
**assumes** *unit*: $\bigwedge x.\ P\ \{x\}\natural$
**assumes** *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\natural\ ys)$
**shows** $P\ (xs::'a\ convex\text{-}pd)$
⟨*proof*⟩

## 33.5 Monadic bind

**definition**
*convex-bind-basis* ::
$'a\ pd\text{-}basis \Rightarrow ('a \to 'b\ convex\text{-}pd) \to 'b\ convex\text{-}pd$ **where**
*convex-bind-basis* = *fold-pd*
$(\lambda a.\ \Lambda\ f.\ f\cdot(Rep\text{-}compact\text{-}basis\ a))$
$(\lambda x\ y.\ \Lambda\ f.\ x\cdot f \cup\natural\ y\cdot f)$

**lemma** *ACI-convex-bind*:
$semilattice\ (\lambda x\ y.\ \Lambda\ f.\ x\cdot f \cup\natural\ y\cdot f)$
⟨*proof*⟩

**lemma** *convex-bind-basis-simps* [*simp*]:
$convex\text{-}bind\text{-}basis\ (PDUnit\ a) =$
$(\Lambda\ f.\ f\cdot(Rep\text{-}compact\text{-}basis\ a))$
$convex\text{-}bind\text{-}basis\ (PDPlus\ t\ u) =$
$(\Lambda\ f.\ convex\text{-}bind\text{-}basis\ t\cdot f \cup\natural\ convex\text{-}bind\text{-}basis\ u\cdot f)$
⟨*proof*⟩

**lemma** *convex-bind-basis-mono*:
$t \le\natural\ u \Longrightarrow convex\text{-}bind\text{-}basis\ t \sqsubseteq convex\text{-}bind\text{-}basis\ u$
⟨*proof*⟩

**definition**
$convex\text{-}bind :: 'a\ convex\text{-}pd \to ('a \to 'b\ convex\text{-}pd) \to 'b\ convex\text{-}pd$ **where**
*convex-bind* = *convex-pd.extension convex-bind-basis*

**syntax**
$\text{-}convex\text{-}bind :: [logic,\ logic,\ logic] \Rightarrow logic$
$((3\bigcup\natural\text{-}\in\text{-}./\ \text{-})\ [0,\ 0,\ 10]\ 10)$

**translations**
$\bigcup\natural x\in xs.\ e == CONST\ convex\text{-}bind\cdot xs\cdot(\Lambda\ x.\ e)$

**lemma** *convex-bind-principal* [*simp*]:
$convex\text{-}bind\cdot(convex\text{-}principal\ t) = convex\text{-}bind\text{-}basis\ t$
⟨*proof*⟩

**lemma** *convex-bind-unit* [*simp*]:
$convex\text{-}bind\cdot\{x\}\natural\cdot f = f\cdot x$
⟨*proof*⟩

**lemma** *convex-bind-plus* [*simp*]:
  $convex\text{-}bind\cdot(xs \cup\natural\ ys)\cdot f = convex\text{-}bind\cdot xs\cdot f \cup\natural\ convex\text{-}bind\cdot ys\cdot f$
⟨*proof*⟩

**lemma** *convex-bind-strict* [*simp*]: $convex\text{-}bind\cdot\bot\cdot f = f\cdot\bot$
⟨*proof*⟩

**lemma** *convex-bind-bind*:
  $convex\text{-}bind\cdot(convex\text{-}bind\cdot xs\cdot f)\cdot g =$
    $convex\text{-}bind\cdot xs\cdot(\Lambda\ x.\ convex\text{-}bind\cdot(f\cdot x)\cdot g)$
⟨*proof*⟩

## 33.6 Map

**definition**
  $convex\text{-}map :: ('a \to 'b) \to 'a\ convex\text{-}pd \to 'b\ convex\text{-}pd$ **where**
  $convex\text{-}map = (\Lambda\ f\ xs.\ convex\text{-}bind\cdot xs\cdot(\Lambda\ x.\ \{f\cdot x\}\natural))$

**lemma** *convex-map-unit* [*simp*]:
  $convex\text{-}map\cdot f\cdot\{x\}\natural = \{f\cdot x\}\natural$
⟨*proof*⟩

**lemma** *convex-map-plus* [*simp*]:
  $convex\text{-}map\cdot f\cdot(xs \cup\natural\ ys) = convex\text{-}map\cdot f\cdot xs \cup\natural\ convex\text{-}map\cdot f\cdot ys$
⟨*proof*⟩

**lemma** *convex-map-bottom* [*simp*]: $convex\text{-}map\cdot f\cdot\bot = \{f\cdot\bot\}\natural$
⟨*proof*⟩

**lemma** *convex-map-ident*: $convex\text{-}map\cdot(\Lambda\ x.\ x)\cdot xs = xs$
⟨*proof*⟩

**lemma** *convex-map-ID*: $convex\text{-}map\cdot ID = ID$
⟨*proof*⟩

**lemma** *convex-map-map*:
  $convex\text{-}map\cdot f\cdot(convex\text{-}map\cdot g\cdot xs) = convex\text{-}map\cdot(\Lambda\ x.\ f\cdot(g\cdot x))\cdot xs$
⟨*proof*⟩

**lemma** *convex-bind-map*:
  $convex\text{-}bind\cdot(convex\text{-}map\cdot f\cdot xs)\cdot g = convex\text{-}bind\cdot xs\cdot(\Lambda\ x.\ g\cdot(f\cdot x))$
⟨*proof*⟩

**lemma** *convex-map-bind*:
  $convex\text{-}map\cdot f\cdot(convex\text{-}bind\cdot xs\cdot g) = convex\text{-}bind\cdot xs\cdot(\Lambda\ x.\ convex\text{-}map\cdot f\cdot(g\cdot x))$
⟨*proof*⟩

**lemma** *ep-pair-convex-map*: $ep\text{-}pair\ e\ p \Longrightarrow ep\text{-}pair\ (convex\text{-}map\cdot e)\ (convex\text{-}map\cdot p)$
⟨*proof*⟩

**lemma** *deflation-convex-map*: *deflation d* $\Longrightarrow$ *deflation* (*convex-map*·*d*)
⟨*proof*⟩

**lemma** *finite-deflation-convex-map*:
**assumes** *finite-deflation d* **shows** *finite-deflation* (*convex-map*·*d*)
⟨*proof*⟩

## 33.7 Convex powerdomain is bifinite

**lemma** *approx-chain-convex-map*:
**assumes** *approx-chain a*
**shows** *approx-chain* ($\lambda i$. *convex-map*·(*a i*))
⟨*proof*⟩

**instance** *convex-pd* :: (*bifinite*) *bifinite*
⟨*proof*⟩

## 33.8 Join

**definition**
*convex-join* :: *'a convex-pd convex-pd* → *'a convex-pd* **where**
*convex-join* = ($\Lambda$ *xss*. *convex-bind*·*xss*·($\Lambda$ *xs*. *xs*))

**lemma** *convex-join-unit* [*simp*]:
*convex-join*·{*xs*}♮ = *xs*
⟨*proof*⟩

**lemma** *convex-join-plus* [*simp*]:
*convex-join*·(*xss* ∪♮ *yss*) = *convex-join*·*xss* ∪♮ *convex-join*·*yss*
⟨*proof*⟩

**lemma** *convex-join-bottom* [*simp*]: *convex-join*·⊥ = ⊥
⟨*proof*⟩

**lemma** *convex-join-map-unit*:
*convex-join*·(*convex-map*·*convex-unit*·*xs*) = *xs*
⟨*proof*⟩

**lemma** *convex-join-map-join*:
*convex-join*·(*convex-map*·*convex-join*·*xsss*) = *convex-join*·(*convex-join*·*xsss*)
⟨*proof*⟩

**lemma** *convex-join-map-map*:
*convex-join*·(*convex-map*·(*convex-map*·*f*)·*xss*) =
*convex-map*·*f*·(*convex-join*·*xss*)
⟨*proof*⟩

## 33.9 Conversions to other powerdomains

Convex to upper

**lemma** *convex-le-imp-upper-le*: $t \leq\natural\ u \Longrightarrow t \leq\sharp\ u$
⟨*proof*⟩

**definition**
*convex-to-upper* :: *'a convex-pd* → *'a upper-pd* **where**
*convex-to-upper* = *convex-pd.extension upper-principal*

**lemma** *convex-to-upper-principal* [*simp*]:
*convex-to-upper*·(*convex-principal t*) = *upper-principal t*
⟨*proof*⟩

**lemma** *convex-to-upper-unit* [*simp*]:
*convex-to-upper*·$\{x\}\natural$ = $\{x\}\sharp$
⟨*proof*⟩

**lemma** *convex-to-upper-plus* [*simp*]:
*convex-to-upper*·(*xs* $\cup\natural$ *ys*) = *convex-to-upper*·*xs* $\cup\sharp$ *convex-to-upper*·*ys*
⟨*proof*⟩

**lemma** *convex-to-upper-bind* [*simp*]:
*convex-to-upper*·(*convex-bind*·*xs*·*f*) =
*upper-bind*·(*convex-to-upper*·*xs*)·(*convex-to-upper oo f*)
⟨*proof*⟩

**lemma** *convex-to-upper-map* [*simp*]:
*convex-to-upper*·(*convex-map*·*f*·*xs*) = *upper-map*·*f*·(*convex-to-upper*·*xs*)
⟨*proof*⟩

**lemma** *convex-to-upper-join* [*simp*]:
*convex-to-upper*·(*convex-join*·*xss*) =
*upper-bind*·(*convex-to-upper*·*xss*)·*convex-to-upper*
⟨*proof*⟩

Convex to lower

**lemma** *convex-le-imp-lower-le*: $t \leq\natural\ u \Longrightarrow t \leq\flat\ u$
⟨*proof*⟩

**definition**
*convex-to-lower* :: *'a convex-pd* → *'a lower-pd* **where**
*convex-to-lower* = *convex-pd.extension lower-principal*

**lemma** *convex-to-lower-principal* [*simp*]:
*convex-to-lower*·(*convex-principal t*) = *lower-principal t*
⟨*proof*⟩

**lemma** *convex-to-lower-unit* [*simp*]:

*convex-to-lower*·{*x*}♮ = {*x*}♭
⟨*proof*⟩

**lemma** *convex-to-lower-plus* [*simp*]:
*convex-to-lower*·(*xs* ∪♮ *ys*) = *convex-to-lower*·*xs* ∪♭ *convex-to-lower*·*ys*
⟨*proof*⟩

**lemma** *convex-to-lower-bind* [*simp*]:
*convex-to-lower*·(*convex-bind*·*xs*·*f*) =
*lower-bind*·(*convex-to-lower*·*xs*)·(*convex-to-lower oo f*)
⟨*proof*⟩

**lemma** *convex-to-lower-map* [*simp*]:
*convex-to-lower*·(*convex-map*·*f*·*xs*) = *lower-map*·*f*·(*convex-to-lower*·*xs*)
⟨*proof*⟩

**lemma** *convex-to-lower-join* [*simp*]:
*convex-to-lower*·(*convex-join*·*xss*) =
*lower-bind*·(*convex-to-lower*·*xss*)·*convex-to-lower*
⟨*proof*⟩

Ordering property

**lemma** *convex-pd-below-iff*:
(*xs* ⊑ *ys*) =
(*convex-to-upper*·*xs* ⊑ *convex-to-upper*·*ys* ∧
*convex-to-lower*·*xs* ⊑ *convex-to-lower*·*ys*)
⟨*proof*⟩

**lemmas** *convex-plus-below-plus-iff* =
*convex-pd-below-iff* [**where** *xs*=*xs* ∪♮ *ys* **and** *ys*=*zs* ∪♮ *ws*]
**for** *xs ys zs ws*

**lemmas** *convex-pd-below-simps* =
*convex-unit-below-plus-iff*
*convex-plus-below-unit-iff*
*convex-plus-below-plus-iff*
*convex-unit-below-iff*
*convex-to-upper-unit*
*convex-to-upper-plus*
*convex-to-lower-unit*
*convex-to-lower-plus*
*upper-pd-below-simps*
*lower-pd-below-simps*

**end**

# 34 Powerdomains

**theory** *Powerdomains*

**imports** *ConvexPD Domain*
**begin**

## 34.1 Universal domain embeddings

**definition** *upper-emb* = *udom-emb* (λ*i*. *upper-map*·(*udom-approx i*))
**definition** *upper-prj* = *udom-prj* (λ*i*. *upper-map*·(*udom-approx i*))

**definition** *lower-emb* = *udom-emb* (λ*i*. *lower-map*·(*udom-approx i*))
**definition** *lower-prj* = *udom-prj* (λ*i*. *lower-map*·(*udom-approx i*))

**definition** *convex-emb* = *udom-emb* (λ*i*. *convex-map*·(*udom-approx i*))
**definition** *convex-prj* = *udom-prj* (λ*i*. *convex-map*·(*udom-approx i*))

**lemma** *ep-pair-upper*: *ep-pair upper-emb upper-prj*
⟨*proof*⟩

**lemma** *ep-pair-lower*: *ep-pair lower-emb lower-prj*
⟨*proof*⟩

**lemma** *ep-pair-convex*: *ep-pair convex-emb convex-prj*
⟨*proof*⟩

## 34.2 Deflation combinators

**definition** *upper-defl* :: *udom defl* → *udom defl*
**where** *upper-defl* = *defl-fun1 upper-emb upper-prj upper-map*

**definition** *lower-defl* :: *udom defl* → *udom defl*
**where** *lower-defl* = *defl-fun1 lower-emb lower-prj lower-map*

**definition** *convex-defl* :: *udom defl* → *udom defl*
**where** *convex-defl* = *defl-fun1 convex-emb convex-prj convex-map*

**lemma** *cast-upper-defl*:
*cast*·(*upper-defl*·*A*) = *upper-emb oo upper-map*·(*cast*·*A*) *oo upper-prj*
⟨*proof*⟩

**lemma** *cast-lower-defl*:
*cast*·(*lower-defl*·*A*) = *lower-emb oo lower-map*·(*cast*·*A*) *oo lower-prj*
⟨*proof*⟩

**lemma** *cast-convex-defl*:
*cast*·(*convex-defl*·*A*) = *convex-emb oo convex-map*·(*cast*·*A*) *oo convex-prj*
⟨*proof*⟩

## 34.3 Domain class instances

**instantiation** *upper-pd* :: (*domain*) *domain*
**begin**

**definition**
*emb = upper-emb oo upper-map·emb*

**definition**
*prj = upper-map·prj oo upper-prj*

**definition**
*defl (t::'a upper-pd itself) = upper-defl·DEFL('a)*

**definition**
*(liftemb :: 'a upper-pd u → udom u) = u-map·emb*

**definition**
*(liftprj :: udom u → 'a upper-pd u) = u-map·prj*

**definition**
*liftdefl (t::'a upper-pd itself) = liftdefl-of·DEFL('a upper-pd)*

**instance** ⟨*proof*⟩

**end**

**instantiation** *lower-pd* :: (*domain*) *domain*
**begin**

**definition**
*emb = lower-emb oo lower-map·emb*

**definition**
*prj = lower-map·prj oo lower-prj*

**definition**
*defl (t::'a lower-pd itself) = lower-defl·DEFL('a)*

**definition**
*(liftemb :: 'a lower-pd u → udom u) = u-map·emb*

**definition**
*(liftprj :: udom u → 'a lower-pd u) = u-map·prj*

**definition**
*liftdefl (t::'a lower-pd itself) = liftdefl-of·DEFL('a lower-pd)*

**instance** ⟨*proof*⟩

**end**

**instantiation** *convex-pd* :: (*domain*) *domain*

**begin**

**definition**
*emb* = *convex-emb oo convex-map·emb*

**definition**
*prj* = *convex-map·prj oo convex-prj*

**definition**
*defl* (*t*::*'a convex-pd itself*) = *convex-defl·DEFL*(*'a*)

**definition**
(*liftemb* :: *'a convex-pd u* → *udom u*) = *u-map·emb*

**definition**
(*liftprj* :: *udom u* → *'a convex-pd u*) = *u-map·prj*

**definition**
*liftdefl* (*t*::*'a convex-pd itself*) = *liftdefl-of·DEFL*(*'a convex-pd*)

**instance** ⟨*proof*⟩

**end**

**lemma** *DEFL-upper*: *DEFL*(*'a*::*domain upper-pd*) = *upper-defl·DEFL*(*'a*)
⟨*proof*⟩

**lemma** *DEFL-lower*: *DEFL*(*'a*::*domain lower-pd*) = *lower-defl·DEFL*(*'a*)
⟨*proof*⟩

**lemma** *DEFL-convex*: *DEFL*(*'a*::*domain convex-pd*) = *convex-defl·DEFL*(*'a*)
⟨*proof*⟩

## 34.4 Isomorphic deflations

**lemma** *isodefl-upper*:
*isodefl d t* ⟹ *isodefl* (*upper-map·d*) (*upper-defl·t*)
⟨*proof*⟩

**lemma** *isodefl-lower*:
*isodefl d t* ⟹ *isodefl* (*lower-map·d*) (*lower-defl·t*)
⟨*proof*⟩

**lemma** *isodefl-convex*:
*isodefl d t* ⟹ *isodefl* (*convex-map·d*) (*convex-defl·t*)
⟨*proof*⟩

## 34.5 Domain package setup for powerdomains

**lemmas** [*domain-defl-simps*] = *DEFL-upper DEFL-lower DEFL-convex*

**lemmas** [*domain-map-ID*] = *upper-map-ID lower-map-ID convex-map-ID*
**lemmas** [*domain-isodefl*] = *isodefl-upper isodefl-lower isodefl-convex*

**lemmas** [*domain-deflation*] =
*deflation-upper-map deflation-lower-map deflation-convex-map*

⟨*ML*⟩

**end**

**theory** *HOLCF*
**imports**
*Main*
*Domain*
*Powerdomains*
**begin**

**default-sort** *domain*

**end**