Some results of number theory

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April 17, 2016

Abstract

This is a collection of formalized proofs of many results of number theory. The proofs of the Chinese Remainder Theorem and Wilson's Theorem are due to Rasmussen. The proof of Gauss's law of quadratic reciprocity is due to Avigad, Gray and Kramer. Proofs can be found in most introductory number theory textbooks; Goldman's *The Queen of Mathematics: a Historically Motivated Guide to Number Theory* provides some historical context.

Avigad, Gray and Kramer have also provided library theories dealing with finite sets and finite sums, divisibility and congruences, parity and residues. The authors are engaged in redesigning and polishing these theories for more serious use. For the latest information in this respect, please see the web page http://www.andrew.cmu.edu/~avigad/isabelle. Other theories contain proofs of Euler's criteria, Gauss' lemma, and the law of quadratic reciprocity. The formalization follows Eisenstein's proof, which is the one most commonly found in introductory textbooks; in particular, it follows the presentation in Niven and Zuckerman, *The Theory of Numbers*.

To avoid having to count roots of polynomials, however, we relied on a trick previously used by David Russinoff in formalizing quadratic reciprocity for the Boyer-Moore theorem prover; see Russinoff, David, "A mechanical proof of quadratic reciprocity," *Journal of Automated Reasoning* 8:3-21, 1992. We are grateful to Larry Paulson for calling our attention to this reference.

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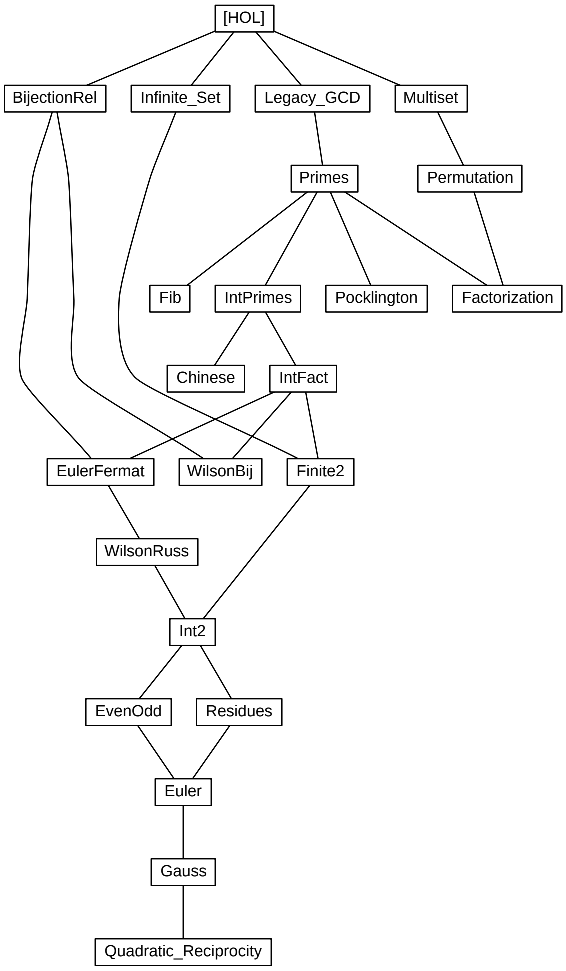
[HOL]
BijectionRel
Infinite_Set
Legacy_GCD
Multiset
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1 The Greatest Common Divisor

theory *Legacy-GCD*
imports *Main*
begin

See [1].

1.1 Specification of GCD on nats

definition
  is-gcd :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *bool* **where** — *gcd* as a relation
  is-gcd m n p ⟷ *p dvd m* ∧ *p dvd n* ∧
    (∀ *d*. *d dvd m* ⟶ *d dvd n* ⟶ *d dvd p*)

Uniqueness

lemma *is-gcd-unique*: *is-gcd a b m* ⟹ *is-gcd a b n* ⟹ *m* = *n*
  by (*simp add*: *is-gcd-def*) (*blast intro*: *dvd-antisym*)

Connection to divides relation

lemma *is-gcd-dvd*: *is-gcd a b m* ⟹ *k dvd a* ⟹ *k dvd b* ⟹ *k dvd m*
  by (*auto simp add*: *is-gcd-def*)

Commutativity

lemma *is-gcd-commute*: *is-gcd m n k* = *is-gcd n m k*
  by (*auto simp add*: *is-gcd-def*)

1.2 GCD on nat by Euclid's algorithm

fun *gcd* :: *nat* => *nat* => *nat*
  where *gcd m n* = (*if n* = *0 then m else gcd n* (*m mod n*))

lemma *gcd-induct* [*case-names 0 rec*]:
  fixes *m n* :: *nat*
  assumes ⋀*m*. *P m 0*
    and ⋀*m n*. *0* < *n* ⟹ *P n* (*m mod n*) ⟹ *P m n*
  shows *P m n*
proof (*induct m n rule*: *gcd.induct*)
  case (*1 m n*)
  with *assms* **show** *?case* **by** (*cases n* = *0*) *simp-all*
qed

lemma *gcd-0* [*simp*, *algebra*]: *gcd m 0* = *m*
  by *simp*

lemma *gcd-0-left* [*simp*,*algebra*]: *gcd 0 m* = *m*
  by *simp*

lemma *gcd-non-0*: $n > 0 \Longrightarrow gcd\ m\ n = gcd\ n\ (m\ mod\ n)$
  by *simp*

lemma *gcd-1* [*simp*, *algebra*]: $gcd\ m\ (Suc\ 0) = Suc\ 0$
  by *simp*

lemma *nat-gcd-1-right* [*simp*, *algebra*]: $gcd\ m\ 1 = 1$
  unfolding *One-nat-def* **by** (*rule gcd-1*)

declare *gcd.simps* [*simp del*]

gcd m n divides *m* and *n*. The conjunctions don't seem provable separately.

lemma *gcd-dvd1* [*iff*, *algebra*]: *gcd m n dvd m*
  and *gcd-dvd2* [*iff*, *algebra*]: *gcd m n dvd n*
  apply (*induct m n rule*: *gcd-induct*)
    apply (*simp-all add*: *gcd-non-0*)
  apply (*blast dest*: *dvd-mod-imp-dvd*)
  done

Maximality: for all *m*, *n*, *k* naturals, if *k* divides *m* and *k* divides *n* then *k* divides *gcd m n*.

lemma *gcd-greatest*: $k\ dvd\ m \Longrightarrow k\ dvd\ n \Longrightarrow k\ dvd\ gcd\ m\ n$
  by (*induct m n rule*: *gcd-induct*) (*simp-all add*: *gcd-non-0 dvd-mod*)

Function gcd yields the Greatest Common Divisor.

lemma *is-gcd*: *is-gcd m n* (*gcd m n*)
  by (*simp add*: *is-gcd-def gcd-greatest*)

1.3 Derived laws for GCD

lemma *gcd-greatest-iff* [*iff*, *algebra*]: $k\ dvd\ gcd\ m\ n \longleftrightarrow k\ dvd\ m \wedge k\ dvd\ n$
  by (*blast intro*!: *gcd-greatest intro*: *dvd-trans*)

lemma *gcd-zero*[*algebra*]: $gcd\ m\ n = 0 \longleftrightarrow m = 0 \wedge n = 0$
  by (*simp only*: *dvd-0-left-iff* [*symmetric*] *gcd-greatest-iff*)

lemma *gcd-commute*: $gcd\ m\ n = gcd\ n\ m$
  apply (*rule is-gcd-unique*)
   apply (*rule is-gcd*)
  apply (*subst is-gcd-commute*)
  apply (*simp add*: *is-gcd*)
  done

lemma *gcd-assoc*: $gcd\ (gcd\ k\ m)\ n = gcd\ k\ (gcd\ m\ n)$
  apply (*rule is-gcd-unique*)
   apply (*rule is-gcd*)
  apply (*simp add*: *is-gcd-def*)

apply (*blast intro*: *dvd-trans*)
done

lemma *gcd-1-left* [*simp*, *algebra*]: *gcd* (*Suc 0*) *m* = *Suc 0*
by (*simp add*: *gcd-commute*)

lemma *nat-gcd-1-left* [*simp*, *algebra*]: *gcd 1 m* = *1*
unfolding *One-nat-def* **by** (*rule gcd-1-left*)

Multiplication laws

lemma *gcd-mult-distrib2*: *k* ∗ *gcd m n* = *gcd* (*k* ∗ *m*) (*k* ∗ *n*)
— [1, page 27]
apply (*induct m n rule*: *gcd-induct*)
apply *simp*
apply (*case-tac k* = *0*)
apply (*simp-all add*: *gcd-non-0*)
done

lemma *gcd-mult* [*simp*, *algebra*]: *gcd k* (*k* ∗ *n*) = *k*
apply (*rule gcd-mult-distrib2* [*of k 1 n*, *simplified*, *symmetric*])
done

lemma *gcd-self* [*simp*, *algebra*]: *gcd k k* = *k*
apply (*rule gcd-mult* [*of k 1*, *simplified*])
done

lemma *relprime-dvd-mult*: *gcd k n* = *1* ==> *k dvd m* ∗ *n* ==> *k dvd m*
apply (*insert gcd-mult-distrib2* [*of m k n*])
apply *simp*
apply (*erule-tac t* = *m* **in** *ssubst*)
apply *simp*
done

lemma *relprime-dvd-mult-iff*: *gcd k n* = *1* ==> (*k dvd m* ∗ *n*) = (*k dvd m*)
by (*auto intro*: *relprime-dvd-mult dvd-mult2*)

lemma *gcd-mult-cancel*: *gcd k n* = *1* ==> *gcd* (*k* ∗ *m*) *n* = *gcd m n*
apply (*rule dvd-antisym*)
apply (*rule gcd-greatest*)
apply (*rule-tac n* = *k* **in** *relprime-dvd-mult*)
apply (*simp add*: *gcd-assoc*)
apply (*simp add*: *gcd-commute*)
apply (*simp-all add*: *mult.commute*)
done

Addition laws

lemma *gcd-add1* [*simp*, *algebra*]: *gcd* (*m* + *n*) *n* = *gcd m n*
by (*cases n* = *0*) (*auto simp add*: *gcd-non-0*)

lemma *gcd-add2* [*simp*, *algebra*]: *gcd m* (*m* + *n*) = *gcd m n*
proof −
  have *gcd m* (*m* + *n*) = *gcd* (*m* + *n*) *m* **by** (*rule gcd-commute*)
  also have ... = *gcd* (*n* + *m*) *m* **by** (*simp add*: *add.commute*)
  also have ... = *gcd n m* **by** *simp*
  also have ... = *gcd m n* **by** (*rule gcd-commute*)
  finally show *?thesis* **.**
qed

lemma *gcd-add2'* [*simp*, *algebra*]: *gcd m* (*n* + *m*) = *gcd m n*
  apply (*subst add.commute*)
  apply (*rule gcd-add2*)
  done

lemma *gcd-add-mult*[*algebra*]: *gcd m* (*k* ∗ *m* + *n*) = *gcd m n*
  by (*induct k*) (*simp-all add*: *add.assoc*)

lemma *gcd-dvd-prod*: *gcd m n dvd m* ∗ *n*
  using *mult-dvd-mono* [*of 1*] **by** *auto*

Division by gcd yields rrelatively primes.

lemma *div-gcd-relprime*:
  assumes *nz*: *a* ≠ *0* ∨ *b* ≠ *0*
  shows *gcd* (*a div gcd a b*) (*b div gcd a b*) = *1*
proof −
  let *?g* = *gcd a b*
  let *?a'* = *a div ?g*
  let *?b'* = *b div ?g*
  let *?g'* = *gcd ?a' ?b'*
  have *dvdg*: *?g dvd a ?g dvd b* **by** *simp-all*
  have *dvdg'*: *?g' dvd ?a' ?g' dvd ?b'* **by** *simp-all*
  from *dvdg dvdg'* **obtain** *ka kb ka' kb'* **where**
    kab: *a* = *?g* ∗ *ka b* = *?g* ∗ *kb ?a'* = *?g'* ∗ *ka' ?b'* = *?g'* ∗ *kb'*
   unfolding *dvd-def* **by** *blast*
  from *this*(*3*−*4*) [*symmetric*] **have** *?g* ∗ *?a'* = (*?g* ∗ *?g'*) ∗ *ka' ?g* ∗ *?b'* = (*?g* ∗ *?g'*) ∗ *kb'*
   by (*simp-all only*: *ac-simps mult.left-commute* [*of - gcd a b*])
  then have *dvdgg'*:*?g* ∗ *?g' dvd a ?g*∗ *?g' dvd b*
   by (*auto simp add*: *dvd-mult-div-cancel* [*OF dvdg*(*1*)]
    dvd-mult-div-cancel [*OF dvdg*(*2*)] *dvd-def*)
  have *?g* ≠ *0* **using** *nz* **by** (*simp add*: *gcd-zero*)
  then have *gp*: *?g* > *0* **by** *simp*
  from *gcd-greatest* [*OF dvdgg'*] **have** *?g* ∗ *?g' dvd ?g* **.**
  with *dvd-mult-cancel1* [*OF gp*] **show** *?g'* = *1* **by** *simp*
qed

lemma *gcd-unique*: *d dvd a*∧*d dvd b* ∧ (∀ *e*. *e dvd a* ∧ *e dvd b* ⟶ *e dvd d*) ⟷

d = *gcd a b*
proof(*auto*)
  assume *H*: *d dvd a d dvd b* ∀ *e*. *e dvd a* ∧ *e dvd b* ⟶ *e dvd d*
  from *H*(*3*)[*rule-format*] *gcd-dvd1*[*of a b*] *gcd-dvd2*[*of a b*]
  have *th*: *gcd a b dvd d* **by** *blast*
  from *dvd-antisym*[*OF th gcd-greatest*[*OF H*(*1*,*2*)]] **show** *d* = *gcd a b* **by** *blast*
qed

lemma *gcd-eq*: **assumes** *H*: ∀ *d*. *d dvd x* ∧ *d dvd y* ⟷ *d dvd u* ∧ *d dvd v*
  shows *gcd x y* = *gcd u v*
proof−
  from *H* **have** ∀ *d*. *d dvd x* ∧ *d dvd y* ⟷ *d dvd gcd u v* **by** *simp*
  with *gcd-unique*[*of gcd u v x y*] **show** *?thesis* **by** *auto*
qed

lemma *ind-euclid*:
  assumes *c*: ∀ *a b*. *P* (*a*::*nat*) *b* ⟷ *P b a* **and** *z*: ∀ *a*. *P a 0*
  and *add*: ∀ *a b*. *P a b* ⟶ *P a* (*a* + *b*)
  shows *P a b*
proof(*induct a* + *b arbitrary*: *a b rule*: *less-induct*)
  case *less*
  have *a* = *b* ∨ *a* < *b* ∨ *b* < *a* **by** *arith*
  moreover {**assume** *eq*: *a*= *b*
    from *add*[*rule-format*, *OF z*[*rule-format*, *of a*]] **have** *P a b* **using** *eq*
    by *simp*}
  moreover
  {**assume** *lt*: *a* < *b*
    hence *a* + *b* − *a* < *a* + *b* ∨ *a* = *0* **by** *arith*
    moreover
    {**assume** *a* =*0* **with** *z c* **have** *P a b* **by** *blast* }
    moreover
    {**assume** *a* + *b* − *a* < *a* + *b*
      also have *th0*: *a* + *b* − *a* = *a* + (*b* − *a*) **using** *lt* **by** *arith*
      finally have *a* + (*b* − *a*) < *a* + *b* **.**
      then have *P a* (*a* + (*b* − *a*)) **by** (*rule add*[*rule-format*, *OF less*])
      then have *P a b* **by** (*simp add*: *th0*[*symmetric*])}
    ultimately have *P a b* **by** *blast*}
  moreover
  {**assume** *lt*: *a* > *b*
    hence *b* + *a* − *b* < *a* + *b* ∨ *b* = *0* **by** *arith*
    moreover
    {**assume** *b* =*0* **with** *z c* **have** *P a b* **by** *blast* }
    moreover
    {**assume** *b* + *a* − *b* < *a* + *b*
      also have *th0*: *b* + *a* − *b* = *b* + (*a* − *b*) **using** *lt* **by** *arith*
      finally have *b* + (*a* − *b*) < *a* + *b* **.**
      then have *P b* (*b* + (*a* − *b*)) **by** (*rule add*[*rule-format*, *OF less*])
      then have *P b a* **by** (*simp add*: *th0*[*symmetric*])
      hence *P a b* **using** *c* **by** *blast* }

ultimately have $P\ a\ b$ **by** *blast*}
ultimately show $P\ a\ b$ **by** *blast*
qed

lemma *bezout-lemma*:
assumes *ex*: $\exists (d::nat)\ x\ y.\ d\ dvd\ a \wedge d\ dvd\ b \wedge (a * x = b * y + d \vee b * x = a * y + d)$
shows $\exists\, d\ x\ y.\ d\ dvd\ a \wedge d\ dvd\ a + b \wedge (a * x = (a + b) * y + d \vee (a + b) * x = a * y + d)$
using *ex*
apply *clarsimp*
apply (*rule-tac* $x{=}d$ **in** *exI*, *simp*)
apply (*case-tac* $a * x = b * y + d$, *simp-all*)
apply (*rule-tac* $x{=}x + y$ **in** *exI*)
apply (*rule-tac* $x{=}y$ **in** *exI*)
apply *algebra*
apply (*rule-tac* $x{=}x$ **in** *exI*)
apply (*rule-tac* $x{=}x + y$ **in** *exI*)
apply *algebra*
done

lemma *bezout-add*: $\exists (d::nat)\ x\ y.\ d\ dvd\ a \wedge d\ dvd\ b \wedge (a * x = b * y + d \vee b * x = a * y + d)$
apply(*induct a b rule*: *ind-euclid*)
apply *blast*
apply *clarify*
apply (*rule-tac* $x{=}a$ **in** *exI*, *simp*)
apply *clarsimp*
apply (*rule-tac* $x{=}d$ **in** *exI*)
apply (*case-tac* $a * x = b * y + d$, *simp-all*)
apply (*rule-tac* $x{=}x{+}y$ **in** *exI*)
apply (*rule-tac* $x{=}y$ **in** *exI*)
apply *algebra*
apply (*rule-tac* $x{=}x$ **in** *exI*)
apply (*rule-tac* $x{=}x{+}y$ **in** *exI*)
apply *algebra*
done

lemma *bezout*: $\exists (d::nat)\ x\ y.\ d\ dvd\ a \wedge d\ dvd\ b \wedge (a * x - b * y = d \vee b * x - a * y = d)$
using *bezout-add*[*of a b*]
apply *clarsimp*
apply (*rule-tac* $x{=}d$ **in** *exI*, *simp*)
apply (*rule-tac* $x{=}x$ **in** *exI*)
apply (*rule-tac* $x{=}y$ **in** *exI*)
apply *auto*
done

We can get a stronger version with a nonzeroness assumption.

```
lemma divides-le: m dvd n ==> m <= n ∨ n = (0::nat) by (auto simp add:
dvd-def)

lemma bezout-add-strong: assumes nz: a ≠ (0::nat)
  shows ∃ d x y. d dvd a ∧ d dvd b ∧ a * x = b * y + d
proof−
  from nz have ap: a > 0 by simp
 from bezout-add[of a b]
 have (∃ d x y. d dvd a ∧ d dvd b ∧ a * x = b * y + d) ∨ (∃ d x y. d dvd a ∧ d
dvd b ∧ b * x = a * y + d) by blast
 moreover
 {fix d x y assume H: d dvd a d dvd b a * x = b * y + d
   from H have ?thesis by blast }
 moreover
 {fix d x y assume H: d dvd a d dvd b b * x = a * y + d
   {assume b0: b = 0 with H  have ?thesis by simp}
   moreover
   {assume b: b ≠ 0 hence bp: b > 0 by simp
     from divides-le[OF H(2)] b have d < b ∨ d = b using le-less by blast
     moreover
     {assume db: d=b
       from nz H db have ?thesis apply simp
         apply (rule exI[where x = b], simp)
         apply (rule exI[where x = b])
        by (rule exI[where x = a − 1], simp add: diff-mult-distrib2)}
    moreover
    {assume db: d < b
        {assume x=0 hence ?thesis using nz H by simp }
        moreover
        {assume x0: x ≠ 0 hence xp: x > 0 by simp

          from db have d ≤ b − 1 by simp
          hence d*b ≤ b*(b − 1) by simp
          with xp mult-mono[of 1 x d*b b*(b − 1)]
          have dble: d*b ≤ x*b*(b − 1) using bp by simp
          from H (3) have a * ((b − 1) * y) + d * (b − 1 + 1) = d + x*b*(b −
1) by algebra
          hence a * ((b − 1) * y) = d + x*b*(b − 1) − d*b using bp by simp
          hence a * ((b − 1) * y) = d + (x*b*(b − 1) − d*b)
            by (simp only: diff-add-assoc[OF dble, of d, symmetric])
          hence a * ((b − 1) * y) = b*(x*(b − 1) − d) + d
            by (simp only: diff-mult-distrib2 ac-simps)
          hence ?thesis using H(1,2)
            apply −
            apply (rule exI[where x=d], simp)
            apply (rule exI[where x=(b − 1) * y])
            by (rule exI[where x=x*(b − 1) − d], simp)}
        ultimately have ?thesis by blast}
    ultimately have ?thesis by blast}
```

ultimately have *?thesis* **by** *blast*}
ultimately show *?thesis* **by** *blast*
qed

lemma *bezout-gcd*: $\exists x\ y.\ a * x - b * y = gcd\ a\ b \lor b * x - a * y = gcd\ a\ b$
proof−
let $?g = gcd\ a\ b$
from *bezout*[*of a b*] **obtain** $d\ x\ y$ **where** d: $d\ dvd\ a\ d\ dvd\ b\ a * x - b * y = d \lor b * x - a * y = d$ **by** *blast*
from $d(1,2)$ **have** $d\ dvd\ ?g$ **by** *simp*
then obtain k **where** k: $?g = d{*}k$ **unfolding** *dvd-def* **by** *blast*
from $d(3)$ **have** $(a * x - b * y){*}k = d{*}k \lor (b * x - a * y){*}k = d{*}k$ **by** *blast*
hence $a * x * k - b * y{*}k = d{*}k \lor b * x * k - a * y{*}k = d{*}k$
by (*algebra add*: *diff-mult-distrib*)
hence $a * (x * k) - b * (y{*}k) = ?g \lor b * (x * k) - a * (y{*}k) = ?g$
by (*simp add*: *k mult.assoc*)
thus *?thesis* **by** *blast*
qed

lemma *bezout-gcd-strong*: **assumes** a: $a \neq 0$
shows $\exists x\ y.\ a * x = b * y + gcd\ a\ b$
proof−
let $?g = gcd\ a\ b$
from *bezout-add-strong*[*OF a*, *of b*]
obtain $d\ x\ y$ **where** d: $d\ dvd\ a\ d\ dvd\ b\ a * x = b * y + d$ **by** *blast*
from $d(1,2)$ **have** $d\ dvd\ ?g$ **by** *simp*
then obtain k **where** k: $?g = d{*}k$ **unfolding** *dvd-def* **by** *blast*
from $d(3)$ **have** $a * x * k = (b * y + d) {*}k$ **by** *algebra*
hence $a * (x * k) = b * (y{*}k) + ?g$ **by** (*algebra add*: *k*)
thus *?thesis* **by** *blast*
qed

lemma *gcd-mult-distrib*: $gcd(a * c)\ (b * c) = c * gcd\ a\ b$
by(*simp add*: *gcd-mult-distrib2 mult.commute*)

lemma *gcd-bezout*: $(\exists x\ y.\ a * x - b * y = d \lor b * x - a * y = d) \longleftrightarrow gcd\ a\ b\ dvd\ d$
(**is** $?lhs \longleftrightarrow ?rhs$)
proof−
let $?g = gcd\ a\ b$
{**assume** H: $?rhs$ **then obtain** k **where** k: $d = ?g{*}k$ **unfolding** *dvd-def* **by** *blast*
from *bezout-gcd*[*of a b*] **obtain** $x\ y$ **where** xy: $a * x - b * y = ?g \lor b * x - a * y = ?g$
by *blast*
hence $(a * x - b * y){*}k = ?g{*}k \lor (b * x - a * y){*}k = ?g{*}k$ **by** *auto*
hence $a * x{*}k - b * y{*}k = ?g{*}k \lor b * x * k - a * y{*}k = ?g{*}k$
by (*simp only*: *diff-mult-distrib*)

hence *a* ∗ (*x*∗*k*) − *b* ∗ (*y*∗*k*) = *d* ∨ *b* ∗ (*x* ∗ *k*) − *a* ∗ (*y*∗*k*) = *d*
by (*simp add*: *k*[*symmetric*] *mult.assoc*)
hence *?lhs* **by** *blast*}
moreover
{**fix** *x y* **assume** *H*: *a* ∗ *x* − *b* ∗ *y* = *d* ∨ *b* ∗ *x* − *a* ∗ *y* = *d*
have *dv*: *?g dvd a*∗*x ?g dvd b* ∗ *y ?g dvd b*∗*x ?g dvd a* ∗ *y*
using *dvd-mult2*[*OF gcd-dvd1*[*of a b*]] *dvd-mult2*[*OF gcd-dvd2*[*of a b*]] **by** *simp-all*
from *dvd-diff-nat*[*OF dv*(*1*,*2*)] *dvd-diff-nat*[*OF dv*(*3*,*4*)] *H*
have *?rhs* **by** *auto*}
ultimately show *?thesis* **by** *blast*
qed

lemma *gcd-bezout-sum*: **assumes** *H*:*a* ∗ *x* + *b* ∗ *y* = *d* **shows** *gcd a b dvd d*
proof−
let *?g* = *gcd a b*
have *dv*: *?g dvd a*∗*x ?g dvd b* ∗ *y*
using *dvd-mult2*[*OF gcd-dvd1*[*of a b*]] *dvd-mult2*[*OF gcd-dvd2*[*of a b*]] **by** *simp-all*
from *dvd-add*[*OF dv*] *H*
show *?thesis* **by** *auto*
qed

lemma *gcd-mult′*: *gcd b* (*a* ∗ *b*) = *b*
by (*simp add*: *mult.commute*[*of a b*])

lemma *gcd-add*: *gcd*(*a* + *b*) *b* = *gcd a b*
gcd(*b* + *a*) *b* = *gcd a b gcd a* (*a* + *b*) = *gcd a b gcd a* (*b* + *a*) = *gcd a b*
by (*simp-all add*: *gcd-commute*)

lemma *gcd-sub*: *b* <= *a* ==> *gcd*(*a* − *b*) *b* = *gcd a b a* <= *b* ==> *gcd a* (*b* − *a*) = *gcd a b*
proof−
{**fix** *a b* **assume** *H*: *b* ≤ (*a*::*nat*)
hence *th*: *a* − *b* + *b* = *a* **by** *arith*
from *gcd-add*(*1*)[*of a* − *b b*] *th* **have** *gcd*(*a* − *b*) *b* = *gcd a b* **by** *simp*}
note *th* = *this*
{
assume *ab*: *b* ≤ *a*
from *th*[*OF ab*] **show** *gcd* (*a* − *b*) *b* = *gcd a b* **by** *blast*
next
assume *ab*: *a* ≤ *b*
from *th*[*OF ab*] **show** *gcd a* (*b* − *a*) = *gcd a b*
by (*simp add*: *gcd-commute*)}
qed

1.4 LCM defined by GCD

definition

```
  lcm :: nat ⇒ nat ⇒ nat
where
  lcm-def: lcm m n = m * n div gcd m n

lemma prod-gcd-lcm:
  m * n = gcd m n * lcm m n
  unfolding lcm-def by (simp add: dvd-mult-div-cancel [OF gcd-dvd-prod])

lemma lcm-0 [simp]: lcm m 0 = 0
  unfolding lcm-def by simp

lemma lcm-1 [simp]: lcm m 1 = m
  unfolding lcm-def by simp

lemma lcm-0-left [simp]: lcm 0 n = 0
  unfolding lcm-def by simp

lemma lcm-1-left [simp]: lcm 1 m = m
  unfolding lcm-def by simp

lemma dvd-pos:
  fixes n m :: nat
  assumes n > 0 and m dvd n
  shows m > 0
using assms by (cases m) auto

lemma lcm-least:
  assumes m dvd k and n dvd k
  shows lcm m n dvd k
proof (cases k)
  case 0 then show ?thesis by auto
next
  case (Suc -) then have pos-k: k > 0 by auto
  from assms dvd-pos [OF this] have pos-mn: m > 0 n > 0 by auto
  with gcd-zero [of m n] have pos-gcd: gcd m n > 0 by simp
  from assms obtain p where k-m: k = m * p using dvd-def by blast
  from assms obtain q where k-n: k = n * q using dvd-def by blast
  from pos-k k-m have pos-p: p > 0 by auto
  from pos-k k-n have pos-q: q > 0 by auto
  have k * k * gcd q p = k * gcd (k * q) (k * p)
    by (simp add: ac-simps gcd-mult-distrib2)
  also have ... = k * gcd (m * p * q) (n * q * p)
    by (simp add: k-m [symmetric] k-n [symmetric])
  also have ... = k * p * q * gcd m n
    by (simp add: ac-simps gcd-mult-distrib2)
  finally have (m * p) * (n * q) * gcd q p = k * p * q * gcd m n
    by (simp only: k-m [symmetric] k-n [symmetric])
  then have p * q * m * n * gcd q p = p * q * k * gcd m n
    by (simp add: ac-simps)
```

```
  with pos-p pos-q have m * n * gcd q p = k * gcd m n
    by simp
  with prod-gcd-lcm [of m n]
  have lcm m n * gcd q p * gcd m n = k * gcd m n
    by (simp add: ac-simps)
  with pos-gcd have lcm m n * gcd q p = k by simp
  then show ?thesis using dvd-def by auto
qed

lemma lcm-dvd1 [iff]:
  m dvd lcm m n
proof (cases m)
  case 0 then show ?thesis by simp
next
  case (Suc -)
  then have mpos: m > 0 by simp
  show ?thesis
  proof (cases n)
    case 0 then show ?thesis by simp
  next
    case (Suc -)
    then have npos: n > 0 by simp
    have gcd m n dvd n by simp
    then obtain k where n = gcd m n * k using dvd-def by auto
    then have m * n div gcd m n = m * (gcd m n * k) div gcd m n by (simp add:
ac-simps)
    also have ... = m * k using mpos npos gcd-zero by simp
    finally show ?thesis by (simp add: lcm-def)
  qed
qed

lemma lcm-dvd2 [iff]:
  n dvd lcm m n
proof (cases n)
  case 0 then show ?thesis by simp
next
  case (Suc -)
  then have npos: n > 0 by simp
  show ?thesis
  proof (cases m)
    case 0 then show ?thesis by simp
  next
    case (Suc -)
    then have mpos: m > 0 by simp
    have gcd m n dvd m by simp
    then obtain k where m = gcd m n * k using dvd-def by auto
    then have m * n div gcd m n = (gcd m n * k) * n div gcd m n by (simp add:
ac-simps)
    also have ... = n * k using mpos npos gcd-zero by simp
```

finally show *?thesis* **by** (*simp add*: *lcm-def*)
qed
qed

lemma *gcd-add1-eq*: *gcd* (*m* + *k*) *k* = *gcd* (*m* + *k*) *m*
by (*simp add*: *gcd-commute*)

lemma *gcd-diff2*: *m* ≤ *n* ==> *gcd n* (*n* − *m*) = *gcd n m*
apply (*subgoal-tac n* = *m* + (*n* − *m*))
apply (*erule ssubst*, *rule gcd-add1-eq*, *simp*)
done

1.5 GCD and LCM on integers

definition
zgcd :: *int* ⇒ *int* ⇒ *int* **where**
zgcd i j = *int* (*gcd* (*nat* |*i*|) (*nat* |*j*|))

lemma *zgcd-zdvd1* [*iff*, *algebra*]: *zgcd i j dvd i*
by (*simp add*: *zgcd-def int-dvd-iff*)

lemma *zgcd-zdvd2* [*iff*, *algebra*]: *zgcd i j dvd j*
by (*simp add*: *zgcd-def int-dvd-iff*)

lemma *zgcd-pos*: *zgcd i j* ≥ *0*
by (*simp add*: *zgcd-def*)

lemma *zgcd0* [*simp*,*algebra*]: (*zgcd i j* = *0*) = (*i* = *0* ∧ *j* = *0*)
by (*simp add*: *zgcd-def gcd-zero*)

lemma *zgcd-commute*: *zgcd i j* = *zgcd j i*
unfolding *zgcd-def* **by** (*simp add*: *gcd-commute*)

lemma *zgcd-zminus* [*simp*, *algebra*]: *zgcd* (− *i*) *j* = *zgcd i j*
unfolding *zgcd-def* **by** *simp*

lemma *zgcd-zminus2* [*simp*, *algebra*]: *zgcd i* (− *j*) = *zgcd i j*
unfolding *zgcd-def* **by** *simp*

lemma *zrelprime-dvd-mult*: *zgcd i j* = *1* ⟹ *i dvd k* ∗ *j* ⟹ *i dvd k*
unfolding *zgcd-def*
proof −
assume *int* (*gcd* (*nat* |*i*|) (*nat* |*j*|)) = *1 i dvd k* ∗ *j*
then have *g*: *gcd* (*nat* |*i*|) (*nat* |*j*|) = *1* **by** *simp*
from ‹*i dvd k* ∗ *j*› **obtain** *h* **where** *h*: *k*∗*j* = *i*∗*h* **unfolding** *dvd-def* **by** *blast*
have *th*: *nat* |*i*| *dvd nat* |*k*| ∗ *nat* |*j*|
unfolding *dvd-def*
by (*rule-tac x*= *nat* |*h*| **in** *exI*, *simp add*: *h nat-abs-mult-distrib* [*symmetric*])

from *relprime-dvd-mult* [*OF g th*] **obtain** *h'* **where** *h'*: *nat* |*k*| = *nat* |*i*| ∗ *h'*
  unfolding *dvd-def* **by** *blast*
from *h'* **have** *int* (*nat* |*k*|) = *int* (*nat* |*i*| ∗ *h'*) **by** *simp*
then have |*k*| = |*i*| ∗ *int h'* **by** (*simp add*: *of-nat-mult*)
then show *?thesis*
  apply (*subst abs-dvd-iff* [*symmetric*])
  apply (*subst dvd-abs-iff* [*symmetric*])
  apply (*unfold dvd-def*)
  apply (*rule-tac x* = *int h'* **in** *exI*, *simp*)
  done
qed

lemma *int-nat-abs*: *int* (*nat* |*x*|) = |*x*| **by** *arith*

lemma *zgcd-greatest*:
  assumes *k dvd m* **and** *k dvd n*
  shows *k dvd zgcd m n*
proof −
  let *?k'* = *nat* |*k*|
  let *?m'* = *nat* |*m*|
  let *?n'* = *nat* |*n*|
  from ‹*k dvd m*› **and** ‹*k dvd n*› **have** *dvd'*: *?k' dvd ?m' ?k' dvd ?n'*
    unfolding *zdvd-int* **by** (*simp-all only*: *int-nat-abs abs-dvd-iff dvd-abs-iff*)
  from *gcd-greatest* [*OF dvd'*] **have** *int* (*nat* |*k*|) *dvd zgcd m n*
    unfolding *zgcd-def* **by** (*simp only*: *zdvd-int*)
  then have |*k*| *dvd zgcd m n* **by** (*simp only*: *int-nat-abs*)
  then show *k dvd zgcd m n* **by** *simp*
qed

lemma *div-zgcd-relprime*:
  assumes *nz*: *a* ≠ *0* ∨ *b* ≠ *0*
  shows *zgcd* (*a div* (*zgcd a b*)) (*b div* (*zgcd a b*)) = *1*
proof −
  from *nz* **have** *nz'*: *nat* |*a*| ≠ *0* ∨ *nat* |*b*| ≠ *0* **by** *arith*
  let *?g* = *zgcd a b*
  let *?a'* = *a div ?g*
  let *?b'* = *b div ?g*
  let *?g'* = *zgcd ?a' ?b'*
  have *dvdg*: *?g dvd a ?g dvd b* **by** *simp-all*
  have *dvdg'*: *?g' dvd ?a' ?g' dvd ?b'* **by** *simp-all*
  from *dvdg dvdg'* **obtain** *ka kb ka' kb'* **where**
   kab: *a* = *?g*∗*ka b* = *?g*∗*kb ?a'* = *?g'*∗*ka' ?b'* = *?g'* ∗ *kb'*
    unfolding *dvd-def* **by** *blast*
  from *this*(*3*−*4*) [*symmetric*] **have** *?g*∗ *?a'* = (*?g* ∗ *?g'*) ∗ *ka' ?g*∗ *?b'* = (*?g* ∗ *?g'*) ∗ *kb'*
    by (*simp-all only*: *ac-simps mult.left-commute* [*of - zgcd a b*])
  then have *dvdgg'*:*?g* ∗ *?g' dvd a ?g*∗ *?g' dvd b*
    by (*auto simp add*: *dvd-mult-div-cancel* [*OF dvdg*(*1*)]
      dvd-mult-div-cancel [*OF dvdg*(*2*)] *dvd-def*)

have *?g* ≠ *0* **using** *nz* **by** *simp*
then have *gp*: *?g* ≠ *0* **using** *zgcd-pos*[**where** *i*=*a* **and** *j*=*b*] **by** *arith*
from *zgcd-greatest* [*OF dvdgg'*] **have** *?g* ∗ *?g'* *dvd* *?g* **.**
with *zdvd-mult-cancel1* [*OF gp*] **have** |*?g'*| = *1* **by** *simp*
with *zgcd-pos* **show** *?g'* = *1* **by** *simp*
qed

lemma *zgcd-0* [*simp*, *algebra*]: *zgcd m 0* = |*m*|
by (*simp add*: *zgcd-def abs-if*)

lemma *zgcd-0-left* [*simp*, *algebra*]: *zgcd 0 m* = |*m*|
by (*simp add*: *zgcd-def abs-if*)

lemma *zgcd-non-0*: *0* < *n* ==> *zgcd m n* = *zgcd n* (*m mod n*)
apply (*frule-tac b* = *n* **and** *a* = *m* **in** *pos-mod-sign*)
apply (*simp del*: *pos-mod-sign add*: *zgcd-def abs-if nat-mod-distrib*)
apply (*auto simp add*: *gcd-non-0 nat-mod-distrib* [*symmetric*] *zmod-zminus1-eq-if*)
apply (*frule-tac a* = *m* **in** *pos-mod-bound*)
apply (*simp del*: *pos-mod-bound add*: *algebra-simps nat-diff-distrib gcd-diff2 nat-le-eq-zle*)
apply (*metis dual-order.strict-implies-order gcd.simps gcd-0-left gcd-diff2 mod-by-0 nat-mono*)
done

lemma *zgcd-eq*: *zgcd m n* = *zgcd n* (*m mod n*)
apply (*cases n* = *0*, *simp*)
apply (*auto simp add*: *linorder-neq-iff zgcd-non-0*)
apply (*cut-tac m* = −*m* **and** *n* = −*n* **in** *zgcd-non-0*, *auto*)
done

lemma *zgcd-1* [*simp*, *algebra*]: *zgcd m 1* = *1*
by (*simp add*: *zgcd-def abs-if*)

lemma *zgcd-0-1-iff* [*simp*, *algebra*]: *zgcd 0 m* = *1* ⟷ |*m*| = *1*
by (*simp add*: *zgcd-def abs-if*)

lemma *zgcd-greatest-iff* [*algebra*]: *k dvd zgcd m n* = (*k dvd m* ∧ *k dvd n*)
by (*simp add*: *zgcd-def abs-if int-dvd-iff dvd-int-iff nat-dvd-iff*)

lemma *zgcd-1-left* [*simp*, *algebra*]: *zgcd 1 m* = *1*
by (*simp add*: *zgcd-def*)

lemma *zgcd-assoc*: *zgcd* (*zgcd k m*) *n* = *zgcd k* (*zgcd m n*)
by (*simp add*: *zgcd-def gcd-assoc*)

lemma *zgcd-left-commute*: *zgcd k* (*zgcd m n*) = *zgcd m* (*zgcd k n*)
apply (*rule zgcd-commute* [*THEN trans*])
apply (*rule zgcd-assoc* [*THEN trans*])
apply (*rule zgcd-commute* [*THEN arg-cong*])
done

lemmas *zgcd-ac* = *zgcd-assoc zgcd-commute zgcd-left-commute*
— addition is an AC-operator

lemma *zgcd-zmult-distrib2*: *0* ≤ *k* ==> *k* ∗ *zgcd m n* = *zgcd* (*k* ∗ *m*) (*k* ∗ *n*)
by (*simp del*: *minus-mult-right* [*symmetric*]
add: *minus-mult-right nat-mult-distrib zgcd-def abs-if*
mult-less-0-iff gcd-mult-distrib2 [*symmetric*] *of-nat-mult*)

lemma *zgcd-zmult-distrib2-abs*: *zgcd* (*k* ∗ *m*) (*k* ∗ *n*) = |*k*| ∗ *zgcd m n*
by (*simp add*: *abs-if zgcd-zmult-distrib2*)

lemma *zgcd-self* [*simp*]: *0* ≤ *m* ==> *zgcd m m* = *m*
by (*cut-tac k* = *m* **and** *m* = *1* **and** *n* = *1* **in** *zgcd-zmult-distrib2*, *simp-all*)

lemma *zgcd-zmult-eq-self* [*simp*]: *0* ≤ *k* ==> *zgcd k* (*k* ∗ *n*) = *k*
by (*cut-tac k* = *k* **and** *m* = *1* **and** *n* = *n* **in** *zgcd-zmult-distrib2*, *simp-all*)

lemma *zgcd-zmult-eq-self2* [*simp*]: *0* ≤ *k* ==> *zgcd* (*k* ∗ *n*) *k* = *k*
by (*cut-tac k* = *k* **and** *m* = *n* **and** *n* = *1* **in** *zgcd-zmult-distrib2*, *simp-all*)

definition *zlcm i j* = *int* (*lcm* (*nat* |*i*|) (*nat* |*j*|))

lemma *dvd-zlcm-self1*[*simp*, *algebra*]: *i dvd zlcm i j*
by(*simp add*:*zlcm-def dvd-int-iff*)

lemma *dvd-zlcm-self2*[*simp*, *algebra*]: *j dvd zlcm i j*
by(*simp add*:*zlcm-def dvd-int-iff*)

lemma *dvd-imp-dvd-zlcm1*:
assumes *k dvd i* **shows** *k dvd* (*zlcm i j*)
proof −
have *nat* |*k*| *dvd nat* |*i*| **using** ‹*k dvd i*›
by(*simp add*:*int-dvd-iff*[*symmetric*] *dvd-int-iff*[*symmetric*])
thus *?thesis* **by**(*simp add*:*zlcm-def dvd-int-iff*)(*blast intro*: *dvd-trans*)
qed

lemma *dvd-imp-dvd-zlcm2*:
assumes *k dvd j* **shows** *k dvd* (*zlcm i j*)
proof −
have *nat* |*k*| *dvd nat* |*j*| **using** ‹*k dvd j*›
by(*simp add*:*int-dvd-iff*[*symmetric*] *dvd-int-iff*[*symmetric*])
thus *?thesis* **by**(*simp add*:*zlcm-def dvd-int-iff*)(*blast intro*: *dvd-trans*)
qed

lemma *zdvd-self-abs1*: (*d*::*int*) *dvd* |*d*|

by (*case-tac d <0*, *simp-all*)

lemma *zdvd-self-abs2*: |*d*::*int*| *dvd d*
by (*case-tac d<0*, *simp-all*)

lemma *lcm-pos*:
  assumes *mpos*: *m > 0*
    and *npos*: *n>0*
  shows *lcm m n > 0*
proof (*rule ccontr*, *simp add*: *lcm-def gcd-zero*)
  assume *h*:*m∗n div gcd m n = 0*
  from *mpos npos* **have** *gcd m n ≠ 0* **using** *gcd-zero* **by** *simp*
  hence *gcdp*: *gcd m n > 0* **by** *simp*
  with *h*
  have *m∗n < gcd m n*
    by (*cases m ∗ n < gcd m n*) (*auto simp add*: *div-if*[*OF gcdp*, **where** *m=m∗n*])
  moreover
  have *gcd m n dvd m* **by** *simp*
  with *mpos dvd-imp-le* **have** *t1*:*gcd m n ≤ m* **by** *simp*
  with *npos* **have** *t1*:*gcd m n ∗n ≤ m∗n* **by** *simp*
  have *gcd m n ≤ gcd m n∗n* **using** *npos* **by** *simp*
  with *t1* **have** *gcd m n ≤ m∗n* **by** *arith*
  ultimately show *False* **by** *simp*
qed

lemma *zlcm-pos*:
  assumes *anz*: *a ≠ 0*
  and *bnz*: *b ≠ 0*
  shows *0 < zlcm a b*
proof−
  let *?na = nat* |*a*|
  let *?nb = nat* |*b*|
  have *nap*: *?na >0* **using** *anz* **by** *simp*
  have *nbp*: *?nb >0* **using** *bnz* **by** *simp*
  have *0 < lcm ?na ?nb* **by** (*rule lcm-pos*[*OF nap nbp*])
  thus *?thesis* **by** (*simp add*: *zlcm-def*)
qed

lemma *zgcd-code* [*code*]:
  zgcd k l = |*if l = 0 then k else zgcd l* (|*k*| *mod* |*l*|)|
  by (*simp add*: *zgcd-def gcd.simps* [*of nat* |*k*|] *nat-mod-distrib*)

end

2 Primality on nat

theory *Primes*

```
imports Complex-Main Legacy-GCD
begin

definition coprime :: nat => nat => bool
  where coprime m n ⟷ gcd m n = 1

definition prime :: nat ⇒ bool
  where prime p ⟷ (1 < p ∧ (∀m. m dvd p --> m = 1 ∨ m = p))


lemma two-is-prime: prime 2
  apply (auto simp add: prime-def)
  apply (case-tac m)
   apply (auto dest!: dvd-imp-le)
  done

lemma prime-imp-relprime: prime p ==> ¬ p dvd n ==> gcd p n = 1
  apply (auto simp add: prime-def)
  apply (metis gcd-dvd1 gcd-dvd2)
  done
```

This theorem leads immediately to a proof of the uniqueness of factorization. If p divides a product of primes then it is one of those primes.

```
lemma prime-dvd-mult: prime p ==> p dvd m * n ==> p dvd m ∨ p dvd n
  by (blast intro: relprime-dvd-mult prime-imp-relprime)

lemma prime-dvd-square: prime p ==> p dvd m^Suc (Suc 0) ==> p dvd m
  by (auto dest: prime-dvd-mult)

lemma prime-dvd-power-two: prime p ==> p dvd m² ==> p dvd m
  by (rule prime-dvd-square) (simp-all add: power2-eq-square)


lemma exp-eq-1:(x::nat)^n = 1 ⟷ x = 1 ∨ n = 0
by (induct n, auto)

lemma exp-mono-lt: (x::nat) ^ (Suc n) < y ^ (Suc n) ⟷ x < y
by(metis linorder-not-less not-less0 power-le-imp-le-base power-less-imp-less-base)

lemma exp-mono-le: (x::nat) ^ (Suc n) ≤ y ^ (Suc n) ⟷ x ≤ y
by (simp only: linorder-not-less[symmetric] exp-mono-lt)

lemma exp-mono-eq: (x::nat) ^ Suc n = y ^ Suc n ⟷ x = y
using power-inject-base[of x n y] by auto


lemma even-square: assumes e: even (n::nat) shows ∃x. n² = 4*x
proof−
  from e have 2 dvd n by presburger
```

then obtain *k* **where** *k*: *n* = *2*∗*k* **using** *dvd-def* **by** *auto*
hence n^2 = *4* ∗ k^2 **by** (*simp add*: *power2-eq-square*)
thus *?thesis* **by** *blast*
qed

lemma *odd-square*: **assumes** *e*: *odd* (*n*::*nat*) **shows** ∃ *x*. n^2 = *4*∗*x* + *1*
proof−
from *e* **have** *np*: *n* > *0* **by** *presburger*
from *e* **have** *2 dvd* (*n* − *1*) **by** *presburger*
then obtain *k* **where** *n* − *1* = *2* ∗ *k* **..**
hence *k*: *n* = *2*∗*k* + *1* **using** *e* **by** *presburger*
hence n^2 = *4*∗ (k^2 + *k*) + *1* **by** *algebra*
thus *?thesis* **by** *blast*
qed

lemma *diff-square*: $(x::nat)^2$ − y^2 = (*x*+*y*)∗(*x* − *y*)
proof−
have *x* ≤ *y* ∨ *y* ≤ *x* **by** (*rule nat-le-linear*)
moreover
{**assume** *le*: *x* ≤ *y*
hence x^2 ≤ y^2 **by** (*simp only*: *numeral-2-eq-2 exp-mono-le Let-def*)
with *le* **have** *?thesis* **by** *simp* }
moreover
{**assume** *le*: *y* ≤ *x*
hence *le2*: y^2 ≤ x^2 **by** (*simp only*: *numeral-2-eq-2 exp-mono-le Let-def*)
from *le* **have** ∃ *z*. *y* + *z* = *x* **by** *presburger*
then obtain *z* **where** *z*: *x* = *y* + *z* **by** *blast*
from *le2* **have** ∃ *z*. x^2 = y^2 + *z* **by** *presburger*
then obtain *z2* **where** *z2*: x^2 = y^2 + *z2* **by** *blast*
from *z z2* **have** *?thesis* **by** *simp algebra* }
ultimately show *?thesis* **by** *blast*
qed

Elementary theory of divisibility

lemma *divides-ge*: (*a*::*nat*) *dvd b* ⟹ *b* = *0* ∨ *a* ≤ *b* **unfolding** *dvd-def* **by** *auto*
lemma *divides-antisym*: (*x*::*nat*) *dvd y* ∧ *y dvd x* ⟷ *x* = *y*
using *dvd-antisym*[*of x y*] **by** *auto*

lemma *divides-add-revr*: **assumes** *da*: (*d*::*nat*) *dvd a* **and** *dab*:*d dvd* (*a* + *b*)
shows *d dvd b*
proof−
from *da* **obtain** *k* **where** *k*:*a* = *d*∗*k* **by** (*auto simp add*: *dvd-def*)
from *dab* **obtain** *k'* **where** *k'*: *a* + *b* = *d*∗*k'* **by** (*auto simp add*: *dvd-def*)
from *k k'* **have** *b* = *d* ∗(*k'* − *k*) **by** (*simp add* : *diff-mult-distrib2*)
thus *?thesis* **unfolding** *dvd-def* **by** *blast*
qed

declare *nat-mult-dvd-cancel-disj*[*presburger*]
lemma *nat-mult-dvd-cancel-disj'*[*presburger*]:

```
(m::nat)*k dvd n*k ⟷ k = 0 ∨ m dvd n unfolding mult.commute[of m k]
mult.commute[of n k] by presburger

lemma divides-mul-l: (a::nat) dvd b ==> (c * a) dvd (c * b)
  by presburger

lemma divides-mul-r: (a::nat) dvd b ==> (a * c) dvd (b * c) by presburger
lemma divides-cases: (n::nat) dvd m ==> m = 0 ∨ m = n ∨ 2 * n <= m
  by (auto simp add: dvd-def)

lemma divides-div-not: (x::nat) = (q * n) + r ⟹ 0 < r ⟹ r < n ==> ~(n
dvd x)
proof(auto simp add: dvd-def)
  fix k assume H: 0 < r r < n q * n + r = n * k
 from H(3) have r: r = n* (k −q) by(simp add: diff-mult-distrib2 mult.commute)
  {assume k − q = 0 with r H(1) have False by simp}
  moreover
  {assume k − q ≠ 0 with r have r ≥ n by auto
    with H(2) have False by simp}
  ultimately show False by blast
qed
lemma divides-exp: (x::nat) dvd y ==> x ^ n dvd y ^ n
  by (auto simp add: power-mult-distrib dvd-def)

lemma divides-exp2: n ≠ 0 ⟹ (x::nat) ^ n dvd y ⟹ x dvd y
  by (induct n ,auto simp add: dvd-def)

fun fact :: nat ⇒ nat where
  fact 0 = 1
| fact (Suc n) = Suc n * fact n

lemma fact-lt: 0 < fact n by(induct n, simp-all)
lemma fact-le: fact n ≥ 1 using fact-lt[of n] by simp
lemma fact-mono: assumes le: m ≤ n shows fact m ≤ fact n
proof−
  from le have ∃ i. n = m+i by presburger
  then obtain i where i: n = m+i by blast
  have fact m ≤ fact (m + i)
  proof(induct m)
    case 0 thus ?case using fact-le[of i] by simp
  next
    case (Suc m)
    have fact (Suc m) = Suc m * fact m by simp
    have th1: Suc m ≤ Suc (m + i) by simp
    from mult-le-mono[of Suc m Suc (m+i) fact m fact (m+i), OF th1 Suc.hyps]
    show ?case by simp
  qed
  thus ?thesis using i by simp
qed
```

lemma *divides-fact*: *1* <= *p* ⟹ *p* <= *n* ==> *p dvd fact n*
proof(*induct n arbitrary*: *p*)
  case *0* **thus** *?case* **by** *simp*
next
  case (*Suc n p*)
  from *Suc.prems* **have** $p = Suc\ n \vee p \leq n$ **by** *presburger*
  moreover
  {**assume** $p = Suc\ n$ **hence** *?case* **by** (*simp only*: *fact.simps dvd-triv-left*)}
  moreover
  {**assume** $p \leq n$
    with *Suc.prems*(*1*) *Suc.hyps* **have** *th*: *p dvd fact n* **by** *simp*
    from *dvd-mult*[*OF th*] **have** *?case* **by** (*simp only*: *fact.simps*) }
  ultimately show *?case* **by** *blast*
qed

declare *dvd-triv-left*[*presburger*]
declare *dvd-triv-right*[*presburger*]
lemma *divides-rexp*:
  x dvd y ⟹ (*x*::*nat*) *dvd* (*y* ^(*Suc n*)) **by** (*simp add*: *dvd-mult2*[*of x y*])

Coprimality

lemma *coprime*: *coprime a b* ⟷ (∀ *d*. *d dvd a* ∧ *d dvd b* ⟷ *d* = *1*)
using *gcd-unique*[*of 1 a b*, *simplified*] **by** (*auto simp add*: *coprime-def*)
lemma *coprime-commute*: *coprime a b* ⟷ *coprime b a* **by** (*simp add*: *coprime-def gcd-commute*)

lemma *coprime-bezout*: *coprime a b* ⟷ (∃ *x y*. $a * x - b * y = 1 \vee b * x - a * y = 1$)
using *coprime-def gcd-bezout* **by** *auto*

lemma *coprime-divprod*: *d dvd a* ∗ *b* ⟹ *coprime d a* ⟹ *d dvd b*
 using *relprime-dvd-mult-iff* [*of d a b*] **by** (*auto simp add*: *coprime-def mult.commute*)

lemma *coprime-1*[*simp*]: *coprime a 1* **by** (*simp add*: *coprime-def*)
lemma *coprime-1'*[*simp*]: *coprime 1 a* **by** (*simp add*: *coprime-def*)
lemma *coprime-Suc0*[*simp*]: *coprime a* (*Suc 0*) **by** (*simp add*: *coprime-def*)
lemma *coprime-Suc0'*[*simp*]: *coprime* (*Suc 0*) *a* **by** (*simp add*: *coprime-def*)

lemma *gcd-coprime*:
  assumes *z*: *gcd a b* ≠ *0* **and** *a*: $a = a' * gcd\ a\ b$ **and** *b*: $b = b' * gcd\ a\ b$
  shows   *coprime a' b'*
proof−
  let *?g* = *gcd a b*
 {**assume** *bz*: *a* = *0* **from** *b bz z a* **have** *?thesis* **by** (*simp add*: *gcd-zero coprime-def*)}
  moreover
 {**assume** *az*: *a*≠ *0*
    from *z* **have** *z'*: *?g* > *0* **by** *simp*
    from *bezout-gcd-strong*[*OF az*, *of b*]

obtain *x y* **where** *xy*: $a{*}x = b{*}y + ?g$ **by** *blast*
from *xy a b* **have** $?g * a'{*}x = ?g * (b'{*}y + 1)$ **by** (*simp add*: *algebra-simps*)
hence $?g * (a'{*}x) = ?g * (b'{*}y + 1)$ **by** (*simp add*: *mult.assoc*)
hence $a'{*}x = (b'{*}y + 1)$
by (*simp only*: *nat-mult-eq-cancel1*[*OF z'*])
hence $a'{*}x - b'{*}y = 1$ **by** *simp*
with *coprime-bezout*[*of a' b'*] **have** *?thesis* **by** *auto*}
ultimately show *?thesis* **by** *blast*
qed
lemma *coprime-0*: *coprime d 0* $\longleftrightarrow$ $d = 1$ **by** (*simp add*: *coprime-def*)
lemma *coprime-mul*: **assumes** *da*: *coprime d a* **and** *db*: *coprime d b*
shows *coprime d* $(a * b)$
proof−
from *da* **have** *th*: *gcd a d* $= 1$ **by** (*simp add*: *coprime-def gcd-commute*)
from *gcd-mult-cancel*[*of a d b*, *OF th*] *db*[*unfolded coprime-def*] **have** *gcd d* $(a{*}b)$ $= 1$
by (*simp add*: *gcd-commute*)
thus *?thesis* **unfolding** *coprime-def* **.**
qed
lemma *coprime-lmul2*: **assumes** *dab*: *coprime d* $(a * b)$ **shows** *coprime d b*
using *dab* **unfolding** *coprime-bezout*
apply *clarsimp*
apply (*case-tac* $d * x - a * b * y = Suc\ 0$, *simp-all*)
apply (*rule-tac* $x{=}x$ **in** *exI*)
apply (*rule-tac* $x{=}a{*}y$ **in** *exI*)
apply (*simp add*: *ac-simps*)
apply (*rule-tac* $x{=}a{*}x$ **in** *exI*)
apply (*rule-tac* $x{=}y$ **in** *exI*)
apply (*simp add*: *ac-simps*)
done

lemma *coprime-rmul2*: *coprime d* $(a * b) \Longrightarrow$ *coprime d a*
unfolding *coprime-bezout*
apply *clarsimp*
apply (*case-tac* $d * x - a * b * y = Suc\ 0$, *simp-all*)
apply (*rule-tac* $x{=}x$ **in** *exI*)
apply (*rule-tac* $x{=}b{*}y$ **in** *exI*)
apply (*simp add*: *ac-simps*)
apply (*rule-tac* $x{=}b{*}x$ **in** *exI*)
apply (*rule-tac* $x{=}y$ **in** *exI*)
apply (*simp add*: *ac-simps*)
done
lemma *coprime-mul-eq*: *coprime d* $(a * b) \longleftrightarrow$ *coprime d a* $\wedge$ *coprime d b*
using *coprime-rmul2*[*of d a b*] *coprime-lmul2*[*of d a b*] *coprime-mul*[*of d a b*]
by *blast*

lemma *gcd-coprime-exists*:
assumes *nz*: *gcd a b* $\neq 0$
shows $\exists\, a'\ b'.\ a = a' * gcd\ a\ b \wedge b = b' * gcd\ a\ b \wedge coprime\ a'\ b'$

proof−
  let *?g* = *gcd a b*
  from *gcd-dvd1*[*of a b*] *gcd-dvd2*[*of a b*]
  obtain *a′ b′* **where** *a* = *?g∗a′* *b* = *?g∗b′* **unfolding** *dvd-def* **by** *blast*
  hence *ab′*: *a* = *a′∗?g b* = *b′∗?g* **by** *algebra+*
  from *ab′ gcd-coprime*[*OF nz ab′*] **show** *?thesis* **by** *blast*
qed

lemma *coprime-exp*: *coprime d a* ==> *coprime d* (*a*ˆ*n*)
  by(*induct n*, *simp-all add*: *coprime-mul*)

lemma *coprime-exp-imp*: *coprime a b* ==> *coprime* (*a* ˆ*n*) (*b* ˆ*n*)
  by (*induct n*, *simp-all add*: *coprime-mul-eq coprime-commute coprime-exp*)
lemma *coprime-refl*[*simp*]: *coprime n n* ⟷ *n* = *1* **by** (*simp add*: *coprime-def*)
lemma *coprime-plus1*[*simp*]: *coprime* (*n* + *1*) *n*
  apply (*simp add*: *coprime-bezout*)
  apply (*rule exI*[**where** *x*=*1*])
  apply (*rule exI*[**where** *x*=*1*])
  apply *simp*
  done
lemma *coprime-minus1*: *n* ≠ *0* ==> *coprime* (*n* − *1*) *n*
  using *coprime-plus1*[*of n* − *1*] *coprime-commute*[*of n* − *1 n*] **by** *auto*

lemma *bezout-gcd-pow*: ∃ *x y*. *a* ˆ*n* ∗ *x* − *b* ˆ *n* ∗ *y* = *gcd a b* ˆ *n* ∨ *b* ˆ *n* ∗ *x* − *a* ˆ *n* ∗ *y* = *gcd a b* ˆ *n*
proof−
  let *?g* = *gcd a b*
  {**assume** *z*: *?g* = *0* **hence** *?thesis*
    apply (*cases n*, *simp*)
    apply *arith*
    apply (*simp only*: *z power-0-Suc*)
    apply (*rule exI*[**where** *x*=*0*])
    apply (*rule exI*[**where** *x*=*0*])
    apply *simp*
    done }
  moreover
  {**assume** *z*: *?g* ≠ *0*
   from *gcd-dvd1*[*of a b*] *gcd-dvd2*[*of a b*] **obtain** *a′ b′* **where**
     ab′: *a* = *a′∗?g b* = *b′∗?g* **unfolding** *dvd-def* **by** (*auto simp add*: *ac-simps*)
   hence *ab″*: *?g∗a′* = *a ?g* ∗ *b′* = *b* **by** *algebra+*
   from *coprime-exp-imp*[*OF gcd-coprime*[*OF z ab′*], *unfolded coprime-bezout*, *of n*]
   obtain *x y* **where** *a′*ˆ*n* ∗ *x* − *b′*ˆ*n* ∗ *y* = *1* ∨ *b′*ˆ*n* ∗ *x* − *a′*ˆ*n* ∗ *y* = *1* **by** *blast*
   hence *?g*ˆ*n* ∗ (*a′*ˆ*n* ∗ *x* − *b′*ˆ*n* ∗ *y*) = *?g*ˆ*n* ∨ *?g*ˆ*n*∗(*b′*ˆ*n* ∗ *x* − *a′*ˆ*n* ∗ *y*) = *?g*ˆ*n*
     using *z* **by** *auto*
   then have *a*ˆ*n* ∗ *x* − *b*ˆ*n* ∗ *y* = *?g*ˆ*n* ∨ *b*ˆ*n* ∗ *x* − *a*ˆ*n* ∗ *y* = *?g*ˆ*n*
     using *z ab″* **by** (*simp only*: *power-mult-distrib*[*symmetric*]

diff-mult-distrib2 mult.assoc[*symmetric*])
hence *?thesis* **by** *blast* **}**
ultimately show *?thesis* **by** *blast*
qed

lemma *gcd-exp*: *gcd* (*a*ˆ*n*) (*b*ˆ*n*) = *gcd a b*ˆ*n*
proof−
let *?g* = *gcd* (*a*ˆ*n*) (*b*ˆ*n*)
let *?gn* = *gcd a b*ˆ*n*
{fix *e* **assume** *H*: *e dvd a*ˆ*n e dvd b*ˆ*n*
from *bezout-gcd-pow*[*of a n b*] **obtain** *x y*
where *xy*: *a* ˆ *n* ∗ *x* − *b* ˆ *n* ∗ *y* = *?gn* ∨ *b* ˆ *n* ∗ *x* − *a* ˆ *n* ∗ *y* = *?gn* **by**
blast
from *dvd-diff-nat* [*OF dvd-mult2*[*OF H*(*1*), *of x*] *dvd-mult2*[*OF H*(*2*), *of y*]]
dvd-diff-nat [*OF dvd-mult2*[*OF H*(*2*), *of x*] *dvd-mult2*[*OF H*(*1*), *of y*]] *xy*
have *e dvd ?gn* **by** (*cases a* ˆ *n* ∗ *x* − *b* ˆ *n* ∗ *y* = *gcd a b* ˆ *n*, *simp-all*)**}**
hence *th*: ∀ *e*. *e dvd a*ˆ*n* ∧ *e dvd b*ˆ*n* ⟶ *e dvd ?gn* **by** *blast*
from *divides-exp*[*OF gcd-dvd1*[*of a b*], *of n*] *divides-exp*[*OF gcd-dvd2*[*of a b*], *of*
n] *th*
gcd-unique **have** *?gn* = *?g* **by** *blast* **thus** *?thesis* **by** *simp*
qed

lemma *coprime-exp2*: *coprime* (*a* ˆ *Suc n*) (*b*ˆ *Suc n*) ⟷ *coprime a b*
by (*simp only*: *coprime-def gcd-exp exp-eq-1*) *simp*

lemma *division-decomp*: **assumes** *dc*: (*a*::*nat*) *dvd b* ∗ *c*
shows ∃ *b*′ *c*′. *a* = *b*′ ∗ *c*′ ∧ *b*′ *dvd b* ∧ *c*′ *dvd c*
proof−
let *?g* = *gcd a b*
{assume *?g* = *0* **with** *dc* **have** *?thesis* **apply** (*simp add*: *gcd-zero*)
apply (*rule exI*[**where** *x*=*0*])
by (*rule exI*[**where** *x*=*c*], *simp*)**}**
moreover
{assume *z*: *?g* ≠ *0*
from *gcd-coprime-exists*[*OF z*]
obtain *a*′ *b*′ **where** *ab*′: *a* = *a*′ ∗ *?g b* = *b*′ ∗ *?g coprime a*′ *b*′ **by** *blast*
from *gcd-dvd2*[*of a b*] **have** *thb*: *?g dvd b* **.**
from *ab*′(*1*) **have** *a*′ *dvd a* **unfolding** *dvd-def* **by** *blast*
with *dc* **have** *th0*: *a*′ *dvd b*∗*c* **using** *dvd-trans*[*of a*′ *a b*∗*c*] **by** *simp*
from *dc ab*′(*1*,*2*) **have** *a*′∗*?g dvd* (*b*′∗*?g*) ∗*c* **by** *auto*
hence *?g*∗*a*′ *dvd ?g* ∗ (*b*′ ∗ *c*) **by** (*simp add*: *mult.assoc*)
with *z* **have** *th-1*: *a*′ *dvd b*′∗*c* **by** *simp*
from *coprime-divprod*[*OF th-1 ab*′(*3*)] **have** *thc*: *a*′ *dvd c* **.**
from *ab*′ **have** *a* = *?g*∗*a*′ **by** *algebra*
with *thb thc* **have** *?thesis* **by** *blast* **}**
ultimately show *?thesis* **by** *blast*
qed

lemma *nat-power-eq-0-iff*: (*m*::*nat*) ˆ *n* = *0* ⟷ *n* ≠ *0* ∧ *m* = *0* **by** (*induct n*, *auto*)

lemma *divides-rev*: **assumes** *ab*: (*a*::*nat*) ^ *n dvd b* ^*n* **and** *n*:*n* ≠ *0* **shows** *a dvd b*
proof−
  let *?g* = *gcd a b*
  from *n* **obtain** *m* **where** *m*: *n* = *Suc m* **by** (*cases n*, *simp-all*)
  {**assume** *?g* = *0* **with** *ab n* **have** *?thesis* **by** (*simp add*: *gcd-zero*)}
  moreover
  {**assume** *z*: *?g* ≠ *0*
    hence *zn*: *?g* ^ *n* ≠ *0* **using** *n* **by** *simp*
    from *gcd-coprime-exists*[*OF z*]
    obtain *a′ b′* **where** *ab′*: *a* = *a′* ∗ *?g b* = *b′* ∗ *?g coprime a′ b′* **by** *blast*
   from *ab* **have** (*a′* ∗ *?g*) ^ *n dvd* (*b′* ∗ *?g*) ^*n* **by** (*simp add*: *ab′*(*1*,*2*)[*symmetric*])
   hence *?g*^*n*∗*a′*^*n dvd ?g*^*n* ∗*b′*^*n* **by** (*simp only*: *power-mult-distrib mult.commute*)
    with *zn z n* **have** *th0*:*a′*^*n dvd b′*^*n* **by** (*auto simp add*: *nat-power-eq-0-iff*)
    have *a′ dvd a′*^*n* **by** (*simp add*: *m*)
    with *th0* **have** *a′ dvd b′*^*n* **using** *dvd-trans*[*of a′ a′*^*n b′*^*n*] **by** *simp*
    hence *th1*: *a′ dvd b′*^*m* ∗ *b′* **by** (*simp add*: *m mult.commute*)
    from *coprime-divprod*[*OF th1 coprime-exp*[*OF ab′*(*3*), *of m*]]
    have *a′ dvd b′* **.**
    hence *a′*∗*?g dvd b′*∗*?g* **by** *simp*
    with *ab′*(*1*,*2*) **have** *?thesis* **by** *simp* }
  ultimately show *?thesis* **by** *blast*
qed

lemma *divides-mul*: **assumes** *mr*: *m dvd r* **and** *nr*: *n dvd r* **and** *mn*:*coprime m n*
  shows *m* ∗ *n dvd r*
proof−
  from *mr nr* **obtain** *m′ n′* **where** *m′*: *r* = *m*∗*m′* **and** *n′*: *r* = *n*∗*n′*
    unfolding *dvd-def* **by** *blast*
  from *mr n′* **have** *m dvd n′*∗*n* **by** (*simp add*: *mult.commute*)
  hence *m dvd n′* **using** *relprime-dvd-mult-iff*[*OF mn*[*unfolded coprime-def*]] **by** *simp*
  then obtain *k* **where** *k*: *n′* = *m*∗*k* **unfolding** *dvd-def* **by** *blast*
  from *n′ k* **show** *?thesis* **unfolding** *dvd-def* **by** *auto*
qed

A binary form of the Chinese Remainder Theorem.

lemma *chinese-remainder*: **assumes** *ab*: *coprime a b* **and** *a*:*a* ≠ *0* **and** *b*:*b* ≠ *0*
  shows ∃ *x q1 q2*. *x* = *u* + *q1* ∗ *a* ∧ *x* = *v* + *q2* ∗ *b*
proof−
  from *bezout-add-strong*[*OF a*, *of b*] *bezout-add-strong*[*OF b*, *of a*]
  obtain *d1 x1 y1 d2 x2 y2* **where** *dxy1*: *d1 dvd a d1 dvd b a* ∗ *x1* = *b* ∗ *y1* + *d1*
    and *dxy2*: *d2 dvd b d2 dvd a b* ∗ *x2* = *a* ∗ *y2* + *d2* **by** *blast*
  from *gcd-unique*[*of 1 a b*, *simplified ab*[*unfolded coprime-def*], *simplified*]
    dxy1(*1*,*2*) *dxy2*(*1*,*2*) **have** *d12*: *d1* = *1 d2* =*1* **by** *auto*
  let *?x* = *v* ∗ *a* ∗ *x1* + *u* ∗ *b* ∗ *x2*

let *?q1* = *v* ∗ *x1* + *u* ∗ *y2*
let *?q2* = *v* ∗ *y1* + *u* ∗ *x2*
from *dxy2*(*3*)[*simplified d12*] *dxy1*(*3*)[*simplified d12*]
have *?x* = *u* + *?q1* ∗ *a* *?x* = *v* + *?q2* ∗ *b* **by** *algebra+*
thus *?thesis* **by** *blast*
qed

Primality

A few useful theorems about primes

lemma *prime-0*[*simp*]: ~*prime 0* **by** (*simp add*: *prime-def*)
lemma *prime-1*[*simp*]: ~ *prime 1* **by** (*simp add*: *prime-def*)
lemma *prime-Suc0*[*simp*]: ~ *prime* (*Suc 0*) **by** (*simp add*: *prime-def*)

lemma *prime-ge-2*: *prime p* ==> *p* ≥ *2* **by** (*simp add*: *prime-def*)
lemma *prime-factor*: **assumes** *n*: *n* ≠ *1* **shows** ∃ *p*. *prime p* ∧ *p dvd n*
using *n*
proof(*induct n rule*: *nat-less-induct*)
fix *n*
assume *H*: ∀ *m*<*n*. *m* ≠ *1* ⟶ (∃ *p*. *prime p* ∧ *p dvd m*) *n* ≠ *1*
let *?ths* = ∃ *p*. *prime p* ∧ *p dvd n*
{assume *n=0* **hence** *?ths* **using** *two-is-prime* **by** *auto***}**
moreover
{assume *nz*: *n*≠*0*
{assume *prime n* **hence** *?ths* **by** − (*rule exI*[**where** *x=n*], *simp*)**}**
moreover
{assume *n*: ¬ *prime n*
with *nz H*(*2*)
obtain *k* **where** *k*:*k dvd n k* ≠ *1 k* ≠ *n* **by** (*auto simp add*: *prime-def*)
from *dvd-imp-le*[*OF k*(*1*)] *nz k*(*3*) **have** *kn*: *k* < *n* **by** *simp*
from *H*(*1*)[*rule-format*, *OF kn k*(*2*)] **obtain** *p* **where** *p*: *prime p p dvd k* **by** *blast*
from *dvd-trans*[*OF p*(*2*) *k*(*1*)] *p*(*1*) **have** *?ths* **by** *blast***}**
ultimately have *?ths* **by** *blast***}**
ultimately show *?ths* **by** *blast*
qed

lemma *prime-factor-lt*: **assumes** *p*: *prime p* **and** *n*: *n* ≠ *0* **and** *npm*:*n* = *p* ∗ *m*
shows *m* < *n*
proof−
{assume *m=0* **with** *n* **have** *?thesis* **by** *simp***}**
moreover
{assume *m*: *m* ≠ *0*
from *npm* **have** *mn*: *m dvd n* **unfolding** *dvd-def* **by** *auto*
from *npm m* **have** *n* ≠ *m* **using** *p* **by** *auto*
with *dvd-imp-le*[*OF mn*] *n* **have** *?thesis* **by** *simp***}**
ultimately show *?thesis* **by** *blast*
qed

lemma *euclid-bound*: ∃ *p*. *prime p* ∧ *n* < *p* ∧ *p* <= *Suc* (*fact n*)

proof−
  have *f1*: *fact n* + *1* ≠ *1* **using** *fact-le*[*of n*] **by** *arith*
  from *prime-factor*[*OF f1*] **obtain** *p* **where** *p*: *prime p p dvd fact n* + *1* **by** *blast*
  from *dvd-imp-le*[*OF p*(*2*)] **have** *pfn*: *p* ≤ *fact n* + *1* **by** *simp*
  {**assume** *np*: *p* ≤ *n*
    from *p*(*1*) **have** *p1*: *p* ≥ *1* **by** (*cases p*, *simp-all*)
    from *divides-fact*[*OF p1 np*] **have** *pfn'*: *p dvd fact n* **.**
    from *divides-add-revr*[*OF pfn' p*(*2*)] *p*(*1*) **have** *False* **by** *simp*}
  hence *n* < *p* **by** *arith*
  with *p*(*1*) *pfn* **show** *?thesis* **by** *auto*
qed

lemma *euclid*: ∃ *p*. *prime p* ∧ *p* > *n* **using** *euclid-bound* **by** *auto*

lemma *primes-infinite*: ¬ (*finite* {*p*. *prime p*})
apply(*simp add*: *finite-nat-set-iff-bounded-le*)
apply (*metis euclid linorder-not-le*)
done

lemma *coprime-prime*: **assumes** *ab*: *coprime a b*
  shows ~(*prime p* ∧ *p dvd a* ∧ *p dvd b*)
proof
  assume *prime p* ∧ *p dvd a* ∧ *p dvd b*
  thus *False* **using** *ab gcd-greatest*[*of p a b*] **by** (*simp add*: *coprime-def*)
qed
lemma *coprime-prime-eq*: *coprime a b* ⟷ (∀ *p*. ~(*prime p* ∧ *p dvd a* ∧ *p dvd b*))

  (**is** *?lhs* = *?rhs*)
proof−
  {**assume** *?lhs* **with** *coprime-prime* **have** *?rhs* **by** *blast*}
  moreover
  {**assume** *r*: *?rhs* **and** *c*: ¬ *?lhs*
    then obtain *g* **where** *g*: *g*≠*1 g dvd a g dvd b* **unfolding** *coprime-def* **by** *blast*
    from *prime-factor*[*OF g*(*1*)] **obtain** *p* **where** *p*: *prime p p dvd g* **by** *blast*
    from *dvd-trans* [*OF p*(*2*) *g*(*2*)] *dvd-trans* [*OF p*(*2*) *g*(*3*)]
    have *p dvd a p dvd b* **. with** *p*(*1*) *r* **have** *False* **by** *blast*}
  ultimately show *?thesis* **by** *blast*
qed

lemma *prime-coprime*: **assumes** *p*: *prime p*
  shows *n* = *1* ∨ *p dvd n* ∨ *coprime p n*
using *p prime-imp-relprime*[*of p n*] **by** (*auto simp add*: *coprime-def*)

lemma *prime-coprime-strong*: *prime p* ⟹ *p dvd n* ∨ *coprime p n*
  using *prime-coprime*[*of p n*] **by** *auto*

declare *coprime-0*[*simp*]

lemma *coprime-0'*[*simp*]: *coprime 0 d* ⟷ *d* = *1* **by** (*simp add*: *coprime-commute*[*of*

0 d])
lemma *coprime-bezout-strong*: **assumes** *ab*: *coprime a b* **and** *b*: $b \neq 1$
shows $\exists x\ y.\ a * x = b * y + 1$
proof−
from *ab b* **have** *az*: $a \neq 0$ **by** − (*rule ccontr*, *auto*)
from *bezout-gcd-strong*[*OF az*, *of b*] *ab*[*unfolded coprime-def*]
show *?thesis* **by** *auto*
qed

lemma *bezout-prime*: **assumes** *p*: *prime p* **and** *pa*: ¬ *p dvd a*
shows $\exists x\ y.\ a{*}x = p{*}y + 1$
proof−
from *p* **have** *p1*: $p \neq 1$ **using** *prime-1* **by** *blast*
from *prime-coprime*[*OF p*, *of a*] *p1 pa* **have** *ap*: *coprime a p*
by (*auto simp add*: *coprime-commute*)
from *coprime-bezout-strong*[*OF ap p1*] **show** *?thesis* **.**
qed
lemma *prime-divprod*: **assumes** *p*: *prime p* **and** *pab*: *p dvd a*∗*b*
shows *p dvd a* ∨ *p dvd b*
proof−
{assume *a=1* **hence** *?thesis* **using** *pab* **by** *simp* **}**
moreover
{assume *p dvd a* **hence** *?thesis* **by** *blast***}**
moreover
{assume *pa*: *coprime p a* **from** *coprime-divprod*[*OF pab pa*] **have** *?thesis* **.. }**
ultimately show *?thesis* **using** *prime-coprime*[*OF p*, *of a*] **by** *blast*
qed

lemma *prime-divprod-eq*: **assumes** *p*: *prime p*
shows *p dvd a*∗*b* ⟷ *p dvd a* ∨ *p dvd b*
using *p prime-divprod dvd-mult dvd-mult2* **by** *auto*

lemma *prime-divexp*: **assumes** *p*:*prime p* **and** *px*: *p dvd x*ˆ*n*
shows *p dvd x*
using *px*
proof(*induct n*)
case *0* **thus** *?case* **by** *simp*
next
case (*Suc n*)
hence *th*: *p dvd x*∗*x*ˆ*n* **by** *simp*
{assume *H*: *p dvd x*ˆ*n*
from *Suc.hyps*[*OF H*] **have** *?case* **.}**
with *prime-divprod*[*OF p th*] **show** *?case* **by** *blast*
qed

lemma *prime-divexp-n*: *prime p* ⟹ *p dvd x*ˆ*n* ⟹ *p*ˆ*n dvd x*ˆ*n*
using *prime-divexp*[*of p x n*] *divides-exp*[*of p x n*] **by** *blast*

lemma *coprime-prime-dvd-ex*: **assumes** *xy*: ¬*coprime x y*

shows $\exists p.$ *prime p* $\wedge$ *p dvd x* $\wedge$ *p dvd y*
proof−
from *xy*[*unfolded coprime-def*] **obtain** *g* **where** *g*: $g \neq 1$ *g dvd x g dvd y*
by *blast*
from *prime-factor*[*OF g*(*1*)] **obtain** *p* **where** *p*: *prime p p dvd g* **by** *blast*
from *g*(*2*,*3*) *dvd-trans*[*OF p*(*2*)] *p*(*1*) **show** *?thesis* **by** *auto*
qed
lemma *coprime-sos*: **assumes** *xy*: *coprime x y*
shows *coprime* $(x * y)$ $(x^2 + y^2)$
proof−
{**assume** *c*: ¬ *coprime* $(x * y)$ $(x^2 + y^2)$
from *coprime-prime-dvd-ex*[*OF c*] **obtain** *p*
where *p*: *prime p p dvd x*∗*y p dvd* $x^2 + y^2$ **by** *blast*
{**assume** *px*: *p dvd x*
from *dvd-mult*[*OF px*, *of x*] *p*(*3*)
obtain *r s* **where** $x * x = p * r$ **and** $x^2 + y^2 = p * s$
by (*auto elim*!: *dvdE*)
then have $y^2 = p * (s - r)$
by (*auto simp add*: *power2-eq-square diff-mult-distrib2*)
then have *p dvd* y^2 **..**
with *prime-divexp*[*OF p*(*1*), *of y 2*] **have** *py*: *p dvd y* **.**
from *p*(*1*) *px py xy*[*unfolded coprime*, *rule-format*, *of p*] *prime-1*
have *False* **by** *simp* }
moreover
{**assume** *py*: *p dvd y*
from *dvd-mult*[*OF py*, *of y*] *p*(*3*)
obtain *r s* **where** $y * y = p * r$ **and** $x^2 + y^2 = p * s$
by (*auto elim*!: *dvdE*)
then have $x^2 = p * (s - r)$
by (*auto simp add*: *power2-eq-square diff-mult-distrib2*)
then have *p dvd* x^2 **..**
with *prime-divexp*[*OF p*(*1*), *of x 2*] **have** *px*: *p dvd x* **.**
from *p*(*1*) *px py xy*[*unfolded coprime*, *rule-format*, *of p*] *prime-1*
have *False* **by** *simp* }
ultimately have *False* **using** *prime-divprod*[*OF p*(*1*,*2*)] **by** *blast*}
thus *?thesis* **by** *blast*
qed

lemma *distinct-prime-coprime*: *prime p* $\Longrightarrow$ *prime q* $\Longrightarrow$ $p \neq q$ $\Longrightarrow$ *coprime p q*
unfolding *prime-def coprime-prime-eq* **by** *blast*

lemma *prime-coprime-lt*: **assumes** *p*: *prime p* **and** *x*: $0 < x$ **and** *xp*: $x < p$
shows *coprime x p*
proof−
{**assume** *c*: ¬ *coprime x p*
then obtain *g* **where** *g*: $g \neq 1$ *g dvd x g dvd p* **unfolding** *coprime-def* **by** *blast*
from *dvd-imp-le*[*OF g*(*2*)] *x xp* **have** *gp*: $g < p$ **by** *arith*
from *g*(*2*) *x* **have** $g \neq 0$ **by** − (*rule ccontr*, *simp*)

```
  with g gp p[unfolded prime-def] have False by blast}
thus ?thesis by blast
qed

lemma prime-odd: prime p ⟹ p = 2 ∨ odd p unfolding prime-def by auto
```

One property of coprimality is easier to prove via prime factors.

```
lemma prime-divprod-pow:
  assumes p: prime p and ab: coprime a b and pab: p^n dvd a * b
  shows p^n dvd a ∨ p^n dvd b
proof−
  {assume n = 0 ∨ a = 1 ∨ b = 1 with pab have ?thesis
      apply (cases n=0, simp-all)
      apply (cases a=1, simp-all) done}
  moreover
  {assume n: n ≠ 0 and a: a≠1 and b: b≠1
    then obtain m where m: n = Suc m by (cases n, auto)
    from divides-exp2[OF n pab] have pab′: p dvd a∗b .
    from prime-divprod[OF p pab′]
    have p dvd a ∨ p dvd b .
    moreover
    {assume pa: p dvd a
      have pnba: p^n dvd b∗a using pab by (simp add: mult.commute)
      from coprime-prime[OF ab, of p] p pa have ¬ p dvd b by blast
      with prime-coprime[OF p, of b] b
      have cpb: coprime b p using coprime-commute by blast
      from coprime-exp[OF cpb] have pnb: coprime (p^n) b
        by (simp add: coprime-commute)
      from coprime-divprod[OF pnba pnb] have ?thesis by blast }
    moreover
    {assume pb: p dvd b
      have pnba: p^n dvd b∗a using pab by (simp add: mult.commute)
      from coprime-prime[OF ab, of p] p pb have ¬ p dvd a by blast
      with prime-coprime[OF p, of a] a
      have cpb: coprime a p using coprime-commute by blast
      from coprime-exp[OF cpb] have pnb: coprime (p^n) a
        by (simp add: coprime-commute)
      from coprime-divprod[OF pab pnb] have ?thesis by blast }
    ultimately have ?thesis by blast}
  ultimately show ?thesis by blast
qed

lemma nat-mult-eq-one: (n::nat) ∗ m = 1 ⟷ n = 1 ∧ m = 1 (is ?lhs ⟷
?rhs)
proof
  assume H: ?lhs
  hence n dvd 1 m dvd 1 unfolding dvd-def by (auto simp add: mult.commute)
  thus ?rhs by auto
next
```

assume *?rhs* **then show** *?lhs* **by** *auto*
qed

lemma *power-Suc0*: *Suc 0 ^ n = Suc 0*
unfolding *One-nat-def* [*symmetric*] *power-one* **..**

lemma *coprime-pow*: **assumes** *ab*: *coprime a b* **and** *abcn*: $a * b = c \,\hat{}\, n$
shows $\exists\, r\, s.\ a = r\hat{}n \ \wedge\ b = s\ \hat{}n$
using *ab abcn*
proof(*induct c arbitrary*: *a b rule*: *nat-less-induct*)
fix *c a b*
assume *H*: $\forall\, m{<}c.\ \forall\, a\ b.$ *coprime a b* $\longrightarrow a * b = m\ \hat{}\ n \longrightarrow (\exists\, r\ s.\ a = r\ \hat{}\ n$ $\wedge\ b = s\ \hat{}\ n)$ *coprime a b* $a * b = c\ \hat{}\ n$
let *?ths* = $\exists\, r\ s.\ a = r\hat{}n \ \wedge\ b = s\ \hat{}n$
{**assume** *n*: *n = 0*
with *H*(*3*) *power-one* **have** *a*∗*b = 1* **by** *simp*
hence $a = 1 \wedge b = 1$ **by** *simp*
hence *?ths*
apply −
apply (*rule exI*[**where** *x=1*])
apply (*rule exI*[**where** *x=1*])
using *power-one*[*of n*]
by *simp*}
moreover
{**assume** *n*: $n \neq 0$ **then obtain** *m* **where** *m*: *n = Suc m* **by** (*cases n*, *auto*)
{**assume** *c*: *c = 0*
with *H*(*3*) *m H*(*2*) **have** *?ths* **apply** *simp*
apply (*cases a=0*, *simp-all*)
apply (*rule exI*[**where** *x=0*], *simp*)
apply (*rule exI*[**where** *x=0*], *simp*)
done}
moreover
{**assume** *c=1* **with** *H*(*3*) *power-one* **have** *a*∗*b = 1* **by** *simp*
hence $a = 1 \wedge b = 1$ **by** *simp*
hence *?ths*
apply −
apply (*rule exI*[**where** *x=1*])
apply (*rule exI*[**where** *x=1*])
using *power-one*[*of n*]
by *simp*}
moreover
{**assume** *c*: $c{\neq}1$ $c \neq 0$
from *prime-factor*[*OF c*(*1*)] **obtain** *p* **where** *p*: *prime p p dvd c* **by** *blast*
from *prime-divprod-pow*[*OF p*(*1*) *H*(*2*), *unfolded H*(*3*), *OF divides-exp*[*OF p*(*2*), *of n*]]
have *pnab*: *p ^ n dvd a* ∨ *p^n dvd b* **.**
from *p*(*2*) **obtain** *l* **where** *l*: *c = p*∗*l* **unfolding** *dvd-def* **by** *blast*
have *pn0*: $p\hat{}n \neq 0$ **using** *n prime-ge-2* [*OF p*(*1*)] **by** *simp*
{**assume** *pa*: *p^n dvd a*

then obtain *k* **where** *k*: $a = p \hat{} n * k$ **unfolding** *dvd-def* **by** *blast*
from *l* **have** *l dvd c* **by** *auto*
with *dvd-imp-le*[*of l c*] *c* **have** $l \leq c$ **by** *auto*
moreover {**assume** $l = c$ **with** *l c* **have** $p = 1$ **by** *simp* **with** *p* **have** *False* **by** *simp*}
ultimately have *lc*: $l < c$ **by** *arith*
from *coprime-lmul2* [*OF H*(*2*)[*unfolded k coprime-commute*[*of p^n*∗*k b*]]]
have *kb*: *coprime k b* **by** (*simp add*: *coprime-commute*)
from *H*(*3*) *l k pn0* **have** *kbln*: $k * b = l \hat{} n$
by (*auto simp add*: *power-mult-distrib*)
from *H*(*1*)[*rule-format*, *OF lc kb kbln*]
obtain *r s* **where** *rs*: $k = r \hat{} n$ $b = s \hat{} n$ **by** *blast*
from *k rs*(*1*) **have** $a = (p*r) \hat{} n$ **by** (*simp add*: *power-mult-distrib*)
with *rs*(*2*) **have** *?ths* **by** *blast* }
moreover
{**assume** *pb*: *p^n dvd b*
then obtain *k* **where** *k*: $b = p \hat{} n * k$ **unfolding** *dvd-def* **by** *blast*
from *l* **have** *l dvd c* **by** *auto*
with *dvd-imp-le*[*of l c*] *c* **have** $l \leq c$ **by** *auto*
moreover {**assume** $l = c$ **with** *l c* **have** $p = 1$ **by** *simp* **with** *p* **have** *False* **by** *simp*}
ultimately have *lc*: $l < c$ **by** *arith*
from *coprime-lmul2* [*OF H*(*2*)[*unfolded k coprime-commute*[*of p^n*∗*k a*]]]
have *kb*: *coprime k a* **by** (*simp add*: *coprime-commute*)
from *H*(*3*) *l k pn0 n* **have** *kbln*: $k * a = l \hat{} n$
by (*simp add*: *power-mult-distrib mult.commute*)
from *H*(*1*)[*rule-format*, *OF lc kb kbln*]
obtain *r s* **where** *rs*: $k = r \hat{} n$ $a = s \hat{} n$ **by** *blast*
from *k rs*(*1*) **have** $b = (p*r) \hat{} n$ **by** (*simp add*: *power-mult-distrib*)
with *rs*(*2*) **have** *?ths* **by** *blast* }
ultimately have *?ths* **using** *pnab* **by** *blast*}
ultimately have *?ths* **by** *blast*}
ultimately show *?ths* **by** *blast*
qed

More useful lemmas.

lemma *prime-product*:
assumes *prime* $(p * q)$
shows $p = 1 \lor q = 1$
proof −
from *assms* **have**
$1 < p * q$ **and** *P*: $\bigwedge m.\ m \ dvd\ p * q \Longrightarrow m = 1 \lor m = p * q$
unfolding *prime-def* **by** *auto*
from ‹$1 < p * q$› **have** $p \neq 0$ **by** (*cases p*) *auto*
then have *Q*: $p = p * q \longleftrightarrow q = 1$ **by** *auto*
have *p dvd p* ∗ *q* **by** *simp*
then have $p = 1 \lor p = p * q$ **by** (*rule P*)
then show *?thesis* **by** (*simp add*: *Q*)
qed

```
lemma prime-exp: prime (p^n) ⟷ prime p ∧ n = 1
proof(induct n)
  case 0 thus ?case by simp
next
  case (Suc n)
  {assume p = 0 hence ?case by simp}
  moreover
  {assume p=1 hence ?case by simp}
  moreover
  {assume p: p ≠ 0 p≠1
    {assume pp: prime (p^Suc n)
      hence p = 1 ∨ p^n = 1 using prime-product[of p p^n] by simp
      with p have n: n = 0
        by (simp only: exp-eq-1 ) simp
      with pp have prime p ∧ Suc n = 1 by simp}
    moreover
    {assume n: prime p ∧ Suc n = 1 hence prime (p^Suc n) by simp}
    ultimately have ?case by blast}
  ultimately show ?case by blast
qed

lemma prime-power-mult:
  assumes p: prime p and xy: x * y = p ^ k
  shows ∃ i j. x = p ^i ∧ y = p^ j
  using xy
proof(induct k arbitrary: x y)
  case 0 thus ?case apply simp by (rule exI[where x=0], simp)
next
  case (Suc k x y)
  from Suc.prems have pxy: p dvd x*y by auto
  from prime-divprod[OF p pxy] have pxyc: p dvd x ∨ p dvd y .
  from p have p0: p ≠ 0 by − (rule ccontr, simp)
  {assume px: p dvd x
    then obtain d where d: x = p*d unfolding dvd-def by blast
    from Suc.prems d have p*d*y = p^Suc k by simp
    hence th: d*y = p^k using p0 by simp
    from Suc.hyps[OF th] obtain i j where ij: d = p^i y = p^j by blast
    with d have x = p^Suc i by simp
    with ij(2) have ?case by blast}
  moreover
  {assume px: p dvd y
    then obtain d where d: y = p*d unfolding dvd-def by blast
    from Suc.prems d have p*d*x = p^Suc k by (simp add: mult.commute)
    hence th: d*x = p^k using p0 by simp
    from Suc.hyps[OF th] obtain i j where ij: d = p^i x = p^j by blast
    with d have y = p^Suc i by simp
    with ij(2) have ?case by blast}
  ultimately show ?case using pxyc by blast
```

qed

lemma *prime-power-exp*: **assumes** *p*: *prime p* **and** *n*:$n \neq 0$
  and *xn*: $x\hat{\ }n = p\hat{\ }k$ **shows** $\exists\, i.\ x = p\hat{\ }i$
  using *n xn*
proof(*induct n arbitrary*: *k*)
  case *0* **thus** *?case* **by** *simp*
next
  case (*Suc n k*) **hence** *th*: $x{*}x\hat{\ }n = p\hat{\ }k$ **by** *simp*
  {**assume** $n = 0$ **with** *Suc* **have** *?case* **by** *simp* (*rule exI*[**where** *x*=*k*], *simp*)}
  moreover
  {**assume** *n*: $n \neq 0$
    from *prime-power-mult*[*OF p th*]
    obtain *i j* **where** *ij*: $x = p\hat{\ }i$ $x\hat{\ }n = p\hat{\ }j$**by** *blast*
    from *Suc.hyps*[*OF n ij*(*2*)] **have** *?case* **.**}
  ultimately show *?case* **by** *blast*
qed

lemma *divides-primepow*: **assumes** *p*: *prime p*
  shows $d\ dvd\ p\hat{\ }k \longleftrightarrow (\exists\ i.\ i \leq k \wedge d = p\ \hat{\ }i)$
proof
  assume *H*: $d\ dvd\ p\hat{\ }k$ **then obtain** *e* **where** *e*: $d{*}e = p\hat{\ }k$
    unfolding *dvd-def* **apply** (*auto simp add*: *mult.commute*) **by** *blast*
  from *prime-power-mult*[*OF p e*] **obtain** *i j* **where** *ij*: $d = p\hat{\ }i$ $e{=}p\hat{\ }j$ **by** *blast*
  from *prime-ge-2*[*OF p*] **have** *p1*: $p > 1$ **by** *arith*
  from *e ij* **have** $p\hat{\ }(i + j) = p\hat{\ }k$ **by** (*simp add*: *power-add*)
  hence $i + j = k$ **using** *power-inject-exp*[*of p i+j k*, *OF p1*] **by** *simp*
  hence $i \leq k$ **by** *arith*
  with *ij*(*1*) **show** $\exists\, i{\leq}k.\ d = p\ \hat{\ }\ i$ **by** *blast*
next
  {**fix** *i* **assume** *H*: $i \leq k$ $d = p\hat{\ }i$
    hence $\exists\, j.\ k = i + j$ **by** *arith*
    then obtain *j* **where** *j*: $k = i + j$ **by** *blast*
    hence $p\hat{\ }k = p\hat{\ }j{*}d$ **using** *H*(*2*) **by** (*simp add*: *power-add*)
    hence $d\ dvd\ p\hat{\ }k$ **unfolding** *dvd-def* **by** *auto*}
  thus $\exists\, i{\leq}k.\ d = p\ \hat{\ }\ i \Longrightarrow d\ dvd\ p\ \hat{\ }\ k$ **by** *blast*
qed

lemma *coprime-divisors*: $d\ dvd\ a \Longrightarrow e\ dvd\ b \Longrightarrow coprime\ a\ b \Longrightarrow coprime\ d\ e$
  by (*auto simp add*: *dvd-def coprime*)

lemma *mult-inj-if-coprime-nat*:
  $inj\text{-}on\ f\ A \Longrightarrow inj\text{-}on\ g\ B \Longrightarrow \forall\, a{\in}A.\ \forall\, b{\in}B.\ Primes.coprime\ (f\ a)\ (g\ b) \Longrightarrow$
    $inj\text{-}on\ (\lambda(a,\ b).\ f\ a * g\ b)\ (A \times B)$
apply (*auto simp add*: *inj-on-def*)
apply (*metis coprime-def dvd-antisym dvd-triv-left relprime-dvd-mult-iff*)
apply (*metis coprime-commute coprime-divprod dvd-antisym dvd-triv-right*)
done

declare *power-Suc0*[*simp del*]

end

3 The Fibonacci function

theory *Fib*
imports *Primes*
begin

Fibonacci numbers: proofs of laws taken from: R. L. Graham, D. E. Knuth, O. Patashnik. Concrete Mathematics. (Addison-Wesley, 1989)

fun *fib* :: *nat* ⇒ *nat*
where
  fib 0 = *0*
| *fib* (*Suc 0*) = *1*
| *fib-2*: *fib* (*Suc* (*Suc n*)) = *fib n* + *fib* (*Suc n*)

The difficulty in these proofs is to ensure that the induction hypotheses are applied before the definition of *fib*. Towards this end, the *fib* equations are not declared to the Simplifier and are applied very selectively at first.

We disable *fib.fib-2fib-2* for simplification ...

declare *fib-2* [*simp del*]

...then prove a version that has a more restrictive pattern.

lemma *fib-Suc3*: *fib* (*Suc* (*Suc* (*Suc n*))) = *fib* (*Suc n*) + *fib* (*Suc* (*Suc n*))
  by (*rule fib-2*)

Concrete Mathematics, page 280

lemma *fib-add*: *fib* (*Suc* (*n* + *k*)) = *fib* (*Suc k*) ∗ *fib* (*Suc n*) + *fib k* ∗ *fib n*
proof (*induct n rule*: *fib.induct*)
  case *1* **show** *?case* **by** *simp*
next
  case *2* **show** *?case* **by** (*simp add*: *fib-2*)
next
  case *3* **thus** *?case* **by** (*simp add*: *fib-2 add-mult-distrib2*)
qed

lemma *fib-Suc-neq-0*: *fib* (*Suc n*) ≠ *0*
  apply (*induct n rule*: *fib.induct*)
    apply (*simp-all add*: *fib-2*)
  done

lemma *fib-Suc-gr-0*: *0* < *fib* (*Suc n*)
  by (*insert fib-Suc-neq-0* [*of n*], *simp*)

```
lemma fib-gr-0: 0 < n ==> 0 < fib n
  by (case-tac n, auto simp add: fib-Suc-gr-0)
```

Concrete Mathematics, page 278: Cassini's identity. The proof is much easier using integers, not natural numbers!

```
lemma fib-Cassini-int:
 int (fib (Suc (Suc n)) * fib n) =
  (if n mod 2 = 0 then int (fib (Suc n) * fib (Suc n)) − 1
   else int (fib (Suc n) * fib (Suc n)) + 1)
proof(induct n rule: fib.induct)
  case 1 thus ?case by (simp add: fib-2)
next
  case 2 thus ?case by (simp add: fib-2 mod-Suc)
next
  case (3 x)
  have Suc 0 ≠ x mod 2 ⟶ x mod 2 = 0 by presburger
  with 3.hyps show ?case by (simp add: fib.simps add-mult-distrib add-mult-distrib2)
qed
```

We now obtain a version for the natural numbers via the coercion function *int*.

```
theorem fib-Cassini:
 fib (Suc (Suc n)) * fib n =
  (if n mod 2 = 0 then fib (Suc n) * fib (Suc n) − 1
   else fib (Suc n) * fib (Suc n) + 1)
  apply (rule of-nat-eq-iff [where 'a = int, THEN iffD1])
  using fib-Cassini-int apply (auto simp add: Suc-leI fib-Suc-gr-0 of-nat-diff)
  done
```

Toward Law 6.111 of Concrete Mathematics

```
lemma gcd-fib-Suc-eq-1: gcd (fib n) (fib (Suc n)) = Suc 0
  apply (induct n rule: fib.induct)
    prefer 3
    apply (simp add: gcd-commute fib-Suc3)
   apply (simp-all add: fib-2)
  done

lemma gcd-fib-add: gcd (fib m) (fib (n + m)) = gcd (fib m) (fib n)
  apply (simp add: gcd-commute [of fib m])
  apply (case-tac m)
   apply simp
  apply (simp add: fib-add)
  apply (simp add: add.commute gcd-non-0 [OF fib-Suc-gr-0])
  apply (simp add: gcd-non-0 [OF fib-Suc-gr-0, symmetric])
  apply (simp add: gcd-fib-Suc-eq-1 gcd-mult-cancel)
  done
```

lemma *gcd-fib-diff*: $m \le n$ ==> *gcd* (*fib m*) (*fib* ($n - m$)) = *gcd* (*fib m*) (*fib n*)
  by (*simp add*: *gcd-fib-add* [*symmetric*, *of* - $n-m$])

lemma *gcd-fib-mod*: $0 < m$ ==> *gcd* (*fib m*) (*fib* (*n mod m*)) = *gcd* (*fib m*) (*fib n*)
proof (*induct n rule*: *less-induct*)
  case (*less n*)
  from *less.prems* **have** *pos-m*: $0 < m$ **.**
  show *gcd* (*fib m*) (*fib* (*n mod m*)) = *gcd* (*fib m*) (*fib n*)
  proof (*cases* $m < n$)
    case *True* **note** *m-n* = *True*
    then have *m-n'*: $m \le n$ **by** *auto*
    with *pos-m* **have** *pos-n*: $0 < n$ **by** *auto*
    with *pos-m m-n* **have** *diff*: $n - m < n$ **by** *auto*
    have *gcd* (*fib m*) (*fib* (*n mod m*)) = *gcd* (*fib m*) (*fib* (($n - m$) *mod m*))
    by (*simp add*: *mod-if* [*of n*]) (*insert m-n*, *auto*)
    also have ... = *gcd* (*fib m*) (*fib* ($n - m$)) **by** (*simp add*: *less.hyps diff pos-m*)
    also have ... = *gcd* (*fib m*) (*fib n*) **by** (*simp add*: *gcd-fib-diff m-n'*)
    finally show *gcd* (*fib m*) (*fib* (*n mod m*)) = *gcd* (*fib m*) (*fib n*) **.**
  next
    case *False* **then show** *gcd* (*fib m*) (*fib* (*n mod m*)) = *gcd* (*fib m*) (*fib n*)
    by (*cases* $m = n$) *auto*
  qed
qed

lemma *fib-gcd*: *fib* (*gcd m n*) = *gcd* (*fib m*) (*fib n*) — Law 6.111
  apply (*induct m n rule*: *gcd-induct*)
  apply (*simp-all add*: *gcd-non-0 gcd-commute gcd-fib-mod*)
  done

theorem *fib-mult-eq-setsum*:
    fib (*Suc n*) $*$ *fib n* = ($\sum k \in \{..n\}$. *fib k* $*$ *fib k*)
  apply (*induct n rule*: *fib.induct*)
    apply (*auto simp add*: *atMost-Suc fib-2*)
  apply (*simp add*: *add-mult-distrib add-mult-distrib2*)
  done

end

4 Fundamental Theorem of Arithmetic (unique factorization into primes)

theory *Factorization*
imports *Primes* ~~/*src*/*HOL*/*Library*/*Permutation*
begin

4.1 Definitions

definition *primel* :: *nat list* => *bool*
  where *primel xs* = (∀ *p* ∈ *set xs*. *prime p*)

primrec *nondec* :: *nat list* => *bool*
where
  nondec [] = *True*
| *nondec* (*x* # *xs*) = (*case xs of* [] => *True* | *y* # *ys* => *x* ≤ *y* ∧ *nondec xs*)

primrec *prod* :: *nat list* => *nat*
where
  prod [] = *Suc 0*
| *prod* (*x* # *xs*) = *x* ∗ *prod xs*

primrec *oinsert* :: *nat* => *nat list* => *nat list*
where
  oinsert x [] = [*x*]
| *oinsert x* (*y* # *ys*) = (*if x* ≤ *y then x* # *y* # *ys else y* # *oinsert x ys*)

primrec *sort* :: *nat list* => *nat list*
where
  sort [] = []
| *sort* (*x* # *xs*) = *oinsert x* (*sort xs*)

4.2 Arithmetic

lemma *one-less-m*: (*m*::*nat*) ≠ *m* ∗ *k* ==> *m* ≠ *Suc 0* ==> *Suc 0* < *m*
  apply (*cases m*)
   apply *auto*
  done

lemma *one-less-k*: (*m*::*nat*) ≠ *m* ∗ *k* ==> *Suc 0* < *m* ∗ *k* ==> *Suc 0* < *k*
  apply (*cases k*)
   apply *auto*
  done

lemma *mult-left-cancel*: (*0*::*nat*) < *k* ==> *k* ∗ *n* = *k* ∗ *m* ==> *n* = *m*
  apply *auto*
  done

lemma *mn-eq-m-one*: (*0*::*nat*) < *m* ==> *m* ∗ *n* = *m* ==> *n* = *Suc 0*
  apply (*cases n*)
   apply *auto*
  done

lemma *prod-mn-less-k*:
    (*0*::*nat*) < *n* ==> *0* < *k* ==> *Suc 0* < *m* ==> *m* ∗ *n* = *k* ==> *n* < *k*
  apply (*induct m*)
   apply *auto*

```
  done
```

4.3 Prime list and product

```
lemma prod-append: prod (xs @ ys) = prod xs * prod ys
  apply (induct xs)
   apply (simp-all add: mult.assoc)
  done

lemma prod-xy-prod:
    prod (x # xs) = prod (y # ys) ==> x * prod xs = y * prod ys
  apply auto
  done

lemma primel-append: primel (xs @ ys) = (primel xs ∧ primel ys)
  apply (unfold primel-def)
  apply auto
  done

lemma prime-primel: prime n ==> primel [n] ∧ prod [n] = n
  apply (unfold primel-def)
  apply auto
  done

lemma prime-nd-one: prime p ==> ¬ p dvd Suc 0
  apply (unfold prime-def dvd-def)
  apply auto
  done

lemma hd-dvd-prod: prod (x # xs) = prod ys ==> x dvd (prod ys)
  by (metis dvd-mult-left dvd-refl prod.simps(2))

lemma primel-tl: primel (x # xs) ==> primel xs
  apply (unfold primel-def)
  apply auto
  done

lemma primel-hd-tl: (primel (x # xs)) = (prime x ∧ primel xs)
  apply (unfold primel-def)
  apply auto
  done

lemma primes-eq: prime p ==> prime q ==> p dvd q ==> p = q
  apply (unfold prime-def)
  apply auto
  done

lemma primel-one-empty: primel xs ==> prod xs = Suc 0 ==> xs = []
  apply (cases xs)
```

apply (*simp-all add*: *primel-def prime-def*)
done

lemma *prime-g-one*: *prime p* ==> *Suc 0* < *p*
apply (*unfold prime-def*)
apply *auto*
done

lemma *prime-g-zero*: *prime p* ==> *0* < *p*
apply (*unfold prime-def*)
apply *auto*
done

lemma *primel-nempty-g-one*:
primel xs $\Longrightarrow$ *xs* $\neq$ [] $\Longrightarrow$ *Suc 0* < *prod xs*
apply (*induct xs*)
apply *simp*
apply (*fastforce simp*: *primel-def prime-def elim*: *one-less-mult*)
done

lemma *primel-prod-gz*: *primel xs* ==> *0* < *prod xs*
apply (*induct xs*)
apply (*auto simp*: *primel-def prime-def*)
done

4.4 Sorting

lemma *nondec-oinsert*: *nondec xs* $\Longrightarrow$ *nondec* (*oinsert x xs*)
apply (*induct xs*)
apply *simp*
apply (*case-tac xs*)
apply (*simp-all cong del*: *list.case-cong-weak*)
done

lemma *nondec-sort*: *nondec* (*sort xs*)
apply (*induct xs*)
apply *simp-all*
apply (*erule nondec-oinsert*)
done

lemma *x-less-y-oinsert*: *x* $\leq$ *y* ==> *l* = *y* # *ys* ==> *x* # *l* = *oinsert x l*
apply *simp-all*
done

lemma *nondec-sort-eq* [*rule-format*]: *nondec xs* $\longrightarrow$ *xs* = *sort xs*
apply (*induct xs*)
apply *safe*
apply *simp-all*
apply (*case-tac xs*)

```
    apply simp-all
  apply (case-tac xs)
   apply simp
  apply (rule-tac y = aa and ys = list in x-less-y-oinsert)
   apply simp-all
  done

lemma oinsert-x-y: oinsert x (oinsert y l) = oinsert y (oinsert x l)
  apply (induct l)
  apply auto
  done
```

4.5 Permutation

```
lemma perm-primel [rule-format]: xs <~~> ys ==> primel xs --> primel ys
  apply (unfold primel-def)
  apply (induct set: perm)
     apply simp
    apply simp
   apply (simp (no-asm))
   apply blast
  apply blast
  done

lemma perm-prod: xs <~~> ys ==> prod xs = prod ys
  apply (induct set: perm)
     apply (simp-all add: ac-simps)
  done

lemma perm-subst-oinsert: xs <~~> ys ==> oinsert a xs <~~> oinsert a ys
  apply (induct set: perm)
     apply auto
  done

lemma perm-oinsert: x # xs <~~> oinsert x xs
  apply (induct xs)
   apply auto
  done

lemma perm-sort: xs <~~> sort xs
  apply (induct xs)
  apply (auto intro: perm-oinsert elim: perm-subst-oinsert)
  done

lemma perm-sort-eq: xs <~~> ys ==> sort xs = sort ys
  apply (induct set: perm)
     apply (simp-all add: oinsert-x-y)
  done
```

4.6 Existence

lemma *ex-nondec-lemma*:
  primel xs ==> ∃ ys. primel ys ∧ nondec ys ∧ prod ys = prod xs
 apply (*blast intro*: *nondec-sort perm-prod perm-primel perm-sort perm-sym*)
 done

lemma *not-prime-ex-mk*:
 Suc 0 < n ∧ ¬ prime n ⟹
  ∃ m k. Suc 0 < m ∧ Suc 0 < k ∧ m < n ∧ k < n ∧ n = m ∗ k
 apply (*unfold prime-def dvd-def*)
 apply (*auto intro*: *n-less-m-mult-n n-less-n-mult-m one-less-m one-less-k*)
 using *n-less-m-mult-n n-less-n-mult-m one-less-m one-less-k*
 apply (*metis Suc-lessD Suc-lessI mult.commute*)
 done

lemma *split-primel*:
 primel xs ⟹ primel ys ⟹ ∃ l. primel l ∧ prod l = prod xs ∗ prod ys
 apply (*rule exI*)
 apply *safe*
  apply (*rule-tac* [*2*] *prod-append*)
 apply (*simp add*: *primel-append*)
 done

lemma *factor-exists* [*rule-format*]: *Suc 0 < n −−> (∃ l. primel l ∧ prod l = n)*
 apply (*induct n rule*: *nat-less-induct*)
 apply (*rule impI*)
 apply (*case-tac prime n*)
  apply (*rule exI*)
  apply (*erule prime-primel*)
 apply (*cut-tac n = n* **in** *not-prime-ex-mk*)
  apply (*auto intro*!: *split-primel*)
 done

lemma *nondec-factor-exists*: *Suc 0 < n ==> ∃ l. primel l ∧ nondec l ∧ prod l = n*
 apply (*erule factor-exists* [*THEN exE*])
 apply (*blast intro*!: *ex-nondec-lemma*)
 done

4.7 Uniqueness

lemma *prime-dvd-mult-list* [*rule-format*]:
  prime p ==> p dvd (prod xs) −−> (∃ m. m:set xs ∧ p dvd m)
 apply (*induct xs*)
  apply (*force simp add*: *prime-def*)
  apply (*force dest*: *prime-dvd-mult*)
 done

lemma *hd-xs-dvd-prod*:

```
  primel (x # xs) ==> primel ys ==> prod (x # xs) = prod ys
    ==> ∃ m. m ∈ set ys ∧ x dvd m
  apply (rule prime-dvd-mult-list)
   apply (simp add: primel-hd-tl)
  apply (erule hd-dvd-prod)
  done

lemma prime-dvd-eq: primel (x # xs) ==> primel ys ==> m ∈ set ys ==> x
dvd m ==> x = m
  apply (rule primes-eq)
    apply (auto simp add: primel-def primel-hd-tl)
  done

lemma hd-xs-eq-prod:
  primel (x # xs) ==>
    primel ys ==> prod (x # xs) = prod ys ==> x ∈ set ys
  apply (frule hd-xs-dvd-prod)
    apply auto
  apply (drule prime-dvd-eq)
     apply auto
  done

lemma perm-primel-ex:
  primel (x # xs) ==>
    primel ys ==> prod (x # xs) = prod ys ==> ∃ l. ys <~~> (x # l)
  apply (rule exI)
  apply (rule perm-remove)
  apply (erule hd-xs-eq-prod)
   apply simp-all
  done

lemma primel-prod-less:
  primel (x # xs) ==>
    primel ys ==> prod (x # xs) = prod ys ==> prod xs < prod ys
  by (metis less-asym linorder-neqE-nat mult-less-cancel2 nat-0-less-mult-iff
    nat-less-le nat-mult-1 prime-def primel-hd-tl primel-prod-gz prod.simps(2))

lemma prod-one-empty:
    primel xs ==> p * prod xs = p ==> prime p ==> xs = []
  apply (auto intro: primel-one-empty simp add: prime-def)
  done

lemma uniq-ex-aux:
  ∀ m. m < prod ys --> (∀ xs ys. primel xs ∧ primel ys ∧
      prod xs = prod ys ∧ prod xs = m --> xs <~~> ys) ==>
    primel list ==> primel x ==> prod list = prod x ==> prod x < prod ys
    ==> x <~~> list
  apply simp
  done
```

lemma *factor-unique* [*rule-format*]:
$\forall$ *xs ys*. *primel xs* $\wedge$ *primel ys* $\wedge$ *prod xs* = *prod ys* $\wedge$ *prod xs* = *n*
−−> *xs* <~~> *ys*
apply (*induct n rule*: *nat-less-induct*)
apply *safe*
apply (*case-tac xs*)
apply (*force intro*: *primel-one-empty*)
apply (*rule perm-primel-ex* [*THEN exE*])
apply *simp-all*
apply (*rule perm.trans* [*THEN perm-sym*])
apply *assumption*
apply (*rule perm.Cons*)
apply (*case-tac x* = [])
apply (*metis perm-prod perm-refl prime-primel primel-hd-tl primel-tl prod-one-empty*)
apply (*metis nat-0-less-mult-iff nat-mult-eq-cancel1 perm-primel perm-prod primel-prod-gz primel-prod-less primel-tl prod.simps*(*2*))
done

lemma *perm-nondec-unique*:
xs <~~> *ys* ==> *nondec xs* ==> *nondec ys* ==> *xs* = *ys*
by (*metis nondec-sort-eq perm-sort-eq*)

theorem *unique-prime-factorization* [*rule-format*]:
$\forall$ *n*. *Suc 0* < *n* −−> ($\exists$!*l*. *primel l* $\wedge$ *nondec l* $\wedge$ *prod l* = *n*)
by (*metis factor-unique nondec-factor-exists perm-nondec-unique*)

end

5 Divisibility and prime numbers (on integers)

theory *IntPrimes*
imports *Primes*
begin

The *dvd* relation, GCD, Euclid's extended algorithm, primes, congruences (all on the Integers). Comparable to theory *Primes*, but *dvd* is included here as it is not present in main HOL. Also includes extended GCD and congruences not present in *Primes*.

5.1 Definitions

fun *xzgcda* :: *int* $\Rightarrow$ *int* $\Rightarrow$ *int* $\Rightarrow$ *int* $\Rightarrow$ *int* $\Rightarrow$ *int* $\Rightarrow$ *int* $\Rightarrow$ *int* => (*int* $*$ *int* $*$ *int*)
where
xzgcda m n r' r s' s t' t =
(*if r* $\leq$ *0 then* (*r'*, *s'*, *t'*)
else xzgcda m n r (*r' mod r*)

s (*s′* − (*r′ div r*) ∗ *s*)
t (*t′* − (*r′ div r*) ∗ *t*))

definition *zprime* :: *int* ⇒ *bool*
where *zprime p* = (*1* < *p* ∧ (∀ *m*. *0* <= *m* & *m dvd p* −−> *m* = *1* ∨ *m* = *p*))

definition *xzgcd* :: *int* => *int* => *int* ∗ *int* ∗ *int*
where *xzgcd m n* = *xzgcda m n m n 1 0 0 1*

definition *zcong* :: *int* => *int* => *int* => *bool* ((*1*[- = -] ′(*mod* -′)))
where [*a* = *b*] (*mod m*) = (*m dvd* (*a* − *b*))

5.2 Euclid's Algorithm and GCD

lemma *zrelprime-zdvd-zmult-aux*:
zgcd n k = *1* ==> *k dvd m* ∗ *n* ==> *0* ≤ *m* ==> *k dvd m*
by (*metis abs-of-nonneg dvd-triv-right zgcd-greatest-iff zgcd-zmult-distrib2-abs mult-1-right*)

lemma *zrelprime-zdvd-zmult*: *zgcd n k* = *1* ==> *k dvd m* ∗ *n* ==> *k dvd m*
apply (*case-tac 0* ≤ *m*)
apply (*blast intro*: *zrelprime-zdvd-zmult-aux*)
apply (*subgoal-tac k dvd* −*m*)
apply (*rule-tac* [*2*] *zrelprime-zdvd-zmult-aux*, *auto*)
done

lemma *zgcd-geq-zero*: *0* <= *zgcd x y*
by (*auto simp add*: *zgcd-def*)

This is merely a sanity check on zprime, since the previous version denoted the empty set.

lemma *zprime 2*
apply (*auto simp add*: *zprime-def*)
apply (*frule zdvd-imp-le*, *simp*)
apply (*auto simp add*: *order-le-less dvd-def*)
done

lemma *zprime-imp-zrelprime*:
zprime p ==> ¬ *p dvd n* ==> *zgcd n p* = *1*
apply (*auto simp add*: *zprime-def*)
apply (*metis zgcd-geq-zero zgcd-zdvd1 zgcd-zdvd2*)
done

lemma *zless-zprime-imp-zrelprime*:
zprime p ==> *0* < *n* ==> *n* < *p* ==> *zgcd n p* = *1*
apply (*erule zprime-imp-zrelprime*)
apply (*erule zdvd-not-zless*, *assumption*)
done

lemma *zprime-zdvd-zmult*:
  $0 \le (m::int) ==> zprime\ p ==> p\ dvd\ m * n ==> p\ dvd\ m \lor p\ dvd\ n$
 by (*metis zgcd-zdvd1 zgcd-zdvd2 zgcd-pos zprime-def zrelprime-dvd-mult*)

lemma *zgcd-zadd-zmult* [*simp*]: $zgcd\ (m + n * k)\ n = zgcd\ m\ n$
 apply (*rule zgcd-eq* [*THEN trans*])
 apply (*simp add*: *mod-add-eq*)
 apply (*rule zgcd-eq* [*symmetric*])
 done

lemma *zgcd-zdvd-zgcd-zmult*: $zgcd\ m\ n\ dvd\ zgcd\ (k * m)\ n$
by (*simp add*: *zgcd-greatest-iff*)

lemma *zgcd-zmult-zdvd-zgcd*:
  $zgcd\ k\ n = 1 ==> zgcd\ (k * m)\ n\ dvd\ zgcd\ m\ n$
 apply (*simp add*: *zgcd-greatest-iff*)
 apply (*rule-tac* $n = k$ **in** *zrelprime-zdvd-zmult*)
  prefer *2*
  apply (*simp add*: *mult.commute*)
 apply (*metis zgcd-1 zgcd-commute zgcd-left-commute*)
 done

lemma *zgcd-zmult-cancel*: $zgcd\ k\ n = 1 ==> zgcd\ (k * m)\ n = zgcd\ m\ n$
 by (*simp add*: *zgcd-def nat-abs-mult-distrib gcd-mult-cancel*)

lemma *zgcd-zgcd-zmult*:
  $zgcd\ k\ m = 1 ==> zgcd\ n\ m = 1 ==> zgcd\ (k * n)\ m = 1$
 by (*simp add*: *zgcd-zmult-cancel*)

lemma *zdvd-iff-zgcd*: $0 < m ==> m\ dvd\ n \longleftrightarrow zgcd\ n\ m = m$
 by (*metis abs-of-pos dvd-mult-div-cancel zgcd-0 zgcd-commute zgcd-geq-zero zgcd-zdvd2 zgcd-zmult-eq-self*)

5.3 Congruences

lemma *zcong-1* [*simp*]: $[a = b]\ (mod\ 1)$
 by (*unfold zcong-def*, *auto*)

lemma *zcong-refl* [*simp*]: $[k = k]\ (mod\ m)$
 by (*unfold zcong-def*, *auto*)

lemma *zcong-sym*: $[a = b]\ (mod\ m) = [b = a]\ (mod\ m)$
 unfolding *zcong-def minus-diff-eq* [*of a*, *symmetric*] *dvd-minus-iff* **..**

lemma *zcong-zadd*:
  $[a = b]\ (mod\ m) ==> [c = d]\ (mod\ m) ==> [a + c = b + d]\ (mod\ m)$
 apply (*unfold zcong-def*)
 apply (*rule-tac* $s = (a - b) + (c - d)$ **in** *subst*)
  apply (*rule-tac* [*2*] *dvd-add*, *auto*)

```
  done

lemma zcong-zdiff:
    [a = b] (mod m) ==> [c = d] (mod m) ==> [a − c = b − d] (mod m)
  apply (unfold zcong-def)
  apply (rule-tac s = (a − b) − (c − d) in subst)
   apply (rule-tac [2] dvd-diff, auto)
  done

lemma zcong-trans:
  [a = b] (mod m) ==> [b = c] (mod m) ==> [a = c] (mod m)
unfolding zcong-def by (auto elim!: dvdE simp add: algebra-simps)

lemma zcong-zmult:
    [a = b] (mod m) ==> [c = d] (mod m) ==> [a ∗ c = b ∗ d] (mod m)
  apply (rule-tac b = b ∗ c in zcong-trans)
   apply (unfold zcong-def)
  apply (metis right-diff-distrib dvd-mult mult.commute)
  apply (metis right-diff-distrib dvd-mult)
  done

lemma zcong-scalar: [a = b] (mod m) ==> [a ∗ k = b ∗ k] (mod m)
  by (rule zcong-zmult, simp-all)

lemma zcong-scalar2: [a = b] (mod m) ==> [k ∗ a = k ∗ b] (mod m)
  by (rule zcong-zmult, simp-all)

lemma zcong-zmult-self: [a ∗ m = b ∗ m] (mod m)
  apply (unfold zcong-def)
  apply (rule dvd-diff, simp-all)
  done

lemma zcong-square:
   [| zprime p;  0 < a;  [a ∗ a = 1] (mod p)|]
    ==> [a = 1] (mod p) ∨ [a = p − 1] (mod p)
  apply (unfold zcong-def)
  apply (rule zprime-zdvd-zmult)
    apply (rule-tac [3] s = a ∗ a − 1 + p ∗ (1 − a) in subst)
     prefer 4
     apply (simp add: zdvd-reduce)
    apply (simp-all add: left-diff-distrib mult.commute right-diff-distrib)
  done

lemma zcong-cancel:
  0 ≤ m ==>
    zgcd k m = 1 ==> [a ∗ k = b ∗ k] (mod m) = [a = b] (mod m)
  apply safe
   prefer 2
   apply (blast intro: zcong-scalar)
```

apply (*case-tac b < a*)
prefer *2*
apply (*subst zcong-sym*)
apply (*unfold zcong-def*)
apply (*rule-tac* [!] *zrelprime-zdvd-zmult*)
apply (*simp-all add*: *left-diff-distrib*)
apply (*subgoal-tac m dvd* (−(*a* ∗ *k* − *b* ∗ *k*)))
apply *simp*
apply (*subst dvd-minus-iff*, *assumption*)
done

lemma *zcong-cancel2*:
0 ≤ *m* ==>
zgcd k m = *1* ==> [*k* ∗ *a* = *k* ∗ *b*] (*mod m*) = [*a* = *b*] (*mod m*)
by (*simp add*: *mult.commute zcong-cancel*)

lemma *zcong-zgcd-zmult-zmod*:
[*a* = *b*] (*mod m*) ==> [*a* = *b*] (*mod n*) ==> *zgcd m n* = *1*
==> [*a* = *b*] (*mod m* ∗ *n*)
apply (*auto simp add*: *zcong-def dvd-def*)
apply (*subgoal-tac m dvd n* ∗ *ka*)
apply (*subgoal-tac m dvd ka*)
apply (*case-tac* [*2*] *0* ≤ *ka*)
apply (*metis dvd-mult-div-cancel dvd-refl dvd-mult-left mult.commute zrelprime-zdvd-zmult*)
apply (*metis abs-dvd-iff abs-of-nonneg add-0 zgcd-0-left zgcd-commute zgcd-zadd-zmult zgcd-zdvd-zgcd-zmult zgcd-zmult-distrib2-abs mult-1-right mult.commute*)
apply (*metis mult-le-0-iff zdvd-mono zdvd-mult-cancel dvd-triv-left zero-le-mult-iff order-antisym linorder-linear order-refl mult.commute zrelprime-zdvd-zmult*)
apply (*metis dvd-triv-left*)
done

lemma *zcong-zless-imp-eq*:
0 ≤ *a* ==>
a < *m* ==> *0* ≤ *b* ==> *b* < *m* ==> [*a* = *b*] (*mod m*) ==> *a* = *b*
apply (*unfold zcong-def dvd-def*, *auto*)
apply (*drule-tac f* = *λz. z mod m* **in** *arg-cong*)
apply (*metis diff-add-cancel mod-pos-pos-trivial add-0 add.commute zmod-eq-0-iff mod-add-right-eq*)
done

lemma *zcong-square-zless*:
zprime p ==> *0* < *a* ==> *a* < *p* ==>
[*a* ∗ *a* = *1*] (*mod p*) ==> *a* = *1* ∨ *a* = *p* − *1*
apply (*cut-tac p* = *p* **and** *a* = *a* **in** *zcong-square*)
apply (*simp add*: *zprime-def*)
apply (*auto intro*: *zcong-zless-imp-eq*)
done

lemma *zcong-not*:

```
  0 < a ==> a < m ==> 0 < b ==> b < a ==> ¬ [a = b] (mod m)
 apply (unfold zcong-def)
 apply (rule zdvd-not-zless, auto)
 done

lemma zcong-zless-0:
  0 ≤ a ==> a < m ==> [a = 0] (mod m) ==> a = 0
 apply (unfold zcong-def dvd-def, auto)
 apply (metis div-pos-pos-trivial linorder-not-less div-mult-self1-is-id)
 done

lemma zcong-zless-unique:
  0 < m ==> (∃!b. 0 ≤ b ∧ b < m ∧ [a = b] (mod m))
 apply auto
  prefer 2 apply (metis zcong-sym zcong-trans zcong-zless-imp-eq)
 apply (unfold zcong-def dvd-def)
 apply (rule-tac x = a mod m in exI, auto)
 apply (metis zmult-div-cancel)
 done

lemma zcong-iff-lin: ([a = b] (mod m)) = (∃k. b = a + m * k)
 unfolding zcong-def
 apply (auto elim!: dvdE simp add: algebra-simps)
 apply (rule-tac x = -k in exI) apply simp
 done

lemma zgcd-zcong-zgcd:
 0 < m ==>
  zgcd a m = 1 ==> [a = b] (mod m) ==> zgcd b m = 1
 by (auto simp add: zcong-iff-lin)

lemma zcong-zmod-aux:
   a - b = (m::int) * (a div m - b div m) + (a mod m - b mod m)
 by(simp add: right-diff-distrib add-diff-eq eq-diff-eq ac-simps)

lemma zcong-zmod: [a = b] (mod m) = [a mod m = b mod m] (mod m)
 apply (unfold zcong-def)
 apply (rule-tac t = a - b in ssubst)
 apply (rule-tac m = m in zcong-zmod-aux)
 apply (rule trans)
  apply (rule-tac [2] k = m and m = a div m - b div m in zdvd-reduce)
 apply (simp add: add.commute)
 done

lemma zcong-zmod-eq: 0 < m ==> [a = b] (mod m) = (a mod m = b mod m)
 apply auto
 apply (metis pos-mod-conj zcong-zless-imp-eq zcong-zmod)
 apply (metis zcong-refl zcong-zmod)
 done
```

lemma *zcong-zminus* [*iff*]: $[a = b]$ $(mod\ -m) = [a = b]$ $(mod\ m)$
by (*auto simp add*: *zcong-def*)

lemma *zcong-zero* [*iff*]: $[a = b]$ $(mod\ 0) = (a = b)$
by (*auto simp add*: *zcong-def*)

lemma $[a = b]$ $(mod\ m) = (a\ mod\ m = b\ mod\ m)$
apply (*cases* $m = 0$, *simp*)
apply (*simp add*: *linorder-neq-iff*)
apply (*erule disjE*)
prefer *2* **apply** (*simp add*: *zcong-zmod-eq*)

Remainding case: $m < 0$

apply (*rule-tac* $t = m$ **in** *minus-minus* [*THEN subst*])
apply (*subst zcong-zminus*)
apply (*subst zcong-zmod-eq*, *arith*)
apply (*frule neg-mod-bound* [*of - a*], *frule neg-mod-bound* [*of - b*])
apply (*simp add*: *zmod-zminus2-eq-if del*: *neg-mod-bound*)
done

5.4 Modulo

lemma *zmod-zdvd-zmod*:
$0 < (m::int) ==> m\ dvd\ b ==> (a\ mod\ b\ mod\ m) = (a\ mod\ m)$
by (*rule mod-mod-cancel*)

5.5 Extended GCD

declare *xzgcda.simps* [*simp del*]

lemma *xzgcd-correct-aux1*:
$zgcd\ r'\ r = k\ --> 0 < r\ -->$
$(\exists\, sn\ tn.\ xzgcda\ m\ n\ r'\ r\ s'\ s\ t'\ t = (k,\ sn,\ tn))$
apply (*induct* $m\ n\ r'\ r\ s'\ s\ t'\ t$ *rule*: *xzgcda.induct*)
apply (*subst zgcd-eq*)
apply (*subst xzgcda.simps*, *auto*)
apply (*case-tac* $r'\ mod\ r = 0$)
prefer *2*
apply (*frule-tac* $a = r'$ **in** *pos-mod-sign*, *auto*)
apply (*rule exI*)
apply (*rule exI*)
apply (*subst xzgcda.simps*, *auto*)
done

lemma *xzgcd-correct-aux2*:
$(\exists\, sn\ tn.\ xzgcda\ m\ n\ r'\ r\ s'\ s\ t'\ t = (k,\ sn,\ tn))\ --> 0 < r\ -->$
$zgcd\ r'\ r = k$
apply (*induct* $m\ n\ r'\ r\ s'\ s\ t'\ t$ *rule*: *xzgcda.induct*)
apply (*subst zgcd-eq*)

```
  apply (subst xzgcda.simps)
  apply (auto simp add: linorder-not-le)
  apply (case-tac r′ mod r = 0)
   prefer 2
   apply (frule-tac a = r′ in pos-mod-sign, auto)
  apply (metis prod.inject xzgcda.simps order-refl)
  done

lemma xzgcd-correct:
    0 < n ==> (zgcd m n = k) = (∃ s t. xzgcd m n = (k, s, t))
  apply (unfold xzgcd-def)
  apply (rule iffI)
   apply (rule-tac [2] xzgcd-correct-aux2 [THEN mp, THEN mp])
    apply (rule xzgcd-correct-aux1 [THEN mp, THEN mp], auto)
  done
```

xzgcd linear

```
lemma xzgcda-linear-aux1:
  (a − r ∗ b) ∗ m + (c − r ∗ d) ∗ (n::int) =
   (a ∗ m + c ∗ n) − r ∗ (b ∗ m + d ∗ n)
  by (simp add: left-diff-distrib distrib-left mult.assoc)

lemma xzgcda-linear-aux2:
  r′ = s′ ∗ m + t′ ∗ n ==> r = s ∗ m + t ∗ n
    ==> (r′ mod r) = (s′ − (r′ div r) ∗ s) ∗ m + (t′ − (r′ div r) ∗ t) ∗ (n::int)
  apply (rule trans)
   apply (rule-tac [2] xzgcda-linear-aux1 [symmetric])
  apply (simp add: eq-diff-eq mult.commute)
  done

lemma order-le-neq-implies-less: (x::′a::order) ≤ y ==> x ≠ y ==> x < y
  by (rule iffD2 [OF order-less-le conjI])

lemma xzgcda-linear [rule-format]:
  0 < r −−> xzgcda m n r′ r s′ s t′ t = (rn, sn, tn) −−>
    r′ = s′ ∗ m + t′ ∗ n −−>  r = s ∗ m + t ∗ n −−> rn = sn ∗ m + tn ∗ n
  apply (induct m n r′ r s′ s t′ t rule: xzgcda.induct)
  apply (subst xzgcda.simps)
  apply (simp (no-asm))
  apply (rule impI)+
  apply (case-tac r′ mod r = 0)
   apply (simp add: xzgcda.simps, clarify)
  apply (subgoal-tac 0 < r′ mod r)
   apply (rule-tac [2] order-le-neq-implies-less)
   apply (rule-tac [2] pos-mod-sign)
    apply (cut-tac m = m and n = n and r′ = r′ and r = r and s′ = s′ and
      s = s and t′ = t′ and t = t in xzgcda-linear-aux2, auto)
  done
```

lemma *xzgcd-linear*:
  $0 < n ==> xzgcd\ m\ n = (r,\ s,\ t) ==> r = s * m + t * n$
 apply (*unfold xzgcd-def*)
 apply (*erule xzgcda-linear*, *assumption*, *auto*)
 done

lemma *zgcd-ex-linear*:
  $0 < n ==> zgcd\ m\ n = k ==> (\exists\, s\ t.\ k = s * m + t * n)$
 apply (*simp add*: *xzgcd-correct*, *safe*)
 apply (*rule exI*)+
 apply (*erule xzgcd-linear*, *auto*)
 done

lemma *zcong-lineq-ex*:
  $0 < n ==> zgcd\ a\ n = 1 ==> \exists\, x.\ [a * x = 1]\ (mod\ n)$
 apply (*cut-tac m = a* **and** *n = n* **and** *k = 1* **in** *zgcd-ex-linear*, *safe*)
 apply (*rule-tac x = s* **in** *exI*)
 apply (*rule-tac b = s * a + t * n* **in** *zcong-trans*)
  prefer *2*
  apply *simp*
 apply (*unfold zcong-def*)
 apply (*simp* (*no-asm*) *add*: *mult.commute*)
 done

lemma *zcong-lineq-unique*:
 $0 < n ==>$
  $zgcd\ a\ n = 1 ==> \exists!x.\ 0 \le x \wedge x < n \wedge [a * x = b]\ (mod\ n)$
 apply *auto*
  apply (*rule-tac* [*2*] *zcong-zless-imp-eq*)
      apply (*tactic* ‹*stac* @{*context*} (@{*thm zcong-cancel2*} *RS sym*) *6*›)
        apply (*rule-tac* [*8*] *zcong-trans*)
         apply (*simp-all* (*no-asm-simp*))
  prefer *2*
  apply (*simp add*: *zcong-sym*)
 apply (*cut-tac a = a* **and** *n = n* **in** *zcong-lineq-ex*, *auto*)
 apply (*rule-tac x = x * b mod n* **in** *exI*, *safe*)
   apply (*simp-all* (*no-asm-simp*))
 apply (*metis zcong-scalar zcong-zmod mod-mult-right-eq mult-1 mult.assoc*)
 done

end

6 The Chinese Remainder Theorem

theory *Chinese*
imports *IntPrimes*
begin

The Chinese Remainder Theorem for an arbitrary finite number of equations. (The one-equation case is included in theory *IntPrimes*. Uses functions for indexing.[1]

6.1 Definitions

primrec *funprod* :: (*nat* => *int*) => *nat* => *nat* => *int*
where
  funprod f i 0 = *f i*
| *funprod f i* (*Suc n*) = *f* (*Suc* (*i* + *n*)) * *funprod f i n*

primrec *funsum* :: (*nat* => *int*) => *nat* => *nat* => *int*
where
  funsum f i 0 = *f i*
| *funsum f i* (*Suc n*) = *f* (*Suc* (*i* + *n*)) + *funsum f i n*

definition
  m-cond :: *nat* => (*nat* => *int*) => *bool* **where**
  m-cond n mf =
    ((∀ *i*. *i* ≤ *n* −−> *0* < *mf i*) ∧
      (∀ *i j*. *i* ≤ *n* ∧ *j* ≤ *n* ∧ *i* ≠ *j* −−> *zgcd* (*mf i*) (*mf j*) = *1*))

definition
  km-cond :: *nat* => (*nat* => *int*) => (*nat* => *int*) => *bool* **where**
  km-cond n kf mf = (∀ *i*. *i* ≤ *n* −−> *zgcd* (*kf i*) (*mf i*) = *1*)

definition
  lincong-sol ::
    nat => (*nat* => *int*) => (*nat* => *int*) => (*nat* => *int*) => *int* => *bool*
where
  lincong-sol n kf bf mf x = (∀ *i*. *i* ≤ *n* −−> *zcong* (*kf i* * *x*) (*bf i*) (*mf i*))

definition
  mhf :: (*nat* => *int*) => *nat* => *nat* => *int* **where**
  mhf mf n i =
    (*if i* = *0 then funprod mf* (*Suc 0*) (*n* − *Suc 0*)
     else if i = *n then funprod mf 0* (*n* − *Suc 0*)
     else funprod mf 0 (*i* − *Suc 0*) * *funprod mf* (*Suc i*) (*n* − *Suc 0* − *i*))

definition
  xilin-sol ::
    nat => *nat* => (*nat* => *int*) => (*nat* => *int*) => (*nat* => *int*) => *int*
where
  xilin-sol i n kf bf mf =
    (*if 0* < *n* ∧ *i* ≤ *n* ∧ *m-cond n mf* ∧ *km-cond n kf mf then*
      (*SOME x*. *0* ≤ *x* ∧ *x* < *mf i* ∧ *zcong* (*kf i* * *mhf mf n i* * *x*) (*bf i*) (*mf i*))
     else 0)

[1] Maybe *funprod* and *funsum* should be based on general *fold* on indices?

definition
x-sol :: *nat* => (*nat* => *int*) => (*nat* => *int*) => (*nat* => *int*) => *int* **where**
x-sol n kf bf mf = *funsum* (λ*i*. *xilin-sol i n kf bf mf* ∗ *mhf mf n i*) *0 n*

funprod and *funsum*

lemma *funprod-pos*: (∀ *i*. *i* ≤ *n* −−> *0* < *mf i*) ==> *0* < *funprod mf 0 n*
by (*induct n*) *auto*

lemma *funprod-zgcd* [*rule-format* (*no-asm*)]:
(∀ *i*. *k* ≤ *i* ∧ *i* ≤ *k* + *l* −−> *zgcd* (*mf i*) (*mf m*) = *1*) −−>
zgcd (*funprod mf k l*) (*mf m*) = *1*
apply (*induct l*)
apply *simp-all*
apply (*rule impI*)+
apply (*subst zgcd-zmult-cancel*)
apply *auto*
done

lemma *funprod-zdvd* [*rule-format*]:
k ≤ *i* −−> *i* ≤ *k* + *l* −−> *mf i dvd funprod mf k l*
apply (*induct l*)
apply *auto*
apply (*subgoal-tac i* = *Suc* (*k* + *l*))
apply (*simp-all* (*no-asm-simp*))
done

lemma *funsum-mod*:
funsum f k l mod m = *funsum* (λ*i*. (*f i*) *mod m*) *k l mod m*
apply (*induct l*)
apply *auto*
apply (*rule trans*)
apply (*rule mod-add-eq*)
apply *simp*
apply (*rule mod-add-right-eq* [*symmetric*])
done

lemma *funsum-zero* [*rule-format* (*no-asm*)]:
(∀ *i*. *k* ≤ *i* ∧ *i* ≤ *k* + *l* −−> *f i* = *0*) −−> (*funsum f k l*) = *0*
apply (*induct l*)
apply *auto*
done

lemma *funsum-oneelem* [*rule-format* (*no-asm*)]:
k ≤ *j* −−> *j* ≤ *k* + *l* −−>
(∀ *i*. *k* ≤ *i* ∧ *i* ≤ *k* + *l* ∧ *i* ≠ *j* −−> *f i* = *0*) −−>
funsum f k l = *f j*
apply (*induct l*)
prefer *2*
apply *clarify*

```
  defer
  apply clarify
  apply (subgoal-tac k = j)
   apply (simp-all (no-asm-simp))
 apply (case-tac Suc (k + l) = j)
  apply (subgoal-tac funsum f k l = 0)
   apply (rule-tac [2] funsum-zero)
   apply (subgoal-tac [3] f (Suc (k + l)) = 0)
    apply (subgoal-tac [3] j ≤ k + l)
     prefer 4
     apply arith
    apply auto
 done
```

6.2 Chinese: uniqueness

```
lemma zcong-funprod-aux:
 m-cond n mf ==> km-cond n kf mf
   ==> lincong-sol n kf bf mf x ==> lincong-sol n kf bf mf y
   ==> [x = y] (mod mf n)
 apply (unfold m-cond-def km-cond-def lincong-sol-def)
 apply (rule iffD1)
  apply (rule-tac k = kf n in zcong-cancel2)
   apply (rule-tac [3] b = bf n in zcong-trans)
    prefer 4
    apply (subst zcong-sym)
    defer
    apply (rule order-less-imp-le)
    apply simp-all
 done

lemma zcong-funprod [rule-format]:
 m-cond n mf --> km-cond n kf mf -->
   lincong-sol n kf bf mf x --> lincong-sol n kf bf mf y -->
   [x = y] (mod funprod mf 0 n)
 apply (induct n)
  apply (simp-all (no-asm))
  apply (blast intro: zcong-funprod-aux)
 apply (rule impI)+
 apply (rule zcong-zgcd-zmult-zmod)
   apply (blast intro: zcong-funprod-aux)
   prefer 2
   apply (subst zgcd-commute)
   apply (rule funprod-zgcd)
  apply (auto simp add: m-cond-def km-cond-def lincong-sol-def)
 done
```

6.3 Chinese: existence

```
lemma unique-xi-sol:
```

```
  0 < n ==> i ≤ n ==> m-cond n mf ==> km-cond n kf mf
    ==> ∃!x. 0 ≤ x ∧ x < mf i ∧ [kf i * mhf mf n i * x = bf i] (mod mf i)
  apply (rule zcong-lineq-unique)
   apply (tactic ‹stac @{context} @{thm zgcd-zmult-cancel} 2›)
    apply (unfold m-cond-def km-cond-def mhf-def)
    apply (simp-all (no-asm-simp))
  apply safe
    apply (tactic ‹stac @{context} @{thm zgcd-zmult-cancel} 3›)
     apply (rule-tac [!] funprod-zgcd)
     apply safe
     apply simp-all
   apply (subgoal-tac i<n)
    prefer 2
    apply arith
   apply (case-tac [2] i)
    apply simp-all
  done

lemma x-sol-lin-aux:
    0 < n ==> i ≤ n ==> j ≤ n ==> j ≠ i ==> mf j dvd mhf mf n i
  apply (unfold mhf-def)
  apply (case-tac i = 0)
   apply (case-tac [2] i = n)
    apply (simp-all (no-asm-simp))
    apply (case-tac [3] j < i)
     apply (rule-tac [3] dvd-mult2)
     apply (rule-tac [4] dvd-mult)
     apply (rule-tac [!] funprod-zdvd)
     apply arith
     apply arith
     apply arith
     apply arith
     apply arith
     apply arith
     apply arith
     apply arith
  done

lemma x-sol-lin:
  0 < n ==> i ≤ n
    ==> x-sol n kf bf mf mod mf i =
      xilin-sol i n kf bf mf * mhf mf n i mod mf i
  apply (unfold x-sol-def)
  apply (subst funsum-mod)
  apply (subst funsum-oneelem)
     apply auto
  apply (subst dvd-eq-mod-eq-0 [symmetric])
  apply (rule dvd-mult)
  apply (rule x-sol-lin-aux)
```

apply *auto*
done

6.4 Chinese

lemma *chinese-remainder*:
0 < n ==> m-cond n mf ==> km-cond n kf mf
==> $\exists!x$. *0* $\leq$ *x* $\wedge$ *x < funprod mf 0 n* $\wedge$ *lincong-sol n kf bf mf x*
apply *safe*
apply (*rule-tac* [*2*] *m = funprod mf 0 n* **in** *zcong-zless-imp-eq*)
apply (*rule-tac* [*6*] *zcong-funprod*)
apply *auto*
apply (*rule-tac x = x-sol n kf bf mf mod funprod mf 0 n* **in** *exI*)
apply (*unfold lincong-sol-def*)
apply *safe*
apply (*tactic* ‹*stac* @{*context*} @{*thm zcong-zmod*} *3*›)
apply (*tactic* ‹*stac* @{*context*} @{*thm mod-mult-eq*} *3*›)
apply (*tactic* ‹*stac* @{*context*} @{*thm mod-mod-cancel*} *3*›)
apply (*tactic* ‹*stac* @{*context*} @{*thm x-sol-lin*} *4*›)
apply (*tactic* ‹*stac* @{*context*} (@{*thm mod-mult-eq*} *RS sym*) *6*›)
apply (*tactic* ‹*stac* @{*context*} (@{*thm zcong-zmod*} *RS sym*) *6*›)
apply (*subgoal-tac* [*6*]
0 $\leq$ *xilin-sol i n kf bf mf* $\wedge$ *xilin-sol i n kf bf mf < mf i*
$\wedge$ [*kf i* $*$ *mhf mf n i* $*$ *xilin-sol i n kf bf mf = bf i*] (*mod mf i*))
prefer *6*
apply (*simp add*: *ac-simps*)
apply (*unfold xilin-sol-def*)
apply (*tactic* ‹*asm-simp-tac* @{*context*} *6*›)
apply (*rule-tac* [*6*] *ex1-implies-ex* [*THEN someI-ex*])
apply (*rule-tac* [*6*] *unique-xi-sol*)
apply (*rule-tac* [*3*] *funprod-zdvd*)
apply (*unfold m-cond-def*)
apply (*rule funprod-pos* [*THEN pos-mod-sign*])
apply (*rule-tac* [*2*] *funprod-pos* [*THEN pos-mod-bound*])
apply *auto*
done

end

7 Bijections between sets

theory *BijectionRel*
imports *Main*
begin

Inductive definitions of bijections between two different sets and between the same set. Theorem for relating the two definitions.

inductive-set

bijR :: (*'a* => *'b* => *bool*) => (*'a set* ∗ *'b set*) *set*
for *P* :: *'a* => *'b* => *bool*
where
empty [*simp*]: ({}, {}) ∈ *bijR P*
| *insert*: *P a b* ==> *a* ∉ *A* ==> *b* ∉ *B* ==> (*A*, *B*) ∈ *bijR P*
==> (*insert a A*, *insert b B*) ∈ *bijR P*

Add extra condition to *insert*: ∀ *b*∈*B*. ¬ *P a b* (and similar for *A*).

definition
bijP :: (*'a* => *'a* => *bool*) => *'a set* => *bool* **where**
bijP P F = (∀ *a b*. *a* ∈ *F* ∧ *P a b* −−> *b* ∈ *F*)

definition
uniqP :: (*'a* => *'a* => *bool*) => *bool* **where**
uniqP P = (∀ *a b c d*. *P a b* ∧ *P c d* −−> (*a* = *c*) = (*b* = *d*))

definition
symP :: (*'a* => *'a* => *bool*) => *bool* **where**
symP P = (∀ *a b*. *P a b* = *P b a*)

inductive-set
bijER :: (*'a* => *'a* => *bool*) => *'a set set*
for *P* :: *'a* => *'a* => *bool*
where
empty [*simp*]: {} ∈ *bijER P*
| *insert1*: *P a a* ==> *a* ∉ *A* ==> *A* ∈ *bijER P* ==> *insert a A* ∈ *bijER P*
| *insert2*: *P a b* ==> *a* ≠ *b* ==> *a* ∉ *A* ==> *b* ∉ *A* ==> *A* ∈ *bijER P*
==> *insert a* (*insert b A*) ∈ *bijER P*

bijR

lemma *fin-bijRl*: (*A*, *B*) ∈ *bijR P* ==> *finite A*
apply (*erule bijR.induct*)
apply *auto*
done

lemma *fin-bijRr*: (*A*, *B*) ∈ *bijR P* ==> *finite B*
apply (*erule bijR.induct*)
apply *auto*
done

lemma *aux-induct*:
assumes *major*: *finite F*
and *subs*: *F* ⊆ *A*
and *cases*: *P* {}
!!*F a*. *F* ⊆ *A* ==> *a* ∈ *A* ==> *a* ∉ *F* ==> *P F* ==> *P* (*insert a F*)
shows *P F*
using *major subs*
apply (*induct set*: *finite*)
apply (*blast intro*: *cases*)+

done

lemma *inj-func-bijR-aux1*:
A ⊆ *B* ==> *a* ∉ *A* ==> *a* ∈ *B* ==> *inj-on f B* ==> *f a* ∉ *f* ‘ *A*
apply (*unfold inj-on-def*)
apply *auto*
done

lemma *inj-func-bijR-aux2*:
∀ *a*. *a* ∈ *A* −−> *P a* (*f a*) ==> *inj-on f A* ==> *finite A* ==> *F* <= *A*
==> (*F*, *f* ‘ *F*) ∈ *bijR P*
apply (*rule-tac F* = *F* **and** *A* = *A* **in** *aux-induct*)
apply (*rule finite-subset*)
apply *auto*
apply (*rule bijR.insert*)
apply (*rule-tac* [*3*] *inj-func-bijR-aux1*)
apply *auto*
done

lemma *inj-func-bijR*:
∀ *a*. *a* ∈ *A* −−> *P a* (*f a*) ==> *inj-on f A* ==> *finite A*
==> (*A*, *f* ‘ *A*) ∈ *bijR P*
apply (*rule inj-func-bijR-aux2*)
apply *auto*
done

bijER

lemma *fin-bijER*: *A* ∈ *bijER P* ==> *finite A*
apply (*erule bijER.induct*)
apply *auto*
done

lemma *aux1*:
a ∉ *A* ==> *a* ∉ *B* ==> *F* ⊆ *insert a A* ==> *F* ⊆ *insert a B* ==> *a* ∈ *F*
==> ∃ *C*. *F* = *insert a C* ∧ *a* ∉ *C* ∧ *C* <= *A* ∧ *C* <= *B*
apply (*rule-tac x* = *F* − {*a*} **in** *exI*)
apply *auto*
done

lemma *aux2*: *a* ≠ *b* ==> *a* ∉ *A* ==> *b* ∉ *B* ==> *a* ∈ *F* ==> *b* ∈ *F*
==> *F* ⊆ *insert a A* ==> *F* ⊆ *insert b B*
==> ∃ *C*. *F* = *insert a* (*insert b C*) ∧ *a* ∉ *C* ∧ *b* ∉ *C* ∧ *C* ⊆ *A* ∧ *C* ⊆ *B*
apply (*rule-tac x* = *F* − {*a*, *b*} **in** *exI*)
apply *auto*
done

lemma *aux-uniq*: *uniqP P* ==> *P a b* ==> *P c d* ==> (*a* = *c*) = (*b* = *d*)
apply (*unfold uniqP-def*)

```
  apply auto
  done

lemma aux-sym: symP P ==> P a b = P b a
  apply (unfold symP-def)
  apply auto
  done

lemma aux-in1:
    uniqP P ==> b ∉ C ==> P b b ==> bijP P (insert b C) ==> bijP P C
  apply (unfold bijP-def)
  apply auto
  apply (subgoal-tac b ≠ a)
   prefer 2
   apply clarify
  apply (simp add: aux-uniq)
  apply auto
  done

lemma aux-in2:
  symP P ==> uniqP P ==> a ∉ C ==> b ∉ C ==> a ≠ b ==> P a b
    ==> bijP P (insert a (insert b C)) ==> bijP P C
  apply (unfold bijP-def)
  apply auto
  apply (subgoal-tac aa ≠ a)
   prefer 2
   apply clarify
  apply (subgoal-tac aa ≠ b)
   prefer 2
   apply clarify
  apply (simp add: aux-uniq)
  apply (subgoal-tac ba ≠ a)
   apply auto
  apply (subgoal-tac P a aa)
   prefer 2
   apply (simp add: aux-sym)
  apply (subgoal-tac b = aa)
   apply (rule-tac [2] iffD1)
    apply (rule-tac [2] a = a and c = a and P = P in aux-uniq)
      apply auto
  done

lemma aux-foo: ∀ a b. Q a ∧ P a b --> R b ==> P a b ==> Q a ==> R b
  apply auto
  done

lemma aux-bij: bijP P F ==> symP P ==> P a b ==> (a ∈ F) = (b ∈ F)
  apply (unfold bijP-def)
  apply (rule iffI)
```

```
apply (erule-tac [!] aux-foo)
    apply simp-all
apply (rule iffD2)
 apply (rule-tac P = P in aux-sym)
 apply simp-all
done


lemma aux-bijRER:
  (A, B) ∈ bijR P ==> uniqP P ==> symP P
   ==> ∀F. bijP P F ∧ F ⊆ A ∧ F ⊆ B --> F ∈ bijER P
apply (erule bijR.induct)
 apply simp
apply (case-tac a = b)
 apply clarify
 apply (case-tac b ∈ F)
  prefer 2
  apply (simp add: subset-insert)
 apply (cut-tac F = F and a = b and A = A and B = B in aux1)
      prefer 6
      apply clarify
      apply (rule bijER.insert1)
        apply simp-all
 apply (subgoal-tac bijP P C)
  apply simp
 apply (rule aux-in1)
    apply simp-all
apply clarify
apply (case-tac a ∈ F)
 apply (case-tac [!] b ∈ F)
   apply (cut-tac F = F and a = a and b = b and A = A and B = B
     in aux2)
         apply (simp-all add: subset-insert)
  apply clarify
  apply (rule bijER.insert2)
      apply simp-all
  apply (subgoal-tac bijP P C)
   apply simp
  apply (rule aux-in2)
        apply simp-all
 apply (subgoal-tac b ∈ F)
  apply (rule-tac [2] iffD1)
   apply (rule-tac [2] a = a and F = F and P = P in aux-bij)
     apply (simp-all (no-asm-simp))
 apply (subgoal-tac [2] a ∈ F)
  apply (rule-tac [3] iffD2)
   apply (rule-tac [3] b = b and F = F and P = P in aux-bij)
     apply auto
done
```

```
lemma bijR-bijER:
  (A, A) ∈ bijR P ==>
    bijP P A ==> uniqP P ==> symP P ==> A ∈ bijER P
  apply (cut-tac A = A and B = A and P = P in aux-bijRER)
     apply auto
  done

end
```

8 Factorial on integers

```
theory IntFact
imports IntPrimes
begin
```

Factorial on integers and recursively defined set including all Integers from *2* up to *a*. Plus definition of product of finite set.

```
fun zfact :: int => int
  where zfact n = (if n ≤ 0 then 1 else n ∗ zfact (n − 1))

fun d22set :: int => int set
  where d22set a = (if 1 < a then insert a (d22set (a − 1)) else {})
```

d22set — recursively defined set including all integers from *2* up to *a*

```
declare d22set.simps [simp del]


lemma d22set-induct:
  assumes !!a. P {} a
    and !!a. 1 < (a::int) ==> P (d22set (a − 1)) (a − 1) ==> P (d22set a) a
  shows P (d22set u) u
  apply (rule d22set.induct)
  apply (case-tac 1 < a)
   apply (rule-tac assms)
    apply (simp-all (no-asm-simp))
  apply (simp-all (no-asm-simp) add: d22set.simps assms)
  done

lemma d22set-g-1 [rule-format]: b ∈ d22set a −−> 1 < b
  apply (induct a rule: d22set-induct)
   apply simp
  apply (subst d22set.simps)
  apply auto
  done

lemma d22set-le [rule-format]: b ∈ d22set a −−> b ≤ a
```

apply (*induct a rule*: *d22set-induct*)
apply *simp*
 apply (*subst d22set.simps*)
 apply *auto*
done

lemma *d22set-le-swap*: *a* < *b* ==> *b* ∉ *d22set a*
by (*auto dest*: *d22set-le*)

lemma *d22set-mem*: $1 < b \Longrightarrow b \leq a \Longrightarrow b \in$ *d22set a*
apply (*induct a rule*: *d22set.induct*)
apply *auto*
apply (*subst d22set.simps*)
apply (*case-tac b* < *a*, *auto*)
done

lemma *d22set-fin*: *finite* (*d22set a*)
apply (*induct a rule*: *d22set-induct*)
 prefer *2*
 apply (*subst d22set.simps*)
 apply *auto*
done

declare *zfact.simps* [*simp del*]

lemma *d22set-prod-zfact*: $\prod$ (*d22set a*) = *zfact a*
apply (*induct a rule*: *d22set.induct*)
apply (*subst d22set.simps*)
apply (*subst zfact.simps*)
apply (*case-tac 1* < *a*)
 prefer *2*
 apply (*simp add*: *d22set.simps zfact.simps*)
apply (*simp add*: *d22set-fin d22set-le-swap*)
done

end

9 Fermat's Little Theorem extended to Euler's Totient function

theory *EulerFermat*
imports *BijectionRel IntFact*
begin

Fermat's Little Theorem extended to Euler's Totient function. More abstract approach than Boyer-Moore (which seems necessary to achieve the extended version).

9.1 Definitions and lemmas

inductive-set *RsetR* :: *int* => *int set set* **for** *m* :: *int*
where
  empty [*simp*]: {} ∈ *RsetR m*
| *insert*: *A* ∈ *RsetR m* ==> *zgcd a m* = *1* ==>
    ∀ *a'*. *a'* ∈ *A* −−> ¬ *zcong a a' m* ==> *insert a A* ∈ *RsetR m*

fun *BnorRset* :: *int* ⇒ *int* => *int set* **where**
  BnorRset a m =
   (*if 0* < *a then*
    let na = *BnorRset* (*a* − *1*) *m*
    in (*if zgcd a m* = *1 then insert a na else na*)
    else {})

definition *norRRset* :: *int* => *int set*
  where *norRRset m* = *BnorRset* (*m* − *1*) *m*

definition *noXRRset* :: *int* => *int* => *int set*
  where *noXRRset m x* = (λ*a*. *a* ∗ *x*) ‘ *norRRset m*

definition *phi* :: *int* => *nat*
  where *phi m* = *card* (*norRRset m*)

definition *is-RRset* :: *int set* => *int* => *bool*
  where *is-RRset A m* = (*A* ∈ *RsetR m* ∧ *card A* = *phi m*)

definition *RRset2norRR* :: *int set* => *int* => *int* => *int*
  where
    RRset2norRR A m a =
       (*if 1* < *m* ∧ *is-RRset A m* ∧ *a* ∈ *A then*
          SOME b. *zcong a b m* ∧ *b* ∈ *norRRset m*
        else 0)

definition *zcongm* :: *int* => *int* => *int* => *bool*
  where *zcongm m* = (λ*a b*. *zcong a b m*)

lemma *abs-eq-1-iff* [*iff*]: (|*z*| = (*1*::*int*)) = (*z* = *1* ∨ *z* = −*1*)
  — LCP: not sure why this lemma is needed now
  by (*auto simp add*: *abs-if*)

norRRset

declare *BnorRset.simps* [*simp del*]

lemma *BnorRset-induct*:
  assumes !!*a m*. *P* {} *a m*
    and !!*a m* :: *int*. *0* < *a* ==> *P* (*BnorRset* (*a* − *1*) *m*) (*a* − *1*) *m*
      ==> *P* (*BnorRset a m*) *a m*
  shows *P* (*BnorRset u v*) *u v*

```
  apply (rule BnorRset.induct)
   apply (case-tac 0 < a)
    apply (rule-tac assms)
     apply simp-all
   apply (simp-all add: BnorRset.simps assms)
  done

lemma Bnor-mem-zle [rule-format]: b ∈ BnorRset a m ⟶ b ≤ a
  apply (induct a m rule: BnorRset-induct)
   apply simp
  apply (subst BnorRset.simps)
   apply (unfold Let-def, auto)
  done

lemma Bnor-mem-zle-swap: a < b ==> b ∉ BnorRset a m
  by (auto dest: Bnor-mem-zle)

lemma Bnor-mem-zg [rule-format]: b ∈ BnorRset a m --> 0 < b
  apply (induct a m rule: BnorRset-induct)
   prefer 2
   apply (subst BnorRset.simps)
   apply (unfold Let-def, auto)
  done

lemma Bnor-mem-if [rule-format]:
    zgcd b m = 1 --> 0 < b --> b ≤ a --> b ∈ BnorRset a m
  apply (induct a m rule: BnorRset.induct, auto)
   apply (subst BnorRset.simps)
   defer
   apply (subst BnorRset.simps)
   apply (unfold Let-def, auto)
  done

lemma Bnor-in-RsetR [rule-format]: a < m --> BnorRset a m ∈ RsetR m
  apply (induct a m rule: BnorRset-induct, simp)
  apply (subst BnorRset.simps)
  apply (unfold Let-def, auto)
  apply (rule RsetR.insert)
    apply (rule-tac [3] allI)
    apply (rule-tac [3] impI)
    apply (rule-tac [3] zcong-not)
       apply (subgoal-tac [6] a' ≤ a - 1)
        apply (rule-tac [7] Bnor-mem-zle)
        apply (rule-tac [5] Bnor-mem-zg, auto)
  done

lemma Bnor-fin: finite (BnorRset a m)
  apply (induct a m rule: BnorRset-induct)
   prefer 2
```

apply (*subst BnorRset.simps*)
apply (*unfold Let-def*, *auto*)
done

lemma *norR-mem-unique-aux*: $a \leq b - 1 ==> a < (b::int)$
apply *auto*
done

lemma *norR-mem-unique*:
$1 < m ==>$
zgcd a m $= 1 ==> \exists!b.\ [a = b]\ (mod\ m) \wedge b \in norRRset\ m$
apply (*unfold norRRset-def*)
apply (*cut-tac a* = *a* **and** *m* = *m* **in** *zcong-zless-unique*, *auto*)
apply (*rule-tac* [*2*] *m* = *m* **in** *zcong-zless-imp-eq*)
apply (*auto intro*: *Bnor-mem-zle Bnor-mem-zg zcong-trans order-less-imp-le norR-mem-unique-aux simp add*: *zcong-sym*)
apply (*rule-tac x* = *b* **in** *exI*, *safe*)
apply (*rule Bnor-mem-if*)
apply (*case-tac* [*2*] *b* = *0*)
apply (*auto intro*: *order-less-le* [*THEN iffD2*])
prefer *2*
apply (*simp only*: *zcong-def*)
apply (*subgoal-tac zgcd a m* = *m*)
prefer *2*
apply (*subst zdvd-iff-zgcd* [*symmetric*])
apply (*rule-tac* [*4*] *zgcd-zcong-zgcd*)
apply (*simp-all* (*no-asm-use*) *add*: *zcong-sym*)
done

noXRRset

lemma *RRset-gcd* [*rule-format*]:
is-RRset A m $==> a \in A --> zgcd\ a\ m = 1$
apply (*unfold is-RRset-def*)
apply (*rule RsetR.induct*, *auto*)
done

lemma *RsetR-zmult-mono*:
$A \in RsetR\ m ==>$
$0 < m ==> zgcd\ x\ m = 1 ==> (\lambda a.\ a * x)$ ‘ $A \in RsetR\ m$
apply (*erule RsetR.induct*, *simp-all*)
apply (*rule RsetR.insert*, *auto*)
apply (*blast intro*: *zgcd-zgcd-zmult*)
apply (*simp add*: *zcong-cancel*)
done

lemma *card-nor-eq-noX*:
$0 < m ==>$
zgcd x m $= 1 ==>$ *card* (*noXRRset m x*) = *card* (*norRRset m*)
apply (*unfold norRRset-def noXRRset-def*)

apply (*rule card-image*)
 apply (*auto simp add*: *inj-on-def Bnor-fin*)
apply (*simp add*: *BnorRset.simps*)
done

lemma *noX-is-RRset*:
  0 < *m* ==> *zgcd x m* = *1* ==> *is-RRset* (*noXRRset m x*) *m*
apply (*unfold is-RRset-def phi-def*)
apply (*auto simp add*: *card-nor-eq-noX*)
apply (*unfold noXRRset-def norRRset-def*)
apply (*rule RsetR-zmult-mono*)
  apply (*rule Bnor-in-RsetR*, *simp-all*)
done

lemma *aux-some*:
1 < *m* ==> *is-RRset A m* ==> *a* ∈ *A*
  ==> *zcong a* (*SOME b*. [*a* = *b*] (*mod m*) ∧ *b* ∈ *norRRset m*) *m* ∧
    (*SOME b*. [*a* = *b*] (*mod m*) ∧ *b* ∈ *norRRset m*) ∈ *norRRset m*
apply (*rule norR-mem-unique* [*THEN ex1-implies-ex*, *THEN someI-ex*])
 apply (*rule-tac* [*2*] *RRset-gcd*, *simp-all*)
done

lemma *RRset2norRR-correct*:
1 < *m* ==> *is-RRset A m* ==> *a* ∈ *A* ==>
  [*a* = *RRset2norRR A m a*] (*mod m*) ∧ *RRset2norRR A m a* ∈ *norRRset m*
apply (*unfold RRset2norRR-def*, *simp*)
apply (*rule aux-some*, *simp-all*)
done

lemmas *RRset2norRR-correct1* = *RRset2norRR-correct* [*THEN conjunct1*]
lemmas *RRset2norRR-correct2* = *RRset2norRR-correct* [*THEN conjunct2*]

lemma *RsetR-fin*: *A* ∈ *RsetR m* ==> *finite A*
by (*induct set*: *RsetR*) *auto*

lemma *RRset-zcong-eq* [*rule-format*]:
1 < *m* ==>
  is-RRset A m ==> [*a* = *b*] (*mod m*) ==> *a* ∈ *A* --> *b* ∈ *A* --> *a* = *b*
apply (*unfold is-RRset-def*)
apply (*rule RsetR.induct*)
  apply (*auto simp add*: *zcong-sym*)
done

lemma *aux*:
P (*SOME a*. *P a*) ==> *Q* (*SOME a*. *Q a*) ==>
  (*SOME a*. *P a*) = (*SOME a*. *Q a*) ==> ∃ *a*. *P a* ∧ *Q a*
apply *auto*
done

lemma *RRset2norRR-inj*:
1 < *m* ==> *is-RRset A m* ==> *inj-on* (*RRset2norRR A m*) *A*
apply (*unfold RRset2norRR-def inj-on-def*, *auto*)
apply (*subgoal-tac* ∃ *b*. ([*x* = *b*] (*mod m*) ∧ *b* ∈ *norRRset m*) ∧
([*y* = *b*] (*mod m*) ∧ *b* ∈ *norRRset m*))
apply (*rule-tac* [*2*] *aux*)
apply (*rule-tac* [*3*] *aux-some*)
apply (*rule-tac* [*2*] *aux-some*)
apply (*rule RRset-zcong-eq*, *auto*)
apply (*rule-tac b* = *b* **in** *zcong-trans*)
apply (*simp-all add*: *zcong-sym*)
done

lemma *RRset2norRR-eq-norR*:
1 < *m* ==> *is-RRset A m* ==> *RRset2norRR A m* ‘ *A* = *norRRset m*
apply (*rule card-seteq*)
prefer *3*
apply (*subst card-image*)
apply (*rule-tac RRset2norRR-inj*, *auto*)
apply (*rule-tac* [*3*] *RRset2norRR-correct2*, *auto*)
apply (*unfold is-RRset-def phi-def norRRset-def*)
apply (*auto simp add*: *Bnor-fin*)
done

lemma *Bnor-prod-power-aux*: *a* ∉ *A* ==> *inj f* ==> *f a* ∉ *f* ‘ *A*
by (*unfold inj-on-def*, *auto*)

lemma *Bnor-prod-power* [*rule-format*]:
x ≠ *0* ==> *a* < *m* −−> ∏((λ*a*. *a* ∗ *x*) ‘ *BnorRset a m*) =
∏(*BnorRset a m*) ∗ *x*ˆ*card* (*BnorRset a m*)
apply (*induct a m rule*: *BnorRset-induct*)
prefer *2*
apply (*simplesubst BnorRset.simps*) — multiple redexes
apply (*unfold Let-def*, *auto*)
apply (*simp add*: *Bnor-fin Bnor-mem-zle-swap*)
apply (*subst setprod.insert*)
apply (*rule-tac* [*2*] *Bnor-prod-power-aux*)
apply (*unfold inj-on-def*)
apply (*simp-all add*: *ac-simps Bnor-fin Bnor-mem-zle-swap*)
done

9.2 Fermat

lemma *bijzcong-zcong-prod*:
(*A*, *B*) ∈ *bijR* (*zcongm m*) ==> [∏ *A* = ∏ *B*] (*mod m*)
apply (*unfold zcongm-def*)
apply (*erule bijR.induct*)
apply (*subgoal-tac* [*2*] *a* ∉ *A* ∧ *b* ∉ *B* ∧ *finite A* ∧ *finite B*)

```
   apply (auto intro: fin-bijRl fin-bijRr zcong-zmult)
  done

lemma Bnor-prod-zgcd [rule-format]:
    a < m --> zgcd (∏(BnorRset a m)) m = 1
  apply (induct a m rule: BnorRset-induct)
   prefer 2
   apply (subst BnorRset.simps)
   apply (unfold Let-def, auto)
  apply (simp add: Bnor-fin Bnor-mem-zle-swap)
  apply (blast intro: zgcd-zgcd-zmult)
  done

theorem Euler-Fermat:
    0 < m ==> zgcd x m = 1 ==> [x^(phi m) = 1] (mod m)
  apply (unfold norRRset-def phi-def)
  apply (case-tac x = 0)
   apply (case-tac [2] m = 1)
    apply (rule-tac [3] iffD1)
     apply (rule-tac [3] k = ∏(BnorRset (m - 1) m)
       in zcong-cancel2)
      prefer 5
      apply (subst Bnor-prod-power [symmetric])
        apply (rule-tac [7] Bnor-prod-zgcd, simp-all)
  apply (rule bijzcong-zcong-prod)
  apply (fold norRRset-def, fold noXRRset-def)
  apply (subst RRset2norRR-eq-norR [symmetric])
    apply (rule-tac [3] inj-func-bijR, auto)
     apply (unfold zcongm-def)
     apply (rule-tac [2] RRset2norRR-correct1)
       apply (rule-tac [5] RRset2norRR-inj)
        apply (auto intro: order-less-le [THEN iffD2]
           simp add: noX-is-RRset)
  apply (unfold noXRRset-def norRRset-def)
  apply (rule finite-imageI)
  apply (rule Bnor-fin)
  done

lemma Bnor-prime:
  ⟦ zprime p; a < p ⟧ ⟹ card (BnorRset a p) = nat a
  apply (induct a p rule: BnorRset.induct)
  apply (subst BnorRset.simps)
  apply (unfold Let-def, auto simp add:zless-zprime-imp-zrelprime)
  apply (subgoal-tac finite (BnorRset (a - 1) m))
   apply (subgoal-tac a ~: BnorRset (a - 1) m)
    apply (auto simp add: card-insert-disjoint Suc-nat-eq-nat-zadd1)
   apply (frule Bnor-mem-zle, arith)
  apply (frule Bnor-fin)
  done
```

```
lemma phi-prime: zprime p ==> phi p = nat (p − 1)
  apply (unfold phi-def norRRset-def)
  apply (rule Bnor-prime, auto)
  done

theorem Little-Fermat:
    zprime p ==> ¬ p dvd x ==> [x^(nat (p − 1)) = 1] (mod p)
  apply (subst phi-prime [symmetric])
   apply (rule-tac [2] Euler-Fermat)
    apply (erule-tac [3] zprime-imp-zrelprime)
    apply (unfold zprime-def, auto)
  done

end
```

10 Wilson's Theorem according to Russinoff

```
theory WilsonRuss
imports EulerFermat
begin
```

Wilson's Theorem following quite closely Russinoff's approach using Boyer-Moore (using finite sets instead of lists, though).

10.1 Definitions and lemmas

```
definition inv :: int => int => int
  where inv p a = (a^(nat (p − 2))) mod p

fun wset :: int ⇒ int => int set where
  wset a p =
    (if 1 < a then
      let ws = wset (a − 1) p
      in (if a ∈ ws then ws else insert a (insert (inv p a) ws)) else {})
```

inv

```
lemma inv-is-inv-aux: 1 < m ==> Suc (nat (m − 2)) = nat (m − 1)
  by simp

lemma inv-is-inv:
    zprime p ⟹ 0 < a ⟹ a < p ==> [a ∗ inv p a = 1] (mod p)
  apply (unfold inv-def)
  apply (subst zcong-zmod)
  apply (subst mod-mult-right-eq [symmetric])
  apply (subst zcong-zmod [symmetric])
  apply (subst power-Suc [symmetric])
  using Little-Fermat inv-is-inv-aux zdvd-not-zless apply auto
```

done

lemma *inv-distinct*:
zprime p ⟹ *1* < *a* ⟹ *a* < *p* − *1* ==> *a* ≠ *inv p a*
apply *safe*
apply (*cut-tac a* = *a* **and** *p* = *p* **in** *zcong-square*)
apply (*cut-tac* [*3*] *a* = *a* **and** *p* = *p* **in** *inv-is-inv*, *auto*)
apply (*subgoal-tac a* = *1*)
apply (*rule-tac* [*2*] *m* = *p* **in** *zcong-zless-imp-eq*)
apply (*subgoal-tac* [*7*] *a* = *p* − *1*)
apply (*rule-tac* [*8*] *m* = *p* **in** *zcong-zless-imp-eq*, *auto*)
done

lemma *inv-not-0*:
zprime p ⟹ *1* < *a* ⟹ *a* < *p* − *1* ==> *inv p a* ≠ *0*
apply *safe*
apply (*cut-tac a* = *a* **and** *p* = *p* **in** *inv-is-inv*)
apply (*unfold zcong-def*, *auto*)
done

lemma *inv-not-1*:
zprime p ⟹ *1* < *a* ⟹ *a* < *p* − *1* ==> *inv p a* ≠ *1*
apply *safe*
apply (*cut-tac a* = *a* **and** *p* = *p* **in** *inv-is-inv*)
prefer *4*
apply *simp*
apply (*subgoal-tac a* = *1*)
apply (*rule-tac* [*2*] *zcong-zless-imp-eq*, *auto*)
done

lemma *inv-not-p-minus-1-aux*:
[*a* ∗ (*p* − *1*) = *1*] (*mod p*) = [*a* = *p* − *1*] (*mod p*)
apply (*unfold zcong-def*)
apply (*simp add*: *diff-diff-eq diff-diff-eq2 right-diff-distrib*)
apply (*rule-tac s* = *p dvd* −((*a* + *1*) + (*p* ∗ −*a*)) **in** *trans*)
apply (*simp add*: *algebra-simps*)
apply (*subst dvd-minus-iff*)
apply (*subst zdvd-reduce*)
apply (*rule-tac s* = *p dvd* (*a* + *1*) + (*p* ∗ −*1*) **in** *trans*)
apply (*subst zdvd-reduce*, *auto*)
done

lemma *inv-not-p-minus-1*:
zprime p ⟹ *1* < *a* ⟹ *a* < *p* − *1* ==> *inv p a* ≠ *p* − *1*
apply *safe*
apply (*cut-tac a* = *a* **and** *p* = *p* **in** *inv-is-inv*, *auto*)
apply (*simp add*: *inv-not-p-minus-1-aux*)
apply (*subgoal-tac a* = *p* − *1*)
apply (*rule-tac* [*2*] *zcong-zless-imp-eq*, *auto*)

done

lemma *inv-g-1*:
$zprime\ p \Longrightarrow 1 < a \Longrightarrow a < p - 1 ==> 1 < inv\ p\ a$
apply (*case-tac* $0 \le inv\ p\ a$)
apply (*subgoal-tac* $inv\ p\ a \neq 1$)
apply (*subgoal-tac* $inv\ p\ a \neq 0$)
apply (*subst order-less-le*)
apply (*subst zle-add1-eq-le* [*symmetric*])
apply (*subst order-less-le*)
apply (*rule-tac* [*2*] *inv-not-0*)
apply (*rule-tac* [*5*] *inv-not-1*, *auto*)
apply (*unfold inv-def zprime-def*, *simp*)
done

lemma *inv-less-p-minus-1*:
$zprime\ p \Longrightarrow 1 < a \Longrightarrow a < p - 1 ==> inv\ p\ a < p - 1$
apply (*case-tac* $inv\ p\ a < p$)
apply (*subst order-less-le*)
apply (*simp add*: *inv-not-p-minus-1*, *auto*)
apply (*unfold inv-def zprime-def*, *simp*)
done

lemma *inv-inv-aux*: $5 \le p ==>$
$nat\ (p - 2) * nat\ (p - 2) = Suc\ (nat\ (p - 1) * nat\ (p - 3))$
apply (*subst of-nat-eq-iff* [**where** *'a = int*, *symmetric*])
apply (*simp add*: *left-diff-distrib right-diff-distrib*)
done

lemma *zcong-zpower-zmult*:
$[x\hat{}y = 1]\ (mod\ p) \Longrightarrow [x\hat{}(y * z) = 1]\ (mod\ p)$
apply (*induct z*)
apply (*auto simp add*: *power-add*)
apply (*subgoal-tac zcong* $(x\hat{}y * x\hat{}(y * z))\ (1 * 1)\ p$)
apply (*rule-tac* [*2*] *zcong-zmult*, *simp-all*)
done

lemma *inv-inv*: $zprime\ p \Longrightarrow$
$5 \le p \Longrightarrow 0 < a \Longrightarrow a < p ==> inv\ p\ (inv\ p\ a) = a$
apply (*unfold inv-def*)
apply (*subst power-mod*)
apply (*subst power-mult* [*symmetric*])
apply (*rule zcong-zless-imp-eq*)
prefer *5*
apply (*subst zcong-zmod*)
apply (*subst mod-mod-trivial*)
apply (*subst zcong-zmod* [*symmetric*])
apply (*subst inv-inv-aux*)
apply (*subgoal-tac* [*2*]

```
      zcong (a * a^(nat (p - 1) * nat (p - 3))) (a * 1) p)
     apply (rule-tac [3] zcong-zmult)
      apply (rule-tac [4] zcong-zpower-zmult)
      apply (erule-tac [4] Little-Fermat)
      apply (rule-tac [4] zdvd-not-zless, simp-all)
  done
```

wset

```
declare wset.simps [simp del]

lemma wset-induct:
  assumes !!a p. P {} a p
    and !!a p. 1 < (a::int) ==>
      P (wset (a - 1) p) (a - 1) p ==> P (wset a p) a p
  shows P (wset u v) u v
  apply (rule wset.induct)
  apply (case-tac 1 < a)
   apply (rule assms)
    apply (simp-all add: wset.simps assms)
  done

lemma wset-mem-imp-or [rule-format]:
  1 < a ==> b ∉ wset (a - 1) p
    ==> b ∈ wset a p --> b = a ∨ b = inv p a
  apply (subst wset.simps)
  apply (unfold Let-def, simp)
  done

lemma wset-mem-mem [simp]: 1 < a ==> a ∈ wset a p
  apply (subst wset.simps)
  apply (unfold Let-def, simp)
  done

lemma wset-subset: 1 < a ==> b ∈ wset (a - 1) p ==> b ∈ wset a p
  apply (subst wset.simps)
  apply (unfold Let-def, auto)
  done

lemma wset-g-1 [rule-format]:
    zprime p --> a < p - 1 --> b ∈ wset a p --> 1 < b
  apply (induct a p rule: wset-induct, auto)
  apply (case-tac b = a)
   apply (case-tac [2] b = inv p a)
    apply (subgoal-tac [3] b = a ∨ b = inv p a)
     apply (rule-tac [4] wset-mem-imp-or)
       prefer 2
       apply simp
       apply (rule inv-g-1, auto)
  done
```

lemma *wset-less* [*rule-format*]:
zprime p --> a < p - 1 --> b ∈ wset a p --> b < p - 1
apply (*induct a p rule*: *wset-induct*, *auto*)
apply (*case-tac b = a*)
apply (*case-tac* [*2*] *b = inv p a*)
apply (*subgoal-tac* [*3*] *b = a ∨ b = inv p a*)
apply (*rule-tac* [*4*] *wset-mem-imp-or*)
prefer *2*
apply *simp*
apply (*rule inv-less-p-minus-1*, *auto*)
done

lemma *wset-mem* [*rule-format*]:
zprime p -->
a < p - 1 --> 1 < b --> b ≤ a --> b ∈ wset a p
apply (*induct a p rule*: *wset.induct*, *auto*)
apply (*rule-tac wset-subset*)
apply (*simp* (*no-asm-simp*))
apply *auto*
done

lemma *wset-mem-inv-mem* [*rule-format*]:
zprime p --> 5 ≤ p --> a < p - 1 --> b ∈ wset a p
--> inv p b ∈ wset a p
apply (*induct a p rule*: *wset-induct*, *auto*)
apply (*case-tac b = a*)
apply (*subst wset.simps*)
apply (*unfold Let-def*)
apply (*rule-tac* [*3*] *wset-subset*, *auto*)
apply (*case-tac b = inv p a*)
apply (*simp* (*no-asm-simp*))
apply (*subst inv-inv*)
apply (*subgoal-tac* [*6*] *b = a ∨ b = inv p a*)
apply (*rule-tac* [*7*] *wset-mem-imp-or*, *auto*)
done

lemma *wset-inv-mem-mem*:
zprime p ⟹ 5 ≤ p ⟹ a < p - 1 ⟹ 1 < b ⟹ b < p - 1
⟹ inv p b ∈ wset a p ⟹ b ∈ wset a p
apply (*rule-tac s = inv p* (*inv p b*) **and** *t = b* **in** *subst*)
apply (*rule-tac* [*2*] *wset-mem-inv-mem*)
apply (*rule inv-inv*, *simp-all*)
done

lemma *wset-fin*: *finite* (*wset a p*)
apply (*induct a p rule*: *wset-induct*)
prefer *2*
apply (*subst wset.simps*)

```
  apply (unfold Let-def, auto)
 done

lemma wset-zcong-prod-1 [rule-format]:
 zprime p -->
   5 ≤ p --> a < p - 1 --> [(∏x∈wset a p. x) = 1] (mod p)
 apply (induct a p rule: wset-induct)
  prefer 2
  apply (subst wset.simps)
  apply (auto, unfold Let-def, auto)
 apply (subst setprod.insert)
   apply (tactic ‹stac @{context} @{thm setprod.insert} 3›)
     apply (subgoal-tac [5]
       zcong (a * inv p a * (∏x∈wset (a - 1) p. x)) (1 * 1) p)
      prefer 5
      apply (simp add: mult.assoc)
     apply (rule-tac [5] zcong-zmult)
      apply (rule-tac [5] inv-is-inv)
        apply (tactic clarify-tac @{context} 4)
        apply (subgoal-tac [4] a ∈ wset (a - 1) p)
         apply (rule-tac [5] wset-inv-mem-mem)
              apply (simp-all add: wset-fin)
 apply (rule inv-distinct, auto)
 done

lemma d22set-eq-wset: zprime p ==> d22set (p - 2) = wset (p - 2) p
 apply safe
  apply (erule wset-mem)
    apply (rule-tac [2] d22set-g-1)
    apply (rule-tac [3] d22set-le)
    apply (rule-tac [4] d22set-mem)
     apply (erule-tac [4] wset-g-1)
      prefer 6
      apply (subst zle-add1-eq-le [symmetric])
      apply (subgoal-tac p - 2 + 1 = p - 1)
       apply (simp (no-asm-simp))
       apply (erule wset-less, auto)
 done
```

10.2 Wilson

```
lemma prime-g-5: zprime p ⟹ p ≠ 2 ⟹ p ≠ 3 ==> 5 ≤ p
 apply (unfold zprime-def dvd-def)
 apply (case-tac p = 4, auto)
  apply (rule notE)
   prefer 2
   apply assumption
  apply (simp (no-asm))
  apply (rule-tac x = 2 in exI)
```

```
  apply (safe, arith)
    apply (rule-tac x = 2 in exI, auto)
  done

theorem Wilson-Russ:
    zprime p ==> [zfact (p - 1) = -1] (mod p)
  apply (subgoal-tac [(p - 1) * zfact (p - 2) = -1 * 1] (mod p))
   apply (rule-tac [2] zcong-zmult)
    apply (simp only: zprime-def)
    apply (subst zfact.simps)
    apply (rule-tac t = p - 1 - 1 and s = p - 2 in subst, auto)
   apply (simp only: zcong-def)
   apply (simp (no-asm-simp))
  apply (case-tac p = 2)
   apply (simp add: zfact.simps)
  apply (case-tac p = 3)
   apply (simp add: zfact.simps)
  apply (subgoal-tac 5 ≤ p)
   apply (erule-tac [2] prime-g-5)
    apply (subst d22set-prod-zfact [symmetric])
    apply (subst d22set-eq-wset)
     apply (rule-tac [2] wset-zcong-prod-1, auto)
  done

end
```

11 Wilson's Theorem using a more abstract approach

```
theory WilsonBij
imports BijectionRel IntFact
begin
```

Wilson's Theorem using a more "abstract" approach based on bijections between sets. Does not use Fermat's Little Theorem (unlike Russinoff).

11.1 Definitions and lemmas

```
definition reciR :: int => int => int => bool
  where reciR p = (λa b. zcong (a * b) 1 p ∧ 1 < a ∧ a < p - 1 ∧ 1 < b ∧ b
< p - 1)

definition inv :: int => int => int where
  inv p a =
    (if zprime p ∧ 0 < a ∧ a < p then
      (SOME x. 0 ≤ x ∧ x < p ∧ zcong (a * x) 1 p)
     else 0)
```

Inverse

```
lemma inv-correct:
  zprime p ==> 0 < a ==> a < p
    ==> 0 ≤ inv p a ∧ inv p a < p ∧ [a ∗ inv p a = 1] (mod p)
  apply (unfold inv-def)
  apply (simp (no-asm-simp))
  apply (rule zcong-lineq-unique [THEN ex1-implies-ex, THEN someI-ex])
   apply (erule-tac [2] zless-zprime-imp-zrelprime)
    apply (unfold zprime-def)
    apply auto
  done

lemmas inv-ge = inv-correct [THEN conjunct1]
lemmas inv-less = inv-correct [THEN conjunct2, THEN conjunct1]
lemmas inv-is-inv = inv-correct [THEN conjunct2, THEN conjunct2]

lemma inv-not-0:
  zprime p ==> 1 < a ==> a < p − 1 ==> inv p a ≠ 0
  — same as WilsonRuss
  apply safe
  apply (cut-tac a = a and p = p in inv-is-inv)
     apply (unfold zcong-def)
     apply auto
  done

lemma inv-not-1:
  zprime p ==> 1 < a ==> a < p − 1 ==> inv p a ≠ 1
  — same as WilsonRuss
  apply safe
  apply (cut-tac a = a and p = p in inv-is-inv)
     prefer 4
     apply simp
     apply (subgoal-tac a = 1)
      apply (rule-tac [2] zcong-zless-imp-eq)
          apply auto
  done

lemma aux: [a ∗ (p − 1) = 1] (mod p) = [a = p − 1] (mod p)
  — same as WilsonRuss
  apply (unfold zcong-def)
  apply (simp add: diff-diff-eq diff-diff-eq2 right-diff-distrib)
  apply (rule-tac s = p dvd −((a + 1) + (p ∗ −a)) in trans)
   apply (simp add: algebra-simps)
  apply (subst dvd-minus-iff)
  apply (subst zdvd-reduce)
  apply (rule-tac s = p dvd (a + 1) + (p ∗ −1) in trans)
   apply (subst zdvd-reduce)
   apply auto
  done
```

lemma *inv-not-p-minus-1*:
zprime p ==> *1* < *a* ==> *a* < *p* − *1* ==> *inv p a* ≠ *p* − *1*
— same as *WilsonRuss*
apply *safe*
apply (*cut-tac a* = *a* **and** *p* = *p* **in** *inv-is-inv*)
apply *auto*
apply (*simp add*: *aux*)
apply (*subgoal-tac a* = *p* − *1*)
apply (*rule-tac* [*2*] *zcong-zless-imp-eq*)
apply *auto*
done

Below is slightly different as we don't expand *inv* but use "*correct*" theorems.

lemma *inv-g-1*: *zprime p* ==> *1* < *a* ==> *a* < *p* − *1* ==> *1* < *inv p a*
apply (*subgoal-tac inv p a* ≠ *1*)
apply (*subgoal-tac inv p a* ≠ *0*)
apply (*subst order-less-le*)
apply (*subst zle-add1-eq-le* [*symmetric*])
apply (*subst order-less-le*)
apply (*rule-tac* [*2*] *inv-not-0*)
apply (*rule-tac* [*5*] *inv-not-1*)
apply *auto*
apply (*rule inv-ge*)
apply *auto*
done

lemma *inv-less-p-minus-1*:
zprime p ==> *1* < *a* ==> *a* < *p* − *1* ==> *inv p a* < *p* − *1*
— ditto
apply (*subst order-less-le*)
apply (*simp add*: *inv-not-p-minus-1 inv-less*)
done

Bijection

lemma *aux1*: *1* < *x* ==> *0* ≤ (*x*::*int*)
apply *auto*
done

lemma *aux2*: *1* < *x* ==> *0* < (*x*::*int*)
apply *auto*
done

lemma *aux3*: *x* ≤ *p* − *2* ==> *x* < (*p*::*int*)
apply *auto*
done

lemma *aux4*: *x* ≤ *p* − *2* ==> *x* < (*p*::*int*) − *1*
apply *auto*

```
  done

lemma inv-inj: zprime p ==> inj-on (inv p) (d22set (p − 2))
  apply (unfold inj-on-def)
  apply auto
  apply (rule zcong-zless-imp-eq)
      apply (tactic ‹stac @{context} (@{thm zcong-cancel} RS sym) 5›)
        apply (rule-tac [7] zcong-trans)
         apply (tactic ‹stac @{context} @{thm zcong-sym} 8›)
         apply (erule-tac [7] inv-is-inv)
          apply (tactic asm-simp-tac @{context} 9)
          apply (erule-tac [9] inv-is-inv)
           apply (rule-tac [6] zless-zprime-imp-zrelprime)
             apply (rule-tac [8] inv-less)
               apply (rule-tac [7] inv-g-1 [THEN aux2])
                 apply (unfold zprime-def)
                 apply (auto intro: d22set-g-1 d22set-le
                   aux1 aux2 aux3 aux4)
  done

lemma inv-d22set-d22set:
    zprime p ==> inv p ‘ d22set (p − 2) = d22set (p − 2)
  apply (rule endo-inj-surj)
    apply (rule d22set-fin)
   apply (erule-tac [2] inv-inj)
  apply auto
  apply (rule d22set-mem)
   apply (erule inv-g-1)
    apply (subgoal-tac [3] inv p xa < p − 1)
     apply (erule-tac [4] inv-less-p-minus-1)
      apply (auto intro: d22set-g-1 d22set-le aux4)
  done

lemma d22set-d22set-bij:
    zprime p ==> (d22set (p − 2), d22set (p − 2)) ∈ bijR (reciR p)
  apply (unfold reciR-def)
  apply (rule-tac s = (d22set (p − 2), inv p ‘ d22set (p − 2)) in subst)
   apply (simp add: inv-d22set-d22set)
  apply (rule inj-func-bijR)
    apply (rule-tac [3] d22set-fin)
   apply (erule-tac [2] inv-inj)
  apply auto
      apply (erule inv-is-inv)
       apply (erule-tac [5] inv-g-1)
        apply (erule-tac [7] inv-less-p-minus-1)
         apply (auto intro: d22set-g-1 d22set-le aux2 aux3 aux4)
  done

lemma reciP-bijP: zprime p ==> bijP (reciR p) (d22set (p − 2))
```

```
  apply (unfold reciR-def bijP-def)
  apply auto
  apply (rule d22set-mem)
   apply auto
  done

lemma reciP-uniq: zprime p ==> uniqP (reciR p)
  apply (unfold reciR-def uniqP-def)
  apply auto
   apply (rule zcong-zless-imp-eq)
       apply (tactic ‹stac @{context} (@{thm zcong-cancel2} RS sym) 5›)
         apply (rule-tac [7] zcong-trans)
          apply (tactic ‹stac @{context} @{thm zcong-sym} 8›)
          apply (rule-tac [6] zless-zprime-imp-zrelprime)
            apply auto
  apply (rule zcong-zless-imp-eq)
      apply (tactic ‹stac @{context} (@{thm zcong-cancel} RS sym) 5›)
        apply (rule-tac [7] zcong-trans)
         apply (tactic ‹stac @{context} @{thm zcong-sym} 8›)
         apply (rule-tac [6] zless-zprime-imp-zrelprime)
           apply auto
  done

lemma reciP-sym: zprime p ==> symP (reciR p)
  apply (unfold reciR-def symP-def)
  apply (simp add: mult.commute)
  apply auto
  done

lemma bijER-d22set: zprime p ==> d22set (p − 2) ∈ bijER (reciR p)
  apply (rule bijR-bijER)
     apply (erule d22set-d22set-bij)
    apply (erule reciP-bijP)
   apply (erule reciP-uniq)
  apply (erule reciP-sym)
  done
```

11.2 Wilson

```
lemma bijER-zcong-prod-1:
    zprime p ==> A ∈ bijER (reciR p) ==> [∏ A = 1] (mod p)
  apply (unfold reciR-def)
  apply (erule bijER.induct)
    apply (subgoal-tac [2] a = 1 ∨ a = p − 1)
     apply (rule-tac [3] zcong-square-zless)
        apply auto
  apply (subst setprod.insert)
    prefer 3
    apply (subst setprod.insert)
```

```
    apply (auto simp add: fin-bijER)
  apply (subgoal-tac zcong ((a * b) * ∏ A) (1 * 1) p)
   apply (simp add: mult.assoc)
  apply (rule zcong-zmult)
   apply auto
  done

theorem Wilson-Bij: zprime p ==> [zfact (p - 1) = -1] (mod p)
  apply (subgoal-tac zcong ((p - 1) * zfact (p - 2)) (-1 * 1) p)
   apply (rule-tac [2] zcong-zmult)
    apply (simp add: zprime-def)
    apply (subst zfact.simps)
    apply (rule-tac t = p - 1 - 1 and s = p - 2 in subst)
     apply auto
   apply (simp add: zcong-def)
  apply (subst d22set-prod-zfact [symmetric])
  apply (rule bijER-zcong-prod-1)
   apply (rule-tac [2] bijER-d22set)
   apply auto
  done

end
```

12 Finite Sets and Finite Sums

```
theory Finite2
imports IntFact ~~/src/HOL/Library/Infinite-Set
begin
```

These are useful for combinatorial and number-theoretic counting arguments.

12.1 Useful properties of sums and products

```
lemma setsum-same-function-zcong:
  assumes a: ∀ x ∈ S. [f x = g x](mod m)
  shows [setsum f S = setsum g S] (mod m)
proof cases
  assume finite S
  thus ?thesis using a by induct (simp-all add: zcong-zadd)
next
  assume infinite S thus ?thesis by simp
qed

lemma setprod-same-function-zcong:
  assumes a: ∀ x ∈ S. [f x = g x](mod m)
  shows [setprod f S = setprod g S] (mod m)
proof cases
  assume finite S
```

thus *?thesis* **using** *a* **by** *induct* (*simp-all add*: *zcong-zmult*)
next
assume *infinite S* **thus** *?thesis* **by** *simp*
qed

lemma *setsum-const*: *finite X ==> setsum* (%*x*. (*c* :: *int*)) *X* = *c* ∗ *int*(*card X*)
by (*simp add*: *of-nat-mult*)

lemma *setsum-const2*: *finite X ==> int* (*setsum* (%*x*. (*c* :: *nat*)) *X*) =
int(*c*) ∗ *int*(*card X*)
by (*simp add*: *of-nat-mult*)

lemma *setsum-const-mult*: *finite A ==> setsum* (%*x*. *c* ∗ ((*f x*)::*int*)) *A* =
c ∗ *setsum f A*
by (*induct set*: *finite*) (*auto simp add*: *distrib-left*)

12.2 Cardinality of explicit finite sets

lemma *finite-surjI*: [| *B* ⊆ *f* ‘ *A*; *finite A* |] *==> finite B*
by (*simp add*: *finite-subset*)

lemma *bdd-nat-set-l-finite*: *finite* {*y*::*nat* . *y* < *x*}
by (*rule bounded-nat-set-is-finite*) *blast*

lemma *bdd-nat-set-le-finite*: *finite* {*y*::*nat* . *y* ≤ *x*}
proof −
have {*y*::*nat* . *y* ≤ *x*} = {*y*::*nat* . *y* < *Suc x*} **by** *auto*
then show *?thesis* **by** (*auto simp add*: *bdd-nat-set-l-finite*)
qed

lemma *bdd-int-set-l-finite*: *finite* {*x*::*int*. *0* ≤ *x* & *x* < *n*}
apply (*subgoal-tac* {(*x* :: *int*). *0* ≤ *x* & *x* < *n*} ⊆
int ‘ {(*x* :: *nat*). *x* < *nat n*})
apply (*erule finite-surjI*)
apply (*auto simp add*: *bdd-nat-set-l-finite image-def*)
apply (*rule-tac x* = *nat x* **in** *exI*, *simp*)
done

lemma *bdd-int-set-le-finite*: *finite* {*x*::*int*. *0* ≤ *x* & *x* ≤ *n*}
apply (*subgoal-tac* {*x*. *0* ≤ *x* & *x* ≤ *n*} = {*x*. *0* ≤ *x* & *x* < *n* + *1*})
apply (*erule ssubst*)
apply (*rule bdd-int-set-l-finite*)
apply *auto*
done

lemma *bdd-int-set-l-l-finite*: *finite* {*x*::*int*. *0* < *x* & *x* < *n*}
proof −
have {*x*::*int*. *0* < *x* & *x* < *n*} ⊆ {*x*::*int*. *0* ≤ *x* & *x* < *n*}
by *auto*

```
  then show ?thesis by (auto simp add: bdd-int-set-l-finite finite-subset)
qed

lemma bdd-int-set-l-le-finite: finite {x::int. 0 < x & x ≤ n}
proof -
  have {x::int. 0 < x & x ≤ n} ⊆ {x::int. 0 ≤ x & x ≤ n}
    by auto
  then show ?thesis by (auto simp add: bdd-int-set-le-finite finite-subset)
qed

lemma card-bdd-nat-set-l: card {y::nat . y < x} = x
proof (induct x)
  case 0
  show card {y::nat . y < 0} = 0 by simp
next
  case (Suc n)
  have {y. y < Suc n} = insert n {y. y < n}
    by auto
  then have card {y. y < Suc n} = card (insert n {y. y < n})
    by auto
  also have ... = Suc (card {y. y < n})
    by (rule card-insert-disjoint) (auto simp add: bdd-nat-set-l-finite)
  finally show card {y. y < Suc n} = Suc n
    using ‹card {y. y < n} = n› by simp
qed

lemma card-bdd-nat-set-le: card { y::nat. y ≤ x} = Suc x
proof -
  have {y::nat. y ≤ x} = { y::nat. y < Suc x}
    by auto
  then show ?thesis by (auto simp add: card-bdd-nat-set-l)
qed

lemma card-bdd-int-set-l: 0 ≤ (n::int) ==> card {y. 0 ≤ y & y < n} = nat n
proof -
  assume 0 ≤ n
  have inj-on (%y. int y) {y. y < nat n}
    by (auto simp add: inj-on-def)
  hence card (int ‘ {y. y < nat n}) = card {y. y < nat n}
    by (rule card-image)
  also from ‹0 ≤ n› have int ‘ {y. y < nat n} = {y. 0 ≤ y & y < n}
    apply (auto simp add: zless-nat-eq-int-zless image-def)
    apply (rule-tac x = nat x in exI)
    apply (auto simp add: nat-0-le)
    done
  also have card {y. y < nat n} = nat n
    by (rule card-bdd-nat-set-l)
  finally show card {y. 0 ≤ y & y < n} = nat n .
qed
```

lemma *card-bdd-int-set-le*: *0* ≤ (*n*::*int*) ==> *card* {*y*. *0* ≤ *y* & *y* ≤ *n*} =
  nat n + *1*
proof –
  assume *0* ≤ *n*
  moreover have {*y*. *0* ≤ *y* & *y* ≤ *n*} = {*y*. *0* ≤ *y* & *y* < *n*+*1*} **by** *auto*
  ultimately show *?thesis*
    using *card-bdd-int-set-l* [*of n* + *1*]
    by (*auto simp add*: *nat-add-distrib*)
qed

lemma *card-bdd-int-set-l-le*: *0* ≤ (*n*::*int*) ==>
    card {*x*. *0* < *x* & *x* ≤ *n*} = *nat n*
proof –
  assume *0* ≤ *n*
  have *inj-on* (%*x*. *x*+*1*) {*x*. *0* ≤ *x* & *x* < *n*}
    by (*auto simp add*: *inj-on-def*)
  hence *card* ((%*x*. *x*+*1*) ' {*x*. *0* ≤ *x* & *x* < *n*}) =
     card {*x*. *0* ≤ *x* & *x* < *n*}
    by (*rule card-image*)
  also from ‹*0* ≤ *n*› **have** ... = *nat n*
    by (*rule card-bdd-int-set-l*)
  also have (%*x*. *x* + *1*) ' {*x*. *0* ≤ *x* & *x* < *n*} = {*x*. *0* < *x* & *x*<= *n*}
    apply (*auto simp add*: *image-def*)
    apply (*rule-tac x* = *x* – *1* **in** *exI*)
    apply *arith*
    done
  finally show *card* {*x*. *0* < *x* & *x* ≤ *n*} = *nat n* **.**
qed

lemma *card-bdd-int-set-l-l*: *0* < (*n*::*int*) ==>
  card {*x*. *0* < *x* & *x* < *n*} = *nat n* – *1*
proof –
  assume *0* < *n*
  moreover have {*x*. *0* < *x* & *x* < *n*} = {*x*. *0* < *x* & *x* ≤ *n* – *1*}
    by *simp*
  ultimately show *?thesis*
    using *insert card-bdd-int-set-l-le* [*of n* – *1*]
    by (*auto simp add*: *nat-diff-distrib*)
qed

lemma *int-card-bdd-int-set-l-l*: *0* < *n* ==>
    int(*card* {*x*. *0* < *x* & *x* < *n*}) = *n* – *1*
  apply (*auto simp add*: *card-bdd-int-set-l-l*)
  done

lemma *int-card-bdd-int-set-l-le*: *0* ≤ *n* ==>
    int(*card* {*x*. *0* < *x* & *x* ≤ *n*}) = *n*
  by (*auto simp add*: *card-bdd-int-set-l-le*)

end

13 Integers: Divisibility and Congruences

```
theory Int2
imports Finite2 WilsonRuss
begin

definition MultInv :: int => int => int
  where MultInv p x = x ^ nat (p - 2)
```

13.1 Useful lemmas about dvd and powers

```
lemma zpower-zdvd-prop1:
  0 < n ⟹ p dvd y ⟹ p dvd ((y::int) ^ n)
  by (induct n) (auto simp add: dvd-mult2 [of p y])

lemma zdvd-bounds: n dvd m ==> m ≤ (0::int) | n ≤ m
proof −
  assume n dvd m
  then have ~(0 < m & m < n)
    using zdvd-not-zless [of m n] by auto
  then show ?thesis by auto
qed

lemma zprime-zdvd-zmult-better: [| zprime p;  p dvd (m ∗ n) |] ==>
    (p dvd m) | (p dvd n)
  apply (cases 0 ≤ m)
  apply (simp add: zprime-zdvd-zmult)
  apply (insert zprime-zdvd-zmult [of −m p n])
  apply auto
  done

lemma zpower-zdvd-prop2:
    zprime p ⟹ p dvd ((y::int) ^ n) ⟹ 0 < n ⟹ p dvd y
  apply (induct n)
   apply simp
  apply (frule zprime-zdvd-zmult-better)
   apply simp
  apply (force simp del:dvd-mult)
  done

lemma div-prop1:
  assumes 0 < z and (x::int) < y ∗ z
  shows x div z < y
proof −
  from ‹0 < z› have modth: x mod z ≥ 0 by simp
```

have $(x\ div\ z) * z \leq (x\ div\ z) * z$ **by** *simp*
then have $(x\ div\ z) * z \leq (x\ div\ z) * z + x\ mod\ z$ **using** *modth* **by** *arith*
also have $\ldots = x$
by (*auto simp add*: *zmod-zdiv-equality* [*symmetric*] *ac-simps*)
also note ‹$x < y * z$›
finally show *?thesis*
apply (*auto simp add*: *mult-less-cancel-right*)
using *assms* **apply** *arith*
done
qed

lemma *div-prop2*:
assumes $0 < z$ **and** $(x::int) < (y * z) + z$
shows $x\ div\ z \leq y$
proof −
from *assms* **have** $x < (y + 1) * z$ **by** (*auto simp add*: *int-distrib*)
then have $x\ div\ z < y + 1$
apply (*rule-tac* $y = y + 1$ **in** *div-prop1*)
apply (*auto simp add*: ‹$0 < z$›)
done
then show *?thesis* **by** *auto*
qed

lemma *zdiv-leq-prop*: **assumes** $0 < y$ **shows** $y * (x\ div\ y) \leq (x::int)$
proof−
from *zmod-zdiv-equality* **have** $x = y * (x\ div\ y) + x\ mod\ y$ **by** *auto*
moreover have $0 \leq x\ mod\ y$ **by** (*auto simp add*: *assms*)
ultimately show *?thesis* **by** *arith*
qed

13.2 Useful properties of congruences

lemma *zcong-eq-zdvd-prop*: $[x = 0](mod\ p) = (p\ dvd\ x)$
by (*auto simp add*: *zcong-def*)

lemma *zcong-id*: $[m = 0]\ (mod\ m)$
by (*auto simp add*: *zcong-def*)

lemma *zcong-shift*: $[a = b]\ (mod\ m) ==> [a + c = b + c]\ (mod\ m)$
by (*auto simp add*: *zcong-zadd*)

lemma *zcong-zpower*: $[x = y](mod\ m) ==> [x\hat{}z = y\hat{}z](mod\ m)$
by (*induct z*) (*auto simp add*: *zcong-zmult*)

lemma *zcong-eq-trans*: $[|\ [a = b](mod\ m);\ b = c;\ [c = d](mod\ m)\ |] ==>$
$[a = d](mod\ m)$
apply (*erule zcong-trans*)
apply *simp*
done

```
lemma aux1: a - b = (c::int) ==> a = c + b
  by auto

lemma zcong-zmult-prop1: [a = b](mod m) ==> ([c = a * d](mod m) =
    [c = b * d] (mod m))
  apply (auto simp add: zcong-def dvd-def)
  apply (rule-tac x = ka + k * d in exI)
  apply (drule aux1)+
  apply (auto simp add: int-distrib)
  apply (rule-tac x = ka - k * d in exI)
  apply (drule aux1)+
  apply (auto simp add: int-distrib)
  done

lemma zcong-zmult-prop2: [a = b](mod m) ==>
    ([c = d * a](mod m) = [c = d * b] (mod m))
  by (auto simp add: ac-simps zcong-zmult-prop1)

lemma zcong-zmult-prop3: [| zprime p; ~[x = 0] (mod p);
    ~[y = 0] (mod p) |] ==> ~[x * y = 0] (mod p)
  apply (auto simp add: zcong-def)
  apply (drule zprime-zdvd-zmult-better, auto)
  done

lemma zcong-less-eq: [| 0 < x; 0 < y; 0 < m; [x = y] (mod m);
    x < m; y < m |] ==> x = y
  by (metis zcong-not zcong-sym less-linear)

lemma zcong-neg-1-impl-ne-1:
  assumes 2 < p and [x = -1] (mod p)
  shows ~([x = 1] (mod p))
proof
  assume [x = 1] (mod p)
  with assms have [1 = -1] (mod p)
    apply (auto simp add: zcong-sym)
    apply (drule zcong-trans, auto)
    done
  then have [1 + 1 = -1 + 1] (mod p)
    by (simp only: zcong-shift)
  then have [2 = 0] (mod p)
    by auto
  then have p dvd 2
    by (auto simp add: dvd-def zcong-def)
  with ‹2 < p› show False
    by (auto simp add: zdvd-not-zless)
qed

lemma zcong-zero-equiv-div: [a = 0] (mod m) = (m dvd a)
```

by (*auto simp add*: *zcong-def*)

lemma *zcong-zprime-prod-zero*: [| *zprime p*; *0* < *a* |] ==>
[*a* ∗ *b* = *0*] (*mod p*) ==> [*a* = *0*] (*mod p*) | [*b* = *0*] (*mod p*)
by (*auto simp add*: *zcong-zero-equiv-div zprime-zdvd-zmult*)

lemma *zcong-zprime-prod-zero-contra*: [| *zprime p*; *0* < *a* |] ==>
~[*a* = *0*](*mod p*) & ~[*b* = *0*](*mod p*) ==> ~[*a* ∗ *b* = *0*] (*mod p*)
apply *auto*
apply (*frule-tac a* = *a* **and** *b* = *b* **and** *p* = *p* **in** *zcong-zprime-prod-zero*)
apply *auto*
done

lemma *zcong-not-zero*: [| *0* < *x*; *x* < *m* |] ==> ~[*x* = *0*] (*mod m*)
by (*auto simp add*: *zcong-zero-equiv-div zdvd-not-zless*)

lemma *zcong-zero*: [| *0* ≤ *x*; *x* < *m*; [*x* = *0*](*mod m*) |] ==> *x* = *0*
apply (*drule order-le-imp-less-or-eq*, *auto*)
apply (*frule-tac m* = *m* **in** *zcong-not-zero*)
apply *auto*
done

lemma *all-relprime-prod-relprime*: [| *finite A*; ∀ *x* ∈ *A*. *zgcd x y* = *1* |]
==> *zgcd* (*setprod id A*) *y* = *1*
by (*induct set*: *finite*) (*auto simp add*: *zgcd-zgcd-zmult*)

13.3 Some properties of MultInv

lemma *MultInv-prop1*: [| *2* < *p*; [*x* = *y*] (*mod p*) |] ==>
[(*MultInv p x*) = (*MultInv p y*)] (*mod p*)
by (*auto simp add*: *MultInv-def zcong-zpower*)

lemma *MultInv-prop2*: [| *2* < *p*; *zprime p*; ~([*x* = *0*](*mod p*)) |] ==>
[(*x* ∗ (*MultInv p x*)) = *1*] (*mod p*)
proof (*simp add*: *MultInv-def zcong-eq-zdvd-prop*)
assume *1*: *2* < *p* **and** *2*: *zprime p* **and** *3*: ~ *p dvd x*
have *x* ∗ *x* ˆ *nat* (*p* − *2*) = *x* ˆ (*nat* (*p* − *2*) + *1*)
by *auto*
also from *1* **have** *nat* (*p* − *2*) + *1* = *nat* (*p* − *2* + *1*)
by (*simp only*: *nat-add-distrib*)
also have *p* − *2* + *1* = *p* − *1* **by** *arith*
finally have [*x* ∗ *x* ˆ *nat* (*p* − *2*) = *x* ˆ *nat* (*p* − *1*)] (*mod p*)
by (*rule ssubst*, *auto*)
also from *2 3* **have** [*x* ˆ *nat* (*p* − *1*) = *1*] (*mod p*)
by (*auto simp add*: *Little-Fermat*)
finally (*zcong-trans*) **show** [*x* ∗ *x* ˆ *nat* (*p* − *2*) = *1*] (*mod p*) **.**
qed

lemma *MultInv-prop2a*: [| *2* < *p*; *zprime p*; ~([*x* = *0*](*mod p*)) |] ==>

```
  [(MultInv p x) * x = 1] (mod p)
  by (auto simp add: MultInv-prop2 ac-simps)

lemma aux-1: 2 < p ==> ((nat p) - 2) = (nat (p - 2))
  by (simp add: nat-diff-distrib)

lemma aux-2: 2 < p ==> 0 < nat (p - 2)
  by auto

lemma MultInv-prop3: [| 2 < p; zprime p; ~([x = 0](mod p)) |] ==>
    ~([MultInv p x = 0](mod p))
  apply (auto simp add: MultInv-def zcong-eq-zdvd-prop aux-1)
  apply (drule aux-2)
  apply (drule zpower-zdvd-prop2, auto)
  done

lemma aux--1: [| 2 < p; zprime p; ~([x = 0](mod p))|] ==>
    [(MultInv p (MultInv p x)) = (x * (MultInv p x) *
      (MultInv p (MultInv p x)))] (mod p)
  apply (drule MultInv-prop2, auto)
  apply (drule-tac k = MultInv p (MultInv p x) in zcong-scalar, auto)
  apply (auto simp add: zcong-sym)
  done

lemma aux--2: [| 2 < p; zprime p; ~([x = 0](mod p))|] ==>
    [(x * (MultInv p x) * (MultInv p (MultInv p x))) = x] (mod p)
  apply (frule MultInv-prop3, auto)
  apply (insert MultInv-prop2 [of p MultInv p x], auto)
  apply (drule MultInv-prop2, auto)
  apply (drule-tac k = x in zcong-scalar2, auto)
  apply (auto simp add: ac-simps)
  done

lemma MultInv-prop4: [| 2 < p; zprime p; ~([x = 0](mod p)) |] ==>
    [(MultInv p (MultInv p x)) = x] (mod p)
  apply (frule aux--1, auto)
  apply (drule aux--2, auto)
  apply (drule zcong-trans, auto)
  done

lemma MultInv-prop5: [| 2 < p; zprime p; ~([x = 0](mod p));
    ~([y = 0](mod p)); [(MultInv p x) = (MultInv p y)] (mod p) |] ==>
    [x = y] (mod p)
  apply (drule-tac a = MultInv p x and b = MultInv p y and
    m = p and k = x in zcong-scalar)
  apply (insert MultInv-prop2 [of p x], simp)
  apply (auto simp only: zcong-sym [of MultInv p x * x])
  apply (auto simp add: ac-simps)
  apply (drule zcong-trans, auto)
```

```
  apply (drule-tac a = x * MultInv p y and k = y in zcong-scalar, auto)
  apply (insert MultInv-prop2a [of p y], auto simp add: ac-simps)
  apply (insert zcong-zmult-prop2 [of y * MultInv p y 1 p y x])
  apply (auto simp add: zcong-sym)
  done

lemma MultInv-zcong-prop1: [| 2 < p; [j = k] (mod p) |] ==>
    [a * MultInv p j = a * MultInv p k] (mod p)
  by (drule MultInv-prop1, auto simp add: zcong-scalar2)

lemma aux---1: [j = a * MultInv p k] (mod p) ==>
    [j * k = a * MultInv p k * k] (mod p)
  by (auto simp add: zcong-scalar)

lemma aux---2: [|2 < p; zprime p; ~([k = 0](mod p));
    [j * k = a * MultInv p k * k] (mod p) |] ==> [j * k = a] (mod p)
  apply (insert MultInv-prop2a [of p k] zcong-zmult-prop2
    [of MultInv p k * k 1 p j * k a])
  apply (auto simp add: ac-simps)
  done

lemma aux---3: [j * k = a] (mod p) ==> [(MultInv p j) * j * k =
     (MultInv p j) * a] (mod p)
  by (auto simp add: mult.assoc zcong-scalar2)

lemma aux---4: [|2 < p; zprime p; ~([j = 0](mod p));
    [(MultInv p j) * j * k = (MultInv p j) * a] (mod p) |]
       ==> [k = a * (MultInv p j)] (mod p)
  apply (insert MultInv-prop2a [of p j] zcong-zmult-prop1
    [of MultInv p j * j 1 p MultInv p j * a k])
  apply (auto simp add: ac-simps zcong-sym)
  done

lemma MultInv-zcong-prop2: [| 2 < p; zprime p; ~([k = 0](mod p));
    ~([j = 0](mod p)); [j = a * MultInv p k] (mod p) |] ==>
    [k = a * MultInv p j] (mod p)
  apply (drule aux---1)
  apply (frule aux---2, auto)
  by (drule aux---3, drule aux---4, auto)

lemma MultInv-zcong-prop3: [| 2 < p; zprime p; ~([a = 0](mod p));
    ~([k = 0](mod p)); ~([j = 0](mod p));
    [a * MultInv p j = a * MultInv p k] (mod p) |] ==>
      [j = k] (mod p)
  apply (auto simp add: zcong-eq-zdvd-prop [of a p])
  apply (frule zprime-imp-zrelprime, auto)
  apply (insert zcong-cancel2 [of p a MultInv p j MultInv p k], auto)
  apply (drule MultInv-prop5, auto)
  done
```

end

14 Residue Sets

theory *Residues*
imports *Int2*
begin

Define the residue of a set, the standard residue, quadratic residues, and prove some basic properties.

definition *ResSet* :: *int => int set => bool*
  where *ResSet m X* = (∀ *y1 y2*. (*y1* ∈ *X* & *y2* ∈ *X* & [*y1* = *y2*] (*mod m*) --> *y1* = *y2*))

definition *StandardRes* :: *int => int => int*
  where *StandardRes m x* = *x mod m*

definition *QuadRes* :: *int => int => bool*
  where *QuadRes m x* = (∃ *y*. ([y^2 = *x*] (*mod m*)))

definition *Legendre* :: *int => int => int* **where**
  Legendre a p = (*if* ([*a* = *0*] (*mod p*)) *then 0*
          else if (*QuadRes p a*) *then 1*
          else −*1*)

definition *SR* :: *int => int set*
  where *SR p* = {*x*. (*0* ≤ *x*) & (*x* < *p*)}

definition *SRStar* :: *int => int set*
  where *SRStar p* = {*x*. (*0* < *x*) & (*x* < *p*)}

14.1 Some useful properties of StandardRes

lemma *StandardRes-prop1*: [*x* = *StandardRes m x*] (*mod m*)
  by (*auto simp add*: *StandardRes-def zcong-zmod*)

lemma *StandardRes-prop2*: *0* < *m* ==> (*StandardRes m x1* = *StandardRes m x2*)
    = ([*x1* = *x2*] (*mod m*))
  by (*auto simp add*: *StandardRes-def zcong-zmod-eq*)

lemma *StandardRes-prop3*: (~[*x* = *0*] (*mod p*)) = (~(*StandardRes p x* = *0*))
  by (*auto simp add*: *StandardRes-def zcong-def dvd-eq-mod-eq-0*)

lemma *StandardRes-prop4*: *2* < *m*
    ==> [*StandardRes m x* ∗ *StandardRes m y* = (*x* ∗ *y*)] (*mod m*)
  by (*auto simp add*: *StandardRes-def zcong-zmod-eq*

mod-mult-eq [*of x y m*])

lemma *StandardRes-lbound*: *0* < *p* ==> *0* ≤ *StandardRes p x*
by (*auto simp add*: *StandardRes-def*)

lemma *StandardRes-ubound*: *0* < *p* ==> *StandardRes p x* < *p*
by (*auto simp add*: *StandardRes-def*)

lemma *StandardRes-eq-zcong*:
(*StandardRes m x* = *0*) = ([*x* = *0*](*mod m*))
by (*auto simp add*: *StandardRes-def zcong-eq-zdvd-prop dvd-def*)

14.2 Relations between StandardRes, SRStar, and SR

lemma *SRStar-SR-prop*: *x* ∈ *SRStar p* ==> *x* ∈ *SR p*
by (*auto simp add*: *SRStar-def SR-def*)

lemma *StandardRes-SR-prop*: *x* ∈ *SR p* ==> *StandardRes p x* = *x*
by (*auto simp add*: *SR-def StandardRes-def mod-pos-pos-trivial*)

lemma *StandardRes-SRStar-prop1*: *2* < *p* ==> (*StandardRes p x* ∈ *SRStar p*)
= (~[*x* = *0*] (*mod p*))
apply (*auto simp add*: *StandardRes-prop3 StandardRes-def SRStar-def*)
apply (*subgoal-tac 0* < *p*)
apply (*drule-tac a* = *x* **in** *pos-mod-sign*, *arith*, *simp*)
done

lemma *StandardRes-SRStar-prop1a*: *x* ∈ *SRStar p* ==> ~([*x* = *0*] (*mod p*))
by (*auto simp add*: *SRStar-def zcong-def zdvd-not-zless*)

lemma *StandardRes-SRStar-prop2*: [| *2* < *p*; *zprime p*; *x* ∈ *SRStar p* |]
==> *StandardRes p* (*MultInv p x*) ∈ *SRStar p*
apply (*frule-tac x* = (*MultInv p x*) **in** *StandardRes-SRStar-prop1*, *simp*)
apply (*rule MultInv-prop3*)
apply (*auto simp add*: *SRStar-def zcong-def zdvd-not-zless*)
done

lemma *StandardRes-SRStar-prop3*: *x* ∈ *SRStar p* ==> *StandardRes p x* = *x*
by (*auto simp add*: *SRStar-SR-prop StandardRes-SR-prop*)

lemma *StandardRes-SRStar-prop4*: [| *zprime p*; *2* < *p*; *x* ∈ *SRStar p* |]
==> *StandardRes p x* ∈ *SRStar p*
by (*frule StandardRes-SRStar-prop3*, *auto*)

lemma *SRStar-mult-prop1*: [| *zprime p*; *2* < *p*; *x* ∈ *SRStar p*; *y* ∈ *SRStar p*|]
==> (*StandardRes p* (*x* ∗ *y*)):*SRStar p*
apply (*frule-tac x* = *x* **in** *StandardRes-SRStar-prop4*, *auto*)
apply (*frule-tac x* = *y* **in** *StandardRes-SRStar-prop4*, *auto*)
apply (*auto simp add*: *StandardRes-SRStar-prop1 zcong-zmult-prop3*)

done

lemma *SRStar-mult-prop2*: [| *zprime p*; *2* < *p*; ~([*a* = *0*](*mod p*));
x ∈ *SRStar p* |]
==> *StandardRes p* (*a* ∗ *MultInv p x*) ∈ *SRStar p*
apply (*frule-tac x* = *x* **in** *StandardRes-SRStar-prop2*, *auto*)
apply (*frule-tac x* = *MultInv p x* **in** *StandardRes-SRStar-prop1*)
apply (*auto simp add*: *StandardRes-SRStar-prop1 zcong-zmult-prop3*)
done

lemma *SRStar-card*: *2* < *p* ==> *int*(*card*(*SRStar p*)) = *p* − *1*
by (*auto simp add*: *SRStar-def int-card-bdd-int-set-l-l*)

lemma *SRStar-finite*: *2* < *p* ==> *finite*(*SRStar p*)
by (*auto simp add*: *SRStar-def bdd-int-set-l-l-finite*)

14.3 Properties relating ResSets with StandardRes

lemma *aux*: *x mod m* = *y mod m* ==> [*x* = *y*] (*mod m*)
apply (*subgoal-tac x* = *y* ==> [*x* = *y*](*mod m*))
apply (*subgoal-tac* [*x mod m* = *y mod m*] (*mod m*) ==> [*x* = *y*] (*mod m*))
apply (*auto simp add*: *zcong-zmod* [*of x y m*])
done

lemma *StandardRes-inj-on-ResSet*: *ResSet m X* ==> (*inj-on* (*StandardRes m*) *X*)
apply (*auto simp add*: *ResSet-def StandardRes-def inj-on-def*)
apply (*drule-tac m* = *m* **in** *aux*, *auto*)
done

lemma *StandardRes-Sum*: [| *finite X*; *0* < *m* |]
==> [*setsum f X* = *setsum* (*StandardRes m o f*) *X*](*mod m*)
apply (*rule-tac F* = *X* **in** *finite-induct*)
apply (*auto intro*!: *zcong-zadd simp add*: *StandardRes-prop1*)
done

lemma *SR-pos*: *0* < *m* ==> (*StandardRes m* ‘ *X*) ⊆ {*x*. *0* ≤ *x* & *x* < *m*}
by (*auto simp add*: *StandardRes-ubound StandardRes-lbound*)

lemma *ResSet-finite*: *0* < *m* ==> *ResSet m X* ==> *finite X*
apply (*rule-tac f* = *StandardRes m* **in** *finite-imageD*)
apply (*rule-tac B* = {*x*. (*0* :: *int*) ≤ *x* & *x* < *m*} **in** *finite-subset*)
apply (*auto simp add*: *StandardRes-inj-on-ResSet bdd-int-set-l-finite SR-pos*)
done

lemma *mod-mod-is-mod*: [*x* = *x mod m*](*mod m*)
by (*auto simp add*: *zcong-zmod*)

lemma *StandardRes-prod*: [| *finite X*; *0* < *m* |]

==> [*setprod f X* = *setprod* (*StandardRes m o f*) *X*] (*mod m*)
apply (*rule-tac F* = *X* **in** *finite-induct*)
apply (*auto intro*!: *zcong-zmult simp add*: *StandardRes-prop1*)
done

lemma *ResSet-image*:
[| *0* < *m*; *ResSet m A*; ∀ *x* ∈ *A*. ∀ *y* ∈ *A*. ([*f x* = *f y*](*mod m*) −−> *x* = *y*) |]
==>
ResSet m (*f* ‘ *A*)
by (*auto simp add*: *ResSet-def*)

14.4 Property for SRStar

lemma *ResSet-SRStar-prop*: *ResSet p* (*SRStar p*)
by (*auto simp add*: *SRStar-def ResSet-def zcong-zless-imp-eq*)

end

15 Parity: Even and Odd Integers

theory *EvenOdd*
imports *Int2*
begin

definition *zOdd* :: *int set*
where *zOdd* = {*x*. ∃ *k*. *x* = *2* ∗ *k* + *1*}

definition *zEven* :: *int set*
where *zEven* = {*x*. ∃ *k*. *x* = *2* ∗ *k*}

15.1 Some useful properties about even and odd

lemma *zOddI* [*intro?*]: *x* = *2* ∗ *k* + *1* =⇒ *x* ∈ *zOdd*
and *zOddE* [*elim?*]: *x* ∈ *zOdd* =⇒ (!!*k*. *x* = *2* ∗ *k* + *1* =⇒ *C*) =⇒ *C*
by (*auto simp add*: *zOdd-def*)

lemma *zEvenI* [*intro?*]: *x* = *2* ∗ *k* =⇒ *x* ∈ *zEven*
and *zEvenE* [*elim?*]: *x* ∈ *zEven* =⇒ (!!*k*. *x* = *2* ∗ *k* =⇒ *C*) =⇒ *C*
by (*auto simp add*: *zEven-def*)

lemma *one-not-even*: ~(*1* ∈ *zEven*)
proof
assume *1* ∈ *zEven*
then obtain *k* :: *int* **where** *1* = *2* ∗ *k* **..**
then show *False* **by** *arith*
qed

lemma *even-odd-conj*: ~(*x* ∈ *zOdd* & *x* ∈ *zEven*)
proof −

```
  {
    fix a b
    assume 2 ∗ (a::int) = 2 ∗ (b::int) + 1
    then have 2 ∗ (a::int) − 2 ∗ (b :: int) = 1
      by arith
    then have 2 ∗ (a − b) = 1
      by (auto simp add: left-diff-distrib)
    moreover have (2 ∗ (a − b)):zEven
      by (auto simp only: zEven-def)
    ultimately have False
      by (auto simp add: one-not-even)
  }
  then show ?thesis
    by (auto simp add: zOdd-def zEven-def)
qed

lemma even-odd-disj: (x ∈ zOdd | x ∈ zEven)
  by (simp add: zOdd-def zEven-def) arith

lemma not-odd-impl-even: ~(x ∈ zOdd) ==> x ∈ zEven
  using even-odd-disj by auto

lemma odd-mult-odd-prop: (x∗y):zOdd ==> x ∈ zOdd
proof (rule classical)
  assume ¬ ?thesis
  then have x ∈ zEven by (rule not-odd-impl-even)
  then obtain a where a: x = 2 ∗ a ..
  assume x ∗ y : zOdd
  then obtain b where x ∗ y = 2 ∗ b + 1 ..
  with a have 2 ∗ a ∗ y = 2 ∗ b + 1 by simp
  then have 2 ∗ a ∗ y − 2 ∗ b = 1
    by arith
  then have 2 ∗ (a ∗ y − b) = 1
    by (auto simp add: left-diff-distrib)
  moreover have (2 ∗ (a ∗ y − b)):zEven
    by (auto simp only: zEven-def)
  ultimately have False
    by (auto simp add: one-not-even)
  then show ?thesis ..
qed

lemma odd-minus-one-even: x ∈ zOdd ==> (x − 1):zEven
  by (auto simp add: zOdd-def zEven-def)

lemma even-div-2-prop1: x ∈ zEven ==> (x mod 2) = 0
  by (auto simp add: zEven-def)

lemma even-div-2-prop2: x ∈ zEven ==> (2 ∗ (x div 2)) = x
  by (auto simp add: zEven-def)
```

lemma *even-plus-even*: [| $x \in zEven$; $y \in zEven$ |] ==> $x + y \in zEven$
  apply (*auto simp add*: *zEven-def*)
  apply (*auto simp only*: *distrib-left* [*symmetric*])
  done

lemma *even-times-either*: $x \in zEven$ ==> $x * y \in zEven$
  by (*auto simp add*: *zEven-def*)

lemma *even-minus-even*: [| $x \in zEven$; $y \in zEven$ |] ==> $x - y \in zEven$
  apply (*auto simp add*: *zEven-def*)
  apply (*auto simp only*: *right-diff-distrib* [*symmetric*])
  done

lemma *odd-minus-odd*: [| $x \in zOdd$; $y \in zOdd$ |] ==> $x - y \in zEven$
  apply (*auto simp add*: *zOdd-def zEven-def*)
  apply (*auto simp only*: *right-diff-distrib* [*symmetric*])
  done

lemma *even-minus-odd*: [| $x \in zEven$; $y \in zOdd$ |] ==> $x - y \in zOdd$
  apply (*auto simp add*: *zOdd-def zEven-def*)
  apply (*rule-tac* $x = k - ka - 1$ **in** *exI*)
  apply *auto*
  done

lemma *odd-minus-even*: [| $x \in zOdd$; $y \in zEven$ |] ==> $x - y \in zOdd$
  apply (*auto simp add*: *zOdd-def zEven-def*)
  apply (*auto simp only*: *right-diff-distrib* [*symmetric*])
  done

lemma *odd-times-odd*: [| $x \in zOdd$; $y \in zOdd$ |] ==> $x * y \in zOdd$
  apply (*auto simp add*: *zOdd-def distrib-right distrib-left*)
  apply (*rule-tac* $x = 2 * ka * k + ka + k$ **in** *exI*)
  apply (*auto simp add*: *distrib-right*)
  done

lemma *odd-iff-not-even*: $(x \in zOdd) = (\sim (x \in zEven))$
  using *even-odd-conj even-odd-disj* **by** *auto*

lemma *even-product*: $x * y \in zEven$ ==> $x \in zEven$ | $y \in zEven$
  using *odd-iff-not-even odd-times-odd* **by** *auto*

lemma *even-diff*: $x - y \in zEven = ((x \in zEven) = (y \in zEven))$
proof
  assume *xy*: $x - y \in zEven$
  {
    assume *x*: $x \in zEven$
    have $y \in zEven$
    proof (*rule classical*)

```
      assume ¬ ?thesis
      then have y ∈ zOdd
        by (simp add: odd-iff-not-even)
      with x have x − y ∈ zOdd
        by (simp add: even-minus-odd)
      with xy have False
        by (auto simp add: odd-iff-not-even)
      then show ?thesis ..
    qed
  } moreover {
    assume y: y ∈ zEven
    have x ∈ zEven
    proof (rule classical)
      assume ¬ ?thesis
      then have x ∈ zOdd
        by (auto simp add: odd-iff-not-even)
      with y have x − y ∈ zOdd
        by (simp add: odd-minus-even)
      with xy have False
        by (auto simp add: odd-iff-not-even)
      then show ?thesis ..
    qed
  }
  ultimately show (x ∈ zEven) = (y ∈ zEven)
    by (auto simp add: odd-iff-not-even even-minus-even odd-minus-odd
      even-minus-odd odd-minus-even)
next
  assume (x ∈ zEven) = (y ∈ zEven)
  then show x − y ∈ zEven
    by (auto simp add: odd-iff-not-even even-minus-even odd-minus-odd
      even-minus-odd odd-minus-even)
qed

lemma neg-one-even-power: [| x ∈ zEven; 0 ≤ x |] ==> (−1::int)ˆ(nat x) = 1
proof −
  assume x ∈ zEven and 0 ≤ x
  from ‹x ∈ zEven› obtain a where x = 2 ∗ a ..
  with ‹0 ≤ x› have 0 ≤ a by simp
  from ‹0 ≤ x› and ‹x = 2 ∗ a› have nat x = nat (2 ∗ a)
    by simp
  also from ‹x = 2 ∗ a› have nat (2 ∗ a) = 2 ∗ nat a
    by (simp add: nat-mult-distrib)
  finally have (−1::int)ˆnat x = (−1)ˆ(2 ∗ nat a)
    by simp
  also have ... = (−1::int)^2 ˆ nat a
    by (simp add: power-mult)
  also have (−1::int)^2 = 1
    by simp
  finally show ?thesis
```

by *simp*
qed

lemma *neg-one-odd-power*: [| *x* ∈ *zOdd*; *0* ≤ *x* |] ==> (−*1*::*int*) ^(*nat x*) = −*1*
proof −
assume *x* ∈ *zOdd* **and** *0* ≤ *x*
from ‹*x* ∈ *zOdd*› **obtain** *a* **where** *x* = *2* ∗ *a* + *1* **..**
with ‹*0* ≤ *x*› **have** *a*: *0* ≤ *a* **by** *simp*
with ‹*0* ≤ *x*› **and** ‹*x* = *2* ∗ *a* + *1*› **have** *nat x* = *nat* (*2* ∗ *a* + *1*)
by *simp*
also from *a* **have** *nat* (*2* ∗ *a* + *1*) = *2* ∗ *nat a* + *1*
by (*auto simp add*: *nat-mult-distrib nat-add-distrib*)
finally have (−*1*::*int*) ^ *nat x* = (−*1*) ^(*2* ∗ *nat a* + *1*)
by *simp*
also have ... = ((−*1*::*int*)^2) ^ *nat a* ∗ (−*1*) ^*1*
by (*auto simp add*: *power-mult power-add*)
also have (−*1*::*int*)^2 = *1*
by *simp*
finally show *?thesis*
by *simp*
qed

lemma *neg-one-power-parity*: [| *0* ≤ *x*; *0* ≤ *y*; (*x* ∈ *zEven*) = (*y* ∈ *zEven*) |] ==>
(−*1*::*int*) ^(*nat x*) = (−*1*::*int*) ^(*nat y*)
using *even-odd-disj* [*of x*] *even-odd-disj* [*of y*]
by (*auto simp add*: *neg-one-even-power neg-one-odd-power*)

lemma *one-not-neg-one-mod-m*: *2* < *m* ==> ~([*1* = −*1*] (*mod m*))
by (*auto simp add*: *zcong-def zdvd-not-zless*)

lemma *even-div-2-l*: [| *y* ∈ *zEven*; *x* < *y* |] ==> *x div 2* < *y div 2*
proof −
assume *y* ∈ *zEven* **and** *x* < *y*
from ‹*y* ∈ *zEven*› **obtain** *k* **where** *k*: *y* = *2* ∗ *k* **..**
with ‹*x* < *y*› **have** *x* < *2* ∗ *k* **by** *simp*
then have *x div 2* < *k* **by** (*auto simp add*: *div-prop1*)
also have *k* = (*2* ∗ *k*) *div 2* **by** *simp*
finally have *x div 2* < *2* ∗ *k div 2* **by** *simp*
with *k* **show** *?thesis* **by** *simp*
qed

lemma *even-sum-div-2*: [| *x* ∈ *zEven*; *y* ∈ *zEven* |] ==> (*x* + *y*) *div 2* = *x div 2* + *y div 2*
by (*auto simp add*: *zEven-def*)

lemma *even-prod-div-2*: [| *x* ∈ *zEven* |] ==> (*x* ∗ *y*) *div 2* = (*x div 2*) ∗ *y*
by (*auto simp add*: *zEven-def*)

lemma *zprime-zOdd-eq-grt-2*: *zprime p* ==> (*p* ∈ *zOdd*) = (*2* < *p*)
  apply (*auto simp add*: *zOdd-def zprime-def*)
  apply (*drule-tac x* = *2* **in** *allE*)
  using *odd-iff-not-even* [*of p*]
  apply (*auto simp add*: *zOdd-def zEven-def*)
  done

lemma *neg-one-special*: *finite A* ==>
  ((− *1*) ˆ *card A*) ∗ ((− *1*) ˆ *card A*) = (*1* :: *int*)
  unfolding *power-add* [*symmetric*] **by** *simp*

lemma *neg-one-power*: (−*1*::*int*)ˆ*n* = *1* | (−*1*::*int*)ˆ*n* = −*1*
  by (*induct n*) *auto*

lemma *neg-one-power-eq-mod-m*: [| *2* < *m*; [(−*1*::*int*)ˆ*j* = (−*1*::*int*)ˆ*k*] (*mod m*) |]
    ==> ((−*1*::*int*)ˆ*j* = (−*1*::*int*)ˆ*k*)
  using *neg-one-power* [*of j*] **and** *ListMem.insert neg-one-power* [*of k*]
  by (*auto simp add*: *one-not-neg-one-mod-m zcong-sym*)

end

16 Euler's criterion

theory *Euler*
imports *Residues EvenOdd*
begin

definition *MultInvPair* :: *int* => *int* => *int* => *int set*
  where *MultInvPair a p j* = {*StandardRes p j*, *StandardRes p* (*a* ∗ (*MultInv p j*))}

definition *SetS* :: *int* => *int* => *int set set*
  where *SetS a p* = *MultInvPair a p* ‘ *SRStar p*

16.1 Property for MultInvPair

lemma *MultInvPair-prop1a*:
  [| *zprime p*; *2* < *p*; ~([*a* = *0*](*mod p*));
    X ∈ (*SetS a p*); *Y* ∈ (*SetS a p*);
    ~((*X* ∩ *Y*) = {}) |] ==> *X* = *Y*
  apply (*auto simp add*: *SetS-def*)
  apply (*drule StandardRes-SRStar-prop1a*)+ **defer** *1*
  apply (*drule StandardRes-SRStar-prop1a*)+
  apply (*auto simp add*: *MultInvPair-def StandardRes-prop2 zcong-sym*)

```
  apply (drule notE, rule MultInv-zcong-prop1, auto)[]
  apply (drule notE, rule MultInv-zcong-prop2, auto simp add: zcong-sym)[]
  apply (drule MultInv-zcong-prop2, auto simp add: zcong-sym)[]
  apply (drule MultInv-zcong-prop3, auto simp add: zcong-sym)[]
  apply (drule MultInv-zcong-prop1, auto)[]
  apply (drule MultInv-zcong-prop2, auto simp add: zcong-sym)[]
  apply (drule MultInv-zcong-prop2, auto simp add: zcong-sym)[]
  apply (drule MultInv-zcong-prop3, auto simp add: zcong-sym)[]
  done

lemma MultInvPair-prop1b:
  [| zprime p; 2 < p; ~([a = 0](mod p));
      X ∈ (SetS a p); Y ∈ (SetS a p);
      X ≠ Y |] ==> X ∩ Y = {}
  apply (rule notnotD)
  apply (rule notI)
  apply (drule MultInvPair-prop1a, auto)
  done

lemma MultInvPair-prop1c: [| zprime p; 2 < p; ~([a = 0](mod p)) |] ==>
    ∀ X ∈ SetS a p. ∀ Y ∈ SetS a p. X ≠ Y --> X∩Y = {}
  by (auto simp add: MultInvPair-prop1b)

lemma MultInvPair-prop2: [| zprime p; 2 < p; ~([a = 0](mod p)) |] ==>
                          ⋃(SetS a p) = SRStar p
  apply (auto simp add: SetS-def MultInvPair-def StandardRes-SRStar-prop4
    SRStar-mult-prop2)
  apply (frule StandardRes-SRStar-prop3)
  apply (rule bexI, auto)
  done

lemma MultInvPair-distinct:
  assumes zprime p and 2 < p and
    ~([a = 0] (mod p)) and
    ~([j = 0] (mod p)) and
    ~(QuadRes p a)
  shows ~([j = a * MultInv p j] (mod p))
proof
  assume [j = a * MultInv p j] (mod p)
  then have [j * j = (a * MultInv p j) * j] (mod p)
    by (auto simp add: zcong-scalar)
  then have a:[j * j = a * (MultInv p j * j)] (mod p)
    by (auto simp add: ac-simps)
  have [j * j = a] (mod p)
  proof -
    from assms(1,2,4) have [MultInv p j * j = 1] (mod p)
      by (simp add: MultInv-prop2a)
    from this and a show ?thesis
      by (auto simp add: zcong-zmult-prop2)
```

qed
then have [j^2 = a] (mod p) **by** (simp add: power2-eq-square)
with assms **show** False **by** (simp add: QuadRes-def)
qed

lemma MultInvPair-card-two: [| zprime p; 2 < p; ~([a = 0] (mod p));
~(QuadRes p a); ~([j = 0] (mod p)) |] ==>
card (MultInvPair a p j) = 2
apply (auto simp add: MultInvPair-def)
apply (subgoal-tac ~ (StandardRes p j = StandardRes p (a * MultInv p j)))
apply auto
apply (metis MultInvPair-distinct StandardRes-def aux)
done

16.2 Properties of SetS

lemma SetS-finite: 2 < p ==> finite (SetS a p)
by (auto simp add: SetS-def SRStar-finite [of p])

lemma SetS-elems-finite: ∀ X ∈ SetS a p. finite X
by (auto simp add: SetS-def MultInvPair-def)

lemma SetS-elems-card: [| zprime p; 2 < p; ~([a = 0] (mod p));
~(QuadRes p a) |] ==>
∀ X ∈ SetS a p. card X = 2
apply (auto simp add: SetS-def)
apply (frule StandardRes-SRStar-prop1a)
apply (rule MultInvPair-card-two, auto)
done

lemma Union-SetS-finite: 2 < p ==> finite (⋃(SetS a p))
by (auto simp add: SetS-finite SetS-elems-finite)

lemma card-setsum-aux: [| finite S; ∀ X ∈ S. finite (X::int set);
∀ X ∈ S. card X = n |] ==> setsum card S = setsum (%x. n) S
by (induct set: finite) auto

lemma SetS-card:
assumes zprime p **and** 2 < p **and** ~([a = 0] (mod p)) **and** ~(QuadRes p a)
shows int(card(SetS a p)) = (p − 1) div 2
proof −
have (p − 1) = 2 * int(card(SetS a p))
proof −
have p − 1 = int(card(⋃(SetS a p)))
by (auto simp add: assms MultInvPair-prop2 SRStar-card)
also have ... = int (setsum card (SetS a p))
by (auto simp add: assms SetS-finite SetS-elems-finite
MultInvPair-prop1c [of p a] card-Union-disjoint)
also have ... = int(setsum (%x.2) (SetS a p))

using *assms* **by** (*auto simp add*: *SetS-elems-card SetS-finite SetS-elems-finite card-setsum-aux simp del*: *setsum-constant*)
also have ... = *2* ∗ *int*(*card*(*SetS a p*))
by (*auto simp add*: *assms SetS-finite setsum-const2*)
finally show *?thesis* **.**
qed
then show *?thesis* **by** *auto*
qed

lemma *SetS-setprod-prop*: [| *zprime p*; *2* < *p*; ~([*a* = *0*] (*mod p*));
~(*QuadRes p a*); *x* ∈ (*SetS a p*) |] ==>
[$\prod$ *x* = *a*] (*mod p*)
apply (*auto simp add*: *SetS-def MultInvPair-def*)
apply (*frule StandardRes-SRStar-prop1a*)
apply *hypsubst-thin*
apply (*subgoal-tac StandardRes p x* ≠ *StandardRes p* (*a* ∗ *MultInv p x*))
apply (*auto simp add*: *StandardRes-prop2 MultInvPair-distinct*)
apply (*frule-tac m* = *p* **and** *x* = *x* **and** *y* = (*a* ∗ *MultInv p x*) **in** *StandardRes-prop4*)
apply (*subgoal-tac* [*x* ∗ (*a* ∗ *MultInv p x*) = *a* ∗ (*x* ∗ *MultInv p x*)] (*mod p*))
apply (*drule-tac a* = *StandardRes p x* ∗ *StandardRes p* (*a* ∗ *MultInv p x*) **and**
b = *x* ∗ (*a* ∗ *MultInv p x*) **and**
c = *a* ∗ (*x* ∗ *MultInv p x*) **in** *zcong-trans*, *force*)
apply (*frule-tac p* = *p* **and** *x* = *x* **in** *MultInv-prop2*, *auto*)
apply (*metis StandardRes-SRStar-prop3 mult-1-right mult.commute zcong-sym zcong-zmult-prop1*)
apply (*auto simp add*: *ac-simps*)
done

lemma *aux1*: [| *0* < *x*; (*x*::*int*) < *a*; *x* ≠ (*a* − *1*) |] ==> *x* < *a* − *1*
by *arith*

lemma *aux2*: [| (*a*::*int*) < *c*; *b* < *c* |] ==> (*a* ≤ *b* | *b* ≤ *a*)
by *auto*

lemma *d22set-induct-old*: ($\bigwedge$*a*::*int*. *1* < *a* ⟶ *P* (*a* − *1*) ⟹ *P a*) ⟹ *P x*
using *d22set.induct* **by** *blast*

lemma *SRStar-d22set-prop*: *2* < *p* ⟹ (*SRStar p*) = {*1*} ∪ (*d22set* (*p* − *1*))
apply (*induct p rule*: *d22set-induct-old*)
apply *auto*
apply (*simp add*: *SRStar-def d22set.simps*)
apply (*simp add*: *SRStar-def d22set.simps*, *clarify*)
apply (*frule aux1*)
apply (*frule aux2*, *auto*)
apply (*simp-all add*: *SRStar-def*)
apply (*simp add*: *d22set.simps*)
apply (*frule d22set-le*)
apply (*frule d22set-g-1*, *auto*)
done

lemma *Union-SetS-setprod-prop1*:
  assumes *zprime p* **and** *2* < *p* **and** ~([*a* = *0*] (*mod p*)) **and**
    ~(*QuadRes p a*)
  shows [$\prod$($\bigcup$(*SetS a p*)) = *a* ^ *nat* ((*p* − *1*) *div 2*)] (*mod p*)
proof −
  from *assms* **have** [$\prod$($\bigcup$(*SetS a p*)) = *setprod* (*setprod* (%*x*. *x*)) (*SetS a p*)] (*mod p*)
    by (*auto simp add*: *SetS-finite SetS-elems-finite*
      MultInvPair-prop1c setprod.Union-disjoint)
  also have [*setprod* (*setprod* (%*x*. *x*)) (*SetS a p*) =
      setprod (%*x*. *a*) (*SetS a p*)] (*mod p*)
    by (*rule setprod-same-function-zcong*)
      (*auto simp add*: *assms SetS-setprod-prop SetS-finite*)
  also (*zcong-trans*) **have** [*setprod* (%*x*. *a*) (*SetS a p*) =
      a^(*card* (*SetS a p*))] (*mod p*)
    by (*auto simp add*: *assms SetS-finite setprod-constant*)
  finally (*zcong-trans*) **show** *?thesis*
    apply (*rule zcong-trans*)
    apply (*subgoal-tac card*(*SetS a p*) = *nat*((*p* − *1*) *div 2*), *auto*)
    apply (*subgoal-tac nat*(*int*(*card*(*SetS a p*))) = *nat*((*p* − *1*) *div 2*), *force*)
    apply (*auto simp add*: *assms SetS-card*)
    done
qed

lemma *Union-SetS-setprod-prop2*:
  assumes *zprime p* **and** *2* < *p* **and** ~([*a* = *0*](*mod p*))
  shows $\prod$($\bigcup$(*SetS a p*)) = *zfact* (*p* − *1*)
proof −
  from *assms* **have** $\prod$($\bigcup$(*SetS a p*)) = $\prod$(*SRStar p*)
    by (*auto simp add*: *MultInvPair-prop2*)
  also have ... = $\prod$({*1*} ∪ (*d22set* (*p* − *1*)))
    by (*auto simp add*: *assms SRStar-d22set-prop*)
  also have ... = *zfact*(*p* − *1*)
  proof −
    have ~(*1* ∈ *d22set* (*p* − *1*)) & *finite*(*d22set* (*p* − *1*))
      by (*metis d22set-fin d22set-g-1 linorder-neq-iff*)
    then have $\prod$({*1*} ∪ (*d22set* (*p* − *1*))) = $\prod$(*d22set* (*p* − *1*))
      by *auto*
    then show *?thesis*
      by (*auto simp add*: *d22set-prod-zfact*)
  qed
  finally show *?thesis* **.**
qed

lemma *zfact-prop*: [| *zprime p*; *2* < *p*; ~([*a* = *0*] (*mod p*)); ~(*QuadRes p a*) |] ==>
        [*zfact* (*p* − *1*) = *a* ^ *nat* ((*p* − *1*) *div 2*)] (*mod p*)
  apply (*frule Union-SetS-setprod-prop1*)

```
  apply (auto simp add: Union-SetS-setprod-prop2)
  done
```

Prove the first part of Euler's Criterion:

```
lemma Euler-part1: [| 2 < p; zprime p; ~([x = 0](mod p));
    ~(QuadRes p x) |] ==>
      [x^(nat (((p) - 1) div 2)) = -1](mod p)
  by (metis Wilson-Russ zcong-sym zcong-trans zfact-prop)
```

Prove another part of Euler Criterion:

```
lemma aux-1: 0 < p ==> (a::int) ^ nat (p) = a * a ^ (nat (p) - 1)
proof -
  assume 0 < p
  then have a ^ (nat p) =  a ^ (1 + (nat p - 1))
    by (auto simp add: diff-add-assoc)
  also have ... = (a ^ 1) * a ^ (nat(p) - 1)
    by (simp only: power-add)
  also have ... = a * a ^ (nat(p) - 1)
    by auto
  finally show ?thesis .
qed

lemma aux-2: [| (2::int) < p; p ∈ zOdd |] ==> 0 < ((p - 1) div 2)
proof -
  assume 2 < p and p ∈ zOdd
  then have (p - 1):zEven
    by (auto simp add: zEven-def zOdd-def)
  then have aux-1: 2 * ((p - 1) div 2) = (p - 1)
    by (auto simp add: even-div-2-prop2)
  with ‹2 < p› have 1 < (p - 1)
    by auto
  then have  1 < (2 * ((p - 1) div 2))
    by (auto simp add: aux-1)
  then have 0 < (2 * ((p - 1) div 2)) div 2
    by auto
  then show ?thesis by auto
qed

lemma Euler-part2:
   [| 2 < p; zprime p; [a = 0] (mod p) |] ==> [0 = a ^ nat ((p - 1) div 2)] (mod
p)
  apply (frule zprime-zOdd-eq-grt-2)
  apply (frule aux-2, auto)
  apply (frule-tac a = a in aux-1, auto)
  apply (frule zcong-zmult-prop1, auto)
  done
```

Prove the final part of Euler's Criterion:

lemma *aux--1*: [| ~([*x* = *0*] (*mod p*)); [y^2 = *x*] (*mod p*)|] ==> ~(*p dvd y*)
by (*metis dvdI power2-eq-square zcong-sym zcong-trans zcong-zero-equiv-div dvd-trans*)

lemma *aux--2*: *2* ∗ *nat*((*p* − *1*) *div 2*) = *nat* (*2* ∗ ((*p* − *1*) *div 2*))
by (*auto simp add*: *nat-mult-distrib*)

lemma *Euler-part3*: [| *2* < *p*; *zprime p*; ~([*x* = *0*](*mod p*)); *QuadRes p x* |] ==>

[*x*ˆ(*nat* (((*p*) − *1*) *div 2*)) = *1*](*mod p*)
apply (*subgoal-tac p* ∈ *zOdd*)
apply (*auto simp add*: *QuadRes-def*)
prefer *2*
apply (*metis zprime-zOdd-eq-grt-2*)
apply (*frule aux--1*, *auto*)
apply (*drule-tac z* = *nat* ((*p* − *1*) *div 2*) **in** *zcong-zpower*)
apply (*auto simp add*: *power-mult* [*symmetric*])
apply (*rule zcong-trans*)
apply (*auto simp add*: *zcong-sym* [*of x* ˆ *nat* ((*p* − *1*) *div 2*)])
apply (*metis Little-Fermat even-div-2-prop2 odd-minus-one-even mult-1 aux--2*)
done

Finally show Euler's Criterion:

theorem *Euler-Criterion*: [| *2* < *p*; *zprime p* |] ==> [(*Legendre a p*) =
*a*ˆ(*nat* (((*p*) − *1*) *div 2*))] (*mod p*)
apply (*auto simp add*: *Legendre-def Euler-part2*)
apply (*frule Euler-part3*, *auto simp add*: *zcong-sym*)[]
apply (*frule Euler-part1*, *auto simp add*: *zcong-sym*)[]
done

end

17 Gauss' Lemma

theory *Gauss*
imports *Euler*
begin

locale *GAUSS* =
fixes *p* :: *int*
fixes *a* :: *int*

assumes *p-prime*: *zprime p*
assumes *p-g-2*: *2* < *p*
assumes *p-a-relprime*: ~[*a* = *0*](*mod p*)
assumes *a-nonzero*: *0* < *a*
begin

definition *A* = {(*x*::*int*). *0* < *x* & *x* ≤ ((*p* − *1*) *div 2*)}

definition $B = (\%x.\ x * a)$ ‘ A
definition $C = StandardRes\ p$ ‘ B
definition $D = C \cap \{x.\ x \le ((p - 1)\ div\ 2)\}$
definition $E = C \cap \{x.\ ((p - 1)\ div\ 2) < x\}$
definition $F = (\%x.\ (p - x))$ ‘ E

17.1 Basic properties of p

lemma *p-odd*: $p \in zOdd$
  by (*auto simp add*: *p-prime p-g-2 zprime-zOdd-eq-grt-2*)

lemma *p-g-0*: $0 < p$
  using *p-g-2* **by** *auto*

lemma *int-nat*: $int\ (nat\ ((p - 1)\ div\ 2)) = (p - 1)\ div\ 2$
  using *ListMem.insert p-g-2* **by** (*auto simp add*: *pos-imp-zdiv-nonneg-iff*)

lemma *p-minus-one-l*: $(p - 1)\ div\ 2 < p$
proof −
  have $(p - 1)\ div\ 2 \le (p - 1)\ div\ 1$
    by (*rule zdiv-mono2*) (*auto simp add*: *p-g-0*)
  also have $\ldots = p - 1$ **by** *simp*
  finally show *?thesis* **by** *simp*
qed

lemma *p-eq*: $p = (2 * (p - 1)\ div\ 2) + 1$
  using *div-mult-self1-is-id* [*of* 2 $p - 1$] **by** *auto*

lemma (**in** −) *zodd-imp-zdiv-eq*: $x \in zOdd \Longrightarrow 2 * (x - 1)\ div\ 2 = 2 * ((x - 1)\ div\ 2)$
  apply (*frule odd-minus-one-even*)
  apply (*simp add*: *zEven-def*)
  apply (*subgoal-tac* $2 \ne 0$)
  apply (*frule-tac* $b = 2 :: int$ **and** $a = x - 1$ **in** *div-mult-self1-is-id*)
  apply (*auto simp add*: *even-div-2-prop2*)
  done

lemma *p-eq2*: $p = (2 * ((p - 1)\ div\ 2)) + 1$
  apply (*insert p-eq p-prime p-g-2 zprime-zOdd-eq-grt-2* [*of p*], *auto*)
  apply (*frule zodd-imp-zdiv-eq*, *auto*)
  done

17.2 Basic Properties of the Gauss Sets

lemma *finite-A*: *finite* (A)
by (*auto simp add*: *A-def*)

lemma *finite-B*: *finite* (B)

by (*auto simp add*: *B-def finite-A*)

lemma *finite-C*: *finite* (*C*)
by (*auto simp add*: *C-def finite-B*)

lemma *finite-D*: *finite* (*D*)
by (*auto simp add*: *D-def finite-C*)

lemma *finite-E*: *finite* (*E*)
by (*auto simp add*: *E-def finite-C*)

lemma *finite-F*: *finite* (*F*)
by (*auto simp add*: *F-def finite-E*)

lemma *C-eq*: $C = D \cup E$
by (*auto simp add*: *C-def D-def E-def*)

lemma *A-card-eq*: *card* A = *nat* ((*p* − *1*) *div* *2*)
  apply (*auto simp add*: *A-def*)
  apply (*insert int-nat*)
  apply (*erule subst*)
  apply (*auto simp add*: *card-bdd-int-set-l-le*)
  done

lemma *inj-on-xa-A*: *inj-on* (%*x*. *x* ∗ *a*) *A*
  using *a-nonzero* **by** (*simp add*: *A-def inj-on-def*)

lemma *A-res*: *ResSet p A*
  apply (*auto simp add*: *A-def ResSet-def*)
  apply (*rule-tac m* = *p* **in** *zcong-less-eq*)
  apply (*insert p-g-2*, *auto*)
  done

lemma *B-res*: *ResSet p B*
  apply (*insert p-g-2 p-a-relprime p-minus-one-l*)
  apply (*auto simp add*: *B-def*)
  apply (*rule ResSet-image*)
  apply (*auto simp add*: *A-res*)
  apply (*auto simp add*: *A-def*)
proof −
  fix *x* **fix** *y*
  assume *a*: [*x* ∗ *a* = *y* ∗ *a*] (*mod p*)
  assume *b*: *0* < *x*
  assume *c*: *x* ≤ (*p* − *1*) *div 2*
  assume *d*: *0* < *y*
  assume *e*: *y* ≤ (*p* − *1*) *div 2*
  from *a p-a-relprime p-prime a-nonzero zcong-cancel* [*of p a x y*]
  have [*x* = *y*](*mod p*)
    by (*simp add*: *zprime-imp-zrelprime zcong-def p-g-0 order-le-less*)

```
  with zcong-less-eq [of x y p] p-minus-one-l
      order-le-less-trans [of x (p - 1) div 2 p]
      order-le-less-trans [of y (p - 1) div 2 p] show x = y
    by (simp add: b c d e p-minus-one-l p-g-0)
qed

lemma SR-B-inj: inj-on (StandardRes p) B
  apply (auto simp add: B-def StandardRes-def inj-on-def A-def)
proof -
  fix x fix y
  assume a: x * a mod p = y * a mod p
  assume b: 0 < x
  assume c: x ≤ (p - 1) div 2
  assume d: 0 < y
  assume e: y ≤ (p - 1) div 2
  assume f: x ≠ y
  from a have [x * a = y * a](mod p)
    by (simp add: zcong-zmod-eq p-g-0)
  with p-a-relprime p-prime a-nonzero zcong-cancel [of p a x y]
  have [x = y](mod p)
    by (simp add: zprime-imp-zrelprime zcong-def p-g-0 order-le-less)
  with zcong-less-eq [of x y p] p-minus-one-l
    order-le-less-trans [of x (p - 1) div 2 p]
    order-le-less-trans [of y (p - 1) div 2 p] have x = y
    by (simp add: b c d e p-minus-one-l p-g-0)
  then have False
    by (simp add: f)
  then show a = 0
    by simp
qed

lemma inj-on-pminusx-E: inj-on (%x. p - x) E
  apply (auto simp add: E-def C-def B-def A-def)
  apply (rule-tac g = %x. -1 * (x - p) in inj-on-inverseI)
  apply auto
  done

lemma A-ncong-p: x ∈ A ==> ~[x = 0](mod p)
  apply (auto simp add: A-def)
  apply (frule-tac m = p in zcong-not-zero)
  apply (insert p-minus-one-l)
  apply auto
  done

lemma A-greater-zero: x ∈ A ==> 0 < x
  by (auto simp add: A-def)

lemma B-ncong-p: x ∈ B ==> ~[x = 0](mod p)
  apply (auto simp add: B-def)
```

```
  apply (frule A-ncong-p)
  apply (insert p-a-relprime p-prime a-nonzero)
  apply (frule-tac a = xa and b = a in zcong-zprime-prod-zero-contra)
  apply (auto simp add: A-greater-zero)
  done

lemma B-greater-zero: x ∈ B ==> 0 < x
  using a-nonzero by (auto simp add: B-def A-greater-zero)

lemma C-ncong-p: x ∈ C ==>  ~[x = 0](mod p)
  apply (auto simp add: C-def)
  apply (frule B-ncong-p)
  apply (subgoal-tac [xa = StandardRes p xa](mod p))
  defer apply (simp add: StandardRes-prop1)
  apply (frule-tac a = xa and b = StandardRes p xa and c = 0 in zcong-trans)
  apply auto
  done

lemma C-greater-zero: y ∈ C ==> 0 < y
  apply (auto simp add: C-def)
proof -
  fix x
  assume a: x ∈ B
  from p-g-0 have 0 ≤ StandardRes p x
    by (simp add: StandardRes-lbound)
  moreover have ~[x = 0] (mod p)
    by (simp add: a B-ncong-p)
  then have StandardRes p x ≠ 0
    by (simp add: StandardRes-prop3)
  ultimately show 0 < StandardRes p x
    by (simp add: order-le-less)
qed

lemma D-ncong-p: x ∈ D ==> ~[x = 0](mod p)
  by (auto simp add: D-def C-ncong-p)

lemma E-ncong-p: x ∈ E ==> ~[x = 0](mod p)
  by (auto simp add: E-def C-ncong-p)

lemma F-ncong-p: x ∈ F ==> ~[x = 0](mod p)
  apply (auto simp add: F-def)
proof -
  fix x assume a: x ∈ E assume b: [p - x = 0] (mod p)
  from E-ncong-p have ~[x = 0] (mod p)
    by (simp add: a)
  moreover from a have 0 < x
    by (simp add: a E-def C-greater-zero)
  moreover from a have x < p
    by (auto simp add: E-def C-def p-g-0 StandardRes-ubound)
```

ultimately have ~[p − x = 0] (mod p)
  by (simp add: zcong-not-zero)
from this **show** False **by** (simp add: b)
qed

lemma F-subset: F ⊆ {x. 0 < x & x ≤ ((p − 1) div 2)}
  apply (auto simp add: F-def E-def)
  apply (insert p-g-0)
  apply (frule-tac x = xa **in** StandardRes-ubound)
  apply (frule-tac x = x **in** StandardRes-ubound)
  apply (subgoal-tac xa = StandardRes p xa)
  apply (auto simp add: C-def StandardRes-prop2 StandardRes-prop1)
proof −
  from zodd-imp-zdiv-eq p-prime p-g-2 zprime-zOdd-eq-grt-2 **have**
    2 ∗ (p − 1) div 2 = 2 ∗ ((p − 1) div 2)
    by simp
  with p-eq2 **show** !!x. [| (p − 1) div 2 < StandardRes p x; x ∈ B |]
      ==> p − StandardRes p x ≤ (p − 1) div 2
    by simp
qed

lemma D-subset: D ⊆ {x. 0 < x & x ≤ ((p − 1) div 2)}
  by (auto simp add: D-def C-greater-zero)

lemma F-eq: F = {x. ∃ y ∈ A. (x = p − (StandardRes p (y∗a)) & (p − 1) div 2 < StandardRes p (y∗a))}
  by (auto simp add: F-def E-def D-def C-def B-def A-def)

lemma D-eq: D = {x. ∃ y ∈ A. (x = StandardRes p (y∗a) & StandardRes p (y∗a) ≤ (p − 1) div 2)}
  by (auto simp add: D-def C-def B-def A-def)

lemma D-leq: x ∈ D ==> x ≤ (p − 1) div 2
  by (auto simp add: D-eq)

lemma F-ge: x ∈ F ==> x ≤ (p − 1) div 2
  apply (auto simp add: F-eq A-def)
proof −
  fix y
  assume (p − 1) div 2 < StandardRes p (y ∗ a)
  then have p − StandardRes p (y ∗ a) < p − ((p − 1) div 2)
    by arith
  also from p-eq2 **have** ... = 2 ∗ ((p − 1) div 2) + 1 − ((p − 1) div 2)
    by auto
  also have 2 ∗ ((p − 1) div 2) + 1 − (p − 1) div 2 = (p − 1) div 2 + 1
    by arith
  finally show p − StandardRes p (y ∗ a) ≤ (p − 1) div 2
    using zless-add1-eq [of p − StandardRes p (y ∗ a) (p − 1) div 2] **by** auto
qed

lemma *all-A-relprime*: ∀ *x* ∈ *A*. *zgcd x p* = *1*
using *p-prime p-minus-one-l* **by** (*auto simp add*: *A-def zless-zprime-imp-zrelprime*)

lemma *A-prod-relprime*: *zgcd* (*setprod id A*) *p* = *1*
by(*rule all-relprime-prod-relprime*[*OF finite-A all-A-relprime*])

17.3 Relationships Between Gauss Sets

lemma *B-card-eq-A*: *card B* = *card A*
using *finite-A* **by** (*simp add*: *finite-A B-def inj-on-xa-A card-image*)

lemma *B-card-eq*: *card B* = *nat* ((*p* − *1*) *div 2*)
by (*simp add*: *B-card-eq-A A-card-eq*)

lemma *F-card-eq-E*: *card F* = *card E*
using *finite-E* **by** (*simp add*: *F-def inj-on-pminusx-E card-image*)

lemma *C-card-eq-B*: *card C* = *card B*
apply (*insert finite-B*)
apply (*subgoal-tac inj-on* (*StandardRes p*) *B*)
apply (*simp add*: *B-def C-def card-image*)
apply (*rule StandardRes-inj-on-ResSet*)
apply (*simp add*: *B-res*)
done

lemma *D-E-disj*: *D* ∩ *E* = {}
by (*auto simp add*: *D-def E-def*)

lemma *C-card-eq-D-plus-E*: *card C* = *card D* + *card E*
by (*auto simp add*: *C-eq card-Un-disjoint D-E-disj finite-D finite-E*)

lemma *C-prod-eq-D-times-E*: *setprod id E* ∗ *setprod id D* = *setprod id C*
apply (*insert D-E-disj finite-D finite-E C-eq*)
apply (*frule setprod.union-disjoint* [*of D E id*])
apply *auto*
done

lemma *C-B-zcong-prod*: [*setprod id C* = *setprod id B*] (*mod p*)
apply (*auto simp add*: *C-def*)
apply (*insert finite-B SR-B-inj*)
apply (*frule setprod.reindex* [*of StandardRes p B id*])
apply *auto*
apply (*rule setprod-same-function-zcong*)
apply (*auto simp add*: *StandardRes-prop1 zcong-sym p-g-0*)
done

lemma *F-Un-D-subset*: (*F* ∪ *D*) ⊆ *A*
apply (*rule Un-least*)

apply (*auto simp add*: *A-def F-subset D-subset*)
done

lemma *F-D-disj*: $(F \cap D) = \{\}$
apply (*simp add*: *F-eq D-eq*)
apply (*auto simp add*: *F-eq D-eq*)
proof −
fix *y* **fix** *ya*
assume $p - StandardRes\ p\ (y * a) = StandardRes\ p\ (ya * a)$
then have $p = StandardRes\ p\ (y * a) + StandardRes\ p\ (ya * a)$
by *arith*
moreover have *p dvd p*
by *auto*
ultimately have $p\ dvd\ (StandardRes\ p\ (y * a) + StandardRes\ p\ (ya * a))$
by *auto*
then have *a*: $[StandardRes\ p\ (y * a) + StandardRes\ p\ (ya * a) = 0]\ (mod\ p)$
by (*auto simp add*: *zcong-def*)
have $[y * a = StandardRes\ p\ (y * a)]\ (mod\ p)$
by (*simp only*: *zcong-sym StandardRes-prop1*)
moreover have $[ya * a = StandardRes\ p\ (ya * a)]\ (mod\ p)$
by (*simp only*: *zcong-sym StandardRes-prop1*)
ultimately have $[y * a + ya * a =$
$StandardRes\ p\ (y * a) + StandardRes\ p\ (ya * a)]\ (mod\ p)$
by (*rule zcong-zadd*)
with *a* **have** $[y * a + ya * a = 0]\ (mod\ p)$
apply (*elim zcong-trans*)
by (*simp only*: *zcong-refl*)
also have $y * a + ya * a = a * (y + ya)$
by (*simp add*: *distrib-left mult.commute*)
finally have $[a * (y + ya) = 0]\ (mod\ p)$ **.**
with *p-prime a-nonzero zcong-zprime-prod-zero* [*of p a y + ya*]
p-a-relprime
have *a*: $[y + ya = 0]\ (mod\ p)$
by *auto*
assume *b*: $y \in A$ **and** *c*: *ya*: *A*
with *A-def* **have** $0 < y + ya$
by *auto*
moreover from *b c A-def* **have** $y + ya \leq (p - 1)\ div\ 2 + (p - 1)\ div\ 2$
by *auto*
moreover from *b c p-eq2 A-def* **have** $y + ya < p$
by *auto*
ultimately show *False*
apply *simp*
apply (*frule-tac* $m = p$ **in** *zcong-not-zero*)
apply (*auto simp add*: *a*)
done
qed

lemma *F-Un-D-card*: $card\ (F \cup D) = nat\ ((p - 1)\ div\ 2)$

```
proof -
  have card (F ∪ D) = card E + card D
    by (auto simp add: finite-F finite-D F-D-disj
      card-Un-disjoint F-card-eq-E)
  then have card (F ∪ D) = card C
    by (simp add: C-card-eq-D-plus-E)
  from this show card (F ∪ D) = nat ((p - 1) div 2)
    by (simp add: C-card-eq-B B-card-eq)
qed

lemma F-Un-D-eq-A: F ∪ D = A
 using finite-A F-Un-D-subset A-card-eq F-Un-D-card by (auto simp add: card-seteq)

lemma prod-D-F-eq-prod-A:
    (setprod id D) * (setprod id F) = setprod id A
  apply (insert F-D-disj finite-D finite-F)
  apply (frule setprod.union-disjoint [of F D id])
  apply (auto simp add: F-Un-D-eq-A)
  done

lemma prod-F-zcong:
  [setprod id F = ((-1) ^ (card E)) * (setprod id E)] (mod p)
proof -
  have setprod id F = setprod id (op - p ` E)
    by (auto simp add: F-def)
  then have setprod id F = setprod (op - p) E
    apply simp
    apply (insert finite-E inj-on-pminusx-E)
    apply (frule setprod.reindex [of minus p E id])
    apply auto
    done
  then have one:
    [setprod id F = setprod (StandardRes p o (op - p)) E] (mod p)
    apply simp
    apply (insert p-g-0 finite-E StandardRes-prod)
    by (auto)
  moreover have a: ∀ x ∈ E. [p - x = 0 - x] (mod p)
    apply clarify
    apply (insert zcong-id [of p])
    apply (rule-tac a = p and m = p and c = x and d = x in zcong-zdiff, auto)
    done
  moreover have b: ∀ x ∈ E. [StandardRes p (p - x) = p - x](mod p)
    apply clarify
    apply (simp add: StandardRes-prop1 zcong-sym)
    done
  moreover have ∀ x ∈ E. [StandardRes p (p - x) = - x](mod p)
    apply clarify
    apply (insert a b)
    apply (rule-tac b = p - x in zcong-trans, auto)
```

done
ultimately have *c*:
[*setprod* (*StandardRes p o* (*op* − *p*)) *E* = *setprod* (*uminus*) *E*](*mod p*)
apply *simp*
using *finite-E p-g-0*
setprod-same-function-zcong [*of E StandardRes p o* (*op* − *p*) *uminus p*]
by *auto*
then have *two*: [*setprod id F* = *setprod* (*uminus*) *E*](*mod p*)
apply (*insert one c*)
apply (*rule zcong-trans* [*of setprod id F*
setprod (*StandardRes p o op* − *p*) *E p*
setprod uminus E], *auto*)
done
also have *setprod uminus E* = (*setprod id E*) ∗ (−*1*) ˆ(*card E*)
using *finite-E* **by** (*induct set*: *finite*) *auto*
then have *setprod uminus E* = (−*1*) ˆ (*card E*) ∗ (*setprod id E*)
by (*simp add*: *mult.commute*)
with *two* **show** *?thesis*
by *simp*
qed

17.4 Gauss' Lemma

lemma *aux*: *setprod id A* ∗ (− *1*) ˆ *card E* ∗ *a* ˆ *card A* ∗ (− *1*) ˆ *card E* = *setprod id A* ∗ *a* ˆ *card A*
by (*auto simp add*: *finite-E neg-one-special*)

theorem *pre-gauss-lemma*:
[*a* ˆ *nat*((*p* − *1*) *div 2*) = (−*1*) ˆ (*card E*)] (*mod p*)
proof −
have [*setprod id A* = *setprod id F* ∗ *setprod id D*](*mod p*)
by (*auto simp add*: *prod-D-F-eq-prod-A mult.commute cong del*:*setprod.cong*)
then have [*setprod id A* = ((−*1*) ˆ(*card E*) ∗ *setprod id E*) ∗
setprod id D] (*mod p*)
apply (*rule zcong-trans*)
apply (*auto simp add*: *prod-F-zcong zcong-scalar cong del*: *setprod.cong*)
done
then have [*setprod id A* = ((−*1*) ˆ(*card E*) ∗ *setprod id C*)] (*mod p*)
apply (*rule zcong-trans*)
apply (*insert C-prod-eq-D-times-E*, *erule subst*)
apply (*subst mult.assoc*, *auto*)
done
then have [*setprod id A* = ((−*1*) ˆ(*card E*) ∗ *setprod id B*)] (*mod p*)
apply (*rule zcong-trans*)
apply (*simp add*: *C-B-zcong-prod zcong-scalar2 cong del*:*setprod.cong*)
done
then have [*setprod id A* = ((−*1*) ˆ(*card E*) ∗
(*setprod id* ((%*x*. *x* ∗ *a*) ' *A*)))] (*mod p*)
by (*simp add*: *B-def*)

then have [*setprod id A* = ((−*1*) ˆ(*card E*) ∗ (*setprod* (%*x*. *x* ∗ *a*) *A*))]
(*mod p*)
by (*simp add*:*finite-A inj-on-xa-A setprod.reindex cong del*:*setprod.cong*)
moreover have *setprod* (%*x*. *x* ∗ *a*) *A* =
setprod (%*x*. *a*) *A* ∗ *setprod id A*
using *finite-A* **by** (*induct set*: *finite*) *auto*
ultimately have [*setprod id A* = ((−*1*) ˆ(*card E*) ∗ (*setprod* (%*x*. *a*) *A* ∗
setprod id A))] (*mod p*)
by *simp*
then have [*setprod id A* = ((−*1*) ˆ(*card E*) ∗ *a*ˆ(*card A*) ∗
setprod id A)](*mod p*)
apply (*rule zcong-trans*)
apply (*simp add*: *zcong-scalar2 zcong-scalar finite-A setprod-constant mult.assoc*)
done
then have *a*: [*setprod id A* ∗ (−*1*) ˆ(*card E*) =
((−*1*) ˆ(*card E*) ∗ *a*ˆ(*card A*) ∗ *setprod id A* ∗ (−*1*) ˆ(*card E*))](*mod p*)
by (*rule zcong-scalar*)
then have [*setprod id A* ∗ (−*1*) ˆ(*card E*) = *setprod id A* ∗
(−*1*) ˆ(*card E*) ∗ *a*ˆ(*card A*) ∗ (−*1*) ˆ(*card E*)](*mod p*)
apply (*rule zcong-trans*)
apply (*simp add*: *a mult.commute mult.left-commute*)
done
then have [*setprod id A* ∗ (−*1*) ˆ(*card E*) = *setprod id A* ∗
*a*ˆ(*card A*)](*mod p*)
apply (*rule zcong-trans*)
apply (*simp add*: *aux cong del*:*setprod.cong*)
done
with *this zcong-cancel2* [*of p setprod id A* (− *1*) ˆ *card E a* ˆ *card A*]
p-g-0 A-prod-relprime **have** [(− *1*) ˆ *card E* = *a* ˆ *card A*](*mod p*)
by (*simp add*: *order-less-imp-le*)
from *this* **show** *?thesis*
by (*simp add*: *A-card-eq zcong-sym*)
qed

theorem *gauss-lemma*: (*Legendre a p*) = (−*1*) ˆ (*card E*)
proof −
from *Euler-Criterion p-prime p-g-2* **have**
[(*Legendre a p*) = *a*ˆ(*nat* (((*p*) − *1*) *div 2*))] (*mod p*)
by *auto*
moreover note *pre-gauss-lemma*
ultimately have [(*Legendre a p*) = (−*1*) ˆ (*card E*)] (*mod p*)
by (*rule zcong-trans*)
moreover from *p-a-relprime* **have** (*Legendre a p*) = *1* | (*Legendre a p*) = (−*1*)
by (*auto simp add*: *Legendre-def*)
moreover have (−*1*::*int*) ˆ (*card E*) = *1* | (−*1*::*int*) ˆ (*card E*) = −*1*
by (*rule neg-one-power*)
ultimately show *?thesis*
by (*auto simp add*: *p-g-2 one-not-neg-one-mod-m zcong-sym*)
qed

```
end

end
```

18 The law of Quadratic reciprocity

```
theory Quadratic-Reciprocity
imports Gauss
begin
```

Lemmas leading up to the proof of theorem 3.3 in Niven and Zuckerman's presentation.

```
context GAUSS
begin

lemma QRLemma1: a ∗ setsum id A =
  p ∗ setsum (%x. ((x ∗ a) div p)) A + setsum id D + setsum id E
proof −
  from finite-A have a ∗ setsum id A = setsum (%x. a ∗ x) A
    by (auto simp add: setsum-const-mult id-def)
  also have setsum (%x. a ∗ x) = setsum (%x. x ∗ a)
    by (auto simp add: mult.commute)
  also have setsum (%x. x ∗ a) A = setsum id B
    by (simp add: B-def setsum.reindex [OF inj-on-xa-A])
  also have ... = setsum (%x. p ∗ (x div p) + StandardRes p x) B
    by (auto simp add: StandardRes-def zmod-zdiv-equality)
  also have ... = setsum (%x. p ∗ (x div p)) B + setsum (StandardRes p) B
    by (rule setsum.distrib)
  also have setsum (StandardRes p) B = setsum id C
    by (auto simp add: C-def setsum.reindex [OF SR-B-inj])
  also from C-eq have ... = setsum id (D ∪ E)
    by auto
  also from finite-D finite-E have ... = setsum id D + setsum id E
    by (rule setsum.union-disjoint) (auto simp add: D-def E-def)
  also have setsum (%x. p ∗ (x div p)) B =
      setsum ((%x. p ∗ (x div p)) o (%x. (x ∗ a))) A
    by (auto simp add: B-def setsum.reindex inj-on-xa-A)
  also have ... = setsum (%x. p ∗ ((x ∗ a) div p)) A
    by (auto simp add: o-def)
  also from finite-A have setsum (%x. p ∗ ((x ∗ a) div p)) A =
    p ∗ setsum (%x. ((x ∗ a) div p)) A
    by (auto simp add: setsum-const-mult)
  finally show ?thesis by arith
qed

lemma QRLemma2: setsum id A = p ∗ int (card E) − setsum id E +
  setsum id D
proof −
```

from *F-Un-D-eq-A* **have** *setsum id A* = *setsum id* (*D* ∪ *F*)
  by (*simp add*: *Un-commute*)
also from *F-D-disj finite-D finite-F*
have ... = *setsum id D* + *setsum id F*
  by (*auto simp add*: *Int-commute intro*: *setsum.union-disjoint*)
also from *F-def* **have** *F* = (%*x*. (*p* − *x*)) ‘ *E*
  by *auto*
also from *finite-E inj-on-pminusx-E* **have** *setsum id* ((%*x*. (*p* − *x*)) ‘ *E*) =
    setsum (%*x*. (*p* − *x*)) *E*
  by (*auto simp add*: *setsum.reindex*)
also from *finite-E* **have** *setsum* (*op* − *p*) *E* = *setsum* (%*x*. *p*) *E* − *setsum id E*
  by (*auto simp add*: *setsum-subtractf id-def*)
also from *finite-E* **have** *setsum* (%*x*. *p*) *E* = *p* ∗ *int*(*card E*)
  by (*intro setsum-const*)
finally show *?thesis*
  by *arith*
qed

lemma *QRLemma3*: (*a* − *1*) ∗ *setsum id A* =
  p ∗ (*setsum* (%*x*. ((*x* ∗ *a*) *div p*)) *A* − *int*(*card E*)) + *2* ∗ *setsum id E*
proof −
have (*a* − *1*) ∗ *setsum id A* = *a* ∗ *setsum id A* − *setsum id A*
  by (*auto simp add*: *left-diff-distrib*)
also note *QRLemma1*
also from *QRLemma2* **have** *p* ∗ (∑*x* ∈ *A*. *x* ∗ *a div p*) + *setsum id D* +
   setsum id E − *setsum id A* =
    p ∗ (∑*x* ∈ *A*. *x* ∗ *a div p*) + *setsum id D* +
    setsum id E − (*p* ∗ *int* (*card E*) − *setsum id E* + *setsum id D*)
  by *auto*
also have ... = *p* ∗ (∑*x* ∈ *A*. *x* ∗ *a div p*) −
    p ∗ *int* (*card E*) + *2* ∗ *setsum id E*
  by *arith*
finally show *?thesis*
  by (*auto simp only*: *right-diff-distrib*)
qed

lemma *QRLemma4*: *a* ∈ *zOdd* ==>
  (*setsum* (%*x*. ((*x* ∗ *a*) *div p*)) *A* ∈ *zEven*) = (*int*(*card E*): *zEven*)
proof −
assume *a-odd*: *a* ∈ *zOdd*
from *QRLemma3* **have** *a*: *p* ∗ (*setsum* (%*x*. ((*x* ∗ *a*) *div p*)) *A* − *int*(*card E*))
=
    (*a* − *1*) ∗ *setsum id A* − *2* ∗ *setsum id E*
  by *arith*
from *a-odd* **have** *a* − *1* ∈ *zEven*
  by (*rule odd-minus-one-even*)
hence (*a* − *1*) ∗ *setsum id A* ∈ *zEven*
  by (*rule even-times-either*)
moreover have *2* ∗ *setsum id E* ∈ *zEven*

```
    by (auto simp add: zEven-def)
  ultimately have (a - 1) * setsum id A - 2 * setsum id E ∈ zEven
    by (rule even-minus-even)
  with a have p * (setsum (%x. ((x * a) div p)) A - int(card E)): zEven
    by simp
  hence p ∈ zEven | (setsum (%x. ((x * a) div p)) A - int(card E)): zEven
    by (rule EvenOdd.even-product)
  with p-odd have (setsum (%x. ((x * a) div p)) A - int(card E)): zEven
    by (auto simp add: odd-iff-not-even)
  thus ?thesis
    by (auto simp only: even-diff [symmetric])
qed

lemma QRLemma5: a ∈ zOdd ==>
   (-1::int)^(card E) = (-1::int)^(nat(setsum (%x. ((x * a) div p)) A))
proof -
  assume a ∈ zOdd
  from QRLemma4 [OF this] have
    (int(card E): zEven) = (setsum (%x. ((x * a) div p)) A ∈ zEven) ..
  moreover have 0 ≤ int(card E)
    by auto
  moreover have 0 ≤ setsum (%x. ((x * a) div p)) A
    proof (intro setsum-nonneg)
      show ∀ x ∈ A. 0 ≤ x * a div p
      proof
        fix x
        assume x ∈ A
        then have 0 ≤ x
          by (auto simp add: A-def)
        with a-nonzero have 0 ≤ x * a
          by (auto simp add: zero-le-mult-iff)
        with p-g-2 show 0 ≤ x * a div p
          by (auto simp add: pos-imp-zdiv-nonneg-iff)
      qed
    qed
  ultimately have (-1::int)^nat((int (card E))) =
      (-1)^nat(((∑ x ∈ A. x * a div p)))
    by (intro neg-one-power-parity, auto)
  also have nat (int(card E)) = card E
    by auto
  finally show ?thesis .
qed

end

lemma MainQRLemma: [| a ∈ zOdd; 0 < a; ~([a = 0] (mod p)); zprime p; 2 <
p;
  A = {x. 0 < x & x ≤ (p - 1) div 2} |] ==>
  (Legendre a p) = (-1::int)^(nat(setsum (%x. ((x * a) div p)) A))
```

```
apply (subst GAUSS.gauss-lemma)
apply (auto simp add: GAUSS-def)
apply (subst GAUSS.QRLemma5)
apply (auto simp add: GAUSS-def)
apply (simp add: GAUSS.A-def [OF GAUSS.intro] GAUSS-def)
done
```

18.1 Stuff about S, S1 and S2

```
locale QRTEMP =
  fixes p    :: int
  fixes q    :: int

  assumes p-prime: zprime p
  assumes p-g-2: 2 < p
  assumes q-prime: zprime q
  assumes q-g-2: 2 < q
  assumes p-neq-q:     p ≠ q
begin

definition P-set :: int set
  where P-set = {x. 0 < x & x ≤ ((p − 1) div 2) }

definition Q-set :: int set
  where Q-set = {x. 0 < x & x ≤ ((q − 1) div 2) }

definition S :: (int ∗ int) set
  where S = P-set × Q-set

definition S1 :: (int ∗ int) set
  where S1 = { (x, y). (x, y):S & ((p ∗ y) < (q ∗ x)) }

definition S2 :: (int ∗ int) set
  where S2 = { (x, y). (x, y):S & ((q ∗ x) < (p ∗ y)) }

definition f1 :: int => (int ∗ int) set
  where f1 j = { (j1, y). (j1, y):S & j1 = j & (y ≤ (q ∗ j) div p) }

definition f2 :: int => (int ∗ int) set
  where f2 j = { (x, j1). (x, j1):S & j1 = j & (x ≤ (p ∗ j) div q) }

lemma p-fact: 0 < (p − 1) div 2
proof −
  from p-g-2 have 2 ≤ p − 1 by arith
  then have 2 div 2 ≤ (p − 1) div 2 by (rule zdiv-mono1, auto)
  then show ?thesis by auto
qed

lemma q-fact: 0 < (q − 1) div 2
```

proof −
  from *q-g-2* **have** $2 \le q - 1$ **by** *arith*
  then have $2\ div\ 2 \le (q - 1)\ div\ 2$ **by** (*rule zdiv-mono1*, *auto*)
  then show *?thesis* **by** *auto*
qed

lemma *pb-neq-qa*:
  assumes $1 \le b$ **and** $b \le (q - 1)\ div\ 2$
  shows $p * b \neq q * a$
proof
  assume $p * b = q * a$
  then have $q\ dvd\ (p * b)$ **by** (*auto simp add*: *dvd-def*)
  with *q-prime p-g-2* **have** $q\ dvd\ p \mid q\ dvd\ b$
    by (*auto simp add*: *zprime-zdvd-zmult*)
  moreover have ~ $(q\ dvd\ p)$
  proof
    assume $q\ dvd\ p$
    with *p-prime* **have** $q = 1 \mid q = p$
      apply (*auto simp add*: *zprime-def QRTEMP-def*)
      apply (*drule-tac* $x = q$ **and** $R = False$ **in** *allE*)
      apply (*simp add*: *QRTEMP-def*)
      apply (*subgoal-tac* $0 \le q$, *simp add*: *QRTEMP-def*)
      apply (*insert assms*)
      apply (*auto simp add*: *QRTEMP-def*)
      done
    with *q-g-2 p-neq-q* **show** *False* **by** *auto*
  qed
  ultimately have $q\ dvd\ b$ **by** *auto*
  then have $q \le b$
  proof −
    assume $q\ dvd\ b$
    moreover from *assms* **have** $0 < b$ **by** *auto*
    ultimately show *?thesis* **using** *zdvd-bounds* [*of q b*] **by** *auto*
  qed
  with *assms* **have** $q \le (q - 1)\ div\ 2$ **by** *auto*
  then have $2 * q \le 2 * ((q - 1)\ div\ 2)$ **by** *arith*
  then have $2 * q \le q - 1$
  proof −
    assume *a*: $2 * q \le 2 * ((q - 1)\ div\ 2)$
    with *assms* **have** $q \in zOdd$ **by** (*auto simp add*: *QRTEMP-def zprime-zOdd-eq-grt-2*)
    with *odd-minus-one-even* **have** $(q - 1):zEven$ **by** *auto*
    with *even-div-2-prop2* **have** $(q - 1) = 2 * ((q - 1)\ div\ 2)$ **by** *auto*
    with *a* **show** *?thesis* **by** *auto*
  qed
  then have *p1*: $q \le -1$ **by** *arith*
  with *q-g-2* **show** *False* **by** *auto*
qed

lemma *P-set-finite*: *finite* (*P-set*)

using *p-fact* **by** (*auto simp add*: *P-set-def bdd-int-set-l-le-finite*)

lemma *Q-set-finite*: *finite* (*Q-set*)
using *q-fact* **by** (*auto simp add*: *Q-set-def bdd-int-set-l-le-finite*)

lemma *S-finite*: *finite S*
by (*auto simp add*: *S-def P-set-finite Q-set-finite finite-cartesian-product*)

lemma *S1-finite*: *finite S1*
proof −
have *finite S* **by** (*auto simp add*: *S-finite*)
moreover have *S1* ⊆ *S* **by** (*auto simp add*: *S1-def S-def*)
ultimately show *?thesis* **by** (*auto simp add*: *finite-subset*)
qed

lemma *S2-finite*: *finite S2*
proof −
have *finite S* **by** (*auto simp add*: *S-finite*)
moreover have *S2* ⊆ *S* **by** (*auto simp add*: *S2-def S-def*)
ultimately show *?thesis* **by** (*auto simp add*: *finite-subset*)
qed

lemma *P-set-card*: (*p* − *1*) *div 2* = *int* (*card* (*P-set*))
using *p-fact* **by** (*auto simp add*: *P-set-def card-bdd-int-set-l-le*)

lemma *Q-set-card*: (*q* − *1*) *div 2* = *int* (*card* (*Q-set*))
using *q-fact* **by** (*auto simp add*: *Q-set-def card-bdd-int-set-l-le*)

lemma *S-card*: ((*p* − *1*) *div 2*) ∗ ((*q* − *1*) *div 2*) = *int* (*card*(*S*))
using *P-set-card Q-set-card P-set-finite Q-set-finite*
by (*simp add*: *S-def*)

lemma *S1-Int-S2-prop*: *S1* ∩ *S2* = {}
by (*auto simp add*: *S1-def S2-def*)

lemma *S1-Union-S2-prop*: *S* = *S1* ∪ *S2*
apply (*auto simp add*: *S-def P-set-def Q-set-def S1-def S2-def*)
proof −
fix *a* **and** *b*
assume ~ *q* ∗ *a* < *p* ∗ *b* **and** *b1*: *0* < *b* **and** *b2*: *b* ≤ (*q* − *1*) *div 2*
with *less-linear* **have** (*p* ∗ *b* < *q* ∗ *a*) | (*p* ∗ *b* = *q* ∗ *a*) **by** *auto*
moreover from *pb-neq-qa b1 b2* **have** (*p* ∗ *b* ≠ *q* ∗ *a*) **by** *auto*
ultimately show *p* ∗ *b* < *q* ∗ *a* **by** *auto*
qed

lemma *card-sum-S1-S2*: ((*p* − *1*) *div 2*) ∗ ((*q* − *1*) *div 2*) =
int(*card*(*S1*)) + *int*(*card*(*S2*))
proof −
have ((*p* − *1*) *div 2*) ∗ ((*q* − *1*) *div 2*) = *int* (*card*(*S*))

```
    by (auto simp add: S-card)
  also have ... = int( card(S1) + card(S2))
    apply (insert S1-finite S2-finite S1-Int-S2-prop S1-Union-S2-prop)
    apply (drule card-Un-disjoint, auto)
    done
  also have ... = int(card(S1)) + int(card(S2)) by auto
  finally show ?thesis .
qed

lemma aux1a:
  assumes 0 < a and a ≤ (p - 1) div 2
    and 0 < b and b ≤ (q - 1) div 2
  shows (p * b < q * a) = (b ≤ q * a div p)
proof -
  have p * b < q * a ==> b ≤ q * a div p
  proof -
    assume p * b < q * a
    then have p * b ≤ q * a by auto
    then have (p * b) div p ≤ (q * a) div p
      by (rule zdiv-mono1) (insert p-g-2, auto)
    then show b ≤ (q * a) div p
      apply (subgoal-tac p ≠ 0)
      apply (frule div-mult-self1-is-id, force)
      apply (insert p-g-2, auto)
      done
  qed
  moreover have b ≤ q * a div p ==> p * b < q * a
  proof -
    assume b ≤ q * a div p
    then have p * b ≤ p * ((q * a) div p)
      using p-g-2 by (auto simp add: mult-le-cancel-left)
    also have ... ≤ q * a
      by (rule zdiv-leq-prop) (insert p-g-2, auto)
    finally have p * b ≤ q * a .
    then have p * b < q * a | p * b = q * a
      by (simp only: order-le-imp-less-or-eq)
    moreover have p * b ≠ q * a
      by (rule pb-neq-qa) (insert assms, auto)
    ultimately show ?thesis by auto
  qed
  ultimately show ?thesis ..
qed

lemma aux1b:
  assumes 0 < a and a ≤ (p - 1) div 2
    and 0 < b and b ≤ (q - 1) div 2
  shows (q * a < p * b) = (a ≤ p * b div q)
proof -
  have q * a < p * b ==> a ≤ p * b div q
```

```
  proof –
    assume q ∗ a < p ∗ b
    then have q ∗ a ≤ p ∗ b by auto
    then have (q ∗ a) div q ≤ (p ∗ b) div q
      by (rule zdiv-mono1) (insert q-g-2, auto)
    then show a ≤ (p ∗ b) div q
      apply (subgoal-tac q ≠ 0)
      apply (frule div-mult-self1-is-id, force)
      apply (insert q-g-2, auto)
      done
  qed
  moreover have a ≤ p ∗ b div q ==> q ∗ a < p ∗ b
  proof –
    assume a ≤ p ∗ b div q
    then have q ∗ a ≤ q ∗ ((p ∗ b) div q)
      using q-g-2 by (auto simp add: mult-le-cancel-left)
    also have ... ≤ p ∗ b
      by (rule zdiv-leq-prop) (insert q-g-2, auto)
    finally have q ∗ a ≤ p ∗ b .
    then have q ∗ a < p ∗ b | q ∗ a = p ∗ b
      by (simp only: order-le-imp-less-or-eq)
    moreover have p ∗ b ≠ q ∗ a
      by (rule  pb-neq-qa) (insert assms, auto)
    ultimately show ?thesis by auto
  qed
  ultimately show ?thesis ..
qed

lemma (in –) aux2:
  assumes zprime p and zprime q and 2 < p and 2 < q
  shows (q ∗ ((p – 1) div 2)) div p ≤ (q – 1) div 2
proof–

  from assms have p ∈ zOdd & q ∈ zOdd
    by (auto simp add:  zprime-zOdd-eq-grt-2)
  then have even1: (p – 1):zEven & (q – 1):zEven
    by (auto simp add: odd-minus-one-even)
  then have even2: (2 ∗ p):zEven & ((q – 1) ∗ p):zEven
    by (auto simp add: zEven-def)
  then have even3: (((q – 1) ∗ p) + (2 ∗ p)):zEven
    by (auto simp: EvenOdd.even-plus-even)

  from assms have q ∗ (p – 1) < ((q – 1) ∗ p) + (2 ∗ p)
    by (auto simp add: int-distrib)
  then have ((p – 1) ∗ q) div 2 < (((q – 1) ∗ p) + (2 ∗ p)) div 2
    apply (rule-tac x = ((p – 1) ∗ q) in even-div-2-l)
    by (auto simp add: even3, auto simp add: ac-simps)
  also have ((p – 1) ∗ q) div 2 = q ∗ ((p – 1) div 2)
    by (auto simp add: even1 even-prod-div-2)
```

```
  also have (((q - 1) * p) + (2 * p)) div 2 = (((q - 1) div 2) * p) + p
    by (auto simp add: even1 even2 even-prod-div-2 even-sum-div-2)
  finally show ?thesis
    apply (rule-tac x =  q * ((p - 1) div 2) and
                y = (q - 1) div 2 in div-prop2)
    using assms by auto
qed

lemma aux3a: ∀ j ∈ P-set. int (card (f1 j)) = (q * j) div p
proof
  fix j
  assume j-fact: j ∈ P-set
  have int (card (f1 j)) = int (card {y. y ∈ Q-set & y ≤ (q * j) div p})
  proof -
    have finite (f1 j)
    proof -
      have (f1 j) ⊆ S by (auto simp add: f1-def)
      with S-finite show ?thesis by (auto simp add: finite-subset)
    qed
    moreover have inj-on (%(x,y). y) (f1 j)
      by (auto simp add: f1-def inj-on-def)
    ultimately have card ((%(x,y). y) ` (f1 j)) = card  (f1 j)
      by (auto simp add: f1-def card-image)
    moreover have ((%(x,y). y) ` (f1 j)) = {y. y ∈ Q-set & y ≤ (q * j) div p}
      using j-fact by (auto simp add: f1-def S-def Q-set-def P-set-def image-def)
    ultimately show ?thesis by (auto simp add: f1-def)
  qed
  also have ... = int (card {y. 0 < y & y ≤ (q * j) div p})
  proof -
    have {y. y ∈ Q-set & y ≤ (q * j) div p} =
        {y. 0 < y & y ≤ (q * j) div p}
      apply (auto simp add: Q-set-def)
    proof -
      fix x
      assume x: 0 < x x ≤ q * j div p
      with j-fact P-set-def  have j ≤ (p - 1) div 2 by auto
      with q-g-2 have q * j ≤ q * ((p - 1) div 2)
        by (auto simp add: mult-le-cancel-left)
      with p-g-2 have q * j div p ≤ q * ((p - 1) div 2) div p
        by (auto simp add: zdiv-mono1)
      also from QRTEMP-axioms j-fact P-set-def have ... ≤ (q - 1) div 2
        apply simp
        apply (insert aux2)
        apply (simp add: QRTEMP-def)
        done
      finally show x ≤ (q - 1) div 2 using x by auto
    qed
    then show ?thesis by auto
  qed
```

also have ... = (*q* ∗ *j*) *div p*
proof −
from *j-fact P-set-def* **have** *0* ≤ *j* **by** *auto*
with *q-g-2* **have** *q* ∗ *0* ≤ *q* ∗ *j* **by** (*auto simp only*: *mult-left-mono*)
then have *0* ≤ *q* ∗ *j* **by** *auto*
then have *0 div p* ≤ (*q* ∗ *j*) *div p*
apply (*rule-tac a* = *0* **in** *zdiv-mono1*)
apply (*insert p-g-2*, *auto*)
done
also have *0 div p* = *0* **by** *auto*
finally show *?thesis* **by** (*auto simp add*: *card-bdd-int-set-l-le*)
qed
finally show *int* (*card* (*f1 j*)) = *q* ∗ *j div p* **.**
qed

lemma *aux3b*: ∀ *j* ∈ *Q-set*. *int* (*card* (*f2 j*)) = (*p* ∗ *j*) *div q*
proof
fix *j*
assume *j-fact*: *j* ∈ *Q-set*
have *int* (*card* (*f2 j*)) = *int* (*card* {*y*. *y* ∈ *P-set* & *y* ≤ (*p* ∗ *j*) *div q*})
proof −
have *finite* (*f2 j*)
proof −
have (*f2 j*) ⊆ *S* **by** (*auto simp add*: *f2-def*)
with *S-finite* **show** *?thesis* **by** (*auto simp add*: *finite-subset*)
qed
moreover have *inj-on* (%(*x*,*y*). *x*) (*f2 j*)
by (*auto simp add*: *f2-def inj-on-def*)
ultimately have *card* ((%(*x*,*y*). *x*) ' (*f2 j*)) = *card* (*f2 j*)
by (*auto simp add*: *f2-def card-image*)
moreover have ((%(*x*,*y*). *x*) ' (*f2 j*)) = {*y*. *y* ∈ *P-set* & *y* ≤ (*p* ∗ *j*) *div q*}
using *j-fact* **by** (*auto simp add*: *f2-def S-def Q-set-def P-set-def image-def*)
ultimately show *?thesis* **by** (*auto simp add*: *f2-def*)
qed
also have ... = *int* (*card* {*y*. *0* < *y* & *y* ≤ (*p* ∗ *j*) *div q*})
proof −
have {*y*. *y* ∈ *P-set* & *y* ≤ (*p* ∗ *j*) *div q*} =
{*y*. *0* < *y* & *y* ≤ (*p* ∗ *j*) *div q*}
apply (*auto simp add*: *P-set-def*)
proof −
fix *x*
assume *x*: *0* < *x x* ≤ *p* ∗ *j div q*
with *j-fact Q-set-def* **have** *j* ≤ (*q* − *1*) *div 2* **by** *auto*
with *p-g-2* **have** *p* ∗ *j* ≤ *p* ∗ ((*q* − *1*) *div 2*)
by (*auto simp add*: *mult-le-cancel-left*)
with *q-g-2* **have** *p* ∗ *j div q* ≤ *p* ∗ ((*q* − *1*) *div 2*) *div q*
by (*auto simp add*: *zdiv-mono1*)
also from *QRTEMP-axioms j-fact* **have** ... ≤ (*p* − *1*) *div 2*
by (*auto simp add*: *aux2 QRTEMP-def*)

```
      finally show x ≤ (p − 1) div 2 using x by auto
      qed
    then show ?thesis by auto
  qed
  also have ... = (p ∗ j) div q
  proof −
    from j-fact Q-set-def have 0 ≤ j by auto
    with p-g-2 have p ∗ 0 ≤ p ∗ j by (auto simp only: mult-left-mono)
    then have 0 ≤ p ∗ j by auto
    then have 0 div q ≤ (p ∗ j) div q
      apply (rule-tac a = 0 in zdiv-mono1)
      apply (insert q-g-2, auto)
      done
    also have 0 div q = 0 by auto
    finally show ?thesis by (auto simp add: card-bdd-int-set-l-le)
  qed
  finally show int (card (f2 j)) = p ∗ j div q .
qed

lemma S1-card: int (card(S1)) = setsum (%j. (q ∗ j) div p) P-set
proof −
  have ∀ x ∈ P-set. finite (f1 x)
  proof
    fix x
    have f1 x ⊆ S by (auto simp add: f1-def)
    with S-finite show finite (f1 x) by (auto simp add: finite-subset)
  qed
  moreover have (∀ x ∈ P-set. ∀ y ∈ P-set. x ≠ y −−> (f1 x) ∩ (f1 y) = {})
    by (auto simp add: f1-def)
  moreover note P-set-finite
  ultimately have int(card (UNION P-set f1)) =
      setsum (%x. int(card (f1 x))) P-set
    by(simp add:card-UN-disjoint int-setsum o-def)
  moreover have S1 = UNION P-set f1
    by (auto simp add: f1-def S-def S1-def S2-def P-set-def Q-set-def aux1a)
  ultimately have int(card (S1)) = setsum (%j. int(card (f1 j))) P-set
    by auto
  also have ... = setsum (%j. q ∗ j div p) P-set
    using aux3a by(fastforce intro: setsum.cong)
  finally show ?thesis .
qed

lemma S2-card: int (card(S2)) = setsum (%j. (p ∗ j) div q) Q-set
proof −
  have ∀ x ∈ Q-set. finite (f2 x)
  proof
    fix x
    have f2 x ⊆ S by (auto simp add: f2-def)
    with S-finite show finite (f2 x) by (auto simp add: finite-subset)
```

```
  qed
  moreover have (∀ x ∈ Q-set. ∀ y ∈ Q-set. x ≠ y -->
      (f2 x) ∩ (f2 y) = {})
    by (auto simp add: f2-def)
  moreover note Q-set-finite
  ultimately have int(card (UNION Q-set f2)) =
      setsum (%x. int(card (f2 x))) Q-set
    by(simp add:card-UN-disjoint int-setsum o-def)
  moreover have S2 = UNION Q-set f2
    by (auto simp add: f2-def S-def S1-def S2-def P-set-def Q-set-def aux1b)
  ultimately have int(card (S2)) = setsum (%j. int(card (f2 j))) Q-set
    by auto
  also have ... = setsum (%j. p * j div q) Q-set
    using aux3b by(fastforce intro: setsum.cong)
  finally show ?thesis .
qed

lemma S1-carda: int (card(S1)) =
    setsum (%j. (j * q) div p) P-set
  by (auto simp add: S1-card ac-simps)

lemma S2-carda: int (card(S2)) =
    setsum (%j. (j * p) div q) Q-set
  by (auto simp add: S2-card ac-simps)

lemma pq-sum-prop: (setsum (%j. (j * p) div q) Q-set) +
    (setsum (%j. (j * q) div p) P-set) = ((p - 1) div 2) * ((q - 1) div 2)
proof -
  have (setsum (%j. (j * p) div q) Q-set) +
      (setsum (%j. (j * q) div p) P-set) = int (card S2) + int (card S1)
    by (auto simp add: S1-carda S2-carda)
  also have ... = int (card S1) + int (card S2)
    by auto
  also have ... = ((p - 1) div 2) * ((q - 1) div 2)
    by (auto simp add: card-sum-S1-S2)
  finally show ?thesis .
qed


lemma (in -) pq-prime-neq: [| zprime p; zprime q; p ≠ q |] ==> (~[p = 0] (mod
q))
  apply (auto simp add: zcong-eq-zdvd-prop zprime-def)
  apply (drule-tac x = q in allE)
  apply (drule-tac x = p in allE)
  apply auto
  done


lemma QR-short: (Legendre p q) * (Legendre q p) =
```

```
    (-1::int)^nat(((p - 1) div 2)*((q - 1) div 2))
proof -
  from QRTEMP-axioms have ~([p = 0] (mod q))
    by (auto simp add: pq-prime-neq QRTEMP-def)
  with QRTEMP-axioms Q-set-def have a1: (Legendre p q) = (-1::int) ^
      nat(setsum (%x. ((x * p) div q)) Q-set)
    apply (rule-tac p = q in  MainQRLemma)
    apply (auto simp add: zprime-zOdd-eq-grt-2 QRTEMP-def)
    done
  from QRTEMP-axioms have ~([q = 0] (mod p))
    apply (rule-tac p = q and q = p in pq-prime-neq)
    apply (simp add: QRTEMP-def)+
    done
  with QRTEMP-axioms P-set-def have a2: (Legendre q p) =
      (-1::int) ^ nat(setsum (%x. ((x * q) div p)) P-set)
    apply (rule-tac p = p in  MainQRLemma)
    apply (auto simp add: zprime-zOdd-eq-grt-2 QRTEMP-def)
    done
  from a1 a2 have (Legendre p q) * (Legendre q p) =
      (-1::int) ^ nat(setsum (%x. ((x * p) div q)) Q-set) *
        (-1::int) ^ nat(setsum (%x. ((x * q) div p)) P-set)
    by auto
  also have ... = (-1::int) ^ (nat(setsum (%x. ((x * p) div q)) Q-set) +
                   nat(setsum (%x. ((x * q) div p)) P-set))
    by (auto simp add: power-add)
  also have nat(setsum (%x. ((x * p) div q)) Q-set) +
      nat(setsum (%x. ((x * q) div p)) P-set) =
        nat((setsum (%x. ((x * p) div q)) Q-set) +
          (setsum (%x. ((x * q) div p)) P-set))
    apply (rule-tac z = setsum (%x. ((x * p) div q)) Q-set in
      nat-add-distrib [symmetric])
    apply (auto simp add: S1-carda [symmetric] S2-carda [symmetric])
    done
  also have ... = nat(((p - 1) div 2) * ((q - 1) div 2))
    by (auto simp add: pq-sum-prop)
  finally show ?thesis .
qed

end

theorem Quadratic-Reciprocity:
     [| p ∈ zOdd; zprime p; q ∈ zOdd; zprime q;
        p ≠ q |]
      ==> (Legendre p q) * (Legendre q p) =
          (-1::int)^nat(((p - 1) div 2)*((q - 1) div 2))
  by (auto simp add: QRTEMP.QR-short zprime-zOdd-eq-grt-2 [symmetric]
                     QRTEMP-def)

end
```

19 Pocklington's Theorem for Primes

theory *Pocklington*
imports *Primes*
begin

definition *modeq*:: *nat => nat => nat => bool* ((*1*[- = -] ′(*mod* -′)))
  where [*a* = *b*] (*mod p*) == ((*a mod p*) = (*b mod p*))

definition *modneq*:: *nat => nat => nat => bool* ((*1*[- ≠ -] ′(*mod* -′)))
  where [*a* ≠ *b*] (*mod p*) == ((*a mod p*) ≠ (*b mod p*))

lemma *modeq-trans*:
  ⟦ [*a* = *b*] (*mod p*); [*b* = *c*] (*mod p*) ⟧ ⟹ [*a* = *c*] (*mod p*)
  by (*simp add*:*modeq-def*)

lemma *modeq-sym*[*sym*]:
  [*a* = *b*] (*mod p*) ⟹ [*b* = *a*] (*mod p*)
  unfolding *modeq-def* **by** *simp*

lemma *modneq-sym*[*sym*]:
  [*a* ≠ *b*] (*mod p*) ⟹ [*b* ≠ *a*] (*mod p*)
  by (*simp add*: *modneq-def*)

lemma *nat-mod-lemma*: **assumes** *xyn*: [*x* = *y*] (*mod n*) **and** *xy*:*y* ≤ *x*
  shows ∃ *q*. *x* = *y* + *n* ∗ *q*
using *xyn xy* **unfolding** *modeq-def* **using** *nat-mod-eq-lemma* **by** *blast*

lemma *nat-mod*[*algebra*]: [*x* = *y*] (*mod n*) ⟷ (∃ *q1 q2*. *x* + *n* ∗ *q1* = *y* + *n* ∗ *q2*)
unfolding *modeq-def nat-mod-eq-iff* **..**

lemma *prime*: *prime p* ⟷ *p* ≠ *0* ∧ *p*≠*1* ∧ (∀ *m*. *0* < *m* ∧ *m* < *p* ⟶ *coprime p m*)
  (**is** *?lhs* ⟷ *?rhs*)
proof−
  {**assume** *p*=*0* ∨ *p*=*1* **hence** *?thesis* **using** *prime-0 prime-1* **by** (*cases p*=*0*, *simp-all*)}
  moreover
  {**assume** *p0*: *p*≠*0 p*≠*1*
    {**assume** *H*: *?lhs*
      {**fix** *m* **assume** *m*: *m* > *0 m* < *p*
        {**assume** *m*=*1* **hence** *coprime p m* **by** *simp*}
        moreover
        {**assume** *p dvd m* **hence** *p* ≤ *m* **using** *dvd-imp-le m* **by** *blast* **with** *m*(*2*)
          have *coprime p m* **by** *simp*}
       ultimately have *coprime p m* **using** *prime-coprime*[*OF H*, *of m*] **by** *blast*}

hence *?rhs* **using** *p0* **by** *auto*}
moreover
{ **assume** *H*: ∀ *m*. *0* < *m* ∧ *m* < *p* ⟶ *coprime p m*
from *prime-factor*[*OF p0*(*2*)] **obtain** *q* **where** *q*: *prime q q dvd p* **by** *blast*
from *prime-ge-2*[*OF q*(*1*)] **have** *q0*: *q* > *0* **by** *arith*
from *dvd-imp-le*[*OF q*(*2*)] *p0* **have** *qp*: *q* ≤ *p* **by** *arith*
{**assume** *q* = *p* **hence** *?lhs* **using** *q*(*1*) **by** *blast*}
moreover
{**assume** *q*≠*p* **with** *qp* **have** *qplt*: *q* < *p* **by** *arith*
from *H*[*rule-format*, *of q*] *qplt q0* **have** *coprime p q* **by** *arith*
with *coprime-prime*[*of p q q*] *q* **have** *False* **by** *simp* **hence** *?lhs* **by** *blast*}
ultimately have *?lhs* **by** *blast*}
ultimately have *?thesis* **by** *blast*}
ultimately show *?thesis* **by** (*casesp=0* ∨ *p=1*, *auto*)
qed

lemma *finite-number-segment*: *card* { *m*. *0* < *m* ∧ *m* < *n* } = *n* − *1*
proof−
have { *m*. *0* < *m* ∧ *m* < *n* } = {*1*..<*n*} **by** *auto*
thus *?thesis* **by** *simp*
qed

lemma *coprime-mod*: **assumes** *n*: *n* ≠ *0* **shows** *coprime* (*a mod n*) *n* ⟷ *coprime a n*
using *n dvd-mod-iff*[*of - n a*] **by** (*auto simp add*: *coprime*)

lemma *cong-mod-01*[*simp*,*presburger*]:
[*x* = *y*] (*mod 0*) ⟷ *x* = *y* [*x* = *y*] (*mod 1*) [*x* = *0*] (*mod n*) ⟷ *n dvd x*
by (*simp-all add*: *modeq-def*, *presburger*)

lemma *cong-sub-cases*:
[*x* = *y*] (*mod n*) ⟷ (*if x* <= *y then* [*y* − *x* = *0*] (*mod n*) *else* [*x* − *y* = *0*] (*mod n*))
apply (*auto simp add*: *nat-mod*)
apply (*rule-tac x*=*q2* **in** *exI*)
apply (*rule-tac x*=*q1* **in** *exI*, *simp*)
apply (*rule-tac x*=*q2* **in** *exI*)
apply (*rule-tac x*=*q1* **in** *exI*, *simp*)
apply (*rule-tac x*=*q1* **in** *exI*)
apply (*rule-tac x*=*q2* **in** *exI*, *simp*)
apply (*rule-tac x*=*q1* **in** *exI*)
apply (*rule-tac x*=*q2* **in** *exI*, *simp*)
done

lemma *cong-mult-lcancel*: **assumes** *an*: *coprime a n* **and** *axy*:[*a* ∗ *x* = *a* ∗ *y*] (*mod n*)
shows [*x* = *y*] (*mod n*)

proof−
{**assume** *a* = *0* **with** *an axy coprime-0′*[*of n*] **have** *?thesis* **by** (*simp add*: *modeq-def*) }
moreover
{**assume** *az*: *a*≠*0*
{**assume** *xy*: *x* ≤ *y* **hence** *axy′*: *a*∗*x* ≤ *a*∗*y* **by** *simp*
with *axy cong-sub-cases*[*of a*∗*x a*∗*y n*] **have** [*a*∗(*y* − *x*) = *0*] (*mod n*)
by (*simp only*: *if-True diff-mult-distrib2*)
hence *th*: *n dvd a*∗(*y* −*x*) **by** *simp*
from *coprime-divprod*[*OF th*] *an* **have** *n dvd y* − *x*
by (*simp add*: *coprime-commute*)
hence *?thesis* **using** *xy cong-sub-cases*[*of x y n*] **by** *simp*}
moreover
{**assume** *H*: ¬*x* ≤ *y* **hence** *xy*: *y* ≤ *x* **by** *arith*
from *H az* **have** *axy′*: ¬ *a*∗*x* ≤ *a*∗*y* **by** *auto*
with *axy H cong-sub-cases*[*of a*∗*x a*∗*y n*] **have** [*a*∗(*x* − *y*) = *0*] (*mod n*)
by (*simp only*: *if-False diff-mult-distrib2*)
hence *th*: *n dvd a*∗(*x* − *y*) **by** *simp*
from *coprime-divprod*[*OF th*] *an* **have** *n dvd x* − *y*
by (*simp add*: *coprime-commute*)
hence *?thesis* **using** *xy cong-sub-cases*[*of x y n*] **by** *simp*}
ultimately have *?thesis* **by** *blast*}
ultimately show *?thesis* **by** *blast*
qed

lemma *cong-mult-rcancel*: **assumes** *an*: *coprime a n* **and** *axy*:[*x*∗*a* = *y*∗*a*] (*mod n*)
shows [*x* = *y*] (*mod n*)
using *cong-mult-lcancel*[*OF an axy*[*unfolded mult.commute*[*of -a*]]] **.**

lemma *cong-refl*: [*x* = *x*] (*mod n*) **by** (*simp add*: *modeq-def*)

lemma *eq-imp-cong*: *a* = *b* ⟹ [*a* = *b*] (*mod n*) **by** (*simp add*: *cong-refl*)

lemma *cong-commute*: [*x* = *y*] (*mod n*) ⟷ [*y* = *x*] (*mod n*)
by (*auto simp add*: *modeq-def*)

lemma *cong-trans*[*trans*]: [*x* = *y*] (*mod n*) ⟹ [*y* = *z*] (*mod n*) ⟹ [*x* = *z*] (*mod n*)
by (*simp add*: *modeq-def*)

lemma *cong-add*: **assumes** *xx′*: [*x* = *x′*] (*mod n*) **and** *yy′*:[*y* = *y′*] (*mod n*)
shows [*x* + *y* = *x′* + *y′*] (*mod n*)
proof−
have (*x* + *y*) *mod n* = (*x mod n* + *y mod n*) *mod n*
by (*simp add*: *mod-add-left-eq*[*of x y n*] *mod-add-right-eq*[*of x mod n y n*])
also have ... = (*x′ mod n* + *y′ mod n*) *mod n* **using** *xx′ yy′ modeq-def* **by** *simp*
also have ... = (*x′* + *y′*) *mod n*
by (*simp add*: *mod-add-left-eq*[*of x′ y′ n*] *mod-add-right-eq*[*of x′ mod n y′ n*])

finally show *?thesis* **unfolding** *modeq-def* **.**
qed

lemma *cong-mult*: **assumes** *xx'*: $[x = x']$ $(mod\ n)$ **and** *yy'*:$[y = y']$ $(mod\ n)$
shows $[x * y = x' * y']$ $(mod\ n)$
proof−
have $(x * y)$ *mod* $n = (x$ *mod* $n) * (y$ *mod* $n)$ *mod* n
by (*simp add*: *mod-mult-left-eq*[*of x y n*] *mod-mult-right-eq*[*of x mod n y n*])
also have $\ldots = (x'$ *mod* $n) * (y'$ *mod* $n)$ *mod* n **using** *xx'*[*unfolded modeq-def*] *yy'*[*unfolded modeq-def*] **by** *simp*
also have $\ldots = (x' * y')$ *mod* n
by (*simp add*: *mod-mult-left-eq*[*of x' y' n*] *mod-mult-right-eq*[*of x' mod n y' n*])
finally show *?thesis* **unfolding** *modeq-def* **.**
qed

lemma *cong-exp*: $[x = y]$ $(mod\ n) \Longrightarrow [x\hat{}k = y\hat{}k]$ $(mod\ n)$
by (*induct k*, *auto simp add*: *cong-refl cong-mult*)
lemma *cong-sub*: **assumes** *xx'*: $[x = x']$ $(mod\ n)$ **and** *yy'*: $[y = y']$ $(mod\ n)$
and *yx*: $y <= x$ **and** *yx'*: $y' <= x'$
shows $[x - y = x' - y']$ $(mod\ n)$
proof−
{ fix $x\ a\ x'\ a'\ y\ b\ y'\ b'$
have $(x::nat) + a = x' + a' \Longrightarrow y + b = y' + b' \Longrightarrow y <= x \Longrightarrow y' <= x'$
$\Longrightarrow (x - y) + (a + b') = (x' - y') + (a' + b)$ **by** *arith***}**
note *th* = *this*
from *xx' yy'* **obtain** *q1 q2 q1' q2'* **where** *q12*: $x + n*q1 = x'+n*q2$
and *q12'*: $y + n*q1' = y'+n*q2'$ **unfolding** *nat-mod* **by** *blast+*
from *th*[*OF q12 q12' yx yx'*]
have $(x - y) + n*(q1 + q2') = (x' - y') + n*(q2 + q1')$
by (*simp add*: *distrib-left*)
thus *?thesis* **unfolding** *nat-mod* **by** *blast*
qed

lemma *cong-mult-lcancel-eq*: **assumes** *an*: *coprime a n*
shows $[a * x = a * y]$ $(mod\ n) \longleftrightarrow [x = y]$ $(mod\ n)$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *H*: *?rhs* **from** *cong-mult*[*OF cong-refl*[*of a n*] *H*] **show** *?lhs* **.**
next
assume *H*: *?lhs* **hence** *H'*: $[x*a = y*a]$ $(mod\ n)$ **by** (*simp add*: *mult.commute*)
from *cong-mult-rcancel*[*OF an H'*] **show** *?rhs* **.**
qed

lemma *cong-mult-rcancel-eq*: **assumes** *an*: *coprime a n*
shows $[x * a = y * a]$ $(mod\ n) \longleftrightarrow [x = y]$ $(mod\ n)$
using *cong-mult-lcancel-eq*[*OF an*, *of x y*] **by** (*simp add*: *mult.commute*)

lemma *cong-add-lcancel-eq*: $[a + x = a + y]$ $(mod\ n) \longleftrightarrow [x = y]$ $(mod\ n)$
by (*simp add*: *nat-mod*)

lemma *cong-add-rcancel-eq*: $[x + a = y + a]$ $(mod\ n) \longleftrightarrow [x = y]$ $(mod\ n)$
  by (*simp add*: *nat-mod*)

lemma *cong-add-rcancel*: $[x + a = y + a]$ $(mod\ n) \Longrightarrow [x = y]$ $(mod\ n)$
  by (*simp add*: *nat-mod*)

lemma *cong-add-lcancel*: $[a + x = a + y]$ $(mod\ n) \Longrightarrow [x = y]$ $(mod\ n)$
  by (*simp add*: *nat-mod*)

lemma *cong-add-lcancel-eq-0*: $[a + x = a]$ $(mod\ n) \longleftrightarrow [x = 0]$ $(mod\ n)$
  by (*simp add*: *nat-mod*)

lemma *cong-add-rcancel-eq-0*: $[x + a = a]$ $(mod\ n) \longleftrightarrow [x = 0]$ $(mod\ n)$
  by (*simp add*: *nat-mod*)

lemma *cong-imp-eq*: **assumes** *xn*: $x < n$ **and** *yn*: $y < n$ **and** *xy*: $[x = y]$ $(mod\ n)$
  shows $x = y$
  using *xy*[*unfolded modeq-def mod-less*[*OF xn*] *mod-less*[*OF yn*]] **.**

lemma *cong-divides-modulus*: $[x = y]$ $(mod\ m) \Longrightarrow n\ dvd\ m ==> [x = y]$ $(mod\ n)$
  apply (*auto simp add*: *nat-mod dvd-def*)
  apply (*rule-tac* $x=k*q1$ **in** *exI*)
  apply (*rule-tac* $x=k*q2$ **in** *exI*)
  by *simp*

lemma *cong-0-divides*: $[x = 0]$ $(mod\ n) \longleftrightarrow n\ dvd\ x$ **by** *simp*

lemma *cong-1-divides*:$[x = 1]$ $(mod\ n) ==> n\ dvd\ x - 1$
  apply (*cases* $x \le 1$, *simp-all*)
  using *cong-sub-cases*[*of x 1 n*] **by** *auto*

lemma *cong-divides*: $[x = y]$ $(mod\ n) \Longrightarrow n\ dvd\ x \longleftrightarrow n\ dvd\ y$
apply (*auto simp add*: *nat-mod dvd-def*)
apply (*rule-tac* $x=k + q1 - q2$ **in** *exI*, *simp add*: *add-mult-distrib2 diff-mult-distrib2*)
apply (*rule-tac* $x=k + q2 - q1$ **in** *exI*, *simp add*: *add-mult-distrib2 diff-mult-distrib2*)
done

lemma *cong-coprime*: **assumes** *xy*: $[x = y]$ $(mod\ n)$
  shows *coprime n x* $\longleftrightarrow$ *coprime n y*
proof−
  {assume $n=0$ **hence** *?thesis* **using** *xy* **by** *simp***}**
  moreover
  {assume *nz*: $n \neq 0$
  have *coprime n x* $\longleftrightarrow$ *coprime* $(x\ mod\ n)$ *n*
    by (*simp add*: *coprime-mod*[*OF nz*, *of x*] *coprime-commute*[*of n x*])
  also have ... $\longleftrightarrow$ *coprime* $(y\ mod\ n)$ *n* **using** *xy*[*unfolded modeq-def*] **by** *simp*
  also have ... $\longleftrightarrow$ *coprime y n* **by** (*simp add*: *coprime-mod*[*OF nz*, *of y*])

finally have *?thesis* **by** (*simp add*: *coprime-commute*) **}**
ultimately show *?thesis* **by** *blast*
qed

lemma *cong-mod*: $\~(n = 0) \Longrightarrow [a\ mod\ n = a]\ (mod\ n)$ **by** (*simp add*: *modeq-def*)

lemma *mod-mult-cong*: $\~(a = 0) \Longrightarrow \~(b = 0)$
$\Longrightarrow [x\ mod\ (a * b) = y]\ (mod\ a) \longleftrightarrow [x = y]\ (mod\ a)$
by (*simp add*: *modeq-def mod-mult2-eq mod-add-left-eq*)

lemma *cong-mod-mult*: $[x = y]\ (mod\ n) \Longrightarrow m\ dvd\ n \Longrightarrow [x = y]\ (mod\ m)$
apply (*auto simp add*: *nat-mod dvd-def*)
apply (*rule-tac x=k∗q1* **in** *exI*)
apply (*rule-tac x=k∗q2* **in** *exI*, *simp*)
done

lemma *cong-le*: $y <= x \Longrightarrow [x = y]\ (mod\ n) \longleftrightarrow (\exists\, q.\ x = q * n + y)$
using *nat-mod-lemma*[*of x y n*]
apply *auto*
apply (*simp add*: *nat-mod*)
apply (*rule-tac x=q* **in** *exI*)
apply (*rule-tac x=q + q* **in** *exI*)
by (*auto simp*: *algebra-simps*)

lemma *cong-to-1*: $[a = 1]\ (mod\ n) \longleftrightarrow a = 0 \wedge n = 1 \vee (\exists\, m.\ a = 1 + m * n)$
proof−
{assume $n = 0 \vee n = 1 \vee a = 0 \vee a = 1$ **hence** *?thesis*
apply (*cases n=0*, *simp-all add*: *cong-commute*)
apply (*cases n=1*, *simp-all add*: *cong-commute modeq-def*)
apply *arith*
apply (*cases a=1*)
apply (*simp-all add*: *modeq-def cong-commute*)
done }
moreover
{assume *n*: $n \neq 0$ $n \neq 1$ **and** *a*:$a \neq 0$ $a \neq 1$ **hence** *a'*: $a \geq 1$ **by** *simp*
hence *?thesis* **using** *cong-le*[*OF a'*, *of n*] **by** *auto* **}**
ultimately show *?thesis* **by** *auto*
qed

lemma *cong-solve*: **assumes** *an*: *coprime a n* **shows** $\exists\, x.\ [a * x = b]\ (mod\ n)$
proof−
{assume *a=0* **hence** *?thesis* **using** *an* **by** (*simp add*: *modeq-def*)**}**
moreover
{assume *az*: $a \neq 0$

from *bezout-add-strong*[*OF az*, *of n*]
obtain *d x y* **where** *dxy*: *d dvd a d dvd n a*∗*x* = *n*∗*y* + *d* **by** *blast*
from *an*[*unfolded coprime*, *rule-format*, *of d*] *dxy*(*1*,*2*) **have** *d1*: *d* = *1* **by** *blast*
hence *a*∗*x*∗*b* = (*n*∗*y* + *1*)∗*b* **using** *dxy*(*3*) **by** *simp*
hence *a*∗(*x*∗*b*) = *n*∗(*y*∗*b*) + *b* **by** *algebra*
hence *a*∗(*x*∗*b*) *mod n* = (*n*∗(*y*∗*b*) + *b*) *mod n* **by** *simp*
hence *a*∗(*x*∗*b*) *mod n* = *b mod n* **by** (*simp add*: *mod-add-left-eq*)
hence [*a*∗(*x*∗*b*) = *b*] (*mod n*) **unfolding** *modeq-def* **.**
hence *?thesis* **by** *blast*}
ultimately **show** *?thesis* **by** *blast*
qed

lemma *cong-solve-unique*: **assumes** *an*: *coprime a n* **and** *nz*: *n* ≠ *0*
shows ∃!*x*. *x* < *n* ∧ [*a* ∗ *x* = *b*] (*mod n*)
proof−
let *?P* = λ*x*. *x* < *n* ∧ [*a* ∗ *x* = *b*] (*mod n*)
from *cong-solve*[*OF an*] **obtain** *x* **where** *x*: [*a*∗*x* = *b*] (*mod n*) **by** *blast*
let *?x* = *x mod n*
from *x* **have** *th*: [*a* ∗ *?x* = *b*] (*mod n*)
by (*simp add*: *modeq-def mod-mult-right-eq*[*of a x n*])
from *mod-less-divisor*[*of n x*] *nz th* **have** *Px*: *?P ?x* **by** *simp*
{**fix** *y* **assume** *Py*: *y* < *n* [*a* ∗ *y* = *b*] (*mod n*)
from *Py*(*2*) *th* **have** [*a* ∗ *y* = *a*∗*?x*] (*mod n*) **by** (*simp add*: *modeq-def*)
hence [*y* = *?x*] (*mod n*) **by** (*simp add*: *cong-mult-lcancel-eq*[*OF an*])
with *mod-less*[*OF Py*(*1*)] *mod-less-divisor*[*of n x*] *nz*
have *y* = *?x* **by** (*simp add*: *modeq-def*)}
with *Px* **show** *?thesis* **by** *blast*
qed

lemma *cong-solve-unique-nontrivial*:
assumes *p*: *prime p* **and** *pa*: *coprime p a* **and** *x0*: *0* < *x* **and** *xp*: *x* < *p*
shows ∃!*y*. *0* < *y* ∧ *y* < *p* ∧ [*x* ∗ *y* = *a*] (*mod p*)
proof−
from *p* **have** *p1*: *p* > *1* **using** *prime-ge-2*[*OF p*] **by** *arith*
hence *p01*: *p* ≠ *0 p* ≠ *1* **by** *arith*+
from *pa* **have** *ap*: *coprime a p* **by** (*simp add*: *coprime-commute*)
from *prime-coprime*[*OF p*, *of x*] *dvd-imp-le*[*of p x*] *x0 xp* **have** *px*:*coprime x p*
by (*auto simp add*: *coprime-commute*)
from *cong-solve-unique*[*OF px p01*(*1*)]
obtain *y* **where** *y*: *y* < *p* [*x* ∗ *y* = *a*] (*mod p*) ∀ *z*. *z* < *p* ∧ [*x* ∗ *z* = *a*] (*mod p*) ⟶ *z* = *y* **by** *blast*
{**assume** *y0*: *y* = *0*
with *y*(*2*) **have** *th*: *p dvd a* **by** (*simp add*: *cong-commute*[*of 0 a p*])
with *p coprime-prime*[*OF pa*, *of p*] **have** *False* **by** *simp*}
with *y* **show** *?thesis* **unfolding** *Ex1-def* **using** *neq0-conv* **by** *blast*
qed
lemma *cong-unique-inverse-prime*:
assumes *p*: *prime p* **and** *x0*: *0* < *x* **and** *xp*: *x* < *p*
shows ∃!*y*. *0* < *y* ∧ *y* < *p* ∧ [*x* ∗ *y* = *1*] (*mod p*)

using *cong-solve-unique-nontrivial*[*OF p coprime-1*[*of p*] *x0 xp*] **.**

lemma *cong-chinese*:
assumes *ab*: *coprime a b* **and** *xya*: [$x = y$] (*mod a*)
and *xyb*: [$x = y$] (*mod b*)
shows [$x = y$] (*mod a*$*$*b*)
using *ab xya xyb*
by (*simp add*: *cong-sub-cases*[*of x y a*] *cong-sub-cases*[*of x y b*]
cong-sub-cases[*of x y a*$*$*b*])
(*cases* $x \leq y$, *simp-all add*: *divides-mul*[*of a - b*])

lemma *chinese-remainder-unique*:
assumes *ab*: *coprime a b* **and** *az*: $a \neq 0$ **and** *bz*: $b \neq 0$
shows $\exists!x.\ x < a * b \wedge [x = m]\ (mod\ a) \wedge [x = n]\ (mod\ b)$
proof−
from *az bz* **have** *abpos*: $a*b > 0$ **by** *simp*
from *chinese-remainder*[*OF ab az bz*] **obtain** *x q1 q2* **where**
xq12: $x = m + q1 * a$ $x = n + q2 * b$ **by** *blast*
let $?w = x\ mod\ (a*b)$
have *wab*: $?w < a*b$ **by** (*simp add*: *mod-less-divisor*[*OF abpos*])
from *xq12*(*1*) **have** $?w\ mod\ a = ((m + q1 * a)\ mod\ (a*b))\ mod\ a$ **by** *simp*
also have $\ldots = m\ mod\ a$ **by** (*simp add*: *mod-mult2-eq*)
finally have *th1*: [$?w = m$] (*mod a*) **by** (*simp add*: *modeq-def*)
from *xq12*(*2*) **have** $?w\ mod\ b = ((n + q2 * b)\ mod\ (a*b))\ mod\ b$ **by** *simp*
also have $\ldots = ((n + q2 * b)\ mod\ (b*a))\ mod\ b$ **by** (*simp add*: *mult.commute*)
also have $\ldots = n\ mod\ b$ **by** (*simp add*: *mod-mult2-eq*)
finally have *th2*: [$?w = n$] (*mod b*) **by** (*simp add*: *modeq-def*)
{fix *y* **assume** *H*: $y < a*b$ [$y = m$] (*mod a*) [$y = n$] (*mod b*)
with *th1 th2* **have** *H'*: [$y = ?w$] (*mod a*) [$y = ?w$] (*mod b*)
by (*simp-all add*: *modeq-def*)
from *cong-chinese*[*OF ab H'*] *mod-less*[*OF H*(*1*)] *mod-less*[*OF wab*]
have $y = ?w$ **by** (*simp add*: *modeq-def*)**}**
with *th1 th2 wab* **show** *?thesis* **by** *blast*
qed

lemma *chinese-remainder-coprime-unique*:
assumes *ab*: *coprime a b* **and** *az*: $a \neq 0$ **and** *bz*: $b \neq 0$
and *ma*: *coprime m a* **and** *nb*: *coprime n b*
shows $\exists!x.$ *coprime x* $(a * b) \wedge x < a * b \wedge [x = m]\ (mod\ a) \wedge [x = n]\ (mod\ b)$
proof−
let $?P = \lambda x.\ x < a * b \wedge [x = m]\ (mod\ a) \wedge [x = n]\ (mod\ b)$
from *chinese-remainder-unique*[*OF ab az bz*]
obtain *x* **where** *x*: $x < a * b$ [$x = m$] (*mod a*) [$x = n$] (*mod b*)
$\forall y.\ ?P\ y \longrightarrow y = x$ **by** *blast*
from *ma nb cong-coprime*[*OF x*(*2*)] *cong-coprime*[*OF x*(*3*)]
have *coprime x a coprime x b* **by** (*simp-all add*: *coprime-commute*)

```
  with coprime-mul[of x a b] have coprime x (a∗b) by simp
  with x show ?thesis by blast
qed


definition phi-def: φ n = card { m. 0 < m ∧ m <= n ∧ coprime m n }

lemma phi-0[simp]: φ 0 = 0
  unfolding phi-def by auto

lemma phi-finite[simp]: finite ({ m. 0 < m ∧ m <= n ∧ coprime m n })
proof−
  have { m. 0 < m ∧ m <= n ∧ coprime m n } ⊆ {0..n} by auto
  thus ?thesis by (auto intro: finite-subset)
qed

declare coprime-1[presburger]
lemma phi-1[simp]: φ 1 = 1
proof−
  {fix m
    have 0 < m ∧ m <= 1 ∧ coprime m 1 ⟷ m = 1 by presburger }
  thus ?thesis by (simp add: phi-def)
qed

lemma [simp]: φ (Suc 0) = Suc 0 using phi-1 by simp

lemma phi-alt: φ(n) = card { m. coprime m n ∧ m < n}
proof−
  {assume n=0 ∨ n=1 hence ?thesis by (cases n=0, simp-all)}
  moreover
  {assume n: n≠0 n≠1
    {fix m
      from n have 0 < m ∧ m <= n ∧ coprime m n ⟷ coprime m n ∧ m < n
        apply (cases m = 0, simp-all)
        apply (cases m = 1, simp-all)
        apply (cases m = n, auto)
        done }
    hence ?thesis unfolding phi-def by simp}
  ultimately show ?thesis by auto
qed

lemma phi-finite-lemma[simp]: finite {m. coprime m n ∧  m < n} (is finite ?S)
  by (rule finite-subset[of ?S {0..n}], auto)

lemma phi-another: assumes n: n≠1
  shows φ n = card {m. 0 < m ∧ m < n ∧ coprime m n }
proof−
  {fix m
```

from *n* **have** $0 < m \wedge m < n \wedge coprime\ m\ n \longleftrightarrow coprime\ m\ n \wedge m < n$
by (*cases m=0*, *auto*)**}**
thus *?thesis* **unfolding** *phi-alt* **by** *auto*
qed

lemma *phi-limit*: $\varphi\ n \leq n$
proof−
have $\{ m.\ coprime\ m\ n \wedge m < n\} \subseteq \{0\ ..<n\}$ **by** *auto*
with *card-mono*[*of* $\{0\ ..<n\}$ $\{ m.\ coprime\ m\ n \wedge m < n\}$]
show *?thesis* **unfolding** *phi-alt* **by** *auto*
qed

lemma *stupid*[*simp*]: $\{m.\ (0::nat) < m \wedge m < n\} = \{1..<n\}$
by *auto*

lemma *phi-limit-strong*: **assumes** *n*: $n \neq 1$
shows $\varphi(n) \leq n - 1$
proof−
show *?thesis*
unfolding *phi-another*[*OF n*] *finite-number-segment*[*of n*, *symmetric*]
by (*rule card-mono*[*of* $\{m.\ 0 < m \wedge m < n\}$ $\{m.\ 0 < m \wedge m < n \wedge coprime$
$m\ n\}$], *auto*)
qed

lemma *phi-lowerbound-1-strong*: **assumes** *n*: $n \geq 1$
shows $\varphi(n) \geq 1$
proof−
let $?S = \{ m.\ 0 < m \wedge m <= n \wedge coprime\ m\ n \}$
from *card-0-eq*[*of ?S*] *n* **have** $\varphi\ n \neq 0$ **unfolding** *phi-alt*
apply *auto*
apply (*cases n=1*, *simp-all*)
apply (*rule exI*[**where** *x=1*], *simp*)
done
thus *?thesis* **by** *arith*
qed

lemma *phi-lowerbound-1*: $2 <= n ==> 1 <= \varphi(n)$
using *phi-lowerbound-1-strong*[*of n*] **by** *auto*

lemma *phi-lowerbound-2*: **assumes** *n*: $3 <= n$ **shows** $2 <= \varphi\ (n)$
proof−
let $?S = \{ m.\ 0 < m \wedge m <= n \wedge coprime\ m\ n \}$
have *inS*: $\{1,\ n - 1\} \subseteq ?S$ **using** *n coprime-plus1*[*of* $n - 1$]
by (*auto simp add*: *coprime-commute*)
from *n* **have** *c2*: *card* $\{1,\ n - 1\} = 2$ **by** (*auto simp add*: *card-insert-if*)
from *card-mono*[*of ?S* $\{1,\ n - 1\}$, *simplified inS c2*] **show** *?thesis*
unfolding *phi-def* **by** *auto*
qed

```
lemma phi-prime: φ n = n − 1 ∧ n≠0 ∧ n≠1 ⟷ prime n
proof−
  {assume n=0 ∨ n=1 hence ?thesis by (cases n=1, simp-all)}
  moreover
  {assume n: n≠0 n≠1
    let ?S = {m. 0 < m ∧ m < n}
    have fS: finite ?S by simp
    let ?S′ = {m. 0 < m ∧ m < n ∧ coprime m n}
    have fS′:finite ?S′ apply (rule finite-subset[of ?S′ ?S]) by auto
    {assume H: φ n = n − 1 ∧ n≠0 ∧ n≠1
      hence ceq: card ?S′ = card ?S
      using n finite-number-segment[of n] phi-another[OF n(2)] by simp
      {fix m assume m: 0 < m m < n ¬ coprime m n
        hence mS′: m ∉ ?S′ by auto
        have insert m ?S′ ≤ ?S using m by auto
        from m have card (insert m ?S′) ≤ card ?S
          by − (rule card-mono[of ?S insert m ?S′], auto)
        hence False
          unfolding card-insert-disjoint[of ?S′ m, OF fS′ mS′] ceq
          by simp }
      hence ∀ m. 0 <m ∧ m < n ⟶ coprime m n by blast
     hence prime n unfolding prime using n by (simp add: coprime-commute)}
    moreover
    {assume H: prime n
      hence ?S = ?S′ unfolding prime using n
        by (auto simp add: coprime-commute)
      hence card ?S = card ?S′ by simp
      hence φ n = n − 1 unfolding phi-another[OF n(2)] by simp}
    ultimately have ?thesis using n by blast}
  ultimately show ?thesis by (cases n=0) blast+
qed
```

```
lemma phi-multiplicative: assumes ab: coprime a b
  shows φ (a ∗ b) = φ a ∗ φ b
proof−
  {assume a = 0 ∨ b = 0 ∨ a = 1 ∨ b = 1
    hence ?thesis
      by (cases a=0, simp, cases b=0, simp, casesa=1, simp-all) }
  moreover
  {assume a: a≠0 a≠1 and b: b≠0 b≠1
    hence ab0: a∗b ≠ 0 by simp
    let ?S = λk. {m. coprime m k ∧ m < k}
    let ?f = λx. (x mod a, x mod b)
    have eq: ?f ‘ (?S (a∗b)) = (?S a × ?S b)
    proof−
      {fix x assume x:x ∈ ?S (a∗b)
        hence x′: coprime x (a∗b) x < a∗b by simp-all
```

    hence *xab*: *coprime x a coprime x b* **by** (*simp-all add*: *coprime-mul-eq*)
    from *mod-less-divisor a b* **have** *xab′*:*x mod a < a x mod b < b* **by** *auto*
    from *xab xab′* **have** *?f x* ∈ (*?S a* × *?S b*)
      by (*simp add*: *coprime-mod*[*OF a*(*1*)] *coprime-mod*[*OF b*(*1*)])}
  moreover
  {**fix** *x y* **assume** *x*: *x* ∈ *?S a* **and** *y*: *y* ∈ *?S b*
    hence *x′*: *coprime x a x < a* **and** *y′*: *coprime y b y < b* **by** *simp-all*
    from *chinese-remainder-coprime-unique*[*OF ab a*(*1*) *b*(*1*) *x′*(*1*) *y′*(*1*)]
    obtain *z* **where** *z*: *coprime z* (*a* ∗ *b*) *z < a* ∗ *b* [*z* = *x*] (*mod a*)
      [*z* = *y*] (*mod b*) **by** *blast*
    hence (*x*,*y*) ∈ *?f* ‘ (*?S* (*a*∗*b*))
      using *y′*(*2*) *mod-less-divisor*[*of b y*] *x′*(*2*) *mod-less-divisor*[*of a x*]
      by (*auto simp add*: *image-iff modeq-def*)}
  ultimately show *?thesis* **by** *auto*
qed
have *finj*: *inj-on ?f* (*?S* (*a*∗*b*))
  unfolding *inj-on-def*
proof(*clarify*)
  fix *x y* **assume** *H*: *coprime x* (*a* ∗ *b*) *x < a* ∗ *b coprime y* (*a* ∗ *b*)
    y < a ∗ *b x mod a* = *y mod a x mod b* = *y mod b*
  hence *cp*: *coprime x a coprime x b coprime y a coprime y b*
    by (*simp-all add*: *coprime-mul-eq*)
  from *chinese-remainder-coprime-unique*[*OF ab a*(*1*) *b*(*1*) *cp*(*3*,*4*)] *H*
  show *x* = *y* **unfolding** *modeq-def* **by** *blast*
qed
from *card-image*[*OF finj*, *unfolded eq*] **have** *?thesis*
  unfolding *phi-alt* **by** *simp* }
ultimately show *?thesis* **by** *auto*
qed

lemma *nproduct-mod*:
  assumes *fS*: *finite S* **and** *n0*: *n* ≠ *0*
  shows [*setprod* (λ*m*. *a*(*m*) *mod n*) *S* = *setprod a S*] (*mod n*)
proof−
  have *th1*:[*1* = *1*] (*mod n*) **by** (*simp add*: *modeq-def*)
  from *cong-mult*
  have *th3*:∀ *x1 y1 x2 y2*.
    [*x1* = *x2*] (*mod n*) ∧ [*y1* = *y2*] (*mod n*) ⟶ [*x1* ∗ *y1* = *x2* ∗ *y2*] (*mod n*)
    by *blast*
  have *th4*:∀ *x*∈*S*. [*a x mod n* = *a x*] (*mod n*) **by** (*simp add*: *modeq-def*)
  from *setprod.related* [**where** *h*=(λ*m*. *a*(*m*) *mod n*) **and** *g*=*a*, *OF th1 th3 fS*,
OF th4] **show** *?thesis* **by** (*simp add*: *fS*)
qed

lemma *nproduct-cmul*:
  assumes *fS*:*finite S*

```
  shows setprod (λm. (c::'a::{comm-monoid-mult})* a(m)) S = c ^ (card S) *
setprod a S
unfolding setprod.distrib setprod-constant [of c] ..

lemma coprime-nproduct:
  assumes fS: finite S and Sn: ∀x∈S. coprime n (a x)
  shows coprime n (setprod a S)
  using fS by (rule finite-subset-induct)
    (insert Sn, auto simp add: coprime-mul)

lemma fermat-little: assumes an: coprime a n
  shows [a ^ (φ n) = 1] (mod n)
proof−
  {assume n=0 hence ?thesis by simp}
  moreover
  {assume n=1 hence ?thesis by (simp add: modeq-def)}
  moreover
  {assume nz: n ≠ 0 and n1: n ≠ 1
    let ?S = {m. coprime m n ∧ m < n}
    let ?P = ∏ ?S
    have fS: finite ?S by simp
    have cardfS: φ n = card ?S unfolding phi-alt ..
    {fix m assume m: m ∈ ?S
      hence coprime m n by simp
      with coprime-mul[of n a m] an have coprime (a*m) n
        by (simp add: coprime-commute)}
    hence Sn: ∀m∈ ?S. coprime (a*m) n by blast
    from coprime-nproduct[OF fS, of n λm. m] have nP:coprime ?P n
      by (simp add: coprime-commute)
    have Paphi: [?P*a^ (φ n) = ?P*1] (mod n)
    proof−
      let ?h = λm. (a * m) mod n

      have eq0: (∏i∈?S. ?h i) = (∏i∈?S. i)
      proof (rule setprod.reindex-bij-betw)
        have inj-on (λi. ?h i) ?S
        proof (rule inj-onI)
          fix x y assume ?h x = ?h y
          then have [a * x = a * y] (mod n)
            by (simp add: modeq-def)
          moreover assume x ∈ ?S y ∈ ?S
          ultimately show x = y
            by (auto intro: cong-imp-eq cong-mult-lcancel an)
        qed
        moreover have ?h ` ?S = ?S
        proof safe
          fix y assume coprime y n then show coprime (?h y) n
            by (metis an nz coprime-commute coprime-mod coprime-mul-eq)
        next
```

fix *y* **assume** *y*: *coprime y n y* < *n*
from *cong-solve-unique*[*OF an nz*] **obtain** *x* **where** *x*: *x* < *n* [*a* ∗ *x* = *y*] (*mod n*)
by *blast*
then show *y* ∈ *?h* ' *?S*
using *cong-coprime*[*OF x*(*2*)] *coprime-mul-eq*[*of n a x*] *an y x*
by (*intro image-eqI*[*of - - x*]) (*auto simp add*: *coprime-commute modeq-def*)
qed (*insert nz*, *simp*)
ultimately show *bij-betw ?h ?S ?S*
by (*simp add*: *bij-betw-def*)
qed
from *nproduct-mod*[*OF fS nz*, *of op* ∗ *a*]
have [(∏ *i*∈*?S*. *a* ∗ *i*) = (∏ *i*∈*?S*. *?h i*)] (*mod n*)
by (*simp add*: *cong-commute*)
also have [(∏ *i*∈*?S*. *?h i*) = *?P*] (*mod n*)
using *eq0 fS an* **by** (*simp add*: *setprod-def modeq-def*)
finally show [*?P*∗*a*ˆ (*φ n*) = *?P*∗*1*] (*mod n*)
unfolding *cardfS mult.commute*[*of ?P a*ˆ (*card ?S*)]
nproduct-cmul[*OF fS*, *symmetric*] *mult-1-right* **by** *simp*
qed
from *cong-mult-lcancel*[*OF nP Paphi*] **have** *?thesis* **. }**
ultimately show *?thesis* **by** *blast*
qed

lemma *fermat-little-prime*: **assumes** *p*: *prime p* **and** *ap*: *coprime a p*
shows [*a*ˆ (*p* − *1*) = *1*] (*mod p*)
using *fermat-little*[*OF ap*] *p*[*unfolded phi-prime*[*symmetric*]]
by *simp*

lemma *lucas-coprime-lemma*:
assumes *m*: *m*≠*0* **and** *am*: [*a*ˆ*m* = *1*] (*mod n*)
shows *coprime a n*
proof−
{assume *n*=*1* **hence** *?thesis* **by** *simp***}**
moreover
{assume *n* = *0* **hence** *?thesis* **using** *am m exp-eq-1*[*of a m*] **by** *simp***}**
moreover
{assume *n*: *n*≠*0 n*≠*1*
from *m* **obtain** *m'* **where** *m'*: *m* = *Suc m'* **by** (*cases m*, *blast*+)
{fix *d*
assume *d*: *d dvd a d dvd n*
from *n* **have** *n1*: *1* < *n* **by** *arith*
from *am mod-less*[*OF n1*] **have** *am1*: *a*ˆ*m mod n* = *1* **unfolding** *modeq-def* **by** *simp*
from *dvd-mult2*[*OF d*(*1*), *of a*ˆ*m'*] **have** *dam*:*d dvd a*ˆ*m* **by** (*simp add*: *m'*)
from *dvd-mod-iff*[*OF d*(*2*), *of a*ˆ*m*] *dam am1*

have *d* = *1* **by** *simp* **}**
hence *?thesis* **unfolding** *coprime* **by** *auto*
}
ultimately show *?thesis* **by** *blast*
qed

lemma *lucas-weak*:
assumes *n*: $n \geq 2$ **and** *an*:[$a\hat{}(n - 1) = 1$] (*mod n*)
and *nm*: $\forall m.\ 0 < m \wedge m < n - 1 \longrightarrow \neg$ [$a\hat{}m = 1$] (*mod n*)
shows *prime n*
proof−
from *n* **have** *n1*: $n \neq 1$ $n \neq 0$ $n - 1 \neq 0$ $n - 1 > 0$ $n - 1 < n$ **by** *arith+*
from *lucas-coprime-lemma*[*OF n1*(*3*) *an*] **have** *can*: *coprime a n* **.**
from *fermat-little*[*OF can*] **have** *afn*: [$a\ \hat{}\ \varphi\ n = 1$] (*mod n*) **.**
{assume $\varphi\ n \neq n - 1$
with *phi-limit-strong*[*OF n1*(*1*)] *phi-lowerbound-1*[*OF n*]
have *c*:$\varphi\ n > 0 \wedge \varphi\ n < n - 1$ **by** *arith*
from *nm*[*rule-format*, *OF c*] *afn* **have** *False* **..}**
hence $\varphi\ n = n - 1$ **by** *blast*
with *phi-prime*[*of n*] *n1*(*1*,*2*) **show** *?thesis* **by** *simp*
qed

lemma *nat-exists-least-iff*: ($\exists$(*n*::*nat*). *P n*) $\longleftrightarrow$ ($\exists$ *n*. *P n* $\wedge$ ($\forall$ *m* < *n*. $\neg$ *P m*))
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?rhs* **thus** *?lhs* **by** *blast*
next
assume *H*: *?lhs* **then obtain** *n* **where** *n*: *P n* **by** *blast*
let *?x* = *Least P*
{fix *m* **assume** *m*: *m* < *?x*
from *not-less-Least*[*OF m*] **have** $\neg$ *P m* **.}**
with *LeastI-ex*[*OF H*] **show** *?rhs* **by** *blast*
qed

lemma *nat-exists-least-iff'*: ($\exists$(*n*::*nat*). *P n*) $\longleftrightarrow$ (*P* (*Least P*) $\wedge$ ($\forall$ *m* < (*Least P*). $\neg$ *P m*))
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof−
{assume *?rhs* **hence** *?lhs* **by** *blast***}**
moreover
{ assume *H*: *?lhs* **then obtain** *n* **where** *n*: *P n* **by** *blast*
let *?x* = *Least P*
{fix *m* **assume** *m*: *m* < *?x*
from *not-less-Least*[*OF m*] **have** $\neg$ *P m* **.}**
with *LeastI-ex*[*OF H*] **have** *?rhs* **by** *blast***}**
ultimately show *?thesis* **by** *blast*
qed

lemma *power-mod*: ((*x*::*nat*) *mod m*)$\hat{}$*n mod m* = *x*$\hat{}$*n mod m*

```
proof(induct n)
  case 0 thus ?case by simp
next
  case (Suc n)
   have (x mod m)^(Suc n) mod m = ((x mod m) * (((x mod m) ^ n) mod m))
mod m
    by (simp add: mod-mult-right-eq[symmetric])
  also have ... = ((x mod m) * (x^n mod m)) mod m using Suc.hyps by simp
  also have ... = x^(Suc n) mod m
    by (simp add: mod-mult-left-eq[symmetric] mod-mult-right-eq[symmetric])
  finally show ?case .
qed

lemma lucas:
  assumes n2: n ≥ 2 and an1: [a^(n − 1) = 1] (mod n)
  and pn: ∀ p. prime p ∧ p dvd n − 1 ⟶ ¬ [a^((n − 1) div p) = 1] (mod n)
  shows prime n
proof−
  from n2 have n01: n≠0 n≠1 n − 1 ≠ 0 by arith+
  from mod-less-divisor[of n 1] n01 have onen: 1 mod n = 1 by simp
  from lucas-coprime-lemma[OF n01(3) an1] cong-coprime[OF an1]
 have an: coprime a n coprime (a^(n − 1)) n by (simp-all add: coprime-commute)
  {assume H0: ∃ m. 0 < m ∧ m < n − 1 ∧ [a ^ m = 1] (mod n) (is EX m. ?P
m)
    from H0[unfolded nat-exists-least-iff[of ?P]] obtain m where
      m: 0 < m m < n − 1 [a ^ m = 1] (mod n) ∀ k <m. ¬?P k by blast
    {assume nm1: (n − 1) mod m > 0
      from mod-less-divisor[OF m(1)] have th0:(n − 1) mod m < m by blast
      let ?y = a^ ((n − 1) div m * m)
      note mdeq = mod-div-equality[of (n − 1) m]
      from coprime-exp[OF an(1)[unfolded coprime-commute[of a n]],
        of (n − 1) div m * m]
      have yn: coprime ?y n by (simp add: coprime-commute)
      have ?y mod n = (a^m)^((n − 1) div m) mod n
        by (simp add: algebra-simps power-mult)
      also have ... = (a^m mod n)^((n − 1) div m) mod n
        using power-mod[of a^m n (n − 1) div m] by simp
      also have ... = 1 using m(3)[unfolded modeq-def onen] onen
        by (simp add: power-Suc0)
      finally have th3: ?y mod n = 1  .
      have th2: [?y * a ^ ((n − 1) mod m) = ?y* 1] (mod n)
        using an1[unfolded modeq-def onen] onen
          mod-div-equality[of (n − 1) m, symmetric]
        by (simp add:power-add[symmetric] modeq-def th3 del: One-nat-def)
      from cong-mult-lcancel[of ?y n a^((n − 1) mod m) 1, OF yn th2]
      have th1: [a ^ ((n − 1) mod m) = 1] (mod n)  .
      from m(4)[rule-format, OF th0] nm1
        less-trans[OF mod-less-divisor[OF m(1), of n − 1] m(2)] th1
      have False by blast }
```

hence $(n - 1)$ *mod* $m = 0$ **by** *auto*
then have *mn*: m *dvd* $n - 1$ **by** *presburger*
then obtain r **where** r: $n - 1 = m*r$ **unfolding** *dvd-def* **by** *blast*
from *n01 r m(2)* **have** *r01*: $r \neq 0$ $r \neq 1$ **by** − (*rule ccontr*, *simp*)+
from *prime-factor*[*OF r01(2)*] **obtain** p **where** p: *prime p p dvd r* **by** *blast*
hence *th*: *prime p* ∧ *p dvd n* − *1* **unfolding** *r* **by** (*auto intro*: *dvd-mult*)
have $(a \hat{} ((n - 1)\ div\ p))\ mod\ n = (a\hat{}(m*r\ div\ p))\ mod\ n$ **using** *r*
by (*simp add*: *power-mult*)
also have ... $= (a\hat{}(m*(r\ div\ p)))\ mod\ n$ **using** *div-mult1-eq*[*of m r p*] *p(2)*[*unfolded dvd-eq-mod-eq-0*] **by** *simp*
also have ... $= ((a\hat{}m)\hat{}(r\ div\ p))\ mod\ n$ **by** (*simp add*: *power-mult*)
also have ... $= ((a\hat{}m\ mod\ n)\hat{}(r\ div\ p))\ mod\ n$ **using** *power-mod*[*of a^m n r div p*] **..**
also have ... $= 1$ **using** *m(3) onen* **by** (*simp add*: *modeq-def power-Suc0*)
finally have $[(a \hat{} ((n - 1)\ div\ p))= 1]\ (mod\ n)$
using *onen* **by** (*simp add*: *modeq-def*)
with *pn*[*rule-format*, *OF th*] **have** *False* **by** *blast*}
hence *th*: $\forall m.\ 0 < m \wedge m < n - 1 \longrightarrow \neg\ [a \hat{}\ m = 1]\ (mod\ n)$ **by** *blast*
from *lucas-weak*[*OF n2 an1 th*] **show** *?thesis* **.**
qed

definition $ord\ n\ a = (if\ coprime\ n\ a\ then\ Least\ (\lambda d.\ d > 0 \wedge [a\ \hat{}d = 1]\ (mod\ n))\ else\ 0)$

lemma *coprime-ord*:
assumes *na*: *coprime n a*
shows $ord\ n\ a > 0 \wedge [a\ \hat{}(ord\ n\ a) = 1]\ (mod\ n) \wedge (\forall m.\ 0 < m \wedge m < ord\ n\ a \longrightarrow \neg\ [a\hat{}\ m = 1]\ (mod\ n))$
proof−
let $?P = \lambda d.\ 0 < d \wedge [a\ \hat{}\ d = 1]\ (mod\ n)$
from *euclid*[*of a*] **obtain** p **where** p: *prime p* $a < p$ **by** *blast*
from *na* **have** *o*: *ord n a* = *Least ?P* **by** (*simp add*: *ord-def*)
{**assume** $n=0 \vee n=1$ **with** *na* **have** $\exists m>0.\ ?P\ m$ **apply** *auto* **apply** (*rule exI*[**where** $x=1$]) **by** (*simp add*: *modeq-def*)}
moreover
{**assume** $n\neq 0 \wedge n\neq 1$ **hence** *n2*:$n \geq 2$ **by** *arith*
from *na* **have** *na'*: *coprime a n* **by** (*simp add*: *coprime-commute*)
from *phi-lowerbound-1*[*OF n2*] *fermat-little*[*OF na'*]
have *ex*: $\exists m>0.\ ?P\ m$ **by** − (*rule exI*[**where** $x=\varphi\ n$], *auto*) }
ultimately have *ex*: $\exists m>0.\ ?P\ m$ **by** *blast*
from *nat-exists-least-iff'*[*of ?P*] *ex na* **show** *?thesis*
unfolding *o*[*symmetric*] **by** *auto*
qed

lemma *ord-works*:

[a ^ (ord n a) = 1] (mod n) ∧ (∀m. 0 < m ∧ m < ord n a ⟶ ~[a^ m = 1] (mod n))
apply (cases coprime n a)
using coprime-ord[of n a]
by (blast, simp add: ord-def modeq-def)

lemma ord: [a^(ord n a) = 1] (mod n) **using** ord-works **by** blast
lemma ord-minimal: 0 < m ⟹ m < ord n a ⟹ ~[a^m = 1] (mod n)
  using ord-works **by** blast
lemma ord-eq-0: ord n a = 0 ⟷ ~coprime n a
by (cases coprime n a, simp add: coprime-ord, simp add: ord-def)

lemma ord-divides:
 [a ^ d = 1] (mod n) ⟷ ord n a dvd d (**is** ?lhs ⟷ ?rhs)
proof
  assume rh: ?rhs
  then obtain k **where** d = ord n a * k **unfolding** dvd-def **by** blast
  hence [a ^ d = (a ^ (ord n a) mod n)^k] (mod n)
    by (simp add : modeq-def power-mult power-mod)
  also have [(a ^ (ord n a) mod n)^k = 1] (mod n)
    using ord[of a n, unfolded modeq-def]
    by (simp add: modeq-def power-mod power-Suc0)
  finally show ?lhs **.**
next
  assume lh: ?lhs
  { assume H: ¬ coprime n a
    hence o: ord n a = 0 **by** (simp add: ord-def)
    {assume d: d=0 **with** o H **have** ?rhs **by** (simp add: modeq-def)**}**
    moreover
    {assume d0: d≠0 **then obtain** d′ **where** d′: d = Suc d′ **by** (cases d, auto)
      from H[unfolded coprime]
      obtain p **where** p: p dvd n p dvd a p ≠ 1 **by** auto
      from lh[unfolded nat-mod]
      obtain q1 q2 **where** q12:a ^ d + n * q1 = 1 + n * q2 **by** blast
      hence a ^ d + n * q1 − n * q2 = 1 **by** simp
     with dvd-diff-nat [OF dvd-add [OF divides-rexp[OF p(2), of d′] dvd-mult2[OF p(1), of q1]] dvd-mult2[OF p(1), of q2]] d′ **have** p dvd 1 **by** simp
      with p(3) **have** False **by** simp
      hence ?rhs **..}**
    ultimately have ?rhs **by** blast**}**
  moreover
  {assume H: coprime n a
    let ?o = ord n a
    let ?q = d div ord n a
    let ?r = d mod ord n a
    from cong-exp[OF ord[of a n], of ?q]
    have eqo: [(a^?o)^?q = 1] (mod n) **by** (simp add: modeq-def power-Suc0)
    from H **have** onz: ?o ≠ 0 **by** (simp add: ord-eq-0)
    hence op: ?o > 0 **by** simp

from *mod-div-equality*[*of d ord n a*] *lh*
have [*a*^(*?o*∗*?q* + *?r*) = *1*] (*mod n*) **by** (*simp add*: *modeq-def mult.commute*)
hence [(*a*^*?o*)^*?q* ∗ (*a*^*?r*) = *1*] (*mod n*)
by (*simp add*: *modeq-def power-mult*[*symmetric*] *power-add*[*symmetric*])
hence *th*: [*a*^*?r* = *1*] (*mod n*)
using *eqo mod-mult-left-eq*[*of* (*a*^*?o*)^*?q a*^*?r n*]
apply (*simp add*: *modeq-def del*: *One-nat-def*)
by (*simp add*: *mod-mult-left-eq*[*symmetric*])
{assume *r*: *?r* = *0* **hence** *?rhs* **by** (*simp add*: *dvd-eq-mod-eq-0*)**}**
moreover
{assume *r*: *?r* ≠ *0*
with *mod-less-divisor*[*OF op*, *of d*] **have** *r0o*:*?r* >*0* ∧ *?r* < *?o* **by** *simp*
from *conjunct2*[*OF ord-works*[*of a n*], *rule-format*, *OF r0o*] *th*
have *?rhs* **by** *blast***}**
ultimately have *?rhs* **by** *blast***}**
ultimately show *?rhs* **by** *blast*
qed

lemma *order-divides-phi*: *coprime n a* ⟹ *ord n a dvd φ n*
using *ord-divides fermat-little coprime-commute* **by** *simp*
lemma *order-divides-expdiff*:
assumes *na*: *coprime n a*
shows [*a*^*d* = *a*^*e*] (*mod n*) ⟷ [*d* = *e*] (*mod* (*ord n a*))
proof−
{fix *n a d e*
assume *na*: *coprime n a* **and** *ed*: (*e*::*nat*) ≤ *d*
hence ∃ *c*. *d* = *e* + *c* **by** *arith*
then obtain *c* **where** *c*: *d* = *e* + *c* **by** *arith*
from *na* **have** *an*: *coprime a n* **by** (*simp add*: *coprime-commute*)
from *coprime-exp*[*OF na*, *of e*]
have *aen*: *coprime* (*a*^*e*) *n* **by** (*simp add*: *coprime-commute*)
from *coprime-exp*[*OF na*, *of c*]
have *acn*: *coprime* (*a*^*c*) *n* **by** (*simp add*: *coprime-commute*)
have [*a*^*d* = *a*^*e*] (*mod n*) ⟷ [*a*^(*e* + *c*) = *a*^(*e* + *0*)] (*mod n*)
using *c* **by** *simp*
also have ... ⟷ [*a*^*e*∗ *a*^*c* = *a*^*e* ∗*a*^*0*] (*mod n*) **by** (*simp add*: *power-add*)
also have ... ⟷ [*a* ^ *c* = *1*] (*mod n*)
using *cong-mult-lcancel-eq*[*OF aen*, *of a*^*c a*^*0*] **by** *simp*
also have ... ⟷ *ord n a dvd c* **by** (*simp only*: *ord-divides*)
also have ... ⟷ [*e* + *c* = *e* + *0*] (*mod ord n a*)
using *cong-add-lcancel-eq*[*of e c 0 ord n a*, *simplified cong-0-divides*]
by *simp*
finally have [*a*^*d* = *a*^*e*] (*mod n*) ⟷ [*d* = *e*] (*mod* (*ord n a*))
using *c* **by** *simp* **}**
note *th* = *this*
have *e* ≤ *d* ∨ *d* ≤ *e* **by** *arith*
moreover
{assume *ed*: *e* ≤ *d* **from** *th*[*OF na ed*] **have** *?thesis* **.}**
moreover

```
  {assume de: d ≤ e
    from th[OF na de] have ?thesis by (simp add: cong-commute) }
  ultimately show ?thesis by blast
qed
```

```
lemma prime-prime-factor:
  prime n ⟷ n ≠ 1∧ (∀p. prime p ∧ p dvd n ⟶ p = n)
proof−
  {assume n: n=0 ∨ n=1 hence ?thesis using prime-0 two-is-prime by auto}
  moreover
  {assume n: n≠0 n≠1
    {assume pn: prime n

      from pn[unfolded prime-def] have ∀p. prime p ∧ p dvd n ⟶ p = n
        using n
        apply (cases n = 0 ∨ n=1,simp)
        by (clarsimp, erule-tac x=p in allE, auto)}
    moreover
    {assume H: ∀p. prime p ∧ p dvd n ⟶ p = n
      from n have n1: n > 1 by arith
      {fix m assume m: m dvd n m≠1
        from prime-factor[OF m(2)] obtain p where
          p: prime p p dvd m by blast
        from dvd-trans[OF p(2) m(1)] p(1) H have p = n by blast
        with p(2) have n dvd m  by simp
        hence m=n  using dvd-antisym[OF m(1)] by simp }
      with n1 have prime n  unfolding prime-def by auto }
    ultimately have ?thesis using n by blast}
  ultimately       show ?thesis by auto
qed
```

```
lemma prime-divisor-sqrt:
  prime n ⟷ n ≠ 1 ∧ (∀d. d dvd n ∧ d^2 ≤ n ⟶ d = 1)
proof−
  {assume n=0 ∨ n=1 hence ?thesis using prime-0 prime-1
    by (auto simp add: nat-power-eq-0-iff)}
  moreover
  {assume n: n≠0 n≠1
    hence np: n > 1 by arith
    {fix d assume d: d dvd n d^2 ≤ n and H: ∀m. m dvd n ⟶ m=1 ∨ m=n
      from H d have d1n: d = 1 ∨ d=n by blast
      {assume dn: d=n
        have n^2 > n*1 using n by (simp add: power2-eq-square)
        with dn d(2) have d=1 by simp}
      with d1n have d = 1 by blast  }
    moreover
    {fix d assume d: d dvd n and H: ∀d'. d' dvd n ∧ d'^2 ≤ n ⟶ d' = 1
```

```
      from d n have d ≠ 0 apply − apply (rule ccontr) by simp
      hence dp: d > 0 by simp
      from d[unfolded dvd-def] obtain e where e: n= d∗e by blast
      from n dp e have ep:e > 0 by simp
      have d^2 ≤ n ∨ e^2 ≤ n using dp ep
        by (auto simp add: e power2-eq-square mult-le-cancel-left)
      moreover
      {assume h: d^2 ≤ n
        from H[rule-format, of d] h d have d = 1 by blast}
      moreover
      {assume h: e^2 ≤ n
        from e have e dvd n unfolding dvd-def by (simp add: mult.commute)
        with H[rule-format, of e] h have e=1 by simp
        with e have d = n by simp}
      ultimately have d=1 ∨ d=n  by blast}
    ultimately have ?thesis unfolding prime-def using np n(2) by blast}
  ultimately show ?thesis by auto
qed
lemma prime-prime-factor-sqrt:
  prime n ⟷ n ≠ 0 ∧ n ≠ 1 ∧ ¬ (∃p. prime p ∧ p dvd n ∧ p^2 ≤ n)
  (is ?lhs ⟷?rhs)
proof−
  {assume n=0 ∨ n=1 hence ?thesis using prime-0 prime-1 by auto}
  moreover
  {assume n: n≠0 n≠1
    {assume H: ?lhs
      from H[unfolded prime-divisor-sqrt] n
      have ?rhs
        apply clarsimp
        apply (erule-tac x=p in allE)
        apply simp
        done
    }
    moreover
    {assume H: ?rhs
      {fix d assume d: d dvd n d^2 ≤ n d≠1
        from prime-factor[OF d(3)]
        obtain p where p: prime p p dvd d by blast
        from n have np: n > 0 by arith
        from d(1) n have d ≠ 0 by − (rule ccontr, auto)
        hence dp: d > 0 by arith
        from mult-mono[OF dvd-imp-le[OF p(2) dp] dvd-imp-le[OF p(2) dp]] d(2)
        have p^2 ≤ n unfolding power2-eq-square by arith
        with H n p(1) dvd-trans[OF p(2) d(1)] have False  by blast}
      with n prime-divisor-sqrt  have ?lhs by auto}
    ultimately have ?thesis by blast }
  ultimately show ?thesis by (cases n=0 ∨ n=1, auto)
qed
```

lemma *pocklington-lemma*:
  assumes *n*: $n \geq 2$ **and** *nqr*: $n - 1 = q*r$ **and** *an*: $[a\hat{}\ (n - 1) = 1]\ (mod\ n)$
  and *aq*:$\forall p.\ prime\ p \wedge p\ dvd\ q \longrightarrow coprime\ (a\hat{}\ ((n - 1)\ div\ p) - 1)\ n$
  and *pp*: *prime p* **and** *pn*: *p dvd n*
  shows $[p = 1]\ (mod\ q)$
proof−
  from *pp prime-0 prime-1* **have** *p01*: $p \neq 0\ p \neq 1$ **by** − (*rule ccontr*, *simp*)+
  from *cong-1-divides*[*OF an*, *unfolded nqr*, *unfolded dvd-def*]
  obtain *k* **where** *k*: $a\ \hat{}\ (q * r) - 1 = n*k$ **by** *blast*
  from *pn*[*unfolded dvd-def*] **obtain** *l* **where** *l*: $n = p*l$ **by** *blast*
  {**assume** *a0*: $a = 0$
    hence $a\hat{}\ (n - 1) = 0$ **using** *n* **by** (*simp add*: *power-0-left*)
   with *n an mod-less*[*of 1 n*] **have** *False* **by** (*simp add*: *power-0-left modeq-def*)}
  hence *a0*: $a \neq 0$ **..**
  from *n nqr* **have** *aqr0*: $a\ \hat{}\ (q * r) \neq 0$ **using** *a0* **by** *simp*
  hence $(a\ \hat{}\ (q * r) - 1) + 1 = a\ \hat{}\ (q * r)$ **by** *simp*
  with *k l* **have** $a\ \hat{}\ (q * r) = p*l*k + 1$ **by** *simp*
  hence $a\ \hat{}\ (r * q) + p * 0 = 1 + p * (l*k)$ **by** (*simp add*: *ac-simps*)
  hence *odq*: $ord\ p\ (a\hat{}r)\ dvd\ q$
    unfolding *ord-divides*[*symmetric*] *power-mult*[*symmetric*] *nat-mod* **by** *blast*
  from *odq*[*unfolded dvd-def*] **obtain** *d* **where** *d*: $q = ord\ p\ (a\hat{}r) * d$ **by** *blast*
  {**assume** *d1*: $d \neq 1$
    from *prime-factor*[*OF d1*] **obtain** *P* **where** *P*: *prime P P dvd d* **by** *blast*
    from *d dvd-mult*[*OF P*(*2*), *of ord p* ($a\hat{}r$)] **have** *Pq*: *P dvd q* **by** *simp*
    from *aq P*(*1*) *Pq* **have** *caP*:$coprime\ (a\hat{}\ ((n - 1)\ div\ P) - 1)\ n$ **by** *blast*
    from *Pq* **obtain** *s* **where** *s*: $q = P*s$ **unfolding** *dvd-def* **by** *blast*
    have *P0*: $P \neq 0$ **using** *P*(*1*) *prime-0* **by** − (*rule ccontr*, *simp*)
    from *P*(*2*) **obtain** *t* **where** *t*: $d = P*t$ **unfolding** *dvd-def* **by** *blast*
    from *d s t P0* **have** *s'*: $ord\ p\ (a\hat{}r) * t = s$ **by** *algebra*
    have $ord\ p\ (a\hat{}r) * t*r = r * ord\ p\ (a\hat{}r) * t$ **by** *algebra*
    hence *exps*: $a\hat{}(ord\ p\ (a\hat{}r) * t*r) = ((a\ \hat{}\ r)\ \hat{}\ ord\ p\ (a\hat{}r))\ \hat{}\ t$
      by (*simp only*: *power-mult*)
    have $[((a\ \hat{}\ r)\ \hat{}\ ord\ p\ (a\hat{}r))\ \hat{}\ t= 1\hat{}t]\ (mod\ p)$
      by (*rule cong-exp*, *rule ord*)
    then have *th*: $[((a\ \hat{}\ r)\ \hat{}\ ord\ p\ (a\hat{}r))\ \hat{}\ t= 1]\ (mod\ p)$
      by (*simp add*: *power-Suc0*)
    from *cong-1-divides*[*OF th*] *exps* **have** *pd0*: $p\ dvd\ a\hat{}(ord\ p\ (a\hat{}r) * t*r) - 1$
by *simp*
    from *nqr s s'* **have** $(n - 1)\ div\ P = ord\ p\ (a\hat{}r) * t*r$ **using** *P0* **by** *simp*
    with *caP* **have** $coprime\ (a\hat{}(ord\ p\ (a\hat{}r) * t*r) - 1)\ n$ **by** *simp*
    with *p01 pn pd0* **have** *False* **unfolding** *coprime* **by** *auto*}
  hence *d1*: $d = 1$ **by** *blast*
  hence *o*: $ord\ p\ (a\hat{}r) = q$ **using** *d* **by** *simp*
  from *pp phi-prime*[*of p*] **have** *phip*: $\varphi\ p = p - 1$ **by** *simp*
  {**fix** *d* **assume** *d*: $d\ dvd\ p\ d\ dvd\ a\ d \neq 1$
    from *pp*[*unfolded prime-def*] *d* **have** *dp*: $d = p$ **by** *blast*
    from *n* **have** *n12*:$Suc\ (n - 2) = n - 1$ **by** *arith*
    with *divides-rexp*[*OF d*(*2*)[*unfolded dp*], *of* $n - 2$]

```
    have th0: p dvd a ^ (n - 1) by simp
    from n have n0: n ≠ 0 by simp
    from d(2) an n12[symmetric] have a0: a ≠ 0
      by - (rule ccontr, simp add: modeq-def)
    have th1: a^ (n - 1) ≠ 0 using n d(2) dp a0 by auto
    from coprime-minus1[OF th1, unfolded coprime]
      dvd-trans[OF pn cong-1-divides[OF an]] th0 d(3) dp
    have False by auto}
  hence cpa: coprime p a using coprime by auto
  from coprime-exp[OF cpa, of r] coprime-commute
  have arp: coprime (a^r) p by blast
  from fermat-little[OF arp, simplified ord-divides] o phip
  have q dvd (p - 1) by simp
  then obtain d where d:p - 1 = q * d unfolding dvd-def by blast
  from prime-0 pp have p0:p ≠ 0 by - (rule ccontr, auto)
  from p0 d have p + q * 0 = 1 + q * d by simp
  with nat-mod[of p 1 q, symmetric]
  show ?thesis by blast
qed

lemma pocklington:
  assumes n: n ≥ 2 and nqr: n - 1 = q*r and sqr: n ≤ q^2
  and an: [a^ (n - 1) = 1] (mod n)
  and aq:∀ p. prime p ∧ p dvd q ⟶ coprime (a^ ((n - 1) div p) - 1) n
  shows prime n
unfolding prime-prime-factor-sqrt[of n]
proof-
  let ?ths = n ≠ 0 ∧ n ≠ 1 ∧ ¬ (∃ p. prime p ∧ p dvd n ∧ p^2 ≤ n)
  from n have n01: n≠0 n≠1 by arith+
  {fix p assume p: prime p p dvd n p^2 ≤ n
    from p(3) sqr have p^(Suc 1) ≤ q^(Suc 1) by (simp add: power2-eq-square)
    hence pq: p ≤ q unfolding exp-mono-le .
    from pocklington-lemma[OF n nqr an aq p(1,2)] cong-1-divides
    have th: q dvd p - 1 by blast
    have p - 1 ≠ 0using prime-ge-2[OF p(1)] by arith
    with divides-ge[OF th] pq have False by arith }
  with n01 show ?ths by blast
qed


lemma pocklington-alt:
  assumes n: n ≥ 2 and nqr: n - 1 = q*r and sqr: n ≤ q^2
  and an: [a^ (n - 1) = 1] (mod n)
   and aq:∀ p. prime p ∧ p dvd q ⟶ (∃ b. [a^((n - 1) div p) = b] (mod n) ∧
coprime (b - 1) n)
  shows prime n
proof-
  {fix p assume p: prime p p dvd q
    from aq[rule-format] p obtain b where
```

b: [*a*ˆ((*n* − *1*) *div p*) = *b*] (*mod n*) *coprime* (*b* − *1*) *n* **by** *blast*
{**assume** *a0*: *a*=*0*
from *n an* **have** [*0* = *1*] (*mod n*) **unfolding** *a0 power-0-left* **by** *auto*
hence *False* **using** *n* **by** (*simp add*: *modeq-def dvd-eq-mod-eq-0*[*symmetric*])}
hence *a0*: *a*≠ *0* **..**
hence *a1*: *a* ≥ *1* **by** *arith*
from *one-le-power*[*OF a1*] **have** *ath*: *1* ≤ *a* ˆ ((*n* − *1*) *div p*) **.**
{**assume** *b0*: *b* = *0*
from *p*(*2*) *nqr* **have** (*n* − *1*) *mod p* = *0*
apply (*simp only*: *dvd-eq-mod-eq-0*[*symmetric*]) **by** (*rule dvd-mult2*, *simp*)
with *mod-div-equality*[*of n* − *1 p*]
have (*n* − *1*) *div p* ∗ *p*= *n* − *1* **by** *auto*
hence *eq*: (*a*ˆ((*n* − *1*) *div p*))ˆ*p* = *a*ˆ(*n* − *1*)
by (*simp only*: *power-mult*[*symmetric*])
from *prime-ge-2*[*OF p*(*1*)] **have** *pS*: *Suc* (*p* − *1*) = *p* **by** *arith*
from *b*(*1*) **have** *d*: *n dvd a*ˆ((*n* − *1*) *div p*) **unfolding** *b0 cong-0-divides* **.**
from *divides-rexp*[*OF d*, *of p* − *1*] *pS eq cong-divides*[*OF an*] *n*
have *False* **by** *simp*}
then have *b0*: *b* ≠ *0* **..**
hence *b1*: *b* ≥ *1* **by** *arith*
from *cong-coprime*[*OF cong-sub*[*OF b*(*1*) *cong-refl*[*of 1*] *ath b1*]] *b*(*2*) *nqr*
have *coprime* (*a* ˆ ((*n* − *1*) *div p*) − *1*) *n* **by** (*simp add*: *coprime-commute*)}
hence *th*: ∀ *p*. *prime p* ∧ *p dvd q* ⟶ *coprime* (*a* ˆ ((*n* − *1*) *div p*) − *1*) *n*
by *blast*
from *pocklington*[*OF n nqr sqr an th*] **show** *?thesis* **.**
qed

definition *primefact ps n* = (*foldr op* ∗ *ps* *1* = *n* ∧ (∀ *p*∈ *set ps*. *prime p*))

lemma *primefact*: **assumes** *n*: *n* ≠ *0*
shows ∃ *ps*. *primefact ps n*
using *n*
proof(*induct n rule*: *nat-less-induct*)
fix *n* **assume** *H*: ∀ *m*<*n*. *m* ≠ *0* ⟶ (∃ *ps*. *primefact ps m*) **and** *n*: *n*≠*0*
let *?ths* = ∃ *ps*. *primefact ps n*
{**assume** *n* = *1*
hence *primefact* [] *n* **by** (*simp add*: *primefact-def*)
hence *?ths* **by** *blast* }
moreover
{**assume** *n1*: *n* ≠ *1*
with *n* **have** *n2*: *n* ≥ *2* **by** *arith*
from *prime-factor*[*OF n1*] **obtain** *p* **where** *p*: *prime p p dvd n* **by** *blast*
from *p*(*2*) **obtain** *m* **where** *m*: *n* = *p*∗*m* **unfolding** *dvd-def* **by** *blast*
from *n m* **have** *m0*: *m* > *0* *m*≠*0* **by** *auto*
from *prime-ge-2*[*OF p*(*1*)] **have** *1* < *p* **by** *arith*
with *m0 m* **have** *mn*: *m* < *n* **by** *auto*
from *H*[*rule-format*, *OF mn m0*(*2*)] **obtain** *ps* **where** *ps*: *primefact ps m* **..**

from *ps m p*(*1*) **have** *primefact* (*p#ps*) *n* **by** (*simp add*: *primefact-def*)
hence *?ths* **by** *blast*}
ultimately show *?ths* **by** *blast*
qed

lemma *primefact-contains*:
assumes *pf*: *primefact ps n* **and** *p*: *prime p* **and** *pn*: *p dvd n*
shows $p \in$ *set ps*
using *pf p pn*
proof(*induct ps arbitrary*: *p n*)
case *Nil* **thus** *?case* **by** (*auto simp add*: *primefact-def*)
next
case (*Cons q qs p n*)
from *Cons.prems*[*unfolded primefact-def*]
have *q*: *prime q q* $*$ *foldr op* $*$ *qs 1* $=$ *n* $\forall$ *p* $\in$*set qs*. *prime p* **and** *p*: *prime p p dvd q* $*$ *foldr op* $*$ *qs 1* **by** *simp-all*
{**assume** *p dvd q*
with *p*(*1*) *q*(*1*) **have** *p* $=$ *q* **unfolding** *prime-def* **by** *auto*
hence *?case* **by** *simp*}
moreover
{ **assume** *h*: *p dvd foldr op* $*$ *qs 1*
from *q*(*3*) **have** *pqs*: *primefact qs* (*foldr op* $*$ *qs 1*)
by (*simp add*: *primefact-def*)
from *Cons.hyps*[*OF pqs p*(*1*) *h*] **have** *?case* **by** *simp*}
ultimately show *?case* **using** *prime-divprod*[*OF p*] **by** *blast*
qed

lemma *primefact-variant*: *primefact ps n* $\longleftrightarrow$ *foldr op* $*$ *ps 1* $=$ *n* $\wedge$ *list-all prime ps*
by (*auto simp add*: *primefact-def list-all-iff*)

lemma *lucas-primefact*:
assumes *n*: $n \geq 2$ **and** *an*: [*a*ˆ(*n* $-$ *1*) $=$ *1*] (*mod n*)
and *psn*: *foldr op* $*$ *ps 1* $=$ *n* $-$ *1*
and *psp*: *list-all* (λ*p*. *prime p* $\wedge$ $\neg$ [*a*ˆ((*n* $-$ *1*) *div p*) $=$ *1*] (*mod n*)) *ps*
shows *prime n*
proof$-$
{**fix** *p* **assume** *p*: *prime p p dvd n* $-$ *1* [*a* ˆ ((*n* $-$ *1*) *div p*) $=$ *1*] (*mod n*)
from *psn psp* **have** *psn1*: *primefact ps* (*n* $-$ *1*)
by (*auto simp add*: *list-all-iff primefact-variant*)
from *p*(*3*) *primefact-contains*[*OF psn1 p*(*1*,*2*)] *psp*
have *False* **by** (*induct ps*, *auto*)}
with *lucas*[*OF n an*] **show** *?thesis* **by** *blast*
qed

lemma *mod-le*: **assumes** *n*: $n \neq (0::nat)$ **shows** $m\ mod\ n \leq m$
proof−
  from *mod-div-equality*[*of m n*]
  have $\exists x.\ x + m\ mod\ n = m$ **by** *blast*
  then show *?thesis* **by** *auto*
qed

lemma *pocklington-primefact*:
 assumes *n*: $n \geq 2$ **and** *qrn*: $q*r = n - 1$ **and** *nq2*: $n \leq q^2$
 and *arnb*: $(a\hat{}r)\ mod\ n = b$ **and** *psq*: *foldr op* ∗ *ps 1* = *q*
 and *bqn*: $(b\hat{}q)\ mod\ n = 1$
 and *psp*: *list-all* (λ*p*. *prime p* ∧ *coprime* ((*b*ˆ(*q div p*)) *mod n* − *1*) *n*) *ps*
 shows *prime n*
proof−
 from *bqn psp qrn*
 have *bqn*: *a* ˆ (*n* − *1*) *mod n* = *1*
  and *psp*: *list-all* (λ*p*. *prime p* ∧ *coprime* (*a*ˆ(*r* ∗(*q div p*)) *mod n* − *1*) *n*) *ps*
unfolding *arnb*[*symmetric*] *power-mod*
  by (*simp-all add*: *power-mult*[*symmetric*] *algebra-simps*)
 from *n* **have** *n0*: *n* > *0* **by** *arith*
 from *mod-div-equality*[*of a*ˆ(*n* − *1*) *n*]
  mod-less-divisor[*OF n0*, *of a*ˆ(*n* − *1*)]
 have *an1*: [*a* ˆ (*n* − *1*) = *1*] (*mod n*)
  unfolding *nat-mod bqn*
  apply −
  apply (*rule exI*[**where** *x*=*0*])
  apply (*rule exI*[**where** *x*=*a*ˆ(*n* − *1*) *div n*])
  by (*simp add*: *algebra-simps*)
 {**fix** *p* **assume** *p*: *prime p p dvd q*
  from *psp psq* **have** *pfpsq*: *primefact ps q*
   by (*auto simp add*: *primefact-variant list-all-iff*)
  from *psp primefact-contains*[*OF pfpsq p*]
  have *p'*: *coprime* (*a* ˆ (*r* ∗ (*q div p*)) *mod n* − *1*) *n*
   by (*simp add*: *list-all-iff*)
  from *prime-ge-2*[*OF p*(*1*)] **have** *p01*: *p* ≠ *0* *p* ≠ *1* *p* =*Suc*(*p* − *1*) **by** *arith*+
  from *div-mult1-eq*[*of r q p*] *p*(*2*)
  have *eq1*: *r*∗ (*q div p*) = (*n* − *1*) *div p*
   unfolding *qrn*[*symmetric*] *dvd-eq-mod-eq-0* **by** (*simp add*: *mult.commute*)
  have *ath*: ⋀*a* (*b*::*nat*). *a* <= *b* ⟹ *a* ≠ *0* ==> *1* <= *a* ∧ *1* <= *b* **by** *arith*
  from *n0* **have** *n00*: *n* ≠ *0* **by** *arith*
  from *mod-le*[*OF n00*]
  have *th10*: *a* ˆ ((*n* − *1*) *div p*) *mod n* ≤ *a* ˆ ((*n* − *1*) *div p*) **.**
  {**assume** *a* ˆ ((*n* − *1*) *div p*) *mod n* = *0*
   then obtain *s* **where** *s*: *a* ˆ ((*n* − *1*) *div p*) = *n*∗*s*
    unfolding *mod-eq-0-iff* **by** *blast*
   hence *eq0*: (*a*ˆ((*n* − *1*) *div p*))ˆ*p* = (*n*∗*s*)ˆ*p* **by** *simp*
   from *qrn*[*symmetric*] **have** *qn1*: *q dvd n* − *1* **unfolding** *dvd-def* **by** *auto*
   from *dvd-trans*[*OF p*(*2*) *qn1*] *div-mod-equality'*[*of n* − *1 p*]

```
      have npp: (n − 1) div p ∗ p = n − 1 by (simp add: dvd-eq-mod-eq-0)
      with eq0 have a ˆ (n − 1) = (n∗s) ˆp
        by (simp add: power-mult[symmetric])
      hence 1 = (n∗s) ˆ(Suc (p − 1)) mod n using bqn p01 by simp
      also have ... = 0 by (simp add: mult.assoc)
      finally have False by simp }
      then have th11: a ˆ ((n − 1) div p) mod n ≠ 0 by auto
    have th1: [a ˆ ((n − 1) div p) mod n = a ˆ ((n − 1) div p)] (mod n)
      unfolding modeq-def by simp
    from cong-sub[OF th1 cong-refl[of 1]]  ath[OF th10 th11]
    have th: [a ˆ ((n − 1) div p) mod n − 1 = a ˆ ((n − 1) div p) − 1] (mod n)
      by blast
    from cong-coprime[OF th] p′[unfolded eq1]
    have coprime (a ˆ ((n − 1) div p) − 1) n by (simp add: coprime-commute) }
  with pocklington[OF n qrn[symmetric] nq2 an1]
  show ?thesis by blast
qed

end
```

References

[1] H. Davenport. *The Higher Arithmetic*. Cambridge University Press, 1992.