

Real-Valued Special Functions:
Upper and Lower Bounds

Lawrence C. Paulson

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Abstract

This development proves upper and lower bounds for several familiar real-valued functions. For $\sin$, $\cos$, $\exp$ and the square root function, it defines and verifies infinite families of upper and lower bounds, mostly based on Taylor series expansions. For $\tan^{-1}$, $\ln$ and $\exp$, it verifies a finite collection of upper and lower bounds, originally obtained from the functions' continued fraction expansions using the computer algebra system Maple. A common theme in these proofs is to take the difference between a function and its approximation, which should be zero at one point, and then consider the sign of the derivative.

The immediate purpose of this development is to verify axioms used by MetiTarski [1], an automatic theorem prover for real-valued special functions. Crucial to MetiTarski's operation is the provision of upper and lower bounds for each function of interest.

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Chapter 1

General Lemmas for Proving Function Inequalities

```
theory Bounds-Lemmas
imports Complex-Main
```

```
begin
```

These are for functions that are differentiable over a closed interval.

```
lemma gen-lower-bound-increasing:
```

```
  fixes a :: real
  assumes a ≤ x
    and  $\bigwedge y. a \leq y \implies y \leq x \implies ((\lambda x. f\ x - f\ x) \text{ has-real-derivative } g\ y) \text{ (at } y)$ 
    and  $\bigwedge y. a \leq y \implies y \leq x \implies g\ y \leq 0$ 
    and  $f\ a = f\ a$ 
  shows  $f\ x \leq f\ x$ 
proof -
  have  $f\ x - f\ x \leq f\ a - f\ a$ 
  apply (rule DERIV-nonpos-imp-nonincreasing [where  $f = \lambda x. f\ x - f\ x$ ])
  apply (rule assms)
  apply (intro allI impI exI conjI)
  apply (rule assms | simp)+
  done
  also have  $\dots = 0$ 
  by (simp add: assms)
  finally show ?thesis
  by simp
qed
```

```
lemma gen-lower-bound-decreasing:
```

```
  fixes a :: real
  assumes  $x \leq a$ 
    and  $\bigwedge y. x \leq y \implies y \leq a \implies ((\lambda x. f\ x - f\ x) \text{ has-real-derivative } g\ y) \text{ (at } y)$ 
    and  $\bigwedge y. x \leq y \implies y \leq a \implies g\ y \geq 0$ 
    and  $f\ a = f\ a$ 
```

```

    shows  $f x \leq f x$ 
  proof -
    have  $f (- (-x)) \leq f (- (-x))$ 
    apply (rule gen-lower-bound-increasing [of  $-a -x - \lambda u. - g (-u)$ ])
    apply (auto simp: assms)
    apply (subst DERIV-mirror [symmetric])
    apply (simp add: assms)
    done
  then show ?thesis
    by simp
qed

lemma gen-upper-bound-increasing:
  fixes  $a :: real$ 
  assumes  $a \leq x$ 
    and  $\bigwedge y. a \leq y \implies y \leq x \implies ((\lambda x. fu\ x - f\ x) \text{ has-real-derivative } g\ y) \text{ (at } y)$ 
    and  $\bigwedge y. a \leq y \implies y \leq x \implies g\ y \geq 0$ 
    and  $fu\ a = f\ a$ 
  shows  $f\ x \leq fu\ x$ 
  apply (rule gen-lower-bound-increasing [of  $a\ x\ fu\ \lambda u. - g\ u$ ])
  using assms DERIV-minus [where  $f = \lambda x. fu\ x - f\ x$ ]
  apply auto
  done

lemma gen-upper-bound-decreasing:
  fixes  $a :: real$ 
  assumes  $x \leq a$ 
    and  $\bigwedge y. x \leq y \implies y \leq a \implies ((\lambda x. fu\ x - f\ x) \text{ has-real-derivative } g\ y) \text{ (at } y)$ 
    and  $\bigwedge y. x \leq y \implies y \leq a \implies g\ y \leq 0$ 
    and  $fu\ a = f\ a$ 
  shows  $f\ x \leq fu\ x$ 
  apply (rule gen-lower-bound-decreasing [of  $x\ a - \lambda u. - g\ u$ ])
  using assms DERIV-minus [where  $f = \lambda x. fu\ x - f\ x$ ]
  apply auto
  done

end

```

Chapter 2

Arctan Upper and Lower Bounds

```
theory Atan-CF-Bounds
imports Bounds-Lemmas
        ~~/src/HOL/Library/Sum-of-Squares
```

begin

Covers all bounds used in arctan-upper.ax, arctan-lower.ax and arctan-extended.ax, excepting only arctan-extended2.ax, which is used in two atan-error-analysis problems.

2.1 Upper Bound 1

```
definition arctan-upper-11 :: real  $\Rightarrow$  real
  where arctan-upper-11  $\equiv \lambda x. -(pi/2) - 1/x$ 
```

```
definition diff-delta-arctan-upper-11 :: real  $\Rightarrow$  real
  where diff-delta-arctan-upper-11  $\equiv \lambda x. 1 / (x^2 * (1 + x^2))$ 
```

```
lemma d-delta-arctan-upper-11:  $x \neq 0 \implies$ 
   $((\lambda x. arctan-upper-11\ x - arctan\ x) \text{ has-field-derivative } diff-delta-arctan-upper-11\ x) \text{ (at } x)$ 
unfolding arctan-upper-11-def diff-delta-arctan-upper-11-def
apply (intro derivative-eq-intros | simp)+
apply (simp add: divide-simps add-nonneg-eq-0-iff, algebra)
done
```

```
lemma d-delta-arctan-upper-11-pos:  $x \neq 0 \implies diff-delta-arctan-upper-11\ x > 0$ 
unfolding diff-delta-arctan-upper-11-def
by (simp add: divide-simps zero-less-mult-iff add-pos-pos)
```

Different proof needed here: they coincide not at zero, but at (-) infinity!

```
lemma arctan-upper-11:
```

```

assumes  $x < 0$ 
shows  $\arctan(x) < \arctan\text{-upper-11 } x$ 
proof -
  have  $((\lambda x. \arctan\text{-upper-11 } x - \arctan x) \longrightarrow - (pi / 2) - 0 - (- (pi / 2)))$ 
  at-bot
  unfolding arctan-upper-11-def
  apply (intro tendsto-intros tendsto-arctan-at-bot, auto simp: ext [OF divide-inverse])
  apply (metis tendsto-inverse-0 at-bot-le-at-infinity tendsto-mono)
  done
  then have  $*$ :  $((\lambda x. \arctan\text{-upper-11 } x - \arctan x) \longrightarrow 0)$  at-bot
  by simp
  have  $0 < \arctan\text{-upper-11 } x - \arctan x$ 
  apply (rule DERIV-pos-imp-increasing-at-bot [OF - *])
  apply (metis assms d-delta-arctan-upper-11 d-delta-arctan-upper-11-pos not-le)
  done
  then show ?thesis
  by auto
qed

```

```

definition arctan-upper-12 :: real  $\Rightarrow$  real
  where arctan-upper-12  $\equiv \lambda x. 3*x / (x^2 + 3)$ 

```

```

definition diff-delta-arctan-upper-12 :: real  $\Rightarrow$  real
  where diff-delta-arctan-upper-12  $\equiv \lambda x. -4*x^4 / ((x^2+3)^2 * (1+x^2))$ 

```

```

lemma d-delta-arctan-upper-12:
   $((\lambda x. \arctan\text{-upper-12 } x - \arctan x)$  has-field-derivative diff-delta-arctan-upper-12
   $x)$  (at x)
  unfolding arctan-upper-12-def diff-delta-arctan-upper-12-def
  using assms
  apply (intro derivative-eq-intros, simp-all)
  apply (auto simp: divide-simps add-nonneg-eq-0-iff, algebra)
  done

```

Strict inequalities also possible

```

lemma arctan-upper-12:
  assumes  $x \leq 0$  shows  $\arctan(x) \leq \arctan\text{-upper-12 } x$ 
  apply (rule gen-upper-bound-decreasing [OF assms d-delta-arctan-upper-12])
  apply (auto simp: diff-delta-arctan-upper-12-def arctan-upper-12-def)
  done

```

```

definition arctan-upper-13 :: real  $\Rightarrow$  real
  where arctan-upper-13  $\equiv \lambda x. x$ 

```

```

definition diff-delta-arctan-upper-13 :: real  $\Rightarrow$  real
  where diff-delta-arctan-upper-13  $\equiv \lambda x. x^2 / (1 + x^2)$ 

```

```

lemma d-delta-arctan-upper-13:
   $((\lambda x. \arctan\text{-upper-13 } x - \arctan x)$  has-field-derivative diff-delta-arctan-upper-13
   $x)$ 

```

```

x) (at x)
unfolding arctan-upper-13-def diff-delta-arctan-upper-13-def
apply (intro derivative-eq-intros, simp-all)
apply (simp add: divide-simps add-nonneg-eq-0-iff)
done

```

```

lemma arctan-upper-13:
  assumes  $x \geq 0$  shows  $\arctan(x) \leq \arctan\text{-upper-13 } x$ 
apply (rule gen-upper-bound-increasing [OF assms d-delta-arctan-upper-13])
apply (auto simp: diff-delta-arctan-upper-13-def arctan-upper-13-def)
done

```

```

definition arctan-upper-14 :: real  $\Rightarrow$  real
  where  $\arctan\text{-upper-14} \equiv \lambda x. \pi/2 - 3*x / (1 + 3*x^2)$ 

```

```

definition diff-delta-arctan-upper-14 :: real  $\Rightarrow$  real
  where  $\text{diff-delta-arctan-upper-14} \equiv \lambda x. -4 / ((1 + 3*x^2)^2 * (1+x^2))$ 

```

```

lemma d-delta-arctan-upper-14:
  (( $\lambda x. \arctan\text{-upper-14 } x - \arctan x$ ) has-field-derivative diff-delta-arctan-upper-14
x) (at x)
unfolding arctan-upper-14-def diff-delta-arctan-upper-14-def
apply (intro derivative-eq-intros | simp add: add-nonneg-eq-0-iff)+
apply (simp add: divide-simps add-nonneg-eq-0-iff, algebra)
done

```

```

lemma d-delta-arctan-upper-14-neg: diff-delta-arctan-upper-14  $x < 0$ 
unfolding diff-delta-arctan-upper-14-def
apply (auto simp: divide-simps add-nonneg-eq-0-iff zero-less-mult-iff)
using power2-less-0 [of x]
apply arith
done

```

```

lemma lim14: (( $\lambda x::\text{real}. 3 * x / (1 + 3 * x^2)$ )  $\longrightarrow 0$ ) at-infinity
apply (rule tendsto-0-le [where f = inverse and K=1])
apply (metis tendsto-inverse-0)
apply (simp add: eventually-at-infinity)
apply (rule-tac x=1 in exI)
apply (simp add: power-eq-if abs-if divide-simps add-sign-intros)
done

```

Different proof needed here: they coincide not at zero, but at (+) infinity!

```

lemma arctan-upper-14:
  assumes  $x > 0$ 
  shows  $\arctan(x) < \arctan\text{-upper-14 } x$ 
proof -
  have (( $\lambda x. \arctan\text{-upper-14 } x - \arctan x$ )  $\longrightarrow \pi / 2 - 0 - \pi / 2$ ) at-top
  unfolding arctan-upper-14-def
  apply (intro tendsto-intros tendsto-arctan-at-top)

```

```

    apply (auto simp: tendsto-mono [OF at-top-le-at-infinity lim14])
  done
then have *: ((λx. arctan-upper-14 x - arctan x) ⟶ 0) at-top
  by simp
have 0 < arctan-upper-14 x - arctan x
  apply (rule DERIV-neg-imp-decreasing-at-top [OF - *])
  apply (metis d-delta-arctan-upper-14 d-delta-arctan-upper-14-neg)
  done
then show ?thesis
  by auto
qed

```

2.2 Lower Bound 1

definition *arctan-lower-11* :: *real* $\Rightarrow$ *real*
 where *arctan-lower-11* $\equiv \lambda x. -(\pi/2) - 3*x / (1 + 3*x^2)$

lemma *arctan-lower-11*:
 assumes $x < 0$
 shows $\arctan(x) > \text{arctan-lower-11 } x$
 using *arctan-upper-14* [of $-x$] *assms*
 by (auto simp: *arctan-upper-14-def* *arctan-lower-11-def* *arctan-minus*)

abbreviation *arctan-lower-12* $\equiv \text{arctan-upper-13}$

lemma *arctan-lower-12*:
 assumes $x \leq 0$
 shows $\arctan(x) \geq \text{arctan-lower-12 } x$
 using *arctan-upper-13* [of $-x$] *assms*
 by (auto simp: *arctan-upper-13-def* *arctan-minus*)

abbreviation *arctan-lower-13* $\equiv \text{arctan-upper-12}$

lemma *arctan-lower-13*:
 assumes $x \geq 0$
 shows $\arctan(x) \geq \text{arctan-lower-13 } x$
 using *arctan-upper-12* [of $-x$] *assms*
 by (auto simp: *arctan-upper-12-def* *arctan-minus*)

definition *arctan-lower-14* :: *real* $\Rightarrow$ *real*
 where *arctan-lower-14* $\equiv \lambda x. \pi/2 - 1/x$

lemma *arctan-lower-14*:
 assumes $x > 0$
 shows $\arctan(x) > \text{arctan-lower-14 } x$
 using *arctan-upper-11* [of $-x$] *assms*
 by (auto simp: *arctan-upper-11-def* *arctan-lower-14-def* *arctan-minus*)

2.3 Upper Bound 3

definition *arctan-upper-31* :: *real* $\Rightarrow$ *real*

where *arctan-upper-31* $\equiv \lambda x. -(pi/2) - (64 + 735*x^2 + 945*x^4) / (15*x*(15 + 70*x^2 + 63*x^4))$

definition *diff-delta-arctan-upper-31* :: *real* $\Rightarrow$ *real*

where *diff-delta-arctan-upper-31* $\equiv \lambda x. 64 / (x^2 * (15 + 70*x^2 + 63*x^4)^2 * (1 + x^2))$

lemma *d-delta-arctan-upper-31*:

assumes $x \neq 0$

shows $((\lambda x. \text{arctan-upper-31 } x - \text{arctan } x) \text{ has-field-derivative } \text{diff-delta-arctan-upper-31 } x) \text{ (at } x)$

unfolding *arctan-upper-31-def* *diff-delta-arctan-upper-31-def*

using *assms*

apply (*intro derivative-eq-intros*)

apply (*rule refl* | *simp add: add-nonneg-eq-0-iff*) +

apply (*simp add: divide-simps add-nonneg-eq-0-iff, algebra*)

done

lemma *d-delta-arctan-upper-31-pos*: $x \neq 0 \implies \text{diff-delta-arctan-upper-31 } x > 0$

unfolding *diff-delta-arctan-upper-31-def*

by (*auto simp: divide-simps zero-less-mult-iff add-pos-pos add-nonneg-eq-0-iff*)

lemma *arctan-upper-31*:

assumes $x < 0$

shows $\text{arctan}(x) < \text{arctan-upper-31 } x$

proof –

have *: $\bigwedge x::\text{real}. (15 + 70 * x^2 + 63 * x^4) > 0$

by (*sos* ($((R < 1 + ((R < 1 * ((R < 7/8 * [19/7 * x^2 + 1]^2) + ((R < 4 * [x]^2 + (R < 10/7 * [x^2]^2)))))) + ((A \leq 0 * R < 1) * (R < 1/8 * [1]^2))))))$

then have **: $\bigwedge x::\text{real}. \neg (15 + 70 * x^2 + 63 * x^4) < 0$

by (*simp add: not-less*)

have $((\lambda x::\text{real}. (64 + 735 * x^2 + 945 * x^4) / (15 * x * (15 + 70 * x^2 + 63 * x^4))) \longrightarrow 0) \text{ at-bot}$

apply (*rule tendsto-0-le [where f = inverse and K=2]*)

apply (*metis at-bot-le-at-infinity tendsto-inverse-0 tendsto-mono*)

apply (*simp add: eventually-at-bot-linorder*)

apply (*rule-tac x=-1 in exI*)

apply (*auto simp: divide-simps abs-if zero-less-mult-iff ***)

done

then have $((\lambda x. \text{arctan-upper-31 } x - \text{arctan } x) \longrightarrow - (pi / 2) - 0 - (- (pi / 2))) \text{ at-bot}$

unfolding *arctan-upper-31-def*

apply (*intro tendsto-intros tendsto-arctan-at-bot, auto*)

done

then have *: $((\lambda x. \text{arctan-upper-31 } x - \text{arctan } x) \longrightarrow 0) \text{ at-bot}$

by *simp*

```

have 0 < arctan-upper-31 x - arctan x
  apply (rule DERIV-pos-imp-increasing-at-bot [OF - *])
  apply (metis assms d-delta-arctan-upper-31 d-delta-arctan-upper-31-pos not-le)
done
then show ?thesis
  by auto
qed

```

```

definition arctan-upper-32 :: real  $\Rightarrow$  real
  where arctan-upper-32  $\equiv \lambda x. 7*(33*x^4 + 170*x^2 + 165)*x / (5*(5*x^6 + 105*x^4 + 315*x^2 + 231))$ 

```

```

definition diff-delta-arctan-upper-32 :: real  $\Rightarrow$  real
  where diff-delta-arctan-upper-32  $\equiv \lambda x. -256*x^{12} / ((5*x^6 + 105*x^4 + 315*x^2 + 231)^2*(1+x^2))$ 

```

```

lemma d-delta-arctan-upper-32:
  (( $\lambda x. \text{arctan-upper-32 } x - \text{arctan } x$ ) has-field-derivative diff-delta-arctan-upper-32
  x) (at x)
  unfolding arctan-upper-32-def diff-delta-arctan-upper-32-def
  apply (intro derivative-eq-intros | simp)+
  apply simp-all
  apply (auto simp: add-nonneg-eq-0-iff divide-simps, algebra)
done

```

```

lemma arctan-upper-32:
  assumes  $x \leq 0$  shows  $\text{arctan}(x) \leq \text{arctan-upper-32 } x$ 
  apply (rule gen-upper-bound-decreasing [OF assms d-delta-arctan-upper-32])
  apply (auto simp: diff-delta-arctan-upper-32-def arctan-upper-32-def)
done

```

```

definition arctan-upper-33 :: real  $\Rightarrow$  real
  where arctan-upper-33  $\equiv \lambda x. (64*x^4 + 735*x^2 + 945)*x / (15*(15*x^4 + 70*x^2 + 63))$ 

```

```

definition diff-delta-arctan-upper-33 :: real  $\Rightarrow$  real
  where diff-delta-arctan-upper-33  $\equiv \lambda x. 64*x^{10} / ((15*x^4 + 70*x^2 + 63)^2*(1+x^2))$ 

```

```

lemma d-delta-arctan-upper-33:
  (( $\lambda x. \text{arctan-upper-33 } x - \text{arctan } x$ ) has-field-derivative diff-delta-arctan-upper-33
  x) (at x)
  unfolding arctan-upper-33-def diff-delta-arctan-upper-33-def
  apply (intro derivative-eq-intros, simp-all)
  apply (auto simp: add-nonneg-eq-0-iff divide-simps, algebra)
done

```

```

lemma arctan-upper-33:
  assumes  $x \geq 0$  shows  $\text{arctan}(x) \leq \text{arctan-upper-33 } x$ 
  apply (rule gen-upper-bound-increasing [OF assms d-delta-arctan-upper-33])
  apply (auto simp: diff-delta-arctan-upper-33-def arctan-upper-33-def)
done

```

```

definition arctan-upper-34 :: real ⇒ real
  where arctan-upper-34 ≡
    λx. pi/2 - (33 + 170*x^2 + 165*x^4)*7*x / (5*(5 + 105*x^2 + 315*x^4
    + 231*x^6))

definition diff-delta-arctan-upper-34 :: real ⇒ real
  where diff-delta-arctan-upper-34 ≡ λx. -256 / ((5+105*x^2+315*x^4+231*x^6)^2*(1+x^2))

lemma d-delta-arctan-upper-34:
  ((λx. arctan-upper-34 x - arctan x) has-field-derivative diff-delta-arctan-upper-34
  x) (at x)
unfolding arctan-upper-34-def diff-delta-arctan-upper-34-def
apply (intro derivative-eq-intros | simp add: add-nonneg-eq-0-iff)+
apply (simp add: divide-simps add-nonneg-eq-0-iff, algebra)
done

lemma d-delta-arctan-upper-34-pos: diff-delta-arctan-upper-34 x < 0
unfolding diff-delta-arctan-upper-34-def
apply (simp add: divide-simps add-nonneg-eq-0-iff zero-less-mult-iff)
using power2-less-0 [of x]
apply arith
done

lemma arctan-upper-34:
  assumes x > 0
  shows arctan(x) < arctan-upper-34 x
proof -
  have ((λx. arctan-upper-34 x - arctan x) ⟶ pi / 2 - 0 - pi / 2) at-top
  unfolding arctan-upper-34-def
  apply (intro tendsto-intros tendsto-arctan-at-top, auto)
  apply (rule tendsto-0-le [where f = inverse and K=1])
  apply (metis tendsto-inverse-0 at-top-le-at-infinity tendsto-mono)
  apply (simp add: eventually-at-top-linorder)
  apply (rule-tac x=1 in exI)
  apply (auto simp: divide-simps power-eq-if add-pos-pos algebra-simps)
  done
  then have *: ((λx. arctan-upper-34 x - arctan x) ⟶ 0) at-top
  by simp
  have 0 < arctan-upper-34 x - arctan x
  apply (rule DERIV-neg-imp-decreasing-at-top [OF - *])
  apply (metis d-delta-arctan-upper-34 d-delta-arctan-upper-34-pos)
  done
  then show ?thesis
  by auto
qed

```

2.4 Lower Bound 3

definition *arctan-lower-31* :: *real* $\Rightarrow$ *real*
where *arctan-lower-31* $\equiv \lambda x. -(pi/2) - (33 + 170*x^2 + 165*x^4)*7*x / (5*(5 + 105*x^2 + 315*x^4 + 231*x^6))$

lemma *arctan-lower-31*:
assumes $x < 0$
shows $arctan(x) > arctan-lower-31\ x$
using *arctan-upper-34* [of $-x$] *assms*
by (*auto simp: arctan-upper-34-def arctan-lower-31-def arctan-minus*)

abbreviation *arctan-lower-32* $\equiv arctan-upper-33$

lemma *arctan-lower-32*:
assumes $x \leq 0$
shows $arctan(x) \geq arctan-lower-32\ x$
using *arctan-upper-33* [of $-x$] *assms*
by (*auto simp: arctan-upper-33-def arctan-minus*)

abbreviation *arctan-lower-33* $\equiv arctan-upper-32$

lemma *arctan-lower-33*:
assumes $x \geq 0$
shows $arctan(x) \geq arctan-lower-33\ x$
using *arctan-upper-32* [of $-x$] *assms*
by (*auto simp: arctan-upper-32-def arctan-minus*)

definition *arctan-lower-34* :: *real* $\Rightarrow$ *real*
where *arctan-lower-34* $\equiv \lambda x. pi/2 - (64 + 735*x^2 + 945*x^4) / (15*x*(15 + 70*x^2 + 63*x^4))$

lemma *arctan-lower-34*:
assumes $x > 0$
shows $arctan(x) > arctan-lower-34\ x$
using *arctan-upper-31* [of $-x$] *assms*
by (*auto simp: arctan-upper-31-def arctan-lower-34-def arctan-minus*)

2.5 Upper Bound 4

definition *arctan-upper-41* :: *real* $\Rightarrow$ *real*
where *arctan-upper-41* $\equiv \lambda x. -(pi/2) - (256 + 5943*x^2 + 19250*x^4 + 15015*x^6) / (35*x*(35 + 315*x^2 + 693*x^4 + 429*x^6))$

definition *diff-delta-arctan-upper-41* :: *real* $\Rightarrow$ *real*
where *diff-delta-arctan-upper-41* $\equiv \lambda x. 256 / (x^2*(35+315*x^2+693*x^4+429*x^6)^2*(1+x^2))$

lemma *d-delta-arctan-upper-41*:

```

assumes  $x \neq 0$ 
shows  $((\lambda x. \text{arctan-upper-41 } x - \text{arctan } x) \text{ has-field-derivative diff-delta-arctan-upper-41 } x) \text{ (at } x)$ 
unfolding arctan-upper-41-def diff-delta-arctan-upper-41-def
using assms
apply (intro derivative-eq-intros)
apply (rule refl | simp add: add-nonneg-eq-0-iff)+
apply (simp add: divide-simps add-nonneg-eq-0-iff, algebra)
done

lemma d-delta-arctan-upper-41-pos:  $x \neq 0 \implies \text{diff-delta-arctan-upper-41 } x > 0$ 
unfolding diff-delta-arctan-upper-41-def
by (auto simp: zero-less-mult-iff add-pos-pos add-nonneg-eq-0-iff)

lemma arctan-upper-41:
assumes  $x < 0$ 
shows  $\text{arctan}(x) < \text{arctan-upper-41 } x$ 
proof -
have *:  $\bigwedge x::\text{real}. (35 + 315 * x^2 + 693 * x^4 + 429 * x^6) > 0$ 
by (sos ((R<1 + ((R<1 * ((R<13/8589934592 * [95/26*x^2 + 1]^2) +
 $((R<38654705675/4294967296 * [170080704731/154618822700*x^3 + x]^2) +$ 
 $((R<14271/446676598784 * [x^2]^2) + (R<3631584276674589067439/2656331147370089676800$ 
 $* [x^3]^2)))))) + ((A<=0 * R<1) * (R<245426703/8589934592 * [1]^2))))))$ 
then have **:  $\bigwedge x::\text{real}. x < 0 \implies \neg (35 + 315 * x^2 + 693 * x^4 + 429 * x^6) < 0$ 
by (simp add: not-less)
have  $((\lambda x::\text{real}. (256 + 5943 * x^2 + 19250 * x^4 + 15015 * x^6) / (35 * x * (35 + 315 * x^2 + 693 * x^4 + 429 * x^6))) \longrightarrow 0) \text{ at-bot}$ 
apply (rule tendsto-0-le [where f = inverse and K=2])
apply (metis at-bot-le-at-infinity tendsto-inverse-0 tendsto-mono)
apply (simp add: eventually-at-bot-linorder)
apply (rule-tac x=-1 in exI)
apply (auto simp: ** abs-if divide-simps zero-less-mult-iff)
done
then have  $((\lambda x. \text{arctan-upper-41 } x - \text{arctan } x) \longrightarrow - (pi / 2) - 0 - (- (pi / 2))) \text{ at-bot}$ 
unfolding arctan-upper-41-def
apply (intro tendsto-intros tendsto-arctan-at-bot, auto)
done
then have *:  $((\lambda x. \text{arctan-upper-41 } x - \text{arctan } x) \longrightarrow 0) \text{ at-bot}$ 
by simp
have  $0 < \text{arctan-upper-41 } x - \text{arctan } x$ 
apply (rule DERIV-pos-imp-increasing-at-bot [OF - *])
apply (metis assms d-delta-arctan-upper-41 d-delta-arctan-upper-41-pos not-le)
done
then show ?thesis
by auto
qed

```

definition *arctan-upper-42* :: *real* $\Rightarrow$ *real*

where *arctan-upper-42* $\equiv$

$$\lambda x. (15159*x^6 + 147455*x^4 + 345345*x^2 + 225225)*x / (35*(35*x^8 + 1260*x^6 + 6930*x^4 + 12012*x^2 + 6435))$$

definition *diff-delta-arctan-upper-42* :: *real* $\Rightarrow$ *real*

where *diff-delta-arctan-upper-42* $\equiv$

$$\lambda x. -16384*x^16 / ((35*x^8 + 1260*x^6 + 6930*x^4 + 12012*x^2 + 6435)^2*(1+x^2))$$

lemma *d-delta-arctan-upper-42*:

(($\lambda x. \text{arctan-upper-42 } x - \text{arctan } x$) *has-field-derivative* *diff-delta-arctan-upper-42* *x*) (*at x*)

unfolding *arctan-upper-42-def* *diff-delta-arctan-upper-42-def*

apply (*intro derivative-eq-intros*, *simp-all*)

apply (*auto simp: divide-simps add-nonneg-eq-0-iff*, *algebra*)

done

lemma *arctan-upper-42*:

assumes $x \leq 0$ **shows** $\text{arctan}(x) \leq \text{arctan-upper-42 } x$

apply (*rule gen-upper-bound-decreasing* [*OF* *assms d-delta-arctan-upper-42*])

apply (*auto simp: diff-delta-arctan-upper-42-def arctan-upper-42-def*)

done

definition *arctan-upper-43* :: *real* $\Rightarrow$ *real*

where *arctan-upper-43* $\equiv$

$$\lambda x. (256*x^6 + 5943*x^4 + 19250*x^2 + 15015)*x / (35 * (35*x^6 + 315*x^4 + 693*x^2 + 429))$$

definition *diff-delta-arctan-upper-43* :: *real* $\Rightarrow$ *real*

where *diff-delta-arctan-upper-43* $\equiv \lambda x. 256*x^14 / ((35*x^6 + 315*x^4 + 693*x^2 + 429)^2*(1+x^2))$

lemma *d-delta-arctan-upper-43*:

(($\lambda x. \text{arctan-upper-43 } x - \text{arctan } x$) *has-field-derivative* *diff-delta-arctan-upper-43* *x*) (*at x*)

unfolding *arctan-upper-43-def* *diff-delta-arctan-upper-43-def*

apply (*intro derivative-eq-intros*, *simp-all*)

apply (*auto simp: add-nonneg-eq-0-iff divide-simps*, *algebra*)

done

lemma *arctan-upper-43*:

assumes $x \geq 0$ **shows** $\text{arctan}(x) \leq \text{arctan-upper-43 } x$

apply (*rule gen-upper-bound-increasing* [*OF* *assms d-delta-arctan-upper-43*])

apply (*auto simp: diff-delta-arctan-upper-43-def arctan-upper-43-def*)

done

definition *arctan-upper-44* :: *real* $\Rightarrow$ *real*

where *arctan-upper-44* $\equiv$

$$\lambda x. \pi/2 - (15159 + 147455*x^2 + 345345*x^4 + 225225*x^6)*x / (35*(35 + 1260*x^2 + 6930*x^4 + 12012*x^6 + 6435*x^8))$$

```

definition diff-delta-arctan-upper-44 :: real ⇒ real
  where diff-delta-arctan-upper-44 ≡
    λx. -16384 / ((35+1260*x^2+6930*x^4+12012*x^6+6435*x^8)^2*(1+x^2))

lemma d-delta-arctan-upper-44:
  ((λx. arctan-upper-44 x - arctan x) has-field-derivative diff-delta-arctan-upper-44
  x) (at x)
unfolding arctan-upper-44-def diff-delta-arctan-upper-44-def
apply (intro derivative-eq-intros | simp add: add-nonneg-eq-0-iff)+
apply (simp add: divide-simps add-nonneg-eq-0-iff, algebra)
done

lemma d-delta-arctan-upper-44-pos: diff-delta-arctan-upper-44 x < 0
unfolding diff-delta-arctan-upper-44-def
apply (auto simp: divide-simps add-nonneg-eq-0-iff zero-less-mult-iff)
using power2-less-0 [of x]
apply arith
done

lemma arctan-upper-44:
  assumes x > 0
  shows arctan(x) < arctan-upper-44 x
proof -
  have ((λx. arctan-upper-44 x - arctan x) ⟶ pi / 2 - 0 - pi / 2) at-top
  unfolding arctan-upper-44-def
  apply (intro tendsto-intros tendsto-arctan-at-top, auto)
  apply (rule tendsto-0-le [where f = inverse and K=1])
  apply (metis tendsto-inverse-0 at-top-le-at-infinity tendsto-mono)
  apply (simp add: eventually-at-top-linorder)
  apply (rule-tac x=1 in exI)
  apply (auto simp: zero-le-mult-iff divide-simps not-le[symmetric] power-eq-if
  algebra-simps)
  done
  then have *: ((λx. arctan-upper-44 x - arctan x) ⟶ 0) at-top
  by simp
  have 0 < arctan-upper-44 x - arctan x
  apply (rule DERIV-neg-imp-decreasing-at-top [OF - *])
  apply (metis d-delta-arctan-upper-44 d-delta-arctan-upper-44-pos)
  done
  then show ?thesis
  by auto
qed

```

2.6 Lower Bound 4

```

definition arctan-lower-41 :: real ⇒ real
  where arctan-lower-41 ≡
    λx. -(pi/2) - (15159+147455*x^2+345345*x^4+225225*x^6)*x /
      (35*(35+1260*x^2+6930*x^4+12012*x^6+6435*x^8))

```

```

lemma arctan-lower-41:
  assumes  $x < 0$ 
  shows  $\arctan(x) > \text{arctan-lower-41 } x$ 
  using arctan-upper-44 [of  $-x$ ] assms
  by (auto simp: arctan-upper-44-def arctan-lower-41-def arctan-minus)

abbreviation arctan-lower-42  $\equiv$  arctan-upper-43

lemma arctan-lower-42:
  assumes  $x \leq 0$ 
  shows  $\arctan(x) \geq \text{arctan-lower-42 } x$ 
  using arctan-upper-43 [of  $-x$ ] assms
  by (auto simp: arctan-upper-43-def arctan-minus)

abbreviation arctan-lower-43  $\equiv$  arctan-upper-42

lemma arctan-lower-43:
  assumes  $x \geq 0$ 
  shows  $\arctan(x) \geq \text{arctan-lower-43 } x$ 
  using arctan-upper-42 [of  $-x$ ] assms
  by (auto simp: arctan-upper-42-def arctan-minus)

definition arctan-lower-44 ::  $\text{real} \Rightarrow \text{real}$ 
  where arctan-lower-44  $\equiv$ 
     $\lambda x. \pi/2 - (256 + 5943x^2 + 19250x^4 + 15015x^6) /$ 
     $(35x(35 + 315x^2 + 693x^4 + 429x^6))$ 

lemma arctan-lower-44:
  assumes  $x > 0$ 
  shows  $\arctan(x) > \text{arctan-lower-44 } x$ 
  using arctan-upper-41 [of  $-x$ ] assms
  by (auto simp: arctan-upper-41-def arctan-lower-44-def arctan-minus)

end

```

Chapter 3

Exp Upper and Lower Bounds

```
theory Exp-Bounds
imports Bounds-Lemmas
         ~~/src/HOL/Library/Sum-of-Squares
         ../Sturm-Sequences/Sturm
```

```
begin
```

Covers all bounds used in exp-upper.ax, exp-lower.ax and exp-extended.ax.

3.1 Taylor Series Bounds

exp-positive is the theorem $0 \leq \exp ?x$

exp-lower-taylor-1 is the theorem $1 + ?x \leq \exp ?x$

All even approximants are lower bounds.

```
lemma exp-lower-taylor-even:
```

```
  fixes x::real
```

```
  shows even n  $\implies (\sum m < n. (x ^ m) / (fact m)) \leq \exp x$ 
```

```
  using Maclaurin-exp-le [of x n]
```

```
  by (auto simp add: zero-le-even-power)
```

```
lemma exp-upper-taylor-even:
```

```
  fixes x::real
```

```
  assumes n: even n
```

```
    and pos:  $(\sum m < n. ((-x) ^ m) / (fact m)) > 0$  (is ?sum > 0)
```

```
  shows  $\exp x \leq \text{inverse } ?\text{sum}$ 
```

```
  using exp-lower-taylor-even [OF n, of -x]
```

```
  by (metis exp-minus inverse-inverse-eq le-imp-inverse-le pos)
```

3 if the previous lemma is expressed in terms of $(2::'a) * m$.

```
lemma exp-lower-taylor-3:
```

```

fixes  $x::\text{real}$ 
shows  $1 + x + (1/2)*x^2 + (1/6)*x^3 + (1/24)*x^4 + (1/120)*x^5 \leq \exp$ 
 $x$ 
by (rule order-trans [OF - exp-lower-taylor-even [of 6]])
      (auto simp: lessThan-nat-numeral fact-numeral)

lemma exp-lower-taylor-3-cubed:
  fixes  $x::\text{real}$ 
  shows  $(1 + x/3 + (1/2)*(x/3)^2 + (1/6)*(x/3)^3 + (1/24)*(x/3)^4 +$ 
 $(1/120)*(x/3)^5)^3 \leq \exp x$ 
proof -
  have  $(1 + x/3 + (1/2)*(x/3)^2 + (1/6)*(x/3)^3 + (1/24)*(x/3)^4 +$ 
 $(1/120)*(x/3)^5)^3$ 
     $\leq \exp (x/3)^3$ 
    by (metis power-mono-odd odd-numeral exp-lower-taylor-3)
  also have  $\dots = \exp x$ 
    by (simp add: exp-real-of-nat-mult [symmetric])
  finally show ?thesis .
qed

lemma exp-lower-taylor-2:
  fixes  $x::\text{real}$ 
  shows  $1 + x + (1/2)*x^2 + (1/6)*x^3 \leq \exp x$ 
proof -
  have even ( $4::\text{nat}$ ) by simp
  then have  $(\sum m < 4. x^m / (\text{fact } m)) \leq \exp x$ 
    by (rule exp-lower-taylor-even)
  then show ?thesis by (auto simp add: numeral-eq-Suc)
qed

lemma exp-upper-bound-case-3:
  fixes  $x::\text{real}$ 
  assumes  $x \leq 3.19$ 
  shows  $\exp x \leq 2304 / (-(x^3) + 6*x^2 - 24*x + 48)^2$ 
proof -
  have  $(1/48)*(-(x^3) + 6*x^2 - 24*x + 48) = (1 + (-x/2) + (1/2)*(-x/2)^2$ 
 $+ (1/6)*(-x/2)^3)$ 
    by (simp add: field-simps power2-eq-square power3-eq-cube)
  also have  $\dots \leq \exp (-x/2)$ 
    by (rule exp-lower-taylor-2)
  finally have  $1: (1/48)*(-(x^3) + 6*x^2 - 24*x + 48) \leq \exp (-x/2)$  .
  have  $(-(x^3) + 6*x^2 - 24*x + 48)^2 / 2304 = ((1/48)*(-(x^3) + 6*x^2$ 
 $- 24*x + 48))^2$ 
    by (simp add: field-simps power2-eq-square power3-eq-cube)
  also have  $\dots \leq (\exp (-x/2))^2$ 
    apply (rule power-mono [OF 1])
    apply (simp add: algebra-simps)
  using assms
  apply (sos ((R < 1 + ((R < 1 * ((R < 1323/13 * [~ 15/49*x + 1]^2) + (R < 1/637

```

```

* [x]^2))) + (((A<0 * R<1) * (R<50/13 * [1]^2)) + ((A<=0 * R<1) * ((R<56/13
* [5/56*x + 1]^2) + (R<199/728 * [x]^2)))))))))
done
also have ... = inverse (exp x)
by (metis exp-minus mult-exp-exp power2-eq-square real-sum-of-halves)
finally have 2:  $-(x^3) + 6x^2 - 24x + 48)^2 / 2304 \leq \text{inverse } (\exp x)$  .
have  $6 * x^2 - x^3 - 24 * x + 48 \neq 0$  using assms
by (sos ((R<1 + (([400/13] * A=0) + ((R<1 * ((R<1323/13 * [15/49*x
+ 1]^2) + (R<1/637 * [x]^2))) + ((A<=0 * R<1) * ((R<56/13 * [5/56*x
+ 1]^2) + (R<199/728 * [x]^2))))))))))
then show ?thesis
using Fields.linordered-field-class.le-imp-inverse-le [OF 2]
by simp
qed

```

lemma exp-upper-bound-case-5:

```

fixes x::real
assumes x ≤ 6.36
shows exp x ≤ 21743271936 /  $-(x^3) + 12x^2 - 96x + 384)^4$ 
proof -
have  $(1/384)*(-(x^3) + 12x^2 - 96x + 384) = (1 + (-x/4) + (1/2)*(-x/4)^2 + (1/6)*(-x/4)^3)$ 
by (simp add: field-simps power2-eq-square power3-eq-cube)
also have ... ≤ exp (-x/4)
by (rule exp-lower-taylor-2)
finally have 1:  $(1/384)*(-(x^3) + 12x^2 - 96x + 384) \leq \exp (-x/4)$  .
have  $-(x^3) + 12x^2 - 96x + 384)^4 / 21743271936 = ((1/384)*(-(x^3) + 12x^2 - 96x + 384))^4$ 
by (simp add: divide-simps)
also have ... ≤  $(\exp (-x/4))^4$ 
apply (rule power-mono [OF 1])
apply (simp add: algebra-simps)
using assms
apply (sos ((R<1 + ((R<1 * ((R<1777/32 * [539/3554*x + 1]^2) + (R<907/227456 * [x]^2))) + (((A<0 * R<1) * (R<25/1024 * [1]^2)) + ((A<=0 * R<1) * ((R<49/32 * [2/49*x + 1]^2) + (R<45/1568 * [x]^2))))))))))
done
also have ... = inverse (exp x)
by (simp add: exp-real-of-nat-mult [symmetric] exp-minus [symmetric])
finally have 2:  $-(x^3) + 12x^2 - 96x + 384)^4 / 21743271936 \leq \text{inverse } (\exp x)$  .
have  $12 * x^2 - x^3 - 96 * x + 384 \neq 0$  using assms
by (sos ((R<1 + (([25/32] * A=0) + ((R<1 * ((R<1777/32 * [539/3554*x
+ 1]^2) + (R<907/227456 * [x]^2))) + ((A<=0 * R<1) * ((R<49/32 * [2/49*x
+ 1]^2) + (R<45/1568 * [x]^2))))))))))
then show ?thesis
using Fields.linordered-field-class.le-imp-inverse-le [OF 2]
by simp
qed

```

3.2 Continued Fraction Bound 2

definition *exp-cf2* :: *real* $\Rightarrow$ *real*

where *exp-cf2* $\equiv \lambda x. (x^2 + 6*x + 12) / (x^2 - 6*x + 12)$

lemma *denom-cf2-pos*: **fixes** *x::real* **shows** $x^2 - 6 * x + 12 > 0$

by (*sos* ((*R*<1 + ((*R*<1 * ((*R*<5 * [$\sim 3/10*x + 1$]²) + (*R*<1/20 * [*x*]²))) + ((*A*<=0 * *R*<1) * (*R*<1/2 * [*1*]²))))))

lemma *numer-cf2-pos*: **fixes** *x::real* **shows** $x^2 + 6 * x + 12 > 0$

by (*sos* ((*R*<1 + ((*R*<1 * ((*R*<5 * [$\sim 3/10*x + 1$]²) + (*R*<1/20 * [*x*]²))) + ((*A*<=0 * *R*<1) * (*R*<1/2 * [*1*]²))))))

lemma *exp-cf2-pos*: *exp-cf2* *x* > 0

unfolding *exp-cf2-def*

by (*auto simp add: divide-simps denom-cf2-pos numer-cf2-pos*)

definition *diff-delta-lnexp-cf2* :: *real* $\Rightarrow$ *real*

where *diff-delta-lnexp-cf2* $\equiv \lambda x. - (x^4) / ((x^2 - 6*x + 12) * (x^2 + 6*x + 12))$

lemma *d-delta-lnexp-cf2-nonpos*: *diff-delta-lnexp-cf2* *x* ≤ 0

unfolding *diff-delta-lnexp-cf2-def*

by (*sos* (((*R*<1 + ((*R*<1 * ((*R*<5/4 * [$\sim 3/40*x^2 + 1$]²) + (*R*<11/1280 * [*x*²]²))) + ((*A*<1 * *R*<1) * (*R*<1/64 * [*1*]²)))))) & ((*R*<1 + ((*R*<1 * ((*R*<5/4 * [$\sim 3/40*x^2 + 1$]²) + (*R*<11/1280 * [*x*²]²))) + ((*A*<1 * *R*<1) * (*R*<1/64 * [*1*]²))))))

lemma *d-delta-lnexp-cf2*:

(($\lambda x. \ln (\exp\text{-cf2 } x) - x$) *has-field-derivative* *diff-delta-lnexp-cf2* *x*) (*at* *x*)

unfolding *exp-cf2-def diff-delta-lnexp-cf2-def*

apply (*intro derivative-eq-intros | simp*) +

apply (*metis exp-cf2-def exp-cf2-pos*)

apply (*simp-all add:*)

using *denom-cf2-pos [of x] numer-cf2-pos [of x]*

apply (*auto simp: divide-simps*)

apply *algebra*

done

Upper bound for non-positive x

lemma *ln-exp-cf2-upper-bound-neg*:

assumes $x \leq 0$

shows $x \leq \ln (\exp\text{-cf2 } x)$

by (*rule gen-upper-bound-decreasing [OF assms d-delta-lnexp-cf2 d-delta-lnexp-cf2-nonpos]*)
(*simp add: exp-cf2-def*)

theorem *exp-cf2-upper-bound-neg*: $x \leq 0 \implies \exp(x) \leq \exp\text{-cf2 } x$

by (*metis ln-exp-cf2-upper-bound-neg exp-cf2-pos exp-le-cancel-iff exp-ln-iff*)

Lower bound for non-negative x

lemma *ln-exp-cf2-lower-bound-pos*:

assumes $0 \leq x$

shows $\ln (\exp\text{-cf2 } x) \leq x$

by (*rule* *gen-lower-bound-increasing* [*OF* *assms* *d-delta-lnexp-cf2* *d-delta-lnexp-cf2-nonpos*])
(simp add: exp-cf2-def)

theorem *exp-cf2-lower-bound-pos*: $0 \leq x \implies \exp\text{-cf2 } x \leq \exp x$

by (*metis* *exp-cf2-pos* *exp-le-cancel-iff* *exp-ln* *ln-exp-cf2-lower-bound-pos*)

3.3 Continued Fraction Bound 3

This bound crosses the X-axis twice, causing complications.

definition *numer-cf3* :: *real* $\Rightarrow$ *real*

where *numer-cf3* $\equiv \lambda x. x^3 + 12*x^2 + 60*x + 120$

definition *exp-cf3* :: *real* $\Rightarrow$ *real*

where *exp-cf3* $\equiv \lambda x. \text{numer-cf3 } x / \text{numer-cf3 } (-x)$

lemma *numer-cf3-pos*: $-4.64 \leq x \implies \text{numer-cf3 } x > 0$

unfolding *numer-cf3-def*

by *sturm*

lemma *exp-cf3-pos*: $\text{numer-cf3 } x > 0 \implies \text{numer-cf3 } (-x) > 0 \implies \exp\text{-cf3 } x > 0$

by (*simp add: exp-cf3-def*)

definition *diff-delta-lnexp-cf3* :: *real* $\Rightarrow$ *real*

where *diff-delta-lnexp-cf3* $\equiv \lambda x. (x^6) / (\text{numer-cf3 } (-x) * \text{numer-cf3 } x)$

lemma *d-delta-lnexp-cf3-nonneg*: $\text{numer-cf3 } x > 0 \implies \text{numer-cf3 } (-x) > 0 \implies \text{diff-delta-lnexp-cf3 } x \geq 0$

unfolding *diff-delta-lnexp-cf3-def*

by (*auto simp: mult-less-0-iff intro: divide-nonneg-neg*)

lemma *d-delta-lnexp-cf3*:

assumes $\text{numer-cf3 } x > 0$ $\text{numer-cf3 } (-x) > 0$

shows $((\lambda x. \ln (\exp\text{-cf3 } x) - x) \text{ has-field-derivative } \text{diff-delta-lnexp-cf3 } x) \text{ (at } x)$

unfolding *exp-cf3-def* *numer-cf3-def* *diff-delta-lnexp-cf3-def*

apply (*intro* *derivative-eq-intros* | *simp*)**+**

using *assms* *numer-cf3-pos* [*of* *x*] *numer-cf3-pos* [*of* $-x$]

apply (*auto simp: numer-cf3-def*)

apply (*auto simp add: divide-simps add-nonneg-eq-0-iff*)

apply *algebra*

done

lemma *numer-cf3-mono*: $y \leq x \implies \text{numer-cf3 } y \leq \text{numer-cf3 } x$

unfolding *numer-cf3-def*

by (sos (((A<0 * R<1) + ((A<=0 * R<1) * ((R<60 * [1/10*x + 1/10*y + 1]^2) +

((R<2/5 * [x + ~1/4*y]^2) + (R<3/8 * [y]^2))))))

Upper bound for non-negative x

lemma *ln-exp-cf3-upper-bound-nonneg*:

assumes *x0*: $0 \leq x$ and *xless*: *numer-cf3* ($-x$) > 0

shows $x \leq \ln (\exp\text{-cf3 } x)$

proof –

have *ncf3*: $\bigwedge y. 0 \leq y \implies y \leq x \implies \text{numer-cf3 } (-y) > 0$

by (metis *neg-le-iff-le numer-cf3-mono order.strict-trans2 xless*)

show ?thesis

apply (rule *gen-upper-bound-increasing* [OF *x0 d-delta-lnextp-cf3 d-delta-lnextp-cf3-nonneg*])

apply (auto simp add: *ncf3* assms *numer-cf3-pos*)

apply (simp add: *exp-cf3-def numer-cf3-def*)

done

qed

theorem *exp-cf3-upper-bound-pos*: $0 \leq x \implies \text{numer-cf3 } (-x) > 0 \implies \exp x \leq \exp\text{-cf3 } x$

using *ln-exp-cf3-upper-bound-nonneg* [of *x*] *exp-cf3-pos* [of *x*] *numer-cf3-pos* [of *x*]

by auto (metis *exp-le-cancel-iff exp-ln-iff*)

corollary $0 \leq x \implies x \leq 4.64 \implies \exp x \leq \exp\text{-cf3 } x$

by (metis *numer-cf3-pos neg-le-iff-le exp-cf3-upper-bound-pos*)

Lower bound for negative x, provided $0 < \exp\text{-cf3 } x$

lemma *ln-exp-cf3-lower-bound-neg*:

assumes *x0*: $x \leq 0$ and *xgtr*: *numer-cf3* $x > 0$

shows $\ln (\exp\text{-cf3 } x) \leq x$

proof –

have *ncf3*: $\bigwedge y. x \leq y \implies y \leq 0 \implies \text{numer-cf3 } y > 0$

by (metis *dual-order.strict-trans1 numer-cf3-mono xgtr*)

show ?thesis

apply (rule *gen-lower-bound-decreasing* [OF *x0 d-delta-lnextp-cf3 d-delta-lnextp-cf3-nonneg*])

apply (auto simp add: *ncf3* assms *numer-cf3-pos*)

apply (simp add: *exp-cf3-def numer-cf3-def*)

done

qed

theorem *exp-cf3-lower-bound-pos*:

assumes $x \leq 0$ shows $\exp\text{-cf3 } x \leq \exp x$

proof (cases *numer-cf3* $x > 0$)

case *True*

then have *exp-cf3* $x > 0$

using assms *numer-cf3-pos* [of $-x$]

by (auto simp: *exp-cf3-pos*)

then show ?thesis

```

using ln-exp-cf3-lower-bound-neg [of x] assms True
by auto (metis exp-le-cancel-iff exp-ln-iff)
next
case False
then have exp-cf3  $x \leq 0$ 
using assms numer-cf3-pos [of  $-x$ ]
unfolding exp-cf3-def
by (simp add: divide-nonpos-pos)
then show ?thesis
by (metis exp-ge-zero order.trans)
qed

```

3.4 Continued Fraction Bound 4

Here we have the extended exp continued fraction bounds

```

definition numer-cf4 :: real  $\Rightarrow$  real
  where numer-cf4  $\equiv \lambda x. x^4 + 20*x^3 + 180*x^2 + 840*x + 1680$ 

```

```

definition exp-cf4 :: real  $\Rightarrow$  real
  where exp-cf4  $\equiv \lambda x. \text{numer-cf4 } x / \text{numer-cf4 } (-x)$ 

```

```

lemma numer-cf4-pos: fixes  $x::\text{real}$  shows numer-cf4  $x > 0$ 
unfolding numer-cf4-def
by (sos (( $R < 1 + ((R < 1 * ((R < 4469/256 * [1135/71504*x^2 + 4725/17876*x$ 
 $+ 1]^2) + ((R < 3728645/18305024 * [536265/2982916*x^2 + x]^2) + (R < 106265/24436047872$ 
 $* [x^2]^2)))) + ((A \leq 0 * R < 1) * (R < 45/4096 * [1]^2))))))$ 

```

```

lemma exp-cf4-pos: exp-cf4  $x > 0$ 
unfolding exp-cf4-def
by (auto simp add: divide-simps numer-cf4-pos)

```

```

definition diff-delta-lnexp-cf4 :: real  $\Rightarrow$  real
  where diff-delta-lnexp-cf4  $\equiv \lambda x. -(x^8) / (\text{numer-cf4 } (-x) * \text{numer-cf4 } x)$ 

```

```

lemma d-delta-lnexp-cf4-nonpos: diff-delta-lnexp-cf4  $x \leq 0$ 
unfolding diff-delta-lnexp-cf4-def
using numer-cf4-pos [of  $x$ ] numer-cf4-pos [of  $-x$ ]
by (simp add: zero-le-divide-iff zero-le-mult-iff)

```

```

lemma d-delta-lnexp-cf4:
  (( $\lambda x. \ln (\text{exp-cf4 } x) - x$ ) has-field-derivative diff-delta-lnexp-cf4  $x$ ) (at  $x$ )
unfolding exp-cf4-def numer-cf4-def diff-delta-lnexp-cf4-def
apply (intro derivative-eq-intros | simp)+
using exp-cf4-pos
apply (simp add: exp-cf4-def numer-cf4-def)
apply (simp-all add: )
using numer-cf4-pos [of  $x$ ] numer-cf4-pos [of  $-x$ ]
apply (auto simp: divide-simps numer-cf4-def)

```

apply *algebra*
done

Upper bound for non-positive x

lemma *ln-exp-cf4-upper-bound-neg*:
assumes $x \leq 0$
shows $x \leq \ln (\exp\text{-cf4 } x)$
by (*rule* *gen-upper-bound-decreasing* [*OF* *assms* *d-delta-lnextp-cf4* *d-delta-lnextp-cf4-nonpos*])
(simp add: exp-cf4-def numer-cf4-def)

theorem *exp-cf4-upper-bound-neg*: $x \leq 0 \implies \exp(x) \leq \exp\text{-cf4 } x$
by (*metis* *ln-exp-cf4-upper-bound-neg* *exp-cf4-pos* *exp-le-cancel-iff* *exp-ln-iff*)

Lower bound for non-negative x

lemma *ln-exp-cf4-lower-bound-pos*:
assumes $0 \leq x$
shows $\ln (\exp\text{-cf4 } x) \leq x$
by (*rule* *gen-lower-bound-increasing* [*OF* *assms* *d-delta-lnextp-cf4* *d-delta-lnextp-cf4-nonpos*])
(simp add: exp-cf4-def numer-cf4-def)

theorem *exp-cf4-lower-bound-pos*: $0 \leq x \implies \exp\text{-cf4 } x \leq \exp x$
by (*metis* *exp-cf4-pos* *exp-le-cancel-iff* *exp-ln* *ln-exp-cf4-lower-bound-pos*)

3.5 Continued Fraction Bound 5

This bound crosses the X-axis twice, causing complications.

definition *numer-cf5* :: *real* $\Rightarrow$ *real*
where *numer-cf5* $\equiv \lambda x. x^5 + 30*x^4 + 420*x^3 + 3360*x^2 + 15120*x + 30240$

definition *exp-cf5* :: *real* $\Rightarrow$ *real*
where *exp-cf5* $\equiv \lambda x. \text{numer-cf5 } x / \text{numer-cf5 } (-x)$

lemma *numer-cf5-pos*: $-7.293 \leq x \implies \text{numer-cf5 } x > 0$
unfolding *numer-cf5-def*
by *sturm*

lemma *exp-cf5-pos*: $\text{numer-cf5 } x > 0 \implies \text{numer-cf5 } (-x) > 0 \implies \exp\text{-cf5 } x > 0$
unfolding *exp-cf5-def numer-cf5-def*
by (*simp add: divide-neg-neg*)

definition *diff-delta-lnextp-cf5* :: *real* $\Rightarrow$ *real*
where *diff-delta-lnextp-cf5* $\equiv \lambda x. (x^5 10) / (\text{numer-cf5 } (-x) * \text{numer-cf5 } x)$

lemma *d-delta-lnextp-cf5-nonneg*: $\text{numer-cf5 } x > 0 \implies \text{numer-cf5 } (-x) > 0 \implies \text{diff-delta-lnextp-cf5 } x \geq 0$
unfolding *diff-delta-lnextp-cf5-def*

by (auto simp add: mult-less-0-iff intro: divide-nonneg-neg)

lemma *d-delta-lnexp-cf5*:

assumes *numer-cf5* $x > 0$ *numer-cf5* $(-x) > 0$
 shows $((\lambda x. \ln (\exp\text{-cf5 } x) - x) \text{ has-field-derivative } \text{diff-delta-lnexp-cf5 } x) \text{ (at } x)$
 unfolding *exp-cf5-def* *numer-cf5-def* *diff-delta-lnexp-cf5-def*
 apply (intro derivative-eq-intros | simp)+
 using *assms* *numer-cf5-pos* [of x] *numer-cf5-pos* [of $-x$]
 apply (auto simp: *numer-cf5-def*)
 apply (auto simp add: divide-simps add-nonneg-eq-0-iff)
 apply algebra
 done

3.5.1 Proving monotonicity via a non-negative derivative

definition *numer-cf5-deriv* :: *real* $\Rightarrow$ *real*

where *numer-cf5-deriv* $\equiv \lambda x. 5*x^4 + 120*x^3 + 1260*x^2 + 6720*x + 15120$

lemma *numer-cf5-deriv*:

shows $(\text{numer-cf5 has-field-derivative } \text{numer-cf5-deriv } x) \text{ (at } x)$
 unfolding *numer-cf5-def* *numer-cf5-deriv-def*
 by (intro derivative-eq-intros | simp)+

lemma *numer-cf5-deriv-pos*: *numer-cf5-deriv* $x \geq 0$

unfolding *numer-cf5-deriv-def*
 by (sos (($R < 1 + ((R < 1 * ((R < 185533/8192 * [73459/5937056*x^2 + 43050/185533*x + 1]^2) + ((R < 4641265253/24318181376 * [700850925/4641265253*x^2 + x]^2) + (R < 38142496079/38933754831437824 * [x^2]^2)))) + ((A < 0 * R < 1) * (R < 205/131072 * [1]^2))))))$)))

lemma *numer-cf5-mono*: $y \leq x \implies \text{numer-cf5 } y \leq \text{numer-cf5 } x$

by (auto intro: DERIV-nonneg-imp-nondecreasing *numer-cf5-deriv* *numer-cf5-deriv-pos*)

3.5.2 Results

Upper bound for non-negative x

lemma *ln-exp-cf5-upper-bound-nonneg*:

assumes $x0: 0 \leq x$ and *xless*: *numer-cf5* $(-x) > 0$
 shows $x \leq \ln (\exp\text{-cf5 } x)$

proof –

have *ncf5*: $\bigwedge y. 0 \leq y \implies y \leq x \implies \text{numer-cf5 } (-y) > 0$

by (metis *neg-le-iff-le* *numer-cf5-mono* *order.strict-trans2* *xless*)

show ?thesis

apply (rule *gen-upper-bound-increasing* [OF $x0$ *d-delta-lnexp-cf5* *d-delta-lnexp-cf5-nonneg*])

apply (auto simp add: *ncf5* *assms* *numer-cf5-pos*)

apply (simp add: *exp-cf5-def* *numer-cf5-def*)

done

qed

theorem *exp-cf5-upper-bound-pos*: $0 \leq x \implies \text{numer-cf5 } (-x) > 0 \implies \text{exp } x \leq \text{exp-cf5 } x$
using *ln-exp-cf5-upper-bound-nonneg* [of *x*] *exp-cf5-pos* [of *x*] *numer-cf5-pos* [of *x*]
by *auto* (*metis exp-le-cancel-iff exp-ln-iff*)

corollary $0 \leq x \implies x \leq 7.293 \implies \text{exp } x \leq \text{exp-cf5 } x$
by (*metis neg-le-iff-le numer-cf5-pos exp-cf5-upper-bound-pos*)

Lower bound for negative x, provided $0 < \text{exp-cf5 } x$

lemma *ln-exp-cf5-lower-bound-neg*:
assumes *x0*: $x \leq 0$ **and** *xgtr*: $\text{numer-cf5 } x > 0$
shows $\ln (\text{exp-cf5 } x) \leq x$
proof –
have *ncf5*: $\bigwedge y. x \leq y \implies y \leq 0 \implies \text{numer-cf5 } y > 0$
by (*metis dual-order.strict-trans1 numer-cf5-mono xgtr*)
show *?thesis*
apply (*rule gen-lower-bound-decreasing* [*OF* *x0 d-delta-lnexp-cf5 d-delta-lnexp-cf5-nonneg*])
apply (*auto simp add: ncf5 assms numer-cf5-pos*)
apply (*simp add: exp-cf5-def numer-cf5-def*)
done

qed

theorem *exp-cf5-lower-bound-pos*:
assumes $x \leq 0$ **shows** $\text{exp-cf5 } x \leq \text{exp } x$
proof (*cases numer-cf5 x > 0*)
case *True*
then have $\text{exp-cf5 } x > 0$
using *assms numer-cf5-pos* [of $-x$]
by (*auto simp: exp-cf5-pos*)
then show *?thesis*
using *ln-exp-cf5-lower-bound-neg* [of *x*] *assms True*
by *auto* (*metis exp-le-cancel-iff exp-ln-iff*)
next
case *False*
then have $\text{exp-cf5 } x \leq 0$
using *assms numer-cf5-pos* [of $-x$]
unfolding *exp-cf5-def numer-cf5-def*
by (*simp add: divide-nonpos-pos*)
then show *?thesis*
by (*metis exp-ge-zero order.trans*)
qed

3.6 Continued Fraction Bound 6

definition *numer-cf6* :: *real* $\Rightarrow$ *real*
where *numer-cf6* $\equiv \lambda x. x^6 + 42*x^5 + 840*x^4 + 10080*x^3 + 75600*x^2 + 332640*x + 665280$

definition $\text{exp-cf6} :: \text{real} \Rightarrow \text{real}$

where $\text{exp-cf6} \equiv \lambda x. \text{numer-cf6 } x / \text{numer-cf6 } (-x)$

lemma numer-cf6-pos : **fixes** $x :: \text{real}$ **shows** $\text{numer-cf6 } x > 0$

unfolding numer-cf6-def

by sturm

lemma exp-cf6-pos : $\text{exp-cf6 } x > 0$

unfolding exp-cf6-def

by $(\text{auto simp add: divide-simps numer-cf6-pos})$

definition $\text{diff-delta-lnexp-cf6} :: \text{real} \Rightarrow \text{real}$

where $\text{diff-delta-lnexp-cf6} \equiv \lambda x. -(x^{12}) / (\text{numer-cf6 } (-x) * \text{numer-cf6 } x)$

lemma $\text{d-delta-lnexp-cf6-nonpos}$: $\text{diff-delta-lnexp-cf6 } x \leq 0$

unfolding $\text{diff-delta-lnexp-cf6-def}$

using $\text{numer-cf6-pos [of } x] \text{ numer-cf6-pos [of } -x]$

by $(\text{simp add: zero-le-divide-iff zero-le-mult-iff})$

lemma d-delta-lnexp-cf6 :

$((\lambda x. \ln (\text{exp-cf6 } x) - x) \text{ has-field-derivative } \text{diff-delta-lnexp-cf6 } x) \text{ (at } x)$

unfolding $\text{exp-cf6-def diff-delta-lnexp-cf6-def numer-cf6-def}$

apply $(\text{intro derivative-eq-intros | simp})+$

using exp-cf6-pos

apply $(\text{simp add: exp-cf6-def numer-cf6-def})$

apply (simp-all add:)

using $\text{numer-cf6-pos [of } x] \text{ numer-cf6-pos [of } -x]$

apply $(\text{auto simp: divide-simps numer-cf6-def})$

apply algebra

done

Upper bound for non-positive x

lemma $\text{ln-exp-cf6-upper-bound-neg}$:

assumes $x \leq 0$

shows $x \leq \ln (\text{exp-cf6 } x)$

by $(\text{rule gen-upper-bound-decreasing [OF assms d-delta-lnexp-cf6 d-delta-lnexp-cf6-nonpos]})$

$(\text{simp add: exp-cf6-def numer-cf6-def})$

theorem $\text{exp-cf6-upper-bound-neg}$: $x \leq 0 \implies \exp(x) \leq \text{exp-cf6 } x$

by $(\text{metis ln-exp-cf6-upper-bound-neg exp-cf6-pos exp-le-cancel-iff exp-ln-iff})$

Lower bound for non-negative x

lemma $\text{ln-exp-cf6-lower-bound-pos}$:

assumes $0 \leq x$

shows $\ln (\text{exp-cf6 } x) \leq x$

by $(\text{rule gen-lower-bound-increasing [OF assms d-delta-lnexp-cf6 d-delta-lnexp-cf6-nonpos]})$

$(\text{simp add: exp-cf6-def numer-cf6-def})$

theorem $\text{exp-cf6-lower-bound-pos}$: $0 \leq x \implies \text{exp-cf6 } x \leq \exp x$

by $(\text{metis exp-cf6-pos exp-le-cancel-iff exp-ln ln-exp-cf6-lower-bound-pos})$

3.7 Continued Fraction Bound 7

This bound crosses the X-axis twice, causing complications.

definition *numer-cf7* :: *real* $\Rightarrow$ *real*

where *numer-cf7* $\equiv \lambda x. x^7 + 56*x^6 + 1512*x^5 + 25200*x^4 + 277200*x^3 + 1995840*x^2 + 8648640*x + 17297280$

definition *exp-cf7* :: *real* $\Rightarrow$ *real*

where *exp-cf7* $\equiv \lambda x. \text{numer-cf7 } x / \text{numer-cf7 } (-x)$

lemma *numer-cf7-pos*: $-9.943 \leq x \implies \text{numer-cf7 } x > 0$

unfolding *numer-cf7-def*

by *sturm*

lemma *exp-cf7-pos*: $\text{numer-cf7 } x > 0 \implies \text{numer-cf7 } (-x) > 0 \implies \text{exp-cf7 } x > 0$

by (*simp add: exp-cf7-def*)

definition *diff-delta-lexp-cf7* :: *real* $\Rightarrow$ *real*

where *diff-delta-lexp-cf7* $\equiv \lambda x. (x^7 - 14) / (\text{numer-cf7 } (-x) * \text{numer-cf7 } x)$

lemma *d-delta-lexp-cf7-nonneg*: $\text{numer-cf7 } x > 0 \implies \text{numer-cf7 } (-x) > 0 \implies \text{diff-delta-lexp-cf7 } x \geq 0$

unfolding *diff-delta-lexp-cf7-def*

by (*auto simp: mult-less-0-iff intro: divide-nonneg-neg*)

lemma *d-delta-lexp-cf7*:

assumes $\text{numer-cf7 } x > 0 \text{ numer-cf7 } (-x) > 0$

shows $((\lambda x. \ln (\text{exp-cf7 } x) - x) \text{ has-field-derivative } \text{diff-delta-lexp-cf7 } x) \text{ (at } x)$

unfolding *exp-cf7-def numer-cf7-def diff-delta-lexp-cf7-def*

apply (*intro derivative-eq-intros | simp*) +

using *assms numer-cf7-pos [of x] numer-cf7-pos [of -x]*

apply (*auto simp: numer-cf7-def*)

apply (*auto simp: divide-simps add-nonneg-eq-0-iff*)

apply *algebra*

done

3.7.1 Proving monotonicity via a non-negative derivative

definition *numer-cf7-deriv* :: *real* $\Rightarrow$ *real*

where *numer-cf7-deriv* $\equiv \lambda x. 7*x^6 + 336*x^5 + 7560*x^4 + 100800*x^3 + 831600*x^2 + 3991680*x + 8648640$

lemma *numer-cf7-deriv*:

shows $(\text{numer-cf7} \text{ has-field-derivative } \text{numer-cf7-deriv } x) \text{ (at } x)$

unfolding *numer-cf7-def numer-cf7-deriv-def*

by (*intro derivative-eq-intros | simp*) +

lemma *numer-cf7-deriv-pos*: $\text{numer-cf7-deriv } x \geq 0$

unfolding *numer-cf7-deriv-def*
apply (*rule order.strict-implies-order*) — FIXME should not be necessary
by *sturm*

lemma *numer-cf7-mono*: $y \leq x \implies \text{numer-cf7 } y \leq \text{numer-cf7 } x$
by (*auto intro: DERIV-nonneg-imp-nondecreasing numer-cf7-deriv numer-cf7-deriv-pos*)

3.7.2 Results

Upper bound for non-negative x

lemma *ln-exp-cf7-upper-bound-nonneg*:
assumes *x0*: $0 \leq x$ **and** *xless*: $\text{numer-cf7 } (-x) > 0$
shows $x \leq \ln (\text{exp-cf7 } x)$
proof –
have *ncf7*: $\bigwedge y. 0 \leq y \implies y \leq x \implies \text{numer-cf7 } (-y) > 0$
by (*metis neg-le-iff-le numer-cf7-mono order.strict-trans2 xless*)
show ?thesis
apply (*rule gen-upper-bound-increasing [OF x0 d-delta-lnexp-cf7 d-delta-lnexp-cf7-nonneg]*)
apply (*auto simp add: ncf7 assms numer-cf7-pos*)
apply (*simp add: exp-cf7-def numer-cf7-def*)
done
qed

theorem *exp-cf7-upper-bound-pos*: $0 \leq x \implies \text{numer-cf7 } (-x) > 0 \implies \text{exp } x \leq \text{exp-cf7 } x$
using *ln-exp-cf7-upper-bound-nonneg [of x] exp-cf7-pos [of x] numer-cf7-pos [of x]*
by *auto (metis exp-le-cancel-iff exp-ln-iff)*

corollary $0 \leq x \implies x \leq 9.943 \implies \text{exp } x \leq \text{exp-cf7 } x$
by (*metis neg-le-iff-le numer-cf7-pos exp-cf7-upper-bound-pos*)

Lower bound for negative x, provided $0 < \text{exp-cf7 } x$

lemma *ln-exp-cf7-lower-bound-neg*:
assumes *x0*: $x \leq 0$ **and** *xgtr*: $\text{numer-cf7 } x > 0$
shows $\ln (\text{exp-cf7 } x) \leq x$
proof –
have *ncf7*: $\bigwedge y. x \leq y \implies y \leq 0 \implies \text{numer-cf7 } y > 0$
by (*metis dual-order.strict-trans1 numer-cf7-mono xgtr*)
show ?thesis
apply (*rule gen-lower-bound-decreasing [OF x0 d-delta-lnexp-cf7 d-delta-lnexp-cf7-nonneg]*)
apply (*auto simp add: ncf7 assms numer-cf7-pos*)
apply (*simp add: exp-cf7-def numer-cf7-def*)
done
qed

theorem *exp-cf7-lower-bound-pos*:
assumes $x \leq 0$ **shows** $\text{exp-cf7 } x \leq \text{exp } x$
proof (*cases numer-cf7 x > 0*)

```

case True
then have  $\exp\text{-cf7 } x > 0$ 
  using assms numer-cf7-pos [of -x]
  by (auto simp: exp-cf7-pos)
then show ?thesis
using ln-exp-cf7-lower-bound-neg [of x] assms True
  by auto (metis exp-le-cancel-iff exp-ln-iff)
next
case False
then have  $\exp\text{-cf7 } x \leq 0$ 
  using assms numer-cf7-pos [of -x]
  unfolding exp-cf7-def
  by (simp add: divide-nonpos-pos)
then show ?thesis
  by (metis exp-ge-zero order.trans)
qed

end

```

Chapter 4

Log Upper and Lower Bounds

theory *Log-CF-Bounds*
imports *Bounds-Lemmas*

begin

theorem *ln-upper-1*: $0 < x \implies \ln(x :: \text{real}) \leq x - 1$
by (*rule ln-le-minus-one*)

definition *ln-lower-1* :: $\text{real} \Rightarrow \text{real}$
where *ln-lower-1* $\equiv \lambda x. 1 - (\text{inverse } x)$

corollary *ln-lower-1*: $0 < x \implies \text{ln-lower-1 } x \leq \ln x$
unfolding *ln-lower-1-def*
by (*metis ln-inverse ln-le-minus-one positive-imp-inverse-positive minus-diff-eq minus-le-iff*)

theorem *ln-lower-1-eq*: $0 < x \implies \text{ln-lower-1 } x = (x - 1)/x$
by (*auto simp: ln-lower-1-def divide-simps*)

4.1 Upper Bound 3

definition *ln-upper-3* :: $\text{real} \Rightarrow \text{real}$
where *ln-upper-3* $\equiv \lambda x. (x + 5) * (x - 1) / (2 * (2 * x + 1))$

definition *diff-delta-ln-upper-3* :: $\text{real} \Rightarrow \text{real}$
where *diff-delta-ln-upper-3* $\equiv \lambda x. (x - 1)^3 / ((2 * x + 1)^2 * x)$

lemma *d-delta-ln-upper-3*: $x > 0 \implies$
 $((\lambda x. \text{ln-upper-3 } x - \ln x) \text{ has-field-derivative } \text{diff-delta-ln-upper-3 } x) \text{ (at } x)$
unfolding *ln-upper-3-def diff-delta-ln-upper-3-def*
apply (*intro derivative-eq-intros | simp*) +

apply (*simp add: divide-simps add-nonneg-eq-0-iff, algebra*)
done

Strict inequalities also possible

lemma *ln-upper-3-pos*:
assumes $1 \leq x$ **shows** $\ln(x) \leq \ln\text{-upper-3 } x$
apply (*rule gen-upper-bound-increasing [OF assms d-delta-ln-upper-3]*)
apply (*auto simp: diff-delta-ln-upper-3-def ln-upper-3-def*)
done

lemma *ln-upper-3-neg*:
assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \ln\text{-upper-3 } x$
apply (*rule gen-upper-bound-decreasing [OF x1 d-delta-ln-upper-3]*)
using *assms*
apply (*auto simp: diff-delta-ln-upper-3-def divide-simps ln-upper-3-def*)
done

theorem *ln-upper-3*: $0 < x \implies \ln(x) \leq \ln\text{-upper-3 } x$
by (*metis le-less-linear less-eq-real-def ln-upper-3-neg ln-upper-3-pos*)

definition *ln-lower-3* :: *real* $\Rightarrow$ *real*
where *ln-lower-3* $\equiv \lambda x. - \ln\text{-upper-3 } (\text{inverse } x)$

corollary *ln-lower-3*: $0 < x \implies \ln\text{-lower-3 } x \leq \ln x$
unfolding *ln-lower-3-def*
by (*metis ln-inverse inverse-positive-iff-positive minus-le-iff ln-upper-3*)

theorem *ln-lower-3-eq*: $0 < x \implies \ln\text{-lower-3 } x = (1/2) * (1 + 5*x) * (x - 1) / (x * (2 + x))$
unfolding *ln-lower-3-def ln-upper-3-def*
by (*simp add: divide-simps algebra*)

4.2 Upper Bound 5

definition *ln-upper-5* :: *real* $\Rightarrow$ *real*
where *ln-upper-5* $x \equiv (x^2 + 19*x + 10) * (x - 1) / (3 * (3*x^2 + 6*x + 1))$

definition *diff-delta-ln-upper-5* :: *real* $\Rightarrow$ *real*
where *diff-delta-ln-upper-5* $\equiv \lambda x. (x - 1)^5 / ((3*x^2 + 6*x + 1)^2 * x)$

lemma *d-delta-ln-upper-5*: $x > 0 \implies$
 ($(\lambda x. \ln\text{-upper-5 } x - \ln x)$ *has-field-derivative* *diff-delta-ln-upper-5* x) (*at* x)
unfolding *ln-upper-5-def diff-delta-ln-upper-5-def*
apply (*intro derivative-eq-intros | simp add: add-nonneg-eq-0-iff*) +
apply (*simp add: divide-simps add-nonneg-eq-0-iff, algebra*)
done

lemma *ln-upper-5-pos*:
assumes $1 \leq x$ **shows** $\ln(x) \leq \ln\text{-upper-5 } x$

apply (rule gen-upper-bound-increasing [OF assms d-delta-ln-upper-5])
apply (auto simp: diff-delta-ln-upper-5-def ln-upper-5-def)
done

lemma ln-upper-5-neg:

assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \ln\text{-upper-5 } x$
apply (rule gen-upper-bound-decreasing [OF x1 d-delta-ln-upper-5])
using assms
apply (auto simp: diff-delta-ln-upper-5-def divide-simps ln-upper-5-def mult-less-0-iff)
done

theorem ln-upper-5: $0 < x \implies \ln(x) \leq \ln\text{-upper-5 } x$
by (metis le-less-linear less-eq-real-def ln-upper-5-neg ln-upper-5-pos)

definition ln-lower-5 :: real $\Rightarrow$ real
where $\ln\text{-lower-5} \equiv \lambda x. - \ln\text{-upper-5 } (inverse\ x)$

corollary ln-lower-5: $0 < x \implies \ln\text{-lower-5 } x \leq \ln x$
unfolding ln-lower-5-def
by (metis ln-inverse inverse-positive-iff-positive minus-le-iff ln-upper-5)

theorem ln-lower-5-eq: $0 < x \implies$
 $\ln\text{-lower-5 } x = (1/3) * (10 * x^2 + 19 * x + 1) * (x - 1) / (x * (x^2 + 6 * x + 3))$
unfolding ln-lower-5-def ln-upper-5-def
by (simp add: zero-less-mult-iff add-pos-pos dual-order.strict-implies-not-eq divide-simps)
 algebra

4.3 Upper Bound 7

definition ln-upper-7 :: real $\Rightarrow$ real
where $\ln\text{-upper-7 } x \equiv (3 * x^3 + 131 * x^2 + 239 * x + 47) * (x - 1) / (12 * (4 * x^3 + 18 * x^2 + 12 * x + 1))$

definition diff-delta-ln-upper-7 :: real $\Rightarrow$ real
where $\text{diff-delta-ln-upper-7} \equiv \lambda x. (x - 1)^7 / ((4 * x^3 + 18 * x^2 + 12 * x + 1)^2 * x)$

lemma d-delta-ln-upper-7: $x > 0 \implies$
 $((\lambda x. \ln\text{-upper-7 } x - \ln x) \text{ has-field-derivative } \text{diff-delta-ln-upper-7 } x) \text{ (at } x)$
unfolding ln-upper-7-def diff-delta-ln-upper-7-def
apply (intro derivative-eq-intros | simp)+
apply auto
apply (auto simp: add-pos-pos dual-order.strict-implies-not-eq divide-simps, algebra)
done

lemma ln-upper-7-pos:

assumes $1 \leq x$ **shows** $\ln(x) \leq \ln\text{-upper-7 } x$
apply (rule gen-upper-bound-increasing [OF assms d-delta-ln-upper-7])

apply (*auto simp: diff-delta-ln-upper-7-def ln-upper-7-def*)
done

lemma *ln-upper-7-neg*:

assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \ln\text{-upper-7 } x$
apply (*rule gen-upper-bound-decreasing [OF x1 d-delta-ln-upper-7]*)
using *assms*
apply (*auto simp: diff-delta-ln-upper-7-def divide-simps ln-upper-7-def mult-less-0-iff*)
done

theorem *ln-upper-7*: $0 < x \implies \ln(x) \leq \ln\text{-upper-7 } x$
by (*metis le-less-linear less-eq-real-def ln-upper-7-neg ln-upper-7-pos*)

definition *ln-lower-7* :: *real* $\Rightarrow$ *real*
where *ln-lower-7* $\equiv \lambda x. - \ln\text{-upper-7 } (\text{inverse } x)$

corollary *ln-lower-7*: $0 < x \implies \ln\text{-lower-7 } x \leq \ln x$
unfolding *ln-lower-7-def*
by (*metis ln-inverse inverse-positive-iff-positive minus-le-iff ln-upper-7*)

theorem *ln-lower-7-eq*: $0 < x \implies$
 $\ln\text{-lower-7 } x = (1/12) * (47 * x^3 + 239 * x^2 + 131 * x + 3) * (x - 1) / (x * (x^3$
 $+ 12 * x^2 + 18 * x + 4))$
unfolding *ln-lower-7-def ln-upper-7-def*
by (*simp add: zero-less-mult-iff add-pos-pos dual-order.strict-implies-not-eq divide-simps*)
algebra

4.4 Upper Bound 9

definition *ln-upper-9* :: *real* $\Rightarrow$ *real*
where *ln-upper-9* $x \equiv (6 * x^4 + 481 * x^3 + 1881 * x^2 + 1281 * x + 131) * (x - 1) /$
 $(30 * (5 * x^4 + 40 * x^3 + 60 * x^2 + 20 * x + 1))$

definition *diff-delta-ln-upper-9* :: *real* $\Rightarrow$ *real*
where *diff-delta-ln-upper-9* $\equiv \lambda x. (x - 1)^9 / (((5 * x^4 + 40 * x^3 + 60 * x^2$
 $+ 20 * x + 1)^2) * x)$

lemma *d-delta-ln-upper-9*: $x > 0 \implies$
 $((\lambda x. \ln\text{-upper-9 } x - \ln x) \text{ has-field-derivative } \text{diff-delta-ln-upper-9 } x) \text{ (at } x)$
unfolding *ln-upper-9-def diff-delta-ln-upper-9-def*
apply (*intro derivative-eq-intros | simp*)
apply *auto*
apply (*auto simp: add-pos-pos dual-order.strict-implies-not-eq divide-simps, algebra*)
done

lemma *ln-upper-9-pos*:
assumes $1 \leq x$ **shows** $\ln(x) \leq \ln\text{-upper-9 } x$

apply (rule gen-upper-bound-increasing [OF assms d-delta-ln-upper-9])
apply (auto simp: diff-delta-ln-upper-9-def ln-upper-9-def)
done

lemma ln-upper-9-neg:
 assumes $0 < x$ and $x1: x \leq 1$ shows $\ln(x) \leq \ln\text{-upper-9 } x$
apply (rule gen-upper-bound-decreasing [OF x1 d-delta-ln-upper-9])
using assms
apply (auto simp: diff-delta-ln-upper-9-def divide-simps ln-upper-9-def mult-less-0-iff)
done

theorem ln-upper-9: $0 < x \implies \ln(x) \leq \ln\text{-upper-9 } x$
by (metis le-less-linear less-eq-real-def ln-upper-9-neg ln-upper-9-pos)

definition ln-lower-9 :: real $\Rightarrow$ real
 where $\ln\text{-lower-9} \equiv \lambda x. - \ln\text{-upper-9 } (\text{inverse } x)$

corollary ln-lower-9: $0 < x \implies \ln\text{-lower-9 } x \leq \ln x$
unfolding ln-lower-9-def
by (metis ln-inverse inverse-positive-iff-positive minus-le-iff ln-upper-9)

theorem ln-lower-9-eq: $0 < x \implies$

$$\ln\text{-lower-9 } x = (1/30) * (6 + 481*x + 1881*x^2 + 1281*x^3 + 131*x^4) * (x - 1) /$$

$$(x * (5 + 40*x + 60*x^2 + 20*x^3 + x^4))$$

unfolding ln-lower-9-def ln-upper-9-def
by (simp add: zero-less-mult-iff add-pos-pos dual-order.strict-implies-not-eq divide-simps) algebra

4.5 Upper Bound 11

Extended bounds start here

definition ln-upper-11 :: real $\Rightarrow$ real
 where $\ln\text{-upper-11 } x \equiv$

$$(5*x^5 + 647*x^4 + 4397*x^3 + 6397*x^2 + 2272*x + 142) * (x - 1) /$$

$$(30 * (6*x^5 + 75*x^4 + 200*x^3 + 150*x^2 + 30*x + 1))$$

definition diff-delta-ln-upper-11 :: real $\Rightarrow$ real
 where $\text{diff-delta-ln-upper-11} \equiv \lambda x. (x - 1)^{11} / ((6*x^5 + 75*x^4 + 200*x^3 + 150*x^2 + 30*x + 1)^2 * x)$

lemma d-delta-ln-upper-11: $x > 0 \implies$
 $((\lambda x. \ln\text{-upper-11 } x - \ln x) \text{ has-field-derivative } \text{diff-delta-ln-upper-11 } x) \text{ (at } x)$
unfolding ln-upper-11-def diff-delta-ln-upper-11-def
apply (intro derivative-eq-intros | simp)+
apply auto

apply (*auto simp: add-pos-pos dual-order.strict-implies-not-eq divide-simps, algebra*)

done

lemma *ln-upper-11-pos*:

assumes $1 \leq x$ **shows** $\ln(x) \leq \ln\text{-upper-11 } x$

apply (*rule gen-upper-bound-increasing [OF assms d-delta-ln-upper-11]*)

apply (*auto simp: diff-delta-ln-upper-11-def ln-upper-11-def*)

done

lemma *ln-upper-11-neg*:

assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \ln\text{-upper-11 } x$

apply (*rule gen-upper-bound-decreasing [OF x1 d-delta-ln-upper-11]*)

using *assms*

apply (*auto simp: diff-delta-ln-upper-11-def divide-simps ln-upper-11-def mult-less-0-iff*)

done

theorem *ln-upper-11*: $0 < x \implies \ln(x) \leq \ln\text{-upper-11 } x$

by (*metis le-less-linear less-eq-real-def ln-upper-11-neg ln-upper-11-pos*)

definition *ln-lower-11* :: *real* $\Rightarrow$ *real*

where *ln-lower-11* $\equiv \lambda x. - \ln\text{-upper-11 } (\text{inverse } x)$

corollary *ln-lower-11*: $0 < x \implies \ln\text{-lower-11 } x \leq \ln x$

unfolding *ln-lower-11-def*

by (*metis ln-inverse inverse-positive-iff-positive minus-le-iff ln-upper-11*)

theorem *ln-lower-11-eq*: $0 < x \implies$

$\ln\text{-lower-11 } x = (1/30) * (142 * x^5 + 2272 * x^4 + 6397 * x^3 + 4397 * x^2 + 647 * x + 5) * (x - 1) /$
 $(x * (x^5 + 30 * x^4 + 150 * x^3 + 200 * x^2 + 75 * x + 6))$

unfolding *ln-lower-11-def ln-upper-11-def*

by (*simp add: zero-less-mult-iff add-pos-pos dual-order.strict-implies-not-eq divide-simps*)
algebra

4.6 Upper Bound 13

definition *ln-upper-13* :: *real* $\Rightarrow$ *real*

where *ln-upper-13* $\equiv (353 + 8389 * x + 20149 * x^4 + 50774 * x^3 + 38524 * x^2 + 1921 * x^5 + 10 * x^6) * (x - 1) /$
 $(70 * (1 + 42 * x + 525 * x^4 + 700 * x^3 + 315 * x^2 + 126 * x^5 + 7 * x^6))$

definition *diff-delta-ln-upper-13* :: *real* $\Rightarrow$ *real*

where *diff-delta-ln-upper-13* $\equiv \lambda x. (x - 1)^{13} /$

$((1 + 42 * x + 525 * x^4 + 700 * x^3 + 315 * x^2 + 126 * x^5 + 7 * x^6)^{2 * x})$

lemma *d-delta-ln-upper-13*: $x > 0 \implies$

$((\lambda x. \text{ln-upper-13 } x - \text{ln } x) \text{ has-field-derivative diff-delta-ln-upper-13 } x) \text{ (at } x)$
unfolding *ln-upper-13-def diff-delta-ln-upper-13-def*
apply (*intro derivative-eq-intros | simp*) +
apply *auto*
apply (*auto simp: add-pos-pos dual-order.strict-implies-not-eq divide-simps, algebra*)
done

lemma *ln-upper-13-pos*:
assumes $1 \leq x$ **shows** $\text{ln}(x) \leq \text{ln-upper-13 } x$
apply (*rule gen-upper-bound-increasing [OF assms d-delta-ln-upper-13]*)
apply (*auto simp: diff-delta-ln-upper-13-def ln-upper-13-def*)
done

lemma *ln-upper-13-neg*:
assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\text{ln}(x) \leq \text{ln-upper-13 } x$
apply (*rule gen-upper-bound-decreasing [OF x1 d-delta-ln-upper-13]*)
using *assms*
apply (*auto simp: diff-delta-ln-upper-13-def divide-simps ln-upper-13-def mult-less-0-iff*)
done

theorem *ln-upper-13*: $0 < x \implies \text{ln}(x) \leq \text{ln-upper-13 } x$
by (*metis le-less-linear less-eq-real-def ln-upper-13-neg ln-upper-13-pos*)

definition *ln-lower-13* :: *real* $\Rightarrow$ *real*
where *ln-lower-13* $\equiv \lambda x. - \text{ln-upper-13 } (\text{inverse } x)$

corollary *ln-lower-13*: $0 < x \implies \text{ln-lower-13 } x \leq \text{ln } x$
unfolding *ln-lower-13-def*
by (*metis ln-inverse inverse-positive-iff-positive minus-le-iff ln-upper-13*)

theorem *ln-lower-13-eq*: $0 < x \implies$

$$\text{ln-lower-13 } x = (1/70) * (10 + 1921 * x + 20149 * x^2 + 50774 * x^3 + 38524 * x^4 + 8389 * x^5 + 353 * x^6) * (x - 1) /$$

$$(x * (7 + 126 * x + 525 * x^2 + 700 * x^3 + 315 * x^4 + 42 * x^5 + x^6))$$

unfolding *ln-lower-13-def ln-upper-13-def*
by (*simp add: zero-less-mult-iff add-pos-pos dual-order.strict-implies-not-eq divide-simps*)
algebra

4.7 Upper Bound 15

definition *ln-upper-15* :: *real* $\Rightarrow$ *real*
where *ln-upper-15* $x \equiv$

$$(1487 + 49199 * x + 547235 * x^4 + 718735 * x^3 + 334575 * x^2 + 141123 * x^5 + 35 * x^7 + 9411 * x^6) * (x - 1) /$$

$$(280 * (1 + 56 * x + 2450 * x^4 + 1960 * x^3 + 588 * x^2 + 1176 * x^5 + 8 * x^7 + 196 * x^6))$$

definition *diff-delta-ln-upper-15* :: *real* $\Rightarrow$ *real*
where *diff-delta-ln-upper-15*
 $\equiv \lambda x. (x - 1)^{15} / ((1 + 56*x + 2450*x^4 + 1960*x^3 + 588*x^2 + 8*x^7 + 196*x^6 + 1176*x^5)^2 * x)$

lemma *d-delta-ln-upper-15*: $x > 0 \implies$
 $((\lambda x. \text{ln-upper-15 } x - \text{ln } x) \text{ has-field-derivative } \text{diff-delta-ln-upper-15 } x) \text{ (at } x)$
unfolding *ln-upper-15-def* *diff-delta-ln-upper-15-def*
apply (*intro derivative-eq-intros* | *simp*) +
apply *auto*
apply (*auto simp: add-pos-pos dual-order.strict-implies-not-eq divide-simps, algebra*)
done

lemma *ln-upper-15-pos*:
assumes $1 \leq x$ **shows** $\text{ln}(x) \leq \text{ln-upper-15 } x$
apply (*rule gen-upper-bound-increasing* [*OF* *assms d-delta-ln-upper-15*])
apply (*auto simp: diff-delta-ln-upper-15-def ln-upper-15-def*)
done

lemma *ln-upper-15-neg*:
assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\text{ln}(x) \leq \text{ln-upper-15 } x$
apply (*rule gen-upper-bound-decreasing* [*OF* $x1$ *d-delta-ln-upper-15*])
using *assms*
apply (*auto simp: diff-delta-ln-upper-15-def divide-simps ln-upper-15-def mult-less-0-iff*)
done

theorem *ln-upper-15*: $0 < x \implies \text{ln}(x) \leq \text{ln-upper-15 } x$
by (*metis le-less-linear less-eq-real-def ln-upper-15-neg ln-upper-15-pos*)

definition *ln-lower-15* :: *real* $\Rightarrow$ *real*
where *ln-lower-15* $\equiv \lambda x. - \text{ln-upper-15 } (\text{inverse } x)$

corollary *ln-lower-15*: $0 < x \implies \text{ln-lower-15 } x \leq \text{ln } x$
unfolding *ln-lower-15-def*
by (*metis ln-inverse inverse-positive-iff-positive minus-le-iff ln-upper-15*)

theorem *ln-lower-15-eq*: $0 < x \implies$
 $\text{ln-lower-15 } x = (1/280) * (35 + 9411*x + 141123*x^2 + 547235*x^3 + 718735*x^4 + 334575*x^5 + 49199*x^6 + 1487*x^7) * (x - 1) /$
 $(x * (8 + 196*x + 1176*x^2 + 2450*x^3 + 1960*x^4 + 588*x^5 + 56*x^6 + x^7))$
unfolding *ln-lower-15-def* *ln-upper-15-def*
by (*simp add: zero-less-mult-iff add-pos-pos dual-order.strict-implies-not-eq divide-simps*) *algebra*

end

Chapter 5

Sine and Cosine Upper and Lower Bounds

```
theory Sin-Cos-Bounds
imports Bounds-Lemmas
```

```
begin
```

5.1 Simple base cases

Upper bound for $(0::'a) \leq x$

```
lemma sin-le-arg:
  fixes  $x :: real$ 
  shows  $0 \leq x \implies \sin x \leq x$ 
  by (fact sin-x-le-x)
```

```
lemma cos-ge-1-arg:
  fixes  $x :: real$ 
  assumes  $0 \leq x$ 
  shows  $1 - x \leq \cos x$ 
  apply (rule gen-lower-bound-increasing [OF assms])
  apply (intro derivative-eq-intros, auto)
  done
```

lemmas *sin-Taylor-0-upper-bound-pos* = *sin-le-arg* — MetiTarski bound

```
lemma cos-Taylor-1-lower-bound:
  fixes  $x :: real$ 
  assumes  $0 \leq x$ 
  shows  $(1 - x^2 / 2) \leq \cos x$ 
  apply (rule gen-lower-bound-increasing [OF assms])
  apply (intro derivative-eq-intros)
  apply (rule refl | simp add: sin-le-arg)
  done
```

```

lemma sin-Taylor-1-lower-bound:
  fixes  $x :: \text{real}$ 
  assumes  $0 \leq x$ 
  shows  $(x - x^3 / 6) \leq \sin x$ 
  apply (rule gen-lower-bound-increasing [OF assms])
  apply (intro derivative-eq-intros)
  apply (rule refl | simp add: cos-Taylor-1-lower-bound) +
  done

```

5.2 Taylor series approximants

```

definition sinpoly ::  $[\text{nat}, \text{real}] \Rightarrow \text{real}$ 
  where sinpoly  $n = (\lambda x. \sum_{k < n}. \sin\text{-coeff } k * x^k)$ 

```

```

definition cospoly ::  $[\text{nat}, \text{real}] \Rightarrow \text{real}$ 
  where cospoly  $n = (\lambda x. \sum_{k < n}. \cos\text{-coeff } k * x^k)$ 

```

```

lemma sinpoly-Suc: sinpoly (Suc  $n$ ) =  $(\lambda x. \sinpoly\ n\ x + \sin\text{-coeff } n * x^n)$ 
  by (simp add: sinpoly-def)

```

```

lemma cospoly-Suc: cospoly (Suc  $n$ ) =  $(\lambda x. \cospoly\ n\ x + \cos\text{-coeff } n * x^n)$ 
  by (simp add: cospoly-def)

```

```

lemma sinpoly-minus [simp]: sinpoly  $n\ (-x) = - \sinpoly\ n\ x$ 
  by (induct n) (auto simp: sinpoly-def sin-coeff-def)

```

```

lemma cospoly-minus [simp]: cospoly  $n\ (-x) = \cospoly\ n\ x$ 
  by (induct n) (auto simp: cospoly-def cos-coeff-def)

```

```

lemma d-sinpoly-cospoly:
  (sinpoly (Suc  $n$ ) has-field-derivative cospoly  $n\ x$ ) (at x)

```

```

proof (induction n)

```

```

  case 0 show ?case

```

```

    by (simp add: sinpoly-def cospoly-def)

```

```

next

```

```

  case (Suc  $n$ ) show ?case

```

```

proof (cases n=0)

```

```

  case True then show ?thesis

```

```

    by (simp add: sinpoly-def sin-coeff-def cospoly-def)

```

```

next

```

```

  case False then

```

```

    have xn:  $x^n - (x^{n-1} * x) = x^n$ 

```

```

      by (metis Suc-pred mult.commute not-gr0 power-Suc)

```

```

    show ?thesis using Suc False

```

```

      apply (simp add: sinpoly-Suc [of Suc n] cospoly-def)

```

```

      apply (intro derivative-eq-intros | simp) +

```

```

    apply (simp add: xn mult.assoc sin-coeff-def cos-coeff-def divide-simps del:
fact-Suc)
    apply (simp add: algebra-simps)
    done
  qed
qed

lemma d-cospoly-sinpoly:
  (cospoly (Suc n) has-field-derivative -sinpoly n x) (at x)
proof (induction n)
  case 0 show ?case
    by (simp add: sinpoly-def cospoly-def)
next
  case (Suc n) show ?case
    proof (cases n=0)
      case True then show ?thesis
        by (simp add: sinpoly-def cospoly-def cos-coeff-def)
    next
      case False then
        have xn:  $x ^ (n - Suc 0) * x = x ^ n$ 
        by (metis Suc-pred mult.commute not-gr0 power-Suc)
        have m1:  $odd\ n \implies (-1 :: real) ^ ((n - Suc 0) div 2) = - ((-1) ^ (Suc\ n\ div\ 2))$ 
        by (cases n) simp-all
        show ?thesis using Suc False
          apply (simp add: cospoly-Suc [of Suc n] sinpoly-def)
          apply (intro derivative-eq-intros | simp)+
          apply (simp add: xn mult.assoc sin-coeff-def cos-coeff-def m1 divide-simps
del: fact-Suc)
          apply (simp add: algebra-simps)
          done
    qed
  qed
qed

```

5.3 Inductive proof of sine inequalities

```

lemma sinpoly-lb-imp-cospoly-ub:
  assumes x0:  $0 \leq x$  and k0:  $k > 0$  and  $\bigwedge x. 0 \leq x \implies sinpoly\ (k - 1)\ x \leq sin\ x$ 
  shows  $cos\ x \leq cospoly\ k\ x$ 
  apply (rule gen-lower-bound-increasing [OF x0])
  apply (intro derivative-eq-intros | simp)+
  using d-cospoly-sinpoly [of k - 1] assms
  apply auto
  apply (simp add: cospoly-def)
  done

```

```

lemma cospoly-ub-imp-sinpoly-ub:
  assumes x0:  $0 \leq x$  and k0:  $k > 0$  and  $\bigwedge x. 0 \leq x \implies cos\ x \leq cospoly\ (k - 1)$ 

```

```

x
  shows  $\sin x \leq \text{sinpoly } k \ x$ 
  apply (rule gen-lower-bound-increasing [OF x0])
  apply (intro derivative-eq-intros | simp)+
  using d-sinpoly-cospoly [of k - 1] assms
  apply auto
  apply (simp add: sinpoly-def)
  done

lemma sinpoly-ub-imp-cospoly-lb:
  assumes x0:  $0 \leq x$  and k0:  $k > 0$  and  $\bigwedge x. 0 \leq x \implies \sin x \leq \text{sinpoly } (k - 1)$ 
x
  shows  $\text{cospoly } k \ x \leq \cos x$ 
  apply (rule gen-lower-bound-increasing [OF x0])
  apply (intro derivative-eq-intros | simp)+
  using d-cospoly-sinpoly [of k - 1] assms
  apply auto
  apply (simp add: cospoly-def)
  done

lemma cospoly-lb-imp-sinpoly-lb:
  assumes x0:  $0 \leq x$  and k0:  $k > 0$  and  $\bigwedge x. 0 \leq x \implies \text{cospoly } (k - 1) \ x \leq \cos$ 
x
  shows  $\text{sinpoly } k \ x \leq \sin x$ 
  apply (rule gen-lower-bound-increasing [OF x0])
  apply (intro derivative-eq-intros | simp)+
  using d-sinpoly-cospoly [of k - 1] assms
  apply auto
  apply (simp add: sinpoly-def)
  done

lemma
  assumes  $0 \leq x$ 
  shows sinpoly-lower-nonneg:  $\text{sinpoly } (4 * \text{Suc } n) \ x \leq \sin x$  (is ?th1)
  and sinpoly-upper-nonneg:  $\sin x \leq \text{sinpoly } (\text{Suc } (\text{Suc } (4 * n))) \ x$  (is ?th2)
proof -
  have  $\text{sinpoly } (4 * \text{Suc } n) \ x \leq \sin x \ \wedge \ \sin x \leq \text{sinpoly } (\text{Suc } (\text{Suc } (4 * n))) \ x$ 
  using assms
  apply (induction n arbitrary: x)
  apply (simp add: sinpoly-def sin-coeff-def sin-Taylor-1-lower-bound sin-Taylor-0-upper-bound-pos
lessThan-nat-numeral fact-numeral)
  apply (auto simp: cospoly-lb-imp-sinpoly-lb sinpoly-ub-imp-cospoly-lb cospoly-ub-imp-sinpoly-ub
sinpoly-lb-imp-cospoly-ub)
  done
  then show ?th1 ?th2
  using assms
  by auto
qed

```

5.4 Collecting the results

corollary *sinpoly-upper-nonpos*:

$x \leq 0 \implies \sin x \leq \text{sinpoly } (4 * \text{Suc } n) x$
using *sinpoly-lower-nonneg* [of $-x n$]
by *simp*

corollary *sinpoly-lower-nonpos*:

$x \leq 0 \implies \text{sinpoly } (\text{Suc } (\text{Suc } (4 * n))) x \leq \sin x$
using *sinpoly-upper-nonneg* [of $-x n$]
by *simp*

corollary *cospoly-lower-nonneg*:

$0 \leq x \implies \text{cospoly } (\text{Suc } (\text{Suc } (\text{Suc } (4 * n)))) x \leq \cos x$
by (*auto simp: sinpoly-upper-nonneg sinpoly-ub-imp-cospoly-lb*)

lemma *cospoly-lower*:

$\text{cospoly } (\text{Suc } (\text{Suc } (\text{Suc } (4 * n)))) x \leq \cos x$
proof (*cases rule: le-cases* [of $0 x$])
case *le* **then show** *?thesis*
by (*simp add: cospoly-lower-nonneg*)
next
case *ge* **then show** *?thesis* **using** *cospoly-lower-nonneg* [of $-x$]
by *simp*
qed

lemma *cospoly-upper-nonneg*:

assumes $0 \leq x$
shows $\cos x \leq \text{cospoly } (\text{Suc } (4 * n)) x$
proof (*cases n*)
case 0 **then show** *?thesis*
by (*simp add: cospoly-def*)
next
case (*Suc m*)
then show *?thesis*
using *sinpoly-lower-nonneg* [of $-m$] *assms*
by (*auto simp: sinpoly-lb-imp-cospoly-ub*)
qed

lemma *cospoly-upper*:

$\cos x \leq \text{cospoly } (\text{Suc } (4 * n)) x$
proof (*cases rule: le-cases* [of $0 x$])
case *le* **then show** *?thesis*
by (*simp add: cospoly-upper-nonneg*)
next
case *ge* **then show** *?thesis* **using** *cospoly-upper-nonneg* [of $-x$]
by *simp*
qed

end

Chapter 6

Square Root Upper and Lower Bounds

```
theory Sqrt-Bounds
imports Bounds-Lemmas
         ~~/src/HOL/Library/Sum-of-Squares
```

```
begin
```

Covers all bounds used in sqrt-upper.ax, sqrt-lower.ax and sqrt-extended.ax.

6.1 Upper bounds

```
primrec sqrtu :: [real,nat]  $\Rightarrow$  real where
  sqrtu x 0 = (x+1)/2
| sqrtu x (Suc n) = (sqrtu x n + x/sqrtu x n) / 2

lemma sqrtu-upper:  $x \leq \text{sqrtu } x \ n \ ^2$ 
using assms
proof (induction n)
  case 0 show ?case
    apply (simp add: power2-eq-square)
    apply (sos (((A<0 * R<1) + (R<1 * (R<1 * [ $\sim$ 1*x + 1] $^2$ ))))))
    done
  next
    case (Suc n)
    have xy:  $\bigwedge y. \llbracket x \leq y * y; y \neq 0 \rrbracket \implies x * (2 * (y * y)) \leq x * x + y * (y * (y * y))$ 
    by (sos ((((((A<0 * A<1) * R<1) + ((A<0 * R<1) * (R<1 * [ $\sim$ 1*y $^2$  + x] $^2$ )))))) &
              (((((A<0 * A<1) * R<1) + ((A<0 * R<1) * (R<1 * [ $\sim$ 1*y $^2$  + x] $^2$ )))))))
    show ?case using Suc
    by (auto simp: power2-eq-square algebra-simps divide-simps xy)
qed
```

lemma *sqrtu-numeral*:
 $\text{sqrtu } x \text{ (numeral } n) = (\text{sqrtu } x \text{ (pred-numeral } n) + x / \text{sqrtu } x \text{ (pred-numeral } n)) / 2$
by (*simp add: numeral-eq-Suc*)

lemma *sqrtu-gt-0*: $x \geq 0 \implies \text{sqrtu } x \text{ } n > 0$
apply (*induct n*)
apply (*auto simp: field-simps*)
by (*metis add-strict-increasing2 mult-zero-left not-real-square-gt-zero*)

theorem *gen-sqrt-upper*: $0 \leq x \implies \text{sqrt } x \leq \text{sqrtu } x \text{ } n$
using *real-sqrt-le-mono* [*OF sqrtu-upper [of x n]*]
by *auto (metis abs-of-nonneg dual-order.strict-iff-order sqrtu-gt-0)*

lemma *sqrt-upper-bound-0*:
assumes $x \geq 0$ **shows** $\text{sqrt } x \leq (x+1)/2$ (**is -** $\leq$ *?rhs*)
proof –
have $\text{sqrt } x \leq \text{sqrtu } x \text{ } 0$
by (*metis assms gen-sqrt-upper*)
also have $\dots = ?rhs$
by (*simp add: divide-simps*)
finally show *?thesis* .
qed

lemma *sqrt-upper-bound-1*:
assumes $x \geq 0$ **shows** $\text{sqrt } x \leq (1/4)*(x^2+6*x+1) / (x+1)$ (**is -** $\leq$ *?rhs*)
proof –
have $\text{sqrt } x \leq \text{sqrtu } x \text{ } 1$
by (*metis assms gen-sqrt-upper*)
also have $\dots = ?rhs$
by (*simp add: divide-simps algebra*)
finally show *?thesis* .
qed

lemma *sqrtu-2-eq*:
 $\text{sqrtu } x \text{ } 2 = (1/8)*(x^4 + 28*x^3 + 70*x^2 + 28*x + 1) / ((x+1)*(x^2 + 6*x + 1))$
by (*simp add: sqrtu-numeral divide-simps algebra*)

lemma *sqrt-upper-bound-2*:
assumes $x \geq 0$
shows $\text{sqrt } x \leq (1/8)*(x^4 + 28*x^3 + 70*x^2 + 28*x + 1) / ((x+1)*(x^2 + 6*x + 1))$
by (*metis assms gen-sqrt-upper sqrtu-2-eq*)

lemma *sqrtu-4-eq*:
 $x \geq 0 \implies$
 $\text{sqrtu } x \text{ } 4 = (1/32)*(225792840*x^6 + 64512240*x^5 + 601080390*x^4 + 471435600*x^3 + 496*x^2 + 1 + 35960*x)$

$$/ ((x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7+1$$

by (*simp add: sqrtu-numeral divide-simps add-nonneg-eq-0-iff*) *algebra*

lemma *sqrt-upper-bound-4*:

assumes $x \geq 0$

shows $\sqrt{x} \leq (1/32)*(225792840*x^6+64512240*x^5+601080390*x^8+471435600*x^7+496*x+1+35$

$$/ ((x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7+1$$

by (*metis assms gen-sqrt-upper sqrtu-4-eq*)

lemma *gen-sqrt-upper-scaled*:

assumes $0 \leq x$ $0 < u$

shows $\sqrt{x} \leq \sqrt[n]{x*u^2}$

proof –

have $\sqrt{x} = \sqrt{x} * \sqrt{u^2} / u$

using *assms*

by *simp*

also have $\dots = \sqrt[n]{x*u^2} / u$

by (*metis real-sqrt-mult-distrib2*)

also have $\dots \leq \sqrt[n]{x*u^2}$

using *assms*

by (*simp add: divide-simps*) (*metis gen-sqrt-upper zero-le-mult-iff zero-le-power2*)

finally show *?thesis* .

qed

lemma *sqrt-upper-bound-2-small*:

assumes $0 \leq x$

shows $\sqrt{x} \leq (1/32)*(65536*x^4 + 114688*x^3 + 17920*x^2 + 448*x +$
 $1) / ((16*x + 1)*(256*x^2 + 96*x + 1))$

apply (*rule order-trans [OF gen-sqrt-upper-scaled [of x 4 2] eq-refl]*)

using *assms*

apply (*auto simp: sqrtu-2-eq*)

apply (*simp add: divide-simps*)

apply *algebra*

done

lemma *sqrt-upper-bound-2-large*:

assumes $0 \leq x$

shows $\sqrt{x} \leq (1/32)*(65536 + 114688*x + 17920*x^2 + 448*x^3 + x^4)$

$$/ ((x + 16)*(256 + 96*x + x^2))$$

apply (*rule order-trans [OF gen-sqrt-upper-scaled [of x 1/4 2] eq-refl]*)

using *assms*

apply (*auto simp: sqrtu-2-eq*)

apply (*simp add: divide-simps*)

apply *algebra*

done

6.2 Lower bounds

lemma *sqrt-lower-bound-id*:

```

    assumes  $0 \leq x \leq 1$ 
    shows  $x \leq \sqrt{x}$ 
  proof -
    have  $x^2 \leq x$  using assms
    by (metis one-le-numeral power-decreasing power-one-right)
    then show ?thesis
    by (metis real-le-rsqrt)
  qed

```

definition *sqrtl* :: $[real, nat] \Rightarrow real$ **where**
sqrtl $x\ n = x / \text{sqrtu}\ x\ n$

```

lemma sqrtl-lower:  $0 \leq x \implies \text{sqrtl}\ x\ n^2 \leq x$ 
  unfolding sqrtl-def using sqrtu-upper [of  $x\ n$ ]
  by (auto simp: power2-eq-square divide-simps mult-left-mono)

```

```

theorem gen-sqrt-lower:  $0 \leq x \implies \text{sqrtl}\ x\ n \leq \sqrt{x}$ 
  using real-sqrt-le-mono [OF sqrtl-lower [of  $x\ n$ ]]
  by auto

```

```

lemma sqrt-lower-bound-0:
  assumes  $x \geq 0$  shows  $2*x/(x+1) \leq \sqrt{x}$  (is ?lhs  $\leq$  -)
  proof -
    have ?lhs = sqrtl  $x\ 0$ 
    by (simp add: sqrtl-def)
    also have  $\dots \leq \sqrt{x}$ 
    by (metis assms gen-sqrt-lower)
    finally show ?thesis .
  qed

```

```

lemma sqrt-lower-bound-1:
  assumes  $x \geq 0$  shows  $4*x*(x+1) / (x^2+6*x+1) \leq \sqrt{x}$  (is ?lhs  $\leq$  -)
  proof -
    have ?lhs = sqrtl  $x\ 1$  using assms
    by (simp add: sqrtl-def power-eq-if algebra-simps divide-simps)
    also have  $\dots \leq \sqrt{x}$ 
    by (metis assms gen-sqrt-lower)
    finally show ?thesis .
  qed

```

```

lemma sqrtl-2-eq: sqrtl  $x\ 2 =$ 
   $8*x*(x+1)*(x^2+6*x+1) / (x^4+28*x^3+70*x^2+28*x+1)$ 
  using sqrtu-gt-0 [of  $x\ 2$ ]
  by (simp add: sqrtl-def sqrtu-2-eq)

```

```

lemma sqrt-lower-bound-2:
  assumes  $x \geq 0$ 
  shows  $8*x*(x+1)*(x^2+6*x+1) / (x^4+28*x^3+70*x^2+28*x+1) \leq \sqrt{x}$ 

```

```

by (metis assms sqrtl-2-eq gen-sqrt-lower)

lemma sqrtl-4-eq:  $x \geq 0 \implies$ 
  sqrtl  $x$  4
  =  $(32*x*(x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7+$ 
    /  $(225792840*x^6+64512240*x^5+601080390*x^8+471435600*x^7+496*x+1+35960*x^2+906192*x^3$ 
  using sqrtu-gt-0 [of  $x$  4]
  by (simp add: sqrtl-def sqrtu-4-eq)

lemma sqrt-lower-bound-4:
  assumes  $x \geq 0$ 
  shows  $(32*x*(x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7+$ 
    /  $(225792840*x^6+64512240*x^5+601080390*x^8+471435600*x^7+496*x+1+35960*x^2+906192*x^3$ 
     $\leq \sqrt{x}$ 
  by (metis assms sqrtl-4-eq gen-sqrt-lower)

lemma gen-sqrt-lower-scaled:
  assumes  $0 \leq x$   $0 < u$ 
  shows sqrtl  $(x*u^2)$   $n$  /  $u \leq \sqrt{x}$ 
proof -
  have sqrtl  $(x*u^2)$   $n$  /  $u \leq \sqrt{x*u^2}$  /  $u$ 
  using assms
  by (simp add: divide-simps) (metis gen-sqrt-lower zero-le-mult-iff zero-le-power2)
  also have ... =  $\sqrt{x} * \sqrt{u^2}$  /  $u$ 
  by (metis real-sqrt-mult-distrib2)
  also have ... =  $\sqrt{x}$ 
  using assms
  by simp
  finally show ?thesis .
qed

lemma sqrt-lower-bound-2-small:
  assumes  $0 \leq x$ 
  shows  $32*x*(16*x+1)*(256*x^2+96*x+1) / (65536*x^4+114688*x^3$ 
    +  $17920*x^2+448*x+1) \leq \sqrt{x}$ 
  apply (rule order-trans [OF eq-refl gen-sqrt-lower-scaled [of  $x$  4 2]])
  using assms
  apply (auto simp: sqrtl-2-eq)
  apply (simp add: divide-simps)
  apply algebra
  done

lemma sqrt-lower-bound-2-large:
  assumes  $0 \leq x$ 
  shows  $32*x*(x+16)*(x^2+96*x+256) / (x^4+448*x^3+17920*x^2$ 
    +  $114688*x+65536) \leq \sqrt{x}$ 
  apply (rule order-trans [OF eq-refl gen-sqrt-lower-scaled [of  $x$  1/4 2]])
  using assms
  apply (auto simp: sqrtl-2-eq)

```

```
apply (simp add: divide-simps)
done

end
```

Bibliography

- [1] B. Akbarpour and L. Paulson. MetiTarski: An automatic theorem prover for real-valued special functions. 44(3):175–205, Mar. 2010.