# The Hurwitz and Riemann $\zeta$ functions

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**Abstract**

This entry builds upon the results about formal and analytic Dirichlet series to define the Hurwitz $\zeta$ function $\zeta(a, s)$ and, based on that, the Riemann $\zeta$ function $\zeta(s)$. This is done by first defining them for $\Re(z) > 1$ and then successively extending the domain to the left using the Euler–MacLaurin formula.

Apart from the most basic facts such as analyticity, the following results are provided:

- the Stieltjes constants and the Laurent expansion of $\zeta(s)$ at $s = 1$
- the non-vanishing of $\zeta(s)$ for $\Re(s) \geq 1$
- the relationship between $\zeta(a, s)$ and $\Gamma$
- the special values at negative integers and positive even integers
- Hurwitz's formula and the reflection formula for $\zeta(s)$
- the Hadjicostas–Chapman formula [3, 4]

The entry also contains Euler's analytic proof of the infinitude of primes, based on the fact that $\zeta(s)$ has a pole at $s = 1$.

# Contents

# 1 Various preliminary material

**theory** *Zeta-Library*
**imports**
  *HOL−Complex-Analysis.Complex-Analysis*
  *HOL−Real-Asymp.Real-Asymp*
  *Dirichlet-Series.Dirichlet-Series-Analysis*
**begin**

## 1.1 Facts about limits

**lemma** *at-within-altdef*:
  $at\ x\ within\ A = (INF\ S{\in}\{S.\ open\ S \wedge x \in S\}.\ principal\ (S \cap (A - \{x\})))$
  ⟨*proof*⟩

**lemma** *tendsto-at-left-realI-sequentially*:
  **fixes** $f :: real \Rightarrow {}'b{::}first\text{-}countable\text{-}topology$
  **assumes** $*$: $\bigwedge X.\ filterlim\ X\ (at\text{-}left\ c)\ sequentially \Longrightarrow (\lambda n.\ f\ (X\ n)) \longrightarrow y$
  **shows** $(f \longrightarrow y)\ (at\text{-}left\ c)$
⟨*proof*⟩

**lemma**
  **shows** *at-right-PInf* [*simp*]: $at\text{-}right\ (\infty :: ereal) = bot$
    **and** *at-left-MInf* [*simp*]: $at\text{-}left\ (-\infty :: ereal) = bot$
⟨*proof*⟩

**lemma** *tendsto-at-left-erealI-sequentially*:
  **fixes** $f :: ereal \Rightarrow {}'b{::}first\text{-}countable\text{-}topology$
  **assumes** $*$: $\bigwedge X.\ filterlim\ X\ (at\text{-}left\ c)\ sequentially \Longrightarrow (\lambda n.\ f\ (X\ n)) \longrightarrow y$
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⟨*proof*⟩

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**lemma** *tendsto-at-right-erealI-sequentially*:
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  **shows** $(f \longrightarrow y)\ (at\text{-}right\ c)$
⟨*proof*⟩

**proposition** *analytic-continuation'*:
  **assumes** *hol*: $f$ *holomorphic-on* $S$ $g$ *holomorphic-on* $S$
    **and** *open* $S$ **and** *connected* $S$
    **and** $U \subseteq S$ **and** $\xi \in S$
    **and** $\xi$ *islimpt* $U$
    **and** *fU0* [*simp*]: $\bigwedge z.\ z \in U \Longrightarrow f\ z = g\ z$

**and** $w \in S$
**shows** $f\ w = g\ w$
⟨*proof*⟩

## 1.2 Various facts about integrals

**lemma** *continuous-on-imp-set-integrable-cbox*:
**fixes** $h$ :: $'a$ :: *euclidean-space* ⇒ $'b$ :: *euclidean-space*
**assumes** *continuous-on* (*cbox a b*) *h*
**shows** *set-integrable lborel* (*cbox a b*) *h*
⟨*proof*⟩

**lemma** *set-nn-integral-cong*:
**assumes** $M = M'$ $A = B$ $\bigwedge x.\ x \in$ *space* $M \cap A \Longrightarrow f\ x = g\ x$
**shows** *set-nn-integral* $M\ A\ f$ = *set-nn-integral* $M'\ B\ g$
⟨*proof*⟩

**lemma** *set-integrable-bound*:
**fixes** $f$ :: $'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
**and** $g$ :: $'a \Rightarrow 'c$::{*banach*, *second-countable-topology*}
**assumes** *set-integrable* $M\ A\ f$ *set-borel-measurable* $M\ A\ g$
**assumes** *AE* $x$ *in* $M$. $x \in A \longrightarrow$ *norm* $(g\ x) \leq$ *norm* $(f\ x)$
**shows** *set-integrable* $M\ A\ g$
⟨*proof*⟩

**lemma** *nn-integral-has-integral-lebesgue*:
**fixes** $f$ :: $'a$::*euclidean-space* ⇒ *real*
**assumes** *nonneg*: $\bigwedge x.\ x \in \Omega \Longrightarrow 0 \leq f\ x$ **and** $I$: ($f$ *has-integral* $I$) $\Omega$
**shows** *integral*$^N$ *lborel* ($\lambda x.$ *indicator* $\Omega\ x * f\ x$) = $I$
⟨*proof*⟩

## 1.3 Uniform convergence of integrals

**lemma** *has-absolute-integral-change-of-variables-1'*:
**fixes** $f$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *real*
**assumes** $S$: $S \in$ *sets lebesgue*
**and** *der-g*: $\bigwedge x.\ x \in S \Longrightarrow$ ($g$ *has-field-derivative* $g'\ x$) (*at* $x$ *within* $S$)
**and** *inj*: *inj-on* $g\ S$
**shows** ($\lambda x.\ |g'\ x| *_R f(g\ x)$) *absolutely-integrable-on* $S$ ∧
*integral* $S$ ($\lambda x.\ |g'\ x| *_R f(g\ x)$) = $b$
⟷ $f$ *absolutely-integrable-on* ($g$ ‘ $S$) ∧ *integral* ($g$ ‘ $S$) $f = b$
⟨*proof*⟩

**lemma** *set-nn-integral-lborel-eq-integral*:
**fixes** $f$::$'a$::*euclidean-space* ⇒ *real*
**assumes** *set-borel-measurable borel* $A\ f$
**assumes** $\bigwedge x.\ x \in A \Longrightarrow 0 \leq f\ x$ $(\int^+ x{\in}A.\ f\ x\ \partial lborel) < \infty$
**shows** $(\int^+ x{\in}A.\ f\ x\ \partial lborel)$ = *integral* $A\ f$
⟨*proof*⟩

**lemma** *nn-integral-has-integral-lebesgue'*:
  **fixes** $f$ :: *'a::euclidean-space* ⇒ *real*
  **assumes** *nonneg*: $\bigwedge x.\ x \in \Omega \Longrightarrow 0 \le f\ x$ **and** $I$: $(f$ *has-integral* $I)\ \Omega$
  **shows** $integral^N$ *lborel* $(\lambda x.$ *ennreal* $(f\ x) *$ *indicator* $\Omega\ x)$ = *ennreal* $I$
⟨*proof*⟩

**lemma** *uniform-limit-set-lebesgue-integral*:
  **fixes** $f$ :: *'a* ⇒ *'b* :: *euclidean-space* ⇒ *'c* :: {*banach*, *second-countable-topology*}
  **assumes** *set-integrable lborel* $X'$ $g$
  **assumes** [*measurable*]: $X' \in$ *sets borel*
  **assumes** [*measurable*]: $\bigwedge y.\ y \in Y \Longrightarrow$ *set-borel-measurable borel* $X'$ $(f\ y)$
  **assumes** $\bigwedge y.\ y \in Y \Longrightarrow$ (*AE* $t \in X'$ *in lborel. norm* $(f\ y\ t) \le g\ t)$
  **assumes** *eventually* $(\lambda x.\ X\ x \in$ *sets borel* $\wedge\ X\ x \subseteq X')$ $F$
  **assumes** *filterlim* $(\lambda x.$ *set-lebesgue-integral lborel* $(X\ x)\ g)$
      (*nhds* (*set-lebesgue-integral lborel* $X'$ $g$)) $F$
  **shows** *uniform-limit* $Y$
      $(\lambda x\ y.$ *set-lebesgue-integral lborel* $(X\ x)\ (f\ y))$
      $(\lambda y.$ *set-lebesgue-integral lborel* $X'$ $(f\ y))$ $F$
⟨*proof*⟩

**lemma** *integral-dominated-convergence-at-right*:
  **fixes** $s$ :: *real* ⇒ *'a* ⇒ *'b*::{*banach*, *second-countable-topology*} **and** $w$ :: *'a* ⇒ *real*
    **and** $f$ :: *'a* ⇒ *'b* **and** $M$ **and** $c$ :: *real*
  **assumes** $f \in$ *borel-measurable* $M$ $\bigwedge t.\ s\ t \in$ *borel-measurable* $M$ *integrable* $M\ w$
  **assumes** *lim*: *AE* $x$ *in* $M$. $((\lambda i.\ s\ i\ x) \longrightarrow f\ x)$ (*at-right* $c$)
  **assumes** *bound*: $\forall_F$ $i$ *in at-right* $c$. *AE* $x$ *in* $M$. *norm* $(s\ i\ x) \le w\ x$
  **shows** $((\lambda t.\ integral^L\ M\ (s\ t)) \longrightarrow integral^L\ M\ f)$ (*at-right* $c$)
⟨*proof*⟩

**lemma** *integral-dominated-convergence-at-left*:
  **fixes** $s$ :: *real* ⇒ *'a* ⇒ *'b*::{*banach*, *second-countable-topology*} **and** $w$ :: *'a* ⇒ *real*
    **and** $f$ :: *'a* ⇒ *'b* **and** $M$ **and** $c$ :: *real*
  **assumes** $f \in$ *borel-measurable* $M$ $\bigwedge t.\ s\ t \in$ *borel-measurable* $M$ *integrable* $M\ w$
  **assumes** *lim*: *AE* $x$ *in* $M$. $((\lambda i.\ s\ i\ x) \longrightarrow f\ x)$ (*at-left* $c$)
  **assumes** *bound*: $\forall_F$ $i$ *in at-left* $c$. *AE* $x$ *in* $M$. *norm* $(s\ i\ x) \le w\ x$
  **shows** $((\lambda t.\ integral^L\ M\ (s\ t)) \longrightarrow integral^L\ M\ f)$ (*at-left* $c$)
⟨*proof*⟩

**lemma** *uniform-limit-interval-integral-right*:
  **fixes** $f$ :: *'a* ⇒ *real* ⇒ *'c* :: {*banach*, *second-countable-topology*}
  **assumes** *interval-lebesgue-integrable lborel* $a\ b\ g$
  **assumes** [*measurable*]: $\bigwedge y.\ y \in Y \Longrightarrow$ *set-borel-measurable borel* (*einterval* $a\ b$) $(f\ y)$
  **assumes** $\bigwedge y.\ y \in Y \Longrightarrow$ (*AE* $t \in$*einterval* $a\ b$ *in lborel. norm* $(f\ y\ t) \le g\ t)$
  **assumes** $a < b$
  **shows** *uniform-limit* $Y$ $(\lambda b'\ y.$ *LBINT* $t$=$a$..$b'$. $f\ y\ t)$ $(\lambda y.$ *LBINT* $t$=$a$..$b$. $f\ y\ t)$ (*at-left* $b$)
⟨*proof*⟩

**lemma** *uniform-limit-interval-integral-left*:
  **fixes** $f :: 'a \Rightarrow real \Rightarrow 'c :: \{banach,\ second\text{-}countable\text{-}topology\}$
  **assumes** *interval-lebesgue-integrable lborel a b g*
  **assumes** [*measurable*]: $\bigwedge y.\ y \in Y \Longrightarrow$ *set-borel-measurable borel* (*einterval a b*) (*f y*)
  **assumes** $\bigwedge y.\ y \in Y \Longrightarrow (AE\ t{\in}einterval\ a\ b\ in\ lborel.\ norm\ (f\ y\ t) \leq g\ t)$
  **assumes** $a < b$
  **shows** *uniform-limit* $Y$ $(\lambda a'\ y.\ LBINT\ t{=}a'..b.\ f\ y\ t)$ $(\lambda y.\ LBINT\ t{=}a..b.\ f\ y\ t)$ (*at-right a*)
⟨*proof*⟩

**lemma** *uniform-limit-interval-integral-sequentially*:
  **fixes** $f :: 'a \Rightarrow real \Rightarrow 'c :: \{banach,\ second\text{-}countable\text{-}topology\}$
  **assumes** *interval-lebesgue-integrable lborel a b g*
  **assumes** [*measurable*]: $\bigwedge y.\ y \in Y \Longrightarrow$ *set-borel-measurable borel* (*einterval a b*) (*f y*)
  **assumes** $\bigwedge y.\ y \in Y \Longrightarrow (AE\ t{\in}einterval\ a\ b\ in\ lborel.\ norm\ (f\ y\ t) \leq g\ t)$
  **assumes** $a'$: *filterlim* $a'$ (*at-right a*) *sequentially*
  **assumes** $b'$: *filterlim* $b'$ (*at-left b*) *sequentially*
  **assumes** $a < b$
  **shows** *uniform-limit* $Y$ $(\lambda n\ y.\ LBINT\ t{=}a'\ n..b'\ n.\ f\ y\ t)$
    $(\lambda y.\ LBINT\ t{=}a..b.\ f\ y\ t)$ *sequentially*
⟨*proof*⟩

**lemma** *interval-lebesgue-integrable-combine*:
  **assumes** *interval-lebesgue-integrable lborel A B f*
  **assumes** *interval-lebesgue-integrable lborel B C f*
  **assumes** *set-borel-measurable borel* (*einterval A C*) *f*
  **assumes** $A \leq B$ $B \leq C$
  **shows** *interval-lebesgue-integrable lborel A C f*
⟨*proof*⟩

**lemma** *interval-lebesgue-integrable-bigo-right*:
  **fixes** $A\ B :: real$
  **fixes** $f :: real \Rightarrow real$
  **assumes** $f \in O[at\text{-}left\ B](g)$
  **assumes** *cont*: *continuous-on* $\{A..{<}B\}$ *f*
  **assumes** *meas*: *set-borel-measurable borel* $\{A{<}..{<}B\}$ *f*
  **assumes** *interval-lebesgue-integrable lborel A B g*
  **assumes** $A < B$
  **shows** *interval-lebesgue-integrable lborel A B f*
⟨*proof*⟩

**lemma** *interval-lebesgue-integrable-bigo-left*:
  **fixes** $A\ B :: real$
  **fixes** $f :: real \Rightarrow real$
  **assumes** $f \in O[at\text{-}right\ A](g)$
  **assumes** *cont*: *continuous-on* $\{A{<}..B\}$ *f*

**assumes** *meas*: *set-borel-measurable borel* $\{A{<}..{<}B\}$ *f*
**assumes** *interval-lebesgue-integrable lborel A B g*
**assumes** $A < B$
**shows** *interval-lebesgue-integrable lborel A B f*
⟨*proof*⟩

## 1.4 Other material

**lemma** *summable-comparison-test-bigo*:
**fixes** *f* :: *nat* ⇒ *real*
**assumes** *summable* ($\lambda n$. *norm* (*g n*)) $f \in O(g)$
**shows** *summable f*
⟨*proof*⟩

**lemma** *fps-expansion-cong*:
**assumes** *eventually* ($\lambda x$. $g\ x = h\ x$) (*nhds x*)
**shows** *fps-expansion g x* = *fps-expansion h x*
⟨*proof*⟩

**lemma** *fps-expansion-eq-zero-iff*:
**assumes** *g holomorphic-on ball z r* $r > 0$
**shows** *fps-expansion g z* = *0* $\longleftrightarrow$ ($\forall z \in$ *ball z r*. $g\ z = 0$)
⟨*proof*⟩

**lemma** *fds-nth-higher-deriv*:
*fds-nth* ((*fds-deriv* ^^ *k*) *F*) = ($\lambda n$. ($-1$) ^ *k* ∗ *of-real* (*ln n*) ^ *k* ∗ *fds-nth F n*)
⟨*proof*⟩

**lemma** *binomial-n-n-minus-one* [*simp*]: $n > 0 \Longrightarrow$ *n choose* ($n$ − *Suc 0*) = *n*
⟨*proof*⟩

**lemma** *has-field-derivative-complex-powr-right*:
$w \neq 0 \Longrightarrow$ (($\lambda z$. *w powr z*) *has-field-derivative Ln w* ∗ *w powr z*) (*at z within A*)
⟨*proof*⟩

**lemmas** *has-field-derivative-complex-powr-right′* =
*has-field-derivative-complex-powr-right*[*THEN DERIV-chain2*]

**end**

# 2 The Hurwitz and Riemann $\zeta$ functions

**theory** *Zeta-Function*
**imports**
*Euler-MacLaurin.Euler-MacLaurin*
*Bernoulli.Bernoulli-Zeta*
*Dirichlet-Series.Dirichlet-Series-Analysis*
*Winding-Number-Eval.Winding-Number-Eval*
*HOL−Real-Asymp.Real-Asymp*

*Zeta-Library*
*Pure−ex.Guess*
**begin**

## 2.1 Preliminary facts

**lemma** *holomorphic-on-extend*:
**assumes** $f$ *holomorphic-on* $S - \{\xi\}$ $\xi \in$ *interior* $S$ $f \in O[at\ \xi](\lambda\text{-}.\ 1)$
**shows** $(\exists\, g.\ g$ *holomorphic-on* $S \wedge (\forall\, z{\in}S - \{\xi\}.\ g\ z = f\ z))$
⟨*proof*⟩

**lemma** *removable-singularities*:
**assumes** *finite* $X$ $X \subseteq$ *interior* $S$ $f$ *holomorphic-on* $(S - X)$
**assumes** $\bigwedge z.\ z \in X \Longrightarrow f \in O[at\ z](\lambda\text{-}.\ 1)$
**shows** $\exists\, g.\ g$ *holomorphic-on* $S \wedge (\forall\, z{\in}S{-}X.\ g\ z = f\ z)$
⟨*proof*⟩

**lemma** *continuous-imp-bigo-1*:
**assumes** *continuous* $(at\ x\ within\ A)\ f$
**shows** $f \in O[at\ x\ within\ A](\lambda\text{-}.\ 1)$
⟨*proof*⟩

**lemma** *taylor-bigo-linear*:
**assumes** $f$ *field-differentiable at x0 within A*
**shows** $(\lambda x.\ f\ x - f\ x0) \in O[at\ x0\ within\ A](\lambda x.\ x - x0)$
⟨*proof*⟩

**lemma** *powr-add-minus-powr-asymptotics*:
**fixes** $a\ z$ :: *complex*
**shows** $((\lambda z.\ ((1 + z)\ powr\ a - 1) / z) \longrightarrow a)\ (at\ 0)$
⟨*proof*⟩

**lemma** *complex-powr-add-minus-powr-asymptotics*:
**fixes** $s$ :: *complex*
**assumes** $a$: $a > 0$ **and** $s$: $Re\ s < 1$
**shows** *filterlim* $(\lambda x.$ *of-real* $(x + a)$ *powr* $s -$ *of-real* $x$ *powr* $s)$ $(nhds\ 0)$ *at-top*
⟨*proof*⟩

**lemma** *summable-zeta*:
**assumes** $Re\ s > 1$
**shows** *summable* $(\lambda n.$ *of-nat* $(Suc\ n)$ *powr* $-s)$
⟨*proof*⟩

**lemma** *summable-zeta-real*:
**assumes** $x > 1$
**shows** *summable* $(\lambda n.$ *real* $(Suc\ n)$ *powr* $-x)$

⟨*proof*⟩

**lemma** *summable-hurwitz-zeta*:
  **assumes** *Re s > 1 a > 0*
  **shows** *summable (λn. (of-nat n + of-real a) powr −s)*
⟨*proof*⟩

**lemma** *summable-hurwitz-zeta-real*:
  **assumes** *x > 1 a > 0*
  **shows** *summable (λn. (real n + a) powr −x)*
⟨*proof*⟩

## 2.2 Definitions

We use the Euler–MacLaurin summation formula to express $\zeta(s, a) - \frac{a^{1-s}}{s-1}$ as a polynomial plus some remainder term, which is an integral over a function of order $O(-1 - 2n - \Re(s))$. It is then clear that this integral converges uniformly to an analytic function in $s$ for all $s$ with $\Re(s) > -2n$.

**definition** *pre-zeta-aux* :: *nat ⇒ real ⇒ complex ⇒ complex* **where**
  *pre-zeta-aux N a s = a powr − s / 2 +*
    *(∑ i=1..N. (bernoulli (2 ∗ i) / fact (2 ∗ i)) ∗<sub>R</sub> (pochhammer s (2∗i − 1) ∗*
            *of-real a powr (− s − of-nat (2∗i − 1)))) +*
    *EM-remainder (Suc (2∗N))*
      *(λx. −(pochhammer s (Suc (2∗N)) ∗ of-real (x + a) powr (− 1 − 2∗N − s))) 0*

By iterating the above construction long enough, we can extend this to the entire complex plane.

**definition** *pre-zeta* :: *real ⇒ complex ⇒ complex* **where**
  *pre-zeta a s = pre-zeta-aux (nat (1 − ⌈Re s / 2⌉)) a s*

We can then obtain the Hurwitz $\zeta$ function by adding back the pole at 1. Note that it is not necessary to trust that this somewhat complicated definition is, in fact, the correct one, since we will later show that this Hurwitz zeta function fulfils

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}$$

and is analytic on $\mathbb{C}\setminus\{1\}$, which uniquely defines the function due to analytic continuation. It is therefore obvious that any alternative definition that is analytic on $\mathbb{C} \setminus \{1\}$ and satisfies the above equation must be equal to our Hurwitz $\zeta$ function.

**definition** *hurwitz-zeta* :: *real ⇒ complex ⇒ complex* **where**
  *hurwitz-zeta a s = (if s = 1 then 0 else pre-zeta a s + of-real a powr (1 − s) / (s − 1))*

The Riemann $\zeta$ function is simply the Hurwitz $\zeta$ function with $a = 1$.

**definition** *zeta* :: *complex* ⇒ *complex* **where**
*zeta* = *hurwitz-zeta 1*

We define the $\zeta$ functions as 0 at their poles. To avoid confusion, these facts are not added as simplification rules by default.

**lemma** *hurwitz-zeta-1*: *hurwitz-zeta c 1 = 0*
⟨*proof*⟩

**lemma** *zeta-1*: *zeta 1 = 0*
⟨*proof*⟩

**lemma** *zeta-minus-pole-eq*: *s* ≠ *1* ⟹ *zeta s* − *1* / (*s* − *1*) = *pre-zeta 1 s*
⟨*proof*⟩

**context**
**begin**

**private lemma** *holomorphic-pre-zeta-aux'*:
**assumes** *a* > *0 bounded U open U U* ⊆ {*s. Re s* > σ} **and** σ: σ > − *2* ∗ *real n*
**shows** *pre-zeta-aux n a holomorphic-on U* ⟨*proof*⟩

**lemma** *analytic-pre-zeta-aux*:
**assumes** *a* > *0*
**shows** *pre-zeta-aux n a analytic-on* {*s. Re s* > − *2* ∗ *real n*}
⟨*proof*⟩
**end**

**context**
**fixes** *s* :: *complex* **and** *N* :: *nat* **and** ζ :: *complex* ⇒ *complex* **and** *a* :: *real*
**assumes** *s*: *Re s* > *1* **and** *a*: *a* > *0*
**defines** ζ ≡ (λ*s*. ∑*n*. (*of-nat n* + *of-real a*) *powr* −*s*)
**begin**

**interpretation** ζ: *euler-maclaurin-nat'*
λ*x*. *of-real* (*x* + *a*) *powr* (*1* − *s*) / (*1* − *s*) λ*x*. *of-real* (*x* + *a*) *powr* −*s*
λ*n x*. (−*1*) ^ *n* ∗ *pochhammer s n* ∗ *of-real* (*x* + *a*) *powr* −(*s* + *n*)
*0 N* ζ *s* {}
⟨*proof*⟩

The pre-$\zeta$ functions agree with the infinite sum that is used to define the $\zeta$ function for $\mathfrak{R}(s) > 1$.

**lemma** *pre-zeta-aux-conv-zeta*:
*pre-zeta-aux N a s* = ζ *s* + *a powr* (*1* − *s*) / (*1* − *s*)
⟨*proof*⟩

**end**

Since all of the partial pre-$\zeta$ functions are analytic and agree in the halfspace with $\Re(s) > 0$, they must agree in their entire domain.

**lemma** *pre-zeta-aux-eq*:
**assumes** $m \leq n$ $a > 0$ $Re\ s > -2 * real\ m$
**shows** $pre\text{-}zeta\text{-}aux\ m\ a\ s = pre\text{-}zeta\text{-}aux\ n\ a\ s$
⟨*proof*⟩

**lemma** *pre-zeta-aux-eq'*:
**assumes** $a > 0$ $Re\ s > -2 * real\ m$ $Re\ s > -2 * real\ n$
**shows** $pre\text{-}zeta\text{-}aux\ m\ a\ s = pre\text{-}zeta\text{-}aux\ n\ a\ s$
⟨*proof*⟩

**lemma** *pre-zeta-aux-eq-pre-zeta*:
**assumes** $Re\ s > -2 * real\ n$ **and** $a > 0$
**shows** $pre\text{-}zeta\text{-}aux\ n\ a\ s = pre\text{-}zeta\ a\ s$
⟨*proof*⟩

This means that the idea of iterating that construction infinitely does yield a well-defined entire function.

**lemma** *analytic-pre-zeta*:
**assumes** $a > 0$
**shows** $pre\text{-}zeta\ a\ analytic\text{-}on\ A$
⟨*proof*⟩

**lemma** *holomorphic-pre-zeta* [*holomorphic-intros*]:
$f\ holomorphic\text{-}on\ A \Longrightarrow a > 0 \Longrightarrow (\lambda z.\ pre\text{-}zeta\ a\ (f\ z))\ holomorphic\text{-}on\ A$
⟨*proof*⟩

**corollary** *continuous-on-pre-zeta*:
$a > 0 \Longrightarrow continuous\text{-}on\ A\ (pre\text{-}zeta\ a)$
⟨*proof*⟩

**corollary** *continuous-on-pre-zeta'* [*continuous-intros*]:
$continuous\text{-}on\ A\ f \Longrightarrow a > 0 \Longrightarrow continuous\text{-}on\ A\ (\lambda x.\ pre\text{-}zeta\ a\ (f\ x))$
⟨*proof*⟩

**corollary** *continuous-pre-zeta* [*continuous-intros*]:
$a > 0 \Longrightarrow continuous\ (at\ s\ within\ A)\ (pre\text{-}zeta\ a)$
⟨*proof*⟩

**corollary** *continuous-pre-zeta'* [*continuous-intros*]:
$a > 0 \Longrightarrow continuous\ (at\ s\ within\ A)\ f \Longrightarrow$
$continuous\ (at\ s\ within\ A)\ (\lambda s.\ pre\text{-}zeta\ a\ (f\ s))$
⟨*proof*⟩

It is now obvious that $\zeta$ is holomorphic everywhere except 1, where it has a simple pole with residue 1, which we can simply read off.

**theorem** *holomorphic-hurwitz-zeta*:

**assumes** $a > 0$ $1 \notin A$
**shows** *hurwitz-zeta a holomorphic-on A*
⟨*proof*⟩

**corollary** *holomorphic-hurwitz-zeta′* [*holomorphic-intros*]:
**assumes** *f holomorphic-on A* **and** $a > 0$ **and** $\bigwedge z.\ z \in A \Longrightarrow f\ z \neq 1$
**shows** $(\lambda x.\ \textit{hurwitz-zeta}\ a\ (f\ x))$ *holomorphic-on A*
⟨*proof*⟩

**theorem** *holomorphic-zeta*: $1 \notin A \Longrightarrow$ *zeta holomorphic-on A*
⟨*proof*⟩

**corollary** *holomorphic-zeta′* [*holomorphic-intros*]:
**assumes** *f holomorphic-on A* **and** $\bigwedge z.\ z \in A \Longrightarrow f\ z \neq 1$
**shows** $(\lambda x.\ \textit{zeta}\ (f\ x))$ *holomorphic-on A*
⟨*proof*⟩

**corollary** *analytic-hurwitz-zeta*:
**assumes** $a > 0$ $1 \notin A$
**shows** *hurwitz-zeta a analytic-on A*
⟨*proof*⟩

**corollary** *analytic-zeta*: $1 \notin A \Longrightarrow$ *zeta analytic-on A*
⟨*proof*⟩

**corollary** *continuous-on-hurwitz-zeta*:
$a > 0 \Longrightarrow 1 \notin A \Longrightarrow$ *continuous-on A* (*hurwitz-zeta a*)
⟨*proof*⟩

**corollary** *continuous-on-hurwitz-zeta′* [*continuous-intros*]:
*continuous-on A f* $\Longrightarrow a > 0 \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 1) \Longrightarrow$
*continuous-on A* $(\lambda x.\ \textit{hurwitz-zeta}\ a\ (f\ x))$
⟨*proof*⟩

**corollary** *continuous-on-zeta*: $1 \notin A \Longrightarrow$ *continuous-on A zeta*
⟨*proof*⟩

**corollary** *continuous-on-zeta′* [*continuous-intros*]:
*continuous-on A f* $\Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 1) \Longrightarrow$
*continuous-on A* $(\lambda x.\ \textit{zeta}\ (f\ x))$
⟨*proof*⟩

**corollary** *continuous-hurwitz-zeta* [*continuous-intros*]:
$a > 0 \Longrightarrow s \neq 1 \Longrightarrow$ *continuous* (*at s within A*) (*hurwitz-zeta a*)
⟨*proof*⟩

**corollary** *continuous-hurwitz-zeta′* [*continuous-intros*]:
$a > 0 \Longrightarrow f\ s \neq 1 \Longrightarrow$ *continuous* (*at s within A*) $f \Longrightarrow$
*continuous* (*at s within A*) $(\lambda s.\ \textit{hurwitz-zeta}\ a\ (f\ s))$

⟨*proof*⟩

**corollary** *continuous-zeta* [*continuous-intros*]:
$s \neq 1 \Longrightarrow$ *continuous* (*at s within A*) *zeta*
⟨*proof*⟩

**corollary** *continuous-zeta'* [*continuous-intros*]:
$f\ s \neq 1 \Longrightarrow$ *continuous* (*at s within A*) $f \Longrightarrow$ *continuous* (*at s within A*) ($\lambda s.$ *zeta* ($f\ s$))
⟨*proof*⟩

**corollary** *field-differentiable-at-zeta*:
**assumes** $s \neq 1$
**shows** *zeta field-differentiable at s*
⟨*proof*⟩

**theorem** *is-pole-hurwitz-zeta*:
**assumes** $a > 0$
**shows** *is-pole* (*hurwitz-zeta a*) *1*
⟨*proof*⟩

**corollary** *is-pole-zeta*: *is-pole zeta 1*
⟨*proof*⟩

**theorem** *zorder-hurwitz-zeta*:
**assumes** $a > 0$
**shows** *zorder* (*hurwitz-zeta a*) $1 = -1$
⟨*proof*⟩

**corollary** *zorder-zeta*: *zorder zeta* $1 = -\ 1$
⟨*proof*⟩

**theorem** *residue-hurwitz-zeta*:
**assumes** $a > 0$
**shows** *residue* (*hurwitz-zeta a*) $1 = 1$
⟨*proof*⟩

**corollary** *residue-zeta*: *residue zeta* $1 = 1$
⟨*proof*⟩

**lemma** *zeta-bigo-at-1*: *zeta* $\in O$[*at 1 within A*]($\lambda x.\ 1\ /\ (x - 1)$)
⟨*proof*⟩

**theorem**
**assumes** $a > 0$ *Re* $s > 1$
**shows** *hurwitz-zeta-conv-suminf*: *hurwitz-zeta a s* = ($\sum n.$ (*of-nat n* + *of-real a*) *powr* $-s$)
**and** *sums-hurwitz-zeta*: ($\lambda n.$ (*of-nat n* + *of-real a*) *powr* $-s$) *sums hurwitz-zeta a s*

⟨*proof*⟩

**corollary**
**assumes** $Re\ s > 1$
**shows** *zeta-conv-suminf*: $zeta\ s = (\sum n.\ of\text{-}nat\ (Suc\ n)\ powr\ -s)$
**and** *sums-zeta*: $(\lambda n.\ of\text{-}nat\ (Suc\ n)\ powr\ -s)\ sums\ zeta\ s$
⟨*proof*⟩

**corollary**
**assumes** $n > 1$
**shows** *zeta-nat-conv-suminf*: $zeta\ (of\text{-}nat\ n) = (\sum k.\ 1\ /\ of\text{-}nat\ (Suc\ k)\ \widehat{}\ n)$
**and** *sums-zeta-nat*: $(\lambda k.\ 1\ /\ of\text{-}nat\ (Suc\ k)\ \widehat{}\ n)\ sums\ zeta\ (of\text{-}nat\ n)$
⟨*proof*⟩

**lemma** *pre-zeta-aux-cnj* [*simp*]:
**assumes** $a > 0$
**shows** $pre\text{-}zeta\text{-}aux\ n\ a\ (cnj\ z) = cnj\ (pre\text{-}zeta\text{-}aux\ n\ a\ z)$
⟨*proof*⟩

**lemma** *pre-zeta-cnj* [*simp*]: $a > 0 \Longrightarrow pre\text{-}zeta\ a\ (cnj\ z) = cnj\ (pre\text{-}zeta\ a\ z)$
⟨*proof*⟩

**lemma** *hurwitz-zeta-cnj* [*simp*]: $a > 0 \Longrightarrow hurwitz\text{-}zeta\ a\ (cnj\ z) = cnj\ (hurwitz\text{-}zeta\ a\ z)$
⟨*proof*⟩

**lemma** *zeta-cnj* [*simp*]: $zeta\ (cnj\ z) = cnj\ (zeta\ z)$
⟨*proof*⟩

**corollary** *hurwitz-zeta-real*: $a > 0 \Longrightarrow hurwitz\text{-}zeta\ a\ (of\text{-}real\ x) \in \mathbb{R}$
⟨*proof*⟩

**corollary** *zeta-real*: $zeta\ (of\text{-}real\ x) \in \mathbb{R}$
⟨*proof*⟩

**corollary** *zeta-real'*: $z \in \mathbb{R} \Longrightarrow zeta\ z \in \mathbb{R}$
⟨*proof*⟩

## 2.3 Connection to Dirichlet series

**lemma** *eval-fds-zeta*: $Re\ s > 1 \Longrightarrow eval\text{-}fds\ fds\text{-}zeta\ s = zeta\ s$
⟨*proof*⟩

**theorem** *euler-product-zeta*:
**assumes** $Re\ s > 1$
**shows** $(\lambda n.\ \prod p{\le}n.\ if\ prime\ p\ then\ inverse\ (1 - 1\ /\ of\text{-}nat\ p\ powr\ s)\ else\ 1) \longrightarrow zeta\ s$
⟨*proof*⟩

**corollary** *euler-product-zeta'*:
**assumes** *Re s > 1*
**shows** $(\lambda n.\ \prod p \mid prime\ p \wedge p \leq n.\ inverse\ (1 - 1\ /\ of\text{-}nat\ p\ powr\ s)) \longrightarrow zeta\ s$
⟨*proof*⟩

**theorem** *zeta-Re-gt-1-nonzero*: $Re\ s > 1 \Longrightarrow zeta\ s \neq 0$
⟨*proof*⟩

**theorem** *tendsto-zeta-Re-going-to-at-top*: $(zeta \longrightarrow 1)\ (Re\ going\text{-}to\ at\text{-}top)$
⟨*proof*⟩

**lemma** *conv-abscissa-zeta* [*simp*]: *conv-abscissa* (*fds-zeta* :: *complex fds*) = *1*
**and** *abs-conv-abscissa-zeta* [*simp*]: *abs-conv-abscissa* (*fds-zeta* :: *complex fds*) = *1*
⟨*proof*⟩

**theorem** *deriv-zeta-sums*:
**assumes** *s*: *Re s > 1*
**shows** $(\lambda n.\ -of\text{-}real\ (ln\ (real\ (Suc\ n)))\ /\ of\text{-}nat\ (Suc\ n)\ powr\ s)\ sums\ deriv\ zeta\ s$
⟨*proof*⟩

**theorem** *inverse-zeta-sums*:
**assumes** *s*: *Re s > 1*
**shows** $(\lambda n.\ moebius\text{-}mu\ (Suc\ n)\ /\ of\text{-}nat\ (Suc\ n)\ powr\ s)\ sums\ inverse\ (zeta\ s)$
⟨*proof*⟩

The following gives an extension of the $\zeta$ functions to the critical strip.

**lemma** *hurwitz-zeta-critical-strip*:
**fixes** *s* :: *complex* **and** *a* :: *real*
**defines** $S \equiv (\lambda n.\ \sum i{<}n.\ (of\text{-}nat\ i + a)\ powr - s)$
**defines** $I' \equiv (\lambda n.\ of\text{-}nat\ n\ powr\ (1 - s)\ /\ (1 - s))$
**assumes** $Re\ s > 0\ s \neq 1$ **and** $a > 0$
**shows** $(\lambda n.\ S\ n - I'\ n) \longrightarrow hurwitz\text{-}zeta\ a\ s$
⟨*proof*⟩

**lemma** *zeta-critical-strip*:
**fixes** *s* :: *complex* **and** *a* :: *real*
**defines** $S \equiv (\lambda n.\ \sum i{=}1..n.\ (of\text{-}nat\ i)\ powr - s)$
**defines** $I \equiv (\lambda n.\ of\text{-}nat\ n\ powr\ (1 - s)\ /\ (1 - s))$
**assumes** *s*: $Re\ s > 0\ s \neq 1$
**shows** $(\lambda n.\ S\ n - I\ n) \longrightarrow zeta\ s$
⟨*proof*⟩

## 2.4 The non-vanishing of $\zeta$ for $\mathfrak{R}(s) \geq 1$

This proof is based on a sketch by Newman [6], which was previously formalised in HOL Light by Harrison [5], albeit in a much more concrete and low-level style.

Our aim here is to reproduce Newman's proof idea cleanly and on the same high level of abstraction.

**theorem** *zeta-Re-ge-1-nonzero*:
**fixes** $s$ **assumes** $Re\ s \geq 1\ s \neq 1$
**shows** $zeta\ s \neq 0$
$\langle proof \rangle$

## 2.5 Special values of the $\zeta$ functions

**theorem** *hurwitz-zeta-neg-of-nat*:
**assumes** $a > 0$
**shows** $hurwitz\text{-}zeta\ a\ (-of\text{-}nat\ n) = -bernpoly\ (Suc\ n)\ a\ /\ of\text{-}nat\ (Suc\ n)$
$\langle proof \rangle$

**lemma** *hurwitz-zeta-0* [*simp*]: $a > 0 \Longrightarrow hurwitz\text{-}zeta\ a\ 0 = 1\ /\ 2 - a$
$\langle proof \rangle$

**lemma** *zeta-0* [*simp*]: $zeta\ 0 = -1\ /\ 2$
$\langle proof \rangle$

**theorem** *zeta-neg-of-nat*:
$zeta\ (-of\text{-}nat\ n) = -of\text{-}real\ (bernoulli'\ (Suc\ n))\ /\ of\text{-}nat\ (Suc\ n)$
$\langle proof \rangle$

**corollary** *zeta-trivial-zero*:
**assumes** $even\ n\ n \neq 0$
**shows** $zeta\ (-of\text{-}nat\ n) = 0$
$\langle proof \rangle$

**theorem** *zeta-even-nat*:
$zeta\ (2 * of\text{-}nat\ n) =$
$of\text{-}real\ ((-1)\ \hat{}\ Suc\ n * bernoulli\ (2 * n) * (2 * pi)\ \hat{}\ (2 * n)\ /\ (2 * fact\ (2 * n)))$
$\langle proof \rangle$

**corollary** *zeta-even-numeral*:
$zeta\ (numeral\ (Num.Bit0\ n)) = of\text{-}real$
$((-\ 1)\ \hat{}\ Suc\ (numeral\ n) * bernoulli\ (numeral\ (num.Bit0\ n)) *$
$(2 * pi)\ \hat{}\ numeral\ (num.Bit0\ n)\ /\ (2 * fact\ (numeral\ (num.Bit0\ n))))$ (**is** - $= ?rhs$)
$\langle proof \rangle$

**corollary** *zeta-neg-even-numeral* [*simp*]: $zeta\ (-numeral\ (Num.Bit0\ n)) = 0$
$\langle proof \rangle$

**corollary** *zeta-neg-numeral*:
$zeta\ (-numeral\ n) =$
$-of\text{-}real\ (bernoulli'\ (numeral\ (Num.inc\ n))\ /\ numeral\ (Num.inc\ n))$

⟨*proof*⟩

**corollary** *zeta-neg1*: *zeta* (−*1*) = − *1* / *12*
⟨*proof*⟩

**corollary** *zeta-neg3*: *zeta* (−*3*) = *1* / *120*
⟨*proof*⟩

**corollary** *zeta-neg5*: *zeta* (−*5*) = − *1* / *252*
⟨*proof*⟩

**corollary** *zeta-2*: *zeta 2* = *pi* ^ *2* / *6*
⟨*proof*⟩

**corollary** *zeta-4*: *zeta 4* = *pi* ^ *4* / *90*
⟨*proof*⟩

**corollary** *zeta-6*: *zeta 6* = *pi* ^ *6* / *945*
⟨*proof*⟩

**corollary** *zeta-8*: *zeta 8* = *pi* ^ *8* / *9450*
⟨*proof*⟩

## 2.6 Integral relation between Γ and ζ function

**lemma**
**assumes** *z*: *Re z* > *0* **and** *a*: *a* > *0*
**shows** *Gamma-hurwitz-zeta-aux-integral*:
*Gamma z* / (*of-nat n* + *of-real a*) *powr z* =
(∫ *s*∈{*0*<..}. (*s powr* (*z* − *1*) / *exp* ((*n*+*a*) ∗ *s*)) ∂*lebesgue*)
**and** *Gamma-hurwitz-zeta-aux-integrable*:
*set-integrable lebesgue* {*0*<..} (λ*s*. *s powr* (*z* − *1*) / *exp* ((*n*+*a*) ∗ *s*))
⟨*proof*⟩

**lemma**
**assumes** *x*: *x* > *0* **and** *a* > *0*
**shows** *Gamma-hurwitz-zeta-aux-integral-real*:
*Gamma x* / (*real n* + *a*) *powr x* =
*set-lebesgue-integral lebesgue* {*0*<..}
(λ*s*. *s powr* (*x* − *1*) / *exp* ((*real n* + *a*) ∗ *s*))
**and** *Gamma-hurwitz-zeta-aux-integrable-real*:
*set-integrable lebesgue* {*0*<..} (λ*s*. *s powr* (*x* − *1*) / *exp* ((*real n* + *a*) ∗
*s*))
⟨*proof*⟩

**theorem**
**assumes** *Re z* > *1* **and** *a* > (*0*::*real*)
**shows** *Gamma-times-hurwitz-zeta-integral*: *Gamma z* ∗ *hurwitz-zeta a z* =
(∫ *x*∈{*0*<..}. (*of-real x powr* (*z* − *1*) ∗ *of-real* (*exp* (−*a*∗*x*) / (*1* − *exp*

$(-x)))) \partial lebesgue)$
**and** *Gamma-times-hurwitz-zeta-integrable*:
*set-integrable lebesgue* $\{0<..\}$
$(\lambda x.\ \mathit{of\text{-}real}\ x\ \mathit{powr}\ (z - 1) * \mathit{of\text{-}real}\ (\mathit{exp}\ (-a*x)\ /\ (1 - \mathit{exp}\ (-x))))$
⟨*proof*⟩

**corollary**
**assumes** *Re* $z > 1$
**shows** *Gamma-times-zeta-integral*: *Gamma* $z *$ *zeta* $z =$
$(\int x\in\{0<..\}.\ (\mathit{of\text{-}real}\ x\ \mathit{powr}\ (z - 1)\ /\ \mathit{of\text{-}real}\ (\mathit{exp}\ x - 1))\ \partial lebesgue)$
(**is** *?th1*)
**and** *Gamma-times-zeta-integrable*:
*set-integrable lebesgue* $\{0<..\}$
$(\lambda x.\ \mathit{of\text{-}real}\ x\ \mathit{powr}\ (z - 1)\ /\ \mathit{of\text{-}real}\ (\mathit{exp}\ x - 1))$ (**is** *?th2*)
⟨*proof*⟩

**corollary** *hurwitz-zeta-integral-Gamma-def*:
**assumes** *Re* $z > 1$ $a > 0$
**shows** *hurwitz-zeta* $a\ z =$
*rGamma* $z * (\int x\in\{0<..\}.\ (\mathit{of\text{-}real}\ x\ \mathit{powr}\ (z - 1) *$
$\mathit{of\text{-}real}\ (\mathit{exp}\ (-a * x)\ /\ (1 - \mathit{exp}\ (-x))))\ \partial lebesgue)$
⟨*proof*⟩

**corollary** *zeta-integral-Gamma-def*:
**assumes** *Re* $z > 1$
**shows** *zeta* $z =$
*rGamma* $z * (\int x\in\{0<..\}.\ (\mathit{of\text{-}real}\ x\ \mathit{powr}\ (z - 1)\ /\ \mathit{of\text{-}real}\ (\mathit{exp}\ x - 1))\ \partial lebesgue)$
⟨*proof*⟩

**lemma** *Gamma-times-zeta-has-integral*:
**assumes** *Re* $z > 1$
**shows** $((\lambda x.\ x\ \mathit{powr}\ (z - 1)\ /\ (\mathit{of\text{-}real}\ (\mathit{exp}\ x) - 1))$ *has-integral* (*Gamma* $z *$ *zeta* $z$)) $\{0<..\}$
(**is** (*?f has-integral* -) -)
⟨*proof*⟩

**lemma** *Gamma-times-zeta-has-integral-real*:
**fixes** $z$ :: *real*
**assumes** $z > 1$
**shows** $((\lambda x.\ x\ \mathit{powr}\ (z - 1)\ /\ (\mathit{exp}\ x - 1))$ *has-integral* (*Gamma* $z *$ *Re* (*zeta* $z$))) $\{0<..\}$
⟨*proof*⟩

**lemma** *Gamma-integral-real'*:
**assumes** $x$: $x > (0$ :: *real*)
**shows** $((\lambda t.\ t\ \mathit{powr}\ (x - 1)\ /\ \mathit{exp}\ t)$ *has-integral Gamma* $x$) $\{0<..\}$
⟨*proof*⟩

## 2.7 An analytic proof of the infinitude of primes

We can now also do an analytic proof of the infinitude of primes.

**lemma** *primes-infinite-analytic*: *infinite* $\{p :: nat.\ prime\ p\}$
$\langle proof \rangle$

## 2.8 The periodic zeta function

The periodic zeta function $F(s, q)$ (as described e. g. by Apostol [1] is related to the Hurwitz zeta function. It is periodic in $q$ with period 1 and it can be represented by a Dirichlet series that is absolutely convergent for $\mathfrak{R}(s) > 1$. If $q \notin \mathbb{Z}$, it furthermore convergent for $\mathfrak{R}(s) > 0$.

It is clear that for integer $q$, we have $F(s, q) = F(s, 0) = \zeta(s)$. Moreover, for non-integer $q$, $F(s, q)$ can be analytically continued to an entire function.

**definition** *fds-perzeta* :: *real* ⇒ *complex fds* **where**
*fds-perzeta q* = *fds* (λ*m. exp* (*2* ∗ *pi* ∗ i ∗ *m* ∗ *q*))

The definition of the periodic zeta function on the full domain is a bit unwieldy. The precise reasoning for this definition will be given later, and, in any case, it is probably more instructive to look at the derived "alternative" definitions later.

**definition** *perzeta* :: *real* ⇒ *complex* ⇒ *complex* **where**
*perzeta q′ s* =
(*if q′* ∈ ℤ *then zeta s*
*else let q* = *frac q′ in*
*if s* = *0 then* i / (*2* ∗ *pi*) ∗ (*pre-zeta q 1* − *pre-zeta* (*1* − *q*) *1* +
*ln* (*1* − *q*) − *ln q* + *pi* ∗ i)
*else if s* ∈ ℕ *then eval-fds* (*fds-perzeta q*) *s*
*else complex-of-real* (*2* ∗ *pi*) *powr* (*s* − *1*) ∗ i ∗ *Gamma* (*1* − *s*) ∗
(i *powr* (−*s*) ∗ *hurwitz-zeta q* (*1* − *s*) −
i *powr s* ∗ *hurwitz-zeta* (*1* − *q*) (*1* − *s*)))

**interpretation** *fds-perzeta*: *periodic-fun-simple′ fds-perzeta*
$\langle proof \rangle$

**interpretation** *perzeta*: *periodic-fun-simple′ perzeta*
$\langle proof \rangle$

**lemma** *perzeta-frac* [*simp*]: *perzeta* (*frac q*) = *perzeta q*
$\langle proof \rangle$

**lemma** *fds-perzeta-frac* [*simp*]: *fds-perzeta* (*frac q*) = *fds-perzeta q*
$\langle proof \rangle$

**lemma** *abs-conv-abscissa-perzeta*: *abs-conv-abscissa* (*fds-perzeta q*) ≤ *1*
$\langle proof \rangle$

**lemma** *conv-abscissa-perzeta*: *conv-abscissa* (*fds-perzeta q*) $\leq$ *1*
⟨*proof*⟩

**lemma** *fds-perzeta--left-0* [*simp*]: *fds-perzeta 0* = *fds-zeta*
⟨*proof*⟩

**lemma** *perzeta-0-left* [*simp*]: *perzeta 0 s* = *zeta s*
⟨*proof*⟩

**lemma** *perzeta-int*: $q \in \mathbb{Z} \Longrightarrow$ *perzeta q* = *zeta*
⟨*proof*⟩

**lemma** *fds-perzeta-int*: $q \in \mathbb{Z} \Longrightarrow$ *fds-perzeta q* = *fds-zeta*
⟨*proof*⟩

**lemma** *sums-fds-perzeta*:
**assumes** *Re s* > *1*
**shows** ($\lambda m$. *exp* (*2* ∗ *pi* ∗ i ∗ *Suc m* ∗ *q*) / *of-nat* (*Suc m*) *powr s*) *sums* *eval-fds* (*fds-perzeta q*) *s*
⟨*proof*⟩

**lemma** *sum-tendsto-fds-perzeta*:
**assumes** *Re s* > *1*
**shows** ($\lambda n$. $\sum$ *k*∈{*0*<..*n*}. *exp* (*2* ∗ *real k* ∗ *pi* ∗ *q* ∗ i) ∗ *of-nat k powr* − *s*) $\longrightarrow$ *eval-fds* (*fds-perzeta q*) *s*
⟨*proof*⟩

Using the geometric series, it is easy to see that the Dirichlet series for $F(s,q)$ has bounded partial sums for non-integer $q$, so it must converge for any $s$ with $\Re(s) > 0$.

**lemma** *conv-abscissa-perzeta'*:
**assumes** $q \notin \mathbb{Z}$
**shows** *conv-abscissa* (*fds-perzeta q*) $\leq$ *0*
⟨*proof*⟩

**lemma** *fds-perzeta-one-half*: *fds-perzeta* (*1* / *2*) = *fds* ($\lambda n$. (−*1*) ^ *n*)
⟨*proof*⟩

**lemma** *perzeta-one-half-1* [*simp*]: *perzeta* (*1* / *2*) *1* = −*ln 2*
⟨*proof*⟩

## 2.9 Hurwitz's formula

We now move on to prove Hurwitz's formula relating the Hurwitz zeta function and the periodic zeta function. We mostly follow Apostol's proof, although we do make some small changes in order to make the proof more amenable to Isabelle's complex analysis library.

The big difference is that Apostol integrates along a circle with a slit, where the two sides of the slit lie on different branches of the integrand. This makes sense when looking as the integrand as a Riemann surface, but we do not have a notion of Riemann surfaces in Isabelle.

It is therefore much easier to simply cut the circle into an upper and a lower half. In fact, the integral on the lower half can be reduced to the one on the upper half easily by symmetry, so we really only need to handle the integral on the upper half. The integration contour that we will use is therefore a semi-annulus in the upper half of the complex plane, centred around the origin.

Now, first of all, we prove the existence of an important improper integral that we will need later.

**lemma** *set-integrable-bigo*:
  **fixes** $f\ g$ :: $real \Rightarrow 'a$ :: $\{banach,\ real\text{-}normed\text{-}field,\ second\text{-}countable\text{-}topology\}$
  **assumes** $f \in O(\lambda x.\ g\ x)$ **and** *set-integrable lborel* $\{a..\}\ g$
  **assumes** $\bigwedge b.\ b \geq a \Longrightarrow$ *set-integrable lborel* $\{a..{<}b\}\ f$
  **assumes** [*measurable*]: *set-borel-measurable borel* $\{a..\}\ f$
  **shows** *set-integrable lborel* $\{a..\}\ f$
⟨*proof*⟩

**lemma** *set-integrable-Gamma-hurwitz-aux2-real*:
  **fixes** $s\ a$ :: *real*
  **assumes** $r > 0$ **and** $a > 0$
  **shows** *set-integrable lborel* $\{r..\}\ (\lambda x.\ x\ powr\ s * (exp\ (-a * x))\ /\ (1 - exp\ (-x)))$
    (**is** *set-integrable - - ?g*)
⟨*proof*⟩

**lemma** *set-integrable-Gamma-hurwitz-aux2*:
  **fixes** $s$ :: *complex* **and** $a$ :: *real*
  **assumes** $r > 0$ **and** $a > 0$
  **shows** *set-integrable lborel* $\{r..\}\ (\lambda x.\ x\ powr\ -s * (exp\ (-\ a * x))\ /\ (1 - exp\ (-\ x)))$
    (**is** *set-integrable - - ?g*)
⟨*proof*⟩

**lemma** *closed-neg-Im-slit*: *closed* $\{z.\ Re\ z = 0 \wedge Im\ z \leq 0\}$
⟨*proof*⟩

We define tour semi-annulus path. When this path is mirrored into the lower half of the complex plane and subtracted from the original path and the outer radius tends to $\infty$, this becomes a Hankel contour extending to $-\infty$.

**definition** *hankel-semiannulus* :: $real \Rightarrow nat \Rightarrow real \Rightarrow complex$ **where**
 *hankel-semiannulus* $r\ N = (let\ R = (2 * N + 1) * pi\ in$
   *part-circlepath* $0\ R\ 0\ pi$ +++ — Big half circle
   *linepath* (*of-real* $(-R)$) (*of-real* $(-r)$) +++ — Line on the negative real axis
   *part-circlepath* $0\ r\ pi\ 0$ +++ — Small half circle
   *linepath* (*of-real* $r$) (*of-real* $R$)) — Line on the positive real axis

**lemma** *path-hankel-semiannulus* [*simp*, *intro*]: *path* (*hankel-semiannulus r R*)
**and** *valid-path-hankel-semiannulus* [*simp*, *intro*]: *valid-path* (*hankel-semiannulus r R*)
**and** *pathfinish-hankel-semiannulus* [*simp*, *intro*]:
*pathfinish* (*hankel-semiannulus r R*) = *pathstart* (*hankel-semiannulus r R*)
⟨*proof*⟩

We set the stage for an application of the Residue Theorem. We define a function

$$f(s,z) = z^{-s} \frac{\exp(az)}{1 - \exp(-z)} \ ,$$

which will be the integrand. However, the principal branch of $z^{-s}$ has a branch cut along the non-positive real axis, which is bad because a part of our integration path also lies on the non-positive real axis. We therefore choose a slightly different branch of $z^{-s}$ by moving the logarithm branch along by 90° so that the branch cut lies on the non-positive imaginary axis instead.

**context**
**fixes** *a* :: *real*
**fixes** *f* :: *complex* ⇒ *complex* ⇒ *complex*
**and** *g* :: *complex* ⇒ *real* ⇒ *complex*
**and** *h* :: *real* ⇒ *complex* ⇒ *real* ⇒ *complex*
**and** *Res* :: *complex* ⇒ *nat* ⇒ *complex*
**and** *Ln'* :: *complex* ⇒ *complex*
**and** *F* :: *real* ⇒ *complex* ⇒ *complex*
**assumes** *a*: *a* ∈ {*0*<..*1*}

— Our custom branch of the logarithm
**defines** *Ln'* ≡ (λ*z*. *ln* (−i ∗ *z*) + i ∗ *pi* / *2*)

— The integrand
**defines** *f* ≡ (λ*s z*. *exp* (*Ln' z* ∗ (−*s*) + *of-real a* ∗ *z*) / (*1* − *exp z*))

— The integrand on the negative real axis
**defines** *g* ≡ (λ*s x*. *complex-of-real x powr* −*s* ∗ *of-real* (*exp* (−*a*∗*x*)) / *of-real* (*1* − *exp* (−*x*)))

— The integrand on the circular arcs
**defines** *h* ≡ (λ*r s t*. *r* ∗ i ∗ *cis t* ∗ *exp* (*a* ∗ (*r* ∗ *cis t*) − (*ln r* + i ∗ *t*) ∗ *s*) /
(*1* − *exp* (*r* ∗ *cis t*)))

— The interesting part of the residues
**defines** *Res* ≡ (λ*s k*. *exp* (*of-real* (*2* ∗ *real k* ∗ *pi* ∗ *a*) ∗ i) ∗
*of-real* (*2* ∗ *real k* ∗ *pi*) *powr* (−*s*))

— The periodic zeta function (at least on $\Re(s) > 1$ half-plane)
**defines** *F* ≡ (λ*q*. *eval-fds* (*fds-perzeta q*))

**begin**

First, some basic properties of our custom branch of the logarithm:

**lemma** *Ln′-i*: $Ln'$ i = i $*$ *pi* / *2*
⟨*proof*⟩

**lemma** *Ln′-of-real-pos*:
**assumes** $x > 0$
**shows** $Ln'$ (*of-real* $x$) = *of-real* (*ln* $x$)
⟨*proof*⟩

**lemma** *Ln′-of-real-neg*:
**assumes** $x < 0$
**shows** $Ln'$ (*of-real* $x$) = *of-real* (*ln* $(-x)$) + i $*$ *pi*
⟨*proof*⟩

**lemma** *Ln′-times-of-real*:
$Ln'$ (*of-real* $x * z$) = *of-real* (*ln* $x$) + $Ln'$ $z$ **if** $x > 0$ $z \neq 0$ **for** $z$ $x$
⟨*proof*⟩

**lemma** *Ln′-cis*:
**assumes** $t \in \{-pi\ /\ 2<..3\ /\ 2 * pi\}$
**shows** $Ln'$ (*cis* $t$) = i $*$ $t$
⟨*proof*⟩

Next, we show that the line and circle integrals are holomorphic using Leibniz's rule:

**lemma** *contour-integral-part-circlepath-h*:
**assumes** $r$: $r > 0$
**shows** *contour-integral* (*part-circlepath* *0* $r$ *0* *pi*) ($f$ $s$) = *integral* {*0*..*pi*} ($h$ $r$ $s$)
⟨*proof*⟩

**lemma** *integral-g-holomorphic*:
**assumes** $b > 0$
**shows** ($\lambda s.$ *integral* {$b$..$c$} ($g$ $s$)) *holomorphic-on* $A$
⟨*proof*⟩

**lemma** *integral-h-holomorphic*:
**assumes** $r$: $r \in \{0<..<2\}$
**shows** ($\lambda s.$ *integral* {$b$..$c$} ($h$ $r$ $s$)) *holomorphic-on* $A$
⟨*proof*⟩

We now move on to the core result, which uses the Residue Theorem to relate a contour integral along a semi-annulus to a partial sum of the periodic zeta function.

**lemma** *hurwitz-formula-integral-semiannulus*:
**fixes** $N$ :: *nat* **and** $r$ :: *real* **and** $s$ :: *complex*
**defines** $R \equiv$ *real* $(2 * N + 1) * pi$

**assumes** $r > 0$ **and** $r < 2$
**shows** $exp\ (-\mathrm{i} * pi * s) * integral\ \{r..R\}\ (\lambda x.\ x\ powr\ (-s) * exp\ (-a * x)\ /\ (1 - exp\ (-x))) +$
$integral\ \{r..R\}\ (\lambda x.\ x\ powr\ (-s) * exp\ (a * x)\ /\ (1 - exp\ x)) +$
$contour\text{-}integral\ (part\text{-}circlepath\ 0\ R\ 0\ pi)\ (f\ s) +$
$contour\text{-}integral\ (part\text{-}circlepath\ 0\ r\ pi\ 0)\ (f\ s)$
$= -2 * pi * \mathrm{i} * exp\ (-\ s * of\text{-}real\ pi * \mathrm{i}\ /\ 2) * (\sum k \in \{0<..N\}.\ Res\ s\ k)$
(**is** *?thesis1*)
**and** $f\ s$ *contour-integrable-on hankel-semiannulus r N*
⟨*proof*⟩

Next, we need bounds on the integrands of the two semicircles.

**lemma** *hurwitz-formula-bound1*:
**defines** $H \equiv \lambda z.\ exp\ (complex\text{-}of\text{-}real\ a * z)\ /\ (1 - exp\ z)$
**assumes** $r > 0$
**obtains** $C$ **where** $C \geq 0$ **and** $\bigwedge z.\ z \notin (\bigcup n{::}int.\ ball\ (2 * n * pi * \mathrm{i})\ r) \Longrightarrow norm\ (H\ z) \leq C$
⟨*proof*⟩

**lemma** *hurwitz-formula-bound2*:
**obtains** $C$ **where** $C \geq 0$ **and** $\bigwedge r\ z.\ r > 0 \Longrightarrow r < pi \Longrightarrow z \in sphere\ 0\ r \Longrightarrow$
$norm\ (f\ s\ z) \leq C * r\ powr\ (-Re\ s - 1)$
⟨*proof*⟩

We can now relate the integral along a partial Hankel contour that is cut off at $-\pi$ to $\zeta(1-s,a)/\Gamma(s)$.

**lemma** *rGamma-hurwitz-zeta-eq-contour-integral*:
**fixes** $s$ :: *complex* **and** $r$ :: *real*
**assumes** $s \neq 0$ **and** $r$: $r \in \{1..<2\}$ **and** $a$: $a > 0$
**defines** $err1 \equiv (\lambda s\ r.\ contour\text{-}integral\ (part\text{-}circlepath\ 0\ r\ pi\ 0)\ (f\ s))$
**defines** $err2 \equiv (\lambda s\ r.\ cnj\ (contour\text{-}integral\ (part\text{-}circlepath\ 0\ r\ pi\ 0)\ (f\ (cnj\ s))))$
**shows** $2 * \mathrm{i} * pi * rGamma\ s * hurwitz\text{-}zeta\ a\ (1 - s) =$
$err2\ s\ r - err1\ s\ r + 2 * \mathrm{i} * sin\ (pi * s) * (CLBINT\ x{:}\{r..\}.\ g\ s\ x)$
(**is** $?f\ s = ?g\ s$)
⟨*proof*⟩

Finally, we obtain Hurwitz's formula by letting the radius of the outer circle tend to $\infty$.

**lemma** *hurwitz-zeta-formula-aux*:
**fixes** $s$ :: *complex*
**assumes** $s$: $Re\ s > 1$
**shows** $rGamma\ s * hurwitz\text{-}zeta\ a\ (1 - s) = (2 * pi)\ powr\ -s *$
$(\mathrm{i}\ powr\ (-s) * F\ a\ s + \mathrm{i}\ powr\ s * F\ (-a)\ s)$
⟨*proof*⟩

**end**

We can now use Hurwitz's formula to prove the following nice formula that expresses the periodic zeta function in terms of the Hurwitz zeta function:

$$F(s,a) = (2\pi)^{s-1} i\Gamma(1-s)\left(i^{-s}\zeta(1-s,a) - i^{s}\zeta(1-s,1-a)\right)$$

This holds for all $s$ with \mathfrak{R}(s) > 0 as long as $a \notin \mathbb{Z}$. For convenience, we move the $\Gamma$ function to the left-hand side in order to avoid having to account for its poles.

**lemma** *perzeta-conv-hurwitz-zeta-aux*:
**fixes** *a* :: *real* **and** *s* :: *complex*
**assumes** *a*: *a* ∈ {*0*<..<*1*} **and** *s*: *Re s* > *0*
**shows** *rGamma* (*1* − *s*) ∗ *eval-fds* (*fds-perzeta a*) *s* = (*2* ∗ *pi*) *powr* (*s* − *1*) ∗ i ∗
(i *powr* −*s* ∗ *hurwitz-zeta a* (*1* − *s*) −
i *powr s* ∗ *hurwitz-zeta* (*1* − *a*) (*1* − *s*))
(**is** *?lhs s* = *?rhs s*)
⟨*proof*⟩

We can now use the above equation as a defining equation to continue the periodic zeta function $F$ to the entire complex plane except at non-negative integer values for $s$. However, the positive integers are already covered by the original Dirichlet series definition of $F$, so we only need to take care of $s = 0$. We do this by cancelling the pole of $\Gamma$ at $0$ with the zero of $i^{-s}\zeta(1-s,a) - i^{s}\zeta(1-s,1-a)$.

**lemma**
**assumes** *q'* ∉ ℤ
**shows** *holomorphic-perzeta'*: *perzeta q' holomorphic-on A*
**and** *perzeta-altdef2*: *Re s* > *0* ⟹ *perzeta q' s* = *eval-fds* (*fds-perzeta q'*) *s*
⟨*proof*⟩

**lemma** *perzeta-altdef1*: *Re s* > *1* ⟹ *perzeta q' s* = *eval-fds* (*fds-perzeta q'*) *s*
⟨*proof*⟩

**lemma** *holomorphic-perzeta*: *q* ∉ ℤ ∨ *1* ∉ *A* ⟹ *perzeta q holomorphic-on A*
⟨*proof*⟩

**lemma** *holomorphic-perzeta''* [*holomorphic-intros*]:
**assumes** *f holomorphic-on A* **and** *q* ∉ ℤ ∨ (∀ *x*∈*A*. *f x* ≠ *1*)
**shows** (λ*x*. *perzeta q* (*f x*)) *holomorphic-on A*
⟨*proof*⟩

Using this analytic continuation of the periodic zeta function, Hurwitz's formula now holds (almost) on the entire complex plane.

**theorem** *hurwitz-zeta-formula*:
**fixes** *a* :: *real* **and** *s* :: *complex*
**assumes** *a* ∈ {*0*<..*1*} **and** *s* ≠ *0* **and** *a* ≠ *1* ∨ *s* ≠ *1*
**shows** *rGamma s* ∗ *hurwitz-zeta a* (*1* − *s*) =
(*2* ∗ *pi*) *powr* − *s* ∗ (i *powr* − *s* ∗ *perzeta a s* + i *powr s* ∗ *perzeta* (−*a*) *s*)

(**is** $?f\ s = ?g\ s$)
⟨*proof*⟩

The equation expressing the periodic zeta function in terms of the Hurwitz zeta function can be extened similarly.

**theorem** *perzeta-conv-hurwitz-zeta*:
**fixes** $a$ :: *real* **and** $s$ :: *complex*
**assumes** $a \in \{0<..<1\}$ **and** $s \neq 0$
**shows** $rGamma\ (1 - s) * perzeta\ a\ s =$
$(2 * pi)\ powr\ (s - 1) * \mathrm{i} * (\mathrm{i}\ powr\ (-s) * hurwitz\text{-}zeta\ a\ (1 - s) -$
$\mathrm{i}\ powr\ s * hurwitz\text{-}zeta\ (1 - a)\ (1 - s))$
(**is** $?f\ s = ?g\ s$)
⟨*proof*⟩

As a simple corollary, we derive the reflection formula for the Riemann zeta function:

**corollary** *zeta-reflect*:
**fixes** $s$ :: *complex*
**assumes** $s \neq 0$ $s \neq 1$
**shows** $rGamma\ s * zeta\ (1 - s) = 2 * (2 * pi)\ powr\ -s * cos\ (s * pi\ /\ 2) * zeta\ s$
⟨*proof*⟩

**corollary** *zeta-reflect'*:
**fixes** $s$ :: *complex*
**assumes** $s \neq 0$ $s \neq 1$
**shows** $rGamma\ (1 - s) * zeta\ s = 2 * (2 * pi)\ powr\ (s - 1) * sin\ (s * pi\ /\ 2) * zeta\ (1 - s)$
⟨*proof*⟩

It is now easy to see that all the non-trivial zeroes of the Riemann zeta function must lie the critical strip $(0; 1)$, and they must be symmetric around the $\Re(z) = \frac{1}{2}$ line.

**corollary** *zeta-zeroD*:
**assumes** $zeta\ s = 0$ $s \neq 1$
**shows** $Re\ s \in \{0<..<1\} \vee (\exists\, n::nat.\ n > 0 \wedge even\ n \wedge s = -real\ n)$
⟨*proof*⟩

**lemma** *zeta-zero-reflect*:
**assumes** $Re\ s \in \{0<..<1\}$ **and** $zeta\ s = 0$
**shows** $zeta\ (1 - s) = 0$
⟨*proof*⟩

**corollary** *zeta-zero-reflect-iff*:
**assumes** $Re\ s \in \{0<..<1\}$
**shows** $zeta\ (1 - s) = 0 \longleftrightarrow zeta\ s = 0$
⟨*proof*⟩

## 2.10 More functional equations

**lemma** *perzeta-conv-hurwitz-zeta-multiplication*:
  **fixes** $k$ :: *nat* **and** $a$ :: *int* **and** $s$ :: *complex*
  **assumes** $k > 0$ $s \neq 1$
  **shows** $k$ *powr* $s * perzeta\ (a\ /\ k)\ s =$
    $(\sum n{=}1..k.\ exp\ (2 * pi * n * a\ /\ k * \mathrm{i}) * hurwitz\text{-}zeta\ (n\ /\ k)\ s)$
  (**is** *?lhs s = ?rhs s*)
⟨*proof*⟩

**lemma** *perzeta-conv-hurwitz-zeta-multiplication'*:
  **fixes** $k$ :: *nat* **and** $a$ :: *int* **and** $s$ :: *complex*
  **assumes** $k > 0$ $s \neq 1$
  **shows** $perzeta\ (a\ /\ k)\ s = k\ powr\ {-s} *$
    $(\sum n{=}1..k.\ exp\ (2 * pi * n * a\ /\ k * \mathrm{i}) * hurwitz\text{-}zeta\ (n\ /\ k)\ s)$
  ⟨*proof*⟩

**lemma** *zeta-conv-hurwitz-zeta-multiplication*:
  **fixes** $k$ $a$ :: *nat* **and** $s$ :: *complex*
  **assumes** $k > 0$ $s \neq 1$
  **shows** $k\ powr\ s * zeta\ s = (\sum n{=}1..k.\ hurwitz\text{-}zeta\ (n\ /\ k)\ s)$
  ⟨*proof*⟩

**lemma** *hurwitz-zeta-one-half-left*:
  **assumes** $s \neq 1$
  **shows** $hurwitz\text{-}zeta\ (1\ /\ 2)\ s = (2\ powr\ s - 1) * zeta\ s$
  ⟨*proof*⟩

**theorem** *hurwitz-zeta-functional-equation*:
  **fixes** $h$ $k$ :: *nat* **and** $s$ :: *complex*
  **assumes** *hk*: $k > 0$ $h \in \{0{<}..k\}$ **and** *s*: $s \notin \{0,\ 1\}$
  **defines** $a \equiv real\ h\ /\ real\ k$
  **shows** $rGamma\ s * hurwitz\text{-}zeta\ a\ (1 - s) =$
    $2 * (2 * pi * k)\ powr\ {-s} *$
    $(\sum n{=}1..k.\ cos\ (s{*}pi/2 - 2{*}pi{*}n{*}h/k) * hurwitz\text{-}zeta\ (n\ /\ k)\ s)$
⟨*proof*⟩

**lemma** *perzeta-one-half-left*: $s \neq 1 \Longrightarrow perzeta\ (1\ /\ 2)\ s = (2\ powr\ (1{-}s) - 1)$
$* zeta\ s$
  ⟨*proof*⟩

**lemma** *perzeta-one-half-left'*:
  $perzeta\ (1\ /\ 2)\ s =$
    $(if\ s = 1\ then\ {-ln}\ 2\ else\ (2\ powr\ (1 - s) - 1)\ /\ (s - 1)) * ((s - 1) *$
$pre\text{-}zeta\ 1\ s + 1)$
  ⟨*proof*⟩

**end**

# 3 The Laurent series expansion of $\zeta$ at 1

**theory** *Zeta-Laurent-Expansion*
  **imports** *Zeta-Function*
**begin**

In this section, we shall derive the Laurent series expansion of $\zeta(s)$ at $s = 1$, which is of the form

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n$$

where the $\gamma_n$ are the *Stieltjes constants*. Notably, $\gamma_0$ is equal to the Euler–Mascheroni constant $\gamma$.

## 3.1 Definition of the Stieltjes constants

We define the Stieltjes constants by their infinite series form, since it is fairly easy to show the convergence of the series by the comparison test.

**definition** *stieltjes-gamma* :: *nat* ⇒ *'a* :: *real-algebra-1* **where**
  *stieltjes-gamma n* =
    *of-real* (∑*k*. *ln* (*k*+*1*) ^ *n* / (*k*+*1*) − (*ln* (*k*+*2*) ^(*n*+*1*) − *ln* (*k*+*1*) ^(*n* + *1*)) / (*n* + *1*))

**lemma** *stieltjes-gamma-0* [*simp*]: *stieltjes-gamma 0* = *euler-mascheroni*
  ⟨*proof*⟩

**lemma** *stieltjes-gamma-summable*:
  *summable* (λ*k*. *ln* (*k*+*1*) ^ *n* / (*k*+*1*) − (*ln* (*k*+*2*) ^(*n*+*1*) − *ln* (*k*+*1*) ^(*n* + *1*)) / (*n* + *1*))
    (**is** *summable ?f*)
⟨*proof*⟩

**lemma** *of-real-stieltjes-gamma* [*simp*]: *of-real* (*stieltjes-gamma k*) = *stieltjes-gamma k*
  ⟨*proof*⟩

**lemma** *sums-stieltjes-gamma*:
  (λ*k*. *ln* (*k*+*1*) ^ *n* / (*k*+*1*) − (*ln* (*k*+*2*) ^(*n*+*1*) − *ln* (*k*+*1*) ^(*n* + *1*)) / (*n* + *1*))
    *sums stieltjes-gamma n*
  ⟨*proof*⟩

We can now derive the alternative definition of the Stieltjes constants as a limit. This limit can also be written in the Euler–MacLaurin-style form

$$\lim_{m\to\infty} \left( \sum_{k=1}^{m} \frac{\ln^n k}{k} - \int_1^m \frac{\ln^n x}{x}\,\mathrm{d}x \right) ,$$

which is perhaps a bit more illuminating.

**lemma** *stieltjes-gamma-real-limit-form*:
$(\lambda m.\ (\sum k = 1..m.\ ln\ (real\ k)\ \hat{}\ n\ /\ real\ k) - ln\ (real\ m)\ \hat{}\ (n + 1)\ /\ real\ (n + 1))$
$\longrightarrow$ *stieltjes-gamma n*
⟨*proof*⟩

**lemma** *stieltjes-gamma-limit-form*:
$(\lambda m.\ of\text{-}real\ ((\sum k=1..m.\ ln\ (real\ k)\ \hat{}\ n\ /\ real\ k) - ln\ (real\ m)\ \hat{}\ (n + 1)\ /\ real\ (n + 1)))$
$\longrightarrow$ $(stieltjes\text{-}gamma\ n :: 'a :: real\text{-}normed\text{-}algebra\text{-}1)$
⟨*proof*⟩

**lemma** *stieltjes-gamma-real-altdef*:
$(stieltjes\text{-}gamma\ n :: real) =$
$lim\ (\lambda m.\ (\sum k = 1..m.\ ln\ (real\ k)\ \hat{}\ n\ /\ real\ k) -$
$ln\ (real\ m)\ \hat{}\ (n + 1)\ /\ real\ (n + 1))$
⟨*proof*⟩

## 3.2 Proof of the Laurent expansion

We shall follow the proof by Briggs and Chowla [2], which examines the entire function $g(s) = (2^{1-s} - 1)\zeta(s)$. They determine the value of $g^{(k)}(1)$ in two different ways: First by the Dirichlet series of $g$ and then by its power series expansion around 1. We shall do the same here.

**context**
**fixes** *g* **and** *G1 G2 G2' G* :: *complex fps* **and** *A* :: *nat* ⇒ *complex*
**defines** *g* ≡ *perzeta (1 / 2)*
**defines** *G1* ≡ *fps-shift 1 (fps-exp (−ln 2 :: complex) − 1)*
**defines** *G2* ≡ *fps-expansion (λs. (s − 1) ∗ pre-zeta 1 s + 1) 1*
**defines** *G2'* ≡ *fps-expansion (pre-zeta 1) 1*
**defines** *G* ≡ *G1 ∗ G2*
**defines** *A* ≡ *fps-nth G2*
**begin**

*G1*, *G2*, *G2'*, and *G2* are the formal power series expansions of functions around *s = 1* of the entire functions

- $(2^{1-s} - 1)/(s - 1)$,
- $(s - 1)\zeta(s)$,
- $\zeta(s) - \frac{1}{s-1}$,
- $(2^{1-s} - 1)\zeta(s)$,

respectively.

Our goal is to determine the coefficients of $G2'$, and we shall do so by determining the coefficients of $G2$ (which are the same, but shifted by 1). This in turn will be done by determining the coefficients of $G = G1 * G2$. Note that $(2^{1-s} - 1)\zeta(s)$ is written as *perzeta (1 / 2)* in Isabelle (using the periodic $\zeta$ function) and the analytic continuation of $\zeta(s) - \frac{1}{s-1}$ is written as *pre-zeta 1 s* (*pre-zeta* is an artefact from the definition of *zeta*, which comes in useful here).

**lemma** *stieltjes-gamma-aux1*: (λ*n*. (−*1*)^(*n*+*1*) ∗ *ln*(*n*+*1*)^*k* / (*n*+*1*)) *sums* ((−*1*)^*k* ∗ (*deriv*^^*k*) *g 1*)
⟨*proof*⟩

**lemma** *stieltjes-gamma-aux2*: (*deriv*^^*k*) *g 1* = *fact k* ∗ *fps-nth G k*
  **and** *stieltjes-gamma-aux3*: *G2* = *fps-X* ∗ *G2′* + *1*
⟨*proof*⟩

**lemma** *stieltjes-gamma-aux4*: *fps-nth G k* = (∑ *i*=*1*..*k*+*1*. (−*ln 2*)^*i* ∗ *A* (*k*−(*i*−*1*)) / *fact i*)
⟨*proof*⟩

**lemma** *stieltjes-gamma-aux5*: (∑ *t*<*k*. (*k choose t*) ∗ *Ln 2* ^ (*k* − *t*) ∗ *stieltjes-gamma t*) −
    *ln 2* ^(*k*+*1*) / *of-nat* (*k*+*1*) = (−*1*) ^ *k* ∗ (*deriv* ^^ *k*) *g 1*
⟨*proof*⟩

**lemma** *stieltjes-gamma-aux6*: (∑ *t*<*k*. (*k choose t*) ∗ *Ln 2* ^ (*k* − *t*) ∗ *stieltjes-gamma t*) −
    *Ln 2* ^(*k* + *1*) / *of-nat* (*k* + *1*) =
    (−*1*)^*k* ∗ *fact k* ∗ (∑ *i*=*1*..*k*+*1*. (−*Ln 2*) ^ *i* ∗ *A* (*k*−(*i*−*1*)) / *fact i*)
⟨*proof*⟩

**theorem** *higher-deriv-pre-zeta-1-1*: (*deriv* ^^ *k*) (*pre-zeta 1*) *1* = (−*1*) ^ *k* ∗ *stieltjes-gamma k*
⟨*proof*⟩

**corollary** *pre-zeta-1-1* [*simp*]: *pre-zeta 1 1* = *euler-mascheroni*
  ⟨*proof*⟩

**corollary** *zeta-minus-pole-limit*: (λ*s*. *zeta s* − *1* / (*s* − *1*)) −*1*→ *euler-mascheroni*
⟨*proof*⟩

**corollary** *fps-expansion-pre-zeta-1-1*:
  *fps-expansion* (*pre-zeta 1*) *1* = *Abs-fps* (λ*n*. (−*1*)^*n* ∗ *stieltjes-gamma n* / *fact n*)
  ⟨*proof*⟩

**end**

**end**

# 4 The Hadjicostas–Chapman formula

**theory** *Hadjicostas-Chapman*
  **imports** *Zeta-Laurent-Expansion*
**begin**

In this section, we will derive a formula for the $\zeta$ function that was conjectured by Hadjicostas [4] and proven shortly afterwards by Chapman [3]. The formula is:

$$\begin{aligned}\int_0^1 \int_0^1 & \frac{(-\ln(xy))^z(1-x)}{1-xy}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \frac{(-\ln u)^z(-\ln u + u - 1)}{1-u}\,\mathrm{d}u \\ &= \Gamma(z+2)\left(\zeta(z+2) - \frac{1}{z+1}\right)\end{aligned}$$

for any $z$ with $\Re(z) > -2$. In particular, setting $z = 1$, we can derive the following formula for the Euler–Mascheroni constant $\gamma$:

$$-\int_0^1 \int_0^1 \frac{1-x}{(1-xy)\ln(xy)}\,\mathrm{d}x\,\mathrm{d}y = \gamma$$

This formula was first proven by Sondow [7].

## 4.1 The real case

We first define the integral for real $z > -2$. This is then a non-negative integral, so that we can ignore the issue of integrability and use the Lebesgue integral on the extended non-negative reals

We first show the equivalence of the single-integral and the double-integral form.

**definition** *Hadjicostas-nn-integral* :: *real* ⇒ *ennreal* **where**
  *Hadjicostas-nn-integral z* =
    *set-nn-integral lborel* {*0*<..<*1*}
      (λ*u*. *ennreal* ((−*ln u*) *powr z* / (*1* − *u*) ∗ (−*ln u* + *u* − *1*)))

**definition** *Hadjicostas-integral* :: *complex* ⇒ *complex* **where**
  *Hadjicostas-integral z* =
    (*LBINT u*=*0*..*1*. *of-real* (−*ln u*) *powr z* / *of-real* (*1* − *u*) ∗ *of-real* (−*ln u* + *u* − *1*))

**lemma** *Hadjicostas-nn-integral-altdef*:
  *Hadjicostas-nn-integral z* =
    (∫⁺(*x*,*y*)∈{*0*<..<*1*}×{*0*<..<*1*}. ((−*ln* (*x*∗*y*)) *powr z* ∗ (*1*−*x*) / (*1*−*x*∗*y*)) ∂*lborel*)
⟨*proof*⟩

We now solve the single integral for real $z > -1$.

**lemma** *Hadjicostas-Chapman-aux*:
**fixes** $z$ :: *real*
**assumes** $z$: $z > -1$
**defines** $f \equiv (\lambda u.\ ((-ln\ u)\ powr\ z\ /\ (1 - u) * (-ln\ u + u - 1)))$
**shows** $(f\ has\text{-}integral\ (Gamma\ (z + 2) * (Re\ (zeta\ (z + 2)) - 1\ /\ (z + 1))))$ $\{0{<}..{<}1\}$
⟨*proof*⟩

**lemma** *real-zeta-ge-one-over-minus-one*:
**fixes** $z$ :: *real*
**assumes** $z$: $z > 1$
**shows** $Re\ (zeta\ (complex\text{-}of\text{-}real\ z)) \geq 1\ /\ (z - 1)$
⟨*proof*⟩

We now have the formula for real $z > -1$.

**lemma** *Hadjicostas-Chapman-formula-real*:
**fixes** $z$ :: *real*
**assumes** $z$: $z > -1$
**shows** $Hadjicostas\text{-}nn\text{-}integral\ z =$
$ennreal\ (Gamma\ (z + 2) * (Re\ (zeta\ (z + 2)) - 1\ /\ (z + 1)))$
⟨*proof*⟩

### 4.2 Analyticity of the integral

To extend the formula to its full domain of validity (any complex $z$ with $\Re(z) > -2$), we will use analytic continuation. To do this, we first have to show that the integral is an analytic function of $z$ on that domain. This is unfortunately somewhat involved, since the integral is an improper one and we first need to show uniform convergence so that we can pull the derivative inside the integral sign.

We will use the single-integral form so that we only have to deal with one integral and not two.

**context**
**fixes** $f$ :: *complex* ⇒ *real* ⇒ *complex*
**defines** $f \equiv (\lambda z\ u.\ of\text{-}real\ (-ln\ u)\ powr\ z\ /\ of\text{-}real\ (1 - u) * of\text{-}real\ (-ln\ u + u - 1))$
**begin**

**context**
**fixes** $x\ y$ :: *real* **and** $g1\ g2$ :: *real* ⇒ *real*
**assumes** $x > -2$
**defines** $g1 \equiv (\lambda x.\ (-\ ln\ x)\ powr\ y * (x - ln\ x - 1)\ /\ (1 - x))$
**defines** $g2 \equiv (\lambda u.\ (-ln\ u)\ powr\ x * (u - ln\ u - 1)\ /\ (1 - u))$
**begin**

**lemma** *integrable-bound1*:

*interval-lebesgue-integrable lborel 0* (*ereal* (*exp* (− *1*))) *g1*
⟨*proof*⟩

**lemma** *integrable-bound2*:
*interval-lebesgue-integrable lborel* (*exp* (−*1*)) *1 g2*
⟨*proof*⟩

**lemma** *bound2*:
*norm* (*f z u*) ≤ *g2 u* **if** *z*: *Re z* ∈ {*x*..*y*} **and** *u*: *u* ∈ {*exp* (−*1*)<..<*1*} **for** *z u*
⟨*proof*⟩

**lemma** *integrable2-aux*: *interval-lebesgue-integrable lborel* (*exp* (−*1*)) *1* (*f z*)
**if** *z*: *Re z* ∈ {*x*..*y*} **for** *z*
⟨*proof*⟩

**lemma** *uniform-limit2*:
*uniform-limit* {*z*. *Re z* ∈ {*x*..*y*}}
(λ*a z*. *LBINT u*=*exp* (−*1*)..*a*. *f z u*)
(λ*z*. *LBINT u*=*exp* (−*1*)..*1*. *f z u*) (*at-left 1*)
⟨*proof*⟩

**lemma** *uniform-limit2'*:
*uniform-limit* {*z*. *Re z* ∈ {*x*..*y*}}
(λ*n z*. *LBINT u*=*exp* (−*1*)..*ereal* (*1*−(*1*/*2*)^*Suc n*). *f z u*)
(λ*z*. *LBINT u*=*exp* (−*1*)..*1*. *f z u*) *sequentially*
⟨*proof*⟩

**lemma** *bound1*: *norm* (*f z u*) ≤ *g1 u* **if** *z*: *Re z* ∈ {*x*..*y*} **and** *u*: *u* ∈ {*0*<..<*exp* (−*1*)} **for** *z u*
⟨*proof*⟩

**lemma** *integrable1-aux*: *interval-lebesgue-integrable lborel 0* (*exp* (−*1*)) (*f z*)
**if** *z*: *Re z* ∈ {*x*..*y*} **for** *z*
⟨*proof*⟩

**lemma** *uniform-limit1*:
*uniform-limit* {*z*. *Re z* ∈ {*x*..*y*}}
(λ*a z*. *LBINT u*=*a*..*exp* (−*1*). *f z u*)
(λ*z*. *LBINT u*=*0*..*exp* (−*1*). *f z u*) (*at-right 0*)
⟨*proof*⟩

**lemma** *uniform-limit1'*:
*uniform-limit* {*z*. *Re z* ∈ {*x*..*y*}}
(λ*n z*. *LBINT u*=*ereal* ((*1*/*2*)^*Suc n*)..*exp* (−*1*). *f z u*)
(λ*z*. *LBINT u*=*0*..*exp* (−*1*). *f z u*) *sequentially*
⟨*proof*⟩

**end**

With all of the above bounds, we have shown that the integral exists for any

$z$ with $\Re(z) > -2$.

**theorem** *Hadjicostas-integral-integrable*: *interval-lebesgue-integrable lborel 0 1* (*f z*)
**if** *z*: *Re z* $>$ $-2$
⟨*proof*⟩

**lemma** *integral-holo-aux*:
**assumes** *ab*: *a* $>$ *0 a* $\leq$ *b b* $<$ *1*
**shows** ($\lambda z$. *LBINT u=ereal a..ereal b. f z u*) *holomorphic-on A*
⟨*proof*⟩

**lemma** *integral-holo*:
**assumes** *ab*: *min a b* $>$ *0 max a b* $<$ *1*
**shows** ($\lambda z$. *LBINT u=ereal a..ereal b. f z u*) *holomorphic-on A*
⟨*proof*⟩

**lemma** *holo1*: ($\lambda z$. *LBINT u=0..exp* ($-$*1*). *f z u*) *holomorphic-on* {*z. Re z* $>$ $-$*2*}
⟨*proof*⟩

**lemma** *holo2*: ($\lambda z$. *LBINT u=exp* ($-$*1*)..*1*. *f z u*) *holomorphic-on* {*z. Re z* $>$ $-$*2*}
⟨*proof*⟩

Finally, we have shown that Hadjicostas's integral is an analytic function of $z$ in the domain $\Re(z) > -2$.

**lemma** *holomorphic-Hadjicostas-integral*:
*Hadjicostas-integral holomorphic-on* {*z. Re z* $>$ $-$*2*}
⟨*proof*⟩

**lemma** *analytic-Hadjicostas-integral*:
*Hadjicostas-integral analytic-on* {*z. Re z* $>$ $-$*2*}
⟨*proof*⟩

**end**

## 4.3 Analytic continuation and main result

Since we have already shown the formula for any real *z* $>$ $-$*1* and e. g. 0 is a limit point of that set, it extends to the full domain by analytic continuation. As a caveat, note that $\zeta(s)$ is *not* analytic at *z* = *1*, so that we use an analytic continuation of $\zeta(z) - \frac{1}{z-1}$ to state the formula. This continuation is *pre-zeta 1*.

**lemma** *Hadjicostas-Chapman-formula-aux*:
**assumes** *z*: *Re z* $>$ $-$*2*
**shows** *Hadjicostas-integral z* = *Gamma* (*z* + *2*) $*$ *pre-zeta 1* (*z* + *2*)
(**is** - *z* = *?f z*)
⟨*proof*⟩

The following form and the corollary are perhaps a bit nicer to read.

**theorem** *Hadjicostas-Chapman-formula*:
**assumes** *z*: *Re* $z > -2$ $z \neq -1$
**shows** *Hadjicostas-integral* $z$ = *Gamma* $(z + 2) * ($*zeta* $(z + 2) - 1 / (z + 1))$
⟨*proof*⟩

**corollary** *euler-mascheroni-integral-form*:
*Hadjicostas-integral* $(-1)$ = *euler-mascheroni*
⟨*proof*⟩

**end**

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