Zeckendorf's Theorem

Christian Dalvit

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Abstract

This work formalizes Zeckendorf's theorem. The theorem states that every positive integer can be uniquely represented as a sum of one or more non-consecutive Fibonacci numbers. More precisely, if N is a positive integer, there exist unique positive integers $c_i \geq 2$ with $c_{i+1} > c_i + 1$, such that

$$N = \sum_{i=0}^{k} F_{c_i}$$

where F_n is the n-th Fibonacci number. This entry formalizes the proof from Gerrit Lekkerkerker's paper [1].

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1.1 Definitions

Formulate auxiliary definitions. An increasing sequence is a predicate of a function f together with a set I. f is an increasing sequence on I, if $f(x) + 1 < f(x + 1)$ for all $x \in I$. This definition is used to ensure that the Fibonacci numbers in the sum are non-consecutive.

definition *is-fib* :: *nat* ⇒ *bool* **where**
is-fib $n = (\exists\ i.\ n = fib\ i)$

definition *le-fib-idx-set* :: *nat* ⇒ *nat set* **where**
le-fib-idx-set $n = \{i\ .fib\ i < n\}$

definition *inc-seq-on* :: (*nat* ⇒ *nat*) ⇒ *nat set* ⇒ *bool* **where**
inc-seq-on $f\ I = (\forall\ n \in I.\ f(Suc\ n) > Suc(f\ n))$

definition *fib-idx-set* :: *nat* ⇒ *nat set* **where**
fib-idx-set $n = \{i.\ fib\ i = n\}$

1.2 Auxiliary Lemmas

lemma *fib-values*[*simp*]:
$fib\ 3 = 2$
$fib\ 4 = 3$
$fib\ 5 = 5$
$fib\ 6 = 8$
⟨*proof*⟩

lemma *fib-strict-mono*: $i \geq 2 \Longrightarrow fib\ i < fib\ (Suc\ i)$
⟨*proof*⟩

lemma *smaller-index-implies-fib-le*: $i < j \Longrightarrow fib(Suc\ i) \leq fib\ j$
⟨*proof*⟩

lemma *fib-index-strict-mono* : $i \geq 2 \Longrightarrow j > i \Longrightarrow fib\ j > fib\ i$
⟨*proof*⟩

lemma *fib-implies-is-fib*: $fib\ i = n \Longrightarrow$ *is-fib* n
⟨*proof*⟩

lemma *zero-fib-unique-idx*: $n = fib\ i \Longrightarrow n = fib\ 0 \Longrightarrow i = 0$
⟨*proof*⟩

lemma *zero-fib-equiv*: $fib\ i = 0 \longleftrightarrow i = 0$
⟨*proof*⟩

lemma *one-fib-idxs*: $fib\ i = Suc\ 0 \Longrightarrow i = Suc\ 0 \lor i = Suc(Suc\ 0)$
⟨*proof*⟩

lemma *ge-two-eq-fib-implies-eq-idx*: $n \geq 2 \Longrightarrow n = fib\ i \Longrightarrow n = fib\ j \Longrightarrow i = j$
⟨*proof*⟩

lemma *ge-two-fib-unique-idx*: $fib\ i \geq 2 \Longrightarrow fib\ i = fib\ j \Longrightarrow i = j$
⟨*proof*⟩

lemma *no-fib-lower-bound*: $\neg$ *is-fib* $n \Longrightarrow n \geq 4$

⟨*proof*⟩

lemma *pos-fib-has-idx-ge-two*: $n > 0 \Longrightarrow$ *is-fib* $n \Longrightarrow (\exists\ i.\ i \geq 2 \wedge fib\ i = n)$
⟨*proof*⟩

lemma *finite-fib0-idx*: *finite*$(\{i.\ fib\ i = 0\})$
⟨*proof*⟩

lemma *finite-fib1-idx*: *finite*$(\{i.\ fib\ i = 1\})$
⟨*proof*⟩

lemma *finite-fib-ge-two-idx*: $n \geq 2 \Longrightarrow$ *finite*$(\{i.\ fib\ i = n\})$
⟨*proof*⟩

lemma *finite-fib-index*: *finite*$(\{i.\ fib\ i = n\})$
⟨*proof*⟩

lemma *no-fib-implies-zero-in-le-idx-set*: $\neg$ *is-fib* $n \Longrightarrow 0 \in \{i.\ fib\ i < n\}$
⟨*proof*⟩

lemma *no-fib-implies-le-fib-idx-set*: $\neg$ *is-fib* $n \Longrightarrow \{i.\ fib\ i < n\} \neq \{\}$
⟨*proof*⟩

lemma *finite-smaller-fibs*: *finite*$(\{i.\ fib\ i < n\})$
⟨*proof*⟩

lemma *nat-ge-2-fib-idx-bound*: $2 \leq n \Longrightarrow fib\ i \leq n \Longrightarrow n < fib\ (Suc\ i) \Longrightarrow 2 \leq i$
⟨*proof*⟩

lemma *inc-seq-on-aux*: *inc-seq-on* $c\ \{0..k - 1\} \Longrightarrow n - fib\ i < fib\ (i-1) \Longrightarrow fib\ (c\ k) < fib\ i \Longrightarrow$

$$(n - fib\ i) = (\sum\ i=0..k.\ fib\ (c\ i)) \Longrightarrow Suc\ (c\ k) < i$$

⟨*proof*⟩

lemma *inc-seq-zero-at-start*: *inc-seq-on* $c\ \{0..k-1\} \Longrightarrow c\ k = 0 \Longrightarrow k = 0$
⟨*proof*⟩

lemma *fib-sum-zero-equiv*: $(\sum\ i=n..m::nat\ .\ fib\ (c\ i)) = 0 \longleftrightarrow (\forall\ i\in\{n..m\}.\ c\ i = 0)$
⟨*proof*⟩

lemma *fib-idx-ge-two-fib-sum-not-zero*: $n \leq m \Longrightarrow \forall i\in\{n..m::nat\}.\ c\ i \geq 2 \Longrightarrow \neg\ (\sum\ i=n..m.\ fib\ (c\ i)) = 0$
⟨*proof*⟩

lemma *one-unique-fib-sum*: *inc-seq-on* $c\ \{0..k-1\} \Longrightarrow \forall i\in\{0..k\}.\ c\ i \geq 2 \Longrightarrow (\sum\ i=0..k.\ fib\ (c\ i)) = 1 \longleftrightarrow k = 0 \wedge c\ 0 = 2$
⟨*proof*⟩

lemma *no-fib-betw-fibs*:
  assumes $\neg$ *is-fib n*
  shows $\exists$ *i. fib i* $<$ *n* $\wedge$ *n* $<$ *fib (Suc i)*
⟨*proof*⟩

lemma *betw-fibs*:
  shows $\exists$ *i. fib i* $\leq$ *n* $\wedge$ *fib(Suc i)* $>$ *n*
⟨*proof*⟩

Proof that the sum of non-consecutive Fibonacci numbers with largest member F_i is strictly less then F_{i+1}. This lemma is used for the uniqueness proof.

lemma *fib-sum-upper-bound*:
  assumes *inc-seq-on c* {*0..k*−*1*} $\forall$ *i*$\in${*0..k*}. *c i* $\geq$ *2*
  shows ($\sum$ *i=0..k. fib (c i)*) $<$ *fib (Suc (c k))*
⟨*proof*⟩

lemma *last-fib-sum-index-constraint*:
  assumes *n* $\geq$ *2 n* = ($\sum$ *i=0..k. fib (c i)*) *inc-seq-on c* {*0..k*−*1*}
  assumes $\forall$ *i*$\in${*0..k*}. *c i* $\geq$ *2 fib i* $\leq$ *n fib(Suc i)* $>$ *n*
  shows *c k* = *i*
⟨*proof*⟩

1.3 Theorem

Now, both parts of Zeckendorf's Theorem can be proven. Firstly, the existence of an increasing sequence for a positive integer N such that the corresponding Fibonacci numbers sum up to N is proven. Then, the uniqueness of such an increasing sequence is proven.

lemma *fib-implies-zeckendorf*:
  assumes *is-fib n n* $>$ *0*
  shows $\exists$ *c k. n* = ($\sum$ *i=0..k. fib(c i)*) $\wedge$ *inc-seq-on c* {*0..k*−*1*} $\wedge$ ($\forall$ *i*$\in${*0..k*}. *c i* $\geq$ *2*)
⟨*proof*⟩

theorem *zeckendorf-existence*:
  assumes *n* $>$ *0*
  shows $\exists$ *c k. n* = ($\sum$ *i=0..k. fib (c i)*) $\wedge$ *inc-seq-on c* {*0..k*−*1*} $\wedge$ ($\forall$ *i*$\in${*0..k*}. *c i* $\geq$ *2*)
  ⟨*proof*⟩

lemma *fib-unique-fib-sum*:
  fixes *k* :: *nat*
  assumes *n* $\geq$ *2 inc-seq-on c* {*0..k*−*1*} $\forall$ *i*$\in${*0..k*}. *c i* $\geq$ *2*
  assumes *n* = *fib i*
  shows *n* = ($\sum$ *i=0..k. fib (c i)*) $\longleftrightarrow$ *k* = *0* $\wedge$ *c 0* = *i*
⟨*proof*⟩

theorem *zeckendorf-unique*:

assumes $n > 0$
assumes $n = (\sum\ i{=}0..k.\ fib\ (c\ i))$ *inc-seq-on* $c\ \{0..k{-}1\}\ \forall i{\in}\{0..k\}.\ c\ i \geq 2$
assumes $n = (\sum\ i{=}0..k'.\ fib\ (c'\ i))$ *inc-seq-on* $c'\ \{0..k'{-}1\}\ \forall i{\in}\{0..k'\}.\ c'\ i \geq$
2
shows $k = k' \wedge (\forall\ i \in \{0..k\}.\ c\ i = c'\ i)$
$\langle proof \rangle$

end

References

[1] C. G. Lekkerkerker. Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci. *Stichting Mathematisch Centrum. Zuivere Wiskunde*, (ZW 30/51), 1951.