Zermelo Fraenkel Set Theory in Higher-Order Logic

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Abstract

This entry is a new formalisation of ZFC set theory in Isabelle/HOL. It is logically equivalent to Obua's HOLZF [2]; the point is to have the closest possible integration with the rest of Isabelle/HOL, minimising the amount of new notations and exploiting type classes.

There is a type V of sets and a function $elts :: V \Rightarrow V\ set$ mapping a set to its elements. Classes simply have type $V\ set$, and the predicate *small* identifies those classes that correspond to actual sets. Type classes connected with orders and lattices are used to minimise the amount of new notation for concepts such as the subset relation, union and intersection. Basic concepts are formalised: Cartesian products, disjoint sums, natural numbers, functions, etc.

More advanced set-theoretic concepts, such as transfinite induction, ordinals, cardinals and the transitive closure of a set, are also provided. The definition of addition and multiplication for general sets (not just ordinals) follows Kirby [1]. The development includes essential results about cardinal arithmetic. It also develops ordinal exponentiation, Cantor normal form and the concept of indecomposable ordinals. There are numerous results about order types.

The theory provides two type classes with the aim of facilitating developments that combine V with other Isabelle/HOL types: *embeddable*, the class of types that can be injected into V (including V itself as well as $V{*}V$, $V\ list$, etc.), and *small*, the class of types that correspond to some ZF set.

Contents

1 The ZF Axioms, Ordinals and Transfinite Recursion **5**
- 1.1 Syntax and axioms . . . 5
- 1.2 Type classes and other basic setup . . . 10
- 1.3 Successor function . . . 17
- 1.4 Ordinals . . . 17
  - 1.4.1 Transitive sets . . . 18
  - 1.4.2 Zero, successor, sups . . . 18
  - 1.4.3 Induction, Linearity, etc. . . . 20
  - 1.4.4 The natural numbers . . . 22
  - 1.4.5 Limit ordinals . . . 24
  - 1.4.6 Properties of LEAST for ordinals . . . 26
- 1.5 Transfinite Recursion and the V-levels . . . 29

2 Cartesian products, Disjoint Sums, Ranks, Cardinals **32**
- 2.1 Ordered Pairs . . . 32
- 2.2 Generalized Cartesian product . . . 34
- 2.3 Disjoint Sum . . . 36
  - 2.3.1 Equivalences for the injections and an elimination rule 36
  - 2.3.2 Injection and freeness equivalences, for rewriting . . . 36
  - 2.3.3 Applications of disjoint sums and pairs: general union theorems for small sets . . . 38
- 2.4 Generalised function space and lambda . . . 39
- 2.5 Transitive closure of a set . . . 41
- 2.6 Rank of a set . . . 45
- 2.7 Cardinal Numbers . . . 49
  - 2.7.1 Transitive Closure and VWO . . . 49
  - 2.7.2 Relation VWF . . . 51
- 2.8 Order types . . . 52
  - 2.8.1 *ordermap* preserves the orderings in both directions . 53
- 2.9 More advanced *ordertype* and *ordermap* results . . . 55
- 2.10 Cardinality of an arbitrary HOL set . . . 71
- 2.11 Cardinality of a set . . . 71
- 2.12 Cardinality of a set . . . 74
- 2.13 Transfinite recursion for definitions based on the three cases of ordinals . . . 75
- 2.14 Cardinal Addition . . . 76
  - 2.14.1 Cardinal addition is commutative . . . 76
  - 2.14.2 Cardinal addition is associative . . . 76
- 2.15 Cardinal multiplication . . . 77
  - 2.15.1 Cardinal multiplication is commutative . . . 78
  - 2.15.2 Cardinal multiplication is associative . . . 78
  - 2.15.3 Cardinal multiplication distributes over addition . . . 79

2.16 Some inequalities for multiplication 79
2.17 The finite cardinals 81
2.18 Infinite cardinals 82
2.19 Toward's Kunen's Corollary 10.13 (1) 86
2.20 The Aleph-seqence 87
2.21 The ordinal $\omega 1$ 90

3 Addition and Multiplication of Sets 91
3.1 Generalised Addition 91
3.1.1 Addition is a monoid 91
3.1.2 Deeper properties of addition 95
3.1.3 Cancellation / set subtraction 99
3.2 Generalised Difference 103
3.3 Generalised Multiplication 105
3.3.1 Proposition 4.3 106
3.3.2 Proposition 4.4-5 109
3.3.3 Theorem 4.6 111
3.3.4 Theorem 4.7 115
3.4 Ordertype properties 123

4 Exponentiation of ordinals 126

5 Cantor Normal Form 138
5.1 Cantor normal form 138
5.2 Simplified Cantor normal form 148
5.3 Indecomposable ordinals 154
5.4 From ordinals to order types 167

6 Type Classes for ZFC 169
6.1 The class of embeddable types 169
6.2 The class of small types 172

7 ZF sets corresponding to $\mathbb{R}$ and $\mathbb{C}$ and the cardinality of the continuum 175
7.1 Making the embedding from the type class explicit 176
7.2 The cardinality of the continuum 176
7.3 Countable and uncountable sets 178

8 Acknowledgements 180

theory *ZFC-Library*
imports *HOL−Library.Countable-Set HOL−Library.Equipollence HOL−Cardinals.Cardinals*

begin

Equipollence and Lists.

lemma *countable-iff-lepoll*: *countable* $A \longleftrightarrow A \lesssim$ (*UNIV* :: *nat set*)
by (*auto simp*: *countable-def lepoll-def*)

lemma *infinite-times-eqpoll-self*:
assumes *infinite* A **shows** $A \times A \approx A$
by (*simp add*: *Times-same-infinite-bij-betw assms eqpoll-def*)

lemma *infinite-finite-times-lepoll-self*:
assumes *infinite* A *finite* B **shows** $A \times B \lesssim A$
proof −
have $B \lesssim A$
by (*simp add*: *assms finite-lepoll-infinite*)
then have $A \times B \lesssim A \times A$
by (*simp add*: *subset-imp-lepoll times-lepoll-mono*)
also have ... $\approx A$
by (*simp add*: ‹*infinite* A› *infinite-times-eqpoll-self*)
finally show *?thesis* **.**
qed

lemma *lists-n-lepoll-self*:
assumes *infinite* A **shows** $\{l \in lists\ A.\ length\ l = n\} \lesssim A$
proof (*induction* n)
case *0*
have $\{l \in lists\ A.\ length\ l = 0\} = \{[]\}$
by *auto*
then show *?case*
by (*metis Set.set-insert assms ex-in-conv finite.emptyI singleton-lepoll*)
next
case (*Suc* n)
have $\{l \in lists\ A.\ length\ l = Suc\ n\} = (\bigcup x \in A.\ \bigcup l \in \{l \in lists\ A.\ length\ l = n\}.\ \{x\#l\})$
by (*auto simp*: *length-Suc-conv*)
also have ... $\lesssim A \times \{l \in lists\ A.\ length\ l = n\}$
unfolding *lepoll-iff*
by (*rule-tac* $x=\lambda(x,l).\ Cons\ x\ l$ **in** *exI*) *auto*
also have ... $\lesssim A$
proof (*cases finite* $\{l \in lists\ A.\ length\ l = n\}$)
case *True*
then show *?thesis*
using *assms infinite-finite-times-lepoll-self* **by** *blast*
next
case *False*
have $A \times \{l \in lists\ A.\ length\ l = n\} \lesssim A \times A$

by (*simp add*: *Suc.IH subset-imp-lepoll times-lepoll-mono*)
also have $\ldots \approx A$
by (*simp add*: *assms infinite-times-eqpoll-self*)
finally show *?thesis* **.**
qed
finally show *?case* **.**
qed

lemma *infinite-eqpoll-lists*:
assumes *infinite A* **shows** *lists* $A \approx A$
proof −
have *lists* $A \lesssim$ *Sigma UNIV* ($\lambda n.$ $\{l \in$ *lists A*. *length* $l = n\}$)
unfolding *lepoll-iff*
by (*rule-tac x=snd* **in** *exI*) (*auto simp*: *in-listsI snd-image-Sigma*)
also have $\ldots \lesssim$ (*UNIV*::*nat set*) $\times A$
by (*rule Sigma-lepoll-mono*) (*auto simp*: *lists-n-lepoll-self assms*)
also have $\ldots \lesssim A \times A$
by (*metis assms infinite-le-lepoll order-refl subset-imp-lepoll times-lepoll-mono*)
also have $\ldots \approx A$
by (*simp add*: *assms infinite-times-eqpoll-self*)
finally show *?thesis*
by (*simp add*: *lepoll-antisym lepoll-lists*)
qed

end

1 The ZF Axioms, Ordinals and Transfinite Recursion

theory *ZFC-in-HOL*
imports *ZFC-Library*

begin

1.1 Syntax and axioms

hide-const (**open**) *list.set Sum subset*

unbundle *lattice-syntax*

typedecl *V*

Presentation refined by Dmitriy Traytel

axiomatization *elts* :: $V \Rightarrow V$ *set*
where *ext* [*intro?*]: *elts* $x =$ *elts* $y \Longrightarrow x=y$
and *down-raw*: $Y \subseteq$ *elts* $x \Longrightarrow Y \in$ *range elts*
and *Union-raw*: $X \in$ *range elts* $\Longrightarrow$ *Union* (*elts* ‘ X) $\in$ *range elts*
and *Pow-raw*: $X \in$ *range elts* $\Longrightarrow$ *inv elts* ‘ *Pow* $X \in$ *range elts*

and *replacement-raw*: $X \in range\ elts \Longrightarrow f\ `\ X \in range\ elts$
and *inf-raw*: $range\ (g :: nat \Rightarrow V) \in range\ elts$
and *foundation*: $wf\ \{(x,y).\ x \in elts\ y\}$

lemma *mem-not-refl* [*simp*]: $i \notin elts\ i$
using *wf-not-refl* [*OF foundation*] **by** *force*

lemma *mem-not-sym*: $\neg\ (x \in elts\ y \wedge y \in elts\ x)$
using *wf-not-sym* [*OF foundation*] **by** *force*

A set is small if it can be injected into the extension of a V-set.

definition *small* :: $'a\ set \Rightarrow bool$
where $small\ X \equiv \exists\ V\text{-}of :: 'a \Rightarrow V.\ inj\text{-}on\ V\text{-}of\ X \wedge V\text{-}of\ `\ X \in range\ elts$

lemma *small-empty* [*iff*]: $small\ \{\}$
by (*simp add*: *small-def down-raw*)

lemma *small-iff-range*: $small\ X \longleftrightarrow X \in range\ elts$
apply (*simp add*: *small-def*)
by (*metis inj-on-id2 replacement-raw the-inv-into-onto*)

lemma *small-eqpoll*: $small\ A \longleftrightarrow (\exists\, x.\ elts\ x \approx A)$
unfolding *small-def* **by** (*metis UNIV-I bij-betw-def eqpoll-def eqpoll-sym imageE image-eqI*)

Small classes can be mapped to sets.

definition *set* :: $V\ set \Rightarrow V$
where $set\ X \equiv (if\ small\ X\ then\ inv\ elts\ X\ else\ inv\ elts\ \{\})$

lemma *set-of-elts* [*simp*]: $set\ (elts\ x) = x$
by (*force simp add*: *ext set-def f-inv-into-f small-def*)

lemma *elts-of-set* [*simp*]: $elts\ (set\ X) = (if\ small\ X\ then\ X\ else\ \{\})$
by (*simp add*: *ZFC-in-HOL.set-def down-raw f-inv-into-f small-iff-range*)

lemma *down*: $Y \subseteq elts\ x \Longrightarrow small\ Y$
by (*simp add*: *down-raw small-iff-range*)

lemma *Union* [*intro*]: $small\ X \Longrightarrow small\ (Union\ (elts\ `\ X))$
by (*simp add*: *Union-raw small-iff-range*)

lemma *Pow*: $small\ X \Longrightarrow small\ (set\ `\ Pow\ X)$
unfolding *small-iff-range* **using** *Pow-raw set-def down* **by** *force*

declare *replacement-raw* [*intro*,*simp*]

lemma *replacement* [*intro*,*simp*]:
assumes *small X*
shows $small\ (f\ `\ X)$

```
proof -
  let ?A = inv-into X f ‘ (f ‘ X)
  have AX: ?A ⊆ X
    by (simp add: image-subsetI inv-into-into)
  have inj: inj-on f ?A
    by (simp add: f-inv-into-f inj-on-def)
  have injo: inj-on (inv-into X f) (f ‘ X)
    using inj-on-inv-into by blast
  have ∃ V-of. inj-on V-of (f ‘ X) ∧ V-of ‘ f ‘ X ∈ range elts
    if inj-on V-of X and V-of ‘ X = elts x
    for V-of :: 'a ⇒ V and x
  proof (intro exI conjI)
    show inj-on (V-of ∘ inv-into X f) (f ‘ X)
      by (meson ‹inv-into X f ‘ f ‘ X ⊆ X› comp-inj-on inj-on-subset injo that)
    have (λx. V-of (inv-into X f (f x))) ‘ X = elts (set (V-of ‘ ?A))
      by (metis AX down elts-of-set image-image image-mono that(2))
    then show (V-of ∘ inv-into X f) ‘ f ‘ X ∈ range elts
      by (metis image-comp image-image rangeI)
  qed
  then show ?thesis
    using assms by (auto simp: small-def)
qed

lemma small-image-iff [simp]: inj-on f A ⟹ small (f ‘ A) ⟷ small A
  by (metis replacement the-inv-into-onto)
```

A little bootstrapping is needed to characterise *small* for sets of arbitrary type.

```
lemma inf: small (range (g :: nat ⇒ V))
  by (simp add: inf-raw small-iff-range)

lemma small-image-nat-V [simp]: small (g ‘ N) for g :: nat ⇒ V
 by (metis (mono-tags, opaque-lifting) down elts-of-set image-iff inf rangeI subsetI)

lemma Finite-V:
  fixes X :: V set
  assumes finite X shows small X
 using ex-bij-betw-nat-finite [OF assms] unfolding bij-betw-def by (metis small-image-nat-V)

lemma small-insert-V:
  fixes X :: V set
  assumes small X
  shows small (insert a X)
proof (cases finite X)
  case True
  then show ?thesis
    by (simp add: Finite-V)
next
  case False
```

```
  show ?thesis
    using infinite-imp-bij-betw2 [OF False]
   by (metis replacement Un-insert-right assms bij-betw-imp-surj-on sup-bot.right-neutral)
qed

lemma small-UN-V [simp,intro]:
  fixes B :: 'a ⇒ V set
  assumes X: small X and B: ⋀x. x ∈ X ⟹ small (B x)
  shows small (⋃x∈X. B x)
proof −
  have (⋃ (elts ` (λx. ZFC-in-HOL.set (B x)) ` X)) = (⋃ (B ` X))
    using B by force
  then show ?thesis
    using Union [OF replacement [OF X, of λx. ZFC-in-HOL.set (B x)]] by simp
qed

definition vinsert where vinsert x y ≡ set (insert x (elts y))

lemma elts-vinsert [simp]: elts (vinsert x y) = insert x (elts y)
  using down small-insert-V vinsert-def by auto

definition succ where succ x ≡ vinsert x x

lemma elts-succ [simp]: elts (succ x) = insert x (elts x)
  by (simp add: succ-def)

lemma finite-imp-small:
  assumes finite X shows small X
  using assms
proof induction
  case empty
  then show ?case
    by simp
next
  case (insert a X)
  then obtain V-of u where u: inj-on V-of X V-of ` X = elts u
    by (meson small-def image-iff)
  show ?case
    unfolding small-def
  proof (intro exI conjI)
    show inj-on (V-of(a:=u)) (insert a X)
      using u
      apply (clarsimp simp add: inj-on-def)
      by (metis image-eqI mem-not-refl)
    have (V-of(a:=u)) ` insert a X = elts (vinsert u u)
      using insert.hyps(2) u(2) by auto
    then show (V-of(a:=u)) ` insert a X ∈ range elts
      by (blast intro:  elim: )
  qed
```

qed

lemma *small-insert*:
  assumes *small X*
  shows *small* (*insert a X*)
proof (*cases finite X*)
  case *True*
  then show *?thesis*
    by (*simp add*: *finite-imp-small*)
next
  case *False*
  show *?thesis*
    using *infinite-imp-bij-betw2* [*OF False*]
   by (*metis replacement Un-insert-right assms bij-betw-imp-surj-on sup-bot.right-neutral*)
qed

lemma *smaller-than-small*:
  assumes *small A B* $\subseteq$ *A* **shows** *small B*
  using *assms*
  by (*metis down elts-of-set image-mono small-def small-iff-range subset-inj-on*)

lemma *small-insert-iff* [*iff*]: *small* (*insert a X*) $\longleftrightarrow$ *small X*
  by (*meson small-insert smaller-than-small subset-insertI*)

lemma *small-iff*: *small X* $\longleftrightarrow$ ($\exists$ *x*. *X* = *elts x*)
  by (*metis down elts-of-set subset-refl*)

lemma *small-elts* [*iff*]: *small* (*elts x*)
  by (*auto simp*: *small-iff*)

lemma *small-diff* [*iff*]: *small* (*elts a* $-$ *X*)
  by (*meson Diff-subset down*)

lemma *small-set* [*simp*]: *small* (*list.set xs*)
  by (*simp add*: *ZFC-in-HOL.finite-imp-small*)

lemma *small-upair*: *small* {*x*,*y*}
  by *simp*

lemma *small-Un-elts*: *small* (*elts x* $\cup$ *elts y*)
  using *Union* [*OF small-upair*] **by** *auto*

lemma *small-eqcong*: $[\![$*small X*; *X* $\approx$ *Y*$]\!]$ $\Longrightarrow$ *small Y*
  by (*metis bij-betw-imp-surj-on eqpoll-def replacement*)

lemma *lepoll-small*: $[\![$*small Y*; *X* $\lesssim$ *Y*$]\!]$ $\Longrightarrow$ *small X*
    by (*meson lepoll-iff replacement smaller-than-small*)

lemma *big-UNIV* [*simp*]: $\neg$ *small* (*UNIV*::*V set*) (**is** $\neg$ *small ?U*)

proof
assume *small ?U*
then have *small A* **for** $A :: V\ set$
by (*metis* (*full-types*) *UNIV-I down small-iff subsetI*)
then have *range elts* = *UNIV*
by (*meson small-iff surj-def*)
then show *False*
by (*metis Cantors-theorem Pow-UNIV*)
qed

lemma *inj-on-set*: *inj-on set* (*Collect small*)
by (*metis elts-of-set inj-onI mem-Collect-eq*)

lemma *set-injective* [*simp*]: $[\![small\ X;\ small\ Y]\!] \Longrightarrow set\ X = set\ Y \longleftrightarrow X{=}Y$
by (*metis elts-of-set*)

1.2 Type classes and other basic setup

instantiation $V :: zero$
begin
definition *zero-V* **where** $0 \equiv set\ \{\}$
instance ..
end

lemma *elts-0* [*simp*]: $elts\ 0 = \{\}$
by (*simp add*: *zero-V-def*)

lemma *set-empty* [*simp*]: $set\ \{\} = 0$
by (*simp add*: *zero-V-def*)

instantiation $V :: one$
begin
definition *one-V* **where** $1 \equiv succ\ 0$
instance ..
end

lemma *elts-1* [*simp*]: $elts\ 1 = \{0\}$
by (*simp add*: *one-V-def*)

lemma *insert-neq-0* [*simp*]: $set\ (insert\ a\ X) = 0 \longleftrightarrow \neg\ small\ X$
unfolding *zero-V-def*
by (*metis elts-of-set empty-not-insert set-of-elts small-insert-iff*)

lemma *elts-eq-empty-iff* [*simp*]: $elts\ x = \{\} \longleftrightarrow x{=}0$
by (*auto simp*: *ZFC-in-HOL.ext*)

instantiation $V :: distrib\text{-}lattice$
begin

definition *inf-V* **where** $inf\text{-}V\ x\ y \equiv set\ (elts\ x \cap elts\ y)$

definition *sup-V* **where** $sup\text{-}V\ x\ y \equiv set\ (elts\ x \cup elts\ y)$

definition *less-eq-V* **where** $less\text{-}eq\text{-}V\ x\ y \equiv elts\ x \subseteq elts\ y$

definition *less-V* **where** $less\text{-}V\ x\ y \equiv less\text{-}eq\ x\ y \wedge x \neq (y{::}V)$

instance
proof
  show $(x < y) = (x \leq y \wedge \neg\ y \leq x)$ **for** $x :: V$ **and** $y :: V$
    using *ext less-V-def less-eq-V-def* **by** *auto*
  show $x \leq x$ **for** $x :: V$
    by (*simp add*: *less-eq-V-def*)
  show $x \leq z$ **if** $x \leq y$ $y \leq z$ **for** $x\ y\ z :: V$
    using *that* **by** (*auto simp*: *less-eq-V-def*)
  show $x = y$ **if** $x \leq y$ $y \leq x$ **for** $x\ y :: V$
    using *that* **by** (*simp add*: *ext less-eq-V-def*)
  show $inf\ x\ y \leq x$ **for** $x\ y :: V$
    by (*metis down elts-of-set inf-V-def inf-sup-ord(1) less-eq-V-def*)
  show $inf\ x\ y \leq y$ **for** $x\ y :: V$
    by (*metis Int-lower2 down elts-of-set inf-V-def less-eq-V-def*)
  show $x \leq inf\ y\ z$ **if** $x \leq y$ $x \leq z$ **for** $x\ y\ z :: V$
  proof −
    have *small* (*elts y* ∩ *elts z*)
      by (*meson down inf.cobounded1*)
    then show *?thesis*
      using *elts-of-set inf-V-def less-eq-V-def that* **by** *auto*
  qed
  show $x \leq x \sqcup y$ $y \leq x \sqcup y$ **for** $x\ y :: V$
    by (*simp-all add*: *less-eq-V-def small-Un-elts sup-V-def*)
  show $sup\ y\ z \leq x$ **if** $y \leq x$ $z \leq x$ **for** $x\ y\ z :: V$
    using *less-eq-V-def sup-V-def that* **by** *auto*
  show $sup\ x\ (inf\ y\ z) = inf\ (x \sqcup y)\ (sup\ x\ z)$ **for** $x\ y\ z :: V$
  proof −
    have *small* (*elts y* ∩ *elts z*)
      by (*meson down inf.cobounded2*)
    then show *?thesis*
      by (*simp add*: *Un-Int-distrib inf-V-def small-Un-elts sup-V-def*)
  qed
qed
end

lemma *V-equalityI* [*intro*]: $(\bigwedge x.\ x \in elts\ a \longleftrightarrow x \in elts\ b) \Longrightarrow a = b$
  by (*meson dual-order.antisym less-eq-V-def subsetI*)

lemma *vsubsetI* [*intro*!]: $(\bigwedge x.\ x \in elts\ a \Longrightarrow x \in elts\ b) \Longrightarrow a \leq b$
  by (*simp add*: *less-eq-V-def subsetI*)

lemma *vsubsetD* [*elim*, *intro?*]: $a \le b \Longrightarrow c \in elts\ a \Longrightarrow c \in elts\ b$
  using *less-eq-V-def* **by** *auto*

lemma *rev-vsubsetD*: $c \in elts\ a \Longrightarrow a \le b \Longrightarrow c \in elts\ b$
  — The same, with reversed premises for use with *erule* – cf. $[\![?P;\ ?P \longrightarrow ?Q]\!] \Longrightarrow ?Q$.
  by (*rule vsubsetD*)

lemma *vsubsetCE* [*elim*,*no-atp*]: $a \le b \Longrightarrow (c \notin elts\ a \Longrightarrow P) \Longrightarrow (c \in elts\ b \Longrightarrow P) \Longrightarrow P$
  — Classical elimination rule.
  using *vsubsetD* **by** *blast*

lemma *set-image-le-iff*: $small\ A \Longrightarrow set\ (f\ `\ A) \le B \longleftrightarrow (\forall x \in A.\ f\ x \in elts\ B)$
  by *auto*

lemma *eq0-iff*: $x = 0 \longleftrightarrow (\forall y.\ y \notin elts\ x)$
  by *auto*

lemma *less-eq-V-0-iff* [*simp*]: $x \le 0 \longleftrightarrow x = 0$ **for** $x{::}V$
  by *auto*

lemma *subset-iff-less-eq-V*:
  assumes *small B* **shows** $A \subseteq B \longleftrightarrow set\ A \le set\ B \wedge small\ A$
  using *assms down small-iff* **by** *auto*

lemma *small-Collect* [*simp*]: $small\ A \Longrightarrow small\ \{x \in A.\ P\ x\}$
  by (*simp add*: *smaller-than-small*)

lemma *small-Union-iff*: $small\ (\bigcup (elts\ `\ X)) \longleftrightarrow small\ X$
  proof
  show *small X*
    if $small\ (\bigcup\ (elts\ `\ X))$
  proof −
    have $X \subseteq set\ `\ Pow\ (\bigcup\ (elts\ `\ X))$
      by *fastforce*
    then show *?thesis*
      using *Pow subset-iff-less-eq-V that* **by** *auto*
  qed
qed *auto*

lemma *not-less-0* [*iff*]:
  fixes $x{::}V$ **shows** $\neg\ x < 0$
  by (*simp add*: *less-eq-V-def less-le-not-le*)

lemma *le-0* [*iff*]:
  fixes $x{::}V$ **shows** $0 \le x$
  by *auto*

lemma *min-0L* [*simp*]: $min\ 0\ n = 0$ **for** $n :: V$
  by (*simp add*: *min-absorb1*)

lemma *min-0R* [*simp*]: $min\ n\ 0 = 0$ **for** $n :: V$
  by (*simp add*: *min-absorb2*)

lemma *neq0-conv*: $\bigwedge n::V.\ n \neq 0 \longleftrightarrow 0 < n$
  by (*simp add*: *less-V-def*)

definition *VPow* :: $V \Rightarrow V$
  where $VPow\ x \equiv set\ (set\ `\ Pow\ (elts\ x))$

lemma *VPow-iff* [*iff*]: $y \in elts\ (VPow\ x) \longleftrightarrow y \leq x$
  using *down Pow*
  apply (*auto simp*: *VPow-def less-eq-V-def*)
  using *less-eq-V-def* **apply** *fastforce*
  done

lemma *VPow-le-VPow-iff* [*simp*]: $VPow\ a \leq VPow\ b \longleftrightarrow a \leq b$
  by *auto*

lemma *elts-VPow*: $elts\ (VPow\ x) = set\ `\ Pow\ (elts\ x)$
  by (*auto simp*: *VPow-def Pow*)

lemma *small-sup-iff* [*simp*]: $small\ (X \cup Y) \longleftrightarrow small\ X \wedge small\ Y$ **for** $X::V\ set$
  by (*metis down elts-of-set small-Un-elts sup-ge1 sup-ge2*)

lemma *elts-sup-iff* [*simp*]: $elts\ (x \sqcup y) = elts\ x \cup elts\ y$
  by (*simp add*: *sup-V-def*)

lemma *trad-foundation*:
  assumes z: $z \neq 0$ **shows** $\exists w.\ w \in elts\ z \wedge w \sqcap z = 0$
  using *foundation assms*
  by (*simp add*: *wf-eq-minimal*) (*metis Int-emptyI equals0I inf-V-def set-of-elts zero-V-def*)

instantiation V :: *Sup*
begin
definition *Sup-V* **where** $Sup\text{-}V\ X \equiv if\ small\ X\ then\ set\ (Union\ (elts\ `\ X))\ else\ 0$
instance ..
end

instantiation V :: *Inf*
begin
definition *Inf-V* **where** $Inf\text{-}V\ X \equiv if\ X = \{\}\ then\ 0\ else\ set\ (Inter\ (elts\ `\ X))$
instance ..
end

lemma *V-disjoint-iff*: $x \sqcap y = 0 \longleftrightarrow elts\ x \cap elts\ y = \{\}$
  by (*metis down elts-of-set inf-V-def inf-le1 zero-V-def*)

I've no idea why *bdd-above* is treated differently from *bdd-below*, but anyway

lemma *bdd-above-iff-small* [*simp*]: *bdd-above X* = *small X* **for** *X*::*V set*
proof
  show *small X* **if** *bdd-above X*
  proof −
    obtain *a* **where** $\forall x \in X.\ x \le a$
      using *that* ‹*bdd-above X*› *bdd-above-def* **by** *blast*
    then show *small X*
      by (*meson VPow-iff* ‹$\forall x \in X.\ x \le a$› *down subsetI*)
  qed
  show *bdd-above X*
    if *small X*
  proof −
    have $\forall x \in X.\ x \le \bigsqcup X$
      by (*simp add*: *SUP-upper Sup-V-def Union less-eq-V-def that*)
    then show *?thesis*
      by (*meson bdd-above-def*)
  qed
qed

instantiation *V* :: *conditionally-complete-lattice*
begin

definition *bdd-below-V* **where** *bdd-below-V X* $\equiv X \neq \{\}$

instance
  proof
  show $\bigsqcap X \le x$ **if** $x \in X$ *bdd-below X*
    for *x* :: *V* **and** *X* :: *V set*
    using *that* **by** (*auto simp*: *bdd-below-V-def Inf-V-def split*: *if-split-asm*)
  show $z \le \bigsqcap X$
    if $X \neq \{\}$ $\bigwedge x.\ x \in X \Longrightarrow z \le x$
    for *X* :: *V set* **and** *z* :: *V*
    using *that*
   apply (*clarsimp simp add*: *bdd-below-V-def Inf-V-def less-eq-V-def split*: *if-split-asm*)
    by (*meson INT-subset-iff down eq-refl equals0I*)
  show $x \le \bigsqcup X$ **if** $x \in X$ **and** *bdd-above X* **for** *x* :: *V* **and** *X* :: *V set*
    using *that Sup-V-def* **by** *auto*
  show $\bigsqcup X \le (z::V)$ **if** $X \neq \{\}$ $\bigwedge x.\ x \in X \Longrightarrow x \le z$ **for** *X* :: *V set* **and** *z* :: *V*
    using *that* **by** (*simp add*: *SUP-least Sup-V-def less-eq-V-def*)
qed
end

lemma *Sup-upper*: $[\![x \in A;\ small\ A]\!] \Longrightarrow x \le \bigsqcup A$ **for** *A*::*V set*

by (*auto simp*: *Sup-V-def SUP-upper Union less-eq-V-def*)

lemma *Sup-least*:
  fixes $z::V$ **shows** $(\bigwedge x.\ x \in A \Longrightarrow x \le z) \Longrightarrow \bigsqcup A \le z$
  by (*auto simp*: *Sup-V-def SUP-least less-eq-V-def*)

lemma *Sup-empty* [*simp*]: $\bigsqcup \{\} = (0::V)$
  using *Sup-V-def* **by** *auto*

lemma *elts-Sup* [*simp*]: $small\ X \Longrightarrow elts\ (\bigsqcup\ X) = \bigcup (elts\ `\ X)$
  by (*auto simp*: *Sup-V-def*)

lemma *sup-V-0-left* [*simp*]: $0 \sqcup a = a$ **and** *sup-V-0-right* [*simp*]: $a \sqcup 0 = a$ **for** $a::V$
  by *auto*

lemma *Sup-V-insert*:
  fixes $x::V$ **assumes** *small A* **shows** $\bigsqcup (insert\ x\ A) = x \sqcup \bigsqcup A$
  by (*simp add*: *assms cSup-insert-If*)

lemma *Sup-Un-distrib*: $[\![small\ A;\ small\ B]\!] \Longrightarrow \bigsqcup (A \cup B) = \bigsqcup A \sqcup \bigsqcup B$ **for** $A::V\ set$
  by *auto*

lemma *SUP-sup-distrib*:
  fixes $f :: V \Rightarrow V$
  shows $small\ A \Longrightarrow (\bigsqcup x{\in}A.\ f\ x \sqcup g\ x) = \bigsqcup\ (f\ `\ A) \sqcup \bigsqcup\ (g\ `\ A)$
  by (*force simp*:)

lemma *SUP-const* [*simp*]: $(\bigsqcup y \in A.\ a) = (if\ A = \{\}\ then\ (0::V)\ else\ a)$
  by *simp*

lemma *cSUP-subset-mono*:
  fixes $f :: 'a \Rightarrow V\ set$ **and** $g :: 'a \Rightarrow V\ set$
  shows $[\![A \subseteq B;\ \bigwedge x.\ x \in A \Longrightarrow f\ x \le g\ x]\!] \Longrightarrow \bigsqcup\ (f\ `\ A) \le \bigsqcup\ (g\ `\ B)$
  by (*simp add*: *SUP-subset-mono*)

lemma *mem-Sup-iff* [*iff*]: $x \in elts\ (\bigsqcup X) \longleftrightarrow x \in \bigcup\ (elts\ `\ X) \wedge small\ X$
  using *Sup-V-def* **by** *auto*

lemma *cSUP-UNION*:
  fixes $B :: V \Rightarrow V\ set$ **and** $f :: V \Rightarrow V$
  assumes *ne*: *small A* **and** *bdd-UN*: $small\ (\bigcup x{\in}A.\ f\ `\ B\ x)$
  shows $\bigsqcup (f\ `\ (\bigcup x{\in}A.\ B\ x)) = \bigsqcup ((\lambda x.\ \bigsqcup (f\ `\ B\ x))\ `\ A)$
proof −
  have *bdd*: $\bigwedge x.\ x \in A \Longrightarrow small\ (f\ `\ B\ x)$
    using *bdd-UN subset-iff-less-eq-V*
    by (*meson SUP-upper smaller-than-small*)
  then have *bdd2*: $small\ ((\lambda x.\ \bigsqcup (f\ `\ B\ x))\ `\ A)$

using *ne*(*1*) **by** *blast*
have $\bigsqcup(f \text{ ` } (\bigcup x{\in}A.\ B\ x)) \le \bigsqcup((\lambda x.\ \bigsqcup(f \text{ ` } B\ x)) \text{ ` } A)$
using *assms* **by** (*fastforce simp add*: *intro*!: *cSUP-least intro*: *cSUP-upper2 simp*: *bdd2 bdd*)
moreover have $\bigsqcup((\lambda x.\ \bigsqcup(f \text{ ` } B\ x)) \text{ ` } A) \le \bigsqcup(f \text{ ` } (\bigcup x{\in}A.\ B\ x))$
using *assms* **by** (*fastforce simp add*: *intro*!: *cSUP-least intro*: *cSUP-upper simp*: *image-UN bdd-UN*)
ultimately show *?thesis*
by (*rule order-antisym*)
qed

lemma *Sup-subset-mono*: $small\ B \Longrightarrow A \subseteq B \Longrightarrow Sup\ A \le Sup\ B$ **for** $A{::}V\ set$
by *auto*

lemma *Sup-le-iff*: $small\ A \Longrightarrow Sup\ A \le a \longleftrightarrow (\forall x{\in}A.\ x \le a)$ **for** $A{::}V\ set$
by *auto*

lemma *SUP-le-iff*: $small\ (f \text{ ` } A) \Longrightarrow \bigsqcup(f \text{ ` } A) \le u \longleftrightarrow (\forall x{\in}A.\ f\ x \le u)$ **for** $f :: V \Rightarrow V$
by *blast*

lemma *Sup-eq-0-iff* [*simp*]: $\bigsqcup A = 0 \longleftrightarrow A \subseteq \{0\} \lor \neg\ small\ A$ **for** $A :: V\ set$
using *Sup-upper* **by** *fastforce*

lemma *Sup-Union-commute*:
fixes $f :: V \Rightarrow V\ set$
assumes $small\ A\ \bigwedge x.\ x{\in}A \Longrightarrow small\ (f\ x)$
shows $\bigsqcup\ (\bigcup x{\in}A.\ f\ x) = (\bigsqcup x{\in}A.\ \bigsqcup\ (f\ x))$
using *assms*
by (*force simp*: *subset-iff-less-eq-V intro*!: *antisym*)

lemma *Sup-eq-Sup*:
fixes $B :: V\ set$
assumes $B \subseteq A$ $small\ A$ **and** $*$: $\bigwedge x.\ x \in A \Longrightarrow \exists y \in B.\ x \le y$
shows $Sup\ A = Sup\ B$
proof −
have *small B*
using *assms subset-iff-less-eq-V* **by** *auto*
moreover have $\exists y{\in}B.\ u \in elts\ y$
if $x \in A$ $u \in elts\ x$ **for** *u x*
using *that* $*$ **by** *blast*
moreover have $\exists x{\in}A.\ v \in elts\ x$
if $y \in B$ $v \in elts\ y$ **for** *v y*
using *that* ‹$B \subseteq A$› **by** *blast*
ultimately show *?thesis*
using *assms* **by** *auto*
qed

1.3 Successor function

lemma *vinsert-not-empty* [*simp*]: *vinsert a A* $\neq$ *0*
and *empty-not-vinsert* [*simp*]: *0* $\neq$ *vinsert a A*
by (*auto simp*: *vinsert-def*)

lemma *succ-not-0* [*simp*]: *succ n* $\neq$ *0* **and** *zero-not-succ* [*simp*]: *0* $\neq$ *succ n*
by (*auto simp*: *succ-def*)

instantiation *V* :: *zero-neq-one*
begin
instance
by *intro-classes* (*metis elts-0 elts-succ empty-iff insert-iff one-V-def set-of-elts*)
end

instantiation *V* :: *zero-less-one*
begin
instance
by *intro-classes* (*simp add*: *less-V-def*)
end

lemma *succ-ne-self* [*simp*]: *i* $\neq$ *succ i*
by (*metis elts-succ insertI1 mem-not-refl*)

lemma *succ-notin-self*: *succ i* $\notin$ *elts i*
using *elts-succ mem-not-refl* **by** *blast*

lemma *le-succE*: *succ i* $\leq$ *succ j* $\Longrightarrow$ *i* $\leq$ *j*
using *less-eq-V-def mem-not-sym* **by** *auto*

lemma *succ-inject-iff* [*iff*]: *succ i* = *succ j* $\longleftrightarrow$ *i* = *j*
by (*simp add*: *dual-order.antisym le-succE*)

lemma *inj-succ*: *inj succ*
by (*simp add*: *inj-def*)

lemma *succ-neq-zero*: *succ x* $\neq$ *0*
by (*metis elts-0 elts-succ insert-not-empty*)

definition *pred* **where** *pred i* $\equiv$ *THE j. i* = *succ j*

lemma *pred-succ* [*simp*]: *pred* (*succ i*) = *i*
by (*simp add*: *pred-def*)

1.4 Ordinals

definition *Transset* **where** *Transset x* $\equiv$ $\forall$ *y* $\in$ *elts x. y* $\leq$ *x*

definition *Ord* **where** *Ord x* $\equiv$ *Transset x* $\wedge$ ($\forall$ *y* $\in$ *elts x. Transset y*)

abbreviation *ON* **where** $ON \equiv$ *Collect Ord*

1.4.1 Transitive sets

lemma *Transset-0* [*iff*]: *Transset 0*
  by (*auto simp*: *Transset-def*)

lemma *Transset-succ* [*intro*]:
  assumes *Transset x* **shows** *Transset* (*succ x*)
  using *assms*
  by (*auto simp*: *Transset-def succ-def less-eq-V-def*)

lemma *Transset-Sup*:
  assumes $\bigwedge x.\ x \in X \Longrightarrow$ *Transset x* **shows** *Transset* ($\bigsqcup X$)
proof (*cases small X*)
  case *True*
  with *assms* **show** *?thesis*
    by (*simp add*: *Transset-def*) (*meson Sup-upper assms dual-order.trans*)
qed (*simp add*: *Sup-V-def*)

lemma *Transset-sup*:
  assumes *Transset x Transset y* **shows** *Transset* ($x \sqcup y$)
  using *Transset-def assms* **by** *fastforce*

lemma *Transset-inf*: $[\![$*Transset i*; *Transset j*$]\!] \Longrightarrow$ *Transset* ($i \sqcap j$)
  by (*simp add*: *Transset-def rev-vsubsetD*)

lemma *Transset-VPow*: *Transset*(i) $\Longrightarrow$ *Transset*(*VPow*(i))
  by (*auto simp*: *Transset-def*)

lemma *Transset-Inf*: ($\bigwedge i.\ i \in A \Longrightarrow$ *Transset i*) $\Longrightarrow$ *Transset* ($\bigsqcap A$)
  by (*force simp*: *Transset-def Inf-V-def*)

lemma *Transset-SUP*: ($\bigwedge x.\ x \in A \Longrightarrow$ *Transset* ($B\ x$)) $\Longrightarrow$ *Transset* ($\bigsqcup$ (B ‘ A))
  by (*metis Transset-Sup imageE*)

lemma *Transset-INT*: ($\bigwedge x.\ x \in A \Longrightarrow$ *Transset* ($B\ x$)) $\Longrightarrow$ *Transset* ($\bigsqcap$ (B ‘ A))
  by (*metis Transset-Inf imageE*)

1.4.2 Zero, successor, sups

lemma *Ord-0* [*iff*]: *Ord 0*
  by (*auto simp*: *Ord-def*)

lemma *Ord-succ* [*intro*]:
  assumes *Ord x* **shows** *Ord* (*succ x*)
  using *assms* **by** (*auto simp*: *Ord-def*)

lemma *Ord-Sup*:
  assumes $\bigwedge x.\ x \in X \Longrightarrow$ *Ord x* **shows** *Ord* ($\bigsqcup X$)

proof (*cases small X*)
  case *True*
  with *assms* **show** *?thesis*
    by (*auto simp*: *Ord-def Transset-Sup*)
qed (*simp add*: *Sup-V-def*)

lemma *Ord-Union*:
  assumes $\bigwedge x.\ x \in X \Longrightarrow$ *Ord x small X* **shows** *Ord* (*set* ($\bigcup$ (*elts* ‘ *X*)))
  by (*metis Ord-Sup Sup-V-def assms*)

lemma *Ord-sup*:
  assumes *Ord x Ord y* **shows** *Ord* ($x \sqcup y$)
  using *assms*
  proof (*clarsimp simp*: *Ord-def*)
  show *Transset* ($x \sqcup y$) $\wedge$ ($\forall y \in$*elts* $x \cup$ *elts y*. *Transset y*)
    if *Transset x Transset y* $\forall y \in$*elts x*. *Transset y* $\forall y \in$*elts y*. *Transset y*
    using *Ord-def Transset-sup assms* **by** *auto*
qed

lemma *big-ON* [*simp*]: $\neg$ *small ON*
proof
  assume *small ON*
  then have *set ON* $\in$ *ON*
   by (*metis Ord-Union Ord-succ Sup-upper elts-Sup elts-succ insertI1 mem-Collect-eq mem-not-refl set-of-elts vsubsetD*)
  then show *False*
    by (*metis* ‹*small ON*› *elts-of-set mem-not-refl*)
qed

lemma *Ord-1* [*iff*]: *Ord 1*
  using *Ord-succ one-V-def succ-def vinsert-def* **by** *fastforce*

lemma *OrdmemD*: *Ord k* $\Longrightarrow j \in$ *elts k* $\Longrightarrow j < k$
  using *Ord-def Transset-def less-V-def* **by** *auto*

lemma *Ord-trans*: $[\![\ i \in$ *elts j*; $\ j \in$ *elts k*; *Ord k* $]\!] \Longrightarrow i \in$ *elts k*
  using *Ord-def Transset-def* **by** *blast*

lemma *mem-0-Ord*:
  assumes *k*: *Ord k* **and** *knz*: $k \neq 0$ **shows** $0 \in$ *elts k*
  by (*metis Ord-def Transset-def inf.orderE k knz trad-foundation*)

lemma *Ord-in-Ord*: $[\![$ *Ord k*; $m \in$ *elts k* $]\!] \Longrightarrow$ *Ord m*
  using *Ord-def Ord-trans* **by** *blast*

lemma *OrdI*: $[\![$*Transset i*; $\bigwedge x.\ x \in$ *elts i* $\Longrightarrow$ *Transset x*$]\!] \Longrightarrow$ *Ord i*
  by (*simp add*: *Ord-def*)

lemma *Ord-is-Transset*: *Ord i* $\Longrightarrow$ *Transset i*

by (*simp add*: *Ord-def*)

lemma *Ord-contains-Transset*: $\llbracket Ord\ i;\ j \in elts\ i\rrbracket \Longrightarrow Transset\ j$
  using *Ord-def* **by** *blast*

lemma *ON-imp-Ord*:
  assumes $H \subseteq ON\ x \in H$
  shows *Ord x*
  using *assms* **by** *blast*

lemma *elts-subset-ON*: $Ord\ \alpha \Longrightarrow elts\ \alpha \subseteq ON$
  using *Ord-in-Ord* **by** *blast*

lemma *Transset-pred* [*simp*]: $Transset\ x \Longrightarrow \bigsqcup(elts\ (succ\ x)) = x$
  by (*fastforce simp*: *Transset-def*)

lemma *Ord-pred* [*simp*]: $Ord\ \beta \Longrightarrow \bigsqcup\ (insert\ \beta\ (elts\ \beta)) = \beta$
  using *Ord-def Transset-pred* **by** *auto*

1.4.3 Induction, Linearity, etc.

lemma *Ord-induct* [*consumes 1*, *case-names step*]:
  assumes *k*: *Ord k*
    and *step*: $\bigwedge x.\llbracket\ Ord\ x;\ \bigwedge y.\ y \in elts\ x \Longrightarrow P\ y\ \rrbracket\ \Longrightarrow P\ x$
   shows *P k*
  using *foundation k*
proof (*induction k rule*: *wf-induct-rule*)
  case (*less x*)
  then show *?case*
   using *Ord-in-Ord local.step* **by** *auto*
qed

Comparability of ordinals

lemma *Ord-linear*: $Ord\ k \Longrightarrow Ord\ l \Longrightarrow k \in elts\ l \lor k{=}l \lor l \in elts\ k$
proof (*induct k arbitrary*: *l rule*: *Ord-induct*)
  case (*step k*)
  note *step-k* = *step*
  show *?case* **using** ‹*Ord l*›
   proof (*induct l rule*: *Ord-induct*)
    case (*step l*)
    thus *?case* **using** *step-k*
     by (*metis Ord-trans V-equalityI*)
   qed
qed

The trichotomy law for ordinals

lemma *Ord-linear-lt*:
  assumes *Ord k Ord l*
  obtains (*lt*) $k < l$ | (*eq*) $k{=}l$ | (*gt*) $l < k$

using *Ord-linear OrdmemD assms* **by** *blast*

lemma *Ord-linear2*:
assumes *Ord k Ord l*
obtains (*lt*) $k < l$ | (*ge*) $l \leq k$
by (*metis Ord-linear-lt eq-refl assms order.strict-implies-order*)

lemma *Ord-linear-le*:
assumes *Ord k Ord l*
obtains (*le*) $k \leq l$ | (*ge*) $l \leq k$
by (*meson Ord-linear2 le-less assms*)

lemma *union-less-iff* [*simp*]: $[\![Ord\ i;\ Ord\ j]\!] \Longrightarrow i \sqcup j < k \longleftrightarrow i{<}k \wedge j{<}k$
by (*metis Ord-linear-le le-iff-sup sup.order-iff sup.strict-boundedE*)

lemma *Ord-mem-iff-lt*: $Ord\ k \Longrightarrow Ord\ l \Longrightarrow k \in elts\ l \longleftrightarrow k < l$
by (*metis Ord-linear OrdmemD less-le-not-le*)

lemma *Ord-Collect-lt*: $Ord\ \alpha \Longrightarrow \{\xi.\ Ord\ \xi \wedge \xi < \alpha\} = elts\ \alpha$
by (*auto simp flip*: *Ord-mem-iff-lt elim*: *Ord-in-Ord OrdmemD*)

lemma *Ord-not-less*: $[\![Ord\ x;\ Ord\ y]\!] \Longrightarrow \neg\ x < y \longleftrightarrow y \leq x$
by (*metis* (*no-types*) *Ord-linear2 leD*)

lemma *Ord-not-le*: $[\![Ord\ x;\ Ord\ y]\!] \Longrightarrow \neg\ x \leq y \longleftrightarrow y < x$
by (*metis* (*no-types*) *Ord-linear2 leD*)

lemma *le-succ-iff*: $Ord\ i \Longrightarrow Ord\ j \Longrightarrow succ\ i \leq succ\ j \longleftrightarrow i \leq j$
by (*metis Ord-linear-le Ord-succ le-succE order-antisym*)

lemma *succ-le-iff*: $Ord\ i \Longrightarrow Ord\ j \Longrightarrow succ\ i \leq j \longleftrightarrow i < j$
using *Ord-mem-iff-lt dual-order.strict-implies-order less-eq-V-def* **by** *fastforce*

lemma *succ-in-Sup-Ord*:
assumes *eq*: $succ\ \beta = \bigsqcup A$ **and** *small A* $A \subseteq ON$ *Ord* β
shows $succ\ \beta \in A$
proof −
have $\neg \bigsqcup A \leq \beta$
using *eq* ‹*Ord* β› *succ-le-iff* **by** *fastforce*
then show *?thesis*
using *assms* **by** (*metis Ord-linear2 Sup-least Sup-upper eq-iff mem-Collect-eq subsetD succ-le-iff*)
qed

lemma *in-succ-iff*: $Ord\ i \Longrightarrow j \in elts\ (ZFC\text{-}in\text{-}HOL.succ\ i) \longleftrightarrow Ord\ j \wedge j \leq i$
by (*metis Ord-in-Ord Ord-mem-iff-lt Ord-not-le Ord-succ succ-le-iff*)

lemma *zero-in-succ* [*simp,intro*]: $Ord\ i \Longrightarrow 0 \in elts\ (succ\ i)$
using *mem-0-Ord* **by** *auto*

lemma *less-succ-self*: $x < succ\ x$
  by (*simp add*: *less-eq-V-def order-neq-le-trans subset-insertI*)

lemma *Ord-finite-Sup*: $[\![finite\ A;\ A \subseteq ON;\ A \neq \{\}]\!] \Longrightarrow \bigsqcup A \in A$
proof (*induction A rule*: *finite-induct*)
  case (*insert x A*)
  then have $*$: *small A A* $\subseteq$ *ON Ord x*
    by (*auto simp add*: *ZFC-in-HOL.finite-imp-small insert.hyps*)
  show *?case*
  proof (*cases* $A = \{\}$)
    case *False*
    then have $\bigsqcup A \in A$
      using *insert* **by** *blast*
    then have $\bigsqcup A \leq x$ **if** $x \sqcup \bigsqcup A \notin A$
      using $*$ **by** (*metis ON-imp-Ord Ord-linear-le sup.absorb2 that*)
    then show *?thesis*
      by (*fastforce simp*: ‹*small A*› *Sup-V-insert*)
  qed *auto*
qed *auto*

1.4.4 The natural numbers

primrec *ord-of-nat* :: *nat* $\Rightarrow$ *V* **where**
  ord-of-nat $0 = 0$
| *ord-of-nat* (*Suc n*) = *succ* (*ord-of-nat n*)

lemma *ord-of-nat-eq-initial*: *ord-of-nat n* = *set* (*ord-of-nat* ‘ $\{..<n\}$)
  by (*induction n*) (*auto simp*: *lessThan-Suc succ-def*)

lemma *mem-ord-of-nat-iff* [*simp*]: $x \in$ *elts* (*ord-of-nat n*) $\longleftrightarrow$ ($\exists\, m<n.\ x =$ *ord-of-nat m*)
  by (*subst ord-of-nat-eq-initial*) *auto*

lemma *elts-ord-of-nat*: *elts* (*ord-of-nat k*) = *ord-of-nat* ‘ $\{..<k\}$
  by *auto*

lemma *Ord-equality*: *Ord i* $\Longrightarrow i = \bigsqcup$ (*succ* ‘ *elts i*)
  by (*force intro*: *Ord-trans*)

lemma *Ord-ord-of-nat* [*simp*]: *Ord* (*ord-of-nat k*)
  by (*induct k*, *auto*)

lemma *ord-of-nat-equality*: *ord-of-nat n* = $\bigsqcup$ ((*succ* $\circ$ *ord-of-nat*) ‘ $\{..<n\}$)
  by (*metis Ord-equality Ord-ord-of-nat elts-of-set image-comp small-image-nat-V ord-of-nat-eq-initial*)

definition ω :: *V* **where** $\omega \equiv$ *set* (*range ord-of-nat*)

lemma *elts-ω*: *elts* ω = $\{\alpha.\ \exists\, n.\ \alpha = \textit{ord-of-nat}\ n\}$
  by (*auto simp*: *ω-def image-iff*)

lemma *nat-into-Ord* [*simp*]: $n \in \textit{elts}\ \omega \Longrightarrow \textit{Ord}\ n$
  by (*metis Ord-ord-of-nat ω-def elts-of-set image-iff inf*)

lemma *Sup-ω*: $\bigsqcup(\textit{elts}\ \omega) = \omega$
  unfolding *ω-def* **by** *force*

lemma *Ord-ω* [*iff*]: *Ord* ω
  by (*metis Ord-Sup Sup-ω nat-into-Ord*)

lemma *zero-in-omega* [*iff*]: $0 \in \textit{elts}\ \omega$
  by (*metis ω-def elts-of-set inf ord-of-nat.simps*(*1*) *rangeI*)

lemma *succ-in-omega* [*simp*]: $n \in \textit{elts}\ \omega \Longrightarrow \textit{succ}\ n \in \textit{elts}\ \omega$
  by (*metis ω-def elts-of-set image-iff small-image-nat-V ord-of-nat.simps*(*2*) *rangeI*)

lemma *ord-of-eq-0*: $\textit{ord-of-nat}\ j = 0 \Longrightarrow j = 0$
  by (*induct j*) (*auto simp*: *succ-neq-zero*)

lemma *ord-of-nat-le-omega*: $\textit{ord-of-nat}\ n \le \omega$
  by (*metis Sup-ω ZFC-in-HOL.Sup-upper ω-def elts-of-set inf rangeI*)

lemma *ord-of-eq-0-iff* [*simp*]: $\textit{ord-of-nat}\ n = 0 \longleftrightarrow n=0$
  by (*auto simp*: *ord-of-eq-0*)

lemma *ord-of-nat-inject* [*iff*]: $\textit{ord-of-nat}\ i = \textit{ord-of-nat}\ j \longleftrightarrow i=j$
proof (*induct i arbitrary*: *j*)
  case *0* **show** *?case*
    using *ord-of-eq-0* **by** *auto*
next
  case (*Suc i*) **then show** *?case*
   by *auto* (*metis elts-0 elts-succ insert-not-empty not0-implies-Suc ord-of-nat.simps succ-inject-iff*)
qed

corollary *inj-ord-of-nat*: *inj ord-of-nat*
  by (*simp add*: *linorder-injI*)

corollary *countable*:
  assumes *countable X* **shows** *small X*
proof −
  have $X \subseteq \textit{range}\ (\textit{from-nat-into}\ X)$
    by (*simp add*: *assms subset-range-from-nat-into*)
  then show *?thesis*
    by (*meson inf-raw inj-ord-of-nat replacement small-def smaller-than-small*)
qed

corollary *infinite-ω*: *infinite* (*elts ω*)
  using *range-inj-infinite* [*of ord-of-nat*]
  by (*simp add*: *ω-def inj-ord-of-nat*)

corollary *ord-of-nat-mono-iff* [*iff*]: *ord-of-nat* $i \leq$ *ord-of-nat* $j \longleftrightarrow i \leq j$
  by (*metis Ord-def Ord-ord-of-nat Transset-def eq-iff mem-ord-of-nat-iff not-less ord-of-nat-inject*)

corollary *ord-of-nat-strict-mono-iff* [*iff*]: *ord-of-nat* $i <$ *ord-of-nat* $j \longleftrightarrow i < j$
  by (*simp add*: *less-le-not-le*)

lemma *small-image-nat* [*simp*]:
  fixes *N* :: *nat set* **shows** *small* (*g* ‘ *N*)
  by (*simp add*: *countable*)

lemma *finite-Ord-omega*: $\alpha \in$ *elts* $\omega \Longrightarrow$ *finite* (*elts* α)
  proof (*clarsimp simp add*: *ω-def*)
  show *finite* (*elts* (*ord-of-nat n*)) **if** $\alpha =$ *ord-of-nat n* **for** *n*
    using *that* **by** (*simp add*: *ord-of-nat-eq-initial* [*of n*])
qed

lemma *infinite-Ord-omega*: *Ord* $\alpha \Longrightarrow$ *infinite* (*elts* α) $\Longrightarrow \omega \leq \alpha$
  by (*meson Ord-ω Ord-linear2 Ord-mem-iff-lt finite-Ord-omega*)

lemma *ord-of-minus-1*: $n > 0 \Longrightarrow$ *ord-of-nat* $n =$ *succ* (*ord-of-nat* $(n - 1)$)
  by (*metis Suc-diff-1 ord-of-nat.simps*(*2*))

lemma *card-ord-of-nat* [*simp*]: *card* (*elts* (*ord-of-nat m*)) = *m*
  by (*induction m*) (*auto simp*: *ω-def finite-Ord-omega*)

lemma *ord-of-nat-ω* [*iff*]:*ord-of-nat* $n \in$ *elts* ω
  by (*simp add*: *ω-def*)

lemma *succ-ω-iff* [*iff*]: *succ* $n \in$ *elts* $\omega \longleftrightarrow n \in$ *elts* ω
  by (*metis Ord-ω OrdmemD elts-vinsert insert-iff less-V-def succ-def succ-in-omega vsubsetD*)

lemma *ω-gt0* [*simp*]: $\omega > 0$
  by (*simp add*: *OrdmemD*)

lemma *ω-gt1* [*simp*]: $\omega > 1$
  by (*simp add*: *OrdmemD one-V-def*)

1.4.5 Limit ordinals

definition *Limit* :: $V \Rightarrow bool$
  where *Limit* $i \equiv$ *Ord* $i \land 0 \in$ *elts* $i \land (\forall y.\ y \in$ *elts* $i \longrightarrow$ *succ* $y \in$ *elts* $i)$

lemma *zero-not-Limit* [*iff*]: ¬ *Limit 0*

by (*simp add*: *Limit-def*)

lemma *not-succ-Limit* [*simp*]: $\neg$ *Limit*(*succ i*)
by (*metis Limit-def Ord-mem-iff-lt elts-succ insertI1 less-irrefl*)

lemma *Limit-is-Ord*: *Limit* $\xi \Longrightarrow$ *Ord* ξ
by (*simp add*: *Limit-def*)

lemma *succ-in-Limit-iff*: *Limit* $\xi \Longrightarrow$ *succ* $\alpha \in$ *elts* $\xi \longleftrightarrow \alpha \in$ *elts* ξ
by (*metis Limit-def OrdmemD elts-succ insertI1 less-V-def vsubsetD*)

lemma *Limit-eq-Sup-self* [*simp*]: *Limit i* $\Longrightarrow$ *Sup* (*elts i*) = *i*
apply (*rule order-antisym*)
apply (*simp add*: *Limit-def Ord-def Transset-def Sup-least*)
by (*metis Limit-def Ord-equality Sup-V-def SUP-le-iff Sup-upper small-elts*)

lemma *zero-less-Limit*: *Limit* $\beta \Longrightarrow$ *0* $<$ β
by (*simp add*: *Limit-def OrdmemD*)

lemma *non-Limit-ord-of-nat* [*iff*]: $\neg$ *Limit* (*ord-of-nat m*)
by (*metis Limit-def mem-ord-of-nat-iff not-succ-Limit ord-of-eq-0-iff ord-of-minus-1*)

lemma *Limit-omega* [*iff*]: *Limit* ω
by (*simp add*: *Limit-def*)

lemma *omega-nonzero* [*simp*]: $\omega \neq$ *0*
using *Limit-omega* **by** *fastforce*

lemma *Ord-cases-lemma*:
assumes *Ord k* **shows** *k* = *0* $\vee$ ($\exists$ *j*. *k* = *succ j*) $\vee$ *Limit k*
proof (*cases Limit k*)
case *False*
have *succ j* $\in$ *elts k* **if** $\forall$ *j*. *k* $\neq$ *succ j j* $\in$ *elts k* **for** *j*
by (*metis Ord-in-Ord Ord-linear Ord-succ assms elts-succ insertE mem-not-sym that*)
with *assms* **show** *?thesis*
by (*auto simp*: *Limit-def mem-0-Ord*)
qed *auto*

lemma *Ord-cases* [*cases type*: *V*, *case-names 0 succ limit*]:
assumes *Ord k*
obtains *k* = *0* | *l* **where** *Ord l succ l* = *k* | *Limit k*
by (*metis assms Ord-cases-lemma Ord-in-Ord elts-succ insertI1*)

lemma *non-succ-LimitI*:
assumes *i*$\neq$*0* *Ord*(*i*) $\bigwedge$*y*. *succ*(*y*) $\neq$ *i*
shows *Limit*(*i*)
using *Ord-cases-lemma assms* **by** *blast*

lemma *Ord-induct3* [*consumes 1*, *case-names 0 succ Limit*, *induct type*: *V*]:
  assumes α: *Ord* α
    and *P*: *P 0* $\bigwedge\alpha.$ $[\![$*Ord* α; *P* $\alpha]\!] \Longrightarrow$ *P* (*succ* α)
        $\bigwedge\alpha.$ $[\![$*Limit* α; $\bigwedge\xi.$ $\xi \in$ *elts* $\alpha \Longrightarrow$ *P* $\xi]\!] \Longrightarrow$ *P* ($\bigsqcup\xi \in$ *elts* $\alpha.$ ξ)
  shows *P* α
  using α
proof (*induction* α *rule*: *Ord-induct*)
  case (*step* α)
  then show *?case*
    by (*metis Limit-eq-Sup-self Ord-cases P elts-succ image-ident insertI1*)
qed

1.4.6 Properties of LEAST for ordinals

lemma
  assumes *Ord k P k*
  shows *Ord-LeastI*: *P* (*LEAST i. Ord i* $\wedge$ *P i*) **and** *Ord-Least-le*: (*LEAST i. Ord i* $\wedge$ *P i*) $\leq$ *k*
proof $-$
  have *P* (*LEAST i. Ord i* $\wedge$ *P i*) $\wedge$ (*LEAST i. Ord i* $\wedge$ *P i*) $\leq$ *k*
    using *assms*
  proof (*induct k rule*: *Ord-induct*)
    case (*step x*) **then have** *P x* **by** *simp*
    show *?case* **proof** (*rule classical*)
      assume *assm*: $\neg$ (*P* (*LEAST a. Ord a* $\wedge$ *P a*) $\wedge$ (*LEAST a. Ord a* $\wedge$ *P a*) $\leq$ *x*)
      have $\bigwedge y.$ *Ord y* $\wedge$ *P y* $\Longrightarrow$ *x* $\leq$ *y*
      proof (*rule classical*)
        fix *y*
        assume *y*: *Ord y* $\wedge$ *P y* $\neg$ *x* $\leq$ *y*
        with *step* **obtain** *P* (*LEAST a. Ord a* $\wedge$ *P a*) **and** *le*: (*LEAST a. Ord a* $\wedge$ *P a*) $\leq$ *y*
          by (*meson Ord-linear2 Ord-mem-iff-lt*)
        with *assm* **have** *x* $<$ (*LEAST a. Ord a* $\wedge$ *P a*)
          by (*meson Ord-linear-le y order.trans* ‹*Ord x*›)
        then show *x* $\leq$ *y*
          using *le* **by** *auto*
      qed
      then have *Least*: (*LEAST a. Ord a* $\wedge$ *P a*) $=$ *x*
        by (*simp add*: *Least-equality* ‹*Ord x*› *step.prems*)
      with ‹*P x*› **show** *?thesis* **by** *simp*
    qed
  qed
  then show *P* (*LEAST i. Ord i* $\wedge$ *P i*) **and** (*LEAST i. Ord i* $\wedge$ *P i*) $\leq$ *k* **by** *auto*
qed

lemma *Ord-Least*:
  assumes *Ord k P k*
  shows *Ord* (*LEAST i. Ord i* $\wedge$ *P i*)

proof –
  have *Ord* (*LEAST* *i*. *Ord* *i* ∧ (*Ord* *i* ∧ *P* *i*))
    using *Ord-LeastI* [**where** *P* = *λi*. *Ord* *i* ∧ *P* *i*] *assms* **by** *blast*
  then show *?thesis*
    by *simp*
qed

— The following 3 lemmas are due to Brian Huffman
lemma *Ord-LeastI-ex*: ∃ *i*. *Ord* *i* ∧ *P* *i* ⟹ *P* (*LEAST* *i*. *Ord* *i* ∧ *P* *i*)
  using *Ord-LeastI* **by** *blast*

lemma *Ord-LeastI2*:
  ⟦*Ord* *a*; *P* *a*; ⋀*x*. ⟦*Ord* *x*; *P* *x*⟧ ⟹ *Q* *x*⟧ ⟹ *Q* (*LEAST* *i*. *Ord* *i* ∧ *P* *i*)
  by (*blast intro*: *Ord-LeastI Ord-Least*)

lemma *Ord-LeastI2-ex*:
  ∃ *a*. *Ord* *a* ∧ *P* *a* ⟹ (⋀*x*. ⟦*Ord* *x*; *P* *x*⟧ ⟹ *Q* *x*) ⟹ *Q* (*LEAST* *i*. *Ord* *i* ∧ *P* *i*)
  by (*blast intro*: *Ord-LeastI-ex Ord-Least*)

lemma *Ord-LeastI2-wellorder*:
  assumes *Ord* *a* *P* *a*
  and ⋀*a*. ⟦ *P* *a*; ∀ *b*. *Ord* *b* ∧ *P* *b* ⟶ *a* ≤ *b* ⟧ ⟹ *Q* *a*
  shows *Q* (*LEAST* *i*. *Ord* *i* ∧ *P* *i*)
proof (*rule LeastI2-order*)
  show *Ord* (*LEAST* *i*. *Ord* *i* ∧ *P* *i*) ∧ *P* (*LEAST* *i*. *Ord* *i* ∧ *P* *i*)
    using *Ord-Least Ord-LeastI assms* **by** *auto*
next
  fix *y* **assume** *Ord* *y* ∧ *P* *y* **thus** (*LEAST* *i*. *Ord* *i* ∧ *P* *i*) ≤ *y*
    by (*simp add*: *Ord-Least-le*)
next
  fix *x* **assume** *Ord* *x* ∧ *P* *x* ∀ *y*. *Ord* *y* ∧ *P* *y* ⟶ *x* ≤ *y* **thus** *Q* *x*
    by (*simp add*: *assms*(*3*))
qed

lemma *Ord-LeastI2-wellorder-ex*:
  assumes ∃ *x*. *Ord* *x* ∧ *P* *x*
  and ⋀*a*. ⟦ *P* *a*; ∀ *b*. *Ord* *b* ∧ *P* *b* ⟶ *a* ≤ *b* ⟧ ⟹ *Q* *a*
  shows *Q* (*LEAST* *i*. *Ord* *i* ∧ *P* *i*)
using *assms* **by** *clarify* (*blast intro*!: *Ord-LeastI2-wellorder*)

lemma *not-less-Ord-Least*: ⟦*k* < (*LEAST* *x*. *Ord* *x* ∧ *P* *x*); *Ord* *k*⟧ ⟹ ¬ *P* *k*
  using *Ord-Least-le less-le-not-le* **by** *auto*

lemma *exists-Ord-Least-iff*: (∃ *α*. *Ord* *α* ∧ *P* *α*) ⟷ (∃ *α*. *Ord* *α* ∧ *P* *α* ∧ (∀ *β* < *α*. *Ord* *β* ⟶ ¬ *P* *β*)) (**is** *?lhs* ⟷ *?rhs*)
proof
  assume *?rhs* **thus** *?lhs* **by** *blast*
next

assume H: *?lhs* **then obtain** α **where** α: *Ord* α *P* α **by** *blast*
let *?x* = *LEAST* α. *Ord* $\alpha \wedge P\ \alpha$
have *Ord ?x*
by (*metis Ord-Least* α)
moreover
{ **fix** β **assume** m: $\beta <$ *?x Ord* β
from *not-less-Ord-Least*[*OF m*] **have** $\neg$ *P* β . }
ultimately show *?rhs*
using *Ord-LeastI-ex*[*OF H*] **by** *blast*
qed

lemma *Ord-mono-imp-increasing*:
assumes *fun-hD*: $h \in D \to D$
and *mono-h*: *strict-mono-on D h*
and $D \subseteq ON$ **and** ν: $\nu \in D$
shows $\nu \le h\ \nu$
proof (*rule ccontr*)
assume *non*: $\neg\ \nu \le h\ \nu$
define μ **where** $\mu \equiv LEAST\ \mu.\ Ord\ \mu \wedge \neg\ \mu \le h\ \mu \wedge \mu \in D$
have *Ord* ν
using ν ‹$D \subseteq ON$› **by** *blast*
then have μ: $\neg\ \mu \le h\ \mu \wedge \mu \in D$
unfolding μ*-def* **by** (*rule Ord-LeastI*) (*simp add*: ν *non*)
have *Ord* ($h\ \nu$)
using *assms* **by** *auto*
then have *Ord* (h ($h\ \nu$))
by (*meson ON-imp-Ord* ν *assms funcset-mem*)
have *Ord* μ
using μ ‹$D \subseteq ON$› **by** *blast*
then have $h\ \mu < \mu$
by (*metis ON-imp-Ord Ord-linear2 PiE* μ ‹$D \subseteq ON$› *fun-hD*)
then have $\neg\ h\ \mu \le h\ (h\ \mu)$
using μ *fun-hD mono-h* **by** (*force simp*: *strict-mono-on-def*)
moreover have $*$: $h\ \mu \in D$
using μ *fun-hD* **by** *auto*
moreover have *Ord* ($h\ \mu$)
using ‹$D \subseteq ON$› $*$ **by** *blast*
ultimately have $\mu \le h\ \mu$
by (*simp add*: μ*-def Ord-Least-le*)
then show *False*
using μ **by** *blast*
qed

lemma *le-Sup-iff*:
assumes $A \subseteq ON$ *Ord* x *small* A **shows** $x \le \bigsqcup A \longleftrightarrow (\forall\, y \in ON.\ y{<}x \longrightarrow (\exists\, a{\in}A.\ y < a))$
proof (*intro iffI ballI impI*)
show $\exists\, a{\in}A.\ y < a$
if $x \le \bigsqcup\ A\ y \in ON\ y < x$

for y
proof $-$
have $\neg \bigsqcup A \le y$ *Ord y*
using *that* **by** *auto*
then show *?thesis*
by (*metis Ord-linear2 Sup-least* ‹$A \subseteq ON$› *mem-Collect-eq subset-eq*)
qed
show $x \le \bigsqcup A$
if $\forall y \in ON.\ y < x \longrightarrow (\exists a \in A.\ y < a)$
using *that assms*
by (*metis Ord-Sup Ord-linear-le Sup-upper less-le-not-le mem-Collect-eq subsetD*)
qed

lemma *le-SUP-iff*: $[\![f \text{ ` } A \subseteq ON;\ Ord\ x;\ small\ A]\!] \Longrightarrow x \le \bigsqcup(f \text{ ` } A) \longleftrightarrow (\forall y \in ON.\ y<x \longrightarrow (\exists i \in A.\ y < f\ i))$
by (*simp add*: *le-Sup-iff*)

1.5 Transfinite Recursion and the V-levels

definition *transrec* :: $((V \Rightarrow 'a) \Rightarrow V \Rightarrow 'a) \Rightarrow V \Rightarrow 'a$
where *transrec H a* $\equiv$ *wfrec* $\{(x,y).\ x \in elts\ y\}$ *H a*

lemma *transrec*: *transrec H a* $=$ *H* $(\lambda x \in elts\ a.\ transrec\ H\ x)$ *a*
proof $-$
have (*cut* (*wfrec* $\{(x, y).\ x \in elts\ y\}$ *H*) $\{(x, y).\ x \in elts\ y\}$ *a*)
$= (\lambda x \in elts\ a.\ wfrec\ \{(x, y).\ x \in elts\ y\}\ H\ x)$
by (*force simp*: *cut-def*)
then show *?thesis*
unfolding *transrec-def*
by (*simp add*: *foundation wfrec*)
qed

Avoids explosions in proofs; resolve it with a meta-level definition

lemma *def-transrec*:
$[\![\bigwedge x.\ f\ x \equiv transrec\ H\ x]\!] \Longrightarrow f\ a = H(\lambda x \in elts\ a.\ f\ x)\ a$
by (*metis restrict-ext transrec*)

lemma *eps-induct* [*case-names step*]:
assumes $\bigwedge x.\ (\bigwedge y.\ y \in elts\ x \Longrightarrow P\ y) \Longrightarrow P\ x$
shows *P a*
using *wf-induct* [*OF foundation*] *assms* **by** *auto*

definition *Vfrom* :: $[V,V] \Rightarrow V$
where *Vfrom a* $\equiv$ *transrec* $(\lambda f\ x.\ a \sqcup \bigsqcup((\lambda y.\ VPow(f\ y)) \text{ ` } elts\ x))$

abbreviation *Vset* :: $V \Rightarrow V$ **where** *Vset* $\equiv$ *Vfrom 0*

lemma *Vfrom*: $Vfrom\ a\ i = a \sqcup \bigsqcup((\lambda j.\ VPow(Vfrom\ a\ j))\ `\ elts\ i)$
apply (*subst Vfrom-def*)
apply (*subst transrec*)
using *Vfrom-def* **by** *auto*

lemma *Vfrom-0* [*simp*]: $Vfrom\ a\ 0 = a$
by (*subst Vfrom*) *auto*

lemma *Vset*: $Vset\ i = \bigsqcup((\lambda j.\ VPow(Vset\ j))\ `\ elts\ i)$
by (*subst Vfrom*) *auto*

lemma *Vfrom-mono1*:
assumes $a \le b$ **shows** $Vfrom\ a\ i \le Vfrom\ b\ i$
proof (*induction i rule*: *eps-induct*)
case (*step i*)
then have $a \sqcup (\bigsqcup j{\in}elts\ i.\ VPow\ (Vfrom\ a\ j)) \le b \sqcup (\bigsqcup j{\in}elts\ i.\ VPow\ (Vfrom\ b\ j))$
by (*intro sup-mono cSUP-subset-mono* ‹$a \le b$›) *auto*
then show *?case*
by (*metis Vfrom*)
qed

lemma *Vfrom-mono2*: $Vfrom\ a\ i \le Vfrom\ a\ (i \sqcup j)$
proof (*induction arbitrary*: *j rule*: *eps-induct*)
case (*step i*)
then have $a \sqcup (\bigsqcup j{\in}elts\ i.\ VPow\ (Vfrom\ a\ j))$
$\le a \sqcup (\bigsqcup j{\in}elts\ (i \sqcup j).\ VPow\ (Vfrom\ a\ j))$
by (*intro sup-mono cSUP-subset-mono order-refl*) *auto*
then show *?case*
by (*metis Vfrom*)
qed

lemma *Vfrom-mono*: $[\![Ord\ i;\ a{\le}b;\ i{\le}j]\!] \Longrightarrow Vfrom\ a\ i \le Vfrom\ b\ j$
by (*metis* (*no-types*) *Vfrom-mono1 Vfrom-mono2 dual-order.trans sup.absorb-iff2*)

lemma *Transset-Vfrom*: $Transset(A) \Longrightarrow Transset(Vfrom\ A\ i)$
proof (*induction i rule*: *eps-induct*)
case (*step i*)
then show *?case*
by (*metis Transset-SUP Transset-VPow Transset-sup Vfrom*)
qed

lemma *Transset-Vset* [*simp*]: *Transset*(*Vset i*)
by (*simp add*: *Transset-Vfrom*)

lemma *Vfrom-sup*: $Vfrom\ a\ (i \sqcup j) = Vfrom\ a\ i \sqcup Vfrom\ a\ j$
proof (*rule order-antisym*)
show $Vfrom\ a\ (i \sqcup j) \le Vfrom\ a\ i \sqcup Vfrom\ a\ j$
by (*simp add*: *Vfrom* [*of* $a\ i \sqcup j$] *Vfrom* [*of a i*] *Vfrom* [*of a j*] *Sup-Un-distrib*

image-Un sup.assoc sup.left-commute)
  show *Vfrom a i* $\sqcup$ *Vfrom a j* $\leq$ *Vfrom a* (*i* $\sqcup$ *j*)
    by (*metis Vfrom-mono2 le-supI sup-commute*)
qed

lemma *Vfrom-succ-Ord*:
  assumes *Ord i* **shows** *Vfrom a* (*succ i*) = *a* $\sqcup$ *VPow*(*Vfrom a i*)
proof (*cases i* = *0*)
  case *True*
  then show *?thesis*
    by (*simp add*: *Vfrom* [*of - succ 0*])
next
  case *False*
  have ∗: ($\bigsqcup$*x*∈*elts i*. *VPow* (*Vfrom a x*)) $\leq$ *VPow* (*Vfrom a i*)
  proof (*rule cSup-least*)
    show (λ*x*. *VPow* (*Vfrom a x*)) ‘ *elts i* $\neq$ {}
      using *False* **by** *auto*
    show *x* $\leq$ *VPow* (*Vfrom a i*) **if** *x* ∈ (λ*x*. *VPow* (*Vfrom a x*)) ‘ *elts i* **for** *x*
      using *that*
     by *clarsimp* (*meson Ord-in-Ord Ord-linear-le Vfrom-mono assms mem-not-refl order-refl vsubsetD*)
  qed
  show *?thesis*
  proof (*rule Vfrom* [*THEN trans*])
    show *a* $\sqcup$ ($\bigsqcup$*j*∈*elts* (*succ i*). *VPow* (*Vfrom a j*)) = *a* $\sqcup$ *VPow* (*Vfrom a i*)
      using *assms*
      by (*intro sup-mono order-antisym*) (*auto simp*: *Sup-V-insert* ∗)
  qed
qed

lemma *Vset-succ*: *Ord i* $\Longrightarrow$ *Vset*(*succ*(*i*)) = *VPow*(*Vset*(*i*))
  by (*simp add*: *Vfrom-succ-Ord*)

lemma *Vfrom-Sup*:
  assumes *X* $\neq$ {} *small X*
  shows *Vfrom a* (*Sup X*) = ($\bigsqcup$*y*∈*X*. *Vfrom a y*)
proof (*rule order-antisym*)
  have *Vfrom a* ($\bigsqcup$ *X*) = *a* $\sqcup$ ($\bigsqcup$*j*∈*elts* ($\bigsqcup$ *X*). *VPow* (*Vfrom a j*))
    by (*metis Vfrom*)
  also have . . . $\leq$ $\bigsqcup$ (*Vfrom a* ‘ *X*)
  proof −
    have *a* $\leq$ $\bigsqcup$ (*Vfrom a* ‘ *X*)
       by (*metis Vfrom all-not-in-conv assms bdd-above-iff-small cSUP-upper2 replacement sup-ge1*)
    moreover have ($\bigsqcup$*j*∈*elts* ($\bigsqcup$ *X*). *VPow* (*Vfrom a j*)) $\leq$ $\bigsqcup$ (*Vfrom a* ‘ *X*)
    proof −
      have *VPow* (*Vfrom a x*) $\leq$ $\bigsqcup$ (*Vfrom a* ‘ *X*)
        if *y* ∈ *X* *x* ∈ *elts y* **for** *x y*
      proof −

have *VPow* (*Vfrom a x*) ≤ *Vfrom a y*
by (*metis Vfrom bdd-above-iff-small cSUP-upper2 le-supI2 order-refl replacement small-elts that*(*2*))
also have . . . ≤ ⨆ (*Vfrom a* ‘ *X*)
using *assms that* **by** (*force intro*: *cSUP-upper*)
finally show *?thesis* **.**
qed
then show *?thesis*
by (*simp add*: *SUP-le-iff* ‹*small X*›)
qed
ultimately show *?thesis*
by *auto*
qed
finally show *Vfrom a* (⨆ *X*) ≤ ⨆ (*Vfrom a* ‘ *X*) **.**
have ⋀*x*. *x* ∈ *X* ⟹
a ⊔ (⨆*j*∈*elts x*. *VPow* (*Vfrom a j*))
≤ *a* ⊔ (⨆*j*∈*elts* (⨆ *X*). *VPow* (*Vfrom a j*))
using *cSUP-subset-mono* ‹*small X*› **by** *auto*
then show ⨆ (*Vfrom a* ‘ *X*) ≤ *Vfrom a* (⨆ *X*)
by (*metis Vfrom assms*(*1*) *cSUP-least*)
qed

lemma *Limit-Vfrom-eq*:
Limit(*i*) ⟹ *Vfrom a i* = (⨆*y* ∈ *elts i*. *Vfrom a y*)
by (*metis Limit-def Limit-eq-Sup-self Vfrom-Sup ex-in-conv small-elts*)

end

2 Cartesian products, Disjoint Sums, Ranks, Cardinals

theory *ZFC-Cardinals*
imports *ZFC-in-HOL*

begin

declare [[*coercion-enabled*]]
declare [[*coercion ord-of-nat* :: *nat* ⇒ *V*]]

2.1 Ordered Pairs

lemma *singleton-eq-iff* [*iff*]: *set* {*a*} = *set* {*b*} ⟷ *a*=*b*
by *simp*

lemma *doubleton-eq-iff*: *set* {*a*,*b*} = *set* {*c*,*d*} ⟷ (*a*=*c* ∧ *b*=*d*) ∨ (*a*=*d* ∧ *b*=*c*)
by (*simp add*: *Set.doubleton-eq-iff*)

definition *vpair* :: *V* ⇒ *V* ⇒ *V*

where *vpair a b* = *set* {*set* {*a*},*set* {*a*,*b*}}

definition *vfst* :: *V* ⇒ *V*
where *vfst p* ≡ *THE x*. ∃ *y*. *p* = *vpair x y*

definition *vsnd* :: *V* ⇒ *V*
where *vsnd p* ≡ *THE y*. ∃ *x*. *p* = *vpair x y*

definition *vsplit* :: [[*V*, *V*] ⇒ ′*a*, *V*] ⇒ ′*a*::{} — for pattern-matching
where *vsplit c* ≡ λ*p*. *c* (*vfst p*) (*vsnd p*)

nonterminal *Vs*
syntax (*ASCII*)
-*Tuple* :: [*V*, *Vs*] ⇒ *V* (<(-,/ -)>)
-*hpattern* :: [*pttrn*, *patterns*] ⇒ *pttrn* (<-,/ ->)
syntax
:: *V* ⇒ *Vs* (-)
-*Enum* :: [*V*, *Vs*] ⇒ *Vs* (-,/ -)
-*Tuple* :: [*V*, *Vs*] ⇒ *V* (⟨(-,/ -)⟩)
-*hpattern* :: [*pttrn*, *patterns*] ⇒ *pttrn* (⟨-,/ -⟩)
translations
<*x*, *y*, *z*> ⇌ <*x*, <*y*, *z*>>
<*x*, *y*> ⇌ *CONST vpair x y*
<*x*, *y*, *z*> ⇌ <*x*, <*y*, *z*>>
λ<*x*,*y*,*zs*>. *b* ⇌ *CONST vsplit*(λ*x* <*y*,*zs*>. *b*)
λ<*x*,*y*>. *b* ⇌ *CONST vsplit*(λ*x y*. *b*)

lemma *vpair-def* ′: *vpair a b* = *set* {*set* {*a*,*a*},*set* {*a*,*b*}}
by (*simp add*: *vpair-def*)

lemma *vpair-iff* [*simp*]: *vpair a b* = *vpair a*′ *b*′ ⟷ *a*=*a*′ ∧ *b*=*b*′
unfolding *vpair-def* ′ *doubleton-eq-iff* **by** *auto*

lemmas *vpair-inject* = *vpair-iff* [*THEN iffD1*, *THEN conjE*, *elim*!]

lemma *vfst-conv* [*simp*]: *vfst* ⟨*a*,*b*⟩ = *a*
by (*simp add*: *vfst-def*)

lemma *vsnd-conv* [*simp*]: *vsnd* ⟨*a*,*b*⟩ = *b*
by (*simp add*: *vsnd-def*)

lemma *vsplit* [*simp*]: *vsplit c* ⟨*a*,*b*⟩ = *c a b*
by (*simp add*: *vsplit-def*)

lemma *vpair-neq-fst*: ⟨*a*,*b*⟩ ≠ *a*
by (*metis elts-of-set insertI1 mem-not-sym small-upair vpair-def* ′)

lemma *vpair-neq-snd*: ⟨*a*,*b*⟩ ≠ *b*

by (*metis elts-of-set insertI1 mem-not-sym small-upair subsetD subset-insertI vpair-def'*)

lemma *vpair-nonzero* [*simp*]: $\langle x,y\rangle \neq 0$
by (*metis elts-0 elts-of-set empty-not-insert small-upair vpair-def*)

lemma *zero-notin-vpair*: $0 \notin elts\ \langle x,y\rangle$
by (*auto simp*: *vpair-def*)

lemma *inj-on-vpair* [*simp*]: *inj-on* $(\lambda(x,\ y).\ \langle x,\ y\rangle)\ A$
by (*auto simp*: *inj-on-def*)

2.2 Generalized Cartesian product

definition *VSigma* :: $V \Rightarrow (V \Rightarrow V) \Rightarrow V$
where $VSigma\ A\ B \equiv set(\bigcup x \in elts\ A.\ \bigcup y \in elts\ (B\ x).\ \{\langle x,y\rangle\})$

abbreviation *vtimes* **where** $vtimes\ A\ B \equiv VSigma\ A\ (\lambda x.\ B)$

definition *pairs* :: $V \Rightarrow (V * V)set$
where $pairs\ r \equiv \{(x,y).\ \langle x,y\rangle \in elts\ r\}$

lemma *pairs-iff-elts*: $(x,y) \in pairs\ z \longleftrightarrow \langle x,y\rangle \in elts\ z$
by (*simp add*: *pairs-def*)

lemma *VSigma-iff* [*simp*]: $\langle a,b\rangle \in elts\ (VSigma\ A\ B) \longleftrightarrow a \in elts\ A \wedge b \in elts\ (B\ a)$
by (*auto simp*: *VSigma-def UNION-singleton-eq-range*)

lemma *VSigmaI* [*intro*!]: $[\![\ a \in elts\ A;\ \ b \in elts\ (B\ a)]\!] \implies \langle a,b\rangle \in elts\ (VSigma\ A\ B)$
by *simp*

lemmas *VSigmaD1* = *VSigma-iff* [*THEN iffD1*, *THEN conjunct1*]
lemmas *VSigmaD2* = *VSigma-iff* [*THEN iffD1*, *THEN conjunct2*]

The general elimination rule

lemma *VSigmaE* [*elim*!]:
assumes $c \in elts\ (VSigma\ A\ B)$
obtains *x y* **where** $x \in elts\ A\ \ y \in elts\ (B\ x)\ \ c=\langle x,y\rangle$
using *assms* **by** (*auto simp*: *VSigma-def split*: *if-split-asm*)

lemma *VSigmaE2* [*elim*!]:
assumes $\langle a,b\rangle \in elts\ (VSigma\ A\ B)$ **obtains** $a \in elts\ A$ **and** $b \in elts\ (B\ a)$
using *assms* **by** *auto*

lemma *VSigma-empty1* [*simp*]: $VSigma\ 0\ B = 0$
by *auto*

lemma *times-iff* [*simp*]: $\langle a,b\rangle \in$ *elts* (*vtimes A B*) $\longleftrightarrow a \in$ *elts A* $\wedge$ $b \in$ *elts B*
  by *simp*

lemma *timesI* [*intro!*]: $[\![a \in \textit{elts } A;\ \ b \in \textit{elts } B]\!] \Longrightarrow \langle a,b\rangle \in \textit{elts } (\textit{vtimes } A\ B)$
  by *simp*

lemma *times-empty2* [*simp*]: *vtimes A 0 = 0*
  using *elts-0* **by** *blast*

lemma *times-empty-iff*: *VSigma A B = 0* $\longleftrightarrow$ *A=0* $\vee$ ($\forall x \in$ *elts A*. *B x = 0*)
  by (*metis VSigmaE VSigmaI elts-0 empty-iff trad-foundation*)

lemma *elts-VSigma*: *elts* (*VSigma A B*) = ($\lambda(x,y)$. *vpair x y*) ‘ *Sigma* (*elts A*) (λx. *elts* (*B x*))
  by *auto*

lemma *small-Sigma* [*simp*]:
  assumes *A*: *small A* **and** *B*: $\bigwedge x.\ x \in A \Longrightarrow$ *small* (*B x*)
  shows *small* (*Sigma A B*)
proof −
  obtain *a* **where** *elts a* $\approx$ *A*
    by (*meson assms small-eqpoll*)
  then obtain *f* **where** *f*: *bij-betw f* (*elts a*) *A*
    using *eqpoll-def* **by** *blast*
  have $\exists y$. *elts y* $\approx$ *B x* **if** $x \in A$ **for** *x*
    using *B small-eqpoll that* **by** *blast*
  then obtain *g* **where** *g*: $\bigwedge x.\ x \in A \Longrightarrow$ *elts* (*g x*) $\approx$ *B x*
    by *metis*
  with *f* **have** *elts* (*VSigma a* ($g \circ f$)) $\approx$ *Sigma A B*
    by (*simp add*: *elts-VSigma Sigma-eqpoll-cong bij-betwE*)
  then show *?thesis*
    using *small-eqpoll* **by** *blast*
qed

lemma *small-Times* [*simp*]:
  assumes *small A small B* **shows** *small* ($A \times B$)
  by (*simp add*: *assms*)

lemma *small-Times-iff*: *small* ($A \times B$) $\longleftrightarrow$ *small A* $\wedge$ *small B* $\vee$ *A*={} $\vee$ *B*={}
(**is** - = *?rhs*)
proof
  assume $*$: *small* ($A \times B$)
  { have *small A* $\wedge$ *small B* **if** $x \in A$ $y \in B$ **for** *x y*
    proof −
      have $A \subseteq$ *fst* ‘ ($A \times B$) $B \subseteq$ *snd* ‘ ($A \times B$)
        using *that* **by** *auto*
      with *that* **show** *?thesis*
        by (*metis* $*$ *replacement smaller-than-small*)
    qed }

then show *?rhs*
  by (*metis equals0I*)
next
  assume *?rhs*
  then show *small* $(A \times B)$
    by *auto*
qed

2.3 Disjoint Sum

definition *vsum* :: $V \Rightarrow V \Rightarrow V$ (**infixl** $\uplus$ *65*) **where**
$A \uplus B \equiv (VSigma\ (set\ \{0\})\ (\lambda x.\ A)) \sqcup (VSigma\ (set\ \{1\})\ (\lambda x.\ B))$

definition *Inl* :: $V \Rightarrow V$ **where**
  $Inl\ a \equiv \langle 0,a \rangle$

definition *Inr* :: $V \Rightarrow V$ **where**
  $Inr\ b \equiv \langle 1,b \rangle$

lemmas *sum-defs* = *vsum-def Inl-def Inr-def*

lemma *Inl-nonzero* [*simp*]:$Inl\ x \neq 0$
  by (*metis Inl-def vpair-nonzero*)

lemma *Inr-nonzero* [*simp*]:$Inr\ x \neq 0$
  by (*metis Inr-def vpair-nonzero*)

2.3.1 Equivalences for the injections and an elimination rule

lemma *Inl-in-sum-iff* [*iff*]: $Inl\ a \in elts\ (A \uplus B) \longleftrightarrow a \in elts\ A$
  by (*auto simp*: *sum-defs*)

lemma *Inr-in-sum-iff* [*iff*]: $Inr\ b \in elts\ (A \uplus B) \longleftrightarrow b \in elts\ B$
  by (*auto simp*: *sum-defs*)

lemma *sumE* [*elim*!]:
  assumes *u*: $u \in elts\ (A \uplus B)$
  obtains *x* **where** $x \in elts\ A\ u{=}Inl\ x$ | *y* **where** $y \in elts\ B\ u{=}Inr\ y$ **using** *u*
  by (*auto simp*: *sum-defs*)

2.3.2 Injection and freeness equivalences, for rewriting

lemma *Inl-iff* [*iff*]: $Inl\ a{=}Inl\ b \longleftrightarrow a{=}b$
  by (*simp add*: *sum-defs*)

lemma *Inr-iff* [*iff*]: $Inr\ a{=}Inr\ b \longleftrightarrow a{=}b$
  by (*simp add*: *sum-defs*)

lemma *inj-on-Inl* [*simp*]: *inj-on Inl A*
  by (*simp add*: *inj-on-def*)

lemma *inj-on-Inr* [*simp*]: *inj-on Inr A*
  by (*simp add*: *inj-on-def*)

lemma *Inl-Inr-iff* [*iff*]: *Inl a=Inr b* ⟷ *False*
  by (*simp add*: *sum-defs*)

lemma *Inr-Inl-iff* [*iff*]: *Inr b=Inl a* ⟷ *False*
  by (*simp add*: *sum-defs*)

lemma *sum-empty* [*simp*]: *0* ⨄ *0* = *0*
  by *auto*

lemma *elts-vsum*: *elts* (*a* ⨄ *b*) = *Inl* ‘ (*elts a*) ∪ *Inr* ‘ (*elts b*)
  by *auto*

lemma *sum-iff*: *u* ∈ *elts* (*A* ⨄ *B*) ⟷ (∃ *x*. *x* ∈ *elts A* ∧ *u*=*Inl x*) ∨ (∃ *y*. *y* ∈ *elts B* ∧ *u*=*Inr y*)
  by *blast*

lemma *sum-subset-iff*: *A*⨄*B* ≤ *C*⨄*D* ⟷ *A*≤*C* ∧ *B*≤*D*
  by (*auto simp*: *less-eq-V-def*)

lemma *sum-equal-iff*:
  fixes *A* :: *V* **shows** *A*⨄*B* = *C*⨄*D* ⟷ *A*=*C* ∧ *B*=*D*
  by (*simp add*: *eq-iff sum-subset-iff*)

definition *is-sum* :: *V* ⇒ *bool*
  where *is-sum z* = (∃ *x*. *z* = *Inl x* ∨ *z* = *Inr x*)

definition *sum-case* :: (*V* ⇒ *'a*) ⇒ (*V* ⇒ *'a*) ⇒ *V* ⇒ *'a*
  where
  sum-case f g a ≡
    THE z. (∀ *x*. *a* = *Inl x* ⟶ *z* = *f x*) ∧ (∀ *y*. *a* = *Inr y* ⟶ *z* = *g y*) ∧ (¬ *is-sum a* ⟶ *z* = *undefined*)

lemma *sum-case-Inl* [*simp*]: *sum-case f g* (*Inl x*) = *f x*
  by (*simp add*: *sum-case-def is-sum-def*)

lemma *sum-case-Inr* [*simp*]: *sum-case f g* (*Inr y*) = *g y*
  by (*simp add*: *sum-case-def is-sum-def*)

lemma *sum-case-non* [*simp*]: ¬ *is-sum a* ⟹ *sum-case f g a* = *undefined*
  by (*simp add*: *sum-case-def is-sum-def*)

lemma *is-sum-cases*: (∃ *x*. *z* = *Inl x* ∨ *z* = *Inr x*) ∨ ¬ *is-sum z*
  by (*auto simp*: *is-sum-def*)

lemma *sum-case-split*:

P (*sum-case f g a*) $\longleftrightarrow$ ($\forall x.\ a = Inl\ x \longrightarrow P(f\ x)) \wedge (\forall y.\ a = Inr\ y \longrightarrow P(g\ y)) \wedge (\neg$ *is-sum a* $\longrightarrow P$ *undefined*)
by (*cases is-sum a*) (*auto simp*: *is-sum-def*)

lemma *sum-case-split-asm*:
P (*sum-case f g a*) $\longleftrightarrow \neg ((\exists x.\ a = Inl\ x \wedge \neg P(f\ x)) \vee (\exists y.\ a = Inr\ y \wedge \neg P(g\ y)) \vee (\neg$ *is-sum a* $\wedge \neg P$ *undefined*))
by (*auto simp*: *sum-case-split*)

2.3.3 Applications of disjoint sums and pairs: general union theorems for small sets

lemma *small-Un*:
assumes X: *small* X **and** Y: *small* Y
shows *small* $(X \cup Y)$
proof –
obtain $x\ y$ **where** *elts* $x \approx X$ *elts* $y \approx Y$
by (*meson assms small-eqpoll*)
then have $X \cup Y \lesssim$ *Inl* ‘ (*elts x*) $\cup$ *Inr* ‘ (*elts y*)
by (*metis* (*mono-tags*, *lifting*) *Inr-Inl-iff Un-lepoll-mono disjnt-iff eqpoll-imp-lepoll eqpoll-sym f-inv-into-f inj-on-Inl inj-on-Inr inj-on-image-lepoll-2*)
then show *?thesis*
by (*metis lepoll-iff replacement small-elts small-sup-iff smaller-than-small*)
qed

lemma *small-UN* [*simp*,*intro*]:
assumes A: *small* A **and** B: $\bigwedge x.\ x \in A \Longrightarrow$ *small* $(B\ x)$
shows *small* $(\bigcup x{\in}A.\ B\ x)$
proof –
obtain a **where** *elts* $a \approx A$
by (*meson assms small-eqpoll*)
then obtain f **where** f: *bij-betw* f (*elts a*) A
using *eqpoll-def* **by** *blast*
have $\exists y.$ *elts* $y \approx B\ x$ **if** $x \in A$ **for** x
using B *small-eqpoll that* **by** *blast*
then obtain g **where** g: $\bigwedge x.\ x \in A \Longrightarrow$ *elts* $(g\ x) \approx B\ x$
by *metis*
have *sm*: *small* (*Sigma* (*elts a*) (*elts* $\circ\ g \circ f$))
by *simp*
have $(\bigcup x{\in}A.\ B\ x) \lesssim$ *Sigma A B*
by (*metis image-lepoll snd-image-Sigma*)
also have ... $\lesssim$ *Sigma* (*elts a*) (*elts* $\circ\ g \circ f$)
by (*smt* (*verit*) *Sigma-eqpoll-cong bij-betw-iff-bijections comp-apply eqpoll-imp-lepoll eqpoll-sym f g*)
finally show *?thesis*
using *lepoll-small sm* **by** *blast*
qed

lemma *small-Union* [*simp*,*intro*]:

assumes $\mathcal{A} \subseteq$ *Collect small small* $\mathcal{A}$
shows *small* $(\bigcup \mathcal{A})$
using *small-UN* [*of* $\mathcal{A}$ $\lambda x.\ x$] *assms* **by** (*simp add*: *subset-iff*)

2.4 Generalised function space and lambda

definition *VLambda* :: $V \Rightarrow (V \Rightarrow V) \Rightarrow V$
where *VLambda A b* $\equiv$ *set* $((\lambda x.\ \langle x,b\ x\rangle)$ ' *elts A*)

definition *app* :: $[V,V] \Rightarrow V$
where *app f x* $\equiv$ *THE y*. $\langle x,y\rangle \in$ *elts f*

lemma *beta* [*simp*]:
assumes $x \in$ *elts A*
shows *app* (*VLambda A b*) $x = b\ x$
using *assms* **by** (*auto simp*: *VLambda-def app-def*)

definition *VPi* :: $V \Rightarrow (V \Rightarrow V) \Rightarrow V$
where *VPi A B* $\equiv$ *set* $\{f \in$ *elts* (*VPow*(*VSigma A B*)). *elts A* $\leq$ *Domain* (*pairs* *f*) $\wedge$ *single-valued* (*pairs f*)$\}$

lemma *VPi-I*:
assumes $\bigwedge x.\ x \in$ *elts A* $\Longrightarrow b\ x \in$ *elts* $(B\ x)$
shows *VLambda A b* $\in$ *elts* (*VPi A B*)
proof (*clarsimp simp*: *VPi-def*, *intro conjI impI*)
show *VLambda A b* $\leq$ *VSigma A B*
by (*auto simp*: *assms VLambda-def split*: *if-split-asm*)
show *elts A* $\subseteq$ *Domain* (*pairs* (*VLambda A b*))
by (*force simp*: *VLambda-def pairs-iff-elts*)
show *single-valued* (*pairs* (*VLambda A b*))
by (*auto simp*: *VLambda-def single-valued-def pairs-iff-elts*)
show *small* $\{f.\ f \leq$ *VSigma A B* $\wedge$ *elts A* $\subseteq$ *Domain* (*pairs f*) $\wedge$ *single-valued* (*pairs f*)$\}$
by (*metis* (*mono-tags*, *lifting*) *down VPow-iff mem-Collect-eq subsetI*)
qed

lemma *apply-pair*:
assumes *f*: $f \in$ *elts* (*VPi A B*) **and** *x*: $x \in$ *elts A*
shows $\langle x,$ *app f x*$\rangle \in$ *elts f*
proof −
have $x \in$ *Domain* (*pairs f*)
by (*metis* (*no-types*, *lifting*) *VPi-def assms elts-of-set empty-iff mem-Collect-eq* *subsetD*)
then obtain *y* **where** *y*: $\langle x,y\rangle \in$ *elts f*
using *pairs-iff-elts* **by** *auto*
show *?thesis*
unfolding *app-def*
proof (*rule theI*)
show $\langle x,\ y\rangle \in$ *elts f*

by (*rule y*)
show $z = y$ **if** $\langle x,\ z\rangle \in$ *elts f* **for** *z*
using *f* **unfolding** *VPi-def*
by (*metis* (*mono-tags*, *lifting*) *that elts-of-set empty-iff mem-Collect-eq pairs-iff-elts single-valued-def y*)
qed
qed

lemma *VPi-D*:
assumes *f*: $f \in$ *elts* (*VPi A B*) **and** *x*: $x \in$ *elts A*
shows *app f x* $\in$ *elts* (*B x*)
proof −
have $f \leq$ *VSigma A B*
by (*metis* (*no-types*, *lifting*) *VPi-def elts-of-set empty-iff f VPow-iff mem-Collect-eq*)
then show *?thesis*
using *apply-pair* [*OF assms*] **by** *blast*
qed

lemma *VPi-memberD*:
assumes *f*: $f \in$ *elts* (*VPi A B*) **and** *p*: $p \in$ *elts f*
obtains *x* **where** $x \in$ *elts A* $p = \langle x,$ *app f x*$\rangle$
proof −
have $f \leq$ *VSigma A B*
by (*metis* (*no-types*, *lifting*) *VPi-def elts-of-set empty-iff f VPow-iff mem-Collect-eq*)
then obtain *x y* **where** $p = \langle x,y\rangle$ $x \in$ *elts A*
using *p* **by** *blast*
then have $y =$ *app f x*
by (*metis* (*no-types*, *lifting*) *VPi-def apply-pair elts-of-set equals0D f mem-Collect-eq p pairs-iff-elts single-valuedD*)
then show *thesis*
using ‹$p = \langle x,\ y\rangle$› ‹$x \in$ *elts A*› *that* **by** *blast*
qed

lemma *fun-ext*:
assumes $f \in$ *elts* (*VPi A B*) $g \in$ *elts* (*VPi A B*) $\bigwedge x.\ x \in$ *elts A* $\Longrightarrow$ *app f x* = *app g x*
shows $f = g$
by (*metis VPi-memberD V-equalityI apply-pair assms*)

lemma *eta*[*simp*]:
assumes $f \in$ *elts* (*VPi A B*)
shows *VLambda A* ((*app*)*f*) = *f*
proof (*rule fun-ext* [*OF - assms*])
show *VLambda A* (*app f*) $\in$ *elts* (*VPi A B*)
using *VPi-D VPi-I assms* **by** *auto*
qed *auto*

lemma *fst-pairs-VLambda*: *fst* ‘ *pairs* (*VLambda A f*) = *elts A*

by (*force simp*: *VLambda-def pairs-def*)

lemma *snd-pairs-VLambda*: *snd* ‘ *pairs* (*VLambda A f*) = *f* ‘ *elts A*
by (*force simp*: *VLambda-def pairs-def*)

lemma *VLambda-eq-D1*: *VLambda A f* = *VLambda B g* $\Longrightarrow$ *A* = *B*
by (*metis ZFC-in-HOL.ext fst-pairs-VLambda*)

lemma *VLambda-eq-D2*: $[\![$*VLambda A f* = *VLambda A g*; *x* $\in$ *elts A*$]\!]$ $\Longrightarrow$ *f x* = *g x*
by (*metis beta*)

2.5 Transitive closure of a set

definition *TC* :: *V*$\Rightarrow$*V*
where *TC* $\equiv$ *transrec* (λ*f x*. *x* $\sqcup$ $\bigsqcup$ (*f* ‘ *elts x*))

lemma *TC*: *TC a* = *a* $\sqcup$ $\bigsqcup$ (*TC* ‘ *elts a*)
by (*metis* (*no-types*, *lifting*) *SUP-cong TC-def restrict-apply′ transrec*)

lemma *TC-0* [*simp*]: *TC 0* = *0*
by (*metis TC ZFC-in-HOL.Sup-empty elts-0 image-is-empty sup-V-0-left*)

lemma *arg-subset-TC*: *a* $\le$ *TC a*
by (*metis* (*no-types*) *TC sup-ge1*)

lemma *Transset-TC*: *Transset*(*TC a*)
proof (*induction a rule*: *eps-induct*)
case (*step x*)
have *1*: *v* $\in$ *elts* (*TC x*) **if** *v* $\in$ *elts u u* $\in$ *elts x* **for** *u v*
using *that* **unfolding** *TC* [*of x*]
using *arg-subset-TC* **by** *fastforce*
have *2*: *v* $\in$ *elts* (*TC x*) **if** *v* $\in$ *elts u* $\exists$*x*$\in$*elts x*. *u* $\in$ *elts* (*TC x*) **for** *u v*
using *that step* **unfolding** *TC* [*of x*] *Transset-def* **by** *auto*
show *?case*
unfolding *Transset-def*
by (*subst TC*) (*force intro*: *1 2*)
qed

lemma *TC-least*: $[\![$*Transset x*; *a*$\le$*x*$]\!]$ $\Longrightarrow$ *TC a* $\le$ *x*
proof (*induction a rule*: *eps-induct*)
case (*step y*)
show *?case*
proof (*cases y*=*0*)
case *True*
then show *?thesis*
by *auto*
next
case *False*

have $\bigsqcup$ (*TC* ‘ *elts* *y*) ≤ *x*
proof (*rule cSup-least*)
show *TC* ‘ *elts y* ≠ {}
using *False* **by** *auto*
show *z* ≤ *x* **if** *z* ∈ *TC* ‘ *elts y* **for** *z*
using *that* **by** (*metis Transset-def image-iff step.IH step.prems vsubsetD*)
qed
then show *?thesis*
by (*simp add*: *step TC* [*of y*])
qed
qed

definition *less-TC* (**infix** ⊏ *50*)
where *x* ⊏ *y* ≡ *x* ∈ *elts* (*TC y*)

definition *le-TC* (**infix** ⊑ *50*)
where *x* ⊑ *y* ≡ *x* ⊏ *y* ∨ *x*=*y*

lemma *less-TC-imp-not-le*: *x* ⊏ *a* ⟹ ¬ *a* ≤ *x*
proof (*induction a arbitrary*: *x rule*: *eps-induct*)
case (*step a*)
then show *?case*
unfolding *TC*[*of a*] *less-TC-def*
using *Transset-TC Transset-def* **by** *force*
qed

lemma *non-TC-less-0* [*iff*]: ¬ (*x* ⊏ *0*)
using *less-TC-imp-not-le* **by** *blast*

lemma *less-TC-iff*: *x* ⊏ *y* ⟷ (∃ *z* ∈ *elts y*. *x* ⊑ *z*)
by (*auto simp*: *less-TC-def le-TC-def TC* [*of y*])

lemma *nonzero-less-TC*: *x* ≠ *0* ⟹ *0* ⊏ *x*
by (*metis eps-induct le-TC-def less-TC-iff trad-foundation*)

lemma *less-irrefl-TC* [*simp*]: ¬ *x* ⊏ *x*
using *less-TC-imp-not-le* **by** *blast*

lemma *less-asym-TC*: ⟦*x* ⊏ *y*; *y* ⊏ *x*⟧ ⟹ *False*
by (*metis TC-least Transset-TC Transset-def antisym-conv less-TC-def less-TC-imp-not-le order-refl*)

lemma *le-antisym-TC*: ⟦*x* ⊑ *y*; *y* ⊑ *x*⟧ ⟹ *x* = *y*
using *le-TC-def less-asym-TC* **by** *auto*

lemma *less-le-TC*: *x* ⊏ *y* ⟷ *x* ⊑ *y* ∧ *x* ≠ *y*
using *le-TC-def less-asym-TC* **by** *blast*

lemma *less-imp-le-TC* [*iff*]: *x* ⊏ *y* ⟹ *x* ⊑ *y*

by (*simp add*: *le-TC-def*)

lemma *le-TC-refl* [*iff*]: $x \sqsubseteq x$
by (*simp add*: *le-TC-def*)

lemma *le-TC-trans* [*trans*]: $[\![x \sqsubseteq y; y \sqsubseteq z]\!] \Longrightarrow x \sqsubseteq z$
by (*smt* (*verit*, *best*) *TC-least Transset-TC Transset-def le-TC-def less-TC-def vsubsetD*)

context *order*
begin

lemma *nless-le-TC*: $(\neg\ a \sqsubset b) \longleftrightarrow (\neg\ a \sqsubseteq b) \vee a = b$
using *le-TC-def less-asym-TC* **by** *blast*

lemma *eq-refl-TC*: $x = y \Longrightarrow x \sqsubseteq y$
by *simp*

local-setup ‹
HOL-Order-Tac.declare-order {
ops = {*eq* = @{*term* ‹(=) :: V ⇒ V ⇒ bool›}, *le* = @{*term* ‹(⊑)›}, *lt* = @{*term* ‹(⊏)›}},
thms = {*trans* = @{*thm le-TC-trans*}, *refl* = @{*thm le-TC-refl*}, *eqD1* = @{*thm eq-refl-TC*},
eqD2 = @{*thm eq-refl-TC*[*OF sym*]}, *antisym* = @{*thm le-antisym-TC*}, *contr* = @{*thm notE*}},
conv-thms = {*less-le* = @{*thm eq-reflection*[*OF less-le-TC*]},
nless-le = @{*thm eq-reflection*[*OF nless-le-TC*]}}
}
›

end

lemma *less-TC-trans* [*trans*]: $[\![x \sqsubset y; y \sqsubset z]\!] \Longrightarrow x \sqsubset z$
and *less-le-TC-trans*: $[\![x \sqsubset y; y \sqsubseteq z]\!] \Longrightarrow x \sqsubset z$
and *le-less-TC-trans* [*trans*]: $[\![x \sqsubseteq y; y \sqsubset z]\!] \Longrightarrow x \sqsubset z$
by *simp-all*

lemma *TC-sup-distrib*: $TC\ (x \sqcup y) = TC\ x \sqcup TC\ y$
by (*simp add*: *Sup-Un-distrib TC* [*of* $x \sqcup y$] *TC* [*of* x] *TC* [*of* y] *image-Un sup.assoc sup-left-commute*)

lemma *TC-Sup-distrib*:
assumes *small X* **shows** $TC\ (\bigsqcup X) = \bigsqcup (TC\ `\ X)$
proof −
have $\bigsqcup X \le \bigsqcup\ (TC\ `\ X)$
using *arg-subset-TC* **by** *fastforce*
moreover have $\bigsqcup\ (\bigcup x{\in}X.\ TC\ `\ elts\ x) \le \bigsqcup\ (TC\ `\ X)$

using *assms*
by *clarsimp* (*meson TC-least Transset-TC Transset-def arg-subset-TC replacement vsubsetD*)
ultimately
have $\bigsqcup X \sqcup \bigsqcup (\bigcup x \in X.\ TC \text{ ` } elts\ x) \le \bigsqcup (TC \text{ ` } X)$
by *simp*
moreover have $\bigsqcup (TC \text{ ` } X) \le \bigsqcup X \sqcup \bigsqcup (\bigcup x \in X.\ TC \text{ ` } elts\ x)$
proof (*clarsimp simp add*: *Sup-le-iff assms*)
show $\exists x \in X.\ y \in elts\ x$
if $x \in X$ $y \in elts\ (TC\ x)$ $\forall x \in X.\ \forall u \in elts\ x.\ y \notin elts\ (TC\ u)$ **for** $x\ y$
using *that* **by** (*auto simp*: *TC* [*of x*])
qed
ultimately show *?thesis*
using *Sup-Un-distrib TC* [*of* $\bigsqcup X$] *image-Union assms*
by (*simp add*: *image-Union inf-sup-aci*(*5*) *sup.absorb-iff2*)
qed

lemma TC': $TC\ x = x \sqcup TC\ (\bigsqcup (elts\ x))$
by (*simp add*: *TC* [*of x*] *TC-Sup-distrib*)

lemma *TC-eq-0-iff* [*simp*]: $TC\ x = 0 \longleftrightarrow x=0$
using *arg-subset-TC* **by** *fastforce*

A distinctive induction principle

lemma *TC-induct-down-lemma*:
assumes *ab*: $a \sqsubset b$ **and** *base*: $b \le d$
and *step*: $\bigwedge y\ z.\ [\![y \sqsubset b;\ y \in elts\ d;\ z \in elts\ y]\!] \Longrightarrow z \in elts\ d$
shows $a \in elts\ d$
proof −
have *Transset* ($TC\ b \sqcap d$)
using *Transset-TC*
unfolding *Transset-def*
by (*metis inf.bounded-iff less-TC-def less-eq-V-def local.step subsetI vsubsetD*)
moreover have $b \le TC\ b \sqcap d$
by (*simp add*: *arg-subset-TC base*)
ultimately show *?thesis*
using *TC-least* [*THEN vsubsetD*] *ab* **unfolding** *less-TC-def*
by (*meson TC-least le-inf-iff vsubsetD*)
qed

lemma *TC-induct-down* [*consumes 1*, *case-names base step small*]:
assumes $a \sqsubset b$
and $\bigwedge y.\ y \in elts\ b \Longrightarrow P\ y$
and $\bigwedge y\ z.\ [\![y \sqsubset b;\ P\ y;\ z \in elts\ y]\!] \Longrightarrow P\ z$
and *small* (*Collect P*)
shows $P\ a$
using *TC-induct-down-lemma* [*of a b set* (*Collect P*)] *assms*
by (*metis elts-of-set mem-Collect-eq vsubsetI*)

2.6 Rank of a set

definition *rank* :: $V \Rightarrow V$
  where $rank\ a \equiv transrec\ (\lambda f\ x.\ set\ (\bigcup y{\in}elts\ x.\ elts\ (succ(f\ y))))\ a$

lemma *rank*: $rank\ a = set(\bigcup y \in elts\ a.\ elts\ (succ(rank\ y)))$
  by (*subst rank-def* [*THEN def-transrec*], *simp*)

lemma *rank-Sup*: $rank\ a = \bigsqcup((\lambda y.\ succ(rank\ y))\ `\ elts\ a)$
  by (*metis elts-Sup image-image rank replacement set-of-elts small-elts*)

lemma *Ord-rank* [*simp*]: $Ord(rank\ a)$
proof (*induction a rule*: *eps-induct*)
  case (*step x*)
  then show *?case*
    unfolding *rank-Sup* [*of x*]
    by (*metis* (*mono-tags*, *lifting*) *Ord-Sup Ord-succ imageE*)
qed

lemma *rank-of-Ord*: $Ord\ i \Longrightarrow rank\ i = i$
  by (*induction rule*: *Ord-induct*) (*metis* (*no-types*, *lifting*) *Ord-equality SUP-cong rank-Sup*)

lemma *Ord-iff-rank*: $Ord\ x \longleftrightarrow rank\ x = x$
  using *Ord-rank* [*of x*] *rank-of-Ord* **by** *fastforce*

lemma *rank-lt*: $a \in elts\ b \Longrightarrow rank\ a < rank\ b$
  by (*metis Ord-linear2 Ord-rank ZFC-in-HOL.SUP-le-iff rank-Sup replacement small-elts succ-le-iff order.irrefl*)

lemma *rank-0* [*simp*]: $rank\ 0 = 0$
  using *transrec Ord-0 rank-def rank-of-Ord* **by** *presburger*

lemma *rank-succ* [*simp*]: $rank(succ\ x) = succ(rank\ x)$
proof (*rule order-antisym*)
  show $rank\ (succ\ x) \le succ\ (rank\ x)$
    by (*metis* (*no-types*, *lifting*) *Sup-insert elts-of-set elts-succ image-insert rank small-UN small-elts subset-insertI sup.orderE vsubsetI*)
  show $succ\ (rank\ x) \le rank\ (succ\ x)$
    by (*metis* (*mono-tags*, *lifting*) *ZFC-in-HOL.Sup-upper elts-succ image-insert insertI1 rank-Sup replacement small-elts*)
qed

lemma *rank-mono*: $a \le b \Longrightarrow rank\ a \le rank\ b$
  using *rank* [*of a*] *rank* [*of b*] *small-UN* **by** *force*

lemma *VsetI*: $rank\ b \sqsubset i \Longrightarrow b \in elts\ (Vset\ i)$
proof (*induction i arbitrary*: *b rule*: *eps-induct*)
  case (*step x*)
  then consider $rank\ b \in elts\ x \mid (\exists\, y{\in}elts\ x.\ rank\ b \in elts\ (TC\ y))$

using *le-TC-def less-TC-def less-TC-iff* **by** *fastforce*
then have $\exists y \in elts\ x.\ b \le Vset\ y$
proof *cases*
case *1*
then have $b \le Vset\ (rank\ b)$
unfolding *less-eq-V-def subset-iff*
by (*meson Ord-mem-iff-lt Ord-rank le-TC-refl less-TC-iff rank-lt step.IH*)
then show *?thesis*
using *1* **by** *blast*
next
case *2*
then show *?thesis*
using *step.IH*
unfolding *less-eq-V-def subset-iff less-TC-def*
by (*meson Ord-mem-iff-lt Ord-rank Transset-TC Transset-def rank-lt vsubsetD*)
qed
then show *?case*
by (*simp add*: *Vset* [*of x*])
qed

lemma *Ord-VsetI*: $[\![Ord\ i;\ rank\ b < i]\!] \Longrightarrow b \in elts\ (Vset\ i)$
by (*meson Ord-mem-iff-lt Ord-rank VsetI arg-subset-TC less-TC-def vsubsetD*)

lemma *arg-le-Vset-rank*: $a \le Vset(rank\ a)$
by (*simp add*: *Ord-VsetI rank-lt vsubsetI*)

lemma *two-in-Vset*:
obtains α **where** $x \in elts\ (Vset\ \alpha)\ y \in elts\ (Vset\ \alpha)$
by (*metis Ord-rank Ord-VsetI elts-of-set insert-iff rank-lt small-elts small-insert-iff*)

lemma *rank-eq-0-iff* [*simp*]: $rank\ x = 0 \longleftrightarrow x=0$
using *arg-le-Vset-rank* **by** *fastforce*

lemma *small-ranks-imp-small*:
assumes *small* (*rank* ‘ *A*) **shows** *small A*
proof −
define *i* **where** $i \equiv set\ (\bigcup (elts\ `\ (rank\ `\ A)))$
have *Ord i*
unfolding *i-def* **using** *Ord-Union Ord-rank assms imageE* **by** *blast*
have $*$: $Vset\ (rank\ x) \le (Vset\ i)$ **if** $x \in A$ **for** *x*
unfolding *i-def* **by** (*metis Ord-rank Sup-V-def ZFC-in-HOL.Sup-upper Vfrom-mono assms imageI le-less that*)
have $A \subseteq elts\ (VPow\ (Vset\ i))$
by (*meson* $*$ *VPow-iff arg-le-Vset-rank order.trans subsetI*)
then show *?thesis*
using *down* **by** *blast*
qed

lemma *rank-Union*: $rank(\bigsqcup A) = \bigsqcup (rank\ `\ A)$

proof (*rule order-antisym*)
  have *elts* ($\bigsqcup y \in$*elts* ($\bigsqcup$ *A*). *succ* (*rank y*)) $\subseteq$ *elts* ($\bigsqcup$ (*rank* ' *A*))
    by *clarsimp* (*meson Ord-mem-iff-lt Ord-rank less-V-def rank-lt vsubsetD*)
  then show *rank* ($\bigsqcup$ *A*) $\leq$ $\bigsqcup$ (*rank* ' *A*)
    by (*metis less-eq-V-def rank-Sup*)
  show $\bigsqcup$ (*rank* ' *A*) $\leq$ *rank* ($\bigsqcup$ *A*)
  proof (*cases small A*)
    case *True*
    then show *?thesis*
      by (*simp add*: *ZFC-in-HOL.SUP-le-iff ZFC-in-HOL.Sup-upper rank-mono*)
  next
    case *False*
    then have $\neg$ *small* (*rank* ' *A*)
      using *small-ranks-imp-small* **by** *blast*
    then show *?thesis*
      by *blast*
  qed
qed

lemma *small-bounded-rank*: *small* {*x*. *rank x* $\in$ *elts a*}
proof $-$
  have {*x*. *rank x* $\in$ *elts a*} $\subseteq$ {*x*. *rank x* $\sqsubset$ *a*}
    using *less-TC-iff* **by** *auto*
  also have ... $\subseteq$ *elts* (*Vset a*)
    using *VsetI* **by** *blast*
  finally show *?thesis*
    using *down* **by** *simp*
qed

lemma *small-bounded-rank-le*: *small* {*x*. *rank x* $\leq$ *a*}
  using *small-bounded-rank* [*of VPow a*] *VPow-iff* [*of - a*] **by** *simp*

lemma *TC-rank-lt*: *a* $\sqsubset$ *b* $\Longrightarrow$ *rank a* $<$ *rank b*
proof (*induction rule*: *TC-induct-down*)
  case (*base y*)
  then show *?case*
    by (*simp add*: *rank-lt*)
next
  case (*step y z*)
  then show *?case*
    using *less-trans rank-lt* **by** *blast*
next
  case *small*
  show *?case*
    using *smaller-than-small* [*OF small-bounded-rank-le* [*of rank b*]]
    by (*simp add*: *Collect-mono less-V-def*)
qed

lemma *TC-rank-mem*: *x* $\sqsubset$ *y* $\Longrightarrow$ *rank x* $\in$ *elts* (*rank y*) by (*simp add*: *Ord-mem-iff-lt TC-rank-lt*)

lemma *wf-TC-less*: *wf* $\{(x,y).\ x \sqsubset y\}$
proof (*rule wf-subset* [*OF wf-inv-image* [*OF foundation*, *of rank*]])
show $\{(x, y).\ x \sqsubset y\} \subseteq$ *inv-image* $\{(x, y).\ x \in$ *elts* $y\}$ *rank*
by (*auto simp*: *TC-rank-mem inv-image-def*)
qed

lemma *less-TC-minimal*:
assumes *P a*
obtains *x* **where** $P\ x\ x \sqsubseteq a \bigwedge y.\ y \sqsubset x \Longrightarrow \neg\ P\ y$
using *wfE-min'* [*OF wf-TC-less*, *of* $\{x.\ P\ x \wedge x \sqsubseteq a\}$]
by *simp* (*metis le-TC-def less-le-TC-trans assms*)

lemma *Vfrom-rank-eq*: *Vfrom A* (*rank*(*x*)) = *Vfrom A x*
proof (*rule order-antisym*)
show *Vfrom A* (*rank x*) ≤ *Vfrom A x*
proof (*induction x rule*: *eps-induct*)
case (*step x*)
have ($\bigsqcup j \in$*elts* (*rank x*). *VPow* (*Vfrom A j*)) ≤ ($\bigsqcup j \in$*elts x*. *VPow* (*Vfrom A j*))
apply (*rule Sup-least*)
apply (*clarsimp simp add*: *rank* [*of x*])
by (*meson Ord-in-Ord Ord-rank OrdmemD Vfrom-mono order.trans less-imp-le order.refl step*)
then show *?case*
by (*simp add*: *Vfrom* [*of - x*] *Vfrom* [*of - rank*(*x*)] *sup.coboundedI2*)
qed
show *Vfrom A x* ≤ *Vfrom A* (*rank x*)
proof (*induction x rule*: *eps-induct*)
case (*step x*)
have ($\bigsqcup j \in$*elts x*. *VPow* (*Vfrom A j*)) ≤ ($\bigsqcup j \in$*elts* (*rank x*). *VPow* (*Vfrom A j*))
using *step.IH TC-rank-mem less-TC-iff* **by** *force*
then show *?case*
by (*simp add*: *Vfrom* [*of - x*] *Vfrom* [*of - rank*(*x*)] *sup.coboundedI2*)
qed
qed

lemma *Vfrom-succ*: *Vfrom A* (*succ*(*i*)) = *A* ⊔ *VPow*(*Vfrom A i*)
by (*metis Ord-rank Vfrom-rank-eq Vfrom-succ-Ord rank-succ*)

lemma *Vset-succ-TC*:
assumes $x \in$ *elts* (*Vset* (*ZFC-in-HOL.succ k*)) $u \sqsubset x$
shows $u \in$ *elts* (*Vset k*)
using *assms*
using *TC-least Transset-Vfrom Vfrom-succ less-TC-def* **by** *auto*

2.7 Cardinal Numbers

We extend the membership relation to a wellordering

definition *VWO* :: $(V \times V)$ *set*
where $VWO \equiv @r.\ \{(x,y).\ x \in elts\ y\} \subseteq r \wedge Well\text{-}order\ r \wedge Field\ r = UNIV$

lemma *VWO*: $\{(x,y).\ x \in elts\ y\} \subseteq VWO \wedge Well\text{-}order\ VWO \wedge Field\ VWO = UNIV$
unfolding *VWO-def*
by (*metis* (*mono-tags*, *lifting*) *VWO-def foundation someI-ex total-well-order-extension*)

lemma *wf-VWO*: $wf(VWO - Id)$
using *VWO well-order-on-def* **by** *blast*

lemma *wf-Ord-less*: $wf\ \{(x,\ y).\ Ord\ y \wedge x < y\}$
by (*metis* (*no-types*, *lifting*) *Ord-mem-iff-lt eps-induct wfPUNIVI wfP-def*)

lemma *refl-VWO*: *refl VWO*
using *VWO order-on-defs* **by** *fastforce*

lemma *trans-VWO*: *trans VWO*
using *VWO* **by** (*simp add*: *VWO wo-rel.TRANS wo-rel-def*)

lemma *antisym-VWO*: *antisym VWO*
using *VWO* **by** (*simp add*: *VWO wo-rel.ANTISYM wo-rel-def*)

lemma *total-VWO*: *total VWO*
using *VWO* **by** (*metis wo-rel.TOTAL wo-rel.intro*)

lemma *total-VWOId*: $total\ (VWO - Id)$
by (*simp add*: *total-VWO*)

lemma *Linear-order-VWO*: *Linear-order VWO*
using *VWO well-order-on-def* **by** *blast*

lemma *wo-rel-VWO*: *wo-rel VWO*
using *VWO wo-rel-def* **by** *blast*

2.7.1 Transitive Closure and VWO

lemma *mem-imp-VWO*: $x \in elts\ y \Longrightarrow (x,y) \in VWO$
using *VWO* **by** *blast*

lemma *less-TC-imp-VWO*: $x \sqsubset y \Longrightarrow (x,y) \in VWO$
unfolding *less-TC-def*
proof (*induction y arbitrary*: *x rule*: *eps-induct*)
case $(step\ y'\ u)$
then consider $u \in elts\ y' \mid v$ **where** $v \in elts\ y'\ u \in elts\ (TC\ v)$
by (*auto simp*: *TC* [*of* y'])

then show *?case*
proof *cases*
case *2*
then show *?thesis*
by (*meson mem-imp-VWO step.IH transD trans-VWO*)
qed (*use mem-imp-VWO* **in** *blast*)
qed

lemma *le-TC-imp-VWO*: $x \sqsubseteq y \Longrightarrow (x,y) \in VWO$
by (*metis Diff-iff Linear-order-VWO Linear-order-in-diff-Id UNIV-I VWO le-TC-def less-TC-imp-VWO*)

lemma *le-TC-0-iff* [*simp*]: $x \sqsubseteq 0 \longleftrightarrow x = 0$
by (*simp add*: *le-TC-def*)

lemma *less-TC-succ*: $x \sqsubset succ\ \beta \longleftrightarrow x \sqsubset \beta \vee x = \beta$
by (*metis elts-succ insert-iff le-TC-def less-TC-iff*)

lemma *le-TC-succ*: $x \sqsubseteq succ\ \beta \longleftrightarrow x \sqsubseteq \beta \vee x = succ\ \beta$
by (*simp add*: *le-TC-def less-TC-succ*)

lemma *Transset-TC-eq* [*simp*]: $Transset\ x \Longrightarrow TC\ x = x$
by (*simp add*: *TC-least arg-subset-TC eq-iff*)

lemma *Ord-TC-less-iff*: $[\![Ord\ \alpha;\ Ord\ \beta]\!] \Longrightarrow \beta \sqsubset \alpha \longleftrightarrow \beta < \alpha$
by (*metis Ord-def Ord-mem-iff-lt Transset-TC-eq less-TC-def*)

lemma *Ord-mem-iff-less-TC*: $Ord\ l \Longrightarrow k \in elts\ l \longleftrightarrow k \sqsubset l$
by (*simp add*: *Ord-def less-TC-def*)

lemma *le-TC-Ord*: $[\![\beta \sqsubseteq \alpha;\ Ord\ \alpha]\!] \Longrightarrow Ord\ \beta$
by (*metis Ord-def Ord-in-Ord Transset-TC-eq le-TC-def less-TC-def*)

lemma *Ord-less-TC-mem*:
assumes $Ord\ \alpha\ \beta \sqsubset \alpha$ **shows** $\beta \in elts\ \alpha$
using *Ord-def assms less-TC-def* **by** *auto*

lemma *VWO-TC-le*: $[\![Ord\ \alpha;\ Ord\ \beta;\ (\beta,\ \alpha) \in VWO]\!] \Longrightarrow \beta \sqsubseteq \alpha$
proof (*induct* α *arbitrary*: β *rule*: *Ord-induct*)
case (*step* α)
then show *?case*
by (*metis DiffI IdD Linear-order-VWO Linear-order-in-diff-Id Ord-linear Ord-mem-iff-less-TC VWO iso-tuple-UNIV-I le-TC-def mem-imp-VWO*)
qed

lemma *VWO-iff-Ord-le* [*simp*]: $[\![Ord\ \alpha;\ Ord\ \beta]\!] \Longrightarrow (\beta,\ \alpha) \in VWO \longleftrightarrow \beta \leq \alpha$
by (*metis VWO-TC-le Ord-TC-less-iff le-TC-def le-TC-imp-VWO le-less*)

lemma *zero-TC-le* [*iff*]: $0 \sqsubseteq y$

using *le-TC-def nonzero-less-TC* **by** *auto*

lemma *succ-le-TC-iff*: *Ord j* $\Longrightarrow$ *succ i* $\sqsubseteq$ *j* $\longleftrightarrow$ *i* $\sqsubset$ *j*
by (*metis Ord-in-Ord Ord-linear Ord-mem-iff-less-TC Ord-succ le-TC-def less-TC-succ less-asym-TC*)

lemma *VWO-0-iff* [*simp*]: $(x,0) \in VWO \longleftrightarrow x=0$
proof
show $x = 0$ **if** $(x,\ 0) \in VWO$
using *zero-TC-le* [*of x*] *le-TC-imp-VWO that*
by (*metis DiffI Linear-order-VWO Linear-order-in-diff-Id UNIV-I VWO pair-in-Id-conv*)
qed *auto*

lemma *VWO-antisym*:
assumes $(x,y) \in VWO$ $(y,x) \in VWO$ **shows** $x=y$
by (*metis Diff-iff IdD Linear-order-VWO Linear-order-in-diff-Id UNIV-I VWO assms*)

2.7.2 Relation VWF

definition *VWF* **where** $VWF \equiv VWO - Id$

lemma *wf-VWF* [*iff*]: *wf VWF*
by (*simp add*: *VWF-def wf-VWO*)

lemma *trans-VWF* [*iff*]: *trans VWF*
by (*simp add*: *VWF-def antisym-VWO trans-VWO trans-diff-Id*)

lemma *asym-VWF* [*iff*]: *asym VWF*
by (*metis wf-VWF wf-imp-asym*)

lemma *total-VWF* [*iff*]: *total VWF*
using *VWF-def total-VWOId* **by** *auto*

lemma *total-on-VWF* [*iff*]: *total-on A VWF*
by (*meson UNIV-I total-VWF total-on-def*)

lemma *VWF-asym*:
assumes $(x,y) \in VWF$ $(y,x) \in VWF$ **shows** *False*
using *VWF-def assms wf-VWO wf-not-sym* **by** *fastforce*

lemma *VWF-non-refl* [*iff*]: $(x,x) \notin VWF$
by *simp*

lemma *VWF-iff-Ord-less* [*simp*]: $[\![Ord\ \alpha;\ Ord\ \beta]\!] \Longrightarrow (\alpha,\beta) \in VWF \longleftrightarrow \alpha < \beta$
by (*simp add*: *VWF-def less-V-def*)

lemma *mem-imp-VWF*: $x \in elts\ y \Longrightarrow (x,y) \in VWF$
using *VWF-def mem-imp-VWO* **by** *fastforce*

2.8 Order types

definition *ordermap* :: $'a$ *set* ⇒ $('a \times 'a)$ *set* ⇒ $'a \Rightarrow V$
where *ordermap A r* ≡ *wfrec r* (λ*f x*. *set* (*f* ‘ {*y* ∈ *A*. (*y*,*x*) ∈ *r*}))

definition *ordertype* :: $'a$ *set* ⇒ $('a \times 'a)$ *set* ⇒ V
where *ordertype A r* ≡ *set* (*ordermap A r* ‘ *A*)

lemma *ordermap-type*:
small A ⟹ *ordermap A r* ∈ *A* → *elts* (*ordertype A r*)
by (*simp add*: *ordertype-def*)

lemma *ordermap-in-ordertype* [*intro*]: ⟦*a* ∈ *A*; *small A*⟧ ⟹ *ordermap A r a* ∈ *elts* (*ordertype A r*)
by (*simp add*: *ordertype-def*)

lemma *ordermap*: *wf r* ⟹ *ordermap A r a* = *set* (*ordermap A r* ‘ {*y* ∈ *A*. (*y*,*a*) ∈ *r*})
unfolding *ordermap-def*
by (*auto simp*: *wfrec-fixpoint adm-wf-def*)

lemma *wf-Ord-ordermap* [*iff*]: **assumes** *wf r trans r* **shows** *Ord* (*ordermap A r x*)
using ‹*wf r*›
proof (*induction x rule*: *wf-induct-rule*)
case (*less u*)
have *Transset* (*set* (*ordermap A r* ‘ {*y* ∈ *A*. (*y*, *u*) ∈ *r*}))
proof (*clarsimp simp add*: *Transset-def*)
show *x* ∈ *ordermap A r* ‘ {*y* ∈ *A*. (*y*, *u*) ∈ *r*}
if *small* (*ordermap A r* ‘ {*y* ∈ *A*. (*y*, *u*) ∈ *r*})
and *x*: *x* ∈ *elts* (*ordermap A r y*) **and** *y* ∈ *A* (*y*, *u*) ∈ *r* **for** *x y*
proof −
have *ordermap A r y* = *ZFC-in-HOL.set* (*ordermap A r* ‘ {*a* ∈ *A*. (*a*, *y*) ∈ *r*})
using *ordermap assms*(*1*) **by** *force*
then have *x* ∈ *ordermap A r* ‘ {*z* ∈ *A*. (*z*, *y*) ∈ *r*}
by (*metis* (*no-types*, *lifting*) *elts-of-set empty-iff x*)
then have ∃ *v*. *v* ∈ *A* ∧ (*v*, *u*) ∈ *r* ∧ *x* = *ordermap A r v*
using *that transD* [*OF* ‹*trans r*›] **by** *blast*
then show *?thesis*
by *blast*
qed
qed
moreover have *Ord x*
if *x* ∈ *elts* (*set* (*ordermap A r* ‘ {*y* ∈ *A*. (*y*, *u*) ∈ *r*})) **for** *x*
using *that less* **by** (*auto simp*: *split*: *if-split-asm*)
ultimately show *?case*
by (*metis* (*full-types*) *Ord-def ordermap assms*(*1*))
qed

lemma *wf-Ord-ordertype*: **assumes** *wf r trans r* **shows** $Ord(ordertype\ A\ r)$
proof −
  have $y \leq set\ (ordermap\ A\ r\ `\ A)$
    if $y = ordermap\ A\ r\ x\ x \in A\ small\ (ordermap\ A\ r\ `\ A)$ **for** $x\ y$
    using *that* **by** (*auto simp*: *less-eq-V-def ordermap* [*OF* ‹*wf r*›, *of A x*])
  moreover have $z \leq y$ **if** $y \in ordermap\ A\ r\ `\ A\ z \in elts\ y$ **for** $y\ z$
    by (*metis wf-Ord-ordermap OrdmemD assms imageE order.strict-implies-order that*)
  ultimately show *?thesis*
    unfolding *ordertype-def Ord-def Transset-def* **by** *simp*
qed

lemma *Ord-ordertype* [*simp*]: $Ord(ordertype\ A\ VWF)$
  using *wf-Ord-ordertype* **by** *blast*

lemma *Ord-ordermap* [*simp*]: $Ord\ (ordermap\ A\ VWF\ x)$
  by *blast*

lemma *ordertype-singleton* [*simp*]:
  assumes *wf r*
  shows $ordertype\ \{x\}\ r = 1$
proof −
  have †: $\{y.\ y = x \wedge (y,\ x) \in r\} = \{\}$
    using *assms* **by** *auto*
  show *?thesis*
    by (*auto simp add*: *ordertype-def assms* † *ordermap* [**where** *a*=*x*])
qed

2.8.1 *ordermap* preserves the orderings in both directions

lemma *ordermap-mono*:
  assumes *wx*: $(w,\ x) \in r$ **and** *wf r* $w \in A$ *small A*
    shows $ordermap\ A\ r\ w \in elts\ (ordermap\ A\ r\ x)$
proof −
  have $small\ \{a \in A.\ (a,\ x) \in r\} \wedge w \in A \wedge (w,\ x) \in r$
    by (*simp add*: *assms*)
  then show *?thesis*
    using *assms ordermap* [*of r A*]
    by (*metis* (*no-types*, *lifting*) *elts-of-set image-eqI mem-Collect-eq replacement*)
qed

lemma *converse-ordermap-mono*:
  assumes $ordermap\ A\ r\ y \in elts\ (ordermap\ A\ r\ x)$ *wf r total-on A r* $x \in A\ y \in A$ *small A*
  shows $(y,\ x) \in r$
proof (*cases* $x = y$)
  case *True*
  then show *?thesis*
    using *assms*(*1*) *mem-not-refl* **by** *blast*

next
  case *False*
  then consider $(x,y) \in r \mid (y,x) \in r$
    using ‹*total-on A r*› *assms* **by** (*meson UNIV-I total-on-def*)
  then show *?thesis*
    by (*meson ordermap-mono assms mem-not-sym*)
qed

lemma *converse-ordermap-mono-iff*:
  assumes *wf r total-on A r* $x \in A$ $y \in A$ *small A*
  shows *ordermap A r y* $\in$ *elts* (*ordermap A r x*) $\longleftrightarrow$ $(y, x) \in r$
  by (*metis assms converse-ordermap-mono ordermap-mono*)

lemma *ordermap-surj*: *elts* (*ordertype A r*) $\subseteq$ *ordermap A r* ‘ *A*
  unfolding *ordertype-def* **by** *simp*

lemma *ordermap-bij*:
  assumes *wf r total-on A r small A*
  shows *bij-betw* (*ordermap A r*) *A* (*elts* (*ordertype A r*))
  unfolding *bij-betw-def*
  proof (*intro conjI*)
    show *inj-on* (*ordermap A r*) *A*
   unfolding *inj-on-def* **by** (*metis assms mem-not-refl ordermap-mono total-on-def*)
  show *ordermap A r* ‘ *A* = *elts* (*ordertype A r*)
    by (*metis ordertype-def* ‹*small A*› *elts-of-set replacement*)
qed

lemma *ordermap-eq-iff* [*simp*]:
  $[\![x \in A;\ y \in A;$ *wf r*; *total-on A r*; *small A*$]\!] \Longrightarrow$ *ordermap A r x* = *ordermap A r y* $\longleftrightarrow$ $x = y$
  by (*metis bij-betw-iff-bijections ordermap-bij*)

lemma *inv-into-ordermap*: $\alpha \in$ *elts* (*ordertype A r*) $\Longrightarrow$ *inv-into A* (*ordermap A r*) $\alpha \in A$
  by (*meson in-mono inv-into-into ordermap-surj*)

lemma *ordertype-nat-imp-finite*:
  assumes *ordertype A r* = *ord-of-nat m small A wf r total-on A r*
  shows *finite A*
proof −
  have $A \approx$ *elts m*
    using *eqpoll-def assms ordermap-bij* **by** *fastforce*
  then show *?thesis*
    using *eqpoll-finite-iff finite-Ord-omega* **by** *blast*
qed

lemma *wf-ordertype-eqpoll*:
  assumes *wf r total-on A r small A*
  shows *elts* (*ordertype A r*) $\approx A$

using *assms eqpoll-def eqpoll-sym ordermap-bij* **by** *blast*

lemma *ordertype-eqpoll*:
assumes *small A*
shows *elts* (*ordertype A VWF*) ≈ *A*
using *assms wf-ordertype-eqpoll total-VWF wf-VWF*
by (*simp add*: *wf-ordertype-eqpoll total-on-def*)

2.9 More advanced *ordertype* and *ordermap* results

lemma *ordermap-VWF-0* [*simp*]: *ordermap A VWF 0* = *0*
by (*simp add*: *ordermap wf-VWO VWF-def*)

lemma *ordertype-empty* [*simp*]: *ordertype* {} *r* = *0*
by (*simp add*: *ordertype-def*)

lemma *ordertype-eq-0-iff* [*simp*]: ⟦*small X*; *wf r*⟧ ⟹ *ordertype X r* = *0* ⟷ *X* = {}
by (*metis ordertype-def elts-of-set replacement image-is-empty zero-V-def*)

lemma *ordermap-mono-less*:
assumes (*w*, *x*) ∈ *r*
and *wf r trans r*
and *w* ∈ *A x* ∈ *A*
and *small A*
shows *ordermap A r w* < *ordermap A r x*
by (*simp add*: *OrdmemD assms ordermap-mono*)

lemma *ordermap-mono-le*:
assumes (*w*, *x*) ∈ *r* ∨ *w*=*x*
and *wf r trans r*
and *w* ∈ *A x* ∈ *A*
and *small A*
shows *ordermap A r w* ≤ *ordermap A r x*
by (*metis assms dual-order.strict-implies-order eq-refl ordermap-mono-less*)

lemma *converse-ordermap-le-mono*:
assumes *ordermap A r y* ≤ *ordermap A r x wf r total r x* ∈ *A small A*
shows (*y*, *x*) ∈ *r* ∨ *y*=*x*
by (*meson UNIV-I assms mem-not-refl ordermap-mono total-on-def vsubsetD*)

lemma *ordertype-mono*:
assumes *X* ⊆ *Y* **and** *r*: *wf r trans r* **and** *small Y*
shows *ordertype X r* ≤ *ordertype Y r*
proof −
have *small X*
using *assms smaller-than-small* **by** *fastforce*
have ∗: *ordermap X r x* ≤ *ordermap Y r x* **for** *x*
using ‹*wf r*›

proof (*induction x rule*: *wf-induct-rule*)
case (*less x*)
have *ordermap X r z* < *ordermap Y r x* **if** $z \in X$ **and** *zx*: $(z,x) \in r$ **for** *z*
using *less* [*OF zx*] *assms*
by (*meson Ord-linear2 OrdmemD wf-Ord-ordermap ordermap-mono in-mono leD that*(*1*) *vsubsetD zx*)
then show *?case*
by (*auto simp add*: *ordermap* [*of - X x*] ‹*small X*› *Ord-mem-iff-lt set-image-le-iff less-eq-V-def r*)
qed
show *?thesis*
proof −
have *ordermap Y r* ‘ *Y* = *elts* (*ordertype Y r*)
by (*metis ordertype-def* ‹*small Y*› *elts-of-set replacement*)
then have *ordertype Y r* ∉ *ordermap X r* ‘ *X*
using ∗ ‹$X \subseteq Y$› **by** *fastforce*
then show *?thesis*
by (*metis Ord-linear2 Ord-mem-iff-lt ordertype-def wf-Ord-ordertype* ‹*small X*› *elts-of-set replacement r*)
qed
qed

corollary *ordertype-VWF-mono*:
assumes $X \subseteq Y$ *small Y*
shows *ordertype X VWF* ≤ *ordertype Y VWF*
using *assms* **by** (*simp add*: *ordertype-mono*)

lemma *ordertype-UNION-ge*:
assumes $A \in \mathcal{A}$ *wf r trans r* $\mathcal{A} \subseteq$ *Collect small small* $\mathcal{A}$
shows *ordertype A r* ≤ *ordertype* $(\bigcup \mathcal{A})$ *r*
by (*rule ordertype-mono*) (*use assms* **in** *auto*)

lemma *inv-ordermap-mono-less*:
assumes (*inv-into M* (*ordermap M r*) α, *inv-into M* (*ordermap M r*) β) ∈ *r*
and *small M* **and** α: $\alpha \in$ *elts* (*ordertype M r*) **and** β: $\beta \in$ *elts* (*ordertype M r*)
and *wf r trans r*
shows $\alpha < \beta$
proof −
have α = *ordermap M r* (*inv-into M* (*ordermap M r*) α)
by (*metis* α *f-inv-into-f ordermap-surj subset-eq*)
also have … < *ordermap M r* (*inv-into M* (*ordermap M r*) β)
by (*meson* α β *assms in-mono inv-into-into ordermap-mono-less ordermap-surj*)
also have … = β
by (*meson* β *f-inv-into-f in-mono ordermap-surj*)
finally show *?thesis* **.**
qed

lemma *inv-ordermap-mono-eq*:
assumes *inv-into M* (*ordermap M r*) α = *inv-into M* (*ordermap M r*) β

and $\alpha \in$ *elts* (*ordertype M r*) $\beta \in$ *elts* (*ordertype M r*)
shows $\alpha = \beta$
by (*metis assms f-inv-into-f ordermap-surj subsetD*)

lemma *inv-ordermap-VWF-mono-le*:
assumes *inv-into M* (*ordermap M VWF*) $\alpha \le$ *inv-into M* (*ordermap M VWF*) β
and $M \subseteq$ *ON small M* **and** α: $\alpha \in$ *elts* (*ordertype M VWF*) **and** β: $\beta \in$ *elts* (*ordertype M VWF*)
shows $\alpha \le \beta$
proof −
have $\alpha =$ *ordermap M VWF* (*inv-into M* (*ordermap M VWF*) α)
by (*metis* α *f-inv-into-f ordermap-surj subset-eq*)
also have $\ldots \le$ *ordermap M VWF* (*inv-into M* (*ordermap M VWF*) β)
by (*metis ON-imp-Ord VWF-iff-Ord-less assms dual-order.strict-implies-order elts-of-set eq-refl inv-into-into order.not-eq-order-implies-strict ordermap-mono-less ordertype-def replacement trans-VWF wf-VWF*)
also have $\ldots = \beta$
by (*meson* β *f-inv-into-f in-mono ordermap-surj*)
finally show *?thesis* **.**
qed

lemma *inv-ordermap-VWF-mono-iff*:
assumes $M \subseteq$ *ON small M* **and** $\alpha \in$ *elts* (*ordertype M VWF*) **and** $\beta \in$ *elts* (*ordertype M VWF*)
shows *inv-into M* (*ordermap M VWF*) $\alpha \le$ *inv-into M* (*ordermap M VWF*) $\beta \longleftrightarrow \alpha \le \beta$
by (*metis ON-imp-Ord Ord-linear-le assms dual-order.eq-iff inv-into-ordermap inv-ordermap-VWF-mono-le*)

lemma *inv-ordermap-VWF-strict-mono-iff*:
assumes $M \subseteq$ *ON small M* **and** $\alpha \in$ *elts* (*ordertype M VWF*) **and** $\beta \in$ *elts* (*ordertype M VWF*)
shows *inv-into M* (*ordermap M VWF*) $\alpha <$ *inv-into M* (*ordermap M VWF*) $\beta \longleftrightarrow \alpha < \beta$
by (*simp add*: *assms inv-ordermap-VWF-mono-iff less-le-not-le*)

lemma *strict-mono-on-ordertype*:
assumes $M \subseteq$ *ON small M*
obtains *f* **where** $f \in$ *elts* (*ordertype M VWF*) $\to$ *M strict-mono-on* (*elts* (*ordertype M VWF*)) *f*
proof
show *inv-into M* (*ordermap M VWF*) $\in$ *elts* (*ordertype M VWF*) $\to M$
by (*meson Pi-I′ in-mono inv-into-into ordermap-surj*)
show *strict-mono-on* (*elts* (*ordertype M VWF*)) (*inv-into M* (*ordermap M VWF*))
proof (*clarsimp simp*: *strict-mono-on-def*)
fix *x y*
assume $x \in$ *elts* (*ordertype M VWF*) $y \in$ *elts* (*ordertype M VWF*) $x < y$
then show *inv-into M* (*ordermap M VWF*) $x <$ *inv-into M* (*ordermap M*

VWF) *y*
using *assms* **by** (*meson ON-imp-Ord Ord-linear2 inv-into-into inv-ordermap-VWF-mono-le leD ordermap-surj subsetD*)
qed
qed

lemma *ordermap-inc-eq*:
assumes $x \in A$ *small A*
and π: $\bigwedge x\ y.\ [\![x{\in}A;\ y{\in}A;\ (x,y) \in r]\!] \Longrightarrow (\pi\ x,\ \pi\ y) \in s$
and *r*: *wf r total-on A r* **and** *wf s*
shows *ordermap* (π ' *A*) *s* (π *x*) = *ordermap A r x*
using ‹*wf r*› ‹$x \in A$›
proof (*induction x rule*: *wf-induct-rule*)
case (*less x*)
then have *1*: $\{y \in A.\ (y,\ x) \in r\} = A \cap \{y.\ (y,\ x) \in r\}$
using *r* **by** *auto*
have *2*: $\{y \in \pi\ `\ A.\ (y,\ \pi\ x) \in s\} = \pi\ `\ A \cap \{y.\ (y,\ \pi\ x) \in s\}$
by *auto*
have *invπ*: $\bigwedge x\ y.\ [\![x{\in}A;\ y{\in}A;\ (\pi\ x,\ \pi\ y) \in s]\!] \Longrightarrow (x,\ y) \in r$
by (*metis* π ‹*wf s*› ‹*total-on A r*› *total-on-def wf-not-sym*)
have *eq*: $f\ `\ (\pi\ `\ A \cap \{y.\ (y,\ \pi\ x) \in s\}) = (f \circ \pi)\ `\ (A \cap \{y.\ (y,\ x) \in r\})$ **for** *f* :: $'b \Rightarrow V$
using *less* **by** (*auto simp*: *image-subset-iff invπ π*)
show *?case*
using *less*
by (*simp add*: *ordermap* [*OF* ‹*wf r*›, *of - x*] *ordermap* [*OF* ‹*wf s*›, *of -* π *x*] *1 2 eq*)
qed

lemma *ordertype-inc-eq*:
assumes *small A*
and π: $\bigwedge x\ y.\ [\![x{\in}A;\ y{\in}A;\ (x,y) \in r]\!] \Longrightarrow (\pi\ x,\ \pi\ y) \in s$
and *r*: *wf r total-on A r* **and** *wf s*
shows *ordertype* (π ' *A*) *s* = *ordertype A r*
proof −
have *ordermap* (π ' *A*) *s* (π *x*) = *ordermap A r x* **if** $x \in A$ **for** *x*
using *assms that* **by** (*auto simp*: *ordermap-inc-eq*)
then show *?thesis*
unfolding *ordertype-def*
by (*metis* (*no-types*, *lifting*) *image-cong image-image*)
qed

lemma *ordertype-inc-le*:
assumes *small A small B*
and π: $\bigwedge x\ y.\ [\![x{\in}A;\ y{\in}A;\ (x,y) \in r]\!] \Longrightarrow (\pi\ x,\ \pi\ y) \in s$
and *r*: *wf r total-on A r* **and** *wf s trans s*
and $\pi\ `\ A \subseteq B$
shows *ordertype A r* $\leq$ *ordertype B s*
by (*metis assms ordertype-inc-eq ordertype-mono*)

corollary *ordertype-VWF-inc-eq*:
  assumes $A \subseteq ON$ π ‘ $A \subseteq ON$ *small A* **and** $\bigwedge x\ y.\ [\![x{\in}A;\ y{\in}A;\ x{<}y]\!] \Longrightarrow \pi\ x < \pi\ y$
  shows *ordertype* (π ‘ A) *VWF* = *ordertype A VWF*
proof (*rule ordertype-inc-eq*)
  show (π x, π y) $\in$ *VWF*
    if $x \in A$ $y \in A$ $(x, y) \in$ *VWF* **for** *x y*
    using *that ON-imp-Ord assms* **by** *auto*
  show *total-on A VWF*
    by (*meson UNIV-I total-VWF total-on-def*)
qed (*use assms* **in** *auto*)

lemma *ordertype-image-ordermap*:
  assumes *small A* $X \subseteq A$ *wf r trans r total-on X r*
  shows *ordertype* (*ordermap A r* ‘ *X*) *VWF* = *ordertype X r*
proof (*rule ordertype-inc-eq*)
  show *small X*
    by (*meson assms smaller-than-small*)
  show (*ordermap A r x*, *ordermap A r y*) $\in$ *VWF*
    if $x \in X$ $y \in X$ $(x, y) \in r$ **for** *x y*
    by (*meson that wf-Ord-ordermap VWF-iff-Ord-less assms ordermap-mono-less subsetD*)
qed (*use assms* **in** *auto*)

lemma *ordertype-map-image*:
  assumes $B \subseteq A$ *small A*
  shows *ordertype* (*ordermap A VWF* ‘ A − *ordermap A VWF* ‘ B) *VWF* = *ordertype* ($A - B$) *VWF*
proof −
  have *ordermap A VWF* ‘ A − *ordermap A VWF* ‘ B = *ordermap A VWF* ‘ ($A - B$)
    using *assms* **by** *auto*
  then have *ordertype* (*ordermap A VWF* ‘ A − *ordermap A VWF* ‘ B) *VWF* = *ordertype* (*ordermap A VWF* ‘ ($A - B$)) *VWF*
    by *simp*
  also have ... = *ordertype* ($A - B$) *VWF*
    using ‹*small A*› *ordertype-image-ordermap* **by** *fastforce*
  finally show *?thesis* **.**
qed

proposition *ordertype-le-ordertype*:
  assumes *r*: *wf r total-on A r* **and** *small A*
  assumes *s*: *wf s total-on B s trans s* **and** *small B*
  shows *ordertype A r* $\le$ *ordertype B s* $\longleftrightarrow$
        ($\exists f \in A \to B.$ *inj-on f A* $\wedge$ ($\forall x \in A.\ \forall y \in A.$ ((*x*,*y*) $\in r \longrightarrow$ (*f x*, *f y*) $\in s$)))
    (**is** *?lhs* = *?rhs*)
proof

assume L: *?lhs*
define f **where** $f \equiv$ *inv-into* B (*ordermap* B s) $\circ$ *ordermap* A r
show *?rhs*
proof (*intro bexI conjI ballI impI*)
have AB: *elts* (*ordertype* A r) $\subseteq$ *ordermap* B s ' B
by (*metis L assms*(*7*) *ordertype-def replacement set-of-elts small-elts subset-iff-less-eq-V*)
have *bijA*: *bij-betw* (*ordermap* A r) A (*elts* (*ordertype* A r))
using *ordermap-bij* ‹*small A*› r **by** *blast*
have *inv-into* B (*ordermap* B s) (*ordermap* A r i) $\in B$ **if** $i \in A$ **for** i
by (*meson L* ‹*small A*› *inv-into-into ordermap-in-ordertype ordermap-surj subsetD that vsubsetD*)
then show $f \in A \to B$
by (*auto simp*: *Pi-iff f-def*)
show *inj-on* f A
proof (*clarsimp simp add*: *f-def inj-on-def*)
fix x y
assume $x \in A$ $y \in A$
and *inv-into* B (*ordermap* B s) (*ordermap* A r x) = *inv-into* B (*ordermap* B s) (*ordermap* A r y)
then have *ordermap* A r x = *ordermap* A r y
by (*meson AB* ‹*small A*› *inv-into-injective ordermap-in-ordertype subsetD*)
then show $x = y$
by (*metis* ‹$x \in A$› ‹$y \in A$› *bijA bij-betw-inv-into-left*)
qed
next
fix x y
assume $x \in A$ $y \in A$ **and** $(x, y) \in r$
have ‡: *ordermap* A r y $\in$ *ordermap* B s ' B
by (*meson L* ‹$y \in A$› ‹*small A*› *in-mono ordermap-in-ordertype ordermap-surj vsubsetD*)
moreover have †: $\bigwedge x$. *inv-into* B (*ordermap* B s) (*ordermap* A r x) = f x
by (*simp add*: *f-def*)
then have $*$: *ordermap* B s (f y) = *ordermap* A r y
using ‡ **by** (*metis f-inv-into-f*)
moreover have *ordermap* A r x $\in$ *ordermap* B s ' B
by (*meson L* ‹$x \in A$› ‹*small A*› *in-mono ordermap-in-ordertype ordermap-surj vsubsetD*)
moreover have *ordermap* A r x $<$ *ordermap* A r y
using $*$ r s **by** (*metis* (*no-types*) *wf-Ord-ordermap OrdmemD* ‹$(x, y) \in r$› ‹$x \in A$› ‹*small A*› *ordermap-mono*)
ultimately show $(f\,x, f\,y) \in s$
using † s **by** (*metis assms*(*7*) *f-inv-into-f inv-into-into less-asym ordermap-mono-less total-on-def*)
qed
next
assume R: *?rhs*
then obtain f **where** f: $f{\in}A \to B$ *inj-on* f A $\forall x{\in}A.\ \forall y{\in}A.\ (x, y) \in r \longrightarrow (f\,x, f\,y) \in s$

by *blast*
show *?lhs*
by (*rule ordertype-inc-le* [**where** π=*f*]) (*use f assms* **in** *auto*)
qed

lemma *iso-imp-ordertype-eq-ordertype*:
assumes *iso*: *iso r r′ f*
and *wf r*
and *Total r*
and *sm*: *small* (*Field r*)
shows *ordertype* (*Field r*) *r* = *ordertype* (*Field r′*) *r′*
by (*metis* (*no-types*, *lifting*) *iso-forward iso-wf assms iso-Field ordertype-inc-eq sm*)

lemma *ordertype-infinite-ge-ω*:
assumes *infinite A small A*
shows *ordertype A VWF* ≥ ω
proof −
have *inj-on* (*ordermap A VWF*) *A*
by (*meson ordermap-bij* ‹*small A*› *bij-betw-def total-on-VWF wf-VWF*)
then have *infinite* (*ordermap A VWF* ‘ *A*)
using ‹*infinite A*› *finite-image-iff* **by** *blast*
then show *?thesis*
using *Ord-ordertype* ‹*small A*› *infinite-Ord-omega* **by** (*auto simp*: *ordertype-def*)
qed

lemma *ordertype-eqI*:
assumes *wf r total-on A r small A wf s*
bij-betw f A B (∀ *x* ∈ *A*. ∀ *y* ∈ *A*. (*f x*, *f y*) ∈ *s* ⟷ (*x*,*y*) ∈ *r*)
shows *ordertype A r* = *ordertype B s*
by (*metis assms bij-betw-imp-surj-on ordertype-inc-eq*)

lemma *ordermap-eq-self*:
assumes *Ord* α **and** *x*: *x* ∈ *elts* α
shows *ordermap* (*elts* α) *VWF x* = *x*
using *Ord-in-Ord* [*OF assms*] *x*
proof (*induction x rule*: *Ord-induct*)
case (*step x*)
have *1*: {*y* ∈ *elts* α. (*y*, *x*) ∈ *VWF*} = *elts x* (**is** *?A* = -)
proof
show *?A* ⊆ *elts x*
using ‹*Ord* α› **by** *clarify* (*meson Ord-in-Ord Ord-mem-iff-lt VWF-iff-Ord-less step.hyps*)
show *elts x* ⊆ *?A*
using ‹*Ord* α› **by** *clarify* (*meson Ord-in-Ord Ord-trans OrdmemD VWF-iff-Ord-less step.prems*)
qed
show *?case*
using *step*

by (*simp add*: *ordermap* [*OF wf-VWF*, *of* - *x*] *1 Ord-trans* [*of* - - α] *step.prems* ‹*Ord* α› *cong*: *image-cong*)
qed

lemma *ordertype-eq-Ord* [*simp*]:
assumes *Ord* α
shows *ordertype* (*elts* α) *VWF* = α
using *assms ordermap-eq-self* [*OF assms*] **by** (*simp add*: *ordertype-def*)

proposition *ordertype-eq-iff*:
assumes α: *Ord* α **and** *r*: *wf r* **and** *small A total-on A r trans r*
shows *ordertype A r* = $\alpha \longleftrightarrow$
($\exists f$. *bij-betw f A* (*elts* α) $\wedge$ ($\forall x \in A$. $\forall y \in A$. $f\,x < f\,y \longleftrightarrow (x,y) \in r$))
(**is** *?lhs* = *?rhs*)
proof *safe*
assume *eq*: α = *ordertype A r*
show $\exists f$. *bij-betw f A* (*elts* (*ordertype A r*)) $\wedge$ ($\forall x{\in}A$. $\forall y{\in}A$. $f\,x < f\,y \longleftrightarrow ((x, y) \in r)$)
proof (*intro exI conjI ballI*)
show *bij-betw* (*ordermap A r*) *A* (*elts* (*ordertype A r*))
by (*simp add*: *assms ordermap-bij*)
then show *ordermap A r x* < *ordermap A r y* $\longleftrightarrow$ $(x, y) \in r$
if $x \in A$ $y \in A$ **for** *x y*
using *that assms*
by (*metis order.asym ordermap-mono-less total-on-def*)
qed
next
fix *f*
assume *f*: *bij-betw f A* (*elts* α) $\forall x{\in}A$. $\forall y{\in}A$. $f\,x < f\,y \longleftrightarrow (x, y) \in r$
have *ordertype A r* = *ordertype* (*elts* α) *VWF*
proof (*rule ordertype-eqI*)
show $\forall x{\in}A$. $\forall y{\in}A$. $((f\,x, f\,y) \in VWF) = ((x, y) \in r)$
by (*meson Ord-in-Ord VWF-iff-Ord-less* α *bij-betwE f*)
qed (*use assms f* **in** *auto*)
then show *?lhs*
by (*simp add*: α)
qed

corollary *ordertype-VWF-eq-iff*:
assumes *Ord* α *small A*
shows *ordertype A VWF* = $\alpha \longleftrightarrow$
($\exists f$. *bij-betw f A* (*elts* α) $\wedge$ ($\forall x \in A$. $\forall y \in A$. $f\,x < f\,y \longleftrightarrow (x,y) \in VWF$))
by (*metis UNIV-I assms ordertype-eq-iff total-VWF total-on-def trans-VWF wf-VWF*)

lemma *ordertype-le-Ord*:
assumes *Ord* α $X \subseteq$ *elts* α
shows *ordertype X VWF* $\leq \alpha$

by (*metis assms ordertype-VWF-mono ordertype-eq-Ord small-elts*)

lemma *ordertype-inc-le-Ord*:
assumes *small A Ord* α
and π: $\bigwedge x\ y.\ [\![x{\in}A;\ y{\in}A;\ (x,y) \in r]\!] \Longrightarrow \pi\ x < \pi\ y$
and *wf r total-on A r*
and *sub*: π ‘ $A \subseteq$ *elts* α
shows *ordertype A r* $\leq \alpha$
proof −
have $\bigwedge x\ y.\ [\![x{\in}A;\ y{\in}A;\ (x,y) \in r]\!] \Longrightarrow (\pi\ x,\ \pi\ y) \in VWF$
by (*meson Ord-in-Ord VWF-iff-Ord-less* π ‹*Ord* α› *sub image-subset-iff*)
with *assms* **show** *?thesis*
by (*metis ordertype-inc-eq ordertype-le-Ord wf-VWF*)
qed

lemma *le-ordertype-obtains-subset*:
assumes α: $\beta \leq \alpha$ *ordertype H VWF* $= \alpha$ **and** *small H Ord* β
obtains *G* **where** $G \subseteq H$ *ordertype G VWF* $= \beta$
proof (*intro exI conjI that*)
let *?f* = *ordermap H VWF*
show ‡: *inv-into H ?f* ‘ *elts* $\beta \subseteq H$
unfolding *image-subset-iff*
by (*metis* α *inv-into-into ordermap-surj subsetD vsubsetD*)
have $\exists f.$ *bij-betw f* (*inv-into H ?f* ‘ *elts* β) (*elts* β) $\wedge$ ($\forall x{\in}$*inv-into H ?f* ‘ *elts* β.
$\forall y{\in}$*inv-into H ?f* ‘ *elts* β. ($f\ x < f\ y$) = ((x, y) $\in$ *VWF*))
proof (*intro exI conjI ballI iffI*)
show *bij-betw ?f* (*inv-into H ?f* ‘ *elts* β) (*elts* β)
using *ordermap-bij* [*OF wf-VWF total-on-VWF* ‹*small H*›] α
by (*metis bij-betw-inv-into-RIGHT bij-betw-subset less-eq-V-def* ‡)
next
fix *x y*
assume *x*: $x \in$ *inv-into H ?f* ‘ *elts* β
and *y*: $y \in$ *inv-into H ?f* ‘ *elts* β
show *?f x* < *?f y* **if** $(x,y) \in$ *VWF*
using *that* ‡ ‹*small H*› *in-mono ordermap-mono-less x y* **by** *fastforce*
show $(x,y) \in$ *VWF* **if** *?f x* < *?f y*
using *that* ‡ ‹*small H*› *in-mono ordermap-mono-less* [*OF - wf-VWF trans-VWF*]
x y
by (*metis UNIV-I less-imp-not-less total-VWF total-on-def*)
qed
then show *ordertype* (*inv-into H ?f* ‘ *elts* β) *VWF* $= \beta$
by (*subst ordertype-eq-iff*) (*use assms* **in** *auto*)
qed

lemma *ordertype-infinite-*ω:
assumes $A \subseteq$ *elts* ω *infinite A*
shows *ordertype A VWF* $= \omega$
proof (*rule antisym*)
show *ordertype A VWF* $\leq \omega$

by (*simp add*: *assms ordertype-le-Ord*)
show $\omega \le$ *ordertype A VWF*
using *assms down ordertype-infinite-ge-*ω **by** *auto*
qed

For infinite sets of natural numbers

lemma *ordertype-nat-*ω:
assumes *infinite N* **shows** *ordertype N less-than* $= \omega$
proof −
have *small N*
by (*meson inj-on-def ord-of-nat-inject small-def small-iff-range small-image-nat-V*)
have *ordertype* (*ord-of-nat ‘ N*) *VWF* $= \omega$
by (*force simp*: *assms finite-image-iff inj-on-def intro*: *ordertype-infinite-*ω)
moreover have *ordertype* (*ord-of-nat ‘ N*) *VWF* = *ordertype N less-than*
by (*auto intro*: *ordertype-inc-eq* ‹*small N*›)
ultimately show *?thesis*
by *simp*
qed

proposition *ordertype-eq-ordertype*:
assumes *r*: *wf r total-on A r trans r* **and** *small A*
assumes *s*: *wf s total-on B s trans s* **and** *small B*
shows *ordertype A r* = *ordertype B s* $\longleftrightarrow$
($\exists f$. *bij-betw f A B* $\wedge$ ($\forall x \in A$. $\forall y \in A$. $(f\ x,\ f\ y) \in s \longleftrightarrow (x,y) \in r$))
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
define γ **where** γ = *ordertype A r*
have *A*: *bij-betw* (*ordermap A r*) *A* (*ordermap A r ‘ A*)
by (*meson ordermap-bij assms*(*4*) *bij-betw-def r*)
have *B*: *bij-betw* (*ordermap B s*) *B* (*ordermap B s ‘ B*)
by (*meson ordermap-bij assms*(*8*) *bij-betw-def s*)
define *f* **where** $f \equiv$ *inv-into B* (*ordermap B s*) $\circ$ *ordermap A r*
show *?rhs*
proof (*intro exI conjI*)
have *bijA*: *bij-betw* (*ordermap A r*) *A* (*elts* γ)
unfolding γ*-def* **using** *ordermap-bij* ‹*small A*› *r* **by** *blast*
moreover have *bijB*: *bij-betw* (*ordermap B s*) *B* (*elts* γ)
by (*simp add*: *L* γ*-def ordermap-bij* ‹*small B*› *s*)
ultimately show *bij*: *bij-betw f A B*
unfolding *f-def* **using** *bij-betw-comp-iff bij-betw-inv-into* **by** *blast*
have *invB*: $\bigwedge\alpha$. $\alpha \in$ *elts* $\gamma \Longrightarrow$ *ordermap B s* (*inv-into B* (*ordermap B s*) α)
$= \alpha$
by (*meson bijB bij-betw-inv-into-right*)
have *ordermap-A-*γ: $\bigwedge a$. $a \in A \Longrightarrow$ *ordermap A r a* $\in$ *elts* γ
using *bijA bij-betwE* **by** *auto*
have *f-in-B*: $\bigwedge a$. $a \in A \Longrightarrow f\ a \in B$
using *bij bij-betwE* **by** *fastforce*
show $\forall x \in A$. $\forall y \in A$. $(f\ x,\ f\ y) \in s \longleftrightarrow (x,\ y) \in r$

proof (*intro iffI ballI*)
fix *x y*
assume $x \in A$ $y \in A$ **and** *ins*: $(f\ x,\ f\ y) \in s$
then have *ordermap* A r x < *ordermap* A r y
unfolding *o-def*
by (*metis* (*mono-tags*, *lifting*) *f-def* ‹*small B*› *comp-apply f-in-B invB ordermap-A-*γ *ordermap-mono-less* *s*(*1*) *s*(*3*))
then show $(x,\ y) \in r$
by (*metis* ‹$x \in A$› ‹$y \in A$› ‹*small A*› *order.asym ordermap-mono-less r total-on-def*)
next
fix *x y*
assume $x \in A$ $y \in A$ **and** $(x,\ y) \in r$
then have *ordermap* A r x < *ordermap* A r y
by (*simp add*: ‹*small A*› *ordermap-mono-less r*)
then have $(f\ y,\ f\ x) \notin s$
by (*metis* (*mono-tags*, *lifting*) ‹$x \in A$› ‹$y \in A$› ‹*small B*› *comp-apply f-def f-in-B invB order.asym ordermap-A-*γ *ordermap-mono-less* *s*(*1*) *s*(*3*))
moreover have $f\ y \neq f\ x$
by (*metis* ‹$(x,\ y) \in r$› ‹$x \in A$› ‹$y \in A$› *bij bij-betw-inv-into-left* *r*(*1*) *wf-not-sym*)
ultimately show $(f\ x,\ f\ y) \in s$
by (*meson* ‹$x \in A$› ‹$y \in A$› *f-in-B* *s*(*2*) *total-on-def*)
qed
qed
next
assume *?rhs*
then show *?lhs*
using *assms ordertype-eqI* **by** *blast*
qed

corollary *ordertype-eq-ordertype-iso*:
assumes *r*: *wf r total-on A r trans r* **and** *small A* **and** *FA*: *Field r* = *A*
assumes *s*: *wf s total-on B s trans s* **and** *small B* **and** *FB*: *Field s* = *B*
shows *ordertype* A r = *ordertype* B s $\longleftrightarrow$ ($\exists f.$ *iso r s f*)
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then obtain *f* **where** *bij-betw f A B* $\forall x \in A.\ \forall y \in A.\ (f\ x,\ f\ y) \in s \longleftrightarrow (x,y) \in r$
using *assms ordertype-eq-ordertype* **by** *blast*
then show *?rhs*
using *FA FB iso-iff2* **by** *blast*
next
assume *?rhs*
then show *?lhs*
using *FA FB* ‹*small A*› *iso-imp-ordertype-eq-ordertype r* **by** *blast*
qed

lemma *Limit-ordertype-imp-Field-Restr*:
  assumes *Lim*: *Limit* (*ordertype A r*) **and** *r*: *wf r total-on A r* **and** *small A*
  shows *Field* (*Restr r A*) = *A*
proof −
  have ∃ *y*∈*A*. (*x*,*y*) ∈ *r* **if** *x* ∈ *A* **for** *x*
  proof −
    let *?oy* = *succ* (*ordermap A r x*)
    have §: *?oy* ∈ *elts* (*ordertype A r*)
      by (*simp add*: *Lim* ‹*small A*› *ordermap-in-ordertype succ-in-Limit-iff that*)
    then have *A*: *inv-into A* (*ordermap A r*) *?oy* ∈ *A*
      by (*simp add*: *inv-into-ordermap*)
    moreover have (*x*, *inv-into A* (*ordermap A r*) *?oy*) ∈ *r*
    proof −
      have *ordermap A r x* ∈ *elts* (*ordermap A r* (*inv-into A* (*ordermap A r*) *?oy*))
        by (*metis* § *elts-succ f-inv-into-f insert-iff ordermap-surj subsetD*)
      then show *?thesis*
        by (*metis* ‹*small A*› *A converse-ordermap-mono r that*)
    qed
    ultimately show *?thesis* **..**
  qed
  then have *A* ⊆ *Field* (*Restr r A*)
    by (*auto simp*: *Field-def*)
  then show *?thesis*
    by (*simp add*: *Field-Restr-subset subset-antisym*)
qed

lemma *ordertype-Field-Restr*:
  assumes *wf r total-on A r trans r small A Field* (*Restr r A*) = *A*
  shows *ordertype* (*Field* (*Restr r A*)) (*Restr r A*) = *ordertype A r*
  using *assms* **by** (*force simp*: *ordertype-eq-ordertype wf-Int1 total-on-def trans-Restr*)

proposition *ordertype-eq-ordertype-iso-Restr*:
  assumes *r*: *wf r total-on A r trans r* **and** *small A* **and** *FA*: *Field* (*Restr r A*) =
A
  assumes *s*: *wf s total-on B s trans s* **and** *small B* **and** *FB*: *Field* (*Restr s B*) =
B
  shows *ordertype A r* = *ordertype B s* ⟷ (∃ *f*. *iso* (*Restr r A*) (*Restr s B*) *f*)
    (**is** *?lhs* = *?rhs*)
proof
  assume *L*: *?lhs*
  then obtain *f* **where** *bij-betw f A B* ∀ *x* ∈ *A*. ∀ *y* ∈ *A*. (*f x*, *f y*) ∈ *s* ⟷ (*x*,*y*)
∈ *r*
    using *assms ordertype-eq-ordertype* **by** *blast*
  then show *?rhs*
    using *FA FB bij-betwE* **unfolding** *iso-iff2* **by** *fastforce*
next
  assume *?rhs*
  moreover
  have *ordertype* (*Field* (*Restr r A*)) (*Restr r A*) = *ordertype A r*

using *FA* ‹*small A*› *ordertype-Field-Restr r* **by** *blast*
moreover
have *ordertype* (*Field* (*Restr s B*)) (*Restr s B*) = *ordertype B s*
using *FB* ‹*small B*› *ordertype-Field-Restr s* **by** *blast*
ultimately show *?lhs*
using *iso-imp-ordertype-eq-ordertype FA FB* ‹*small A*› *r*
by (*fastforce intro*: *total-on-imp-Total-Restr trans-Restr wf-Int1*)
qed

lemma *ordermap-insert*:
assumes *Ord* α **and** *y*: *Ord y* $y \le \alpha$ **and** *U*: $U \subseteq$ *elts* α
shows *ordermap* (*insert* α *U*) *VWF y* = *ordermap U VWF y*
using *y*
proof (*induction rule*: *Ord-induct*)
case (*step y*)
then have *1*: $\{u \in U.\ (u, y) \in VWF\}$ = *elts y* $\cap$ *U*
apply (*simp add*: *set-eq-iff*)
by (*meson Ord-in-Ord Ord-mem-iff-lt VWF-iff-Ord-less assms subsetD*)
have *2*: $\{u \in$ *insert* α $U.\ (u, y) \in VWF\}$ = *elts y* $\cap$ *U*
apply (*simp add*: *set-eq-iff*)
by (*meson Ord-in-Ord Ord-mem-iff-lt VWF-iff-Ord-less assms leD step.hyps step.prems subsetD*)
show *?case*
using *step*
apply (*simp only*: *ordermap* [*OF wf-VWF*, *of - y*] *1 2*)
by (*meson Int-lower1 Ord-is-Transset Sup.SUP-cong Transset-def assms*(*1*) *in-mono vsubsetD*)
qed

lemma *ordertype-insert*:
assumes *Ord* α **and** *U*: $U \subseteq$ *elts* α
shows *ordertype* (*insert* α *U*) *VWF* = *succ* (*ordertype U VWF*)
proof −
have †: $\{y \in$ *insert* α $U.\ (y, \alpha) \in VWF\} = U$ $\{y \in U.\ (y, \alpha) \in VWF\} = U$
using *Ord-in-Ord OrdmemD assms* **by** *auto*
have *eq*: $\bigwedge x.\ x \in U \Longrightarrow$ *ordermap* (*insert* α *U*) *VWF x* = *ordermap U VWF x*
by (*meson Ord-in-Ord Ord-is-Transset Transset-def U assms*(*1*) *in-mono ordermap-insert*)
have *ordertype* (*insert* α *U*) *VWF* =
ZFC-in-HOL.set (*insert* (*ordermap U VWF* α) (*ordermap U VWF* ‘ *U*))
by (*simp add*: *ordertype-def ordermap-insert assms eq*)
also have . . . = *succ* (*ZFC-in-HOL.set* (*ordermap U VWF* ‘ *U*))
using † *U* **by** (*simp add*: *ordermap* [*OF wf-VWF*, *of -* α] *down succ-def vinsert-def*)
also have . . . = *succ* (*ordertype U VWF*)
by (*simp add*: *ordertype-def*)
finally show *?thesis* **.**
qed

lemma *finite-ordertype-le-card*:
assumes *finite A wf r trans r*
shows *ordertype A r* $\leq$ *ord-of-nat* (*card A*)
proof –
have *Ord* (*ordertype A r*)
by (*simp add*: *wf-Ord-ordertype assms*)
moreover have *ordermap A r ‘ A* = *elts* (*ordertype A r*)
by (*simp add*: *ordertype-def finite-imp-small* ‹*finite A*›)
moreover have *card* (*ordermap A r ‘ A*) $\leq$ *card A*
using ‹*finite A*› *card-image-le* **by** *blast*
ultimately show *?thesis*
by (*metis Ord-linear-le Ord-ord-of-nat* ‹*finite A*› *card-ord-of-nat card-seteq finite-imageI less-eq-V-def*)
qed

lemma *ordertype-VWF-ω*:
assumes *finite A*
shows *ordertype A VWF* $\in$ *elts ω*
proof –
have *finite* (*ordermap A VWF ‘ A*)
using *assms* **by** *blast*
then have *ordertype A VWF* < *ω*
by (*meson Ord-ω OrdmemD trans-VWF wf-VWF assms finite-ordertype-le-card le-less-trans ord-of-nat-ω*)
then show *?thesis*
by (*simp add*: *Ord-mem-iff-lt*)
qed

lemma *ordertype-VWF-finite-nat*:
assumes *finite A*
shows *ordertype A VWF* = *ord-of-nat* (*card A*)
by (*metis finite-imp-small ordermap-bij total-on-VWF wf-VWF ω-def assms bij-betw-same-card card-ord-of-nat elts-of-set f-inv-into-f inf ordertype-VWF-ω*)

lemma *finite-ordertype-eq-card*:
assumes *small A wf r trans r total-on A r*
shows *ordertype A r* = *ord-of-nat m* $\longleftrightarrow$ *finite A* $\wedge$ *card A* = *m*
using *ordermap-bij* [*OF* ‹*wf r*›]
proof –
have ∗: *bij-betw* (*ordermap A r*) *A* (*elts* (*ordertype A r*))
by (*simp add*: *assms ordermap-bij*)
moreover have *card* (*ordermap A r ‘ A*) = *card A*
by (*meson bij-betw-def* ∗ *card-image*)
ultimately show *?thesis*
using *assms bij-betw-finite bij-betw-imp-surj-on finite-Ord-omega ordertype-VWF-finite-nat wf-Ord-ordertype* **by** *fastforce*
qed

```
lemma ex-bij-betw-strict-mono-card:
  assumes finite M M ⊆ ON
  obtains h where bij-betw h {..<card M} M and strict-mono-on {..<card M} h
proof −
  have bij: bij-betw (ordermap M VWF) M (elts (card M))
    using Finite-V ‹finite M› ordermap-bij ordertype-VWF-finite-nat by fastforce
  let ?h = (inv-into M (ordermap M VWF)) ∘ ord-of-nat
  show thesis
  proof
    show bijh: bij-betw ?h {..<card M} M
    proof (rule bij-betw-trans)
      show bij-betw ord-of-nat {..<card M} (elts (card M))
        by (simp add: bij-betw-def elts-ord-of-nat inj-on-def)
      show bij-betw (inv-into M (ordermap M VWF)) (elts (card M)) M
      using Finite-V assms bij-betw-inv-into ordermap-bij ordertype-VWF-finite-nat
by fastforce
    qed
    show strict-mono-on {..<card M} ?h
    proof −
      have ?h m < ?h n
        if m < n n < card M for m n
      proof (rule ccontr)
        obtain mn: m ∈ elts (ordertype M VWF) n ∈ elts (ordertype M VWF)
         using ‹m < n› ‹n < card M› ‹finite M› ordertype-VWF-finite-nat by auto
        have ord: Ord (?h m) Ord (?h n)
          using bijh assms(2) bij-betwE that by fastforce+
        moreover
        assume ¬ ?h m < ?h n
        ultimately consider ?h m = ?h n | ?h m > ?h n
          using Ord-linear-lt by blast
        then show False
        proof cases
          case 1
          then have m = n
            by (metis inv-ordermap-mono-eq mn comp-apply ord-of-nat-inject)
          with ‹m < n› show False by blast
        next
          case 2
          then have ord-of-nat n ≤ ord-of-nat m
               by (metis Finite-V mn assms comp-def inv-ordermap-VWF-mono-le
less-imp-le)
          then show ?thesis
            using leD ‹m < n› by blast
        qed
      qed
      with assms show ?thesis
        by (auto simp: strict-mono-on-def)
    qed
  qed
```

qed

lemma *ordertype-finite-less-than* [*simp*]:
assumes *finite A* **shows** *ordertype A less-than = card A*
proof −
let *?M = ord-of-nat ‘ A*
obtain *M*: *finite ?M ?M ⊆ ON*
using *Ord-ord-of-nat assms* **by** *blast*
have *ordertype A less-than = ordertype ?M VWF*
by (*rule ordertype-inc-eq* [*symmetric*]) (*use assms finite-imp-small total-on-def* **in** ‹*force+*›)
also have . . . = *card A*
proof (*subst ordertype-eq-iff*)
let *?M = ord-of-nat ‘ A*
obtain *h* **where** *bijh*: *bij-betw h* {..<*card A*} *?M* **and** *smh*: *strict-mono-on* {..<*card A*} *h*
by (*metis M card-image ex-bij-betw-strict-mono-card inj-on-def ord-of-nat-inject*)
define *f* **where** *f ≡ ord-of-nat ∘ inv-into* {..<*card A*} *h*
show ∃ *f*. *bij-betw f ?M* (*elts* (*card A*)) ∧ (∀ *x*∈*?M*. ∀ *y*∈*?M*. *f x* < *f y* ⟷ ((*x*, *y*) ∈ *VWF*))
proof (*intro exI conjI ballI*)
have *bij-betw* (*ord-of-nat ∘ inv-into* {..<*card A*} *h*) (*ord-of-nat ‘ A*) (*ord-of-nat ‘* {..<*card A*})
by (*meson UNIV-I bijh bij-betw-def bij-betw-inv-into bij-betw-subset bij-betw-trans inj-ord-of-nat subsetI*)
then show *bij-betw f ?M* (*elts* (*card A*))
by (*metis elts-ord-of-nat f-def*)
next
fix *x y*
assume *xy*: *x ∈ ?M y ∈ ?M*
then obtain *m n* **where** *x = ord-of-nat m y = ord-of-nat n*
by *auto*
have (*f x* < *f y*) ⟷ ((*h ∘ inv-into* {..<*card A*} *h*) *x* < (*h ∘ inv-into* {..<*card A*} *h*) *y*)
unfolding *f-def* **using** *smh bij-betw-imp-surj-on* [*OF bijh*]
apply *simp*
by (*metis* (*mono-tags*, *lifting*) *inv-into-into not-less-iff-gr-or-eq order.asym strict-mono-onD xy*)
also have . . . = (*x* < *y*)
using *bijh*
by (*simp add*: *bij-betw-inv-into-right xy*)
also have . . . ⟷ ((*x*, *y*) ∈ *VWF*)
using *M*(*2*) *ON-imp-Ord xy* **by** *auto*
finally show (*f x* < *f y*) ⟷ ((*x*, *y*) ∈ *VWF*) **.**
qed
qed *auto*
finally show *?thesis* **.**
qed

2.10 Cardinality of an arbitrary HOL set

definition *gcard* :: $'a\ set \Rightarrow V$
  where *gcard X* $\equiv$ *if small X then* (*LEAST i. Ord i* $\wedge$ *elts i* $\approx$ *X*) *else 0*

2.11 Cardinality of a set

definition *vcard* :: $V \Rightarrow V$
  where *vcard a* $\equiv$ (*LEAST i. Ord i* $\wedge$ *elts i* $\approx$ *elts a*)

lemma *gcard-eq-vcard* [*simp*]: *gcard* (*elts x*) = *vcard x*
  by (*simp add*: *vcard-def gcard-def*)

definition *Card*:: $V \Rightarrow bool$
  where *Card i* $\equiv$ *i* = *vcard i*

abbreviation *CARD* **where** *CARD* $\equiv$ *Collect Card*

lemma *cardinal-cong*: *elts x* $\approx$ *elts y* $\Longrightarrow$ *vcard x* = *vcard y*
  unfolding *vcard-def* **by** (*meson eqpoll-sym eqpoll-trans*)

lemma *gcardinal-cong*:
  assumes *X* $\approx$ *Y* **shows** *gcard X* = *gcard Y*
proof −
  have (*LEAST i. Ord i* $\wedge$ *elts i* $\approx$ *X*) = (*LEAST i. Ord i* $\wedge$ *elts i* $\approx$ *Y*)
    by (*meson eqpoll-sym eqpoll-trans assms*)
  then show *?thesis*
    unfolding *gcard-def*
    by (*meson eqpoll-sym small-eqcong assms*)
qed

lemma *vcard-set-image*: *inj-on f* (*elts x*) $\Longrightarrow$ *vcard* (*set* (*f* ‘ *elts x*)) = *vcard x*
  by (*simp add*: *cardinal-cong*)

lemma *gcard-image*: *inj-on f X* $\Longrightarrow$ *gcard* (*f* ‘ *X*) = *gcard X*
  by (*simp add*: *gcardinal-cong*)

lemma *Card-cardinal-eq*: *Card* κ $\Longrightarrow$ *vcard* κ = κ
  by (*simp add*: *Card-def*)

lemma *Card-is-Ord*:
  assumes *Card* κ **shows** *Ord* κ
proof −
  obtain α **where** *Ord* α *elts* α $\approx$ *elts* κ
    using *Ord-ordertype ordertype-eqpoll* **by** *blast*
  then have *Ord* (*LEAST i. Ord i* $\wedge$ *elts i* $\approx$ *elts* κ)
    by (*metis Ord-Least*)
  then show *?thesis*
    using *Card-def vcard-def assms* **by** *auto*
qed

lemma *cardinal-eqpoll*: *elts* (*vcard a*) $\approx$ *elts a*
  unfolding *vcard-def*
  using *ordertype-eqpoll* [*of elts a*] *Ord-LeastI* **by** (*meson Ord-ordertype small-elts*)

lemma *inj-into-vcard*:
  obtains *f* **where** $f \in$ *elts A* $\to$ *elts* (*vcard A*) *inj-on f* (*elts A*)
  using *cardinal-eqpoll* [*of A*] *inj-on-the-inv-into the-inv-into-onto*
  by (*fastforce simp*: *Pi-iff bij-betw-def eqpoll-def*)

lemma *cardinal-idem* [*simp*]: *vcard* (*vcard a*) = *vcard a*
  using *cardinal-cong cardinal-eqpoll* **by** *blast*

lemma *subset-smaller-vcard*:
  assumes $\kappa \leq$ *vcard x Card* κ
  obtains *y* **where** $y \leq x$ *vcard y* $= \kappa$
proof −
  obtain φ **where** φ: *bij-betw* φ (*elts* (*vcard x*)) (*elts x*)
    using *cardinal-eqpoll eqpoll-def* **by** *blast*
  show *thesis*
  proof
    let *?y* = *ZFC-in-HOL.set* (φ ‘ *elts* κ)
    show *?y* $\leq x$
      by (*meson* φ *assms bij-betwE set-image-le-iff small-elts vsubsetD*)
    show *vcard ?y* $= \kappa$
        by (*metis vcard-set-image Card-def assms bij-betw-def bij-betw-subset* φ *less-eq-V-def*)
  qed
qed

every natural number is a (finite) cardinal

lemma *nat-into-Card*:
  assumes $\alpha \in$ *elts* ω **shows** *Card*(α)
proof (*unfold Card-def vcard-def*, *rule sym*)
  obtain *n* **where** *n*: α = *ord-of-nat n*
    by (*metis* ω*-def assms elts-of-set imageE inf*)
  have *Ord*(α) **using** *assms* **by** *auto*
  moreover
  { fix β
    assume $\beta < \alpha$ *Ord* β *elts* $\beta \approx$ *elts* α
    with *n* **have** *elts* $\beta \approx \{..<n\}$
     by (*simp add*: *ord-of-nat-eq-initial* [*of n*] *eqpoll-trans inj-on-def inj-on-image-eqpoll-self*)
    hence *False* **using** *assms n* ‹*Ord* β› ‹$\beta < \alpha$› ‹*Ord*(α)›
     by (*metis* ‹*elts* $\beta \approx$ *elts* α› *card-seteq eqpoll-finite-iff eqpoll-iff-card finite-lessThan less-eq-V-def less-le-not-le order-refl*)
  }
  ultimately
    show (*LEAST i*. *Ord i* $\wedge$ *elts i* $\approx$ *elts* α) $= \alpha$
     by (*metis* (*no-types*, *lifting*) *Least-equality Ord-linear-le eqpoll-refl less-le-not-le*)

qed

lemma *Card-ord-of-nat* [*simp*]: *Card* (*ord-of-nat n*)
by (*simp add*: *ω-def nat-into-Card*)

lemma *Card-0* [*iff*]: *Card 0*
by (*simp add*: *nat-into-Card*)

lemma *CardI*: ⟦*Ord i*; ⋀*j*. ⟦*j* < *i*; *Ord j*⟧ ⟹ ¬ *elts j* ≈ *elts i*⟧ ⟹ *Card i*
unfolding *Card-def vcard-def*
by (*metis Ord-Least Ord-linear-lt cardinal-eqpoll eqpoll-refl not-less-Ord-Least vcard-def*)

lemma *vcard-0* [*simp*]: *vcard 0* = *0*
using *Card-0 Card-def* **by** *auto*

lemma *Ord-cardinal* [*simp*,*intro*!]: *Ord*(*vcard a*)
unfolding *vcard-def* **by** (*metis Card-def Card-is-Ord cardinal-cong cardinal-eqpoll vcard-def*)

lemma *gcard-big-0*: ¬ *small X* ⟹ *gcard X* = *0*
by (*simp add*: *gcard-def*)

lemma *gcard-eq-card*:
assumes *finite X* **shows** *gcard X* = *ord-of-nat* (*card X*)
proof −
have ⋀*y*. *Ord y* ∧ *elts y* ≈ *X* ⟹ *ord-of-nat* (*card X*) ≤ *y*
by (*metis assms eqpoll-finite-iff eqpoll-iff-card order-le-less ordertype-VWF-finite-nat ordertype-eq-Ord*)
then have (*LEAST i*. *Ord i* ∧ *elts i* ≈ *X*) = *ord-of-nat* (*card X*)
by (*simp add*: *assms eqpoll-iff-card finite-Ord-omega Least-equality*)
with *assms* **show** *?thesis*
by (*simp add*: *finite-imp-small gcard-def*)
qed

lemma *gcard-empty-0* [*simp*]: *gcard* {} = *0*
by (*simp add*: *gcard-eq-card*)

lemma *gcard-single-1* [*simp*]: *gcard* {*x*} = *1*
by (*simp add*: *gcard-eq-card one-V-def*)

lemma *Card-gcard* [*iff*]: *Card* (*gcard X*)
by (*metis Card-0 Card-def cardinal-idem gcard-big-0 gcardinal-cong small-eqpoll gcard-eq-vcard*)

lemma *gcard-eqpoll*: *small X* ⟹ *elts* (*gcard X*) ≈ *X*
by (*metis cardinal-eqpoll eqpoll-trans gcard-eq-vcard gcardinal-cong small-eqpoll*)

lemma *lepoll-imp-gcard-le*:

assumes $A \lesssim B$ *small B*
shows *gcard A* $\leq$ *gcard B*
proof −
have *elts* (*gcard A*) $\approx$ *A elts* (*gcard B*) $\approx$ *B*
by (*meson assms gcard-eqpoll lepoll-small*)+
with ‹$A \lesssim B$› **show** *?thesis*
by (*metis Ord-cardinal Ord-linear2 eqpoll-sym gcard-eq-vcard gcardinal-cong lepoll-antisym*
lepoll-trans2 less-V-def less-eq-V-def subset-imp-lepoll)
qed

lemma *gcard-image-le*:
assumes *small A* **shows** *gcard* (*f* ‘ *A*) $\leq$ *gcard A*
using *assms image-lepoll lepoll-imp-gcard-le* **by** *blast*

lemma *subset-imp-gcard-le*:
assumes $A \subseteq B$ *small B*
shows *gcard A* $\leq$ *gcard B*
by (*simp add*: *assms lepoll-imp-gcard-le subset-imp-lepoll*)

lemma *gcard-le-lepoll*: $[\![$*gcard A* $\leq \alpha$; *small A*$]\!] \Longrightarrow A \lesssim$ *elts* α
by (*meson eqpoll-sym gcard-eqpoll lepoll-trans1 less-eq-V-def subset-imp-lepoll*)

2.12 Cardinality of a set

The cardinals are the initial ordinals.

lemma *Card-iff-initial*: *Card* $\kappa \longleftrightarrow$ *Ord* $\kappa \land (\forall \alpha.$ *Ord* $\alpha \land \alpha < \kappa \longrightarrow$ ~ *elts* $\alpha \approx$ *elts* κ)
by (*metis CardI Card-def Card-is-Ord not-less-Ord-Least vcard-def*)

lemma *Card-ω* [*iff*]: *Card* ω
by (*meson CardI Ord-ω eqpoll-finite-iff infinite-Ord-omega infinite-ω leD*)

lemma *lt-Card-imp-lesspoll*: $[\![i < a$; *Card a*; *Ord i*$]\!] \Longrightarrow$ *elts i* $\prec$ *elts a*
by (*meson Card-iff-initial less-eq-V-def less-imp-le lesspoll-def subset-imp-lepoll*)

lemma *lepoll-imp-Card-le*:
assumes *elts a* $\lesssim$ *elts b* **shows** *vcard a* $\leq$ *vcard b*
using *assms lepoll-imp-gcard-le* **by** *fastforce*

lemma *lepoll-cardinal-le*: $[\![$*elts A* $\lesssim$ *elts i*; *Ord i*$]\!] \Longrightarrow$ *vcard A* $\leq i$
by (*metis Ord-Least Ord-linear2 dual-order.trans eqpoll-refl lepoll-imp-Card-le not-less-Ord-Least vcard-def*)

lemma *cardinal-le-lepoll*: *vcard A* $\leq \alpha \Longrightarrow$ *elts A* $\lesssim$ *elts* α
by (*meson cardinal-eqpoll eqpoll-sym lepoll-trans1 less-eq-V-def subset-imp-lepoll*)

lemma *lesspoll-imp-Card-less*:
assumes *elts a* $\prec$ *elts b* **shows** *vcard a* $<$ *vcard b*

by (*metis assms cardinal-eqpoll eqpoll-sym eqpoll-trans lepoll-imp-Card-le less-V-def lesspoll-def*)

lemma *Card-Union* [*simp*,*intro*]:
  assumes *A*: $\bigwedge x.\ x \in A \Longrightarrow Card(x)$ **shows** $Card(\bigsqcup A)$
proof (*rule CardI*)
  show $Ord(\bigsqcup A)$ **using** *A*
    by (*simp add*: *Card-is-Ord Ord-Sup*)
next
  fix *j*
  assume *j*: $j < \bigsqcup A$ *Ord j*
  hence $\exists\, c{\in}A.\ j < c \wedge Card(c)$ **using** *A*
    by (*meson Card-is-Ord Ord-linear2 ZFC-in-HOL.Sup-least leD*)
  then obtain *c* **where** *c*: $c{\in}A$ $j < c$ $Card(c)$
    by *blast*
  hence *jls*: *elts j* $\prec$ *elts c*
    using *j*(*2*) *lt-Card-imp-lesspoll* **by** *blast*
  { assume *eqp*: *elts j* $\approx$ *elts* ($\bigsqcup A$)
    have *elts c* $\lesssim$ *elts* ($\bigsqcup A$) **using** *c*
      by (*metis Card-def Sup-V-def ZFC-in-HOL.Sup-upper cardinal-le-lepoll j*(*1*) *not-less-0*)
    also have ... $\approx$ *elts j* **by** (*rule eqpoll-sym* [*OF eqp*])
    also have ... $\prec$ *elts c* **by** (*rule jls*)
    finally have *elts c* $\prec$ *elts c* **.**
    hence *False*
      by *auto*
  } thus $\neg$ *elts j* $\approx$ *elts* ($\bigsqcup A$) **by** *blast*
qed

lemma *Card-UN*: ($\bigwedge x.\ x \in A \Longrightarrow Card(K\ x)$) ==> *Card*(*Sup* (*K* ‘ *A*))
  by *blast*

2.13 Transfinite recursion for definitions based on the three cases of ordinals

definition
  transrec3 :: $[V, [V,V] \Rightarrow V, [V, V \Rightarrow V] \Rightarrow V, V] \Rightarrow V$ **where**
    transrec3 a b c $\equiv$
      transrec ($\lambda r\ x$.
        if x=0 then a
        else if Limit x then c x ($\lambda y \in$ *elts x. r y*)
        else b(*pred x*) (*r* (*pred x*)))

lemma *transrec3-0* [*simp*]: *transrec3 a b c 0* = *a*
  by (*simp add*: *transrec transrec3-def*)

lemma *transrec3-succ* [*simp*]:
    transrec3 a b c (*succ i*) = *b i* (*transrec3 a b c i*)

by (*simp add*: *transrec transrec3-def*)

lemma *transrec3-Limit* [*simp*]:
  Limit i $\Longrightarrow$ *transrec3 a b c i* = *c i* ($\lambda j \in$ *elts i*. *transrec3 a b c j*)
  unfolding *transrec3-def*
  by (*subst transrec*) *auto*

2.14 Cardinal Addition

definition *cadd* :: [V,V]$\Rightarrow$$V$ (**infixl** ‹$\oplus$› *65*)
  where $\kappa \oplus \mu \equiv$ *vcard* ($\kappa \uplus \mu$)

2.14.1 Cardinal addition is commutative

lemma *vsum-commute-eqpoll*: *elts* ($a \uplus b$) $\approx$ *elts* ($b \uplus a$)
proof −
  have *bij-betw* ($\lambda z \in$ *elts* ($a \uplus b$). *sum-case Inr Inl z*) (*elts* ($a \uplus b$)) (*elts* ($b \uplus a$))
    unfolding *bij-betw-def*
  proof (*intro conjI inj-onI*)
    show *restrict* (*sum-case Inr Inl*) (*elts* ($a \uplus b$)) ‘ *elts* ($a \uplus b$) = *elts* ($b \uplus a$)
     apply *auto*
     apply (*metis* (*no-types*) *imageI sum-case-Inr sum-iff*)
      by (*metis Inl-in-sum-iff imageI sum-case-Inl*)
  qed *auto*
  then show *?thesis*
    using *eqpoll-def* **by** *blast*
qed

lemma *cadd-commute*: $i \oplus j = j \oplus i$
  by (*simp add*: *cadd-def cardinal-cong vsum-commute-eqpoll*)

2.14.2 Cardinal addition is associative

lemma *sum-assoc-bij*:
  bij-betw ($\lambda z \in$ *elts* (($a \uplus b$)$\uplus c$). *sum-case*(*sum-case Inl* (λy. *Inr*(*Inl y*))) (λy. *Inr*(*Inr y*)) *z*)
     (*elts* (($a \uplus b$)$\uplus c$)) (*elts* ($a \uplus$($b \uplus c$)))
  by (*rule-tac* f' = *sum-case* (λx. *Inl* (*Inl x*)) (*sum-case* (λx. *Inl* (*Inr x*)) *Inr*)
     in *bij-betw-byWitness*) *auto*

lemma *sum-assoc-eqpoll*: *elts* (($a \uplus b$)$\uplus c$) $\approx$ *elts* ($a \uplus$($b \uplus c$))
  unfolding *eqpoll-def* **by** (*metis sum-assoc-bij*)

lemma *elts-vcard-vsum-eqpoll*: *elts* (*vcard* ($i \uplus j$)) $\approx$ *Inl* ‘ *elts i* $\cup$ *Inr* ‘ *elts j*
proof −
  have *elts* ($i \uplus j$) $\approx$ *Inl* ‘ *elts i* $\cup$ *Inr* ‘ *elts j*
    by (*simp add*: *elts-vsum*)
  then show *?thesis*
    using *cardinal-eqpoll eqpoll-trans* **by** *blast*
qed

lemma *cadd-assoc*: $(i \oplus j) \oplus k = i \oplus (j \oplus k)$
proof (*unfold cadd-def*, *rule cardinal-cong*)
  have *elts* (*vcard*($i \uplus j$) $\uplus$ k) $\approx$ *elts* (($i \uplus j$) $\uplus$ k)
    by (*auto simp*: *disjnt-def elts-vsum elts-vcard-vsum-eqpoll intro*: *Un-eqpoll-cong*)
  also have ... $\approx$ *elts* ($i \uplus (j \uplus k)$)
    by (*rule sum-assoc-eqpoll*)
  also have ... $\approx$ *elts* ($i \uplus$ *vcard*($j \uplus k$))
    by (*auto simp*: *disjnt-def elts-vsum elts-vcard-vsum-eqpoll* [*THEN eqpoll-sym*]
intro: *Un-eqpoll-cong*)
  finally show *elts* (*vcard* ($i \uplus j$) $\uplus$ k) $\approx$ *elts* ($i \uplus$ *vcard* ($j \uplus k$)) **.**
qed

lemma *cadd-left-commute*: $j \oplus (i \oplus k) = i \oplus (j \oplus k)$
  using *cadd-assoc cadd-commute* **by** *force*

lemmas *cadd-ac* = *cadd-assoc cadd-commute cadd-left-commute*

0 is the identity for addition

lemma *vsum-0-eqpoll*: *elts* ($0 \uplus a$) $\approx$ *elts a*
  by (*simp add*: *elts-vsum*)

lemma *cadd-0* [*simp*]: *Card* $\kappa \Longrightarrow 0 \oplus \kappa = \kappa$
  by (*metis Card-def cadd-def cardinal-cong vsum-0-eqpoll*)

lemma *cadd-0-right* [*simp*]: *Card* $\kappa \Longrightarrow \kappa \oplus 0 = \kappa$
  by (*simp add*: *cadd-commute*)

lemma *vsum-lepoll-self*: *elts a* $\lesssim$ *elts* ($a \uplus b$)
  unfolding *elts-vsum* **by** (*meson Inl-iff Un-upper1 inj-onI lepoll-def*)

lemma *cadd-le-self*:
  assumes κ: *Card* κ **shows** $\kappa \le \kappa \oplus a$
proof (*unfold cadd-def*)
  have $\kappa \le$ *vcard* κ
    using *Card-def* κ **by** *auto*
  also have ... $\le$ *vcard* ($\kappa \uplus a$)
    by (*simp add*: *lepoll-imp-Card-le vsum-lepoll-self*)
  finally show $\kappa \le$ *vcard* ($\kappa \uplus a$) **.**
qed

Monotonicity of addition

lemma *cadd-le-mono*: $[\![\kappa' \le \kappa; \mu' \le \mu]\!] \Longrightarrow \kappa' \oplus \mu' \le \kappa \oplus \mu$
  unfolding *cadd-def*
  by (*metis* (*no-types*) *lepoll-imp-Card-le less-eq-V-def subset-imp-lepoll sum-subset-iff*)

2.15 Cardinal multiplication

definition *cmult* :: $[V,V] \Rightarrow V$ (**infixl** ‹$\otimes$› *70*)

where $\kappa \otimes \mu \equiv$ *vcard* (*VSigma* κ ($\lambda z.\ \mu$))

2.15.1 Cardinal multiplication is commutative

lemma *prod-bij*: $[\![$*bij-betw f A C*; *bij-betw g B D*$]\!]$
$\Longrightarrow$ *bij-betw* ($\lambda(x, y)$. (f x, g y)) ($A \times B$) ($C \times D$)
apply (*rule bij-betw-byWitness* [**where** $f' = \lambda(x,y)$. (*inv-into A f x*, *inv-into B g y*)])
apply (*auto simp*: *bij-betw-inv-into-left bij-betw-inv-into-right bij-betwE*)
using *bij-betwE bij-betw-inv-into* **apply** *blast+*
done

lemma *cmult-commute*: $i \otimes j = j \otimes i$
proof $-$
have ($\lambda(x, y)$. $\langle x, y\rangle$) ‘ (*elts* i $\times$ *elts* j) $\approx$ ($\lambda(x, y)$. $\langle x, y\rangle$) ‘ (*elts* j $\times$ *elts* i)
by (*simp add*: *times-commute-eqpoll*)
then show *?thesis*
unfolding *cmult-def*
using *cardinal-cong elts-VSigma* **by** *auto*
qed

2.15.2 Cardinal multiplication is associative

lemma *elts-vcard-VSigma-eqpoll*: *elts* (*vcard* (*vtimes i j*)) $\approx$ *elts* i $\times$ *elts* j
proof $-$
have *elts* (*vtimes i j*) $\approx$ *elts* i $\times$ *elts* j
by (*simp add*: *elts-VSigma*)
then show *?thesis*
using *cardinal-eqpoll eqpoll-trans* **by** *blast*
qed

lemma *elts-cmult*: *elts* ($\kappa' \otimes \kappa$) $\approx$ *elts* κ' $\times$ *elts* κ
by (*simp add*: *cmult-def elts-vcard-VSigma-eqpoll*)

lemma *cmult-assoc*: $(i \otimes j) \otimes k = i \otimes (j \otimes k)$
unfolding *cmult-def*
proof (*rule cardinal-cong*)
have *elts* (*vcard* (*vtimes i j*)) $\times$ *elts* k $\approx$ (*elts* i $\times$ *elts* j) $\times$ *elts* k
by (*blast intro*: *times-eqpoll-cong elts-vcard-VSigma-eqpoll cardinal-eqpoll*)
also have ... $\approx$ *elts* i $\times$ (*elts* j $\times$ *elts* k)
by (*rule times-assoc-eqpoll*)
also have ... $\approx$ *elts* i $\times$ *elts* (*vcard* (*vtimes j k*))
by (*blast intro*: *times-eqpoll-cong elts-vcard-VSigma-eqpoll cardinal-eqpoll eqpoll-sym*)
finally show *elts* (*VSigma* (*vcard* (*vtimes i j*)) ($\lambda z.\ k$)) $\approx$ *elts* (*VSigma* i ($\lambda z.$ *vcard* (*vtimes j k*)))
by (*simp add*: *elts-VSigma*)
qed

2.15.3 Cardinal multiplication distributes over addition

lemma *cadd-cmult-distrib*: $(i \oplus j) \otimes k = (i \otimes k) \oplus (j \otimes k)$
  unfolding *cadd-def cmult-def*
proof (*rule cardinal-cong*)
have *elts* (*vtimes* (*vcard* (*i* ⨄ *j*)) *k*) ≈ *elts* (*vcard* (*vsum i j*)) × *elts k*
  using *cardinal-eqpoll elts-vcard-VSigma-eqpoll eqpoll-sym eqpoll-trans* **by** *blast*
  also have ... ≈ (*Inl* ‘ *elts i* ∪ *Inr* ‘ *elts j*) × *elts k*
    using *elts-vcard-vsum-eqpoll times-eqpoll-cong* **by** *blast*
  also have ... ≈ (*Inl* ‘ *elts i*) × *elts k* ∪ (*Inr* ‘ *elts j*) × *elts k*
    by (*simp add*: *Sigma-Un-distrib1*)
  also have ... ≈ *elts* (*vtimes i k* ⨄ *vtimes j k*)
    unfolding *Plus-def*
   by (*auto simp*: *elts-vsum elts-VSigma disjnt-iff intro*!: *Un-eqpoll-cong times-eqpoll-cong*)
  also have ... ≈ *elts* (*vcard* (*vtimes i k* ⨄ *vtimes j k*))
    by (*simp add*: *cardinal-eqpoll eqpoll-sym*)
  also have ... ≈ *elts* (*vcard* (*vtimes i k*) ⨄ *vcard* (*vtimes j k*))
   by (*metis cadd-assoc cadd-def cardinal-cong cardinal-eqpoll vsum-0-eqpoll vsum-commute-eqpoll*)
  finally show *elts* (*VSigma* (*vcard* (*i* ⨄ *j*)) ($\lambda z.$ *k*))
            ≈ *elts* (*vcard* (*vtimes i k*) ⨄ *vcard* (*vtimes j k*)) **.**
qed

Multiplication by 0 yields 0

lemma *cmult-0* [*simp*]: $0 \otimes i = 0$
  using *Card-0 Card-def cmult-def* **by** *auto*

1 is the identity for multiplication

lemma *cmult-1* [*simp*]: **assumes** *Card* κ **shows** $1 \otimes \kappa = \kappa$
proof −
  have *elts* (*vtimes* (*set* {*0*}) κ) ≈ *elts* κ
    by (*auto simp*: *elts-VSigma intro*!: *times-singleton-eqpoll*)
  then show *?thesis*
    by (*metis Card-def assms cardinal-cong cmult-def elts-1 set-of-elts*)
qed

2.16 Some inequalities for multiplication

lemma *cmult-square-le*: **assumes** *Card* κ **shows** $\kappa \le \kappa \otimes \kappa$
proof −
  have *elts* $\kappa \lesssim$ *elts* $(\kappa \otimes \kappa)$
   using *times-square-lepoll* [*of elts* κ] *cmult-def elts-vcard-VSigma-eqpoll eqpoll-sym lepoll-trans2*
    by *fastforce*
  then show *?thesis*
    using *Card-def assms cmult-def lepoll-cardinal-le* **by** *fastforce*
qed

Multiplication by a non-empty set

lemma *cmult-le-self*: **assumes** *Card* κ $\alpha \neq 0$ **shows** $\kappa \le \kappa \otimes \alpha$

proof –
  have $\kappa = vcard\ \kappa$
    using *Card-def* ‹*Card* κ› **by** *blast*
  also have $\ldots \leq vcard\ (vtimes\ \kappa\ \alpha)$
    apply (*rule lepoll-imp-Card-le*)
    apply (*simp add*: *elts-VSigma*)
    by (*metis ZFC-in-HOL.ext* ‹$\alpha \neq 0$› *elts-0 lepoll-times1*)
  also have $\ldots = \kappa \otimes \alpha$
    by (*simp add*: *cmult-def*)
  finally show *?thesis* **.**
qed

Monotonicity of multiplication

lemma *cmult-le-mono*: $[\![\kappa' \leq \kappa;\ \mu' \leq \mu]\!] \Longrightarrow \kappa' \otimes \mu' \leq \kappa \otimes \mu$
  unfolding *cmult-def*
  by (*auto simp*: *elts-VSigma intro*!: *lepoll-imp-Card-le times-lepoll-mono subset-imp-lepoll*)

lemma *vcard-Sup-le-cmult*:
  assumes *small U* **and** κ: $\bigwedge x.\ x \in U \Longrightarrow vcard\ x \leq \kappa$
  shows $vcard\ (\bigsqcup U) \leq vcard\ (set\ U) \otimes \kappa$
proof –
  have $\exists f.\ f \in elts\ x \to elts\ \kappa \wedge inj\text{-}on\ f\ (elts\ x)$ **if** $x \in U$ **for** x
    using κ [*OF that*] **by** (*metis cardinal-le-lepoll image-subset-iff-funcset lepoll-def*)
  then obtain φ **where** φ: $\bigwedge x.\ x \in U \Longrightarrow (\varphi\ x) \in elts\ x \to elts\ \kappa \wedge inj\text{-}on\ (\varphi\ x)\ (elts\ x)$
    by *metis*
  define u **where** $u \equiv \lambda y.\ @x.\ x \in U \wedge y \in elts\ x$
  have u: $u\ y \in U \wedge y \in elts\ (u\ y)$ **if** $y \in \bigcup(elts\ `\ U)$ **for** y
    unfolding *u-def* **by** (*metis* (*mono-tags*, *lifting*)*that someI2-ex UN-iff*)
  define ψ **where** $\psi \equiv \lambda y.\ (u\ y,\ \varphi\ (u\ y)\ y)$
  have U: $elts\ (vcard\ (set\ U)) \approx U$
    by (*metis* ‹*small U*› *cardinal-eqpoll elts-of-set*)
  have $elts\ (\bigsqcup U) = \bigcup(elts\ `\ U)$
    using ‹*small U*› **by** *blast*
  also have $\ldots \lesssim U \times elts\ \kappa$
    unfolding *lepoll-def*
  proof (*intro exI conjI*)
    show $inj\text{-}on\ \psi\ (\bigcup\ (elts\ `\ U))$
      using φ u **by** (*smt* (*verit*) ψ*-def inj-on-def prod.inject*)
    show $\psi\ `\bigcup\ (elts\ `\ U) \subseteq U \times elts\ \kappa$
      using φ u **by** (*auto simp*: ψ*-def*)
  qed
  also have $\ldots\ \approx elts\ (vcard\ (set\ U) \otimes \kappa)$
    using *U elts-cmult eqpoll-sym eqpoll-trans times-eqpoll-cong* **by** *blast*
  finally have $elts\ (\bigsqcup\ U) \lesssim elts\ (vcard\ (set\ U) \otimes \kappa)$ **.**
  then show *?thesis*
    by (*simp add*: *cmult-def lepoll-cardinal-le*)
qed

2.17 The finite cardinals

lemma *succ-lepoll-succD*: *elts* (*succ*(*m*)) $\lesssim$ *elts* (*succ*(*n*)) $\Longrightarrow$ *elts m* $\lesssim$ *elts n*
  by (*simp add*: *insert-lepoll-insertD*)

Congruence law for *succ* under equipollence

lemma *succ-eqpoll-cong*: *elts a* $\approx$ *elts b* $\Longrightarrow$ *elts* (*succ*(*a*)) $\approx$ *elts* (*succ*(*b*))
  by (*simp add*: *succ-def insert-eqpoll-cong*)

lemma *sum-succ-eqpoll*: *elts* (*succ a* $\uplus$ *b*) $\approx$ *elts* (*succ*(*a* $\uplus$ *b*))
  unfolding *eqpoll-def*
proof (*rule exI*)
  let *?f* = λ*z*. *if z*=*Inl a then a* $\uplus$ *b else z*
  let *?g* = λ*z*. *if z*=*a* $\uplus$ *b then Inl a else z*
  show *bij-betw ?f* (*elts* (*succ a* $\uplus$ *b*)) (*elts* (*succ* (*a* $\uplus$ *b*)))
    apply (*rule bij-betw-byWitness* [**where** *f'* = *?g*], *auto*)
     apply (*metis Inl-in-sum-iff mem-not-refl*)
    by (*metis Inr-in-sum-iff mem-not-refl*)
qed

lemma *cadd-succ*: *succ m* $\oplus$ *n* = *vcard* (*succ*(*m* $\oplus$ *n*))
proof (*unfold cadd-def*)
  have [*intro*]: *elts* (*m* $\uplus$ *n*) $\approx$ *elts* (*vcard* (*m* $\uplus$ *n*))
    using *cardinal-eqpoll eqpoll-sym* **by** *blast*
  have *vcard* (*succ m* $\uplus$ *n*) = *vcard* (*succ*(*m* $\uplus$ *n*))
    by (*rule sum-succ-eqpoll* [*THEN cardinal-cong*])
  also have ... = *vcard* (*succ*(*vcard* (*m* $\uplus$ *n*)))
    by (*blast intro*: *succ-eqpoll-cong cardinal-cong*)
  finally show *vcard* (*succ m* $\uplus$ *n*) = *vcard* (*succ*(*vcard* (*m* $\uplus$ *n*))) **.**
qed

lemma *nat-cadd-eq-add*: *ord-of-nat m* $\oplus$ *ord-of-nat n* = *ord-of-nat* (*m* + *n*)
proof (*induct m*)
  case (*Suc m*) **thus** *?case*
    by (*metis Card-def Card-ord-of-nat add-Suc cadd-succ ord-of-nat.simps*(*2*))
qed *auto*

lemma *vcard-disjoint-sup*:
  assumes *x* $\sqcap$ *y* = *0* **shows** *vcard* (*x* $\sqcup$ *y*) = *vcard x* $\oplus$ *vcard y*
proof −
  have *elts* (*x* $\sqcup$ *y*) $\approx$ *elts* (*x* $\uplus$ *y*)
    unfolding *eqpoll-def*
  proof (*rule exI*)
    let *?f* = λ*z*. *if z* $\in$ *elts x then Inl z else Inr z*
    let *?g* = *sum-case id id*
    show *bij-betw ?f* (*elts* (*x* $\sqcup$ *y*)) (*elts* (*x* $\uplus$ *y*))
      by (*rule bij-betw-byWitness* [**where** *f'* = *?g*]) (*use assms V-disjoint-iff* **in** *auto*)
  qed
  then show *?thesis*

by (*metis cadd-commute cadd-def cardinal-cong cardinal-idem vsum-0-eqpoll cadd-assoc*)
qed

lemma *vcard-sup*: *vcard* $(x \sqcup y) \leq$ *vcard* $x \oplus$ *vcard* y
proof $-$
have *elts* $(x \sqcup y) \lesssim$ *elts* $(x \uplus y)$
unfolding *lepoll-def*
proof (*intro exI conjI*)
let *?f* $= \lambda z.$ *if* $z \in$ *elts x then Inl z else Inr z*
show *inj-on ?f* (*elts* $(x \sqcup y)$)
by (*simp add*: *inj-on-def*)
show *?f* ' *elts* $(x \sqcup y) \subseteq$ *elts* $(x \uplus y)$
by *force*
qed
then show *?thesis*
using *cadd-ac*
by (*metis cadd-def cardinal-cong cardinal-idem lepoll-imp-Card-le vsum-0-eqpoll*)
qed

2.18 Infinite cardinals

definition *InfCard* :: $V \Rightarrow$ *bool*
where *InfCard* $\kappa \equiv$ *Card* $\kappa \wedge \omega \leq \kappa$

lemma *InfCard-iff*: *InfCard* $\kappa \longleftrightarrow$ *Card* $\kappa \wedge$ *infinite* (*elts* κ)
proof (*cases* $\omega \leq \kappa$)
case *True*
then show *?thesis*
using *inj-ord-of-nat lepoll-def less-eq-V-def*
by (*auto simp*: *InfCard-def* ω*-def infinite-le-lepoll*)
next
case *False*
then show *?thesis*
using *Card-iff-initial InfCard-def infinite-Ord-omega* **by** *blast*
qed

lemma *InfCard-ge-ord-of-nat*:
assumes *InfCard* κ **shows** *ord-of-nat* $n \leq \kappa$
using *InfCard-def assms ord-of-nat-le-omega* **by** *blast*

lemma *InfCard-not-0*[*iff*]: $\neg$ *InfCard 0*
by (*simp add*: *InfCard-iff*)

definition *csucc* :: $V \Rightarrow V$
where *csucc* $\kappa \equiv$ *LEAST* κ'. *Ord* $\kappa' \wedge$ (*Card* $\kappa' \wedge \kappa < \kappa'$)

lemma *less-vcard-VPow*: *vcard* $A <$ *vcard* (*VPow* A)

proof (*rule lesspoll-imp-Card-less*)
  show *elts A* $\prec$ *elts* (*VPow A*)
    by (*simp add*: *elts-VPow down inj-on-def lesspoll-Pow-self*)
qed

lemma *greater-Card*:
  assumes *Card* κ **shows** $\kappa <$ *vcard* (*VPow* κ)
proof −
  have $\kappa =$ *vcard* κ
    using *Card-def assms* **by** *blast*
  also have ... $<$ *vcard* (*VPow* κ)
  proof (*rule lesspoll-imp-Card-less*)
    show *elts* $\kappa \prec$ *elts* (*VPow* κ)
      by (*simp add*: *elts-VPow down inj-on-def lesspoll-Pow-self*)
  qed
  finally show *?thesis* **.**
qed

lemma
  assumes *Card* κ
  shows *Card-csucc* [*simp*]: *Card* (*csucc* κ) **and** *less-csucc* [*simp*]: $\kappa <$ *csucc* κ
proof −
  have *Card* (*csucc* κ) $\wedge$ $\kappa <$ *csucc* κ
    unfolding *csucc-def*
  proof (*rule Ord-LeastI2*)
    show *Card* (*vcard* (*VPow* κ)) $\wedge$ $\kappa <$ (*vcard* (*VPow* κ))
      using *Card-def assms greater-Card* **by** *auto*
  qed *auto*
  then show *Card* (*csucc* κ) $\kappa <$ *csucc* κ
    by *auto*
qed

lemma *le-csucc*:
  assumes *Card* κ **shows** $\kappa \leq$ *csucc* κ
  by (*simp add*: *assms less-csucc less-imp-le*)

lemma *csucc-le*: $[\![$*Card* μ; $\kappa \in$ *elts* $\mu]\!] \Longrightarrow$ *csucc* $\kappa \leq \mu$
  unfolding *csucc-def*
  by (*simp add*: *Card-is-Ord Ord-Least-le OrdmemD*)

lemma *finite-csucc*: $a \in$ *elts* $\omega \Longrightarrow$ *csucc* $a =$ *succ* a
  unfolding *csucc-def*
  proof (*rule Least-equality*)
  show *Ord* (*ZFC-in-HOL.succ a*) $\wedge$ *Card* (*ZFC-in-HOL.succ a*) $\wedge$ $a <$ *ZFC-in-HOL.succ*
a
    if $a \in$ *elts* ω
    using *that* **by** (*auto simp*: *less-V-def less-eq-V-def nat-into-Card*)
  show *ZFC-in-HOL.succ* $a \leq y$

if $a \in$ *elts* ω
and *Ord* $y \wedge$ *Card* $y \wedge a < y$
for y :: V
using *that*
using *Ord-mem-iff-lt dual-order.strict-implies-order* **by** *fastforce*
qed

lemma *Finite-imp-cardinal-cons* [*simp*]:
assumes *FA*: *finite A* **and** *a*: $a \notin A$
shows *vcard* (*set* (*insert a A*)) = *csucc*(*vcard* (*set A*))
proof −
show *?thesis*
unfolding *csucc-def*
proof (*rule Least-equality* [*THEN sym*])
have *small A*
by (*simp add*: *FA Finite-V*)
then have ¬ *elts* (*set A*) ≈ *elts* (*set* (*insert a A*))
using *FA a eqpoll-imp-lepoll eqpoll-sym finite-insert-lepoll* **by** *fastforce*
then show *Ord* (*vcard* (*set* (*insert a A*))) ∧ *Card* (*vcard* (*set* (*insert a A*))) ∧ *vcard* (*set A*) < *vcard* (*set* (*insert a A*))
by (*simp add*: *Card-def lesspoll-imp-Card-less lesspoll-def subset-imp-lepoll subset-insertI*)
show *vcard* (*set* (*insert a A*)) ≤ *i*
if *Ord i* ∧ *Card i* ∧ *vcard* (*set A*) < *i* **for** *i*
proof −
have *elts* (*vcard* (*set A*)) ≈ *A*
by (*metis FA finite-imp-small cardinal-eqpoll elts-of-set*)
then have *less*: *A* ≺ *elts i*
using *eq-lesspoll-trans eqpoll-sym lt-Card-imp-lesspoll that* **by** *blast*
show *?thesis*
using *that less* **by** (*auto simp*: *less-imp-insert-lepoll lepoll-cardinal-le*)
qed
qed
qed

lemma *vcard-finite-set*: *finite A* ⟹ *vcard* (*set A*) = *ord-of-nat* (*card A*)
by (*induction A rule*: *finite-induct*) (*auto simp*: *set-empty* ω*-def finite-csucc*)

lemma *lt-csucc-iff*:
assumes *Ord* α *Card* κ
shows $\alpha <$ *csucc* $\kappa \longleftrightarrow$ *vcard* $\alpha \le \kappa$
proof
show *vcard* $\alpha \le \kappa$ **if** $\alpha <$ *csucc* κ
proof −
have *vcard* $\alpha \le$ *csucc* κ
by (*meson* ‹*Ord* α› *dual-order.trans lepoll-cardinal-le lepoll-refl less-le-not-le that*)
then show *?thesis*
by (*metis* (*no-types*) *Card-def Card-iff-initial Ord-linear2 Ord-mem-iff-lt assms*

cardinal-eqpoll cardinal-idem csucc-le eq-iff eqpoll-sym that)
  qed
  show α < *csucc* κ **if** *vcard* $\alpha \leq \kappa$
  proof −
    have ¬ *csucc* $\kappa \leq \alpha$
      using *that*
        by (*metis Card-csucc Card-def assms*(*2*) *le-less-trans lepoll-imp-Card-le less-csucc less-eq-V-def less-le-not-le subset-imp-lepoll*)
    then show *?thesis*
      by (*meson Card-csucc Card-is-Ord Ord-linear2 assms*)
  qed
qed

lemma *Card-lt-csucc-iff*: ⟦*Card* κ'; *Card* κ⟧ $\Longrightarrow$ (κ' < *csucc* κ) = ($\kappa' \leq \kappa$)
  by (*simp add*: *lt-csucc-iff Card-cardinal-eq Card-is-Ord*)

lemma *csucc-lt-csucc-iff*: ⟦*Card* κ'; *Card* κ⟧ $\Longrightarrow$ (*csucc* κ' < *csucc* κ) = ($\kappa' < \kappa$)
  by (*metis Card-csucc Card-is-Ord Card-lt-csucc-iff Ord-not-less*)

lemma *csucc-le-csucc-iff*: ⟦*Card* κ'; *Card* κ⟧ $\Longrightarrow$ (*csucc* $\kappa' \leq$ *csucc* κ) = ($\kappa' \leq \kappa$)
  by (*metis Card-csucc Card-is-Ord Card-lt-csucc-iff Ord-not-less*)

lemma *csucc-0* [*simp*]: *csucc 0* = *1*
  by (*simp add*: *finite-csucc one-V-def*)

lemma *Card-Un* [*simp*,*intro*]:
  assumes *Card x Card y* **shows** *Card*($x \sqcup y$)
  by (*metis Card-is-Ord Ord-linear-le assms sup.absorb2 sup.orderE*)

lemma *InfCard-csucc*: *InfCard* $\kappa \Longrightarrow$ *InfCard* (*csucc* κ)
  using *InfCard-def le-csucc* **by** *auto*

Kunen's Lemma 10.11

lemma *InfCard-is-Limit*:
  assumes *InfCard* κ **shows** *Limit* κ
  proof (*rule non-succ-LimitI*)
  show $\kappa \neq 0$
    using *InfCard-def assms mem-not-refl* **by** *blast*
  show *Ord* κ
    using *Card-is-Ord InfCard-def assms* **by** *blast*
  show *ZFC-in-HOL.succ* $y \neq \kappa$ **for** *y*
  proof
    assume *succ* $y = \kappa$
    then have *Card* (*succ y*)
      using *InfCard-def assms* **by** *auto*
    moreover have $\omega \leq y$
      by (*metis InfCard-iff Ord-in-Ord* ‹*Ord* κ› ‹*ZFC-in-HOL.succ* $y = \kappa$› *assms elts-succ finite-insert infinite-Ord-omega insertI1*)
    moreover have *elts* $y \approx$ *elts* (*succ y*)

using *InfCard-iff* ‹*ZFC-in-HOL.succ y* = *κ*› *assms eqpoll-sym infinite-insert-eqpoll*
by *fastforce*
ultimately show *False*
by (*metis Card-iff-initial Ord-in-Ord OrdmemD elts-succ insertI1*)
qed
qed

2.19 Toward's Kunen's Corollary 10.13 (1)

Kunen's Theorem 10.12

lemma *InfCard-csquare-eq*:
assumes *InfCard*(*κ*) **shows** $\kappa \otimes \kappa = \kappa$
using *infinite-times-eqpoll-self* [*of elts κ*] *assms*
unfolding *InfCard-iff Card-def*
by (*metis cardinal-cong cardinal-eqpoll cmult-def elts-vcard-VSigma-eqpoll eqpoll-trans*)

lemma *InfCard-le-cmult-eq*:
assumes *InfCard κ* $\mu \leq \kappa$ $\mu \neq 0$
shows $\kappa \otimes \mu = \kappa$
proof (*rule order-antisym*)
have $\kappa \otimes \mu \leq \kappa \otimes \kappa$
by (*simp add*: *assms*(*2*) *cmult-le-mono*)
also have $\ldots \leq \kappa$
by (*simp add*: *InfCard-csquare-eq assms*(*1*))
finally show $\kappa \otimes \mu \leq \kappa$ **.**
show $\kappa \leq \kappa \otimes \mu$
using *InfCard-def assms*(*1*) *assms*(*3*) *cmult-le-self* **by** *auto*
qed

Kunen's Corollary 10.13 (1), for cardinal multiplication

lemma *InfCard-cmult-eq*: $[\![$*InfCard κ*; *InfCard μ*$]\!] \Longrightarrow \kappa \otimes \mu = \kappa \sqcup \mu$
by (*metis Card-is-Ord InfCard-def InfCard-le-cmult-eq Ord-linear-le cmult-commute inf-sup-aci*(*5*) *mem-not-refl sup.orderE sup-V-0-right zero-in-omega*)

lemma *cmult-succ*:
succ(*m*) $\otimes$ *n* = *n* $\oplus$ (*m* $\otimes$ *n*)
unfolding *cmult-def cadd-def*
proof (*rule cardinal-cong*)
have *elts* (*vtimes* (*ZFC-in-HOL.succ m*) *n*) ≈ *elts n* <+> *elts m* × *elts n*
by (*simp add*: *elts-VSigma prod-insert-eqpoll*)
also have . . . ≈ *elts* (*n* ⊎ *vcard* (*vtimes m n*))
unfolding *elts-VSigma elts-vsum Plus-def*
proof (*rule Un-eqpoll-cong*)
show (*Sum-Type.Inr* ‘ (*elts m* × *elts n*)::(*V* + *V* × *V*) *set*) ≈ *Inr* ‘ *elts* (*vcard* (*vtimes m n*))
by (*simp add*: *elts-vcard-VSigma-eqpoll eqpoll-sym*)
qed (*auto simp*: *disjnt-def*)
finally show *elts* (*vtimes* (*ZFC-in-HOL.succ m*) *n*) ≈ *elts* (*n* ⊎ *vcard* (*vtimes m n*)) **.**

qed

lemma *cmult-2*:
assumes *Card n* **shows** *ord-of-nat 2* $\otimes$ *n* = *n* $\oplus$ *n*
proof −
have *ord-of-nat 2* = *succ* (*succ 0*)
by *force*
then show *?thesis*
by (*simp add*: *cmult-succ assms*)
qed

lemma *InfCard-cdouble-eq*:
assumes *InfCard* κ **shows** $\kappa \oplus \kappa = \kappa$
proof −
have $\kappa \oplus \kappa = \kappa \otimes$ *ord-of-nat 2*
using *InfCard-def assms cmult-2 cmult-commute* **by** *auto*
also have ... = κ
by (*simp add*: *InfCard-le-cmult-eq InfCard-ge-ord-of-nat assms*)
finally show *?thesis* **.**
qed

Corollary 10.13 (1), for cardinal addition

lemma *InfCard-le-cadd-eq*: $[\![$*InfCard* κ; $\mu \leq \kappa]\!] \Longrightarrow \kappa \oplus \mu = \kappa$
by (*metis InfCard-cdouble-eq InfCard-def antisym cadd-le-mono cadd-le-self*)

lemma *InfCard-cadd-eq*: $[\![$*InfCard* κ; *InfCard* $\mu]\!] \Longrightarrow \kappa \oplus \mu = \kappa \sqcup \mu$
by (*metis Card-iff-initial InfCard-def InfCard-le-cadd-eq Ord-linear-le cadd-commute sup.absorb2 sup.orderE*)

lemma *csucc-le-Card-iff*: $[\![$*Card* κ'; *Card* $\kappa]\!] \Longrightarrow$ *csucc* $\kappa' \leq \kappa \longleftrightarrow \kappa' < \kappa$
by (*metis Card-csucc Card-is-Ord Card-lt-csucc-iff Ord-not-le*)

lemma *cadd-InfCard-le*:
assumes $\alpha \leq \kappa$ $\beta \leq \kappa$ *InfCard* κ
shows $\alpha \oplus \beta \leq \kappa$
using *assms* **by** (*metis InfCard-cdouble-eq cadd-le-mono*)

lemma *cmult-InfCard-le*:
assumes $\alpha \leq \kappa$ $\beta \leq \kappa$ *InfCard* κ
shows $\alpha \otimes \beta \leq \kappa$
using *assms*
by (*metis InfCard-csquare-eq cmult-le-mono*)

2.20 The Aleph-seqence

This is the well-known transfinite enumeration of the cardinal numbers.

definition *Aleph* :: $V \Rightarrow V$ (‹ℵ-› [*90*] *90*)
where *Aleph* $\equiv$ *transrec* ($\lambda f\ x.\ \omega \sqcup \bigsqcup((\lambda y.\ csucc(f\ y))$ ‘ *elts x*))

lemma *Aleph*: *Aleph* $\alpha = \omega \sqcup (\bigsqcup y{\in}elts\ \alpha.\ csucc\ (Aleph\ y))$
  by (*simp add*: *Aleph-def transrec*[*of* - α])

lemma *InfCard-Aleph* [*simp*, *intro*]: *InfCard*(*Aleph x*)
proof (*induction x rule*: *eps-induct*)
  case (*step x*)
  then show *?case*
    by (*simp add*: *Aleph* [*of x*] *InfCard-def Card-UN step*)
qed

lemma *Card-Aleph* [*simp*, *intro*]: *Card*(*Aleph x*)
  using *InfCard-def* **by** *auto*

lemma *Aleph-0* [*simp*]: $\aleph 0 = \omega$
  by (*simp add*: *Aleph* [*of 0*])

lemma *mem-Aleph-succ*: $\aleph\alpha \in elts\ (Aleph\ (succ\ \alpha))$
  apply (*simp add*: *Aleph* [*of succ* α])
   by (*meson InfCard-Aleph Card-csucc Card-is-Ord InfCard-def Ord-mem-iff-lt less-csucc*)

lemma *Aleph-lt-succD* [*simp*]: $\aleph\alpha < Aleph\ (succ\ \alpha)$
  by (*simp add*: *InfCard-is-Limit Limit-is-Ord OrdmemD mem-Aleph-succ*)

lemma *Aleph-succ* [*simp*]: *Aleph* (*succ x*) = *csucc* (*Aleph x*)
proof (*rule antisym*)
  show *Aleph* (*ZFC-in-HOL.succ x*) $\le$ *csucc* (*Aleph x*)
    apply (*simp add*: *Aleph* [*of succ x*])
    by (*metis Aleph InfCard-Aleph InfCard-def Sup-V-insert le-csucc le-sup-iff order-refl*
         replacement small-elts)
  show *csucc* (*Aleph x*) $\le$ *Aleph* (*ZFC-in-HOL.succ x*)
    by (*force simp add*: *Aleph* [*of succ x*])
qed

lemma *csucc-Aleph-le-Aleph*: $\alpha \in elts\ \beta \Longrightarrow csucc\ (\aleph\alpha) \le \aleph\beta$
  by (*metis Aleph ZFC-in-HOL.SUP-le-iff replacement small-elts sup-ge2*)

lemma *Aleph-in-Aleph*: $\alpha \in elts\ \beta \Longrightarrow \aleph\alpha \in elts\ (\aleph\beta)$
  using *csucc-Aleph-le-Aleph mem-Aleph-succ* **by** *auto*

lemma *Aleph-Limit*:
  assumes *Limit* γ
  shows *Aleph* $\gamma = \bigsqcup\ (Aleph\ `\ elts\ \gamma)$
proof –
  have [*simp*]: $\gamma \neq 0$
    using *assms* **by** *auto*
  show *?thesis*
  proof (*rule antisym*)

show *Aleph* $\gamma \leq \bigsqcup$ (*Aleph* ‘ *elts* γ)
apply (*simp add*: *Aleph* [*of* γ])
by (*metis* (*mono-tags*, *lifting*) *Aleph-0 Aleph-succ Limit-def ZFC-in-HOL.SUP-le-iff ZFC-in-HOL.Sup-upper assms imageI replacement small-elts*)
show $\bigsqcup$ (*Aleph* ‘ *elts* γ) $\leq$ *Aleph* γ
apply (*simp add*: *cSup-le-iff*)
by (*meson InfCard-Aleph InfCard-def csucc-Aleph-le-Aleph le-csucc order-trans*)
qed
qed

lemma *Aleph-increasing*:
assumes *ab*: $\alpha < \beta$ *Ord* α *Ord* β **shows** $\aleph\alpha < \aleph\beta$
by (*meson Aleph-in-Aleph InfCard-Aleph Card-iff-initial InfCard-def Ord-mem-iff-lt assms*)

lemma *countable-iff-le-Aleph0*: *countable* (*elts A*) $\longleftrightarrow$ *vcard A* $\leq \aleph 0$
proof
show *vcard A* $\leq \aleph 0$
if *countable* (*elts A*)
proof (*cases finite* (*elts A*))
case *True*
then show *?thesis*
using *vcard-finite-set* **by** *fastforce*
next
case *False*
then have *elts* $\omega \approx$ *elts A*
using *countableE-infinite* [*OF that*]
by (*simp add*: *eqpoll-def* ω*-def*)
(*meson bij-betw-def bij-betw-inv bij-betw-trans inj-ord-of-nat*)
then show *?thesis*
using *Card-*ω *Card-def cardinal-cong vcard-def* **by** *auto*
qed
show *countable* (*elts A*)
if *vcard A* $\leq$ *Aleph 0*
proof −
have *elts A* $\lesssim$ *elts* ω
using *cardinal-le-lepoll* [*OF that*] **by** *simp*
then show *?thesis*
by (*simp add*: *countable-iff-lepoll* ω*-def inj-ord-of-nat*)
qed
qed

lemma *Aleph-csquare-eq* [*simp*]: $\aleph\alpha \otimes \aleph\alpha = \aleph\alpha$
using *InfCard-csquare-eq* **by** *auto*

lemma *vcard-Aleph* [*simp*]: *vcard* ($\aleph\alpha$) = $\aleph\alpha$
using *Card-def InfCard-Aleph InfCard-def* **by** *auto*

lemma *omega-le-Aleph* [*simp*]: $\omega \leq \aleph\alpha$
  using *InfCard-def* **by** *auto*

lemma *finite-iff-less-Aleph0*: *finite* (*elts* x) $\longleftrightarrow$ *vcard* $x < \omega$
proof
  show *finite* (*elts* x) $\Longrightarrow$ *vcard* $x < \omega$
   by (*metis Card-ω Card-def finite-lesspoll-infinite infinite-ω lesspoll-imp-Card-less*)
  show *vcard* $x < \omega \Longrightarrow$ *finite* (*elts* x)
   by (*meson Ord-cardinal cardinal-eqpoll eqpoll-finite-iff infinite-Ord-omega less-le-not-le*)
qed

lemma *countable-iff-vcard-less1*: *countable* (*elts* x) $\longleftrightarrow$ *vcard* $x < \aleph 1$
  by (*simp add*: *countable-iff-le-Aleph0 lt-csucc-iff one-V-def*)

lemma *countable-infinite-vcard*: *countable* (*elts* x) $\wedge$ *infinite* (*elts* x) $\longleftrightarrow$ *vcard* x
$= \aleph 0$
 by (*metis Aleph-0 countable-iff-le-Aleph0 dual-order.refl finite-iff-less-Aleph0 less-V-def*)

2.21 The ordinal $\omega 1$

abbreviation $\omega 1 \equiv$ *Aleph 1*

lemma *Ord-ω1* [*simp*]: *Ord* $\omega 1$
  by (*metis Ord-cardinal vcard-Aleph*)

lemma *omega-ω1* [*iff*]: $\omega \in$ *elts* $\omega 1$
  by (*metis Aleph-0 mem-Aleph-succ one-V-def*)

lemma *ord-of-nat-ω1* [*iff*]: *ord-of-nat* $n \in$ *elts* $\omega 1$
  using *Ord-ω1 Ord-trans* **by** *blast*

lemma *countable-iff-less-ω1*:
  assumes *Ord* α
  shows *countable* (*elts* α) $\longleftrightarrow \alpha < \omega 1$
  by (*simp add*: *assms countable-iff-le-Aleph0 lt-csucc-iff one-V-def*)

lemma *less-ω1-imp-countable*:
  assumes $\alpha \in$ *elts* $\omega 1$
  shows *countable* (*elts* α)
  using *Ord-ω1 Ord-in-Ord OrdmemD assms countable-iff-less-ω1* **by** *blast*

lemma *ω1-gt0* [*simp*]: $\omega 1 > 0$
  using *Ord-ω1 Ord-trans OrdmemD* **by** *blast*

lemma *ω1-gt1* [*simp*]: $\omega 1 > 1$
  using *Ord-ω1 OrdmemD ω-gt1 less-trans* **by** *blast*

lemma *Limit-ω1* [*simp*]: *Limit* $\omega 1$
  by (*simp add*: *InfCard-def InfCard-is-Limit le-csucc one-V-def*)

end

3 Addition and Multiplication of Sets

theory *Kirby*
  imports *ZFC-Cardinals*

begin

3.1 Generalised Addition

Source: Laurence Kirby, Addition and multiplication of sets Math. Log. Quart. 53, No. 1, 52-65 (2007) / DOI 10.1002/malq.200610026 http://faculty.baruch.cuny.edu/lkirby/mlqarticlejan2007.pdf

3.1.1 Addition is a monoid

instantiation *V* :: *plus*
begin

This definition is credited to Tarski

definition *plus-V* :: $V \Rightarrow V \Rightarrow V$
  where *plus-V* $x \equiv$ *transrec* $(\lambda f\ z.\ x \sqcup$ *set* $(f$ ‘ *elts* $z))$

instance ..
end

definition *lift* :: $V \Rightarrow V \Rightarrow V$
  where *lift* $x\ y \equiv$ *set* (*plus* x ‘ *elts* y)

lemma *plus*: $x + y = x \sqcup$ *set* $((+)x$ ‘ *elts* $y)$
  unfolding *plus-V-def* **by** (*subst transrec*) *auto*

lemma *plus-eq-lift*: $x + y = x \sqcup$ *lift* $x\ y$
  unfolding *lift-def* **using** *plus* **by** *blast*

Lemma 3.2

lemma *lift-sup-distrib*: *lift* x $(a \sqcup b) =$ *lift* $x\ a \sqcup$ *lift* $x\ b$
  by (*simp add*: *image-Un lift-def sup-V-def*)

lemma *lift-Sup-distrib*: *small* $Y \Longrightarrow$ *lift* x $(\bigsqcup\ Y) = \bigsqcup$ (*lift* x ‘ Y)
  by (*auto simp*: *lift-def Sup-V-def image-Union*)

lemma *add-Sup-distrib*:
  fixes x::V **shows** $y \neq 0 \Longrightarrow x + (\bigsqcup z{\in}elts\ y.\ f\ z) = (\bigsqcup z{\in}elts\ y.\ x + f\ z)$
  by (*auto simp*: *plus-eq-lift SUP-sup-distrib lift-Sup-distrib image-image*)

lemma *Limit-add-Sup-distrib*:
  fixes x:: V **shows** *Limit* $\alpha \Longrightarrow x + (\bigsqcup z{\in}elts\ \alpha.\ f\ z) = (\bigsqcup z{\in}elts\ \alpha.\ x + f\ z)$
  using *add-Sup-distrib* **by** *force*

Proposition 3.3(ii)

```
instantiation V :: monoid-add
begin
instance
proof
  show a + b + c = a + (b + c) for a b c :: V
  proof (induction c rule: eps-induct)
    case (step c)
    have (a+b) + c = a + b ⊔ set ((+) (a + b) ` elts c)
      by (metis plus)
    also have ... = a ⊔ lift a b ⊔ set ((λu. a + (b+u)) ` elts c)
      using plus-eq-lift step.IH by auto
    also have ... = a ⊔ lift a (b + c)
    proof –
      have lift a b ⊔ set ((λu. a + (b + u)) ` elts c) = lift a (b + c)
        unfolding lift-def
         by (metis elts-of-set image-image lift-def lift-sup-distrib plus-eq-lift replace-
ment small-elts)
      then show ?thesis
        by (simp add: sup-assoc)
    qed
    also have ... = a + (b + c)
      using plus-eq-lift by auto
    finally show ?case .
  qed
  show 0 + x = x for x :: V
  proof (induction rule: eps-induct)
    case (step x)
    then show ?case
      by (subst plus) auto
  qed
  show x + 0 = x for x :: V
    by (subst plus) auto
qed
end
```

lemma *lift-0* [*simp*]: *lift* $0\ x = x$
  by (*simp add*: *lift-def*)

lemma *lift-by0* [*simp*]: *lift* $x\ 0 = 0$
  by (*simp add*: *lift-def*)

lemma *lift-by1* [*simp*]: *lift* $x\ 1 = set\{x\}$
  by (*simp add*: *lift-def*)

lemma *add-eq-0-iff* [*simp*]:
  fixes *x y*::*V*
  shows $x+y = 0 \longleftrightarrow x=0 \wedge y=0$
  proof *safe*
  show $x = 0$ **if** $x + y = 0$
    by (*metis that le-imp-less-or-eq not-less-0 plus sup-ge1*)
  then show $y = 0$ **if** $x + y = 0$
    using *that* **by** *auto*
qed *auto*

lemma *plus-vinsert*: $x + vinsert\ z\ y = vinsert\ (x+z)\ (x + y)$
proof −
  have *f1*: $elts\ (x + y) = elts\ x \cup (+)\ x\ `\ elts\ y$
   by (*metis elts-of-set lift-def plus-eq-lift replacement small-Un small-elts sup-V-def*)
  moreover have $lift\ x\ (vinsert\ z\ y) = set\ ((+)\ x\ `\ elts\ (set\ (insert\ z\ (elts\ y))))$
    using *vinsert-def lift-def* **by** *presburger*
  ultimately show *?thesis*
    by (*simp add*: *vinsert-def plus-eq-lift sup-V-def*)
qed

lemma *plus-V-succ-right*: $x + succ\ y = succ\ (x + y)$
  by (*metis plus-vinsert succ-def*)

lemma *succ-eq-add1*: $succ\ x = x + 1$
  by (*simp add*: *plus-V-succ-right one-V-def*)

lemma *ord-of-nat-add*: $ord\text{-}of\text{-}nat\ (m+n) = ord\text{-}of\text{-}nat\ m + ord\text{-}of\text{-}nat\ n$
  by (*induction n*) (*auto simp*: *plus-V-succ-right*)

lemma *succ-0-plus-eq* [*simp*]:
  assumes $\alpha \in elts\ \omega$
  shows $succ\ 0 + \alpha = succ\ \alpha$
proof −
  obtain *n* **where** $\alpha = ord\text{-}of\text{-}nat\ n$
    using *assms elts-ω* **by** *blast*
  then show *?thesis*
    by (*metis One-nat-def ord-of-nat.simps ord-of-nat-add plus-1-eq-Suc*)
qed

lemma *omega-closed-add* [*intro*]:
  assumes $\alpha \in elts\ \omega$ $\beta \in elts\ \omega$ **shows** $\alpha+\beta \in elts\ \omega$
proof −
  obtain *m n* **where** $\alpha = ord\text{-}of\text{-}nat\ m$ $\beta = ord\text{-}of\text{-}nat\ n$
    using *assms elts-ω* **by** *auto*
  then have $\alpha+\beta = ord\text{-}of\text{-}nat\ (m+n)$
    using *ord-of-nat-add* **by** *auto*
  then show *?thesis*
    by (*simp add*: *ω-def*)
qed

lemma *mem-plus-V-E*:
  assumes *l*: $l \in elts\ (x + y)$
  obtains $l \in elts\ x \mid z$ **where** $z \in elts\ y\ l = x + z$
  using *l* **by** (*auto simp*: *plus* [*of x y*] *split*: *if-split-asm*)

lemma *not-add-less-right*: **assumes** *Ord y* **shows** $\neg\ (x + y < x)$
  using *assms*
proof (*induction rule*: *Ord-induct*)
  case (*step i*)
  then show *?case*
    by (*metis less-le-not-le plus sup-ge1*)
qed

lemma *not-add-mem-right*: $\neg\ (x + y \in elts\ x)$
  by (*metis sup-ge1 mem-not-refl plus vsubsetD*)

Proposition 3.3(iii)

lemma *add-not-less-TC-self*: $\neg\ x + y \sqsubset x$
proof (*induction y arbitrary*: *x rule*: *eps-induct*)
  case (*step y*)
  then show *?case*
    using *less-TC-imp-not-le plus-eq-lift* **by** *fastforce*
qed

lemma *TC-sup-lift*: $TC\ x \sqcap lift\ x\ y = 0$
proof −
  have $elts\ (TC\ x) \cap elts\ (set\ ((+)\ x\ `\ elts\ y)) = \{\}$
    using *add-not-less-TC-self* **by** (*auto simp*: *less-TC-def*)
  then have $TC\ x \sqcap set\ ((+)\ x\ `\ elts\ y) = set\ \{\}$
    by (*metis inf-V-def*)
  then show *?thesis*
    using *lift-def* **by** *auto*
qed

lemma *lift-lift*: $lift\ x\ (lift\ y\ z) = lift\ (x+y)\ z$
  using *add.assoc* **by** (*auto simp*: *lift-def*)

lemma *lift-self-disjoint*: $x \sqcap lift\ x\ u = 0$
  by (*metis TC-sup-lift arg-subset-TC inf.absorb-iff2 inf-assoc inf-sup-aci*(*3*) *lift-0*)

lemma *sup-lift-eq-lift*:
  assumes $x \sqcup lift\ x\ u = x \sqcup lift\ x\ v$
  shows $lift\ x\ u = lift\ x\ v$
  by (*metis* (*no-types*) *assms inf-sup-absorb inf-sup-distrib2 lift-self-disjoint sup-commute sup-inf-absorb*)

3.1.2 Deeper properties of addition

Proposition 3.4(i)

proposition *lift-eq-lift*: *lift x y* = *lift x z* $\Longrightarrow$ *y* = *z*
proof (*induction y arbitrary*: *z rule*: *eps-induct*)
  case (*step y*)
  show *?case*
  proof (*intro vsubsetI order-antisym*)
    show *u* ∈ *elts z* **if** *u* ∈ *elts y* **for** *u*
    proof −
      have *x*+*u* ∈ *elts* (*lift x z*)
        using *lift-def step.prems that* **by** *fastforce*
      then obtain *v* **where** *v* ∈ *elts z x*+*u* = *x*+*v*
        using *lift-def* **by** *auto*
      then have *lift x u* = *lift x v*
        using *sup-lift-eq-lift* **by** (*simp add*: *plus-eq-lift*)
      then have *u*=*v*
        using *step.IH that* **by** *blast*
      then show *?thesis*
        using ‹*v* ∈ *elts z*› **by** *blast*
    qed
    show *u* ∈ *elts y* **if** *u* ∈ *elts z* **for** *u*
    proof −
      have *x*+*u* ∈ *elts* (*lift x y*)
        using *lift-def step.prems that* **by** *fastforce*
      then obtain *v* **where** *v* ∈ *elts y x*+*u* = *x*+*v*
        using *lift-def* **by** *auto*
      then have *lift x u* = *lift x v*
        using *sup-lift-eq-lift* **by** (*simp add*: *plus-eq-lift*)
      then have *u*=*v*
        using *step.IH* **by** (*metis* ‹*v* ∈ *elts y*›)
      then show *?thesis*
        using ‹*v* ∈ *elts y*› **by** *auto*
    qed
  qed
qed

corollary *inj-lift*: *inj-on* (*lift x*) *A*
  by (*auto simp*: *inj-on-def dest*: *lift-eq-lift*)

corollary *add-right-cancel* [*iff*]:
  fixes *x y z*::*V* **shows** *x*+*y* = *x*+*z* ⟷ *y*=*z*
  by (*metis lift-eq-lift plus-eq-lift sup-lift-eq-lift*)

corollary *add-mem-right-cancel* [*iff*]:
  fixes *x y z*::*V* **shows** *x*+*y* ∈ *elts* (*x*+*z*) ⟷ *y* ∈ *elts z*
  apply *safe*
   apply (*metis mem-plus-V-E not-add-mem-right add-right-cancel*)
  by (*metis ZFC-in-HOL.ext dual-order.antisym elts-vinsert insert-subset order-refl*

plus-vinsert)

corollary *add-le-cancel-left* [*iff*]:
fixes *x y z*::*V* **shows** $x+y \le x+z \longleftrightarrow y \le z$
by *auto* (*metis add-mem-right-cancel mem-plus-V-E plus sup-ge1 vsubsetD*)

corollary *add-less-cancel-left* [*iff*]:
fixes *x y z*::*V* **shows** $x+y < x+z \longleftrightarrow y<z$
by (*simp add*: *less-le-not-le*)

corollary *lift-le-self* [*simp*]: *lift x y* $\le$ *x* $\longleftrightarrow$ *y* = *0*
by (*auto simp*: *inf.absorb-iff2 lift-eq-lift lift-self-disjoint*)

lemma *succ-less-ω-imp*: *succ x* $< \omega \Longrightarrow x < \omega$
by (*metis add-le-cancel-left add.right-neutral le-0 le-less-trans succ-eq-add1*)

Proposition 3.5

lemma *card-lift*: *vcard* (*lift x y*) = *vcard y*
proof (*rule cardinal-cong*)
have *bij-betw* ((+)*x*) (*elts y*) (*elts* (*lift x y*))
unfolding *bij-betw-def*
by (*simp add*: *inj-on-def lift-def*)
then show *elts* (*lift x y*) $\approx$ *elts y*
using *eqpoll-def eqpoll-sym* **by** *blast*
qed

lemma *eqpoll-lift*: *elts* (*lift x y*) $\approx$ *elts y*
by (*metis card-lift cardinal-eqpoll eqpoll-sym eqpoll-trans*)

lemma *vcard-add*: *vcard* (*x* + *y*) = *vcard x* $\oplus$ *vcard y*
using *card-lift* [*of x y*] *lift-self-disjoint* [*of x*]
by (*simp add*: *plus-eq-lift vcard-disjoint-sup*)

lemma *countable-add*:
assumes *countable* (*elts A*) *countable* (*elts B*)
shows *countable* (*elts* (*A*+*B*))
proof −
have *vcard A* $\le \aleph 0$ *vcard B* $\le \aleph 0$
using *assms countable-iff-le-Aleph0* **by** *blast+*
then have *vcard* (*A*+*B*) $\le \aleph 0$
unfolding *vcard-add*
by (*metis Aleph-0 Card-ω InfCard-cdouble-eq InfCard-def cadd-le-mono order-refl*)
then show *?thesis*
by (*simp add*: *countable-iff-le-Aleph0*)
qed

Proposition 3.6

proposition *TC-add*: *TC* (*x* + *y*) = *TC x* $\sqcup$ *lift x* (*TC y*)

proof (*induction y rule*: *eps-induct*)
  case (*step y*)
  have $*$: $\bigsqcup$ (*TC* ‘ (+) *x* ‘ *elts y*) = *TC x* $\sqcup$ ($\bigsqcup$*u*∈*elts y*. *TC* (*set* ((+) *x* ‘ *elts u*)))
    if *elts y* $\neq$ {}
  proof −
    obtain *w* **where** *w* ∈ *elts y*
      using ‹*elts y* $\neq$ {}› **by** *blast*
    then have *TC x* $\leq$ *TC* (*x* + *w*)
      by (*simp add*: *step.IH*)
    then have †: *TC x* $\leq$ ($\bigsqcup$*w*∈*elts y*. *TC* (*x* + *w*))
      using ‹*w* ∈ *elts y*› **by** *blast*
    show *?thesis*
      using *that*
      apply (*intro conjI ballI impI order-antisym*; *clarsimp simp add*: *image-comp* †)
       apply(*metis TC-sup-distrib Un-iff elts-sup-iff plus*)
      by (*metis TC-least Transset-TC arg-subset-TC le-sup-iff plus vsubsetD*)
  qed
  have *TC* (*x* + *y*) = (*x* + *y*) $\sqcup$ $\bigsqcup$ (*TC* ‘ *elts* (*x* + *y*))
    using *TC* **by** *blast*
  also have ... = *x* $\sqcup$ *lift x y* $\sqcup$ $\bigsqcup$ (*TC* ‘ *elts x*) $\sqcup$ $\bigsqcup$ ((λ*u*. *TC* (*x*+*u*)) ‘ *elts y*)
   apply (*simp add*: *plus-eq-lift image-Un Sup-Un-distrib sup.left-commute sup-assoc TC-sup-distrib SUP-sup-distrib*)
    apply (*simp add*: *lift-def sup.commute sup-aci* $*$)
    done
  also have ... = *x* $\sqcup$ $\bigsqcup$ (*TC* ‘ *elts x*) $\sqcup$ *lift x y* $\sqcup$ $\bigsqcup$ ((λ*u*. *TC x* $\sqcup$ *lift x* (*TC u*)) ‘ *elts y*)
    by (*simp add*: *sup-aci step.IH*)
  also have ... = *TC x* $\sqcup$ *lift x y* $\sqcup$ $\bigsqcup$ ((λ*u*. *lift x* (*TC u*)) ‘ *elts y*)
    by (*simp add*: *sup-aci SUP-sup-distrib flip*: *TC* [*of x*])
  also have ... = *TC x* $\sqcup$ *lift x* (*y* $\sqcup$ $\bigsqcup$ (*TC* ‘ *elts y*))
     by (*metis* (*no-types*) *elts-of-set lift-Sup-distrib image-image lift-sup-distrib replacement small-elts sup-assoc*)
  also have ... = *TC x* $\sqcup$ *lift x* (*TC y*)
    by (*simp add*: *TC* [*of y*])
  finally show *?case* **.**
qed

corollary *TC-add'*: *z* $\sqsubset$ *x* + *y* $\longleftrightarrow$ *z* $\sqsubset$ *x* $\vee$ (∃ *v*. *v* $\sqsubset$ *y* $\wedge$ *z* = *x* + *v*)
  using *TC-add* **by** (*force simp*: *less-TC-def lift-def*)

Corollary 3.7

corollary *vcard-TC-add*: *vcard* (*TC* (*x*+*y*)) = *vcard* (*TC x*) $\oplus$ *vcard* (*TC y*)
  by (*simp add*: *TC-add TC-sup-lift card-lift vcard-disjoint-sup*)

Corollary 3.8

corollary *TC-lift*:
  assumes *y* $\neq$ *0*
  shows *TC* (*lift x y*) = *TC x* $\sqcup$ *lift x* (*TC y*)

proof −
  have *TC* (*lift x y*) = *lift x y* ⊔ ⨆ ((λ*u*. *TC*(*x*+*u*)) ‘ *elts y*)
    unfolding *TC* [*of lift x y*] **by** (*simp add*: *lift-def image-image*)
  also have . . . = *lift x y* ⊔ (⨆*u*∈*elts y*. *TC x* ⊔ *lift x* (*TC u*))
    by (*simp add*: *TC-add*)
  also have . . . = *lift x y* ⊔ *TC x* ⊔ (⨆*u*∈*elts y*. *lift x* (*TC u*))
    using *assms* **by** (*auto simp*: *SUP-sup-distrib*)
  also have . . . = *TC x* ⊔ *lift x* (*TC y*)
    by (*simp add*: *TC* [*of y*] *sup-aci image-image lift-sup-distrib lift-Sup-distrib*)
  finally show *?thesis* **.**
qed

proposition *rank-add-distrib*: *rank* (*x*+*y*) = *rank x* + *rank y*
proof (*induction y rule*: *eps-induct*)
  case (*step y*)
  show *?case*
  proof (*cases y*=*0*)
    case *False*
    then obtain *e* **where** *e*: *e* ∈ *elts y*
      by *fastforce*
   have *rank* (*x*+*y*) = (⨆*u*∈*elts* (*x* ⊔ *ZFC-in-HOL.set* ((+) *x* ‘ *elts y*)). *succ* (*rank u*))
      by (*metis plus rank-Sup*)
    also have . . . = (⨆*x*∈*elts x*. *succ* (*rank x*)) ⊔ (⨆*z*∈*elts y*. *succ* (*rank x* + *rank z*))
      apply (*simp add*: *Sup-Un-distrib image-Un image-image*)
      apply (*simp add*: *step cong*: *SUP-cong-simp*)
      done
    also have . . . = (⨆*z* ∈ *elts y*. *rank x* + *succ* (*rank z*))
    proof −
      have *rank x* ≤ (⨆*z*∈*elts y*. *ZFC-in-HOL.succ* (*rank x* + *rank z*))
        using ‹*y* ≠ *0*›
        by (*auto simp*: *plus-eq-lift intro*: *order-trans* [*OF* - *cSUP-upper* [*OF e*]])
      then show *?thesis*
        by (*force simp*: *plus-V-succ-right simp flip*: *rank-Sup* [*of x*] *intro*!: *order-antisym*)
    qed
    also have . . . = *rank x* + (⨆*z* ∈ *elts y*. *succ* (*rank z*))
      by (*simp add*: *add-Sup-distrib False*)
    also have . . . = *rank x* + *rank y*
      by (*simp add*: *rank-Sup* [*of y*])
    finally show *?thesis* **.**
  qed *auto*
qed

lemma *Ord-add* [*simp*]: ⟦*Ord x*; *Ord y*⟧ ⟹ *Ord* (*x*+*y*)
proof (*induction y rule*: *eps-induct*)
  case (*step y*)
  then show *?case*

by (*metis Ord-rank rank-add-distrib rank-of-Ord*)
qed

lemma *add-Sup-distrib-id*: $A \neq 0 \Longrightarrow x + \bigsqcup(elts\ A) = (\bigsqcup z{\in}elts\ A.\ x + z)$
by (*metis add-Sup-distrib image-ident image-image*)

lemma *add-Limit*: $Limit\ \alpha \Longrightarrow x + \alpha = (\bigsqcup z{\in}elts\ \alpha.\ x + z)$
by (*metis Limit-add-Sup-distrib Limit-eq-Sup-self image-ident image-image*)

lemma *add-le-left*:
assumes $Ord\ \alpha\ Ord\ \beta$ **shows** $\beta \leq \alpha{+}\beta$
using ‹$Ord\ \beta$›
proof (*induction rule*: *Ord-induct3*)
case *0*
then show *?case*
by *auto*
next
case (*succ* α)
then show *?case*
by (*auto simp*: *plus-V-succ-right Ord-mem-iff-lt assms*(*1*))
next
case (*Limit* μ)
then have *k*: $\mu = (\bigsqcup \beta \in elts\ \mu.\ \beta)$
by (*simp add*: *Limit-eq-Sup-self*)
also have $\ldots\ \leq (\bigsqcup \beta \in elts\ \mu.\ \alpha + \beta)$
using *Limit.IH* **by** *auto*
also have $\ldots = \alpha + (\bigsqcup \beta \in elts\ \mu.\ \beta)$
using *Limit.hyps Limit-add-Sup-distrib* **by** *presburger*
finally show *?case*
using *k* **by** *simp*
qed

lemma *plus-ω-equals-ω*:
assumes $\alpha \in elts\ \omega$ **shows** $\alpha + \omega = \omega$
proof (*rule antisym*)
show $\alpha + \omega \leq \omega$
using *Ord-trans assms* **by** (*auto simp*: *elim*!: *mem-plus-V-E*)
show $\omega \leq \alpha + \omega$
by (*simp add*: *add-le-left assms*)
qed

lemma *one-plus-ω-equals-ω* [*simp*]: $1 + \omega = \omega$
by (*simp add*: *one-V-def plus-ω-equals-ω*)

3.1.3 Cancellation / set subtraction

definition *vle* :: $V \Rightarrow V \Rightarrow bool$ (**infix** $\trianglelefteq$ *50*)
where $x \trianglelefteq y \equiv \exists z{::}V.\ x{+}z = y$

```
lemma vle-refl [iff]: x ⊴ x
  by (metis (no-types) add.right-neutral vle-def)

lemma vle-antisym: ⟦x ⊴ y; y ⊴ x⟧ ⟹ x = y
  by (metis V-equalityI plus-eq-lift sup-ge1 vle-def vsubsetD)

lemma vle-trans [trans]: ⟦x ⊴ y; y ⊴ z⟧ ⟹ x ⊴ z
  by (metis add.assoc vle-def)

definition vle-comparable :: V ⇒ V ⇒ bool
  where vle-comparable x y ≡ x ⊴ y ∨ y ⊴ x
```

Lemma 3.13

```
lemma comparable:
  assumes a+b = c+d
  shows vle-comparable a c
unfolding vle-comparable-def
proof (rule ccontr)
  assume non: ¬ (a ⊴ c ∨ c ⊴ a)
  let ?φ = λx. ∀z. a+x ≠ c+z
  have ?φ x for x
  proof (induction x rule: eps-induct)
    case (step x)
    show ?case
    proof (cases x=0)
      case True
      with non nonzero-less-TC show ?thesis
        using vle-def by auto
    next
      case False
      then obtain v where v ∈ elts x
        using trad-foundation by blast
      show ?thesis
      proof clarsimp
        fix z
        assume eq: a + x = c + z
        then have z ≠ 0
          using vle-def non by auto
        have av: a+v ∈ elts (a+x)
          by (simp add: ‹v ∈ elts x›)
        moreover have a+x = c ⊔ lift c z
          using eq plus-eq-lift by fastforce
        ultimately have a+v ∈ elts (c ⊔ lift c z)
          by simp
        moreover
        define u where u ≡ set (elts x − {v})
        have u: v ∉ elts u and xeq: x = vinsert v u
          using ‹v ∈ elts x› by (auto simp: u-def intro: order-antisym)
        have case1: a+v ∉ elts c
```

```
      proof
        assume avc: a + v ∈ elts c
        then have a ≤ c
          by (clarify (metis Un-iff elts-sup-iff eq mem-not-sym mem-plus-V-E plus-eq-lift)
        moreover have a ⊔ lift a x = c ⊔ lift c z
          using eq by (simp add: plus-eq-lift)
        ultimately have lift c z ≤ lift a x
          by (metis inf.absorb-iff2 inf-commute inf-sup-absorb inf-sup-distrib2 lift-self-disjoint sup.commute)
        also have ... = vinsert (a+v) (lift a u)
          by (simp add: lift-def vinsert-def xeq)
        finally have ∗: lift c z ≤ vinsert (a + v) (lift a u) .
        have lift c z ≤ lift a u
        proof −
          have a + v ∉ elts (lift c z)
            using lift-self-disjoint [of c z] avc V-disjoint-iff by auto
          then show ?thesis
            using ∗ less-eq-V-def by auto
        qed
        { fix e
          assume e ∈ elts z
          then have c+e ∈ elts (lift c z)
            by (simp add: lift-def)
          then have c+e ∈ elts (lift a u)
            using ‹lift c z ≤ lift a u› by blast
          then obtain y where y ∈ elts u c+e = a+y
            using lift-def by auto
          then have False
            by (metis elts-vinsert insert-iff step.IH xeq)
        }
        then show False
          using ‹z ≠ 0› by fastforce
      qed
      ultimately show False
        by (metis (no-types) ‹v ∈ elts x› av case1 eq mem-plus-V-E step.IH)
    qed
  qed
 qed
 then show False
   using assms by blast
qed

lemma vle1: x ⊴ y ⟹ x ≤ y
  using vle-def plus-eq-lift by auto

lemma vle2: x ⊴ y ⟹ x ⊑ y
 by (metis (full-types) TC-add′ add.right-neutral le-TC-def vle-def nonzero-less-TC)
```

lemma *vle-iff-le-Ord*:
  assumes *Ord* α *Ord* β
  shows $\alpha \trianglelefteq \beta \longleftrightarrow \alpha \leq \beta$
proof
  show $\alpha \leq \beta$ **if** $\alpha \trianglelefteq \beta$
    using *that* **by** (*simp add*: *vle1*)
  show $\alpha \trianglelefteq \beta$ **if** $\alpha \leq \beta$
    using ‹*Ord* α› ‹*Ord* β› *that*
  proof (*induction* α *arbitrary*: β *rule*: *Ord-induct*)
    case (*step* γ)
    then show *?case*
      unfolding *vle-def*
     by (*metis Ord-add Ord-linear add-le-left mem-not-refl mem-plus-V-E vsubsetD*)
  qed
qed

lemma *add-le-cancel-left0* [*iff*]:
  fixes *x*::*V* **shows** $x \leq x+z$
  by (*simp add*: *vle1 vle-def*)

lemma *add-less-cancel-left0* [*iff*]:
  fixes *x*::*V* **shows** $x < x+z \longleftrightarrow 0<z$
  by (*metis add-less-cancel-left add.right-neutral*)

lemma *le-Ord-diff*:
  assumes $\alpha \leq \beta$ *Ord* α *Ord* β
  obtains γ **where** $\alpha+\gamma = \beta$ $\gamma \leq \beta$ *Ord* γ
proof −
  obtain γ **where** γ: $\alpha+\gamma = \beta$ $\gamma \leq \beta$
    by (*metis add-le-cancel-left add-le-left assms vle-def vle-iff-le-Ord*)
  then have *Ord* γ
    using *Ord-def Transset-def* ‹*Ord* β› **by** *force*
  with γ *that* **show** *thesis* **by** *blast*
qed

lemma *plus-Ord-le*:
  assumes $\alpha \in$ *elts* ω *Ord* β **shows** $\alpha+\beta \leq \beta+\alpha$
proof (*cases* $\beta \in$ *elts* ω)
  case *True*
  with *assms* **have** $\alpha+\beta = \beta+\alpha$
    by (*auto simp*: *elts-ω add.commute ord-of-nat-add* [*symmetric*])
  then show *?thesis* **by** *simp*
next
  case *False*
  then have $\omega \leq \beta$
    using *Ord-linear2 Ord-mem-iff-lt* ‹*Ord* β› **by** *auto*
  then obtain γ **where** $\omega+\gamma = \beta$ $\gamma \leq \beta$ *Ord* γ
    using ‹*Ord* β› *le-Ord-diff* **by** *auto*
  then have $\alpha+\beta = \beta$

by (*metis add.assoc assms*(*1*) *plus-ω-equals-ω*)
then show *?thesis*
by *simp*
qed

lemma *add-right-mono*: $[\![\alpha \leq \beta;\ Ord\ \alpha;\ Ord\ \beta;\ Ord\ \gamma]\!] \Longrightarrow \alpha+\gamma \leq \beta+\gamma$
by (*metis add-le-cancel-left add.assoc add-le-left le-Ord-diff*)

lemma *add-strict-mono*: $[\![\alpha < \beta;\ \gamma < \delta;\ Ord\ \alpha;\ Ord\ \beta;\ Ord\ \gamma;\ Ord\ \delta]\!] \Longrightarrow \alpha+\gamma < \beta+\delta$
by (*metis order.strict-implies-order add-less-cancel-left add-right-mono le-less-trans*)

lemma *add-right-strict-mono*: $[\![\alpha \leq \beta;\ \gamma < \delta;\ Ord\ \alpha;\ Ord\ \beta;\ Ord\ \gamma;\ Ord\ \delta]\!] \Longrightarrow \alpha+\gamma < \beta+\delta$
using *add-strict-mono le-imp-less-or-eq* **by** *blast*

lemma *Limit-add-Limit* [*simp*]:
assumes *Limit* μ *Ord* β **shows** *Limit* $(\beta + \mu)$
unfolding *Limit-def*
proof (*intro conjI allI impI*)
show *Ord* $(\beta + \mu)$
using *Limit-def assms* **by** *auto*
show $0 \in elts\ (\beta + \mu)$
using *Limit-def add-le-left assms* **by** *auto*
next
fix γ
assume $\gamma \in elts\ (\beta + \mu)$
then consider $\gamma \in elts\ \beta \mid \xi$ **where** $\xi \in elts\ \mu\ \gamma = \beta + \xi$
using *mem-plus-V-E* **by** *blast*
then show $succ\ \gamma \in elts\ (\beta + \mu)$
proof *cases*
case *1*
then show *?thesis*
by (*metis Kirby.add-strict-mono Limit-def Ord-add Ord-in-Ord Ord-mem-iff-lt assms one-V-def succ-eq-add1*)
next
case *2*
then show *?thesis*
by (*metis Limit-def add-mem-right-cancel assms*(*1*) *plus-V-succ-right*)
qed
qed

3.2 Generalised Difference

definition *odiff* **where** *odiff y x* $\equiv$ *THE* $z{::}V.\ (x+z = y) \vee (z=0 \wedge \neg\ x \trianglelefteq y)$

lemma *vle-imp-odiff-eq*: $x \trianglelefteq y \Longrightarrow x + (odiff\ y\ x) = y$
by (*auto simp*: *vle-def odiff-def*)

lemma *not-vle-imp-odiff-0*: $\neg\, x \trianglelefteq y \Longrightarrow (odiff\ y\ x) = 0$
  by (*auto simp*: *vle-def odiff-def*)

lemma *Ord-odiff-eq*:
  assumes $\alpha \le \beta$ *Ord* α *Ord* β
  shows $\alpha + odiff\ \beta\ \alpha = \beta$
  by (*simp add*: *assms vle-iff-le-Ord vle-imp-odiff-eq*)

lemma *Ord-odiff*:
  assumes *Ord* α *Ord* β **shows** *Ord* $(odiff\ \beta\ \alpha)$
proof (*cases* $\alpha \trianglelefteq \beta$)
  case *True*
  then show *?thesis*
    by (*metis add-right-cancel assms le-Ord-diff vle1 vle-imp-odiff-eq*)
next
  case *False*
  then show *?thesis*
    by (*simp add*: *odiff-def vle-def*)
qed

lemma *Ord-odiff-le*:
  assumes *Ord* α *Ord* β **shows** $odiff\ \beta\ \alpha \le \beta$
proof (*cases* $\alpha \trianglelefteq \beta$)
  case *True*
  then show *?thesis*
    by (*metis add-right-cancel assms le-Ord-diff vle1 vle-imp-odiff-eq*)
next
  case *False*
  then show *?thesis*
    by (*simp add*: *odiff-def vle-def*)
qed

lemma *odiff-0-right* [*simp*]: $odiff\ x\ 0 = x$
  by (*metis add.left-neutral vle-def vle-imp-odiff-eq*)

lemma *odiff-succ*: $y \trianglelefteq x \Longrightarrow odiff\ (succ\ x)\ y = succ\ (odiff\ x\ y)$
  unfolding *odiff-def*
  by (*metis add-right-cancel odiff-def plus-V-succ-right vle-def vle-imp-odiff-eq*)

lemma *odiff-eq-iff*: $z \trianglelefteq x \Longrightarrow odiff\ x\ z = y \longleftrightarrow x = z + y$
  by (*auto simp*: *odiff-def vle-def*)

lemma *odiff-le-iff*: $z \trianglelefteq x \Longrightarrow odiff\ x\ z \le y \longleftrightarrow x \le z + y$
  by (*auto simp*: *odiff-def vle-def*)

lemma *odiff-less-iff*: $z \trianglelefteq x \Longrightarrow odiff\ x\ z < y \longleftrightarrow x < z + y$
  by (*auto simp*: *odiff-def vle-def*)

lemma *odiff-ge-iff*: $z \trianglelefteq x \Longrightarrow$ *odiff* $x\ z \geq y \longleftrightarrow x \geq z + y$
  by (*auto simp*: *odiff-def vle-def*)

lemma *Ord-odiff-le-iff*: $[\![\alpha \leq x;\ Ord\ x;\ Ord\ \alpha]\!] \Longrightarrow$ *odiff* $x\ \alpha \leq y \longleftrightarrow x \leq \alpha + y$
  by (*simp add*: *odiff-le-iff vle-iff-le-Ord*)

lemma *odiff-le-odiff*:
  assumes $x \trianglelefteq y$ **shows** *odiff* $x\ z \leq$ *odiff* $y\ z$
proof (*cases* $z \trianglelefteq x$)
  case *True*
  then show *?thesis*
    using *assms odiff-le-iff vle1 vle-imp-odiff-eq vle-trans* **by** *presburger*
next
  case *False*
  then show *?thesis*
    by (*simp add*: *not-vle-imp-odiff-0*)
qed

lemma *Ord-odiff-le-odiff*: $[\![x \leq y;\ Ord\ x;\ Ord\ y]\!] \Longrightarrow$ *odiff* $x\ \alpha \leq$ *odiff* $y\ \alpha$
  by (*simp add*: *odiff-le-odiff vle-iff-le-Ord*)

lemma *Ord-odiff-less-odiff*: $[\![\alpha \leq x;\ x < y;\ Ord\ x;\ Ord\ y;\ Ord\ \alpha]\!] \Longrightarrow$ *odiff* $x\ \alpha <$ *odiff* $y\ \alpha$
  by (*metis Ord-odiff-eq Ord-odiff-le-odiff dual-order.strict-trans less-V-def*)

lemma *Ord-odiff-less-imp-less*: $[\![$*odiff* $x\ \alpha <$ *odiff* $y\ \alpha;\ Ord\ x;\ Ord\ y]\!] \Longrightarrow x < y$
  by (*meson Ord-linear2 leD odiff-le-odiff vle-iff-le-Ord*)

lemma *odiff-add-cancel* [*simp*]: *odiff* $(x + y)\ x = y$
  by (*simp add*: *odiff-eq-iff vle-def*)

lemma *odiff-add-cancel-0* [*simp*]: *odiff* $x\ x = 0$
  by (*simp add*: *odiff-eq-iff*)

lemma *odiff-add-cancel-both* [*simp*]: *odiff* $(x + y)\ (x + z) =$ *odiff* $y\ z$
  by (*simp add*: *add.assoc odiff-def vle-def*)

3.3 Generalised Multiplication

Credited to Dana Scott

instantiation V :: *times*
begin

This definition is credited to Tarski

definition *times-V* :: $V \Rightarrow V \Rightarrow V$
  where *times-V* $x \equiv$ *transrec* $(\lambda f\ y.\ \bigsqcup\ ((\lambda u.\ lift\ (f\ u)\ x)$ ‘ *elts* $y))$

instance ..
end

lemma *mult*: $x * y = (\bigsqcup u{\in}elts\ y.\ lift\ (x * u)\ x)$
unfolding *times-V-def* **by** (*subst transrec*) (*force simp*:)

lemma *elts-multE*:
assumes $z \in elts\ (x * y)$
obtains *u v* **where** $u \in elts\ x\ v \in elts\ y\ z = x{*}v + u$
using *mult* [*of x y*] *lift-def assms* **by** *auto*

Lemma 4.2

lemma *mult-zero-right* [*simp*]:
fixes $x{::}V$ **shows** $x * 0 = 0$
by (*metis ZFC-in-HOL.Sup-empty elts-0 image-empty mult*)

lemma *mult-insert*: $x * (vinsert\ y\ z) = x{*}z \sqcup lift\ (x{*}y)\ x$
by (*metis* (*no-types*, *lifting*) *elts-vinsert image-insert replacement small-elts sup-commute mult Sup-V-insert*)

lemma *mult-succ*: $x * succ\ y = x{*}y + x$
by (*simp add*: *mult-insert plus-eq-lift succ-def*)

lemma *ord-of-nat-mult*: $ord\text{-}of\text{-}nat\ (m{*}n) = ord\text{-}of\text{-}nat\ m * ord\text{-}of\text{-}nat\ n$
proof (*induction n*)
case (*Suc n*)
then show *?case*
by (*simp add*: *add.commute* [*of m*]) (*simp add*: *ord-of-nat-add mult-succ*)
qed *auto*

lemma *omega-closed-mult* [*intro*]:
assumes $\alpha \in elts\ \omega\ \beta \in elts\ \omega$ **shows** $\alpha{*}\beta \in elts\ \omega$
proof −
obtain *m n* **where** $\alpha = ord\text{-}of\text{-}nat\ m\ \beta = ord\text{-}of\text{-}nat\ n$
using *assms elts-ω* **by** *auto*
then have $\alpha{*}\beta = ord\text{-}of\text{-}nat\ (m{*}n)$
by (*simp add*: *ord-of-nat-mult*)
then show *?thesis*
by (*simp add*: *ω-def*)
qed

lemma *zero-imp-le-mult*: $0 \in elts\ y \Longrightarrow x \le x{*}y$
by (*auto simp*: *mult* [*of x y*])

3.3.1 Proposition 4.3

lemma *mult-zero-left* [*simp*]:
fixes $x{::}V$ **shows** $0 * x = 0$
proof (*induction x rule*: *eps-induct*)
case (*step x*)
then show *?case*

```
    by (subst mult) auto
qed

lemma mult-sup-distrib:
  fixes x::V shows x ∗ (y ⊔ z) = x∗y ⊔ x∗z
  unfolding mult [of x y ⊔ z] mult [of x y] mult [of x z]
  by (simp add: Sup-Un-distrib image-Un)

lemma mult-Sup-distrib: small Y ⟹ x ∗ (⨆ Y) = ⨆ ((∗) x ‘ Y) for Y:: V set
  unfolding mult [of x ⨆ Y]
  by (simp add: cSUP-UNION) (metis mult)

lemma mult-lift-imp-distrib: x ∗ (lift y z) = lift (x∗y) (x∗z) ⟹ x ∗ (y+z) = x∗y
+ x∗z
  by (simp add: mult-sup-distrib plus-eq-lift)

lemma mult-lift: x ∗ (lift y z) = lift (x∗y) (x∗z)
proof (induction z rule: eps-induct)
  case (step z)
  have x ∗ lift y z = (⨆u∈elts (lift y z). lift (x ∗ u) x)
    using mult by blast
  also have . . . = (⨆v∈elts z. lift (x ∗ (y + v)) x)
    using lift-def by auto
  also have . . . = (⨆v∈elts z. lift (x ∗ y + x ∗ v) x)
    using mult-lift-imp-distrib step.IH by auto
  also have . . . = (⨆v∈elts z. lift (x ∗ y) (lift (x ∗ v) x))
    by (simp add: lift-lift)
  also have . . . = lift (x ∗ y) (⨆v∈elts z. lift (x ∗ v) x)
    by (simp add: image-image lift-Sup-distrib)
  also have . . . = lift (x∗y) (x∗z)
    by (metis mult)
  finally show ?case .
qed

lemma mult-Limit: Limit γ ⟹ x ∗ γ = ⨆ ((∗) x ‘ elts γ)
  by (metis Limit-eq-Sup-self mult-Sup-distrib small-elts)

lemma add-mult-distrib: x ∗ (y+z) = x∗y + x∗z for x::V
  by (simp add: mult-lift mult-lift-imp-distrib)

instantiation V :: monoid-mult
begin
instance
proof
  show 1 ∗ x = x for x :: V
  proof (induction x rule: eps-induct)
    case (step x)
    then show ?case
      by (subst mult) auto
```

```
  qed
  show x * 1 = x for x :: V
    by (subst mult) auto
  show (x * y) * z = x * (y * z) for x y z::V
  proof (induction z rule: eps-induct)
    case (step z)
    have (x * y) * z = (⨆u∈elts z. lift (x * y * u) (x * y))
      using mult by blast
    also have ... = (⨆u∈elts z. lift (x * (y * u)) (x * y))
      using step.IH by auto
    also have ... = (⨆u∈elts z. x * lift (y * u) y)
      using mult-lift by auto
    also have ... = x * (⨆u∈elts z. lift (y * u) y)
      by (simp add: image-image mult-Sup-distrib)
    also have ... = x * (y * z)
      by (metis mult)
    finally show ?case .
  qed
qed

end

lemma le-mult:
  assumes Ord β β ≠ 0 shows α ≤ α * β
  using assms
proof (induction rule: Ord-induct3)
  case (succ α)
  then show ?case
    using mult-insert succ-def by fastforce
next
  case (Limit μ)
  have α ∈ (*) α ` elts μ
    using Limit.hyps Limit-def one-V-def by (metis imageI mult.right-neutral)
  then have α ≤ ⨆ ((*) α ` elts μ)
    by auto
  then show ?case
    by (simp add: Limit.hyps mult-Limit)
qed auto

lemma mult-sing-1 [simp]:
  fixes x::V shows x * set{1} = lift x x
  by (subst mult) auto

lemma mult-2-right [simp]:
  fixes x::V shows x * set{0,1} = x+x
  by (subst mult) (auto simp: Sup-V-insert plus-eq-lift)

lemma Ord-mult [simp]: ⟦Ord y; Ord x⟧ ⟹ Ord (x*y)
proof (induction y rule: Ord-induct3)
```

  case *0*
  then show *?case*
    by *auto*
next
  case (*succ k*)
  then show *?case*
    by (*simp add*: *mult-succ*)
next
  case (*Limit k*)
  then have *Ord* ($x * \bigsqcup$ (*elts k*))
    by (*metis Ord-Sup imageE mult-Sup-distrib small-elts*)
  then show *?case*
    using *Limit.hyps Limit-eq-Sup-self* **by** *auto*
qed

3.3.2 Proposition 4.4-5

proposition *rank-mult-distrib*: *rank* ($x*y$) = *rank* x $*$ *rank* y
proof (*induction y rule*: *eps-induct*)
  case (*step y*)
  have *rank* ($x*y$) = ($\bigsqcup$ $y\in$*elts* ($\bigsqcup$ $u\in$*elts* y. *lift* ($x * u$) x). *succ* (*rank* y))
    by (*metis rank-Sup mult*)
  also have ... = ($\bigsqcup$ $u\in$*elts* y. $\bigsqcup$ $r\in$*elts* x. *succ* (*rank* ($x * u + r$)))
    apply (*simp add*: *lift-def image-image image-UN*)
    apply (*simp add*: *Sup-V-def*)
    done
  also have ... = ($\bigsqcup$ $u\in$*elts* y. $\bigsqcup$ $r\in$*elts* x. *succ* (*rank* ($x * u$) + *rank* r))
    using *rank-add-distrib* **by** *auto*
  also have ... = ($\bigsqcup$ $u\in$*elts* y. $\bigsqcup$ $r\in$*elts* x. *succ* (*rank* x $*$ *rank* u + *rank* r))
    using *step arg-cong* [**where** f = *Sup*] **by** *auto*
  also have ... = ($\bigsqcup$ $u\in$*elts* y. *rank* x $*$ *rank* u + *rank* x)
  proof (*rule SUP-cong*)
    show ($\bigsqcup$ $r\in$*elts* x. *succ* (*rank* x $*$ *rank* u + *rank* r)) = *rank* x $*$ *rank* u + *rank* x
      if $u \in$ *elts* y **for** u
    proof (*cases* x=*0*)
      case *False*
       have ($\bigsqcup$ $r\in$*elts* x. *succ* (*rank* x $*$ *rank* u + *rank* r)) = *rank* x $*$ *rank* u + ($\bigsqcup$ $y\in$*elts* x. *succ* (*rank* y))
      proof (*rule order-antisym*)
        show ($\bigsqcup$ $r\in$*elts* x. *succ* (*rank* x $*$ *rank* u + *rank* r)) $\le$ *rank* x $*$ *rank* u + ($\bigsqcup$ $y\in$*elts* x. *succ* (*rank* y))
          by (*auto simp*: *Sup-le-iff simp flip*: *plus-V-succ-right*)
        have *rank* x $*$ *rank* u + ($\bigsqcup$ $y\in$*elts* x. *succ* (*rank* y)) = ($\bigsqcup$ $y\in$*elts* x. *rank* x $*$ *rank* u + *succ* (*rank* y))
          by (*simp add*: *add-Sup-distrib False*)
        also have ... $\le$ ($\bigsqcup$ $r\in$*elts* x. *succ* (*rank* x $*$ *rank* u + *rank* r))
          using *plus-V-succ-right* **by** *auto*
        finally show *rank* x $*$ *rank* u + ($\bigsqcup$ $y\in$*elts* x. *succ* (*rank* y)) $\le$ ($\bigsqcup$ $r\in$*elts* x.

succ (*rank* x $*$ *rank* u + *rank* r)) **.**
      qed
      also have . . . = *rank* x $*$ *rank* u + *rank* x
        by (*metis rank-Sup*)
      finally show *?thesis* **.**
    qed *auto*
  qed *auto*
  also have . . . = *rank* x $*$ *rank* y
    by (*simp add*: *rank-Sup* [*of y*] *mult-Sup-distrib mult-succ image-image*)
  finally show *?case* **.**
qed

lemma *mult-le1*:
  fixes y::V **assumes** $y \neq 0$ **shows** $x \sqsubseteq x * y$
proof (*cases* $x = 0$)
  case *False*
  then obtain r **where** r: $r \in$ *elts* x
    by *fastforce*
  from ‹$y \neq 0$› **show** *?thesis*
  proof (*induction* y *rule*: *eps-induct*)
    case (*step* y)
    show *?case*
    proof (*cases* $y = 1$)
      case *False*
      with ‹$y \neq 0$› **obtain** p **where** p: $p \in$ *elts* y $p \neq 0$
        by (*metis V-equalityI elts-1 insertI1 singletonD trad-foundation*)
      then have $x*p + r \in$ *elts* (*lift* ($x*p$) x)
        by (*simp add*: *lift-def* r)
      moreover have *lift* ($x*p$) $x \le x*y$
        by (*metis bdd-above-iff-small cSUP-upper2 order-refl* ‹$p \in$ *elts* y› *replacement small-elts mult*)
      ultimately have $x*p + r \in$ *elts* ($x*y$)
        by *blast*
      moreover have $x*p \sqsubseteq x*p + r$
          by (*metis TC-add′ V-equalityI add.right-neutral eps-induct le-TC-refl less-TC-iff less-imp-le-TC*)
      ultimately show *?thesis*
        using *step.IH* [*OF p*] *le-TC-trans less-TC-iff* **by** *blast*
    qed *auto*
  qed
qed *auto*

lemma *mult-eq-0-iff* [*simp*]:
  fixes y::V **shows** $x * y = 0 \longleftrightarrow x=0 \lor y=0$
proof
  show $x = 0 \lor y = 0$ **if** $x * y = 0$
    by (*metis le-0 le-TC-def less-TC-imp-not-le mult-le1 that*)
qed *auto*

lemma *lift-lemma*:
  assumes $x \neq 0$ $y \neq 0$ **shows** $\neg$ *lift* $(x * y)$ $x \leq x$
  using *assms mult-le1* [*of concl*: *x y*]
  by (*auto simp*: *le-TC-def TC-lift less-TC-def less-TC-imp-not-le*)

lemma *mult-le2*:
  fixes y:: V **assumes** $x \neq 0$ $y \neq 0$ $y \neq 1$ **shows** $x \sqsubset x * y$
proof −
  obtain *v* **where** *v*: $v \in$ *elts* y $v \neq 0$
    using *assms* **by** *fastforce*
  have $x \neq x * y$
    using *lift-lemma* [*of x v*]
      by (*metis* ‹$x \neq 0$› *bdd-above-iff-small cSUP-upper2 order-refl replacement small-elts mult v*)
  then show *?thesis*
  using *assms mult-le1* [*of y x*]
    by (*auto simp*: *le-TC-def*)
qed

lemma *elts-mult-ωE*:
  assumes $x \in$ *elts* $(y * \omega)$
  obtains *n* **where** $n \neq 0$ $x \in$ *elts* $(y *$ *ord-of-nat* $n)$ $\bigwedge m.\ m < n \Longrightarrow x \notin$ *elts* $(y *$ *ord-of-nat* $m)$
proof −
  obtain *k* **where** *k*: $k \neq 0 \wedge x \in$ *elts* $(y *$ *ord-of-nat* $k)$
    using *assms*
    apply (*simp add*: *mult-Limit elts-ω*)
    by (*metis mult-eq-0-iff elts-0 ex-in-conv ord-of-eq-0-iff that*)
  define *n* **where** $n \equiv$ (*LEAST* $k.\ k \neq 0 \wedge x \in$ *elts* $(y *$ *ord-of-nat* $k))$
  show *thesis*
  proof
    show $n \neq 0$ $x \in$ *elts* $(y *$ *ord-of-nat* $n)$
      unfolding *n-def* **by** (*metis* (*mono-tags*, *lifting*) *LeastI-ex k*)+
    show $\bigwedge m.\ m < n \Longrightarrow x \notin$ *elts* $(y *$ *ord-of-nat* $m)$
      by (*metis* (*mono-tags*, *lifting*) *mult-eq-0-iff elts-0 empty-iff n-def not-less-Least ord-of-eq-0-iff*)
  qed
qed

3.3.3 Theorem 4.6

theorem *mult-eq-imp-0*:
  assumes $a*x = a*y + b$ $b \sqsubset a$
  shows $b=0$
proof (*cases* $a=0 \vee x=0$)
  case *True*
  with *assms* **show** *?thesis*
    by (*metis add-le-cancel-left mult-eq-0-iff eq-iff le-0*)
next

case *False*
then have $a{\neq}0$ $x{\neq}0$
by *auto*
then show *?thesis*
proof (*cases* $y{=}0$)
case *True*
then show *?thesis*
using *assms less-asym-TC mult-le2* **by** *force*
next
case *False*
have $b{=}0$ **if** *Ord* α $x \in$ *elts* (*Vset* α) $y \in$ *elts* (*Vset* α) **for** α
using *that assms*
proof (*induction* α *arbitrary*: x y b *rule*: *Ord-induct3*)
case *0*
then show *?case* **by** *auto*
next
case (*succ* k)
define Φ **where** $\Phi \equiv \lambda x\ y.\ \exists r.\ 0 \sqsubset r \wedge r \sqsubset a \wedge a{*}x = a{*}y + r$
show *?case*
proof (*rule ccontr*)
assume $b \neq 0$
then have $0 \sqsubset b$
by (*metis nonzero-less-TC*)
then have $\Phi\ x\ y$
unfolding Φ*-def* **using** *succ.prems* **by** *blast*
then obtain x' **where** $\Phi\ x'\ y\ x' \sqsubseteq x$ **and** *min*: $\bigwedge x''.\ x'' \sqsubset x' \Longrightarrow \neg\ \Phi\ x''\ y$
using *less-TC-minimal* [*of* $\lambda x.\ \Phi\ x\ y\ x$] **by** *blast*
then obtain b' **where** $0 \sqsubset b'\ b' \sqsubset a$ **and** *eq*: $a{*}x' = a{*}y + b'$
using Φ*-def* **by** *blast*
have $a{*}y \sqsubset a{*}x'$
using *TC-add'* ‹$0 \sqsubset b'$› *eq* **by** *auto*
then obtain p **where** $p \in$ *elts* $(a * x')$ $a * y \sqsubseteq p$
using *less-TC-iff* **by** *blast*
then have $p \notin$ *elts* $(a * y)$
using *less-TC-iff less-irrefl-TC* **by** *blast*
then have $p \in \bigcup$ (*elts* ` ($\lambda v.$ *lift* $(a * v)$ $a$) ` *elts* x')
by (*metis* ‹$p \in$ *elts* $(a * x')$› *elts-Sup replacement small-elts mult*)
then obtain u c **where** $u \in$ *elts* x' $c \in$ *elts* a $p = a{*}u + c$
using *lift-def* **by** *auto*
then have $p \in$ *elts* (*lift* $(a{*}y)$ b')
using ‹$p \in$ *elts* $(a * x')$› ‹$p \notin$ *elts* $(a * y)$› *eq plus-eq-lift* **by** *auto*
then obtain d **where** d: $d \in$ *elts* b' $p = a{*}y + d$ $p = a{*}u + c$
by (*metis* ‹$p = a * u + c$› ‹$p \in$ *elts* $(a * x')$› ‹$p \notin$ *elts* $(a * y)$› *eq mem-plus-V-E*)
have *noteq*: $a{*}y \neq a{*}u$
proof
assume $a{*}y = a{*}u$
then have *lift* $(a{*}y)$ a = *lift* $(a{*}u)$ a
by *metis*

also have ... $\leq$ *a*∗*x*′
unfolding *mult* [*of* - *x*′] **using** ‹*u* ∈ *elts x*′› **by** (*auto intro*: *cSUP-upper*)
also have ... = *a*∗*y* ⊔ *lift* (*a*∗*y*) *b*′
by (*simp add*: *eq plus-eq-lift*)
finally have *lift* (*a*∗*y*) *a* ≤ *a*∗*y* ⊔ *lift* (*a*∗*y*) *b*′ **.**
then have *lift* (*a*∗*y*) *a* ≤ *lift* (*a*∗*y*) *b*′
using *add-le-cancel-left less-TC-imp-not-le plus-eq-lift* ‹*b*′ ⊏ *a*› **by** *auto*
then have *a* ≤ *b*′
by (*simp add*: *le-iff-sup lift-eq-lift lift-sup-distrib*)
then show *False*
using ‹*b*′ ⊏ *a*› *less-TC-imp-not-le* **by** *auto*
qed
consider *a*∗*y* ⊴ *a*∗*u* | *a*∗*u* ⊴ *a*∗*y*
using *d comparable vle-comparable-def* **by** *auto*
then show *False*
proof *cases*
case *1*
then obtain *e* **where** *e*: *a*∗*u* = *a*∗*y* + *e e* ≠ *0*
by (*metis add.right-neutral noteq vle-def*)
moreover have *e* + *c* = *d*
by (*metis e add-right-cancel* ‹*p* = *a* ∗ *u* + *c*› ‹*p* = *a* ∗ *y* + *d*› *add.assoc*)
with ‹*d* ∈ *elts b*′› ‹*b*′ ⊏ *a*› **have** *e* ⊏ *a*
by (*meson less-TC-iff less-TC-trans vle2 vle-def*)
ultimately show *False*
— contradicts minimality of *x*′
using *min* **unfolding** Φ-*def* **by** (*meson* ‹*u* ∈ *elts x*′› *le-TC-def less-TC-iff nonzero-less-TC*)
next
case *2*
then obtain *e* **where** *e*: *a*∗*y* = *a*∗*u* + *e e* ≠ *0*
by (*metis add.right-neutral noteq vle-def*)
moreover have *e* + *d* = *c*
by (*metis e add-right-cancel* ‹*p* = *a* ∗ *u* + *c*› ‹*p* = *a* ∗ *y* + *d*› *add.assoc*)
with ‹*d* ∈ *elts b*′› ‹*b*′ ⊏ *a*› **have** *e* ⊏ *a*
by (*metis* ‹*c* ∈ *elts a*› *less-TC-iff vle2 vle-def*)
ultimately have Φ *y u*
unfolding Φ-*def* **using** *nonzero-less-TC* **by** *blast*
then obtain *y*′ **where** Φ *y*′ *u y*′ ⊑ *y* **and** *min*: ⋀*x*′′. *x*′′ ⊏ *y*′ ⟹ ¬ Φ *x*′′ *u*
using *less-TC-minimal* [*of* λ*x*. Φ *x u y*] **by** *blast*
then obtain *b*′ **where** *0* ⊏ *b*′ *b*′ ⊏ *a* **and** *eq*: *a*∗*y*′ = *a*∗*u* + *b*′
using Φ-*def* **by** *blast*
have *u-k*: *u* ∈ *elts* (*Vset k*)
using ‹*u* ∈ *elts x*′› ‹*x*′ ⊑ *x*› *succ Vset-succ-TC less-TC-iff less-le-TC-trans* **by** *blast*
have *a*∗*u* ⊏ *a*∗*y*′
using *TC-add*′ ‹*0* ⊏ *b*′› *eq* **by** *auto*
then obtain *p* **where** *p* ∈ *elts* (*a* ∗ *y*′) *a* ∗ *u* ⊑ *p*
using *less-TC-iff* **by** *blast*

then have $p \notin$ *elts* $(a * u)$
  using *less-TC-iff less-irrefl-TC* **by** *blast*
then have $p \in \bigcup$ (*elts* ` ($\lambda v.$ *lift* $(a * v)$ $a$) ` *elts* y')
  by (*metis* ‹$p \in$ *elts* $(a * y')$› *elts-Sup replacement small-elts mult*)
then obtain v c **where** $v \in$ *elts* y' $c \in$ *elts* a $p = a{*}v + c$
  using *lift-def* **by** *auto*
then have $p \in$ *elts* (*lift* $(a{*}u)$ b')
  using ‹$p \in$ *elts* $(a * y')$› ‹$p \notin$ *elts* $(a * u)$› *eq plus-eq-lift* **by** *auto*
then obtain d **where** d: $d \in$ *elts* b' $p = a{*}u + d$ $p = a{*}v + c$
  by (*metis* ‹$p = a * v + c$› ‹$p \in$ *elts* $(a * y')$› ‹$p \notin$ *elts* $(a * u)$› *eq mem-plus-V-E*)
have *v-k*: $v \in$ *elts* (*Vset* k)
  using *Vset-succ-TC* ‹$v \in$ *elts* y'› ‹$y' \sqsubseteq y$› *less-TC-iff less-le-TC-trans succ.hyps succ.prems*(*2*) **by** *blast*
have *noteq*: $a{*}u \neq a{*}v$
  proof
    assume $a{*}u = a{*}v$
    then have *lift* $(a{*}v)$ $a \le a{*}y'$
    unfolding *mult* [*of* - y'] **using** ‹$v \in$ *elts* y'› **by** (*auto intro*: *cSUP-upper*)
    also have $\ldots = a{*}u \sqcup$ *lift* $(a{*}u)$ b'
      by (*simp add*: *eq plus-eq-lift*)
    finally have *lift* $(a{*}v)$ $a \le a{*}u \sqcup$ *lift* $(a{*}u)$ b' **.**
    then have *lift* $(a{*}u)$ $a \le$ *lift* $(a{*}u)$ b'
      by (*metis* ‹$a * u = a * v$› *le-iff-sup lift-sup-distrib sup-left-commute sup-lift-eq-lift*)
    then have $a \le b'$
      by (*simp add*: *le-iff-sup lift-eq-lift lift-sup-distrib*)
    then show *False*
      using ‹$b' \sqsubset a$› *less-TC-imp-not-le* **by** *auto*
  qed
consider $a{*}u \trianglelefteq a{*}v$ | $a{*}v \trianglelefteq a{*}u$
  using d *comparable vle-comparable-def* **by** *auto*
then show *False*
proof *cases*
  case *1*
  then obtain e **where** e: $a{*}v = a{*}u + e$ $e \neq 0$
    by (*metis add.right-neutral noteq vle-def*)
  moreover have $e + c = d$
    by (*metis add-right-cancel* ‹$p = a * u + d$› ‹$p = a * v + c$› *add.assoc* e)
  with ‹$d \in$ *elts* b'› ‹$b' \sqsubset a$› **have** $e \sqsubset a$
    by (*meson less-TC-iff less-TC-trans vle2 vle-def*)
  ultimately show *False*
    using *succ.IH u-k v-k* **by** *blast*
next
  case *2*
  then obtain e **where** e: $a{*}u = a{*}v + e$ $e \neq 0$
    by (*metis add.right-neutral noteq vle-def*)
  moreover have $e + d = c$

by (*metis add-right-cancel add.assoc d e*)
with ‹$d \in$ *elts* b'› ‹$b' \sqsubset a$› **have** $e \sqsubset a$
by (*metis* ‹$c \in$ *elts a*› *less-TC-iff vle2 vle-def*)
ultimately show *False*
using *succ.IH u-k v-k* **by** *blast*
qed
qed
qed
next
case (*Limit k*)
obtain *i j* **where** *k*: $i \in$ *elts k* $j \in$ *elts k*
and *x*: $x \in$ *elts* (*Vset i*)
and *y*: $y \in$ *elts* (*Vset j*)
using *that Limit* **by** (*auto simp*: *Limit-Vfrom-eq*)
show *?case*
proof (*rule Limit.IH* [*of* $i \sqcup j$])
show $i \sqcup j \in$ *elts k*
by (*meson k x y Limit.hyps Limit-def Ord-in-Ord Ord-mem-iff-lt Ord-sup union-less-iff*)
show $x \in$ *elts* (*Vset* ($i \sqcup j$)) $y \in$ *elts* (*Vset* ($i \sqcup j$))
using *x y* **by** (*auto simp*: *Vfrom-sup*)
qed (*use Limit.prems* **in** *auto*)
qed
then show *?thesis*
by (*metis two-in-Vset Ord-rank Ord-VsetI rank-lt*)
qed
qed

3.3.4 Theorem 4.7

lemma *mult-cancellation-half*:
assumes $a*x + r \le a*y + s$ $r \sqsubset a$ $s \sqsubset a$
shows $x \le y$
proof −
have $x \le y$ **if** *Ord* α $x \in$ *elts* (*Vset* α) $y \in$ *elts* (*Vset* α) **for** α
using *that assms*
proof (*induction* α *arbitrary*: *x y r s rule*: *Ord-induct3*)
case *0*
then show *?case*
by *auto*
next
case (*succ k*)
show *?case*
proof
fix *u*
assume *u*: $u \in$ *elts x*
have *u-k*: $u \in$ *elts* (*Vset k*)
using *Vset-succ succ.hyps succ.prems*(*1*) *u* **by** *auto*
obtain r' **where** $r' \in$ *elts a* $r \sqsubseteq r'$

using *less-TC-iff succ.prems(4)* **by** *blast*
have $a{*}u + r' \in$ *elts* (*lift* ($a{*}u$) a)
by (*simp add*: ‹$r' \in$ *elts* a› *lift-def*)
also have $\ldots \le$ *elts* ($a{*}x$)
using u **by** (*force simp*: *mult* [*of* - x])
also have $\ldots \le$ *elts* ($a{*}y + s$)
using *plus-eq-lift succ.prems(3)* **by** *auto*
also have $\ldots =$ *elts* ($a{*}y$) $\cup$ *elts* (*lift* ($a{*}y$) s)
by (*simp add*: *plus-eq-lift*)
finally have $a * u + r' \in$ *elts* ($a * y$) $\cup$ *elts* (*lift* ($a * y$) s) **.**
then show $u \in$ *elts* y
proof
assume $*$: $a * u + r' \in$ *elts* ($a * y$)
show $u \in$ *elts* y
proof $-$
obtain v e **where** v: $v \in$ *elts* y $e \in$ *elts* a $a * u + r' = a * v + e$
using $*$ **by** (*auto simp*: *mult* [*of* - y] *lift-def*)
then have *v-k*: $v \in$ *elts* (*Vset* k)
using *Vset-succ-TC less-TC-iff succ.prems(2)* **by** *blast*
then show *?thesis*
by (*metis* ‹$r' \in$ *elts* a› *antisym le-TC-refl less-TC-iff order-refl succ.IH*
u-k v)
qed
next
assume $a * u + r' \in$ *elts* (*lift* ($a * y$) s)
then obtain t **where** $t \in$ *elts* s **and** t: $a * u + r' = a * y + t$
using *lift-def* **by** *auto*
have *noteq*: $a{*}y \neq a{*}u$
proof
assume $a{*}y = a{*}u$
then have *lift* ($a{*}y$) a = *lift* ($a{*}u$) a
by *metis*
also have $\ldots \le a{*}x$
unfolding *mult* [*of* - x] **using** ‹$u \in$ *elts* x› **by** (*auto intro*: *cSUP-upper*)
also have $\ldots \le a{*}y \sqcup$ *lift* ($a{*}y$) s
using ‹*elts* ($a * x$) $\subseteq$ *elts* ($a * y + s$)› *plus-eq-lift* **by** *auto*
finally have *lift* ($a{*}y$) $a \le a{*}y \sqcup$ *lift* ($a{*}y$) s **.**
then have *lift* ($a{*}y$) $a \le$ *lift* ($a{*}y$) s
using *add-le-cancel-left less-TC-imp-not-le plus-eq-lift* ‹$s \sqsubset a$› **by** *auto*
then have $a \le s$
by (*simp add*: *le-iff-sup lift-eq-lift lift-sup-distrib*)
then show *False*
using ‹$s \sqsubset a$› *less-TC-imp-not-le* **by** *auto*
qed
consider $a * u \trianglelefteq a * y \mid a * y \trianglelefteq a * u$
using t *comparable vle-comparable-def* **by** *blast*
then have *False*
proof *cases*
case *1*

```
        then obtain c where a∗y = a∗u + c
          by (metis vle-def)
        then have c+t = r′
          by (metis add-right-cancel add.assoc t)
        then have c ⊏ a
          using ‹r′ ∈ elts a› less-TC-iff vle2 vle-def by force
        moreover have c ≠ 0
          using ‹a ∗ y = a ∗ u + c› noteq by auto
        ultimately show ?thesis
          using ‹a ∗ y = a ∗ u + c› mult-eq-imp-0 by blast
      next
        case 2
        then obtain c where a∗u = a∗y + c
          by (metis vle-def)
        then have c+r′ = t
          by (metis add-right-cancel add.assoc t)
        then have c ⊏ a
          by (metis ‹t ∈ elts s› less-TC-iff less-TC-trans ‹s ⊏ a› vle2 vle-def)
        moreover have c ≠ 0
          using ‹a ∗ u = a ∗ y + c› noteq by auto
        ultimately show ?thesis
          using ‹a ∗ u = a ∗ y + c› mult-eq-imp-0 by blast
      qed
      then show u ∈ elts y ..
    qed
  qed
next
  case (Limit k)
  obtain i j where k: i ∈ elts k j ∈ elts k
    and x: x ∈ elts (Vset i) and y: y ∈ elts (Vset j)
    using that Limit by (auto simp: Limit-Vfrom-eq)
  show ?case
  proof (rule Limit.IH [of i ⊔ j])
    show i ⊔ j ∈ elts k
       by (meson k x y Limit.hyps Limit-def Ord-in-Ord Ord-mem-iff-lt Ord-sup
union-less-iff)
    show x ∈ elts (Vset (i ⊔ j)) y ∈ elts (Vset (i ⊔ j))
      using x y by (auto simp: Vfrom-sup)
    show a ∗ x + r ≤ a ∗ y + s
      by (simp add: Limit.prems)
  qed (auto simp: Limit.prems)
qed
then show ?thesis
  by (metis two-in-Vset Ord-rank Ord-VsetI rank-lt)
qed

theorem mult-cancellation-lemma:
  assumes a∗x + r = a∗y + s r ⊏ a s ⊏ a
  shows x=y ∧ r=s
```

by (*metis assms leD less-V-def mult-cancellation-half odiff-add-cancel order-refl*)

corollary *mult-cancellation* [*simp*]:
fixes a::*V*
assumes $a \neq 0$
shows $a*x = a*y \longleftrightarrow x=y$
by (*metis assms nonzero-less-TC mult-cancellation-lemma*)

corollary *mult-cancellation-less*:
assumes *lt*: $a*x + r < a*y + s$ **and** $r \sqsubset a$ $s \sqsubset a$
obtains $x < y$ | $x = y$ $r < s$
proof −
have $x \leq y$
by (*meson assms dual-order.strict-implies-order mult-cancellation-half*)
then consider $x < y$ | $x = y$
using *less-V-def* **by** *blast*
with *lt that* **show** *?thesis* **by** *blast*
qed

corollary *lift-mult-TC-disjoint*:
fixes x::*V*
assumes $x \neq y$
shows *lift* $(a*x)$ (*TC a*) $\sqcap$ *lift* $(a*y)$ (*TC a*) $= 0$
apply (*rule V-equalityI*)
using *assms*
by (*auto simp*: *less-TC-def inf-V-def lift-def image-iff dest*: *mult-cancellation-lemma*)

corollary *lift-mult-disjoint*:
fixes x::*V*
assumes $x \neq y$
shows *lift* $(a*x)$ a $\sqcap$ *lift* $(a*y)$ $a = 0$
proof −
have *lift* $(a*x)$ a $\sqcap$ *lift* $(a*y)$ $a \leq$ *lift* $(a*x)$ (*TC a*) $\sqcap$ *lift* $(a*y)$ (*TC a*)
by (*metis TC′ inf-mono lift-sup-distrib sup-ge1*)
then show *?thesis*
using *assms lift-mult-TC-disjoint* **by** *auto*
qed

lemma *mult-add-mem*:
assumes $a*x + r \in$ *elts* $(a*y)$ $r \sqsubset a$
shows $x \in$ *elts* y $r \in$ *elts* a
proof −
obtain v s **where** v: $a * x + r = a * v + s$ $v \in$ *elts* y $s \in$ *elts* a
using *assms* **unfolding** *mult* [*of a y*] *lift-def* **by** *auto*
then show $x \in$ *elts* y
by (*metis arg-subset-TC assms*(*2*) *less-TC-def mult-cancellation-lemma vsubsetD*)
show $r \in$ *elts* a
by (*metis arg-subset-TC assms*(*2*) *less-TC-def mult-cancellation-lemma v*(*1*)

v(*3*) *vsubsetD*)
qed

lemma *mult-add-mem-0* [*simp*]: $a{*}x \in elts\ (a{*}y) \longleftrightarrow x \in elts\ y \wedge 0 \in elts\ a$
  proof −
  have $x \in elts\ y$
    if $a * x \in elts\ (a * y) \wedge 0 \in elts\ a$
    using *that* **using** *mult-add-mem* [*of a x 0*]
    using *nonzero-less-TC* **by** *force*
  moreover have $a * x \in elts\ (a * y)$
    if $x \in elts\ y\ 0 \in elts\ a$
    using *that* **by** (*force simp*: *image-iff mult* [*of a y*] *lift-def*)
  ultimately show *?thesis*
    by (*metis mult-eq-0-iff add.right-neutral mult-add-mem*(*2*) *nonzero-less-TC*)
qed

lemma *zero-mem-mult-iff*: $0 \in elts\ (x{*}y) \longleftrightarrow 0 \in elts\ x \wedge 0 \in elts\ y$
  by (*metis Kirby.mult-zero-right mult-add-mem-0*)

lemma *zero-less-mult-iff* [*simp*]: $0 < x{*}y \longleftrightarrow 0 < x \wedge 0 < y$ **if** *Ord x*
  using *Kirby.mult-eq-0-iff ZFC-in-HOL.neq0-conv* **by** *blast*

lemma *mult-cancel-less-iff* [*simp*]:
  $[\![Ord\ \alpha;\ Ord\ \beta;\ Ord\ \gamma]\!] \Longrightarrow \alpha{*}\beta < \alpha{*}\gamma \longleftrightarrow \beta < \gamma \wedge 0 < \alpha$
  using *mult-add-mem-0* [*of* $\alpha\ \beta\ \gamma$]
  by (*meson Ord-0 Ord-mem-iff-lt Ord-mult*)

lemma *mult-cancel-le-iff* [*simp*]:
  $[\![Ord\ \alpha;\ Ord\ \beta;\ Ord\ \gamma]\!] \Longrightarrow \alpha{*}\beta \le \alpha{*}\gamma \longleftrightarrow \beta \le \gamma \vee \alpha{=}0$
  by (*metis Ord-linear2 Ord-mult eq-iff leD mult-cancel-less-iff mult-cancellation*)

lemma *mult-Suc-add-less*: $[\![\alpha < \gamma;\ \beta < \gamma;\ Ord\ \alpha;\ Ord\ \beta;\ Ord\ \gamma]\!] \Longrightarrow \gamma * ord\text{-}of\text{-}nat\ m + \alpha < \gamma * ord\text{-}of\text{-}nat\ (Suc\ m) + \beta$
  apply (*simp add*: *mult-succ add.assoc*)
  by (*meson Ord-add Ord-linear2 le-less-trans not-add-less-right*)

lemma *mult-nat-less-add-less*:
  assumes $m < n\ \alpha < \gamma\ \beta < \gamma$ **and** *ord*: $Ord\ \alpha\ Ord\ \beta\ Ord\ \gamma$
    shows $\gamma * ord\text{-}of\text{-}nat\ m + \alpha < \gamma * ord\text{-}of\text{-}nat\ n + \beta$
proof −
  have $Suc\ m \le n$
    using ‹$m < n$› **by** *auto*
  have $\gamma * ord\text{-}of\text{-}nat\ m + \alpha < \gamma * ord\text{-}of\text{-}nat\ (Suc\ m) + \beta$
    using *assms mult-Suc-add-less* **by** *blast*
  also have $\ldots \le \gamma * ord\text{-}of\text{-}nat\ n + \beta$
    using *Ord-mult Ord-ord-of-nat add-right-mono* ‹$Suc\ m \le n$› *ord mult-cancel-le-iff ord-of-nat-mono-iff* **by** *presburger*
  finally show *?thesis* **.**
qed

lemma *add-mult-less-add-mult*:
  assumes $x < y$ $x \in$ *elts* β $y \in$ *elts* β $\mu \in$ *elts* α $\nu \in$ *elts* α *Ord* α *Ord* β
    shows $\alpha{*}x + \mu < \alpha{*}y + \nu$
proof $-$
  obtain *Ord x Ord y*
    using *Ord-in-Ord assms* **by** *blast*
  then obtain δ **where** $0 \in$ *elts* δ $y = x + \delta$
    by (*metis add.right-neutral* ‹$x < y$› *le-Ord-diff less-V-def mem-0-Ord*)
  then show *?thesis*
    apply (*simp add*: *add-mult-distrib add.assoc*)
     by (*meson OrdmemD add-le-cancel-left0* ‹$\mu \in$ *elts* α› ‹*Ord* α› *less-le-trans zero-imp-le-mult*)
qed

lemma *add-mult-less*:
  assumes $\gamma \in$ *elts* α $\nu \in$ *elts* β *Ord* α *Ord* β
    shows $\alpha * \nu + \gamma \in$ *elts* $(\alpha * \beta)$
proof $-$
  have *Ord* ν
    using *Ord-in-Ord assms* **by** *blast*
  with *assms* **show** *?thesis*
   by (*metis Ord-mem-iff-lt Ord-succ add-mem-right-cancel mult-cancel-le-iff mult-succ succ-le-iff vsubsetD*)
qed

lemma *Ord-add-mult-iff*:
  assumes $\beta \in$ *elts* γ $\beta' \in$ *elts* γ *Ord* α *Ord* α' *Ord* γ
  shows $\gamma * \alpha + \beta \in$ *elts* $(\gamma * \alpha' + \beta') \longleftrightarrow \alpha \in$ *elts* $\alpha' \vee \alpha = \alpha' \wedge \beta \in$ *elts* β' (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
  assume *L*: *?lhs*
  show *?rhs*
  proof (*cases* $\alpha \in$ *elts* α')
    case *False*
    with *assms* **have** $\alpha = \alpha'$
     by (*meson L Ord-linear Ord-mult Ord-trans add-mult-less not-add-mem-right*)
    then show *?thesis*
      using *L less-V-def* **by** *auto*
  qed *auto*
next
  assume *R*: *?rhs*
  then show *?lhs*
  proof
    assume $\alpha \in$ *elts* α'
    then obtain δ **where** $\alpha' = \alpha{+}\delta$
      by (*metis OrdmemD assms*(*3*) *assms*(*4*) *le-Ord-diff less-V-def*)
    show *?lhs*
      using *assms*

by $(meson \ \langle\alpha \in elts\ \alpha'\rangle\ add\text{-}le\text{-}cancel\text{-}left0\ add\text{-}mult\text{-}less\ vsubsetD)$
next
assume $\alpha = \alpha' \wedge \beta \in elts\ \beta'$
then show *?lhs*
using *less-V-def* **by** *auto*
qed
qed

lemma *vcard-mult*: $vcard\ (x * y) = vcard\ x \otimes vcard\ y$
proof $-$
have *1*: $elts\ (lift\ (x * u)\ x) \approx elts\ x$ **if** $u \in elts\ y$ **for** u
by $(metis\ cardinal\text{-}eqpoll\ eqpoll\text{-}sym\ eqpoll\text{-}trans\ card\text{-}lift)$
have *2*: $pairwise\ (\lambda u\ u'.\ disjnt\ (elts\ (lift\ (x * u)\ x))\ (elts\ (lift\ (x * u')\ x)))\ \ (elts\ y)$
by $(simp\ add{:}\ pairwise\text{-}def\ disjnt\text{-}def)\ (metis\ V\text{-}disjoint\text{-}iff\ lift\text{-}mult\text{-}disjoint)$
have $x * y = (\bigsqcup u{\in}elts\ y.\ lift\ (x * u)\ x)$
using *mult* **by** *blast*
then have $elts\ (x * y) \approx (\bigcup u{\in}elts\ y.\ elts\ (lift\ (x * u)\ x))$
by *simp*
also have $\ldots \approx elts\ y \times elts\ x$
using $Union\text{-}eqpoll\text{-}Times\ [OF\ 1\ 2]$ **.**
also have $\ldots \approx elts\ x \times elts\ y$
by $(simp\ add{:}\ times\text{-}commute\text{-}eqpoll)$
also have $\ldots \approx elts\ (vcard\ x) \times elts\ (vcard\ y)$
using *cardinal-eqpoll eqpoll-sym times-eqpoll-cong* **by** *blast*
also have $\ldots \approx elts\ (vcard\ x \otimes vcard\ y)$
by $(simp\ add{:}\ cmult\text{-}def\ elts\text{-}vcard\text{-}VSigma\text{-}eqpoll\ eqpoll\text{-}sym)$
finally have $elts\ (x * y) \approx elts\ (vcard\ x \otimes vcard\ y)$ **.**
then show *?thesis*
by $(metis\ cadd\text{-}cmult\text{-}distrib\ cadd\text{-}def\ cardinal\text{-}cong\ cardinal\text{-}idem\ vsum\text{-}0\text{-}eqpoll)$
qed

proposition *TC-mult*: $TC(x * y) = (\bigsqcup r \in elts\ (TC\ x).\ \bigsqcup u \in elts\ (TC\ y).\ set\{x * u + r\})$
proof $(cases\ x = 0)$
case *False*
have $*$: $TC(x * y) = (\bigsqcup u \in elts\ (TC\ y).\ lift\ (x * u)\ (TC\ x))$ **for** y
proof $(induction\ y\ rule{:}\ eps\text{-}induct)$
case $(step\ y)$
have $TC(x * y) = (\bigsqcup u \in elts\ y.\ TC\ (lift\ (x * u)\ x))$
by $(simp\ add{:}\ mult\ [of\ x\ y]\ TC\text{-}Sup\text{-}distrib\ image\text{-}image)$
also have $\ldots = (\bigsqcup u \in elts\ y.\ TC(x * u) \sqcup lift\ (x * u)\ (TC\ x))$
by $(simp\ add{:}\ TC\text{-}lift\ False)$
also have $\ldots = (\bigsqcup u \in elts\ y.\ (\bigsqcup z \in elts\ (TC\ u).\ lift\ (x * z)\ (TC\ x)) \sqcup lift\ (x * u)\ (TC\ x))$
by $(simp\ add{:}\ step)$
also have $\ldots = (\bigsqcup u \in elts\ (TC\ y).\ lift\ (x * u)\ (TC\ x))$
by $(auto\ simp{:}\ TC'\ [of\ y]\ image\text{-}Un\ Sup\text{-}Un\text{-}distrib\ TC\text{-}Sup\text{-}distrib\ cSUP\text{-}UNION\ SUP\text{-}sup\text{-}distrib)$

finally show *?case* **.**
qed
show *?thesis*
by (*force simp*: $*$ *lift-def*)
qed *auto*

corollary *vcard-TC-mult*: *vcard* $(TC(x * y)) = $ *vcard* $(TC\ x) \otimes$ *vcard* $(TC\ y)$
proof $-$
have $(\bigcup u{\in}elts\ (TC\ x).\ \bigcup v{\in}elts\ (TC\ y).\ \{x * v + u\}) = (\bigcup u{\in}elts\ (TC\ x).\ (\lambda v.\ x * v + u)$ ` $elts\ (TC\ y))$
by (*simp add*: *UNION-singleton-eq-range*)
also have $\ldots \approx (\bigcup x{\in}elts\ (TC\ x).\ elts\ (lift\ (TC\ y * x)\ (TC\ y)))$
proof (*rule UN-eqpoll-UN*)
show $(\lambda v.\ x * v + u)$ ` $elts\ (TC\ y) \approx elts\ (lift\ (TC\ y * u)\ (TC\ y))$
if $u \in elts\ (TC\ x)$ **for** u
proof $-$
have *inj-on* $(\lambda v.\ x * v + u)\ (elts\ (TC\ y))$
by (*meson inj-onI less-TC-def mult-cancellation-lemma that*)
then have $(\lambda v.\ x * v + u)$ ` $elts\ (TC\ y) \approx elts\ (TC\ y)$
by (*rule inj-on-image-eqpoll-self*)
also have $\ldots \approx elts\ (lift\ (TC\ y * u)\ (TC\ y))$
by (*simp add*: *eqpoll-lift eqpoll-sym*)
finally show *?thesis* **.**
qed
show *pairwise* $(\lambda u\ ya.\ disjnt\ ((\lambda v.\ x * v + u)$ ` $elts\ (TC\ y))\ ((\lambda v.\ x * v + ya)$ ` $elts\ (TC\ y)))\ (elts\ (TC\ x))$
apply (*auto simp*: *pairwise-def disjnt-def*)
using *less-TC-def mult-cancellation-lemma* **by** *blast*
show *pairwise* $(\lambda u\ ya.\ disjnt\ (elts\ (lift\ (TC\ y * u)\ (TC\ y)))\ (elts\ (lift\ (TC\ y * ya)\ (TC\ y))))\ (elts\ (TC\ x))$
apply (*auto simp*: *pairwise-def disjnt-def*)
by (*metis Int-iff V-disjoint-iff empty-iff lift-mult-disjoint*)
qed
also have $\ldots = elts\ (TC\ y * TC\ x)$
by (*metis elts-Sup image-image mult replacement small-elts*)
finally have $(\bigcup u{\in}elts\ (TC\ x).\ \bigcup v{\in}elts\ (TC\ y).\ \{x * v + u\}) \approx elts\ (TC\ y * TC\ x)$ **.**
then show *?thesis*
apply (*subst cmult-commute*)
by (*simp add*: *TC-mult cardinal-cong flip*: *vcard-mult*)
qed

lemma *countable-mult*:
assumes *countable* $(elts\ A)$ *countable* $(elts\ B)$
shows *countable* $(elts\ (A{*}B))$
proof $-$
have *vcard* $A \le \aleph 0$ *vcard* $B \le \aleph 0$
using *assms countable-iff-le-Aleph0* **by** *blast+*

then have *vcard* $(A*B) \leq \aleph 0$
unfolding *vcard-mult*
by (*metis InfCard-csquare-eq cmult-le-mono Aleph-0 Card-ω InfCard-def order-refl*)
then show *?thesis*
by (*simp add*: *countable-iff-le-Aleph0*)
qed

3.4 Ordertype properties

lemma *ordertype-image-plus*:
assumes *Ord* α
shows *ordertype* $((+)\ u$ ' *elts* $\alpha)$ *VWF* $= \alpha$
proof (*subst ordertype-VWF-eq-iff*)
have *1*: $(u + x, u + y) \in$ *VWF* **if** $x \in$ *elts* α $y \in$ *elts* α $x < y$ **for** x y
using *that*
by (*meson Ord-in-Ord Ord-mem-iff-lt add-mem-right-cancel assms mem-imp-VWF*)
then have *2*: $x < y$
if $x \in$ *elts* α $y \in$ *elts* α $(u + x, u + y) \in$ *VWF* **for** x y
using *that* **by** (*metis Ord-in-Ord Ord-linear-lt VWF-asym assms*)
show $\exists f.$ *bij-betw* f $((+)\ u$ ' *elts* $\alpha)$ (*elts* α) $\wedge$ $(\forall x \in (+)\ u$ ' *elts* $\alpha.\ \forall y \in (+)\ u$ ' *elts* $\alpha.\ (f\,x < f\,y) = ((x, y) \in$ *VWF*$))$
using *1 2* **unfolding** *bij-betw-def inj-on-def*
by (*rule-tac* $x = \lambda x.$ *odiff* x u **in** *exI*) (*auto simp*: *image-iff*)
qed (*use assms* **in** *auto*)

lemma *ordertype-diff*:
assumes $\beta + \delta = \alpha$ **and** α: $\delta \in$ *elts* α *Ord* α
shows *ordertype* (*elts* α $-$ *elts* β) *VWF* $= \delta$
proof $-$
have $*$: *elts* α $-$ *elts* β $= ((+)\beta)$ ' *elts* δ
proof
show *elts* α $-$ *elts* $\beta \subseteq (+)\ \beta$ ' *elts* δ
by *clarsimp* (*metis assms*(*1*) *image-iff mem-plus-V-E*)
show $(+)\ \beta$ ' *elts* $\delta \subseteq$ *elts* α $-$ *elts* β
using *assms*(*1*) *not-add-mem-right* **by** *force*
qed
have *ordertype* $((+)\ \beta$ ' *elts* $\delta)$ *VWF* $= \delta$
proof (*subst ordertype-VWF-inc-eq*)
show *elts* $\delta \subseteq$ *ON ordertype* (*elts* δ) *VWF* $= \delta$
using α *elts-subset-ON ordertype-eq-Ord* **by** *blast+*
qed (*use* $*$ *assms elts-subset-ON* **in** *auto*)
then show *?thesis*
by (*simp add*: $*$)
qed

lemma *ordertype-interval-eq*:
assumes α: *Ord* α **and** β: *Ord* β
shows *ordertype* $(\{\alpha\ ..< \alpha+\beta\} \cap ON)$ *VWF* $= \beta$

proof −
  have *ON*: (+) α ‘ *elts* $\beta \subseteq$ *ON*
    using *assms Ord-add Ord-in-Ord* **by** *blast*
  have ($\{\alpha$ ..< $\alpha+\beta\} \cap$ *ON*) = (+) α ‘ *elts* β
    using *assms*
    apply (*simp add*: *image-def set-eq-iff*)
    by (*metis add-less-cancel-left Ord-add Ord-in-Ord Ord-linear2 Ord-mem-iff-lt le-Ord-diff not-add-less-right*)
  moreover have *ordertype* (*elts* β) *VWF* = *ordertype* ((+) α ‘ *elts* β) *VWF*
    using *ON* β *elts-subset-ON ordertype-VWF-inc-eq* **by** *auto*
  ultimately show *?thesis*
    using β **by** *auto*
qed

lemma *ordertype-Times*:
  assumes *small A small B* **and** *r*: *wf r trans r total-on A r* **and** *s*: *wf s trans s total-on B s*
  shows *ordertype* ($A \times B$) (*r* <∗*lex*∗> *s*) = *ordertype B s* ∗ *ordertype A r* (**is** - = *?β* ∗ *?α*)
proof (*subst ordertype-eq-iff*)
  show *Ord* (*?β* ∗ *?α*)
    by (*intro wf-Ord-ordertype Ord-mult r s*; *simp*)
  define *f* **where** $f \equiv \lambda(x,y)$. *?β* ∗ *ordermap A r x* + (*ordermap B s y*)
  show $\exists f$. *bij-betw f* ($A \times B$) (*elts* (*?β* ∗ *?α*)) $\wedge$ ($\forall x \in A \times B$. $\forall y \in A \times B$. ($f\ x$ < $f\ y$) = ((x, y) $\in$ (*r* <∗*lex*∗> *s*)))
    unfolding *bij-betw-def*
  proof (*intro exI conjI strip*)
    show *inj-on f* ($A \times B$)
    proof (*clarsimp simp*: *f-def inj-on-def*)
      fix $x\ y\ x'\ y'$
      assume $x \in A$ $y \in B$ $x' \in A$ $y' \in B$
        and *eq*: *?β* ∗ *ordermap A r x* + *ordermap B s y* = *?β* ∗ *ordermap A r x′* + *ordermap B s y′*
      have *ordermap A r x* = *ordermap A r x′* $\wedge$
            ordermap B s y = *ordermap B s y′*
      proof (*rule mult-cancellation-lemma* [*OF eq*])
        show *ordermap B s y* $\sqsubset$ *?β*
          using *ordermap-in-ordertype* [*OF* ‹$y \in B$›, *of s*] *less-TC-iff* ‹*small B*› **by** *blast*
        show *ordermap B s y′* $\sqsubset$ *?β*
          using *ordermap-in-ordertype* [*OF* ‹$y' \in B$›, *of s*] *less-TC-iff* ‹*small B*› **by** *blast*
      qed
      then show $x = x' \wedge y = y'$
        using ‹$x \in A$› ‹$x' \in A$› ‹$y \in B$› ‹$y' \in B$› *r s* ‹*small A*› ‹*small B*› **by** *auto*
    qed
    show *f* ‘ ($A \times B$) = *elts* (*?β* ∗ *?α*) (**is** *?lhs* = *?rhs*)
    proof
      show *f* ‘ ($A \times B$) $\subseteq$ *elts* (*?β* ∗ *?α*)

apply (*auto simp*: *f-def add-mult-less ordermap-in-ordertype wf-Ord-ordertype r s*)
by (*simp add*: *add-mult-less assms ordermap-in-ordertype wf-Ord-ordertype*)
show *elts* (*?β* ∗ *?α*) ⊆ *f* ' (*A* × *B*)
proof (*clarsimp simp*: *f-def image-iff elim* !: *elts-multE split*: *prod.split*)
fix *u v*
assume *u*: *u* ∈ *elts* (*?β*) **and** *v*: *v* ∈ *elts ?α*
have *inv-into B* (*ordermap B s*) *u* ∈ *B*
by (*simp add*: *inv-into-ordermap u*)
moreover have *inv-into A* (*ordermap A r*) *v* ∈ *A*
by (*simp add*: *inv-into-ordermap v*)
ultimately show ∃ *x*∈*A*. ∃ *y*∈*B*. *?β* ∗ *v* + *u* = *?β* ∗ *ordermap A r x* + *ordermap B s y*
by (*metis* ‹*small A*› ‹*small B*› *bij-betw-inv-into-right ordermap-bij r*(*1*) *r*(*3*) *s*(*1*) *s*(*3*) *u v*)
qed
qed
next
fix *p q*
assume *p* ∈ *A* × *B* **and** *q* ∈ *A* × *B*
then obtain *u v x y* **where** §: *p* = (*u*,*v*) *u* ∈ *A* *v* ∈ *B* *q* = (*x*,*y*) *x* ∈ *A* *y* ∈ *B*
by *blast*
show ((*f p*) < *f q*) = ((*p*, *q*) ∈ (*r* <∗*lex*∗> *s*))
proof
assume *f p* < *f q*
with § *assms* **have** (*u*, *x*) ∈ *r* ∨ *u*=*x* ∧ (*v*, *y*) ∈ *s*
apply (*simp add*: *f-def*)
by (*metis Ord-add Ord-add-mult-iff Ord-mem-iff-lt Ord-mult wf-Ord-ordermap converse-ordermap-mono*
ordermap-eq-iff ordermap-in-ordertype wf-Ord-ordertype)
then show (*p*,*q*) ∈ (*r* <∗*lex*∗> *s*)
by (*simp add*: §)
next
assume (*p*,*q*) ∈ (*r* <∗*lex*∗> *s*)
then have (*u*, *x*) ∈ *r* ∨ *u* = *x* ∧ (*v*, *y*) ∈ *s*
by (*simp add*: §)
then show *f p* < *f q*
proof
assume *ux*: (*u*, *x*) ∈ *r*
have *oo*: ⋀*x*. *Ord* (*ordermap A r x*) ⋀*y*. *Ord* (*ordermap B s y*)
by (*simp-all add*: *r s*)
show *f p* < *f q*
proof (*clarsimp simp*: *f-def split*: *prod.split*)
fix *a b a′ b′*
assume *p* = (*a*, *b*) **and** *q* = (*a′*, *b′*)
then have *?β* ∗ *ordermap A r a* + *ordermap B s b* < *?β* ∗ *ordermap A r a′*
using *ux assms* §
by (*metis Ord-mult wf-Ord-ordermap OrdmemD Pair-inject add-mult-less*

ordermap-in-ordertype ordermap-mono wf-Ord-ordertype)
also have ... $\le$ *?β* $*$ *ordermap A r a′* + *ordermap B s b′*
by *simp*
finally show *?β* $*$ *ordermap A r a* + *ordermap B s b* < *?β* $*$ *ordermap A r a′* + *ordermap B s b′* **.**
qed
next
assume $u = x \wedge (v, y) \in s$
then show $f\,p < f\,q$
using § *assms* **by** (*fastforce simp*: *f-def split*: *prod.split intro*: *ordermap-mono-less*)
qed
qed
qed
qed (*use assms small-Times* **in** *auto*)

end

4 Exponentiation of ordinals

theory *Ordinal-Exp*
imports *Kirby*

begin

Source: Schlöder, Julian. Ordinal Arithmetic; available online at http://www.math.uni-bonn.de/ag/logik/teaching/2012WS/Set%20theory/oa.pdf

definition *oexp* :: $[V,V] \Rightarrow V$ (**infixr** ↑ *80*)
where *oexp a b* ≡ *transrec* (λ*f x*. *if x=0 then 1*
else if Limit x then if a=0 then 0 else $\bigsqcup\xi \in$ *elts x*. *f ξ*
else f ($\bigsqcup$(*elts x*)) $*$ *a*) *b*

$0 \uparrow \omega = 1$ if we don't make a special case for Limit ordinals and zero

lemma *oexp-0-right* [*simp*]: $\alpha\uparrow 0 = 1$
by (*simp add*: *def-transrec* [*OF oexp-def*])

lemma *oexp-succ* [*simp*]: *Ord β* $\Longrightarrow$ $\alpha\uparrow(succ\ \beta) = \alpha\uparrow\beta * \alpha$
by (*simp add*: *def-transrec* [*OF oexp-def*])

lemma *oexp-Limit*: *Limit β* $\Longrightarrow$ $\alpha\uparrow\beta =$ (*if α=0 then 0 else* $\bigsqcup\xi \in$ *elts β*. $\alpha\uparrow\xi$)
by (*auto simp*: *def-transrec* [*OF oexp-def*, *of - β*])

lemma *oexp-1-right* [*simp*]: $\alpha\uparrow 1 = \alpha$
using *one-V-def oexp-succ* **by** *fastforce*

lemma *oexp-1* [*simp*]: *Ord α* $\Longrightarrow$ $1\uparrow\alpha = 1$
by (*induction rule*: *Ord-induct3*) (*use Limit-def oexp-Limit* **in** *auto*)

lemma *oexp-0* [*simp*]: *Ord* $\alpha \Longrightarrow 0{\uparrow}\alpha = (if\ \alpha = 0\ then\ 1\ else\ 0)$
  by (*induction rule*: *Ord-induct3*) (*use Limit-def oexp-Limit* **in** *auto*)

lemma *oexp-eq-0-iff* [*simp*]:
  assumes *Ord* β **shows** $\alpha{\uparrow}\beta = 0 \longleftrightarrow \alpha{=}0 \wedge \beta{\neq}0$
  using ‹*Ord* β›
proof (*induction rule*: *Ord-induct3*)
  case (*Limit* μ)
  then show *?case*
    using *Limit-def oexp-Limit* **by** *auto*
qed *auto*

lemma *oexp-gt-0-iff* [*simp*]:
  assumes *Ord* β **shows** $\alpha{\uparrow}\beta > 0 \longleftrightarrow \alpha{>}0 \vee \beta{=}0$
  by (*simp add*: *assms less-V-def*)

lemma *ord-of-nat-oexp*: *ord-of-nat* $(m\hat{\ }n)$ = *ord-of-nat* $m{\uparrow}$*ord-of-nat* n
proof (*induction n*)
  case (*Suc n*)
  then show *?case*
    by (*simp add*: *mult.commute* [*of m*]) (*simp add*: *ord-of-nat-mult*)
qed *auto*

lemma *omega-closed-oexp* [*intro*]:
  assumes $\alpha \in$ *elts* ω $\beta \in$ *elts* ω **shows** $\alpha{\uparrow}\beta \in$ *elts* ω
proof −
  obtain *m n* **where** α = *ord-of-nat* m β = *ord-of-nat* n
    using *assms elts-*ω **by** *auto*
  then have $\alpha{\uparrow}\beta$ = *ord-of-nat* $(m\hat{\ }n)$
    by (*simp add*: *ord-of-nat-oexp*)
  then show *?thesis*
    by (*simp add*: ω*-def*)
qed

lemma *Ord-oexp* [*simp*]:
  assumes *Ord* α *Ord* β **shows** *Ord* $(\alpha{\uparrow}\beta)$
  using ‹*Ord* β›
proof (*induction rule*: *Ord-induct3*)
  case (*Limit* α)
  then show *?case*
    by (*auto simp*: *oexp-Limit image-iff intro*: *Ord-Sup*)
qed (*auto intro*: *Ord-mult assms*)

Lemma 3.19

lemma *le-oexp*:
  assumes *Ord* α *Ord* β $\beta \neq 0$ **shows** $\alpha \le \alpha{\uparrow}\beta$
  using ‹*Ord* β› ‹$\beta \neq 0$›
proof (*induction rule*: *Ord-induct3*)

case (*succ* β)
then show *?case*
by *simp* (*metis* ‹*Ord* α› *le-0 le-mult mult.left-neutral oexp-0-right order-refl order-trans*)
next
case (*Limit* μ)
then show *?case*
by (*metis Limit-def Limit-eq-Sup-self ZFC-in-HOL.Sup-upper eq-iff image-eqI image-ident oexp-1-right oexp-Limit replacement small-elts one-V-def*)
qed *auto*

Lemma 3.20

lemma *le-oexp*$'$:
assumes *Ord* α $1 < \alpha$ *Ord* β **shows** $\beta \leq \alpha\uparrow\beta$
proof (*cases* $\beta = 0$)
case *True*
then show *?thesis*
by *auto*
next
case *False*
show *?thesis*
using ‹*Ord* β›
proof (*induction rule*: *Ord-induct3*)
case *0*
then show *?case*
by *auto*
next
case (*succ* γ)
then have $\alpha\uparrow\gamma * 1 < \alpha\uparrow\gamma * \alpha$
using ‹*Ord* α› ‹$1 < \alpha$›
by (*metis le-mult less-V-def mult.right-neutral mult-cancellation not-less-0 oexp-eq-0-iff succ.hyps*)
then have $\gamma < \alpha\uparrow$*succ* γ
using *succ.IH succ.hyps* **by** *auto*
then show *?case*
using *False* ‹*Ord* α› ‹$1 < \alpha$› *succ*
by (*metis Ord-mem-iff-lt Ord-oexp Ord-succ elts-succ insert-subset less-eq-V-def less-imp-le*)
next
case (*Limit* μ)
with *False* ‹$1 < \alpha$› **show** *?case*
by (*force simp*: *Limit-def oexp-Limit intro*: *elts-succ*)
qed
qed

lemma *oexp-Limit-le*:
assumes $\beta < \gamma$ *Limit* γ *Ord* β $\alpha > 0$ **shows** $\alpha\uparrow\beta \leq \alpha\uparrow\gamma$
proof $-$

  have *Ord* γ
    using *Limit-def assms*(*2*) **by** *blast*
  with *assms* **show** *?thesis*
    using *Ord-mem-iff-lt ZFC-in-HOL.Sup-upper oexp-Limit* **by** *auto*
qed

proposition *oexp-less*:
  assumes β: $\beta \in$ *elts* γ **and** *Ord* γ **and** α: $\alpha > 1$ *Ord* α **shows** $\alpha\uparrow\beta < \alpha\uparrow\gamma$
proof –
  obtain $\beta < \gamma$ *Ord* β
    using *Ord-in-Ord OrdmemD assms* **by** *auto*
  have *gt0*: $\alpha\uparrow\beta > 0$
    using ‹*Ord* β› α *dual-order.order-iff-strict* **by** *auto*
  show *?thesis*
    using ‹*Ord* γ› β
  proof (*induction rule*: *Ord-induct3*)
    case *0*
    then show *?case*
      by *auto*
  next
    case (*succ* δ)
    then consider $\beta = \delta \mid \beta < \delta$
      using *OrdmemD elts-succ* **by** *blast*
    then show *?case*
    proof *cases*
      case *1*
      then have $(\alpha\uparrow\beta) * 1 < (\alpha\uparrow\delta) * \alpha$
        using *Ord-1 Ord-oexp* α *gt0 mult-cancel-less-iff succ.hyps* **by** *metis*
      then show *?thesis*
        by (*simp add*: *succ.hyps*)
    next
      case *2*
      then have $(\alpha\uparrow\delta) * 1 < (\alpha\uparrow\delta) * \alpha$
      by (*meson Ord-1 Ord-mem-iff-lt Ord-oexp* ‹*Ord* β› α *gt0 less-trans mult-cancel-less-iff succ*)
      with *2* **show** *?thesis*
        using *Ord-mem-iff-lt* ‹*Ord* β› *succ* **by** *auto*
    qed
  next
    case (*Limit* γ)
    then obtain *Ord* γ *succ* $\beta < \gamma$
      using *Limit-def Ord-in-Ord OrdmemD assms* **by** *auto*
    have $\alpha\uparrow\beta = (\alpha\uparrow\beta) * 1$
      by *simp*
    also have $\ldots < (\alpha\uparrow\beta) * \alpha$
      using *Ord-oexp* ‹*Ord* β› *assms gt0 mult-cancel-less-iff* **by** *blast*
    also have $\ldots = \alpha\uparrow$*succ* β
      by (*simp add*: ‹*Ord* β›)
    also have $\ldots \le (\bigsqcup\xi \in$ *elts* $\gamma.\ \alpha\uparrow\xi)$

proof −
have *succ* $\beta \in$ *elts* γ
using *Limit.hyps Limit.prems Limit-def* **by** *auto*
then show *?thesis*
by (*simp add*: *ZFC-in-HOL.Sup-upper*)
qed
finally
have $\alpha\uparrow\beta < (\bigsqcup\xi \in$ *elts* $\gamma.\ \alpha\uparrow\xi)$ **.**
then show *?case*
using *Limit.hyps oexp-Limit* ‹$\alpha > 1$› **by** *auto*
qed
qed

corollary *oexp-less-iff*:
assumes $\alpha > 0$ *Ord* α *Ord* β *Ord* γ **shows** $\alpha\uparrow\beta < \alpha\uparrow\gamma \longleftrightarrow \beta \in$ *elts* $\gamma \wedge \alpha > 1$
proof *safe*
show $\beta \in$ *elts* γ $1 < \alpha$
if $\alpha\uparrow\beta < \alpha\uparrow\gamma$
proof −
show $\alpha > 1$
proof (*rule ccontr*)
assume $\neg\ \alpha > 1$
then consider $\alpha=0 \mid \alpha=1$
using ‹*Ord* α› *less-V-def mem-0-Ord* **by** *fastforce*
then show *False*
by *cases* (*use that* ‹$\alpha > 0$› ‹*Ord* β› ‹*Ord* γ› **in** ‹*auto split*: *if-split-asm*›)
qed
show β: $\beta \in$ *elts* γ
proof (*rule ccontr*)
assume $\beta \notin$ *elts* γ
then have $\gamma \le \beta$
by (*meson Ord-linear-le Ord-mem-iff-lt assms less-le-not-le*)
then consider $\gamma = \beta \mid \gamma < \beta$
using *less-V-def* **by** *blast*
then show *False*
proof *cases*
case *1*
then show *?thesis*
using *that* **by** *blast*
next
case *2*
with ‹$\alpha > 1$› **have** $\alpha\uparrow\gamma < \alpha\uparrow\beta$
by (*simp add*: *Ord-mem-iff-lt assms oexp-less*)
with *that* **show** *?thesis*
by *auto*
qed
qed
qed
show $\alpha\uparrow\beta < \alpha\uparrow\gamma$ **if** $\beta \in$ *elts* γ $1 < \alpha$

using *that* **by** (*simp add*: *assms oexp-less*)
qed

lemma *ω-oexp-iff* [*simp*]: $[\![Ord\ \alpha;\ Ord\ \beta]\!] \Longrightarrow \omega\uparrow\alpha = \omega\uparrow\beta \longleftrightarrow \alpha=\beta$
by (*metis Ord-ω Ord-linear ω-gt1 less-irrefl oexp-less*)

lemma *Limit-oexp*:
assumes $Limit\ \gamma\ Ord\ \alpha\ \alpha > 1$ **shows** $Limit\ (\alpha\uparrow\gamma)$
unfolding *Limit-def*
proof *safe*
show $O\alpha\gamma$: $Ord\ (\alpha\uparrow\gamma)$
using *Limit-def Ord-oexp* ‹$Limit\ \gamma$› *assms*(*2*) **by** *blast*
show *0*: $0 \in elts\ (\alpha\uparrow\gamma)$
using *Limit-def oexp-Limit* ‹$Limit\ \gamma$› ‹$\alpha > 1$› **by** *fastforce*
have $Ord\ \gamma$
using *Limit-def* ‹$Limit\ \gamma$› **by** *blast*
fix x
assume x: $x \in elts\ (\alpha\uparrow\gamma)$
with ‹$Limit\ \gamma$› ‹$\alpha > 1$›
obtain β **where** $\beta < \gamma\ Ord\ \beta\ Ord\ x$ **and** $x\beta$: $x \in elts\ (\alpha\uparrow\beta)$
apply (*simp add*: *oexp-Limit split*: *if-split-asm*)
using *Ord-in-Ord OrdmemD* ‹$Ord\ \gamma$› $O\alpha\gamma\ x$ **by** *blast*
then have $O\alpha\beta$: $Ord\ (\alpha\uparrow\beta)$
using *Ord-oexp assms*(*2*) **by** *blast*
have $\beta \in elts\ \gamma$
by (*simp add*: *Ord-mem-iff-lt* ‹$Ord\ \beta$› ‹$Ord\ \gamma$› ‹$\beta < \gamma$›)
moreover have $\alpha \neq 0$
using ‹$\alpha > 1$› **by** *blast*
ultimately have $\alpha\beta\gamma$: $\alpha\uparrow\beta \leq \alpha\uparrow\gamma$
by (*simp add*: *Sup-upper oexp-Limit* ‹$Limit\ \gamma$›)
have $succ\ x \leq \alpha\uparrow\beta$
by (*simp add*: *OrdmemD* $O\alpha\beta$ ‹$Ord\ x$› *succ-le-iff* $x\beta$)
then consider $succ\ x < \alpha\uparrow\beta \mid succ\ x = \alpha\uparrow\beta$
using *le-neq-trans* **by** *blast*
then show $succ\ x \in elts\ (\alpha\uparrow\gamma)$
proof *cases*
case *1*
with $\alpha\beta\gamma$ **show** *?thesis*
using $O\alpha\beta$ *Ord-mem-iff-lt* ‹$Ord\ x$› **by** *blast*
next
case *2*
then have $succ\ \beta < \gamma$
using *Limit-def OrdmemD* ‹$\beta \in elts\ \gamma$› *assms*(*1*) **by** *auto*
have *ge1*: $1 \leq \alpha\uparrow\beta$
by (*metis 2 Ord-0* ‹$Ord\ x$› *le-0 le-succ-iff one-V-def*)
have $succ\ x < succ\ (\alpha\uparrow\beta)$
using *2* $O\alpha\beta$ *succ-le-iff* **by** *auto*
also have $\ldots \leq (\alpha\uparrow\beta) + (\alpha\uparrow\beta)$
using *ge1* **by** (*simp add*: *succ-eq-add1*)

also have ... = $(\alpha\uparrow\beta) * succ\ (succ\ 0)$
by (*simp add*: *mult-succ*)
also have ... $\leq (\alpha\uparrow\beta) * \alpha$
using $O\alpha\beta$ *Ord-succ assms*(2) *assms*(3) *one-V-def succ-le-iff* **by** *auto*
also have ... = $\alpha\uparrow succ\ \beta$
by (*simp add*: ‹*Ord* β›)
also have ... $\leq \alpha\uparrow\gamma$
by (*meson Limit-def* ‹$\beta \in elts\ \gamma$› *assms dual-order.order-iff-strict oexp-less*)
finally show *?thesis*
by (*simp add*: *2* $O\alpha\beta$ $O\alpha\gamma$ *Ord-mem-iff-lt*)
qed
qed

lemma *oexp-mono*:
assumes α: *Ord* α $\alpha \neq 0$ **and** β: *Ord* β $\gamma \sqsubseteq \beta$ **shows** $\alpha\uparrow\gamma \leq \alpha\uparrow\beta$
using β
proof (*induction rule*: *Ord-induct3*)
case *0*
then show *?case*
by *simp*
next
case (*succ* β)
with α *le-mult* **show** *?case*
by (*auto simp*: *le-TC-succ*)
next
case (*Limit* μ)
then have $\alpha\uparrow\gamma \leq \bigsqcup\ ((\uparrow)\ \alpha\ `\ elts\ \mu)$
using *Limit.hyps Ord-less-TC-mem* ‹$\alpha \neq 0$› *le-TC-def* **by** (*auto simp*: *oexp-Limit Limit-def*)
then show *?case*
using α **by** (*simp add*: *oexp-Limit Limit.hyps*)
qed

lemma *oexp-mono-le*:
assumes $\gamma \leq \beta$ $\alpha \neq 0$ *Ord* α *Ord* β *Ord* γ **shows** $\alpha\uparrow\gamma \leq \alpha\uparrow\beta$
by (*simp add*: *assms oexp-mono vle2 vle-iff-le-Ord*)

lemma *oexp-sup*:
assumes $\alpha \neq 0$ *Ord* α *Ord* β *Ord* γ **shows** $\alpha\uparrow(\beta \sqcup \gamma) = \alpha\uparrow\beta \sqcup \alpha\uparrow\gamma$
by (*metis Ord-linear-le assms oexp-mono-le sup.absorb2 sup.orderE*)

lemma *oexp-Sup*:
assumes α: $\alpha \neq 0$ *Ord* α **and** X: $X \subseteq ON$ *small* X $X \neq \{\}$ **shows** $\alpha\uparrow\bigsqcup\ X = \bigsqcup\ ((\uparrow)\ \alpha\ `\ X)$
proof (*rule order-antisym*)
show $\bigsqcup\ ((\uparrow)\ \alpha\ `\ X) \leq \alpha\uparrow\bigsqcup\ X$
by (*metis ON-imp-Ord Ord-Sup ZFC-in-HOL.Sup-upper assms cSUP-least*

```
oexp-mono-le)
next
  have Ord (Sup X)
    using Ord-Sup X by auto
  then show α↑⨆ X ≤ ⨆ ((↑) α ‘ X)
  proof (cases rule: Ord-cases)
    case 0
    then show ?thesis
      using X dual-order.antisym by fastforce
  next
    case (succ β)
    then show ?thesis
      using ZFC-in-HOL.Sup-upper X succ-in-Sup-Ord by auto
  next
    case limit
    show ?thesis
    proof (clarsimp simp: assms oexp-Limit limit)
      fix x y z
      assume x: x ∈ elts (α ↑ y) and z ∈ X y ∈ elts z
      then have α ↑ y ≤ α ↑ z
      by (meson ON-imp-Ord Ord-in-Ord OrdmemD α ‹X ⊆ ON› le-less oexp-mono-le)
      with x have x ∈ elts (α ↑ z) by blast
      then show ∃ u∈X. x ∈ elts (α ↑ u)
        using ‹z ∈ X› by blast
    qed
  qed
qed


lemma omega-le-Limit:
  assumes Limit μ shows ω ≤ μ
proof
  fix ϱ
  assume ϱ ∈ elts ω
  then obtain n where ϱ = ord-of-nat n
    using elts-ω by auto
  have ord-of-nat n ∈ elts μ
    by (induction n) (use Limit-def assms in auto)
  then show ϱ ∈ elts μ
    using ‹ϱ = ord-of-nat n› by auto
qed

lemma finite-omega-power [simp]:
  assumes 1 < n n ∈ elts ω shows n↑ω = ω
proof (rule order-antisym)
  have ⨆ ((↑) (ord-of-nat k) ‘ elts ω) ≤ ω for k
  proof (induction k)
    case 0
    then show ?case
```

by *auto*
next
case (*Suc k*)
then show *?case*
by (*metis Ord-ω OrdmemD Sup-eq-0-iff ZFC-in-HOL.SUP-le-iff le-0 le-less omega-closed-oexp ord-of-nat-ω*)
qed
then show $n\uparrow\omega \le \omega$
using *assms*
by (*simp add*: *elts-ω oexp-Limit*) *metis*
show $\omega \le n\uparrow\omega$
using *Ord-in-Ord assms le-oexp′* **by** *blast*
qed

proposition *oexp-add*:
assumes *Ord* α *Ord* β *Ord* γ **shows** $\alpha\uparrow(\beta + \gamma) = \alpha\uparrow\beta * \alpha\uparrow\gamma$
proof (*cases* ‹$\alpha = 0$›)
case *True*
then show *?thesis*
using *assms* **by** *simp*
next
case *False*
show *?thesis*
using ‹*Ord* γ›
proof (*induction rule*: *Ord-induct3*)
case *0*
then show *?case*
by *auto*
next
case (*succ* ξ)
then show *?case*
using ‹*Ord* β› **by** (*auto simp*: *plus-V-succ-right mult.assoc*)
next
case (*Limit* μ)
have $\alpha\uparrow(\beta + (\bigsqcup\xi\in elts\ \mu.\ \xi)) = (\bigsqcup\xi\in elts\ (\beta + \mu).\ \alpha\uparrow\xi)$
by (*simp add*: *Limit.hyps oexp-Limit assms False*)
also have $\ldots = (\bigsqcup\xi \in \{\xi.\ Ord\ \xi \wedge \beta + \xi < \beta + \mu\}.\ \alpha\uparrow(\beta + \xi))$
proof (*rule Sup-eq-Sup*)
show $(\lambda\xi.\ \alpha\uparrow(\beta + \xi))$ ‘ $\{\xi.\ Ord\ \xi \wedge \beta + \xi < \beta + \mu\} \subseteq (\uparrow)\ \alpha$ ‘ $elts\ (\beta + \mu)$
using *Limit.hyps Limit-def Ord-mem-iff-lt imageI* **by** *blast*
fix x
assume $x \in (\uparrow)\ \alpha$ ‘ $elts\ (\beta + \mu)$
then obtain ξ **where** ξ: $\xi \in elts\ (\beta + \mu)$ **and** x: $x = \alpha\uparrow\xi$
by *auto*
have $\exists\gamma.\ Ord\ \gamma \wedge \gamma < \mu \wedge \alpha\uparrow\xi \le \alpha\uparrow(\beta + \gamma)$
proof (*rule mem-plus-V-E* [*OF* ξ])
assume $\xi \in elts\ \beta$
then have $\alpha\uparrow\xi \le \alpha\uparrow\beta$

```
          by (meson arg-subset-TC assms False le-TC-def less-TC-def oexp-mono
vsubsetD)
        with zero-less-Limit [OF ‹Limit μ›]
        show ∃γ. Ord γ ∧ γ < μ ∧ α↑ξ ≤ α↑(β + γ)
          by force
      next
        fix δ
        assume δ ∈ elts μ and ξ = β + δ
        have Ord δ
          using Limit.hyps Limit-def Ord-in-Ord ‹δ ∈ elts μ› by blast
        moreover have δ < μ
          using Limit.hyps Limit-def OrdmemD ‹δ ∈ elts μ› by auto
        ultimately show ∃γ. Ord γ ∧ γ < μ ∧ α↑ξ ≤ α↑(β + γ)
          using ‹ξ = β + δ› by blast
      qed
      then show ∃y∈(λξ. α↑(β + ξ)) ‘ {ξ. Ord ξ ∧ β + ξ < β + μ}. x ≤ y
        using x by auto
    qed auto
    also have … = (⨆ξ∈elts μ. α↑(β + ξ))
      using ‹Limit μ›
      by (simp add: Ord-Collect-lt Limit-def)
    also have … = (⨆ξ∈elts μ. α↑β ∗ α↑ξ)
      using Limit.IH by auto
    also have … = α↑β ∗ α↑(⨆ξ∈elts μ. ξ)
      using ‹α ≠ 0› Limit.hyps
      by (simp add: image-image oexp-Limit mult-Sup-distrib)
    finally show ?case .
  qed
qed

proposition oexp-mult:
  assumes Ord α Ord β Ord γ shows α↑(β ∗ γ) = (α↑β)↑γ
proof (cases α = 0 ∨ β = 0)
  case True
  then show ?thesis
    by (auto simp: ‹Ord β› ‹Ord γ›)
next
  case False
  show ?thesis
    using ‹Ord γ›
  proof (induction rule: Ord-induct3)
    case 0
    then show ?case
      by auto
  next
    case succ
    then show ?case
      using assms by (auto simp: mult-succ oexp-add)
  next
```

case (*Limit μ*)
have *Lim*: *Limit* ($\bigsqcup$ ((∗) *β* ‘ *elts μ*))
unfolding *Limit-def*
proof (*intro conjI allI impI*)
show *Ord* ($\bigsqcup$ ((∗) *β* ‘ *elts μ*))
using *Limit.hyps Limit-def Ord-in-Ord* ‹*Ord β*› **by** (*auto intro*: *Ord-Sup*)
have *succ 0* ∈ *elts μ*
using *Limit.hyps Limit-def* **by** *blast*
then show *0* ∈ *elts* ($\bigsqcup$ ((∗) *β* ‘ *elts μ*))
using *False* ‹*Ord β*› *mem-0-Ord* **by** *force*
show *succ y* ∈ *elts* ($\bigsqcup$ ((∗) *β* ‘ *elts μ*))
if *y* ∈ *elts* ($\bigsqcup$ ((∗) *β* ‘ *elts μ*)) **for** *y*
using *that False Limit.hyps*
apply (*clarsimp simp*: *Limit-def*)
by (*metis Ord-in-Ord Ord-linear Ord-mem-iff-lt Ord-mult Ord-succ assms*(*2*) *less-V-def mult-cancellation mult-succ not-add-mem-right succ-le-iff succ-ne-self*)
qed
have *α*↑(*β* ∗ ($\bigsqcup$*ξ*∈*elts μ*. *ξ*)) = *α*↑$\bigsqcup$ ((∗) *β* ‘ *elts μ*)
by (*simp add*: *mult-Sup-distrib*)
also have . . . = $\bigsqcup$ ($\bigcup$*x*∈*elts μ*. (↑) *α* ‘ *elts* (*β* ∗ *x*))
using *False Lim oexp-Limit* **by** *fastforce*
also have . . . = ($\bigsqcup$*x*∈*elts μ*. *α*↑(*β* ∗ *x*))
proof (*rule Sup-eq-Sup*)
show (*λx*. *α*↑(*β* ∗ *x*)) ‘ *elts μ* ⊆ ($\bigcup$*x*∈*elts μ*. (↑) *α* ‘ *elts* (*β* ∗ *x*))
using ‹*Ord α*› ‹*Ord β*› *False Limit*
apply *clarsimp*
by (*metis Limit-def elts-succ imageI insertI1 mem-0-Ord mult-add-mem-0*)
show ∃ *y*∈(*λx*. *α*↑(*β* ∗ *x*)) ‘ *elts μ*. *x* ≤ *y*
if *x* ∈ ($\bigcup$*x*∈*elts μ*. (↑) *α* ‘ *elts* (*β* ∗ *x*)) **for** *x*
using *that* ‹*Ord α*› ‹*Ord β*› *False Limit*
by *clarsimp* (*metis Limit-def Ord-in-Ord Ord-mult VWO-TC-le mem-imp-VWO oexp-mono*)
qed *auto*
also have . . . = $\bigsqcup$ ((↑) (*α*↑*β*) ‘ *elts* ($\bigsqcup$*ξ*∈*elts μ*. *ξ*))
using *Limit.IH Limit.hyps* **by** *auto*
also have . . . = (*α*↑*β*)↑($\bigsqcup$*ξ*∈*elts μ*. *ξ*)
using *False Limit.hyps oexp-Limit* ‹*Ord β*› **by** *auto*
finally show *?case* **.**
qed
qed

lemma *Limit-omega-oexp*:
assumes *Ord δ δ* ≠ *0*
shows *Limit* (*ω*↑*δ*)
using *assms*
proof (*cases δ rule*: *Ord-cases*)
case *0*
then show *?thesis*
using *assms*(*2*) **by** *blast*

next
  case (*succ l*)
  have $*$: *succ* $\beta \in$ *elts* $(\omega\uparrow l * n + \omega\uparrow l)$
    if *n*: $n \in$ *elts* ω **and** β: $\beta \in$ *elts* $(\omega\uparrow l * n)$ **for** n β
  proof −
    obtain *Ord n Ord* β
      by (*meson Ord-*ω *Ord-in-Ord Ord-mult Ord-oexp* β *n succ*(*1*))
    obtain *oo*: *Ord* $(\omega\uparrow l)$ *Ord* $(\omega\uparrow l * n)$
      by (*simp add*: ‹*Ord n*› *succ*(*1*))
    moreover have *f4*: $\beta < \omega\uparrow l * n$
      using *oo Ord-mem-iff-lt* ‹*Ord* β› ‹$\beta \in$ *elts* $(\omega\uparrow l * n)$› **by** *blast*
    moreover have *f5*: *Ord* (*succ* β)
      using ‹*Ord* β› **by** *blast*
    moreover have $\omega\uparrow l \neq 0$
      using *oexp-eq-0-iff omega-nonzero succ*(*1*) **by** *blast*
    ultimately show *?thesis*
     by (*metis add-less-cancel-left Ord-*ω *Ord-add Ord-mem-iff-lt OrdmemD* ‹*Ord* β›
add.right-neutral dual-order.strict-trans2 oexp-gt-0-iff succ(*1*) *succ-le-iff zero-in-omega*)
  qed
  show *?thesis*
    using *succ*
    apply (*clarsimp simp*: *Limit-def mem-0-Ord*)
    apply (*simp add*: *mult-Limit*)
    by (*metis* $*$ *mult-succ succ-in-omega*)
next
  case *limit*
  then show *?thesis*
   by (*metis Limit-oexp Ord-*ω *OrdmemD one-V-def succ-in-omega zero-in-omega*)
qed

lemma *oexp-mult-commute*:
  fixes *j*::*nat*
  assumes *Ord* α
  shows $(\alpha \uparrow j) * \alpha = \alpha * (\alpha \uparrow j)$
proof −
  have $(\alpha \uparrow j) * \alpha = \alpha \uparrow (1 + \textit{ord-of-nat}\ j)$
    by (*simp add*: *one-V-def*)
  also have ... $= \alpha * (\alpha \uparrow j)$
    by (*simp add*: *assms oexp-add*)
  finally show *?thesis* **.**
qed

lemma *oexp-*ω*-Limit*: *Limit* $\beta \Longrightarrow \omega\uparrow\beta = (\bigsqcup \xi \in \textit{elts}\ \beta.\ \omega\uparrow\xi)$
  by (*simp add*: *oexp-Limit*)

lemma ω*-power-succ-gtr*: *Ord* $\alpha \Longrightarrow \omega \uparrow \alpha * \textit{ord-of-nat}\ n < \omega \uparrow \textit{succ}\ \alpha$
  by (*simp add*: *OrdmemD*)

lemma *countable-oexp*:

```
  assumes ν: α ∈ elts ω1
  shows ω ↑ α ∈ elts ω1
proof −
  have Ord α
    using Ord-ω1 Ord-in-Ord assms by blast
  then show ?thesis
    using assms
  proof (induction rule: Ord-induct3)
    case 0
    then show ?case
      by (simp add: Ord-mem-iff-lt)
  next
    case (succ α)
    then have countable (elts (ω ↑ α ∗ ω))
      by (simp add: succ-in-Limit-iff countable-mult less-ω1-imp-countable)
    then show ?case
      using Ord-mem-iff-lt countable-iff-less-ω1 succ.hyps by auto
  next
    case (Limit α)
    with Ord-ω1 have countable (⋃β∈elts α. elts (ω ↑ β)) Ord (ω ↑ ⨆ (elts α))
      by (force simp: Limit-def intro: Ord-trans less-ω1-imp-countable)+
    then have ω ↑ ⨆ (elts α) < ω1
      using Limit.hyps countable-iff-less-ω1 oexp-Limit by fastforce
    then show ?case
      using Limit.hyps Limit-def Ord-mem-iff-lt by auto
  qed
qed

end
```

5 Cantor Normal Form

```
theory Cantor-NF
  imports Ordinal-Exp
begin
```

5.1 Cantor normal form

Lemma 5.1

```
lemma cnf-1:
  assumes α: α ∈ elts β Ord β and m > 0
  shows ω↑α ∗ ord-of-nat n < ω↑β ∗ ord-of-nat m
proof −
  have †: ω↑succ α ≤ ω↑β
    using Ord-mem-iff-less-TC assms oexp-mono succ-le-TC-iff by auto
  have ω↑α ∗ ord-of-nat n < ω↑α ∗ ω
    using Ord-in-Ord OrdmemD assms by auto
  also have . . . = ω↑succ α
```

using *Ord-in-Ord* α **by** *auto*
also have ... $\leq \omega\uparrow\beta$
using † **by** *blast*
also have ... $\leq \omega\uparrow\beta *$ *ord-of-nat m*
using ‹$m > 0$› *le-mult* **by** *auto*
finally show *?thesis* **.**
qed

fun *Cantor-sum* **where**
Cantor-sum-Nil: *Cantor-sum* [] *ms* = *0*
| *Cantor-sum-Nil2*: *Cantor-sum* ($\alpha\#\alpha s$) [] = *0*
| *Cantor-sum-Cons*: *Cantor-sum* ($\alpha\#\alpha s$) ($m\#ms$) = ($\omega\uparrow\alpha$) $*$ *ord-of-nat m* + *Cantor-sum* αs *ms*

abbreviation *Cantor-dec* :: *V list* $\Rightarrow$ *bool* **where**
Cantor-dec $\equiv$ *sorted-wrt* (>)

lemma *Ord-Cantor-sum*:
assumes *List.set* $\alpha s \subseteq$ *ON*
shows *Ord* (*Cantor-sum* αs *ms*)
using *assms*
proof (*induction* αs *arbitrary*: *ms*)
case (*Cons a* αs *ms*)
then show *?case*
by (*cases ms*) *auto*
qed *auto*

lemma *Cantor-dec-Cons-iff* [*simp*]: *Cantor-dec* ($\alpha\#\beta\#\beta s$) $\longleftrightarrow \beta < \alpha \wedge$ *Cantor-dec* ($\beta\#\beta s$)
by *auto*

Lemma 5.2. The second and third premises aren't really necessary, but their removal requires quite a lot of work.

lemma *cnf-2*:
assumes *List.set* ($\alpha\#\alpha s$) $\subseteq$ *ON list.set ms* $\subseteq$ {*0*<..} *length* αs = *length ms*
and *Cantor-dec* ($\alpha\#\alpha s$)
shows $\omega\uparrow\alpha >$ *Cantor-sum* αs *ms*
using *assms*
proof (*induction ms arbitrary*: α αs)
case *Nil*
then obtain $\alpha 0$ **where** $\alpha 0$: ($\alpha\#\alpha s$) = [$\alpha 0$]
by (*metis length-0-conv*)
then have *Ord* $\alpha 0$
using *Nil.prems*(*1*) **by** *auto*
then show *?case*
using $\alpha 0$ *zero-less-Limit* **by** *auto*
next

case (*Cons m1 ms*)
then obtain $\alpha 0$ $\alpha 1$ $\alpha s'$ **where** $\alpha 01$: $(\alpha\#\alpha s) = \alpha 0\#\alpha 1\#\alpha s'$
by (*metis* (*no-types*, *lifting*) *Cons.prems*(*3*) *Suc-length-conv*)
then have *Ord* $\alpha 0$ *Ord* $\alpha 1$
using *Cons.prems*(*1*) $\alpha 01$ **by** *auto*
have $*$: $\omega\uparrow\alpha 0 * \textit{ord-of-nat } 1 > \omega\uparrow\alpha 1 * \textit{ord-of-nat } m1$
proof (*rule cnf-1*)
show $\alpha 1 \in \textit{elts } \alpha 0$
using *Cons.prems* $\alpha 01$ **by** (*simp add*: *Ord-mem-iff-lt* ‹*Ord* $\alpha 0$› ‹*Ord* $\alpha 1$›)
qed (*use* ‹*Ord* $\alpha 0$› **in** *auto*)
show *?case*
proof (*cases ms*)
case *Nil*
then show *?thesis*
using $*$ *one-V-def Cons.prems*(*3*) $\alpha 01$ **by** *auto*
next
case (*Cons m2 ms'*)
then obtain $\alpha 2$ $\alpha s''$ **where** $\alpha 02$: $(\alpha\#\alpha s) = \alpha 0\#\alpha 1\#\alpha 2\#\alpha s''$
by (*metis Cons.prems*(*3*) *Suc-length-conv* $\alpha 01$ *length-tl list.sel*(*3*))
then have *Ord* $\alpha 2$
using *Cons.prems*(*1*) **by** *auto*
have $m1 > 0$ $m2 > 0$
using *Cons.prems Cons* **by** *auto*
have $\omega\uparrow\alpha 1 * \textit{ord-of-nat } m1 + \omega\uparrow\alpha 1 * \textit{ord-of-nat } m1 = (\omega\uparrow\alpha 1 * \textit{ord-of-nat } m1) * \textit{ord-of-nat } 2$
by (*simp add*: *mult-succ eval-nat-numeral*)
also have $\ldots < \omega\uparrow\alpha 0$
using *cnf-1* [*of concl*: $\alpha 1$ $m1 * 2$ $\alpha 0$ 1] *Cons.prems* $\alpha 01$ *one-V-def*
by (*simp add*: *mult.assoc ord-of-nat-mult Ord-mem-iff-lt*)
finally have *II*: $\omega\uparrow\alpha 1 * \textit{ord-of-nat } m1 + \omega\uparrow\alpha 1 * \textit{ord-of-nat } m1 < \omega\uparrow\alpha 0$
by *simp*
have *Cantor-sum* (*tl* αs) *ms* $< \omega\uparrow$*hd* αs
proof (*rule Cons.IH*)
show *Cantor-dec* (*hd* αs # *tl* αs)
using ‹*Cantor-dec* $(\alpha\#\alpha s)$› $\alpha 01$ **by** *auto*
qed (*use Cons.prems* $\alpha 01$ **in** *auto*)
then have *Cantor-sum* $(\alpha 2\ \#\ \alpha s'')$ *ms* $< \omega\uparrow\alpha 1$
using $\alpha 02$ **by** *auto*
also have $\ldots \le \omega\uparrow\alpha 1 * \textit{ord-of-nat } m1$
by (*simp add*: ‹$0 < m1$› *le-mult*)
finally show *?thesis*
using *II* $\alpha 02$ *dual-order.strict-trans* **by** *fastforce*
qed
qed

proposition *Cantor-nf-exists*:
assumes *Ord* α
obtains αs *ms* **where** *List.set* $\alpha s \subseteq$ *ON list.set ms* $\subseteq \{0<..\}$ *length* αs = *length ms*

and *Cantor-dec* αs
and α = *Cantor-sum* αs *ms*
using *assms*
proof (*induction* α *arbitrary*: *thesis rule*: *Ord-induct*)
case (*step* α)
show *?case*
proof (*cases* $\alpha = 0$)
case *True*
have *Cantor-sum* [] [] = *0*
by *simp*
with *True* **show** *?thesis*
using *length-pos-if-in-set step.prems subset-eq*
by (*metis length-0-conv not-gr-zero sorted-wrt.simps*(*1*))
next
case *False*
define αhat **where** $\alpha hat \equiv$ *Sup* $\{\gamma \in ON.\ \omega\uparrow\gamma \le \alpha\}$
then have *Ord* αhat
using *Ord-Sup assms* **by** *fastforce*
have $\bigwedge\xi.\ [\![\mathit{Ord}\ \xi;\ \omega\uparrow\xi \le \alpha]\!] \Longrightarrow \xi \le \omega\uparrow\alpha$
by (*metis Ord-ω OrdmemD le-oexp′ order-trans step.hyps one-V-def succ-in-omega zero-in-omega*)
then have $\{\gamma \in ON.\ \omega\uparrow\gamma \le \alpha\} \subseteq$ *elts* (*succ* $(\omega\uparrow\alpha)$)
using *Ord-mem-iff-lt step.hyps* **by** *force*
then have *sma*: *small* $\{\gamma \in ON.\ \omega\uparrow\gamma \le \alpha\}$
by (*meson down*)
have *le*: $\omega\uparrow\alpha hat \le \alpha$
proof (*rule ccontr*)
assume $\neg\ \omega\uparrow\alpha hat \le \alpha$
then have †: $\alpha \in$ *elts* $(\omega\uparrow\alpha hat)$
by (*meson Ord-ω Ord-linear2 Ord-mem-iff-lt Ord-oexp* ‹*Ord* αhat› *step.hyps*)
obtain γ **where** *Ord* γ $\omega\uparrow\gamma \le \alpha$ $\alpha < \gamma$
using ‹*Ord* αhat›
proof (*cases* αhat *rule*: *Ord-cases*)
case *0*
with † **show** *thesis*
by (*auto simp*: *False*)
next
case (*succ* β)
have *succ* $\beta \in \{\gamma \in ON.\ \omega\uparrow\gamma \le \alpha\}$
by (*rule succ-in-Sup-Ord*) (*use succ* α*hat-def sma* **in** *auto*)
then have $\omega\uparrow$*succ* $\beta \le \alpha$
by *blast*
with † **show** *thesis*
using ‹$\neg\ \omega\uparrow\alpha hat \le \alpha$› *succ* **by** *blast*
next
case *limit*
with † **show** *thesis*
apply (*clarsimp simp*: *oexp-Limit* α*hat-def*)
by (*meson Ord-ω Ord-in-Ord Ord-linear-le mem-not-refl oexp-mono-le*

omega-nonzero vsubsetD)
qed
then show *False*
by (*metis Ord-ω OrdmemD leD le-less-trans le-oexp$'$ one-V-def succ-in-omega zero-in-omega*)
qed
have *False* **if** $\nexists M.\ \alpha < \omega\uparrow\alpha hat * ord\text{-}of\text{-}nat\ M$
proof $-$
have $\dagger$: $\omega\uparrow\alpha hat * ord\text{-}of\text{-}nat\ M \le \alpha$ **for** M
by (*meson that Ord-ω Ord-linear2 Ord-mult Ord-oexp Ord-ord-of-nat ‹Ord αhat› step.hyps*)
have $\neg\ \omega\uparrow succ\ \alpha hat \le \alpha$
using *Ord-mem-iff-lt αhat-def ‹Ord αhat› sma elts-succ* **by** *blast*
then have $\alpha < \omega\uparrow succ\ \alpha hat$
by (*meson Ord-ω Ord-linear2 Ord-oexp Ord-succ ‹Ord αhat› step.hyps*)
also have $\ldots = \omega\uparrow\alpha hat * \omega$
using *‹Ord αhat› oexp-succ* **by** *blast*
also have $\ldots = Sup\ (range\ (\lambda m.\ \omega\uparrow\alpha hat * ord\text{-}of\text{-}nat\ m))$
by (*simp add*: *mult-Limit*) (*auto simp*: *ω-def image-image*)
also have $\ldots \le \alpha$
using $\dagger$ **by** *blast*
finally show *False*
by *simp*
qed
then obtain M **where** M: $\omega\uparrow\alpha hat * ord\text{-}of\text{-}nat\ M > \alpha$
by *blast*
have *bound*: $i \le M$ **if** $\omega\uparrow\alpha hat * ord\text{-}of\text{-}nat\ i \le \alpha$ **for** i
proof $-$
have $\omega\uparrow\alpha hat * ord\text{-}of\text{-}nat\ i < \omega\uparrow\alpha hat * ord\text{-}of\text{-}nat\ M$
using *M dual-order.strict-trans2 that* **by** *blast*
then show *?thesis*
using *‹Ord αhat› less-V-def* **by** *auto*
qed
define *mhat* **where** $mhat \equiv Greatest\ (\lambda m.\ \omega\uparrow\alpha hat * ord\text{-}of\text{-}nat\ m \le \alpha)$
have *mhat-ge*: $m \le mhat$ **if** $\omega\uparrow\alpha hat * ord\text{-}of\text{-}nat\ m \le \alpha$ **for** m
unfolding *mhat-def*
by (*metis* (*mono-tags*, *lifting*) *Greatest-le-nat bound that*)
have *mhat*: $\omega\uparrow\alpha hat * ord\text{-}of\text{-}nat\ mhat \le \alpha$
unfolding *mhat-def*
by (*rule GreatestI-nat* [**where** $k{=}0$ **and** $b{=}M$]) (*use bound* **in** *auto*)
then obtain ξ **where** $Ord\ \xi\ \xi \le \alpha$ **and** ξ: $\alpha = \omega\uparrow\alpha hat * ord\text{-}of\text{-}nat\ mhat + \xi$
by (*metis Ord-ω Ord-mult Ord-oexp Ord-ord-of-nat ‹Ord αhat› step.hyps le-Ord-diff*)
have *False* **if** $\xi = \alpha$
proof $-$
have $\xi \ge \omega\uparrow\alpha hat$
by (*simp add*: *le that*)
then obtain ζ **where** $Ord\ \zeta\ \zeta \le \xi$ **and** ζ: $\xi = \omega\uparrow\alpha hat + \zeta$
by (*metis Ord-ω Ord-oexp ‹Ord αhat› ‹Ord ξ› le-Ord-diff*)

then have $\alpha = \omega\uparrow\alpha hat * \mathit{ord\text{-}of\text{-}nat}\ mhat + \omega\uparrow\alpha hat + \zeta$
by (*simp add*: ξ *add.assoc*)
then have $\omega\uparrow\alpha hat * \mathit{ord\text{-}of\text{-}nat}\ (\mathit{Suc}\ mhat) \leq \alpha$
by (*metis add-le-cancel-left add.right-neutral le-0 mult-succ ord-of-nat.simps*(*2*))
then show *False*
using *Suc-n-not-le-n mhat-ge* **by** *blast*
qed
then have $\xi in\alpha$: $\xi \in \mathit{elts}\ \alpha$
using *Ord-mem-iff-lt* ‹*Ord* ξ› ‹$\xi \leq \alpha$› *less-V-def step.hyps* **by** *auto*
show *thesis*
proof (*cases* $\xi = 0$)
case *True*
show *thesis*
proof (*rule step.prems*)
show $\alpha = \mathit{Cantor\text{-}sum}\ [\alpha hat]\ [mhat]$
by (*simp add*: *True* ξ)
qed (*use* ξ *True* ‹$\alpha \neq 0$› ‹*Ord* αhat› **in** *auto*)
next
case *False*
obtain βs *ns* **where** *sub*: *List.set* $\beta s \subseteq ON$ *list.set ns* $\subseteq \{0<..\}$
and *len-eq*: *length* βs = *length ns*
and *dec*: *Cantor-dec* βs
and ξeq: $\xi = \mathit{Cantor\text{-}sum}\ \beta s\ ns$
using *step.IH* [*OF* $\xi in\alpha$] **by** *blast*
then have *length* $\beta s > 0$ *length ns* > 0
using *False Cantor-sum.simps*(*1*) ‹$\xi = \mathit{Cantor\text{-}sum}\ \beta s\ ns$› **by** *auto*
then obtain $\beta 0$ *n0* $\beta s'$ *ns'* **where** $\beta 0$: $\beta s = \beta 0\ \#\ \beta s'$ **and** *Ord* $\beta 0$
and *n0*: $ns = n0\ \#\ ns'$ **and** $n0 > 0$
using *sub* **by** (*auto simp*: *neq-Nil-conv*)
moreover have *False* **if** $\beta 0 > \alpha hat$
proof −
have $\omega\uparrow\beta 0 \leq \omega\uparrow\beta 0 * \mathit{ord\text{-}of\text{-}nat}\ n0 + u$ **for** u
using ‹$n0 > 0$›
by (*metis add-le-cancel-left Ord-ord-of-nat add.right-neutral dual-order.trans gr-implies-not-zero le-0 le-mult ord-of-eq-0-iff*)
moreover have $\omega\uparrow\beta 0 > \alpha$
using *that* ‹*Ord* $\beta 0$›
by (*metis* (*no-types*, *lifting*) *Ord-*ω *Ord-linear2 Ord-oexp Sup-upper* α*hat-def leD mem-Collect-eq sma step.hyps*)
ultimately have $\xi \geq \omega\uparrow\beta 0$
by (*simp add*: ξeq $\beta 0$ *n0*)
then show *?thesis*
using ‹$\alpha < \omega\uparrow\beta 0$› ‹$\xi \leq \alpha$› **by** *auto*
qed
ultimately have $\beta 0 \leq \alpha hat$
using *Ord-linear2* ‹*Ord* αhat› **by** *auto*
then consider $\beta 0 < \alpha hat \mid \beta 0 = \alpha hat$
using *dual-order.order-iff-strict* **by** *auto*
then show *?thesis*

proof *cases*
case *1*
show *?thesis*
proof (*rule step.prems*)
show *list.set* (α*hat*#β*s*) $\subseteq$ *ON*
using *sub* **by** (*auto simp*: ‹*Ord* α*hat*›)
show *list.set* (*mhat*#*ns*) $\subseteq$ {*0*::*nat*<..}
using *sub* **using** ‹$\xi = \alpha \Longrightarrow$ *False*› ξ **by** *fastforce*
show *Cantor-dec* (α*hat*#β*s*)
using *that* ‹β*0* < α*hat*› ‹*Ord* α*hat*› ‹*Ord* β*0*› *Ord-mem-iff-lt* β*0* *dec* *less-Suc-eq-0-disj*
by (*force simp*: β*0* *n0*)
show *length* (α*hat*#β*s*) = *length* (*mhat*#*ns*)
by (*auto simp*: *len-eq*)
show α = *Cantor-sum* (α*hat*#β*s*) (*mhat*#*ns*)
by (*simp add*: ξ ξ*eq* β*0* *n0*)
qed
next
case *2*
show *?thesis*
proof (*rule step.prems*)
show *list.set* β*s* $\subseteq$ *ON*
by (*simp add*: *sub*(*1*))
show *list.set* ((*n0*+*mhat*)#*ns*′) $\subseteq$ {*0*::*nat*<..}
using *n0* *sub*(*2*) **by** *auto*
show *length* (β*s*::*V list*) = *length* ((*n0*+*mhat*)#*ns*′)
by (*simp add*: *len-eq n0*)
show *Cantor-dec* β*s*
using *that* β*0* *dec* **by** *auto*
show α = *Cantor-sum* β*s* ((*n0*+*mhat*)#*ns*′)
using *2*
by (*simp add*: *add-mult-distrib* β*0* ξ ξ*eq* *add.assoc* *add.commute* *n0* *ord-of-nat-add*)
qed
qed
qed
qed
qed

lemma *Cantor-sum-0E*:
assumes *Cantor-sum* α*s* *ms* = *0* *List.set* α*s* $\subseteq$ *ON* *list.set* *ms* $\subseteq$ {*0*<..} *length* α*s* = *length* *ms*
shows α*s* = []
using *assms*
proof (*induction* α*s* *arbitrary*: *ms*)
case *Nil*
then show *?case*
by *auto*
next

case (*Cons a list*)
then obtain *m ms′* **where** *ms* = *m#ms′ m* ≠ *0 list.set ms′* ⊆ {*0*<..}
by *simp* (*metis Suc-length-conv greaterThan-iff insert-subset list.set*(*2*))
with *Cons* **show** *?case* **by** *auto*
qed

lemma *Cantor-nf-unique-aux*:
assumes *Ord α*
and *αsON*: *List.set αs* ⊆ *ON*
and *βsON*: *List.set βs* ⊆ *ON*
and *ms*: *list.set ms* ⊆ {*0*<..}
and *ns*: *list.set ns* ⊆ {*0*<..}
and *mseq*: *length αs* = *length ms*
and *nseq*: *length βs* = *length ns*
and *αsdec*: *Cantor-dec αs*
and *βsdec*: *Cantor-dec βs*
and *αseq*: *α* = *Cantor-sum αs ms*
and *βseq*: *α* = *Cantor-sum βs ns*
shows *αs* = *βs* ∧ *ms* = *ns*
using *assms*
proof (*induction α arbitrary*: *αs ms βs ns rule*: *Ord-induct*)
case (*step α*)
show *?case*
proof (*cases α* = *0*)
case *True*
then show *?thesis*
using *step.prems* **by** (*metis Cantor-sum-0E length-0-conv*)
next
case *False*
then obtain *α0 αs′ β0 βs′* **where** *αs*: *αs* = *α0* # *αs′* **and** *βs*: *βs* = *β0* # *βs′*
by (*metis Cantor-sum.simps*(*1*) *min-list.cases step.prems*(*9*,*10*))
then have *ON*: *Ord α0 list.set αs′* ⊆ *ON Ord β0 list.set βs′* ⊆ *ON*
using *αs βs step.prems*(*1*,*2*) **by** *auto*
then obtain *m0 ms′ n0 ns′* **where** *ms*: *ms* = *m0* # *ms′* **and** *ns*: *ns* = *n0* # *ns′*
by (*metis αs βs length-0-conv list.distinct*(*1*) *list.exhaust step.prems*(*5*,*6*))
then have *nz*: *m0* ≠ *0 list.set ms′* ⊆ {*0*<..} *n0* ≠ *0 list.set ns′* ⊆ {*0*<..}
using *ms ns step.prems*(*3*,*4*) **by** *auto*
have *False* **if** *β0* < *α0*
proof −
have *Ordc*: *Ord* (*Cantor-sum βs ns*) *Ord* (*ω*↑*α0*)
using *Ord-oexp* ‹*Ord α0*› *step.hyps step.prems*(*10*) **by** *blast*+
have ∗: *Cantor-sum βs ns* < *ω*↑*α0*
using *step.prems*(*2*−*6*) ‹*Ord α0*› ‹*Cantor-dec βs*› *that βs cnf-2*
by (*metis Cantor-dec-Cons-iff insert-subset list.set*(*2*) *mem-Collect-eq*)
then show *False*
by (*metis Cantor-sum-Cons Ord-mem-iff-lt Ord-ord-of-nat Ordc αs* ‹*m0* ≠

0› ∗ *le-mult ms not-add-mem-right ord-of-eq-0 step.prems*(*9*,*10*) *vsubsetD*)
  qed
  moreover
  have *False* **if** $\alpha 0 < \beta 0$
  proof −
   have *Ordc*: *Ord* (*Cantor-sum* αs *ms*) *Ord* ($\omega\uparrow\beta 0$)
    using *Ord-oexp* ‹*Ord* $\beta 0$› *step.hyps step.prems*(*9*) **by** *blast*+
   have ∗: *Cantor-sum* αs *ms* $< \omega\uparrow\beta 0$
    using *step.prems*(*1*−*5*) ‹*Ord* $\beta 0$› ‹*Cantor-dec* αs› *that* αs *cnf-2*
    by (*metis Cantor-dec-Cons-iff* βs *insert-subset list.set*(*2*))
   then show *False*
    by (*metis Cantor-sum-Cons Ord-mem-iff-lt Ord-ord-of-nat Ordc* βs ‹*n0* ≠
0› ∗ *le-mult not-add-mem-right ns ord-of-eq-0 step.prems*(*9*,*10*) *vsubsetD*)
  qed
  ultimately have *1*: $\alpha 0 = \beta 0$
   using *Ord-linear-lt* ‹*Ord* $\alpha 0$› ‹*Ord* $\beta 0$› **by** *blast*
  have *False* **if** *m0* < *n0*
  proof −
   have $\omega\uparrow\alpha 0 >$ *Cantor-sum* $\alpha s'$ *ms'*
    using αs ‹*list.set ms'* ⊆ {*0*<..}› *cnf-2 ms step.prems*(*1*,*5*,*7*) **by** *auto*
   then have $\alpha < \omega\uparrow\alpha 0$ ∗ *ord-of-nat m0* + $\omega\uparrow\alpha 0$
    by (*simp add*: αs *ms step.prems*(*9*))
   also have ... = $\omega\uparrow\alpha 0$ ∗ *ord-of-nat* (*Suc m0*)
    by (*simp add*: *mult-succ*)
   also have ... ≤ $\omega\uparrow\alpha 0$ ∗ *ord-of-nat n0*
    by (*meson Ord-ω Ord-oexp Ord-ord-of-nat Suc-leI* ‹*Ord* $\alpha 0$› *mult-cancel-le-iff*
ord-of-nat-mono-iff that)
   also have ... ≤ α
    by (*metis Cantor-sum-Cons add-le-cancel-left* βs ‹$\alpha 0 = \beta 0$› *add.right-neutral*
le-0 ns step.prems(*10*))
   finally show *False*
    by *blast*
  qed
  moreover have *False* **if** *n0* < *m0*
  proof −
   have $\omega\uparrow\beta 0 >$ *Cantor-sum* $\beta s'$ *ns'*
    using βs ‹*list.set ns'* ⊆ {*0*<..}› *cnf-2 ns step.prems*(*2*,*6*,*8*) **by** *auto*
   then have $\alpha < \omega\uparrow\beta 0$ ∗ *ord-of-nat n0* + $\omega\uparrow\beta 0$
    by (*simp add*: βs *ns step.prems*(*10*))
   also have ... = $\omega\uparrow\beta 0$ ∗ *ord-of-nat* (*Suc n0*)
    by (*simp add*: *mult-succ*)
   also have ... ≤ $\omega\uparrow\beta 0$ ∗ *ord-of-nat m0*
    by (*meson Ord-ω Ord-oexp Ord-ord-of-nat Suc-leI* ‹*Ord* $\beta 0$› *mult-cancel-le-iff*
ord-of-nat-mono-iff that)
   also have ... ≤ α
    by (*metis Cantor-sum-Cons add-le-cancel-left* αs ‹$\alpha 0 = \beta 0$› *add.right-neutral*
le-0 ms step.prems(*9*))
   finally show *False*
    by *blast*

qed
ultimately have *2*: *m0 = n0*
using *nat-neq-iff* **by** *blast*
have $\alpha s' = \beta s' \wedge ms' = ns'$
proof (*rule step.IH*)
have *Cantor-sum* $\alpha s'$ *ms'* $< \omega\uparrow\alpha 0$
using αs *cnf-2 ms nz*(*2*) *step.prems*(*1*) *step.prems*(*5*) *step.prems*(*7*) **by** *auto*
also have ... $\leq$ *Cantor-sum* αs *ms*
apply (*simp add*: αs βs *ms ns*)
by (*metis Cantor-sum-Cons add-less-cancel-left ON*(*1*) *Ord-*ω *Ord-linear2 Ord-oexp Ord-ord-of-nat* αs *add.right-neutral dual-order.strict-trans1 le-mult ms not-less-0 nz*(*1*) *ord-of-eq-0 step.hyps step.prems*(*9*))
finally show *Cantor-sum* $\alpha s'$ *ms'* $\in$ *elts* α
using *ON*(*2*) *Ord-Cantor-sum Ord-mem-iff-lt step.hyps step.prems*(*9*) **by** *blast*
show *length* $\alpha s'$ = *length ms' length* $\beta s'$ = *length ns'*
using αs *ms* βs *ns step.prems* **by** *auto*
show *Cantor-dec* $\alpha s'$ *Cantor-dec* $\beta s'$
using αs βs *step.prems*(*7*,*8*) **by** *auto*
have *Cantor-sum* αs *ms* = *Cantor-sum* βs *ns*
using *step.prems*(*9*,*10*) **by** *auto*
then show *Cantor-sum* $\alpha s'$ *ms'* = *Cantor-sum* $\beta s'$ *ns'*
using *1 2* **by** (*simp add*: αs βs *ms ns*)
qed (*use ON nz* **in** *auto*)
then show *?thesis*
using *1 2* **by** (*simp add*: αs βs *ms ns*)
qed
qed

proposition *Cantor-nf-unique*:
assumes *Cantor-sum* αs *ms* = *Cantor-sum* βs *ns*
and αsON: *List.set* $\alpha s \subseteq ON$
and βsON: *List.set* $\beta s \subseteq ON$
and *ms*: *list.set ms* $\subseteq$ {*0*<..}
and *ns*: *list.set ns* $\subseteq$ {*0*<..}
and *mseq*: *length* αs = *length ms*
and *nseq*: *length* βs = *length ns*
and $\alpha sdec$: *Cantor-dec* αs
and $\beta sdec$: *Cantor-dec* βs
shows $\alpha s = \beta s \wedge ms = ns$
using *Cantor-nf-unique-aux Ord-Cantor-sum assms* **by** *auto*

lemma *less-*ω*-power*:
assumes *Ord* $\alpha 1$ *Ord* β
and $\alpha 2$: $\alpha 2 \in$ *elts* $\alpha 1$ **and** β: $\beta < \omega\uparrow\alpha 2$
and *m1* > *0 m2* > *0*
shows $\omega\uparrow\alpha 2 *$ *ord-of-nat m2* $+ \beta < \omega\uparrow\alpha 1 *$ *ord-of-nat m1* + ($\omega\uparrow\alpha 2 *$ *ord-of-nat*

m2 + β)
(**is** *?lhs* < *?rhs*)
proof −
obtain *oo*: *Ord* ($\omega\uparrow\alpha 1$) *Ord* ($\omega\uparrow\alpha 2$)
using *Ord-in-Ord Ord-oexp assms* **by** *blast*
moreover obtain *ord-of-nat m2* $\neq$ *0*
using *assms ord-of-eq-0* **by** *blast*
ultimately have $\beta < \omega\uparrow\alpha 2 *$ *ord-of-nat m2*
by (*meson Ord-ord-of-nat* β *dual-order.strict-trans1 le-mult*)
with *oo assms* **have** *?lhs* $\neq$ *?rhs*
by (*metis Ord-mult Ord-ord-of-nat add-strict-mono add.assoc cnf-1 not-add-less-right oo*)
then show *?thesis*
by (*simp add*: *add-le-left* ‹*Ord* β› *less-V-def oo*)
qed

lemma *Cantor-sum-ge*:
assumes *List.set* ($\alpha\#\alpha s$) $\subseteq$ *ON list.set ms* $\subseteq$ {*0*<..} *length ms* > *0*
shows $\omega \uparrow \alpha \leq$ *Cantor-sum* ($\alpha\#\alpha s$) *ms*
proof −
obtain *m ns* **where** *ms*: *ms* = *Cons m ns*
by (*meson assms*(*3*) *list.set-cases nth-mem*)
then have $\omega \uparrow \alpha \leq \omega \uparrow \alpha *$ *ord-of-nat m*
using *assms*(*2*) *le-mult* **by** *auto*
then show *?thesis*
using *dual-order.trans ms* **by** *auto*
qed

5.2 Simplified Cantor normal form

No coefficients, and the exponents decreasing non-strictly

fun ω-*sum* **where**
ω-*sum-Nil*: ω-*sum* [] = *0*
| ω-*sum-Cons*: ω-*sum* ($\alpha\#\alpha s$) = ($\omega\uparrow\alpha$) + ω-*sum* αs

abbreviation ω-*dec* :: *V list* ⇒ *bool* **where**
ω-*dec* ≡ *sorted-wrt* ($\geq$)

lemma *Ord-*ω*-sum* [*simp*]: *List.set* $\alpha s \subseteq$ *ON* ⟹ *Ord* (ω-*sum* αs)
by (*induction* αs) *auto*

lemma ω-*dec-Cons-iff* [*simp*]: ω-*dec* ($\alpha\#\beta\#\beta s$) ⟷ $\beta \leq \alpha \wedge$ ω-*dec* ($\beta\#\beta s$)
by *auto*

lemma ω-*sum-0E*:
assumes ω-*sum* αs = *0 List.set* $\alpha s \subseteq$ *ON*
shows αs = []
using *assms*
by (*induction* αs) *auto*

fun *ω-of-Cantor* **where**
ω-of-Cantor-Nil: *ω-of-Cantor* [] *ms* = []
| *ω-of-Cantor-Nil2*: *ω-of-Cantor* (*α#αs*) [] = []
| *ω-of-Cantor-Cons*: *ω-of-Cantor* (*α#αs*) (*m#ms*) = *replicate m α* @ *ω-of-Cantor αs ms*

lemma *ω-sum-append* [*simp*]: *ω-sum* (*xs* @ *ys*) = *ω-sum xs* + *ω-sum ys*
by (*induction xs*) (*auto simp*: *add.assoc*)

lemma *ω-sum-replicate* [*simp*]: *ω-sum* (*replicate m a*) = *ω* ↑ *a* ∗ *ord-of-nat m*
by (*induction m*) (*auto simp*: *mult-succ simp flip*: *replicate-append-same*)

lemma *ω-sum-of-Cantor* [*simp*]: *ω-sum* (*ω-of-Cantor αs ms*) = *Cantor-sum αs ms*
proof (*induction αs arbitrary*: *ms*)
case (*Cons a αs ms*)
then show *?case*
by (*cases ms*) *auto*
qed *auto*

lemma *ω-of-Cantor-subset*: *List.set* (*ω-of-Cantor αs ms*) ⊆ *List.set αs*
proof (*induction αs arbitrary*: *ms*)
case (*Cons a αs ms*)
then show *?case*
by (*cases ms*) *auto*
qed *auto*

lemma *ω-dec-replicate*: *ω-dec* (*replicate m α* @ *αs*) = (*if m=0 then ω-dec αs else ω-dec* (*α#αs*))
by (*induction m arbitrary*: *αs*) (*simp-all flip*: *replicate-append-same*)

lemma *ω-dec-of-Cantor-aux*:
assumes *Cantor-dec* (*α#αs*) *length αs* = *length ms*
shows *ω-dec* (*ω-of-Cantor* (*α#αs*) (*m#ms*))
using *assms*
proof (*induction αs arbitrary*: *ms*)
case *Nil*
then show *?case*
using *sorted-wrt-iff-nth-less* **by** *fastforce*
next
case (*Cons a αs ms*)
then obtain *n ns* **where** *ns*: *ms* = *n#ns*
by (*metis length-Suc-conv*)
then have *a* ≤ *α*
using *Cons.prems*(*1*) *order.strict-implies-order* **by** *auto*

moreover have $\forall x{\in}list.set\ (\omega\text{-}of\text{-}Cantor\ \alpha s\ ns).\ x \le a$
using *Cons ns* ‹$a \le \alpha$›
apply (*simp add*: *ω-dec-replicate*)
by (*meson ω-of-Cantor-subset order.strict-implies-order subsetD*)
ultimately show *?case*
using *Cons ns* **by** (*force simp*: *ω-dec-replicate*)
qed

lemma *ω-dec-of-Cantor*:
assumes *Cantor-dec αs length αs = length ms*
shows *ω-dec* (*ω-of-Cantor αs ms*)
proof (*cases αs*)
case *Nil*
then have *ms* = []
using *assms* **by** *auto*
with *Nil* **show** *?thesis*
by *simp*
next
case (*Cons a list*)
then show *?thesis*
by (*metis ω-dec-of-Cantor-aux assms length-Suc-conv*)
qed

proposition *ω-nf-exists*:
assumes *Ord α*
obtains *αs* **where** *List.set αs* ⊆ *ON* **and** *ω-dec αs* **and** *α = ω-sum αs*
proof −
obtain *αs ms* **where** *List.set αs* ⊆ *ON list.set ms* ⊆ {*0<..*} **and** *length*: *length αs = length ms*
and *Cantor-dec αs*
and *α*: *α = Cantor-sum αs ms*
using *Cantor-nf-exists assms* **by** *blast*
then show *thesis*
by (*metis ω-dec-of-Cantor ω-of-Cantor-subset ω-sum-of-Cantor order-trans that*)
qed

lemma *ω-sum-take-drop*: *ω-sum αs = ω-sum* (*take k αs*) + *ω-sum* (*drop k αs*)
proof (*induction k arbitrary*: *αs*)
case *0*
then show *?case*
by *simp*
next
case (*Suc k*)
then show *?case*
proof (*cases αs*)
case *Nil*
then show *?thesis*
by *simp*

next
case (*Cons a list*)
with *Suc.prems* **show** *?thesis*
by (*simp add*: *add.assoc flip*: *Suc.IH*)
qed
qed

lemma *in-elts-ω-sum*:
assumes $\delta \in$ *elts* (*ω-sum αs*)
shows $\exists\, k<$*length αs*. $\exists \gamma\in$*elts* ($\omega \uparrow (\alpha s!k)$). $\delta =$ *ω-sum* (*take k αs*) $+ \gamma$
using *assms*
proof (*induction αs arbitrary*: δ)
case (*Cons α αs*)
then have $\delta \in$ *elts* ($\omega \uparrow \alpha +$ *ω-sum αs*)
by *simp*
then show *?case*
proof (*rule mem-plus-V-E*)
fix η
assume η: $\eta \in$ *elts* (*ω-sum αs*) **and** δ: $\delta = \omega \uparrow \alpha + \eta$
then obtain $k\ \gamma$ **where** $k<$*length αs* $\gamma \in$ *elts* ($\omega \uparrow (\alpha s!k)$) $\eta =$ *ω-sum* (*take k αs*) $+ \gamma$
using *Cons.IH* **by** *blast*
then show *?case*
by (*rule-tac x=Suc k* **in** *exI*) (*simp add*: δ *add.assoc*)
qed *auto*
qed *auto*

lemma *ω-le-ω-sum*: ⟦$k <$ *length αs*; *List.set αs* $\subseteq$ *ON*⟧ $\Longrightarrow \omega \uparrow (\alpha s!k) \le$ *ω-sum αs*
proof (*induction αs arbitrary*: *k*)
case (*Cons a αs*)
then obtain *Ord a list.set αs* $\subseteq$ *ON*
by *simp*
with *Cons.IH* **have** $\bigwedge k\ x.\ k <$ *length αs* $\Longrightarrow \omega \uparrow \alpha s\ !\ k \le \omega \uparrow a +$ *ω-sum αs*
by (*meson Ord-ω Ord-ω-sum Ord-oexp add-le-left order-trans*)
then show *?case*
using *Cons* **by** (*simp add*: *nth-Cons split*: *nat.split*)
qed *auto*

lemma *ω-sum-less-self*:
assumes *List.set* (α#αs) $\subseteq$ *ON* **and** *ω-dec* (α#αs)
shows *ω-sum αs* $< \omega\uparrow\alpha +$ *ω-sum αs*
using *assms*
proof (*induction αs arbitrary*: α)
case *Nil*
then show *?case*
using *ZFC-in-HOL.neq0-conv* **by** *fastforce*
next
case (*Cons α1 αs*)

then show *?case*
by (*simp add*: *add-right-strict-mono oexp-mono-le*)
qed

Something like Lemma 5.2 for *ω-sum*

lemma *ω-sum-less-ω-power*:
assumes *ω-dec* $(\alpha\#\alpha s)$ *List.set* $(\alpha\#\alpha s) \subseteq ON$
shows *ω-sum* $\alpha s < \omega\uparrow\alpha * \omega$
using *assms*
proof (*induction* αs)
case *Nil*
then show *?case*
by (*simp add*: *ω-gt0*)
next
case (*Cons* β αs)
then have *Ord* α
by *auto*
have *ω-sum* $\alpha s < \omega\uparrow\alpha * \omega$
using *Cons* **by** *force*
then have $\omega\uparrow\beta$ + *ω-sum* $\alpha s < \omega\uparrow\alpha + \omega\uparrow\alpha * \omega$
using *Cons.prems add-right-strict-mono oexp-mono-le* **by** *auto*
also have $\ldots = \omega\uparrow\alpha * \omega$
by (*metis Kirby.add-mult-distrib mult.right-neutral one-plus-ω-equals-ω*)
finally show *?case*
by *simp*
qed

lemma *ω-sum-nf-unique-aux*:
assumes *Ord* α
and αsON: *List.set* $\alpha s \subseteq ON$
and βsON: *List.set* $\beta s \subseteq ON$
and $\alpha sdec$: *ω-dec* αs
and $\beta sdec$: *ω-dec* βs
and αseq: $\alpha =$ *ω-sum* αs
and βseq: $\alpha =$ *ω-sum* βs
shows $\alpha s = \beta s$
using *assms*
proof (*induction* α *arbitrary*: αs βs *rule*: *Ord-induct*)
case (*step* α)
show *?case*
proof (*cases* $\alpha = 0$)
case *True*
then show *?thesis*
using *step.prems* **by** (*metis ω-sum-0E*)
next
case *False*
then obtain $\alpha 0$ $\alpha s'$ $\beta 0$ $\beta s'$ **where** αs: $\alpha s = \alpha 0 \# \alpha s'$ **and** βs: $\beta s = \beta 0 \#$
$\beta s'$

```
      by (metis ω-sum.elims step.prems(5,6))
    then have ON: Ord α0 list.set αs' ⊆ ON Ord β0 list.set βs' ⊆ ON
      using αs βs step.prems(1,2) by auto
    have False if β0 < α0
    proof −
      have Ordc: Ord (ω-sum βs) Ord (ω↑α0)
        using Ord-oexp ‹Ord α0›  step.hyps step.prems(6) by blast+
      have ω-sum βs < ω↑β0 ∗ ω
        by (rule ω-sum-less-ω-power) (use βs step.prems ON in auto)
      also have . . . ≤ ω↑α0
       using ON by (metis Ord-ω Ord-succ oexp-mono-le oexp-succ omega-nonzero
succ-le-iff that)
      finally show False
        using αs leD step.prems(5,6) by auto
    qed
    moreover
    have False if α0 < β0
    proof −
      have Ordc: Ord (ω-sum αs) Ord (ω↑β0)
        using Ord-oexp ‹Ord β0›  step.hyps step.prems(5) by blast+
      have ω-sum αs < ω↑α0 ∗ ω
        by (rule ω-sum-less-ω-power) (use αs step.prems ON in auto)
      also have . . . ≤ ω↑β0
       using ON by (metis Ord-ω Ord-succ oexp-mono-le oexp-succ omega-nonzero
succ-le-iff that)
      finally show False
        using βs leD step.prems(5,6)
        by (simp add: ‹α = ω-sum αs› leD)
    qed
    ultimately have †: α0 = β0
      using Ord-linear-lt ‹Ord α0› ‹Ord β0› by blast
    moreover have αs' = βs'
    proof (rule step.IH)
      show ω-sum αs' ∈ elts α
        using step.prems αs
        by (simp add: Ord-mem-iff-lt ω-sum-less-self)
      show ω-dec αs' ω-dec βs'
        using αs βs step.prems(3,4) by auto
      have ω-sum αs = ω-sum βs
        using step.prems(5,6) by auto
      then show ω-sum αs' = ω-sum βs'
        by (simp add: † αs βs)
    qed (use ON in auto)
    ultimately show ?thesis
      by (simp add: αs βs)
  qed
qed
```

5.3 Indecomposable ordinals

Cf exercise 5 on page 43 of Kunen

definition *indecomposable*
  where *indecomposable* $\alpha \equiv$ *Ord* $\alpha \wedge (\forall \beta \in$ *elts* $\alpha. \forall \gamma \in$ *elts* $\alpha. \beta+\gamma \in$ *elts* $\alpha)$

lemma *indecomposableD*:
  $[\![$*indecomposable* α; $\beta < \alpha$; $\gamma < \alpha$; *Ord* β; *Ord* $\gamma]\!] \Longrightarrow \beta+\gamma < \alpha$
  by (*meson Ord-mem-iff-lt OrdmemD indecomposable-def*)

lemma *indecomposable-imp-Ord*:
  indecomposable $\alpha \Longrightarrow$ *Ord* α
  using *indecomposable-def* **by** *blast*

lemma *indecomposable-1*: *indecomposable 1*
  by (*auto simp*: *indecomposable-def*)

lemma *indecomposable-0*: *indecomposable 0*
  by (*auto simp*: *indecomposable-def*)

lemma *indecomposable-succ* [*simp*]: *indecomposable* (*succ* α) $\longleftrightarrow \alpha = 0$
  using *not-add-mem-right*
  apply (*auto simp*: *indecomposable-def*)
  apply (*metis add-right-cancel add.right-neutral*)
  done

lemma *indecomposable-alt*:
  assumes *ord*: *Ord* α *Ord* β **and** β: $\beta < \alpha$ **and** *minor*: $\bigwedge\beta\ \gamma.\ [\![\beta < \alpha; \gamma < \alpha;$
Ord $\gamma]\!] \Longrightarrow \beta+\gamma < \alpha$
  shows $\beta+\alpha = \alpha$
proof −
  have $\neg\ \beta+\alpha < \alpha$
    by (*simp add*: *add-le-left ord leD*)
  moreover have $\neg\ \alpha < \beta+\alpha$
    by (*metis assms le-Ord-diff less-V-def*)
  ultimately show *?thesis*
    by (*simp add*: *add-le-left less-V-def ord*)
qed

lemma *indecomposable-imp-eq*:
  assumes *indecomposable* α *Ord* β $\beta < \alpha$
  shows $\beta+\alpha = \alpha$
  by (*metis assms indecomposableD indecomposable-def le-Ord-diff less-V-def less-irrefl*)

lemma *indecomposable2*:
  assumes *y*: $y < x$ **and** *z*: $z < x$ **and** *minor*: $\bigwedge y{::}V.\ y < x \Longrightarrow y+x = x$
  shows $y+z < x$
  by (*metis add-less-cancel-left y z minor*)

lemma *indecomposable-imp-Limit*:
  assumes *indec*: *indecomposable* α **and** $\alpha > 1$
  shows *Limit* α
  using *indecomposable-imp-Ord* [*OF indec*]
proof (*cases rule*: *Ord-cases*)
  case (*succ* β)
  then show *?thesis*
    using *assms one-V-def* **by** *auto*
qed (*use assms* **in** *auto*)

lemma *eq-imp-indecomposable*:
  assumes *Ord* α $\bigwedge\beta{::}V.\ \beta \in elts\ \alpha \Longrightarrow \beta{+}\alpha = \alpha$
  shows *indecomposable* α
  by (*metis add-mem-right-cancel assms indecomposable-def*)

lemma *indecomposable-ω-power*:
  assumes *Ord* δ
  shows *indecomposable* $(\omega\uparrow\delta)$
  unfolding *indecomposable-def*
proof (*intro conjI ballI*)
  show *Ord* $(\omega\uparrow\delta)$
    by (*simp add*: ‹*Ord* δ›)
next
  fix β γ
  assume *asm*: $\beta \in elts\ (\omega\uparrow\delta)$ $\gamma \in elts\ (\omega\uparrow\delta)$
  then obtain *ord*: *Ord* β *Ord* γ **and** β: $\beta < \omega\uparrow\delta$ **and** γ: $\gamma < \omega\uparrow\delta$
    by (*meson Ord-ω Ord-in-Ord Ord-oexp OrdmemD* ‹*Ord* δ›)
  show $\beta + \gamma \in elts\ (\omega\uparrow\delta)$
    using ‹*Ord* δ›
  proof (*cases* δ *rule*: *Ord-cases*)
    case *0*
    then show *?thesis*
      using ‹*Ord* δ› *asm* **by** *auto*
  next
    case (*succ* l)
    have $\exists x{\in}elts\ \omega.\ \beta + \gamma \in elts\ (\omega\uparrow l * x)$
      if x: $x \in elts\ \omega$ $\beta \in elts\ (\omega\uparrow l * x)$ **and** y: $y \in elts\ \omega$ $\gamma \in elts\ (\omega\uparrow l * y)$
      for x y
    proof −
      obtain *Ord* x *Ord* y *Ord* $(\omega\uparrow l * x)$ *Ord* $(\omega\uparrow l * y)$
        using *Ord-ω Ord-mult Ord-oexp x y nat-into-Ord succ*(*1*) **by** *presburger*
      then have $\beta + \gamma \in elts\ (\omega\uparrow l * (x{+}y))$
        using *add-mult-distrib Ord-add Ord-mem-iff-lt add-strict-mono ord x y* **by**
presburger
      then show *?thesis*
        using *x y* **by** *blast*
    qed
    then show *?thesis*
    using ‹*Ord* δ› *succ ord* β γ **by** (*auto simp*: *mult-Limit simp flip*: *Ord-mem-iff-lt*)

next
case *limit*
have *Ord* $(\omega\uparrow\delta)$
by (*simp add*: ‹*Ord* δ›)
then obtain $x\ y$ **where** x: $x \in elts\ \delta$ *Ord* x $\beta \in elts\ (\omega\uparrow x)$
and y: $y \in elts\ \delta$ *Ord* y $\gamma \in elts\ (\omega\uparrow y)$
using ‹*Ord* δ› *limit ord* β γ *oexp-Limit*
by (*auto simp flip*: *Ord-mem-iff-lt intro*: *Ord-in-Ord*)
then have *succ* $(x \sqcup y) \in elts\ \delta$
by (*metis Limit-def Ord-linear-le limit sup.absorb2 sup.orderE*)
moreover have $\beta + \gamma \in elts\ (\omega\uparrow succ\ (x \sqcup y))$
proof $-$
have *oxy*: *Ord* $(x \sqcup y)$
using *Ord-sup* $x\ y$ **by** *blast*
then obtain $\omega\uparrow x \le \omega\uparrow(x \sqcup y)$ $\omega\uparrow y \le \omega\uparrow(x \sqcup y)$
by (*metis Ord-ω Ord-linear-le Ord-mem-iff-less-TC Ord-mem-iff-lt le-TC-def less-le-not-le oexp-mono omega-nonzero sup.absorb2 sup.orderE* $x(2)$ $y(2)$)
then have $\beta \in elts\ (\omega\uparrow(x \sqcup y))$ $\gamma \in elts\ (\omega\uparrow(x \sqcup y))$
using $x\ y$ **by** *blast+*
then have $\beta + \gamma \in elts\ (\omega\uparrow(x \sqcup y) * succ\ (succ\ 0))$
by (*metis Ord-ω Ord-add Ord-mem-iff-lt Ord-oexp Ord-sup add-strict-mono mult.right-neutral mult-succ ord one-V-def* $x(2)$ $y(2)$)
then show *?thesis*
apply (*simp add*: *oxy*)
using *Ord-ω Ord-mult Ord-oexp Ord-trans mem-0-Ord mult-add-mem-0 oexp-eq-0-iff omega-nonzero oxy succ-in-omega* **by** *presburger*
qed
ultimately show *?thesis*
using *ord* ‹*Ord* $(\omega\uparrow\delta)$› *limit oexp-Limit* **by** *auto*
qed
qed

lemma *ω-power-imp-eq*:
assumes $\beta < \omega\uparrow\delta$ *Ord* β *Ord* δ $\delta \neq 0$
shows $\beta + \omega\uparrow\delta = \omega\uparrow\delta$
by (*simp add*: *assms indecomposable-ω-power indecomposable-imp-eq*)

lemma *mult-oexp-indec*: $[\![Ord\ \alpha; Limit\ \mu; indecomposable\ \mu]\!] \Longrightarrow \alpha * (\alpha \uparrow \mu) = (\alpha \uparrow \mu)$
by (*metis Limit-def Ord-1 OrdmemD indecomposable-imp-eq oexp-1-right oexp-add one-V-def*)

lemma *mult-oexp-ω*: *Ord* $\alpha \Longrightarrow \alpha * (\alpha \uparrow \omega) = (\alpha \uparrow \omega)$
by (*metis Ord-1 Ord-ω oexp-1-right oexp-add one-plus-ω-equals-ω*)

lemma *type-imp-indecomposable*:
assumes α: *Ord* α
and *minor*: $\bigwedge X.\ X \subseteq elts\ \alpha \Longrightarrow ordertype\ X\ VWF = \alpha \vee ordertype\ (elts\ \alpha - X)\ VWF = \alpha$

shows *indecomposable* α
unfolding *indecomposable-def*
proof (*intro conjI ballI*)
fix β γ
assume β: $\beta \in$ *elts* α **and** γ: $\gamma \in$ *elts* α
then obtain $\beta\gamma$: *elts* $\beta \subseteq$ *elts* α *elts* $\gamma \subseteq$ *elts* α *Ord* β *Ord* γ
using α *Ord-in-Ord Ord-trans* **by** *blast*
then have *oeq*: *ordertype* (*elts* β) *VWF* $= \beta$
by *auto*
show $\beta + \gamma \in$ *elts* α
proof (*rule ccontr*)
assume $\beta + \gamma \notin$ *elts* α
then obtain δ **where** δ: *Ord* δ $\beta + \delta = \alpha$
by (*metis Ord-ordertype* $\beta\gamma(1)$ *le-Ord-diff less-eq-V-def minor oeq*)
then have $\delta \in$ *elts* α
using *Ord-linear* $\beta\gamma$ γ ‹$\beta + \gamma \notin$ *elts* α› **by** *blast*
then have *ordertype* (*elts* α $-$ *elts* β) *VWF* $= \delta$
using δ *ordertype-diff Limit-def* α ‹*Ord* β› **by** *blast*
then show *False*
by (*metis* β ‹$\delta \in$ *elts* α› ‹*elts* $\beta \subseteq$ *elts* α› *oeq mem-not-refl minor*)
qed
qed (*use assms* **in** *auto*)

This proof uses Cantor normal form, yet still is rather long

proposition *indecomposable-is-*ω*-power*:
assumes *inc*: *indecomposable* μ
obtains $\mu = 0$ | δ **where** *Ord* δ $\mu = \omega\uparrow\delta$
proof (*cases* $\mu = 0$)
case *True*
then show *thesis*
by (*simp add*: *that*)
next
case *False*
obtain *Ord* μ
using *Limit-def assms indecomposable-def* **by** *blast*
then obtain αs *ms* **where** *Cantor*: *List.set* $\alpha s \subseteq$ *ON list.set ms* $\subseteq \{0<..\}$
length αs = *length ms Cantor-dec* αs
and μ: μ = *Cantor-sum* αs *ms*
using *Cantor-nf-exists* **by** *blast*
consider (*0*) *length* $\alpha s = 0$ | (*1*) *length* $\alpha s = 1$ | (*2*) *length* $\alpha s \geq 2$
by *linarith*
then show *thesis*
proof *cases*
case *0*
then show *?thesis*
using μ *assms False indecomposable-def* **by** *auto*
next
case *1*
then obtain α m **where** αm: $\alpha s = [\alpha]$ $ms = [m]$

by (*metis One-nat-def* ‹*length* αs = *length ms*› *length-0-conv length-Suc-conv*)
then obtain *Ord* α $m \neq 0$ *Ord* $(\omega\uparrow\alpha)$
using ‹*list.set* $\alpha s \subseteq ON$› ‹*list.set ms* $\subseteq \{0<..\}$› **by** *auto*
have μ: $\mu = \omega\uparrow\alpha * $ *ord-of-nat m*
using αm **by** (*simp add*: μ)
moreover have $m = 1$
proof (*rule ccontr*)
assume $m \neq 1$
then have *2*: $m \geq 2$
using ‹$m \neq 0$› **by** *linarith*
then have $m =$ *Suc 0* + *Suc 0* + $(m-2)$
by *simp*
then have *ord-of-nat m* = *1* + *1* + *ord-of-nat* $(m-2)$
by (*metis add.left-neutral mult.left-neutral mult-succ ord-of-nat.simps ord-of-nat-add*)
then have μeq: $\mu = \omega\uparrow\alpha + \omega\uparrow\alpha + \omega\uparrow\alpha *$ *ord-of-nat* $(m-2)$
using μ **by** (*simp add*: *add-mult-distrib*)
moreover have *less*: $\omega\uparrow\alpha < \mu$
by (*metis Ord-*ω *OrdmemD* μeq ‹*Ord* α› *add-le-cancel-left0 add-less-cancel-left0 le-less-trans less-V-def oexp-gt-0-iff zero-in-omega*)
moreover have $\omega\uparrow\alpha + \omega\uparrow\alpha *$ *ord-of-nat* $(m-2) < \mu$
using *2* μ ‹*Ord* α› *assms less indecomposableD less-V-def* **by** *auto*
ultimately show *False*
using *indecomposableD* [*OF inc*, *of* $\omega\uparrow\alpha$ $\omega\uparrow\alpha + \omega\uparrow\alpha *$ *ord-of-nat* $(m-2)$]
by (*simp add*: ‹*Ord* $(\omega\uparrow\alpha)$› *add.assoc*)
qed
moreover have *Ord* α
using ‹*List.set* $\alpha s \subseteq ON$› **by** (*simp add*: ‹$\alpha s = [\alpha]$›)
ultimately show *?thesis*
by (*metis One-nat-def mult.right-neutral ord-of-nat.simps one-V-def that*(*2*))
next
case *2*
then obtain $\alpha 1$ $\alpha 2$ $\alpha s'$ *m1 m2 ms'* **where** αm: $\alpha s = \alpha 1\#\alpha 2\#\alpha s'$ *ms* = *m1#m2#ms'*
by (*metis Cantor*(*3*) *One-nat-def Suc-1 impossible-Cons length-Cons list.size*(*3*) *not-numeral-le-zero remdups-adj.cases*)
then obtain *Ord* $\alpha 1$ *Ord* $\alpha 2$ *m1* $\neq 0$ *m2* $\neq 0$ *Ord* $(\omega\uparrow\alpha 1)$ *Ord* $(\omega\uparrow\alpha 2)$
list.set $\alpha s' \subseteq ON$ *list.set ms'* $\subseteq \{0<..\}$
using ‹*list.set* $\alpha s \subseteq ON$› ‹*list.set ms* $\subseteq \{0<..\}$› **by** *auto*
have *oCs*: *Ord* (*Cantor-sum* $\alpha s'$ *ms'*)
by (*simp add*: *Ord-Cantor-sum* ‹*list.set* $\alpha s' \subseteq ON$›)
have $\alpha 21$: $\alpha 2 \in$ *elts* $\alpha 1$
using *Cantor-dec-Cons-iff* αm(*1*) ‹*Cantor-dec* αs›
by (*simp add*: *Ord-mem-iff-lt* ‹*Ord* $\alpha 1$› ‹*Ord* $\alpha 2$›)
have $\omega\uparrow\alpha 2 \neq 0$
by (*simp add*: ‹*Ord* $\alpha 2$›)
then have $*$: $(\omega\uparrow\alpha 2 *$ *ord-of-nat m2* + *Cantor-sum* $\alpha s'$ *ms'*$) > 0$
by (*simp add*: *OrdmemD* ‹*Ord* $(\omega\uparrow\alpha 2)$› ‹*m2* $\neq 0$› *mem-0-Ord oCs*)
have μ: $\mu = \omega\uparrow\alpha 1 *$ *ord-of-nat m1* + $(\omega\uparrow\alpha 2 *$ *ord-of-nat m2* + *Cantor-sum*

$\alpha s'$ ms')
  (**is** $\mu = ?\alpha + ?\beta$)
  using αm **by** (*simp add*: μ)
 moreover
 have $\omega\uparrow\alpha 2 * ord\text{-}of\text{-}nat\ m2 + Cantor\text{-}sum\ \alpha s'\ ms' < \omega\uparrow\alpha 1 * ord\text{-}of\text{-}nat\ m1 + (\omega\uparrow\alpha 2 * ord\text{-}of\text{-}nat\ m2 + Cantor\text{-}sum\ \alpha s'\ ms')$
  if $\alpha 2 \in elts\ \alpha 1$
 proof (*rule less-ω-power*)
  show $Cantor\text{-}sum\ \alpha s'\ ms' < \omega\uparrow\alpha 2$
   using αm *Cantor cnf-2* **by** *auto*
 qed (*use oCs* ‹*Ord* $\alpha 1$› ‹$m1 \neq 0$› ‹$m2 \neq 0$› *that* **in** *auto*)
 then have $?\beta < \mu$
  using $\alpha 21$ **by** (*simp add*: μ αm)
 moreover have *less*: $?\alpha < \mu$
  using *oCs* **by** (*metis* μ $*$ *add-less-cancel-left add.right-neutral*)
 ultimately have *False*
  using *indecomposableD* [*OF inc*, *of* $?\alpha$ $?\beta$]
  by (*simp add*: ‹*Ord* $(\omega\uparrow\alpha 1)$› ‹*Ord* $(\omega\uparrow\alpha 2)$› *oCs*)
 then show *?thesis* **..**
 qed
qed

corollary *indecomposable-iff-ω-power*:
 indecomposable $\mu \longleftrightarrow \mu = 0 \lor (\exists\delta.\ \mu = \omega\uparrow\delta \land Ord\ \delta)$
 by (*meson indecomposable-0 indecomposable-ω-power indecomposable-is-ω-power*)

theorem *indecomposable-imp-type*:
 fixes $X :: bool \Rightarrow V\ set$
 assumes γ: *indecomposable* γ
  and $\bigwedge b.\ ordertype\ (X\ b)\ VWF \leq \gamma$ $\bigwedge b.\ small\ (X\ b)$ $\bigwedge b.\ X\ b \subseteq ON$
  and $elts\ \gamma \subseteq (UN\ b.\ X\ b)$
 shows $\exists\, b.\ ordertype\ (X\ b)\ VWF = \gamma$
 using γ [*THEN indecomposable-imp-Ord*] *assms*
proof (*induction arbitrary*: X)
 case (*succ* β)
 show *?case*
 proof (*cases* $\beta = 0$)
  case *True*
  then have $\exists\, b.\ 0 \in X\ b$
   using *succ.prems*(5) **by** *blast*
  then have $\exists\, b.\ ordertype\ (X\ b)\ VWF \neq 0$
   using *succ.prems*(3) **by** *auto*
  then have $\exists\, b.\ ordertype\ (X\ b)\ VWF \geq succ\ 0$
   by (*meson Ord-0 Ord-linear2 Ord-ordertype less-eq-V-0-iff succ-le-iff*)
  then show *?thesis*
   using *True succ.prems*(2) **by** *blast*
 next
  case *False*
  then show *?thesis*

using *succ.prems* **by** *auto*
qed
next
case ($Limit\ \gamma$)
then obtain δ **where** δ: $\gamma = \omega\uparrow\delta$ **and** $\delta \neq 0$ $Ord\ \delta$
by (*metis Limit-eq-Sup-self image-ident indecomposable-is-ω-power not-succ-Limit oexp-0-right one-V-def zero-not-Limit*)
show *?case*
proof (*cases Limit* δ)
case *True*
have *ot*: $\exists\, b.\ ordertype\ (X\ b \cap elts\ (\omega\uparrow\alpha))\ VWF = \omega\uparrow\alpha$
if $\alpha \in elts\ \delta$ **for** α
proof (*rule Limit.IH*)
have $Ord\ \alpha$
using *Ord-in-Ord* ‹$Ord\ \delta$› *that* **by** *blast*
then show $\omega\uparrow\alpha \in elts\ \gamma$
by (*simp add*: *Ord-mem-iff-lt* δ *ω-gt1* ‹$Ord\ \delta$› *oexp-less that*)
show *indecomposable* ($\omega\uparrow\alpha$)
using ‹$Ord\ \alpha$› *indecomposable-1 indecomposable-ω-power* **by** *fastforce*
show $small\ (X\ b \cap elts\ (\omega\uparrow\alpha))$ **for** b
by (*meson down inf-le2*)
show $ordertype\ (X\ b \cap elts\ (\omega \uparrow \alpha))\ VWF \leq \omega \uparrow \alpha$ **for** b
by (*simp add*: ‹$Ord\ \alpha$› *ordertype-le-Ord*)
show $X\ b \cap elts\ (\omega \uparrow \alpha) \subseteq ON$ **for** b
by (*simp add*: *Limit.prems inf.coboundedI1*)
show $elts\ (\omega \uparrow \alpha) \subseteq (\bigcup b.\ X\ b \cap elts\ (\omega \uparrow \alpha))$
using *Limit.prems Limit.hyps* ‹$\omega \uparrow \alpha \in elts\ \gamma$›
by *clarsimp* (*metis Ord-trans UN-E indecomposable-imp-Ord subset-eq*)
qed
define A **where** $A \equiv \lambda b.\ \{\alpha \in elts\ \delta.\ ordertype\ (X\ b \cap elts\ (\omega\uparrow\alpha))\ VWF \geq \omega\uparrow\alpha\}$
have *Asmall*: $small\ (A\ b)$ **for** b
by (*simp add*: *A-def*)
have *AON*: $A\ b \subseteq ON$ **for** b
using *A-def* ‹$Ord\ \delta$› *elts-subset-ON* **by** *blast*
have *eq*: $elts\ \delta = (\bigcup\ b.\ A\ b)$
by (*auto simp*: *A-def*) (*metis ot eq-refl*)
then obtain b **where** b: $Sup\ (A\ b) = \delta$
using ‹$Limit\ \delta$›
apply (*auto simp*: *UN-bool-eq*)
by (*metis AON ON-imp-Ord Ord-Sup Ord-linear-le Limit-eq-Sup-self Sup-Un-distrib Asmall sup.absorb2 sup.orderE*)
have $\omega\uparrow\alpha \leq ordertype\ (X\ b)\ VWF$ **if** $\alpha \in A\ b$ **for** α
proof −
have $(\omega\uparrow\alpha) = ordertype\ ((X\ b) \cap elts\ (\omega\uparrow\alpha))\ VWF$
using ‹$Ord\ \delta$› *that* **by** (*simp add*: *A-def Ord-in-Ord dual-order.antisym ordertype-le-Ord*)
also have $\ldots \leq ordertype\ (X\ b)\ VWF$
by (*simp add*: *Limit.prems ordertype-VWF-mono*)

finally show *?thesis* **.**
qed
then have *ordertype* $(X\ b)$ *VWF* $\geq$ *Sup* $((\lambda\alpha.\ \omega\uparrow\alpha)$ ‘ $A\ b)$
by *blast*
moreover have *Sup* $((\lambda\alpha.\ \omega\uparrow\alpha)$ ‘ $A\ b) = \omega \uparrow$ *Sup* $(A\ b)$
by (*metis b Ord-ω ZFC-in-HOL.Sup-empty AON* ‹$\delta \neq 0$› *Asmall oexp-Sup omega-nonzero*)
ultimately show *?thesis*
using *Limit.hyps Limit.prems* δ *b* **by** *auto*
next
case *False*
then obtain β **where** β: $\delta =$ *succ* β *Ord* β
using *Ord-cases* ‹$\delta \neq 0$› ‹*Ord* δ› **by** *auto*
then have *Ordωβ*: *Ord* $(\omega\uparrow\beta)$
using *Ord-oexp* **by** *blast*
have *subX12*: *elts* $(\omega\uparrow\beta * \omega) \subseteq (\bigcup b.\ X\ b)$
using *Limit* β δ **by** *auto*
define E **where** $E \equiv \lambda n.\ \{\omega\uparrow\beta *$ *ord-of-nat* n ..< $\omega\uparrow\beta *$ *ord-of-nat* (*Suc* n)$\}$ $\cap$ *ON*
have *EON*: $E\ n \subseteq$ *ON* **for** n
using *E-def* **by** *blast*
have *E-imp-less*: $x < y$ **if** $i < j$ $x \in E\ i$ $y \in E\ j$ **for** $x\ y\ i\ j$
proof −
have *succ* $(i) \leq$ *ord-of-nat* j
using *that*(*1*) **by** *force*
then have $\neg\ y \leq x$
using *that*
apply (*auto simp*: *E-def*)
by (*metis Ordωβ Ord-ord-of-nat leD mult-cancel-le-iff ord-of-nat.simps*(*2*) *order-trans*)
with *that* **show** *?thesis*
by (*meson EON ON-imp-Ord Ord-linear2*)
qed
then have *djE*: *disjnt* $(E\ i)$ $(E\ j)$ **if** $i \neq j$ **for** $i\ j$
using *that nat-neq-iff* **unfolding** *disjnt-def* **by** *auto*
have *less-imp-E*: $i \leq j$ **if** $x < y$ $x \in E\ i$ $y \in E\ j$ **for** $x\ y\ i\ j$
using *that E-imp-less* [*OF* - ‹$y \in E\ j$› ‹$x \in E\ i$›] *leI less-asym* **by** *blast*
have *inc*: *indecomposable* $(\omega\uparrow\beta)$
using β *indecomposable-1 indecomposable-ω-power* **by** *fastforce*
have *in-En*: $\omega\uparrow\beta *$ *ord-of-nat* $n + x \in E\ n$ **if** $x \in$ *elts* $(\omega\uparrow\beta)$ **for** $x\ n$
using *that Ordωβ Ord-in-Ord* [*OF Ordωβ*] **by** (*auto simp*: *E-def Ordωβ OrdmemD mult-succ*)
have $*$: *elts* $\gamma = \bigcup$(*range* E)
proof
have $\exists\, m.\ \omega\uparrow\beta * m \leq x \wedge x < \omega\uparrow\beta *$ *succ* (*ord-of-nat* m)
if $x \in$ *elts* $(\omega\uparrow\beta *$ *ord-of-nat* $n)$ **for** $x\ n$
using *that*
apply (*clarsimp simp add*: *mult* [*of* - *ord-of-nat* n] *lift-def*)
by (*metis add-less-cancel-left OrdmemD inc indecomposable-imp-Ord mult-succ*

plus sup-ge1)
moreover have *Ord x* **if** $x \in$ *elts* ($\omega\uparrow\beta$ $*$ *ord-of-nat n*) **for** *x n*
by (*meson Ordωβ Ord-in-Ord Ord-mult Ord-ord-of-nat that*)
ultimately show *elts* $\gamma \subseteq \bigcup$(*range E*)
by (*auto simp*: δ β *E-def mult-Limit elts-ω*)
have $x \in$ *elts* ($\omega\uparrow\beta$ $*$ *succ*(*ord-of-nat n*))
if *Ord x* $x < \omega\uparrow\beta$ $*$ *succ* (*n*) **for** *x n*
by (*metis that Ord-mem-iff-lt Ord-mult Ord-ord-of-nat inc indecomposable-imp-Ord ord-of-nat.simps*(*2*))
then show $\bigcup$(*range E*) $\subseteq$ *elts* γ
by (*force simp*: δ β *E-def Limit.prems mult-Limit*)
qed
have *smE*: *small* (*E n*) **for** *n*
by (*metis* $*$ *complete-lattice-class.Sup-upper down rangeI*)
have *otE*: *ordertype* (*E n*) *VWF* $= \omega\uparrow\beta$ **for** *n*
by (*simp add*: *E-def inc indecomposable-imp-Ord mult-succ ordertype-interval-eq*)

define *cut* **where** *cut* $\equiv \lambda n\ x.$ *odiff x* ($\omega\uparrow\beta$ $*$ *ord-of-nat n*)
have *cutON*: *cut n* ‘ $X \subseteq ON$ **if** $X \subseteq ON$ **for** *n X*
using *that* **by** (*simp add*: *image-subset-iff cut-def ON-imp-Ord Ordωβ Ord-odiff*)
have *cut* [*simp*]: *cut n* ($\omega \uparrow \beta$ $*$ *ord-of-nat n* $+ x) = x$ **for** *x n*
by (*auto simp*: *cut-def*)
have *cuteq*: $x \in$ *cut n* ‘ ($X \cap E\ n$) $\longleftrightarrow \omega\uparrow\beta$ $*$ *ord-of-nat n* $+ x \in X$
if *x*: $x \in$ *elts* ($\omega\uparrow\beta$) **for** *x X n*
proof
show $\omega\uparrow\beta$ $*$ *ord-of-nat n* $+ x \in X$ **if** $x \in$ *cut n* ‘ ($X \cap E\ n$)
using *E-def Ordωβ Ord-odiff-eq image-iff local.cut-def that* **by** *auto*
show $x \in$ *cut n* ‘ ($X \cap E\ n$)
if $\omega\uparrow\beta$ $*$ *ord-of-nat n* $+ x \in X$
by (*metis* (*full-types*) *IntI cut image-iff in-En that x*)
qed
have *ot-cuteq*: *ordertype* (*cut n* ‘ ($X \cap E\ n$)) *VWF* $=$ *ordertype* ($X \cap E\ n$) *VWF* **for** *n X*
proof (*rule ordertype-VWF-inc-eq*)
show $X \cap E\ n \subseteq ON$
using *E-def* **by** *blast*
then show *cut n* ‘ ($X \cap E\ n$) $\subseteq ON$
by (*simp add*: *cutON*)
show *small* ($X \cap E\ n$)
by (*meson Int-lower2 smE smaller-than-small*)
show *cut n x* $<$ *cut n y*
if $x \in X \cap E\ n$ $y \in X \cap E\ n$ $x < y$ **for** *x y*
using *that* ‹$X \cap E\ n \subseteq ON$› **by**(*simp add*: *E-def Ordωβ Ord-odiff-less-odiff local.cut-def*)
qed

define *N* **where** $N \equiv \lambda b.$ {*n. ordertype* ($X\ b \cap E\ n$) *VWF* $= \omega\uparrow\beta$}
have $\exists\, b.$ *infinite* (*N b*)
proof (*rule ccontr*)

assume $\nexists b.$ *infinite* $(N\ b)$
then obtain n **where** $\bigwedge b.\ n \notin N\ b$
apply (*simp add*: *ex-bool-eq*)
by (*metis* (*full-types*) *finite-nat-set-iff-bounded not-less-iff-gr-or-eq*)
moreover
have $\exists b.$ *ordertype* (*cut* n ‘ $(X\ b \cap E\ n)$) *VWF* $= \omega\uparrow\beta$
proof (*rule Limit.IH*)
show $\omega\uparrow\beta \in$ *elts* γ
by (*metis Limit.hyps Limit-def Limit-omega Ord-mem-iff-less-TC* β δ *mult-le2 not-succ-Limit oexp-succ omega-nonzero one-V-def*)
show *indecomposable* $(\omega\uparrow\beta)$
by (*simp add*: *inc*)
show *ordertype* (*cut* n ‘ $(X\ b \cap E\ n)$) *VWF* $\leq \omega\uparrow\beta$ **for** b
by (*metis otE inf-le2 ordertype-VWF-mono ot-cuteq smE*)
show *small* (*cut* n ‘ $(X\ b \cap E\ n)$) **for** b
using *smE subset-iff-less-eq-V*
by (*meson inf-le2 replacement*)
show *cut* n ‘ $(X\ b \cap E\ n) \subseteq$ *ON* **for** b
using *E-def cutON* **by** *auto*
have *elts* $(\omega\uparrow\beta *$ *succ* $n) \subseteq \bigcup$(*range* X)
by (*metis Ordωβ Ord-ω Ord-ord-of-nat less-eq-V-def mult-cancel-le-iff ord-of-nat.simps*(*2*) *ord-of-nat-le-omega order-trans subX12*)
then show *elts* $(\omega\uparrow\beta) \subseteq (\bigcup b.$ *cut* n ‘ $(X\ b \cap E\ n))$
by (*auto simp*: *mult-succ mult-Limit UN-subset-iff cuteq UN-bool-eq*)
qed
then have $\exists b.$ *ordertype* $(X\ b \cap E\ n)$ *VWF* $= \omega\uparrow\beta$
by (*simp add*: *ot-cuteq*)
ultimately show *False*
by (*simp add*: *N-def*)
qed
then obtain b **where** b: *infinite* $(N\ b)$
by *blast*
then obtain $\varphi :: nat \Rightarrow nat$ **where** φ: *bij-betw* φ *UNIV* $(N\ b)$ **and** *mono*: *strict-mono* φ
by (*meson bij-enumerate enumerate-mono strict-mono-def*)
then have *ordertype* $(X\ b \cap E\ (\varphi\ n))$ *VWF* $= \omega\uparrow\beta$ **for** n
using *N-def bij-betw-imp-surj-on* **by** *blast*
moreover have *small* $(X\ b \cap E\ (\varphi\ n))$ **for** n
by (*meson inf-le2 smE subset-iff-less-eq-V*)
ultimately have $\exists f.$ *bij-betw* $f\ (X\ b \cap E\ (\varphi\ n))$ (*elts* $(\omega\uparrow\beta)$) $\wedge$ $(\forall x \in X\ b \cap E\ (\varphi\ n).\ \forall y \in X\ b \cap E\ (\varphi\ n).\ f\ x < f\ y \longleftrightarrow (x,y) \in$ *VWF*)
for n **by** (*metis Ord-ordertype ordertype-VWF-eq-iff*)
then obtain F **where** *bijF*: $\bigwedge n.$ *bij-betw* $(F\ n)\ (X\ b \cap E\ (\varphi\ n))$ (*elts* $(\omega\uparrow\beta)$)
and F: $\bigwedge n.\ \forall x \in X\ b \cap E\ (\varphi\ n).\ \forall y \in X\ b \cap E\ (\varphi\ n).\ F\ n\ x < F\ n\ y \longleftrightarrow (x,y) \in$ *VWF*
by *metis*
then have *F-bound*: $\bigwedge n.\ \forall x \in X\ b \cap E\ (\varphi\ n).\ F\ n\ x < \omega\uparrow\beta$
by (*metis Ord-ω Ord-oexp OrdmemD* β(*2*) *bij-betw-imp-surj-on image-eqI*)
have *F-Ord*: $\bigwedge n.\ \forall x \in X\ b \cap E\ (\varphi\ n).$ *Ord* $(F\ n\ x)$

by (*metis otE ON-imp-Ord Ord-ordertype bijF bij-betw-def elts-subset-ON imageI*)

have *inc*: $\varphi\ n \geq n$ **for** n
by (*simp add*: *mono strict-mono-imp-increasing*)

have *djφ*: *disjnt* $(E\ (\varphi\ i))\ (E\ (\varphi\ j))$ **if** $i \neq j$ **for** $i\ j$
by (*rule djE*) (*use* φ *that* **in** ‹*auto simp*: *bij-betw-def inj-def*›)
define Y **where** $Y \equiv (\bigcup n.\ E\ (\varphi\ n))$
have $\exists\, n.\ y \in E\ (\varphi\ n)$ **if** $y \in Y$ **for** y
using *Y-def that* **by** *blast*
then obtain ι **where** ι: $\bigwedge y.\ y \in Y \Longrightarrow y \in E\ (\varphi\ (\iota\ y))$
by *metis*
have $Y \subseteq ON$
by (*auto simp*: *Y-def E-def*)
have *ιle*: $\iota\ x \leq \iota\ y$ **if** $x < y\ x \in Y\ y \in Y$ **for** $x\ y$
using *less-imp-E strict-mono-less-eq that* ι [*OF* ‹$x \in Y$›] ι [*OF* ‹$y \in Y$›] *mono*
unfolding *Y-def* **by** *blast*
have *eqι*: $x \in E\ (\varphi\ k) \Longrightarrow \iota\ x = k$ **for** $x\ k$
using ι **unfolding** *Y-def*
by (*meson UN-I disjnt-iff djφ iso-tuple-UNIV-I*)

have *upper*: $\omega\uparrow\beta *$ *ord-of-nat* $(\iota\ x) \leq x$ **if** $x \in Y$ **for** x
using *that*
proof (*clarsimp simp add*: *Y-def eqι*)
fix $u\ v$
assume u: $u \in$ *elts* $(\omega\uparrow\beta *$ *ord-of-nat* $v)$ **and** v: $x \in E\ (\varphi\ v)$
then have $u < \omega\uparrow\beta *$ *ord-of-nat* v
by (*simp add*: *OrdmemD* $\beta(2)$)
also have $\ldots \leq \omega\uparrow\beta *$ *ord-of-nat* $(\varphi\ v)$
by (*simp add*: $\beta(2)$ *inc*)
also have $\ldots \leq x$
using v **by** (*simp add*: *E-def*)
finally show $u \in$ *elts* x
using ‹$Y \subseteq ON$›
by (*meson ON-imp-Ord Ord-ω Ord-in-Ord Ord-mem-iff-lt Ord-mult Ord-oexp Ord-ord-of-nat* $\beta(2)$ *that* u)
qed

define G **where** $G \equiv \lambda x.\ \omega\uparrow\beta *$ *ord-of-nat* $(\iota\ x) + F\ (\iota\ x)\ x$
have *G-strict-mono*: $G\ x < G\ y$ **if** $x < y\ x \in X\ b \cap Y\ y \in X\ b \cap Y$ **for** $x\ y$
proof (*cases* $\iota\ x = \iota\ y$)
case *True*
then show *?thesis*
using *that* **unfolding** *G-def*
by (*metis F Int-iff add-less-cancel-left Limit.prems*(*4*) *ON-imp-Ord VWF-iff-Ord-less* ι)
next

```
        case False
        then have ι x < ι y
          by (meson IntE ιle le-less that)
        then show ?thesis
       using that by (simp add: G-def F-Ord F-bound Ordωβ ι mult-nat-less-add-less)
      qed

      have ordertype (X b ∩ Y) VWF = (ω↑β) ∗ ω
      proof (rule ordertype-VWF-eq-iff [THEN iffD2])
        show Ord (ω↑β ∗ ω)
          by (simp add: β)
        show small (X b ∩ Y)
          by (meson Limit.prems(3) inf-le1 subset-iff-less-eq-V)
        have bij-betw G (X b ∩ Y) (elts (ω↑β ∗ ω))
        proof (rule bij-betw-imageI)
          show inj-on G (X b ∩ Y)
          proof (rule linorder-inj-onI)
            fix x y
            assume xy: x < y x ∈ (X b ∩ Y) y ∈ (X b ∩ Y)
            show G x ≠ G y
              using G-strict-mono xy by force
          next
            show x ≤ y ∨ y ≤ x
              if x ∈ (X b ∩ Y) y ∈ (X b ∩ Y) for x y
             using that ‹X b ⊆ ON› by (clarsimp simp: Y-def) (metis ON-imp-Ord
Ord-linear Ord-trans)
          qed
          show G ‘ (X b ∩ Y) = elts (ω↑β ∗ ω)
          proof
            show G ‘ (X b ∩ Y) ⊆ elts (ω↑β ∗ ω)
              using ‹X b ⊆ ON›
              apply (clarsimp simp: G-def mult-Limit Y-def eqι)
             by (metis IntI add-mem-right-cancel bijF bij-betw-imp-surj-on image-eqI
mult-succ ord-of-nat-ω succ-in-omega)
            show elts (ω↑β ∗ ω) ⊆ G ‘ (X b ∩ Y)
            proof
              fix x
              assume x: x ∈ elts (ω↑β ∗ ω)
              then obtain k where n: x ∈ elts (ω↑β ∗ ord-of-nat (Suc k))
                          and minim: ⋀m. m < Suc k ⟹ x ∉ elts (ω↑β ∗ ord-of-nat
m)
                using elts-mult-ωE
                by (metis old.nat.exhaust)
             then obtain y where y: y ∈ elts (ω↑β) and xeq: x = ω↑β ∗ ord-of-nat
k + y
                using x by (auto simp: mult-succ elim: mem-plus-V-E)
              then have 1: inv-into (X b ∩ E (φ k)) (F k) y ∈ (X b ∩ E (φ k))
                by (metis bijF bij-betw-def inv-into-into)
              then have (inv-into (X b ∩ E (φ k)) (F k) y) ∈ X b ∩ Y
```

by (*force simp*: *Y-def*)
moreover have G (*inv-into* ($X\ b \cap E\ (\varphi\ k)$) ($F\ k$) y) $= x$
by (*metis 1 G-def Int-iff bijF bij-betw-inv-into-right eqι xeq y*)
ultimately show $x \in G$ ‘ ($X\ b \cap Y$)
by *blast*
qed
qed
qed
moreover have $(x,y) \in VWF$
if $x \in X\ b\ x \in Y\ y \in X\ b\ y \in Y\ G\ x < G\ y$ **for** $x\ y$
proof −
have $x < y$
using *that* **by** (*metis G-strict-mono Int-iff Limit.prems*(*4*) *ON-imp-Ord Ord-linear-lt less-asym*)
then show *?thesis*
using *ON-imp-Ord* ‹$Y \subseteq ON$› *that* **by** *auto*
qed
moreover have $G\ x < G\ y$
if $x \in X\ b\ x \in Y\ y \in X\ b\ y \in Y\ (x,\ y) \in VWF$ **for** $x\ y$
proof −
have $x < y$
using *that ON-imp-Ord* ‹$Y \subseteq ON$› **by** *auto*
then show *?thesis*
by (*simp add*: *G-strict-mono that*)
qed
ultimately show $\exists f.$ *bij-betw* f ($X\ b \cap Y$) (*elts* $(\omega\uparrow\beta * \omega)$) $\wedge$ $(\forall x\in(X\ b \cap Y).\ \forall y\in(X\ b \cap Y).\ f\ x < f\ y \longleftrightarrow ((x,\ y) \in VWF))$
by *blast*
qed
moreover have *ordertype* ($\bigcup n.\ X\ b \cap E\ (\varphi\ n)$) $VWF \le$ *ordertype* ($X\ b$) VWF
using *Limit.prems*(*3*) *ordertype-VWF-mono* **by** *auto*
ultimately have *ordertype* ($X\ b$) $VWF = (\omega\uparrow\beta) * \omega$
using *Limit.hyps Limit.prems*(*2*) $\beta\ \delta$
using *Y-def* **by** *auto*
then show *?thesis*
using *Limit.hyps* $\beta\ \delta$ **by** *auto*
qed
qed *auto*

corollary *indecomposable-imp-type2*:
assumes α: *indecomposable* γ $X \subseteq$ *elts* γ
shows *ordertype* $X\ VWF = \gamma \vee$ *ordertype* (*elts* $\gamma - X$) $VWF = \gamma$
proof −
have *Ord* γ
using *assms indecomposable-imp-Ord* **by** *blast*
have $\exists b.$ *ordertype* (*if b then X else elts* $\gamma - X$) $VWF = \gamma$
proof (*rule indecomposable-imp-type*)
show *ordertype* (*if b then X else elts* $\gamma - X$) $VWF \le \gamma$ **for** b

by (*simp add*: ‹*Ord* γ› *assms ordertype-le-Ord*)
show (*if b then X else elts* γ − *X*) ⊆ *ON* **for** *b*
using ‹*Ord* γ› *assms elts-subset-ON* **by** *auto*
qed (*use assms down* **in** *auto*)
then show *?thesis*
by (*metis* (*full-types*))
qed

5.4 From ordinals to order types

lemma *indecomposable-ordertype-eq*:
assumes *indec*: *indecomposable* α **and** α: *ordertype A VWF* = α **and** *A*: *B* ⊆ *A small A*
shows *ordertype B VWF* = α ∨ *ordertype* (*A*−*B*) *VWF* = α
proof (*rule ccontr*)
assume ¬ (*ordertype B VWF* = α ∨ *ordertype* (*A* − *B*) *VWF* = α)
moreover have *ordertype* (*ordermap A VWF* ‘ *B*) *VWF* = *ordertype B VWF*
using ‹*B* ⊆ *A*› **by** (*auto intro*: *ordertype-image-ordermap* [*OF* ‹*small A*›])
moreover have *ordertype* (*elts* α − *ordermap A VWF* ‘ *B*) *VWF* = *ordertype* (*A* − *B*) *VWF*
by (*metis ordertype-map-image* α *A elts-of-set ordertype-def replacement*)
moreover have *ordermap A VWF* ‘ *B* ⊆ *elts* α
using α *A* **by** *blast*
ultimately show *False*
using *indecomposable-imp-type2* [*OF* ‹*indecomposable* α›] ‹*small A*› **by** *metis*
qed

lemma *indecomposable-ordertype-ge*:
assumes *indec*: *indecomposable* α **and** α: *ordertype A VWF* ≥ α **and** *small*: *small A small B*
shows *ordertype B VWF* ≥ α ∨ *ordertype* (*A*−*B*) *VWF* ≥ α
proof −
obtain *A'* **where** *A'* ⊆ *A ordertype A' VWF* = α
by (*meson* α ‹*small A*› *indec indecomposable-def le-ordertype-obtains-subset*)
then have *ordertype* (*B* ∩ *A'*) *VWF* = α ∨ *ordertype* (*A'*−*B*) *VWF* = α
by (*metis Diff-Diff-Int Diff-subset Int-commute* ‹*small A*› *indecomposable-ordertype-eq indec smaller-than-small*)
moreover have *ordertype* (*B* ∩ *A'*) *VWF* ≤ *ordertype B VWF*
by (*meson Int-lower1 small ordertype-VWF-mono smaller-than-small*)
moreover have *ordertype* (*A'*−*B*) *VWF* ≤ *ordertype* (*A*−*B*) *VWF*
by (*meson Diff-mono Diff-subset* ‹*A'* ⊆ *A*› ‹*small A*› *order-refl ordertype-VWF-mono smaller-than-small*)
ultimately show *?thesis*
by *blast*
qed

now for finite partitions

lemma *indecomposable-ordertype-finite-eq*:
assumes *indecomposable* α

and $\mathcal{A}$: *finite* $\mathcal{A}$ *pairwise disjnt* $\mathcal{A}$ $\bigcup\mathcal{A} = A$ $\mathcal{A} \neq \{\}$ *ordertype* A *VWF* $= \alpha$ *small* A
  shows $\exists X \in \mathcal{A}.$ *ordertype* X *VWF* $= \alpha$
  using $\mathcal{A}$
proof (*induction arbitrary*: A)
  case (*insert* X $\mathcal{A}$)
  show *?case*
  proof (*cases* $\mathcal{A} = \{\}$)
    case *True*
    then show *?thesis*
      using *insert.prems* **by** *blast*
  next
    case *False*
    have *smA*: *small* ($\bigcup\mathcal{A}$)
      using *insert.prems* **by** *auto*
    show *?thesis*
    proof (*cases* $\exists X \in \mathcal{A}.$ *ordertype* X *VWF* $= \alpha$)
      case *True*
      then show *?thesis*
        using *insert.prems* **by** *blast*
    next
      case *False*
      have $X = A - \bigcup\mathcal{A}$
        using *insert.hyps insert.prems* **by** (*auto simp*: *pairwise-insert disjnt-iff*)
      then have *ordertype* X *VWF* $= \alpha$
        using *indecomposable-ordertype-eq assms insert False*
        by (*metis Union-mono cSup-singleton pairwise-insert smA subset-insertI*)
      then show *?thesis*
        using *insert.prems* **by** *blast*
    qed
  qed
qed *auto*

lemma *indecomposable-ordertype-finite-ge*:
  assumes *indec*: *indecomposable* α
    and $\mathcal{A}$: *finite* $\mathcal{A}$ $A \subseteq \bigcup\mathcal{A}$ $\mathcal{A} \neq \{\}$ *ordertype* A *VWF* $\geq \alpha$ *small* ($\bigcup\mathcal{A}$)
  shows $\exists X \in \mathcal{A}.$ *ordertype* X *VWF* $\geq \alpha$
  using $\mathcal{A}$
proof (*induction arbitrary*: A)
  case (*insert* X $\mathcal{A}$)
  show *?case*
  proof (*cases* $\mathcal{A} = \{\}$)
    case *True*
    then have $\alpha \leq$ *ordertype* X *VWF*
      using *insert.prems*
      by (*simp add*: *order.trans ordertype-VWF-mono*)
    then show *?thesis*
      using *insert.prems* **by** *blast*
  next

case *False*
show *?thesis*
proof (*cases* $\exists X \in \mathcal{A}$. *ordertype* X *VWF* $\geq \alpha$)
case *True*
then show *?thesis*
using *insert.prems* **by** *blast*
next
case *False*
moreover have *small* $(X \cup \bigcup \mathcal{A})$
using *insert.prems* **by** *auto*
moreover have *ordertype* $(\bigcup (insert\ X\ \mathcal{A}))$ *VWF* $\geq \alpha$
using *insert.prems ordertype-VWF-mono* **by** *blast*
ultimately have *ordertype* X *VWF* $\geq \alpha$
using *indecomposable-ordertype-ge* [*OF indec*]
by (*metis Diff-subset-conv Sup-insert cSup-singleton insert.IH small-sup-iff subset-refl*)
then show *?thesis*
using *insert.prems* **by** *blast*
qed
qed
qed *auto*

end

6 Type Classes for ZFC

theory *ZFC-Typeclasses*
imports *ZFC-Cardinals Complex-Main*

begin

6.1 The class of embeddable types

class *embeddable* =
assumes *ex-inj*: $\exists$ *V-of* :: $'a \Rightarrow V$. *inj V-of*

context *countable*
begin

subclass *embeddable*
proof −
have *inj* (*ord-of-nat* $\circ$ *to-nat*) **if** *inj to-nat*
for *to-nat* :: $'a \Rightarrow nat$
using *that* **by** (*simp add*: *inj-compose inj-ord-of-nat*)
then show *class.embeddable TYPE*($'a$)
by *intro-classes* (*meson local.ex-inj*)
qed

end

instance *unit* :: *embeddable* **..**
instance *bool* :: *embeddable* **..**
instance *nat* :: *embeddable* **..**
instance *int* :: *embeddable* **..**
instance *rat* :: *embeddable* **..**
instance *char* :: *embeddable* **..**
instance *String.literal* :: *embeddable* **..**
instance *typerep* :: *embeddable* **..**

lemma *embeddable-classI*:
  fixes $f :: 'a \Rightarrow V$
  assumes $\bigwedge x\ y.\ f\ x = f\ y \Longrightarrow x = y$
  shows *OFCLASS*('*a*, *embeddable-class*)
proof (*intro-classes*, *rule exI*)
  show *inj f*
    by (*rule injI* [*OF assms*]) *assumption*
qed

instance *V* :: *embeddable*
  by (*intro-classes*) (*meson inj-on-id*)

instance *prod* :: (*embeddable*,*embeddable*) *embeddable*
proof −
  have *inj* ($\lambda(x,y).\ \langle V\text{-}of1\ x,\ V\text{-}of2\ y\rangle$) **if** *inj V-of1 inj V-of2*
    for $V\text{-}of1 :: 'a \Rightarrow V$ **and** $V\text{-}of2 :: 'b \Rightarrow V$
    using *that* **by** (*auto simp*: *inj-on-def*)
  then show *OFCLASS*($'a \times 'b$, *embeddable-class*)
    by *intro-classes* (*meson embeddable-class.ex-inj*)
qed

instance *sum* :: (*embeddable*,*embeddable*) *embeddable*
proof −
  have *inj* (*case-sum* (*Inl* ∘ *V-of1*) (*Inr* ∘ *V-of2*)) **if** *inj V-of1 inj V-of2*
    for $V\text{-}of1 :: 'a \Rightarrow V$ **and** $V\text{-}of2 :: 'b \Rightarrow V$
    using *that* **by** (*auto simp*: *inj-on-def split*: *sum.split-asm*)
  then show *OFCLASS*($'a + 'b$, *embeddable-class*)
    by *intro-classes* (*meson embeddable-class.ex-inj*)
qed

instance *option* :: (*embeddable*) *embeddable*
proof −
  have *inj* (*case-option 0* (λx. *ZFC-in-HOL.set*{*V-of x*})) **if** *inj V-of*
    for $V\text{-}of :: 'a \Rightarrow V$
    using *that* **by** (*auto simp*: *inj-on-def split*: *option.split-asm*)
  then show *OFCLASS*('*a option*, *embeddable-class*)
    by *intro-classes* (*meson embeddable-class.ex-inj*)
qed

```
primrec V-of-list where
  V-of-list V-of Nil = 0
| V-of-list V-of (x#xs) = ⟨V-of x, V-of-list V-of xs⟩

lemma inj-V-of-list:
  assumes inj V-of
  shows inj (V-of-list V-of)
proof −
  note inj-eq [OF assms, simp]
  have x = y if V-of-list V-of x = V-of-list V-of y for x y
    using that
  proof (induction x arbitrary: y)
    case Nil
    then show ?case
      by (cases y) auto
  next
    case (Cons a x)
    then show ?case
      by (cases y) auto
  qed
  then show ?thesis
    by (auto simp: inj-on-def)
qed

instance list :: (embeddable) embeddable
proof −
  have inj (rec-list 0 (λx xs r. ⟨V-of x, r⟩)) (is inj ?f)
    if V-of: inj V-of for V-of :: 'a ⇒ V
  proof −
    note inj-eq [OF V-of, simp]
    have x = y if ?f x = ?f y for x y
      using that
    proof (induction x arbitrary: y)
      case Nil
      then show ?case
        by (cases y) auto
    next
      case (Cons a x)
      then show ?case
        by (cases y) auto
    qed
    then show ?thesis
      by (auto simp: inj-on-def)
  qed
  then show OFCLASS('a list, embeddable-class)
    by intro-classes (meson embeddable-class.ex-inj)
qed
```

6.2 The class of small types

class *small* =
  assumes *small*: *small* (*UNIV*::′*a set*)
begin

subclass *embeddable*
  by *intro-classes* (*meson local.small small-def*)

lemma *TC-small* [*iff*]:
  fixes *A* :: ′*a set*
  shows *small A*
  using *small smaller-than-small* **by** *blast*

end

context *countable*
begin

subclass *small*
proof −
  have ∗: *inj* (*ord-of-nat* ∘ *to-nat*) **if** *inj to-nat*
    for *to-nat* :: ′*a* ⇒ *nat*
    using *that* **by** (*simp add*: *inj-compose inj-ord-of-nat*)
  then show *class.small TYPE*(′*a*)
    by *intro-classes* (*metis small-image-nat local.ex-inj the-inv-into-onto*)
qed

end

lemma *lepoll-UNIV-imp-small*: $X \lesssim$ (*UNIV*::′*a*::*small set*) ⟹ *small X*
  by (*meson lepoll-iff replacement small smaller-than-small*)

lemma *lepoll-imp-small*:
  fixes *A* :: ′*a*::*small set*
  assumes $X \lesssim A$
  shows *small X*
  by (*metis lepoll-UNIV-imp-small UNIV-I assms lepoll-def subsetI*)

instance *unit* :: *small* **..**
instance *bool* :: *small* **..**
instance *nat* :: *small* **..**
instance *int* :: *small* **..**
instance *rat* :: *small* **..**
instance *char* :: *small* **..**
instance *String.literal* :: *small* **..**
instance *typerep* :: *small* **..**

instance *prod* :: (*small*,*small*) *small*
proof −

have *inj* ($\lambda(x,y)$. ⟨*V-of1 x*, *V-of2 y*⟩)
  range ($\lambda(x,y)$. ⟨*V-of1 x*, *V-of2 y*⟩) $\le$ *elts* (*VSigma A* (λx. *B*))
 if *inj V-of1 inj V-of2 range V-of1* $\le$ *elts A range V-of2* $\le$ *elts B*
 for *V-of1* :: *'a* $\Rightarrow$ *V* **and** *V-of2* :: *'b* $\Rightarrow$ *V* **and** *A B*
 using *that* **by** (*auto simp*: *inj-on-def*)
with *small* [**where** *'a='a*] *small* [**where** *'a='b*]
show *OFCLASS*(*'a* × *'b*, *small-class*)
 by *intro-classes* (*smt* (*verit*) *down-raw f-inv-into-f set-eq-subset small-def*)
qed

instance *sum* :: (*small*,*small*) *small*
proof −
have *inj* (*case-sum* (*Inl* ∘ *V-of1*) (*Inr* ∘ *V-of2*))
  range (*case-sum* (*Inl* ∘ *V-of1*) (*Inr* ∘ *V-of2*)) $\le$ *elts* (*A* ⊎ *B*)
 if *inj V-of1 inj V-of2 range V-of1* $\le$ *elts A range V-of2* $\le$ *elts B*
 for *V-of1* :: *'a* $\Rightarrow$ *V* **and** *V-of2* :: *'b* $\Rightarrow$ *V* **and** *A B*
 using *that* **by** (*force simp*: *inj-on-def split*: *sum.split*)+
with *small* [**where** *'a='a*] *small* [**where** *'a='b*]
show *OFCLASS*(*'a* + *'b*, *small-class*)
 by *intro-classes* (*metis down-raw replacement set-eq-subset small-def small-iff*)
qed

instance *option* :: (*small*) *small*
proof −
have *inj* (λx. *case x of None* $\Rightarrow$ *0* | *Some x* $\Rightarrow$ *ZFC-in-HOL.set* {*V-of x*})
  range (λx. *case x of None* $\Rightarrow$ *0* | *Some x* $\Rightarrow$ *ZFC-in-HOL.set* {*V-of x*}) $\le$ *insert 0* (*elts* (*VPow A*))
 if *inj V-of range V-of* $\le$ *elts A*
 for *V-of* :: *'a* $\Rightarrow$ *V* **and** *A*
 using *that* **by** (*auto simp*: *inj-on-def split*: *option.split-asm*)
with *small* [**where** *'a='a*]
show *OFCLASS*(*'a option*, *small-class*)
 by *intro-classes* (*smt* (*verit*) *down order.refl ex-inj small-iff small-image-iff small-insert*)
qed

instance *list* :: (*small*) *small*
proof −
have *small* (*range* (*V-of-list V-of*))
 if *inj V-of range V-of* $\le$ *elts A*
 for *V-of* :: *'a* $\Rightarrow$ *V* **and** *A*
proof −
 have *range* (*V-of-list V-of*) $\approx$ (*UNIV* :: *'a list set*)
  using *that* **by** (*simp add*: *inj-V-of-list*)
 also have ... $\approx$ *lists* (*UNIV* :: *'a set*)
  by *simp*
 also have ... $\lesssim$ (*UNIV* :: *'a set*) × (*UNIV* :: *nat set*)
 proof (*cases finite* (*range* (*V-of*::*'a* $\Rightarrow$ *V*)))
  case *True*

then have *lists* (*UNIV* :: *'a set*) $\lesssim$ (*UNIV* :: *nat set*)
using *countable-iff-lepoll countable-image-inj-on that*(*1*) *uncountable-infinite* **by** *blast*
then show *?thesis*
by (*blast intro*: *lepoll-trans* [*OF* - *lepoll-times2*])
next
case *False*
then have *lists* (*UNIV* :: *'a set*) $\lesssim$ (*UNIV* :: *'a set*)
using ‹*infinite* (*range V-of*)› *eqpoll-imp-lepoll infinite-eqpoll-lists* **by** *blast*
then show *?thesis*
using *lepoll-times1 lepoll-trans* **by** *fastforce*
qed
finally show *?thesis*
by (*simp add*: *lepoll-imp-small*)
qed
with *small* [**where** *'a='a*]
show *OFCLASS*(*'a list*, *small-class*)
by *intro-classes* (*metis inj-V-of-list order.refl small-def small-iff small-iff-range*)
qed

instance *fun* :: (*small*,*embeddable*) *embeddable*
proof −
have *inj* (λf. *VLambda A* (λx. *V-of2* (*f* (*inv V-of1 x*))))
if ∗: *inj V-of1 inj V-of2 range V-of1* $\le$ *elts A*
for *V-of1* :: *'a* $\Rightarrow$ *V* **and** *V-of2* :: *'b* $\Rightarrow$ *V* **and** *A*
proof −
have *f u* = *f' u*
if *VLambda A* (λu. *V-of2* (*f* (*inv V-of1 u*))) = *VLambda A* (λx. *V-of2* (*f'* (*inv V-of1 x*)))
for *f f'* :: *'a* $\Rightarrow$ *'b* **and** *u*
by (*metis inv-f-f rangeI subsetD VLambda-eq-D2* [*OF that*, *of V-of1 u*] ∗)
then show *?thesis*
by (*auto simp*: *inj-on-def*)
qed
then show *OFCLASS*(*'a* $\Rightarrow$ *'b*, *embeddable-class*)
by *intro-classes* (*metis embeddable-class.ex-inj small order-refl replacement small-iff*)
qed

instance *fun* :: (*small*,*small*) *small*
proof −
have *inj* (λf. *VLambda A* (λx. *V-of2* (*f* (*inv V-of1 x*)))) (**is** *inj ?φ*)
range (λf. *VLambda A* (λx. *V-of2* (*f* (*inv V-of1 x*)))) $\le$ *elts* (*VPi A* (λx. *B*))
if ∗: *inj V-of1 inj V-of2 range V-of1* $\le$ *elts A* **and** *range V-of2* $\le$ *elts B*
for *V-of1* :: *'a* $\Rightarrow$ *V* **and** *V-of2* :: *'b* $\Rightarrow$ *V* **and** *A B*
proof −
have *f u* = *f' u*
if *VLambda A* (λu. *V-of2* (*f* (*inv V-of1 u*))) = *VLambda A* (λx. *V-of2* (*f'* (*inv V-of1 x*)))

for $f\,f'$ :: $'a \Rightarrow 'b$ **and** u
by (*metis inv-f-f rangeI subsetD VLambda-eq-D2* [*OF that*, *of V-of1 u*] ∗)
then show *inj ?φ*
by (*auto simp*: *inj-on-def*)
show *range ?φ* ≤ *elts* (*VPi A* (λ*x*. *B*))
using *that* **by** (*simp add*: *VPi-I subset-eq*)
qed
with *small* [**where** $'a='a$] *small* [**where** $'a='b$]
show *OFCLASS*($'a \Rightarrow 'b$, *small-class*)
by *intro-classes* (*smt* (*verit*, *best*) *down-raw order-refl imageE small-def*)
qed

instance *set* :: (*small*) *small*
proof −
have *1*: *inj* (λ*x*. *ZFC-in-HOL.set* (*V-of* ‘ *x*))
if *inj V-of* **for** *V-of* :: $'a \Rightarrow V$
by (*simp add*: *inj-on-def inj-image-eq-iff* [*OF that*])
have *2*: *range* (λ*x*. *ZFC-in-HOL.set* (*V-of* ‘ *x*)) ≤ *elts* (*VPow A*)
if *range V-of* ≤ *elts A* **for** *V-of* :: $'a \Rightarrow V$ **and** *A*
using *that* **by** (*auto simp*: *inj-on-def image-subset-iff*)
from *small* [**where** $'a='a$]
show *OFCLASS*($'a$ *set*, *small-class*)
by *intro-classes* (*metis 1 2 down-raw imageE small-def order-refl*)
qed

instance *real* :: *small*
proof −
have *small* (*range* (*Rep-real*))
by *simp*
then show *OFCLASS*(*real*, *small-class*)
by *intro-classes* (*metis Rep-real-inverse image-inv-f-f inj-on-def replacement*)
qed

instance *complex* :: *small*
proof −
have $\bigwedge$*c*. *c* ∈ *range* (λ(*x*,*y*). *Complex x y*)
by (*metis case-prod-conv complex.exhaust-sel rangeI*)
then show *OFCLASS*(*complex*, *small-class*)
by *intro-classes* (*meson TC-small replacement smaller-than-small subset-eq*)
qed

end

7 ZF sets corresponding to $\mathbb{R}$ and $\mathbb{C}$ and the cardinality of the contimuum

theory *General-Cardinals*
imports *ZFC-Typeclasses HOL−Analysis.Continuum-Not-Denumerable*

begin

7.1 Making the embedding from the type class explicit

definition *V-of* :: *'a::embeddable* ⇒ *V*
  where *V-of* ≡ *SOME f. inj f*

lemma *inj-V-of*: *inj V-of*
  unfolding *V-of-def* **by** (*metis embeddable-class.ex-inj some-eq-imp*)

declare *inv-f-f* [*OF inj-V-of*, *simp*]

lemma *inv-V-of-image-eq* [*simp*]: *inv V-of* ‘ (*V-of* ‘ *X*) = *X*
  by (*auto simp*: *image-comp*)

lemma *infinite-V-of*: *infinite* (*UNIV*::*'a set*) ⟹ *infinite* (*range* (*V-of*::*'a*::*embeddable*⇒*V*))
  using *finite-imageD inj-V-of* **by** *blast*

lemma *countable-V-of*: *countable* (*range* (*V-of*::*'a*::*countable*⇒*V*))
  by *blast*

lemma *elts-set-V-of*: *small X* ⟹ *elts* (*ZFC-in-HOL.set* (*V-of* ‘ *X*)) ≈ *X*
  by (*metis inj-V-of inj-eq inj-on-def inj-on-image-eqpoll-self replacement set-of-elts small-iff*)

lemma *V-of-image-times*: *V-of* ‘ (*X* × *Y*) ≈ (*V-of* ‘ *X*) × (*V-of* ‘ *Y*)
proof −
  have *V-of* ‘ (*X* × *Y*) ≈ *X* × *Y*
    by (*meson inj-V-of inj-eq inj-onI inj-on-image-eqpoll-self*)
  also have ... ≈ (*V-of* ‘ *X*) × (*V-of* ‘ *Y*)
    by (*metis eqpoll-sym inj-V-of inj-eq inj-onI inj-on-image-eqpoll-self times-eqpoll-cong*)
  finally show *?thesis* **.**
qed

7.2 The cardinality of the continuum

definition *Real-set* ≡ *ZFC-in-HOL.set* (*range* (*V-of*::*real*⇒*V*))
definition *Complex-set* ≡ *ZFC-in-HOL.set* (*range* (*V-of*::*complex*⇒*V*))
definition *C-continuum* ≡ *vcard Real-set*

lemma *V-of-Real-set*: *bij-betw V-of* (*UNIV*::*real set*) (*elts Real-set*)
  by (*simp add*: *Real-set-def bij-betw-def inj-V-of*)

lemma *uncountable-Real-set*: *uncountable* (*elts Real-set*)
  using *V-of-Real-set countable-iff-bij uncountable-UNIV-real* **by** *blast*

lemma *Card C-continuum*
  by (*simp add*: *C-continuum-def Card-def*)

lemma *C-continuum-ge*: *C-continuum* $\geq$ *ℵ1*
by (*metis C-continuum-def Ord-ω1 Ord-cardinal Ord-linear2 countable-iff-vcard-less1*

uncountable-Real-set)

lemma *V-of-Complex-set*: *bij-betw V-of* (*UNIV::complex set*) (*elts Complex-set*)
by (*simp add*: *Complex-set-def bij-betw-def inj-V-of*)

lemma *uncountable-Complex-set*: *uncountable* (*elts Complex-set*)
using *V-of-Complex-set countableI-bij2 uncountable-UNIV-complex* **by** *blast*

lemma *Complex-vcard*: *vcard Complex-set* = *C-continuum*
proof −
define *emb1* **where** *emb1* ≡ *V-of o complex-of-real o inv V-of*
have *elts Real-set* ≈ *elts Complex-set*
proof (*rule lepoll-antisym*)
show *elts Real-set* $\lesssim$ *elts Complex-set*
unfolding *lepoll-def*
proof (*intro conjI exI*)
show *inj-on emb1* (*elts Real-set*)
unfolding *emb1-def Real-set-def*
by (*simp add*: *inj-V-of inj-compose inj-of-real inj-on-imageI*)
show *emb1* ‘ *elts Real-set* ⊆ *elts Complex-set*
by (*simp add*: *emb1-def Real-set-def Complex-set-def image-subset-iff*)
qed
define *emb2* **where** *emb2* ≡ (λ*z*. (*V-of* (*Re z*), *V-of* (*Im z*))) *o inv V-of*
have *elts Complex-set* $\lesssim$ *elts Real-set* × *elts Real-set*
unfolding *lepoll-def*
proof (*intro conjI exI*)
show *inj-on emb2* (*elts Complex-set*)
unfolding *emb2-def Complex-set-def inj-on-def*
by (*simp add*: *complex.expand inj-V-of inj-eq*)
show *emb2* ‘ *elts Complex-set* ⊆ *elts Real-set* × *elts Real-set*
by (*simp add*: *emb2-def Real-set-def Complex-set-def image-subset-iff*)
qed
also have ... ≈ *elts Real-set*
by (*simp add*: *infinite-times-eqpoll-self uncountable-Real-set uncountable-infinite*)
finally show *elts Complex-set* $\lesssim$ *elts Real-set* .
qed
then show *?thesis*
by (*simp add*: *C-continuum-def cardinal-cong*)
qed

lemma *gcard-Union-le-cmult*:
assumes *small U* **and** κ: ⋀*x*. *x* ∈ *U* ⟹ *gcard x* ≤ κ **and** *sm*: ⋀*x*. *x* ∈ *U* ⟹ *small x*
shows *gcard* (⋃ *U*) ≤ *gcard U* ⊗ κ
proof −
have ∃*f*. *f* ∈ *x* → *elts* κ ∧ *inj-on f x* **if** *x* ∈ *U* **for** *x*

using κ [*OF that*] *gcard-le-lepoll* **by** (*metis image-subset-iff-funcset lepoll-def sm that*)
then obtain φ **where** φ: $\bigwedge x.\ x \in U \Longrightarrow (\varphi\ x) \in x \to$ *elts* $\kappa \wedge$ *inj-on* $(\varphi\ x)\ x$
by *metis*
define u **where** $u \equiv \lambda y.\ @x.\ x \in U \wedge y \in x$
have u: $u\ y \in U \wedge y \in\ (u\ y)$ **if** $y \in \bigcup(\ U)$ **for** y
unfolding *u-def* **using** *that* **by** (*fast intro*!: *someI2*)
define ψ **where** $\psi \equiv \lambda y.\ (u\ y,\ \varphi\ (u\ y)\ y)$
have U: *elts* (*gcard* U) $\approx U$
using *assms* **by** (*simp add*: *gcard-eqpoll*)
have $\bigcup U \lesssim U \times$ *elts* κ
unfolding *lepoll-def*
proof (*intro exI conjI*)
show *inj-on* ψ $(\bigcup\ U)$
using φ u **by** (*smt* (*verit*) ψ*-def inj-on-def prod.inject*)
show ψ ` $\bigcup\ U \subseteq U \times$ *elts* κ
using φ u **by** (*auto simp*: ψ*-def*)
qed
also have ... $\approx$ *elts* (*gcard* $U \otimes \kappa$)
using U *elts-cmult eqpoll-sym eqpoll-trans times-eqpoll-cong* **by** *blast*
finally have $(\bigcup U) \lesssim$ *elts* (*gcard* $U \otimes \kappa$) **.**
then show *?thesis*
by (*metis cardinal-idem cmult-def gcard-eq-vcard lepoll-imp-gcard-le small-elts*)
qed

lemma *gcard-Times* [*simp*]: *gcard* $(X \times Y) =$ *gcard* $X \otimes$ *gcard* Y
proof (*cases small* $X \wedge$ *small* Y)
case *True*
have *elts* (*gcard* $(X \times Y)$) $\approx X \times Y$
by (*simp add*: *True gcard-eqpoll*)
also have ... $\approx$ *elts* (*gcard* X) $\times$ *elts* (*gcard* Y)
by (*simp add*: *True eqpoll-sym gcard-eqpoll times-eqpoll-cong*)
also have ... $\approx$ *elts* (*gcard* $X \otimes$ *gcard* Y)
by (*simp add*: *elts-cmult eqpoll-sym*)
finally show *?thesis*
using *Card-cardinal-eq cmult-def gcardinal-cong* **by** *force*
next
case *False*
have *gcard* $(X \times Y) = 0$
by (*metis False Times-empty gcard-big-0 gcard-empty-0 small-Times-iff*)
then show *?thesis*
by (*metis False cmult-0 cmult-commute gcard-big-0*)
qed

7.3 Countable and uncountable sets

lemma *countable-iff-g-le-Aleph0*:
assumes *small* X
shows *countable* $X \longleftrightarrow$ *gcard* $X \leq \aleph 0$

proof −
have *countable* $X \longleftrightarrow X \lesssim$ *elts* ω
by (*simp add*: ω*-def countable-iff-lepoll inj-ord-of-nat*)
also have ... $\longleftrightarrow$ *gcard* $X \leq \aleph 0$
using *Card-*ω *Card-def assms gcard-le-lepoll lepoll-imp-gcard-le* **by** *fastforce*
finally show *?thesis* **.**
qed

lemma *countable-imp-g-le-Aleph0*: *countable* $X \Longrightarrow$ *gcard* $X \leq \aleph 0$
by (*meson countable countable-iff-g-le-Aleph0*)

lemma *finite-iff-g-le-Aleph0*: *small* $X \Longrightarrow$ *finite* $X \longleftrightarrow$ *gcard* $X < \aleph 0$
by (*metis Aleph-0 eqpoll-finite-iff finite-iff-less-Aleph0 gcard-eq-vcard gcard-eqpoll gcardinal-cong*)

lemma *finite-imp-g-le-Aleph0*: *finite* $X \Longrightarrow$ *gcard* $X < \aleph 0$
by (*meson finite-iff-g-le-Aleph0 finite-imp-small*)

lemma *countable-infinite-gcard*: *countable* $X \wedge$ *infinite* $X \longleftrightarrow$ *gcard* $X = \aleph 0$
proof −
have *gcard* $X = \omega$
if *countable* X **and** *infinite* X
using *that countable countable-imp-g-le-Aleph0 finite-iff-g-le-Aleph0 less-V-def* **by** *auto*
moreover have *countable* X **if** *gcard* $X = \omega$
by (*metis Aleph-0 countable-iff-g-le-Aleph0 dual-order.refl gcard-big-0 omega-nonzero that*)
moreover have *False* **if** *gcard* $X = \omega$ **and** *finite* X
by (*metis Aleph-0 dual-order.irrefl finite-iff-g-le-Aleph0 finite-imp-small that*)
ultimately show *?thesis*
by *auto*
qed

lemma *uncountable-gcard*: *small* $X \Longrightarrow$ *uncountable* $X \longleftrightarrow$ *gcard* $X > \aleph 0$
by (*simp add*: *Card-is-Ord Ord-not-le countable-iff-g-le-Aleph0*)

lemma *uncountable-gcard-ge*: *small* $X \Longrightarrow$ *uncountable* $X \longleftrightarrow$ *gcard* $X \geq \aleph 1$
by (*simp add*: *uncountable-gcard csucc-le-Card-iff one-V-def*)

lemma *subset-smaller-gcard*:
assumes κ: $\kappa \leq$ *gcard* X *Card* κ
obtains Y **where** $Y \subseteq X$ *gcard* $Y = \kappa$
proof (*cases small* X)
case *True*
then have *elts* $\kappa \lesssim X$
by (*meson assms*(*1*) *eqpoll-imp-lepoll gcard-eqpoll lepoll-trans less-eq-V-def subset-imp-lepoll*)
then obtain Y **where** $Y \subseteq X$ *elts* $\kappa \approx Y$
by (*metis bij-betw-def eqpoll-def lepoll-def*)

then show *?thesis*
using *Card-def* ‹*Card* κ› *gcardinal-cong that* **by** *force*
next
case *False*
with *assms* **show** *?thesis*
by (*metis antisym gcard-big-0 le-0 order-refl that*)
qed

lemma *Real-gcard*: *gcard* (*UNIV*::*real set*) = *C-continuum*
by (*metis C-continuum-def V-of-Real-set bij-betw-def gcard-eq-vcard gcard-image*)

lemma *Complex-gcard*: *gcard* (*UNIV*::*complex set*) = *C-continuum*
by (*metis Complex-vcard V-of-Complex-set bij-betw-def gcard-eq-vcard gcard-image*)

end

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