Wooley's Discrete Inequality

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Abstract

This is a formalisation of the proof of an inequality by Trevor D. Wooley attesting that when $\lambda > 0$,

$$\min_{r \in \mathbb{N}} (r + \lambda/r) \leq \sqrt{4\lambda + 1}$$

with equality if and only if $\lambda = m(m-1)$ for some positive integer m.

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theory *Wooley-Elementary-Discrete-Inequality*
imports *HOL−Library.Quadratic-Discriminant HOL−Real-Asymp.Real-Asymp*

begin

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$$\min_{r \in \mathbb{N}} (r + \lambda/r) \leq \sqrt{4\lambda + 1}$$

with equality if and only if $\lambda = m(m-1)$ for some positive integer m. The source is the note "An Elementary Discrete Inequality" available on Wooley's webpage [1]: https://www.math.purdue.edu/~twooley/publ/20230410discineq.pdf.

1.1 General elementary technical lemmas

lemma *obtains-nat-in-interval*:
  fixes $x::real$ **assumes** $x \geq 0$
  obtains $c::nat$ **where** $c \in \{x<..\ x+1\}$
⟨*proof*⟩

lemma *obtains-nat-in-interval-greater-leq*:
  fixes $x::real$ **assumes** $x \geq 0$
  obtains $c::nat$ **where** $c > x$ **and** $c \leq x+1$
  ⟨*proof*⟩

lemma *obtains-nat-in-interval-half*:
  fixes $x::real$ **assumes** $x \geq 1/2$
  obtains $c::nat$ **where** $c > x - (1/2)$ **and** $c \leq x+1/2$
  ⟨*proof*⟩

1.2 Trivial case, where we minimise over all positive real values of r

theorem *elementary-ineq-Wooley-real*:
  fixes $l::real$ **and** $g::real \Rightarrow real$
  assumes $l>0$ **and** $\forall\ r \in R.\ g\ r = r+(l/r)$
    and $R=\{r::real.\ r>0\}$
  shows $(\forall\ r \in R.\ g\ r \geq\ 2*\ sqrt(l)) \wedge (\forall\ r \in R.\ g\ (sqrt(l)) \leq g\ r)$
⟨*proof*⟩

1.3 Main result: Inequality for the discrete version

theorem *elementary-discrete-ineq-Wooley*:
  fixes $l::real$ **and** $g::nat \Rightarrow real$
  assumes $l>0$ **and** $R = \{r::nat.\ r>0\}$ **and** $\forall\ r \in R.\ g\ r = r+\ (l/r)$
  shows $(INF\ r \in R.\ g\ r) \leq sqrt(4*l+1)$

⟨*proof*⟩

1.4 Special case: Equality for the discrete version

We will now show a special case of the main result where equality holds instead of inequality.

We will need to make use of the following technical lemma, which will be used so as to guarantee that there exists a $p \in R$ for which the INF of $g(r)$ equals to $g(p)$. To this end, we will show that here the infimum INF can be identified with the minimum Min by restricting to a finite set. As the operator Min in Isabelle is used for finite sets and R is infinite, we used INF in the original formulation, however here Min and INF can be identified.

The following technical lemma is by Larry Paulson:

lemma *restrict-to-min*:

fixes $l::real$ **and** $g::nat \Rightarrow real$
assumes $l>0$ **and** $R\text{-}def$: $R=\{r::nat.\ r>0\}$ **and** $g\text{-}def$: $\forall\ r.\ g\ r = r + (l/r)$
obtains F **where** $finite\ F\ F \subseteq R\ (INF\ r \in R.\ g\ r) = Min\ (g\ `\ F)\ F \neq \{\}$

⟨*proof*⟩

We will make use of the following calculation, which is convenient to formulate separately as a lemma.

lemma *elementary-discrete-ineq-Wooley-quadratic-eq-sol*:
fixes $l::real$ **and** $g::nat \Rightarrow real$
assumes $l>0$ **and** $\forall\ r.\ g\ r = r + (l/r)$ **and** $g\ r = sqrt(4*l+1)$
shows $(r = 1/2 + (1/2)*\ sqrt(\ 4*l\ +1)) \vee (r = -\ 1/2 + (1/2)*\ sqrt(4*l\ +1))$
⟨*proof*⟩

The special case with equality involves a double implication (iff), and we start by showing one direction.

theorem *elementary-discrete-ineq-Wooley-special-case-1*:
fixes $l::real$ **and** $g::nat \Rightarrow real$ **assumes** $l>0$ **and** $R=\{r::nat.\ r>0\}$ **and** $\forall\ r.\ g\ r = r + (l/r)$
and $(INF\ r \in R.\ g\ r) = sqrt(4*l+1)$
shows $\exists\ m::nat.\ l = m*(m-1)$

⟨*proof*⟩

Now we show the other direction.

theorem *elementary-discrete-ineq-Wooley-special-case-2*:
fixes $l::real$ **and** $g::nat \Rightarrow real$
assumes $l>0$ **and** $R=\{r::nat.\ r>0\}$ **and** $\forall\ r.\ g\ r = r + (l/r)$ **and** $\exists\ m::nat.\ l = m*(m-1)$
shows $(INF\ r \in R.\ g\ r) = sqrt(4*l+1)$

⟨*proof*⟩

Finally, for convenience and completeness, we state the special case where equality holds formulated with the double implication and moreover including the values for which the INF (i.e. minimum here as we have seen) is attained as previously calculated.

theorem *elementary-discrete-ineq-Wooley-special-case-iff*:
fixes $l::real$ **and** $g::nat \Rightarrow real$
assumes $l>0$ **and** $R=\{r::nat.\ r>0\}$ **and** $\forall\ r.\ g\ r = r + (l/r)$
shows $((INF\ r \in R.\ g\ r) = sqrt(4*l+1)) \longleftrightarrow (\exists\ m::nat.\ l = m*(m-1))$
and
$g\ p = sqrt(4*l+1) \longrightarrow (p = 1/2 + (1/2)*\ sqrt(\ 4*l\ +1)) \vee (p = -1/2 + (1/2)*\ sqrt(4*l\ +1))$
⟨*proof*⟩

end

References

[1] T. D. Wooley. An elementary discrete inequality. https://www.math.purdue.edu/~twooley/publ/20230410discineq.pdf.