Wlog – Without Loss of Generality

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Abstract

We introduce a new command `wlog` in Isabelle/HOL that allows us to (soundly) assume facts without loss of generality inside a proof.

Contents

1 Introduction

We introduce a command `wlog` for assuming facts without loss of generality inside a proof in Isabelle/HOL. The `wlog` command makes sure this is sound by requiring us to prove that the assumption is indeed made without loss of generality.

A simple example is the following:

```
lemma card_nth_roots_strengthened:
  assumes "c ≠ 0"
  shows   "card {z::complex. z ^ n = c} = n"
proof -
  wlog n_pos: "n > 0"
    using negation by (simp add: infinite_UNIV_char_0)
  have "card {z. z ^ n = c} = card {z::complex. z ^ n = 1}"
    by (rule sym, rule bij_betw_same_card, rule bij_betw_nth_root_unity) fact+
  also have "... = n" by (rule card_roots_unity_eq) fact+
  finally show ?thesis .
qed
```

This proof is exactly like the proof of `Complex.card_nth_roots` in the Isabelle/HOL library, except that the latter uses the additional assumption `n > 0` in the theorem statement. We omit this assumption and instead state that it can be assumed without loss of generality. (`wlog n_pos: "n > 0"`) The next line then shows that this can be assumed without loss of generality.[1]

Of course, we could have shown this theorem also, e.g., by doing a case distinction on whether $n = 0$. But this would additionally clutter the proof; the case $n = 0$ is almost trivial, yet in the proof it will be a separate case on the same level as the main proof. So doing a `wlog` improves readability here by allowing us to focus on the important parts of the proof and reducing boilerplate.

In other cases, a `wlog` argument cannot easily be done as a case distinction. E.g., if we say that we can assume w.l.o.g. that $a \geq b$ because the case $a < b$ can be easily reduced to the $a \geq b$ case. (This is common in symmetric situations.) We give an example of this in the proof of lemma `schur_ineq` below.

The full syntax of the `wlog` command is roughly as follows:

```
wlog wlogassmname: wlogassm1 wlogassm2
     goal G  generalizing x y z  keeping fact1 fact2
  [... your proof ...]
```

(The defaults being: The goal is `?thesis`. And empty lists of variables and facts for `generalizing` and `keeping`.)

This means that we assume w.l.o.g. that the facts `wlogassm1` and `wlogassm2` hold when proving the goal `G`. We say that the assumptions `fact1` and `fact2` (made prior to the `wlog` command) should still be available afterwards. (If we include less assumptions here, the justification for the `wlog` command becomes easier.) And we wish to generalize the variables `x, y, z`; that is, inside the justification of the `wlog`, we want to be allowed to use the theorem that we are proving for other values of `x, y, z` (needed, e.g., in symmetry arguments). And `[... your proof ...]` is a proof of the fact that we can make the w.l.o.g.-assumption, either as an apply-script or as an Isar subproof.

The `wlog` command is realized by translation to existing Isar commands. The above translates roughly to:

```
  presume hypothesis:
    ⋀x y z. wlogassm ⟹ fact1 ⟹ fact2 ⟹ G
  have G if negation: ň (wlogassm1 ∧ wlogassm2)
    [... your proof ...]
  then show G
```

[1] The argument is basically: If $\neg(n > 0)$, then $n = 0$ (since n is a natural number). Then $\{z.\ z^n = c\}$ is infinite, and for infinite sets, the cardinality `card` is defined to be 0 in Isabelle/HOL. Thus that cardinality is 0. This reasoning is done almost automatically by Isabelle.

```
    [... autogenerated proof ...]
next
  fix x y z
  assume fact1: fact1 and fact2: fact2
  assume wlogassmname: wlogassm1 wlogassm2
```

(There are more steps and additional convenience definitions, but this is the main part.)

More examples of how to use `wlog` are given in the theory `Wlog_Examples` below.

2 *Wlog* – Setting up the command

theory *Wlog*
imports *Main*
keywords *wlog* :: *prf-goal* % *proof*
  and *generalizing* **and** *keeping* **and** *goal*
begin

⟨*ML*⟩

For symmetric predicates involving 3–5 variables on a linearly ordered type, the following lemmas are very useful for wlog-proofs.

For two variables, we already have *linorder-wlog*.

lemma *linorder-wlog-3*:
  fixes $x\ y\ z :: \langle 'a :: linorder \rangle$
  assumes $\langle \bigwedge x\ y\ z.\ P\ x\ y\ z \Longrightarrow P\ y\ x\ z \wedge P\ x\ z\ y \rangle$
  assumes $\langle \bigwedge x\ y\ z.\ x \le y \wedge y \le z \Longrightarrow P\ x\ y\ z \rangle$
  shows $\langle P\ x\ y\ z \rangle$
  ⟨*proof*⟩

lemma *linorder-wlog-4*:
  fixes $x\ y\ z\ w :: \langle 'a :: linorder \rangle$
  assumes $\langle \bigwedge x\ y\ z\ w.\ P\ x\ y\ z\ w \Longrightarrow P\ y\ x\ z\ w \wedge P\ x\ z\ y\ w \wedge P\ x\ y\ w\ z \rangle$
  assumes $\langle \bigwedge x\ y\ z\ w.\ x \le y \wedge y \le z \wedge z \le w \Longrightarrow P\ x\ y\ z\ w \rangle$
  shows $\langle P\ x\ y\ z\ w \rangle$
  ⟨*proof*⟩

lemma *linorder-wlog-5*:
  fixes $x\ y\ z\ w\ v :: \langle 'a :: linorder \rangle$
  assumes $\langle \bigwedge x\ y\ z\ w\ v.\ P\ x\ y\ z\ w\ v \Longrightarrow P\ y\ x\ z\ w\ v \wedge P\ x\ z\ y\ w\ v \wedge P\ x\ y\ w\ z\ v \wedge P\ x\ y\ z\ v\ w \rangle$
  assumes $\langle \bigwedge x\ y\ z\ w\ v.\ x \le y \wedge y \le z \wedge z \le w \wedge w \le v \Longrightarrow P\ x\ y\ z\ w\ v \rangle$
  shows $\langle P\ x\ y\ z\ w\ v \rangle$
  ⟨*proof*⟩

end

3 *Wlog-Examples* – Examples how to use `wlog`

theory *Wlog-Examples*
  imports *Wlog Complex-Main*
begin

The theorem *Complex.card-nth-roots* has the additional assumption $0 < n$. We use exactly the same proof except for stating that w.l.o.g., $0 < n$.

lemma *card-nth-roots-strengthened*:
  assumes $c \neq 0$
  shows $card\ \{z::complex.\ z\ \hat{}\ n = c\} = n$
⟨*proof*⟩

This example very roughly follows Harrison [1]:

lemma *schur-ineq*:
  fixes $a\ b\ c$ :: ‹'a :: *linordered-idom*› **and** k :: *nat*
  assumes *a0*: ‹$a \geq 0$› **and** *b0*: ‹$b \geq 0$› **and** *c0*: ‹$c \geq 0$›
  shows ‹$a\hat{}k * (a - b) * (a - c) + b\hat{}k * (b - a) * (b - c) + c\hat{}k * (c - a) * (c - b) \geq 0$›
    (**is** ‹*?lhs* ≥ 0›)
⟨*proof*⟩

The following illustrates how facts already proven before a **wlog** can be still be used after the wlog. The example does not do anything useful.

lemma ‹$A \Longrightarrow B \Longrightarrow A \wedge B$›
⟨*proof*⟩

end

References

[1] J. Harrison. Without loss of generality. In S. Berghofer, T. Nipkow, C. Urban, and M. Wenzel, editors, *Theorem Proving in Higher Order Logics*, pages 43–59, Berlin, Heidelberg, 2009. Springer Berlin Heidelberg. Eprint available at https://www.cl.cam.ac.uk/~jrh13/papers/wlog.pdf.