Wetzel's Problem and the Continuum Hypothesis

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Abstract

Let F be a set of analytic functions on the complex plane such that, for each $z \in \mathbb{C}$, the set $\{f(z) \mid f \in F\}$ is countable; must then F itself be countable? The answer is yes if the Continuum Hypothesis is false, i.e., if the cardinality of $\mathbb{R}$ exceeds $\aleph_1$. But if CH is true then such an F, of cardinality $\aleph_1$, can be constructed by transfinite recursion.

The formal proof illustrates reasoning about complex analysis (analytic and homomorphic functions) and set theory (transfinite cardinalities) in a single setting. The mathematical text comes from *Proofs from THE BOOK* [1, pp. 137–8], by Aigner and Ziegler.

Contents

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1 Wetzel's Problem, Solved by Erdös

Martin Aigner and Günter M. Ziegler. Proofs from THE BOOK. (Springer, 2018). Chapter 19: Sets, functions, and the continuum hypothesis Theorem 5 (pages 137–8)

theory *Wetzels-Problem* **imports**
HOL−Complex-Analysis.Complex-Analysis ZFC-in-HOL.ZFC-Typeclasses

begin

1.1 Added to the developer libraries

lemma *inj-on-restrict-iff*: $A \subseteq B \Longrightarrow$ *inj-on* (*restrict f B*) $A \longleftrightarrow$ *inj-on f A*
⟨*proof*⟩

lemma *Rats-closure-real*: *closure* $\mathbb{Q}$ = (*UNIV*::*real set*)
⟨*proof*⟩

lemma *fsigma-UNIV* [*iff*]: *fsigma* (*UNIV* :: *'a*::*real-inner set*)
⟨*proof*⟩

theorem *complex-non-denum*: $\nexists f$ :: *nat* ⇒ *complex*. *surj f*
⟨*proof*⟩

lemma *uncountable-UNIV-complex*: *uncountable* (*UNIV* :: *complex set*)
⟨*proof*⟩

lemma *analytic-on-prod* [*analytic-intros*]:
($\bigwedge i$. $i \in I \Longrightarrow$ (*f i*) *analytic-on S*) $\Longrightarrow$ (λx. *prod* (λi. *f i x*) *I*) *analytic-on S*
⟨*proof*⟩

lemma *holomorphic-countable-zeros*:
assumes *S*: *f holomorphic-on S open S connected S* **and** *fsigma S*
and ¬ *f constant-on S*
shows *countable* {*z*∈*S*. *f z* = *0*}
⟨*proof*⟩

lemma *holomorphic-countable-equal*:
assumes *f holomorphic-on S g holomorphic-on S open S connected S* **and** *fsigma S*
and *eq*: *uncountable* {*z*∈*S*. *f z* = *g z*}
shows $S \subseteq$ {*z*∈*S*. *f z* = *g z*}
⟨*proof*⟩

lemma *holomorphic-countable-equal-UNIV*:
assumes *fg*: *f holomorphic-on UNIV g holomorphic-on UNIV*
and *eq*: *uncountable* {*z*. *f z* = *g z*}
shows *f*=*g*
⟨*proof*⟩

lemma *finite-iff-less-Aleph0*: *finite* (*elts* x) $\longleftrightarrow$ *vcard* $x < \omega$
⟨*proof*⟩

lemma *cadd-left-commute*: $j \oplus (i \oplus k) = i \oplus (j \oplus k)$
⟨*proof*⟩

lemmas *cadd-ac* = *cadd-assoc cadd-commute cadd-left-commute*

lemma *csucc-lt-csucc-iff*: $[\![$*Card* κ'; *Card* $\kappa]\!] \Longrightarrow$ (*csucc* κ' < *csucc* κ) = ($\kappa' < \kappa$)
⟨*proof*⟩

lemma *csucc-le-csucc-iff*: $[\![$*Card* κ'; *Card* $\kappa]\!] \Longrightarrow$ (*csucc* $\kappa' \leq$ *csucc* κ) = ($\kappa' \leq \kappa$)
⟨*proof*⟩

lemma *Card-Un* [*simp*,*intro*]:
assumes *Card*(x) *Card*(y) **shows** *Card*($x \sqcup y$)
⟨*proof*⟩

lemma *csucc-0* [*simp*]: *csucc* *0* = *1*
⟨*proof*⟩

lemma *InfCard-Aleph* [*simp*, *intro*]:
assumes *Ord* α
shows *InfCard*(*Aleph* α)
⟨*proof*⟩

corollary *Aleph-csquare-eq* [*simp*]: *Ord* $\alpha \Longrightarrow \aleph\alpha \otimes \aleph\alpha = \aleph\alpha$
⟨*proof*⟩

lemma *small-Times-iff*: *small* ($X \times Y$) $\longleftrightarrow$ *small* $X \wedge$ *small* $Y \vee X=\{\} \vee Y=\{\}$
(**is** - = *?rhs*)
⟨*proof*⟩

lemma *lepoll-small*:
assumes $A \lesssim B$ *small* B
shows *small* A
⟨*proof*⟩

lemma *countable-iff-vcard-less1*: *countable* (*elts* x) $\longleftrightarrow$ *vcard* $x < \aleph 1$
⟨*proof*⟩

lemma *countable-infinite-vcard*: *countable* (*elts* x) $\wedge$ *infinite* (*elts* x) $\longleftrightarrow$ *vcard* x = $\aleph 0$
⟨*proof*⟩

lemma *vcard-set-image*: *inj-on* f (*elts* x) $\Longrightarrow$ *vcard* (*ZFC-in-HOL.set* (f ‘ *elts* x)) = *vcard* x
⟨*proof*⟩

definition *transrec* :: $((V \Rightarrow 'a) \Rightarrow V \Rightarrow 'a) \Rightarrow V \Rightarrow 'a$
where $transrec\ H\ a \equiv wfrec\ \{(x,y).\ x \in elts\ y\}\ H\ a$

lemma *transrec*: $transrec\ H\ a = H\ (\lambda x \in elts\ a.\ transrec\ H\ x)\ a$
⟨*proof*⟩

lemma *less-succ-self*: $x < succ\ x$
⟨*proof*⟩

lemma *subset-smaller-vcard*:
assumes $\kappa \le vcard\ x\ Card\ \kappa$
obtains y **where** $y \le x\ vcard\ y = \kappa$
⟨*proof*⟩

lemma *vcard-sup*: $vcard\ (x \sqcup y) \le vcard\ x \oplus vcard\ y$
⟨*proof*⟩

lemma *elts-cmult*: $elts\ (\kappa' \otimes \kappa) \approx elts\ \kappa' \times elts\ \kappa$
⟨*proof*⟩

lemma *vcard-Sup-le-cmult*:
assumes $small\ U$ **and** κ: $\bigwedge x.\ x \in U \Longrightarrow vcard\ x \le \kappa$
shows $vcard\ (\bigsqcup U) \le vcard\ (set\ U) \otimes \kappa$
⟨*proof*⟩

lemma *csucc-le-Card-iff*: $[\![Card\ \kappa';\ Card\ \kappa]\!] \Longrightarrow csucc\ \kappa' \le \kappa \longleftrightarrow \kappa' < \kappa$
⟨*proof*⟩

lemma *cadd-InfCard-le*:
assumes $\alpha \le \kappa\ \beta \le \kappa\ InfCard\ \kappa$
shows $\alpha \oplus \beta \le \kappa$
⟨*proof*⟩

lemma *cmult-InfCard-le*:
assumes $\alpha \le \kappa\ \beta \le \kappa\ InfCard\ \kappa$
shows $\alpha \otimes \beta \le \kappa$
⟨*proof*⟩

lemma *vcard-Aleph* [*simp*]: $Ord\ \alpha \Longrightarrow vcard\ (\aleph\alpha) = \aleph\alpha$
⟨*proof*⟩

lemma *omega-le-Aleph* [*simp*]: $Ord\ \alpha \Longrightarrow \omega \le \aleph\alpha$
⟨*proof*⟩

1.2 Making the embedding explicit

definition *V-of* :: *'a::embeddable* ⇒ *V*
  where *V-of* ≡ *SOME f. inj f*

lemma *inj-V-of*: *inj V-of*
  ⟨*proof*⟩

declare *inv-f-f* [*OF inj-V-of*, *simp*]

lemma *inv-V-of-image-eq* [*simp*]: *inv V-of* ‘ (*V-of* ‘ *X*) = *X*
  ⟨*proof*⟩

lemma *infinite-V-of*: *infinite* (*UNIV*::*'a set*) ⟹ *infinite* (*range* (*V-of*::*'a*::*embeddable*⇒*V*))
  ⟨*proof*⟩

lemma *countable-V-of*: *countable* (*range* (*V-of*::*'a*::*countable*⇒*V*))
  ⟨*proof*⟩

lemma *elts-set-V-of*: *small X* ⟹ *elts* (*ZFC-in-HOL.set* (*V-of* ‘ *X*)) ≈ *X*
  ⟨*proof*⟩

lemma *V-of-image-times*: *V-of* ‘ (*X* × *Y*) ≈ (*V-of* ‘ *X*) × (*V-of* ‘ *Y*)
⟨*proof*⟩

1.3 The cardinality of the continuum

definition *Real-set* ≡ *ZFC-in-HOL.set* (*range* (*V-of*::*real*⇒*V*))
definition *Complex-set* ≡ *ZFC-in-HOL.set* (*range* (*V-of*::*complex*⇒*V*))
definition *C-continuum* ≡ *vcard Real-set*

lemma *V-of-Real-set*: *bij-betw V-of* (*UNIV*::*real set*) (*elts Real-set*)
  ⟨*proof*⟩

lemma *uncountable-Real-set*: *uncountable* (*elts Real-set*)
  ⟨*proof*⟩

lemma *Card C-continuum*
  ⟨*proof*⟩

lemma *C-continuum-ge*: *C-continuum* ≥ ℵ*1*
⟨*proof*⟩

lemma *V-of-Complex-set*: *bij-betw V-of* (*UNIV*::*complex set*) (*elts Complex-set*)
  ⟨*proof*⟩

lemma *uncountable-Complex-set*: *uncountable* (*elts Complex-set*)
  ⟨*proof*⟩

lemma *Complex-vcard*: *vcard Complex-set* = *C-continuum*
⟨*proof*⟩

1.4 Cardinality of an arbitrary HOL set

definition *gcard* :: *'a*::*embeddable set* ⇒ *V*
where *gcard X* ≡ *vcard* (*ZFC-in-HOL.set* (*V-of* ‘ *X*))

lemma *gcard-big-0*: ¬ *small X* ⟹ *gcard X* = *0*
⟨*proof*⟩

lemma *gcard-empty-0* [*simp*]: *gcard* {} = *0*
⟨*proof*⟩

lemma *gcard-single-1* [*simp*]: *gcard* {*x*} = *1*
⟨*proof*⟩

lemma *gcard-finite-set*: ⟦*finite X*; *a* ∉ *X*⟧ ⟹ *gcard* (*insert a X*) = *succ* (*gcard X*)
⟨*proof*⟩

lemma *gcard-eq-card*: *finite X* ⟹ *gcard X* = *ord-of-nat* (*card X*)
⟨*proof*⟩

lemma *Card-gcard* [*iff*]: *Card* (*gcard X*)
⟨*proof*⟩

lemma *gcard-eq-vcard* [*simp*]: *gcard* (*elts x*) = *vcard x*
⟨*proof*⟩

lemma *gcard-eqpoll*: *small X* ⟹ *elts* (*gcard X*) ≈ *X*
⟨*proof*⟩

lemma *gcard-image-le*:
assumes *small A*
shows *gcard* (*f* ‘ *A*) ≤ *gcard A*
⟨*proof*⟩

lemma *gcard-image*: *inj-on f A* ⟹ *gcard* (*f* ‘ *A*) = *gcard A*
⟨*proof*⟩

lemma *lepoll-imp-gcard-le*:
assumes *A* ≲ *B small B*
shows *gcard A* ≤ *gcard B*
⟨*proof*⟩

lemma *subset-imp-gcard-le*:
assumes *A* ⊆ *B small B*
shows *gcard A* ≤ *gcard B*

⟨*proof*⟩

lemma *gcard-le-lepoll*: $[\![$*gcard* $A \leq \alpha$; *small* $A]\!] \Longrightarrow A \lesssim$ *elts* α
⟨*proof*⟩

lemma *gcard-Union-le-cmult*:
assumes *small* U **and** κ: $\bigwedge x.\ x \in U \Longrightarrow$ *gcard* $x \leq \kappa$ **and** *sm*: $\bigwedge x.\ x \in U \Longrightarrow$ *small* x
shows *gcard* $(\bigcup U) \leq$ *gcard* $U \otimes \kappa$
⟨*proof*⟩

lemma *countable-iff-g-le-Aleph0*: *small* $X \Longrightarrow$ *countable* $X \longleftrightarrow$ *gcard* $X \leq \aleph 0$
⟨*proof*⟩

lemma *countable-imp-g-le-Aleph0*: *countable* $X \Longrightarrow$ *gcard* $X \leq \aleph 0$
⟨*proof*⟩

lemma *finite-iff-g-le-Aleph0*: *small* $X \Longrightarrow$ *finite* $X \longleftrightarrow$ *gcard* $X < \aleph 0$
⟨*proof*⟩

lemma *finite-imp-g-le-Aleph0*: *finite* $X \Longrightarrow$ *gcard* $X < \aleph 0$
⟨*proof*⟩

lemma *countable-infinite-gcard*: *countable* $X \wedge$ *infinite* $X \longleftrightarrow$ *gcard* $X = \aleph 0$
⟨*proof*⟩

lemma *uncountable-gcard*: *small* $X \Longrightarrow$ *uncountable* $X \longleftrightarrow$ *gcard* $X > \aleph 0$
⟨*proof*⟩

lemma *uncountable-gcard-ge*: *small* $X \Longrightarrow$ *uncountable* $X \longleftrightarrow$ *gcard* $X \geq \aleph 1$
⟨*proof*⟩

lemma *subset-smaller-gcard*:
assumes κ: $\kappa \leq$ *gcard* X *Card* κ
obtains Y **where** $Y \subseteq X$ *gcard* $Y = \kappa$
⟨*proof*⟩

lemma *Real-gcard*: *gcard* (*UNIV*::*real set*) = *C-continuum*
⟨*proof*⟩

lemma *Complex-gcard*: *gcard* (*UNIV*::*complex set*) = *C-continuum*
⟨*proof*⟩

lemma *gcard-Times* [*simp*]: *gcard* $(X \times Y) =$ *gcard* $X \otimes$ *gcard* Y
⟨*proof*⟩

1.5 Wetzel's problem

definition *Wetzel* :: (*complex* ⇒ *complex*) *set* ⇒ *bool*
where $Wetzel \equiv \lambda F.\ (\forall f{\in}F.\ f\ analytic\text{-}on\ UNIV) \wedge (\forall z.\ countable((\lambda f.\ f\ z)\ `\ F))$

1.5.1 When the continuum hypothesis is false

proposition *Erdos-Wetzel-nonCH*:
assumes *W*: *Wetzel F* **and** *NCH*: $C\text{-}continuum > \aleph 1$ **and** *small F*
shows *countable F*
⟨*proof*⟩

1.5.2 When the continuum hypothesis is true

lemma *Rats-closure-real2*: $closure\ (\mathbb{Q}\times\mathbb{Q}) = (UNIV::real\ set)\times(UNIV::real\ set)$
⟨*proof*⟩

proposition *Erdos-Wetzel-CH*:
assumes *CH*: $C\text{-}continuum = \aleph 1$
obtains *F* **where** *Wetzel F* **and** *uncountable F*
⟨*proof*⟩

theorem *Erdos-Wetzel*: $C\text{-}continuum = \aleph 1 \longleftrightarrow (\exists F.\ Wetzel\ F \wedge uncountable\ F)$
⟨*proof*⟩

end

References

[1] M. Aigner and G. M. Ziegler. *Proofs from THE BOOK*. Springer, 6th edition, 2018.