Pólya's Proof of the Weighted Arithmetic–Geometric Mean Inequality

Manuel Eberl

March 17, 2025

Abstract

This article provides a formalisation of the Weighted Arithmetic– Geometric Mean Inequality: given non-negative reals a_1, \ldots, a_n and non-negative weights w_1, \ldots, w_n such that $w_1 + \ldots + w_n = 1$, we have

$$\prod_{i=1}^n a_i^{w_i} \le \sum_{i=1}^n w_i a_i \; .$$

If the weights are additionally all non-zero, equality holds if and only if $a_1 = \ldots = a_n$.

As a corollary with $w_1 = \ldots = w_n = \frac{1}{n}$, the regular arithmeticgeometric mean inequality follows, namely that

$$\sqrt[n]{a_1 \dots a_n} \leq \frac{1}{n}(a_1 + \dots + a_n)$$
.

I follow Pólya's elegant proof, which uses the inequality $1 + x \le e^x$ as a starting point. Pólya claims that this proof came to him in a dream, and that it was 'the best mathematics he had ever dreamt." [1, pp. 22–26]

Contents

1	The	e Weighted Arithmetic–Geometric Mean Inequality
	1.1	Auxiliary Facts
	1.2	The Inequality
	1.3	The Equality Case
	1.4	The Binary Version

1 The Weighted Arithmetic–Geometric Mean Inequality

theory Weighted-Arithmetic-Geometric-Mean imports Complex-Main begin

1.1 Auxiliary Facts

lemma root-powr-inverse': $0 < n \implies 0 \le x \implies root \ n \ x = x \ powr \ (1/n)$ $\langle proof \rangle$ **lemma** *powr-sum-distrib-real-right*: assumes $a \neq 0$ **shows** $(\prod x \in X. \ a \ powr \ e \ x :: real) = a \ powr \ (\sum x \in X. \ e \ x)$ $\langle proof \rangle$ **lemma** *powr-sum-distrib-real-left*: assumes $\bigwedge x. \ x \in X \Longrightarrow a \ x \ge 0$ **shows** $(\prod x \in X. \ a \ x \ powr \ e \ :: \ real) = (\prod x \in X. \ a \ x) \ powr \ e$ $\langle proof \rangle$ **lemma** prod-ge-pointwise-le-imp-pointwise-eq: fixes $f :: 'a \Rightarrow real$ assumes finite X**assumes** ge: prod $f X \ge prod g X$ assumes nonneg: $\bigwedge x. \ x \in X \Longrightarrow f \ x \ge 0$ assumes pos: $\bigwedge x. \ x \in X \Longrightarrow g \ x > 0$ assumes le: $\bigwedge x. \ x \in X \Longrightarrow f \ x \leq g \ x \text{ and } x: \ x \in X$ shows f x = g x $\langle proof \rangle$ **lemma** powr-right-real-eq-iff: assumes $a \ge (0 :: real)$ shows a powr x = a powr $y \leftrightarrow a = 0 \lor a = 1 \lor x = y$ $\langle proof \rangle$ **lemma** powr-left-real-eq-iff: assumes $a \ge (0 :: real)$ $b \ge 0$ $x \ne 0$ **shows** a powr x = b powr $x \leftrightarrow a = b$ $\langle proof \rangle$ **lemma** exp-real-eq-one-plus-iff: fixes x :: realshows $exp \ x = 1 + x \leftrightarrow x = 0$ $\langle proof \rangle$

1.2 The Inequality

We first prove the equality under the assumption that all the a_i and w_i are positive.

```
lemma weighted-arithmetic-geometric-mean-pos:

fixes a w :: a \Rightarrow real

assumes finite X

assumes pos1: \Lambda x. x \in X \Longrightarrow a x > 0

assumes pos2: \Lambda x. x \in X \Longrightarrow w x > 0

assumes sum-weights: (\sum x \in X. w x) = 1

shows (\prod x \in X. a x powr w x) \le (\sum x \in X. w x * a x)

\langle proof \rangle
```

We can now relax the positivity assumptions to non-negativity: if one of the a_i is zero, the theorem becomes trivial (note that $0^0 = 0$ by convention for the real-valued power operator (powr)).

Otherwise, we can simply remove all the indices that have weight 0 and apply the above auxiliary version of the theorem.

```
theorem weighted-arithmetic-geometric-mean:

fixes a w :: 'a \Rightarrow real

assumes finite X

assumes nonneg1: \bigwedge x. \ x \in X \Longrightarrow a \ x \ge 0

assumes nonneg2: \bigwedge x. \ x \in X \Longrightarrow w \ x \ge 0

assumes sum-weights: (\sum x \in X. \ w \ x) = 1

shows (\prod x \in X. \ a \ x \ powr \ w \ x) \le (\sum x \in X. \ w \ x \ast a \ x)

\langle proof \rangle
```

We can derive the regular arithmetic/geometric mean inequality from this by simply setting all the weights to $\frac{1}{n}$:

```
corollary arithmetic-geometric-mean:

fixes a :: 'a \Rightarrow real

assumes finite X

defines n \equiv card X

assumes nonneg: \bigwedge x. \ x \in X \Longrightarrow a \ x \ge 0

shows root n (\prod x \in X. \ a \ x) \le (\sum x \in X. \ a \ x) / n

\langle proof \rangle
```

1.3 The Equality Case

Next, we show that weighted arithmetic and geometric mean are equal if and only if all the a_i are equal.

We first prove the more difficult direction as a lemmas and again first assume positivity of all a_i and w_i and will relax this somewhat later.

```
lemma weighted-arithmetic-geometric-mean-eq-iff-pos:
fixes a w :: 'a \Rightarrow real
assumes finite X
```

assumes $pos1: \bigwedge x. x \in X \implies a \ x > 0$ assumes $pos2: \bigwedge x. x \in X \implies w \ x > 0$ assumes $sum-weights: (\sum x \in X. \ w \ x) = 1$ assumes $eq: (\prod x \in X. \ a \ x \ powr \ w \ x) = (\sum x \in X. \ w \ x \ \ast \ a \ x)$ shows $\forall x \in X. \ \forall \ y \in X. \ a \ x = a \ y$ $\langle proof \rangle$

We can now show the full theorem and relax the positivity condition on the a_i to non-negativity. This is possible because if some a_i is zero and the two means coincide, then the product is obviously 0, but the sum can only be 0 if *all* the a_i are 0.

theorem weighted-arithmetic-geometric-mean-eq-iff: **fixes** $a w :: a \Rightarrow real$ **assumes** finite X **assumes** nonneg1: $\bigwedge x. \ x \in X \Longrightarrow a \ x \ge 0$ **assumes** pos2: $\bigwedge x. \ x \in X \Longrightarrow w \ x > 0$ **assumes** sum-weights: $(\sum x \in X. \ w \ x) = 1$ **shows** $(\prod x \in X. \ a \ x \ powr \ w \ x) = (\sum x \in X. \ w \ x \ * \ a \ x) \longleftrightarrow X \neq \{\} \land (\forall x \in X. \ \forall y \in X. \ a \ x = a \ y)$ $\langle proof \rangle$

Again, we derive a version for the unweighted arithmetic/geometric mean.

corollary arithmetic-geometric-mean-eq-iff: **fixes** $a :: 'a \Rightarrow real$ **assumes** finite X **defines** $n \equiv card X$ **assumes** nonneg: $\bigwedge x. \ x \in X \implies a \ x \ge 0$ **shows** root $n (\prod x \in X. \ a \ x) = (\sum x \in X. \ a \ x) / n \iff (\forall x \in X. \ \forall y \in X. \ a \ x = a \ y)$ $\langle proof \rangle$

1.4 The Binary Version

For convenience, we also derive versions for only two numbers:

corollary weighted-arithmetic-geometric-mean-binary: fixes $w1 \ w2 \ x1 \ x2 \ :: real$ assumes $x1 \ge 0 \ x2 \ge 0 \ w1 \ge 0 \ w2 \ge 0 \ w1 + w2 = 1$ shows $x1 \ powr \ w1 \ * x2 \ powr \ w2 \le w1 \ * x1 + w2 \ * x2$ $\langle proof \rangle$ corollary weighted-arithmetic-geometric-mean-eq-iff-binary: fixes $w1 \ w2 \ x1 \ x2 \ :: real$ assumes $x1 \ge 0 \ x2 \ge 0 \ w1 > 0 \ w2 > 0 \ w1 + w2 = 1$ shows $x1 \ powr \ w1 \ * x2 \ powr \ w2 = w1 \ * x1 + w2 \ * x2 \ \longleftrightarrow \ x1 = x2$ $\langle proof \rangle$ corollary arithmetic-geometric-mean-binary:

fixes x1 x2 :: real

assumes $x1 \ge 0$ $x2 \ge 0$ **shows** $sqrt(x1 * x2) \le (x1 + x2) / 2$ $\langle proof \rangle$ **corollary** arithmetic-geometric-mean-eq-iff-binary: **fixes** x1 x2 :: real **assumes** $x1 \ge 0 x2 \ge 0$ **shows** $sqrt(x1 * x2) = (x1 + x2) / 2 \iff x1 = x2$ $\langle proof \rangle$

 \mathbf{end}

References

[1] J. M. Steele. The Cauchy–Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities. Cambridge University Press, 2004.