An Isabelle Correctness Proof for the Volpano/Smith Security Typing System

Gregor Snelting and Daniel Wasserrab
IPD Snelting
Universität Karlsruhe (TH)

March 17, 2025

Abstract

The Volpano/Smith/Irvine security type systems [2] requires that variables are annotated as high (secret) or low (public), and provides typing rules which guarantee that secret values cannot leak to public output ports. This property of a program is called confidentiality.

For a simple while-language without threads, our proof shows that typeability in the Volpano/Smith system guarantees noninterference. Noninterference means that if two initial states for program execution are low-equivalent, then the final states are low-equivalent as well. This indeed implies that secret values cannot leak to public ports. For more details on noninterference and security typing systems, see [1].

The proof defines an abstract syntax and operational semantics for programs, formalizes noninterference, and then proceeds by rule induction on the operational semantics. The mathematically most intricate part is the treatment of implicit flows. Note that the Volpano/Smith system is not flow-sensitive and thus quite unprecise, resulting in false alarms. However, due to the correctness property, all potential breaks of confidentiality are discovered.

Contents

1 The Language **3**
1.1 Variables and Values 3
1.2 Expressions and Commands 3
1.3 State 4
1.4 Small Step Semantics 4

2 Security types **10**
2.1 Security definitions 10
2.2 Lemmas concerning expressions 11
2.3 Noninterference definitions 15
2.3.1 Low Equivalence 15
2.3.2 Non Interference 15

3 Executing the small step semantics **25**
3.1 An example taken from Volpano, Smith, Irvine 27

theory *Semantics*
imports *Main*
begin

1 The Language

1.1 Variables and Values

type-synonym *vname* = *string* — names for variables

datatype *val*
= *Bool bool* — Boolean value
| *Intg int* — integer value

abbreviation *true* == *Bool True*
abbreviation *false* == *Bool False*

1.2 Expressions and Commands

datatype *bop* = *Eq* | *And* | *Less* | *Add* | *Sub* — names of binary operations

datatype *expr*
= *Val val* — value
| *Var vname* — local variable
| *BinOp expr bop expr* (‹- «-» -› [*80*,*0*,*81*] *80*) — binary operation

Note: we assume that only type correct expressions are regarded as later proofs fail if expressions evaluate to None due to type errors. However there is [yet] no typing system

fun *binop* :: *bop* ⇒ *val* ⇒ *val* ⇒ *val option*
where *binop Eq* v_1 v_2 = *Some*(*Bool*(v_1 = v_2))
| *binop And* (*Bool* b_1) (*Bool* b_2) = *Some*(*Bool*(b_1 ∧ b_2))
| *binop Less* (*Intg* i_1) (*Intg* i_2) = *Some*(*Bool*(i_1 < i_2))
| *binop Add* (*Intg* i_1) (*Intg* i_2) = *Some*(*Intg*(i_1 + i_2))
| *binop Sub* (*Intg* i_1) (*Intg* i_2) = *Some*(*Intg*(i_1 − i_2))

| *binop bop* v_1 v_2 = *Some*(*Intg*(*0*))

datatype *com*
= *Skip*
| *LAss vname expr* (‹-:=-› [*70*,*70*] *70*) — local assignment
| *Seq com com* (‹-;;/ -› [*61*,*60*] *60*)
| *Cond expr com com* (‹*if* '(-') -/ *else* -› [*80*,*79*,*79*] *70*)
| *While expr com* (‹*while* '(-') -› [*80*,*79*] *70*)

fun *fv* :: *expr* ⇒ *vname set* — free variables in an expression
where

FVc: *fv* (*Val V*) = {}
| *FVv*: *fv* (*Var V*) = {*V*}
| *FVe*: *fv* (*e1* «*bop*» *e2*) = *fv e1* ∪ *fv e2*

1.3 State

type-synonym *state* = *vname* ⇀ *val*

interpret silently assumes type correct expressions, i.e. no expression evaluates to None

fun *interpret* :: *expr* ⇒ *state* ⇒ *val option* (‹⟦-⟧-›)
where *Val*: ⟦*Val v*⟧ *s* = *Some v*
| *Var*: ⟦*Var V*⟧ *s* = *s V*
| *BinOp*: ⟦e_1«*bop*»e_2⟧ *s* = (*case* ⟦e_1⟧ *s of None* ⇒ *None*
| *Some* v_1 ⇒ (*case* ⟦e_2⟧ *s of None* ⇒ *None*
| *Some* v_2 ⇒ *binop bop* v_1 v_2))

1.4 Small Step Semantics

inductive *red* :: *com* ∗ *state* ⇒ *com* ∗ *state* ⇒ *bool*
and *red′* :: *com* ⇒ *state* ⇒ *com* ⇒ *state* ⇒ *bool*
(‹((*1*⟨-,/-⟩) →/ (*1*⟨-,/-⟩))› [*0*,*0*,*0*,*0*] *81*)
where
⟨*c1*,*s1*⟩ → ⟨*c2*,*s2*⟩ == *red* (*c1*,*s1*) (*c2*,*s2*) |
RedLAss:
⟨*V*:=*e*,*s*⟩ → ⟨*Skip*,*s*(*V*:=(⟦*e*⟧ *s*))⟩

| *SeqRed*:
⟨c_1,*s*⟩ → ⟨c_1′,*s*′⟩ ⟹ ⟨c_1;;c_2,*s*⟩ → ⟨c_1′;;c_2,*s*′⟩

| *RedSeq*:
⟨*Skip*;;c_2,*s*⟩ → ⟨c_2,*s*⟩

| *RedCondTrue*:
⟦*b*⟧ *s* = *Some true* ⟹ ⟨*if* (*b*) c_1 *else* c_2,*s*⟩ → ⟨c_1,*s*⟩

| *RedCondFalse*:
⟦*b*⟧ *s* = *Some false* ⟹ ⟨*if* (*b*) c_1 *else* c_2,*s*⟩ → ⟨c_2,*s*⟩

| *RedWhileTrue*:
⟦*b*⟧ *s* = *Some true* ⟹ ⟨*while* (*b*) *c*,*s*⟩ → ⟨*c*;;*while* (*b*) *c*,*s*⟩

| *RedWhileFalse*:
⟦*b*⟧ *s* = *Some false* ⟹ ⟨*while* (*b*) *c*,*s*⟩ → ⟨*Skip*,*s*⟩

lemmas *red-induct* = *red.induct*[*split-format* (*complete*)]

abbreviation *reds* ::*com* ⇒ *state* ⇒ *com* ⇒ *state* ⇒ *bool*
(‹((*1*⟨-,/-⟩) →∗/ (*1*⟨-,/-⟩))› [*0*,*0*,*0*,*0*] *81*) **where**
⟨*c*,*s*⟩ →∗ ⟨*c′*,*s′*⟩ == *red*∗∗ (*c*,*s*) (*c′*,*s′*)

lemma *Skip-reds*:
$\langle Skip,s\rangle \rightarrow* \langle c',s'\rangle \Longrightarrow s = s' \wedge c' = Skip$
by(*blast elim*:*converse-rtranclpE red.cases*)

lemma *LAss-reds*:
$\langle V{:=}e,s\rangle \rightarrow* \langle Skip,s'\rangle \Longrightarrow s' = s(V{:=}[\![e]\!]\ s)$
proof(*induct* $V{:=}e$ *s rule*:*converse-rtranclp-induct2*)
case (*step s* c'' s'')
hence $c'' = Skip$ **and** $s'' = s(V{:=}([\![e]\!]\ s))$ **by**(*auto elim*:*red.cases*)
with $\langle\langle c'',s''\rangle \rightarrow* \langle Skip,s'\rangle\rangle$ **show** *?case* **by**(*auto dest*:*Skip-reds*)
qed

lemma *Seq2-reds*:
$\langle Skip;;c_2,s\rangle \rightarrow* \langle Skip,s'\rangle \Longrightarrow \langle c_2,s\rangle \rightarrow* \langle Skip,s'\rangle$
by(*induct* $c{=}{=}Skip;;c_2$ *s rule*:*converse-rtranclp-induct2*)(*auto elim*:*red.cases*)

lemma *Seq-reds*:
assumes $\langle c_1;;c_2,s\rangle \rightarrow* \langle Skip,s'\rangle$
obtains s'' **where** $\langle c_1,s\rangle \rightarrow* \langle Skip,s''\rangle$ **and** $\langle c_2,s''\rangle \rightarrow* \langle Skip,s'\rangle$
proof −
have $\exists\, s''.\ \langle c_1,s\rangle \rightarrow* \langle Skip,s''\rangle \wedge \langle c_2,s''\rangle \rightarrow* \langle Skip,s'\rangle$
proof −
{ **fix** $c\ c'$
assume $\langle c,s\rangle \rightarrow* \langle c',s'\rangle$ **and** $c = c_1;;c_2$ **and** $c' = Skip$
hence $\exists\, s''.\ \langle c_1,s\rangle \rightarrow* \langle Skip,s''\rangle \wedge \langle c_2,s''\rangle \rightarrow* \langle Skip,s'\rangle$
proof(*induct arbitrary*:c_1 *rule*:*converse-rtranclp-induct2*)
case *refl* **thus** *?case* **by** *simp*
next
case (*step c s* c'' s'')
note *IH* = $\langle\bigwedge c_1.\ [\![c'' = c_1;;c_2;\ c' = Skip]\!]$
$\Longrightarrow \exists\, sx.\ \langle c_1,s''\rangle \rightarrow* \langle Skip,sx\rangle \wedge \langle c_2,sx\rangle \rightarrow* \langle Skip,s'\rangle\rangle$
from *step*
have $\langle c_1;;c_2,s\rangle \rightarrow \langle c'',s''\rangle$ **by** *simp*
hence $(c_1 = Skip \wedge c'' = c_2 \wedge s = s'') \vee$
$(\exists\, c_1'.\ \langle c_1,s\rangle \rightarrow \langle c_1',s''\rangle \wedge c'' = c_1';;c_2)$
by(*auto elim*:*red.cases*)
thus *?case*
proof
assume $c_1 = Skip \wedge c'' = c_2 \wedge s = s''$
with $\langle\langle c'',s''\rangle \rightarrow* \langle c',s'\rangle\rangle$ $\langle c' = Skip\rangle$
show *?thesis* **by** *auto*
next
assume $\exists\, c_1'.\ \langle c_1,s\rangle \rightarrow \langle c_1',s''\rangle \wedge c'' = c_1';;c_2$
then obtain c_1' **where** $\langle c_1,s\rangle \rightarrow \langle c_1',s''\rangle$ **and** $c'' = c_1';;c_2$ **by** *blast*
from *IH*[*OF* $\langle c'' = c_1';;c_2\rangle$ $\langle c' = Skip\rangle$]
obtain *sx* **where** $\langle c_1',s''\rangle \rightarrow* \langle Skip,sx\rangle$ **and** $\langle c_2,sx\rangle \rightarrow* \langle Skip,s'\rangle$
by *blast*
from $\langle\langle c_1,s\rangle \rightarrow \langle c_1',s''\rangle\rangle$ $\langle\langle c_1',s''\rangle \rightarrow* \langle Skip,sx\rangle\rangle$

have $\langle c_1,s\rangle \rightarrow* \langle Skip,sx\rangle$ **by**(*auto intro*:*converse-rtranclp-into-rtranclp*)
with $\langle\langle c_2,sx\rangle \rightarrow* \langle Skip,s'\rangle\rangle$ **show** *?thesis* **by** *auto*
qed
qed }
with $\langle\langle c_1;;c_2,s\rangle \rightarrow* \langle Skip,s'\rangle\rangle$ **show** *?thesis* **by** *simp*
qed
with *that* **show** *?thesis* **by** *blast*
qed

lemma *Cond-True-or-False*:
$\langle$*if* (*b*) c_1 *else* c_2,*s*$\rangle \rightarrow* \langle Skip,s'\rangle \Longrightarrow [\![b]\!]\ s = Some\ true \vee [\![b]\!]\ s = Some\ false$
by(*induct c*==*if* (*b*) c_1 *else* c_2 *s rule*:*converse-rtranclp-induct2*)(*auto elim*:*red.cases*)

lemma *CondTrue-reds*:
$\langle$*if* (*b*) c_1 *else* c_2,*s*$\rangle \rightarrow* \langle Skip,s'\rangle \Longrightarrow [\![b]\!]\ s = Some\ true \Longrightarrow \langle c_1,s\rangle \rightarrow* \langle Skip,s'\rangle$
by(*induct c*==*if* (*b*) c_1 *else* c_2 *s rule*:*converse-rtranclp-induct2*)(*auto elim*:*red.cases*)

lemma *CondFalse-reds*:
$\langle$*if* (*b*) c_1 *else* c_2,*s*$\rangle \rightarrow* \langle Skip,s'\rangle \Longrightarrow [\![b]\!]\ s = Some\ false \Longrightarrow \langle c_2,s\rangle \rightarrow* \langle Skip,s'\rangle$
by(*induct c*==*if* (*b*) c_1 *else* c_2 *s rule*:*converse-rtranclp-induct2*)(*auto elim*:*red.cases*)

lemma *WhileFalse-reds*:
$\langle$*while* (*b*) *cx*,*s*$\rangle \rightarrow* \langle Skip,s'\rangle \Longrightarrow [\![b]\!]\ s = Some\ false \Longrightarrow s = s'$
proof(*induct while* (*b*) *cx s rule*:*converse-rtranclp-induct2*)
case *step* **thus** *?case* **by**(*auto elim*:*red.cases dest*: *Skip-reds*)
qed

lemma *WhileTrue-reds*:
$\langle$*while* (*b*) *cx*,*s*$\rangle \rightarrow* \langle Skip,s'\rangle \Longrightarrow [\![b]\!]\ s = Some\ true$
$\Longrightarrow \exists\, sx.\ \langle cx,s\rangle \rightarrow* \langle Skip,sx\rangle \wedge \langle$*while* (*b*) *cx*,*sx*$\rangle \rightarrow* \langle Skip,s'\rangle$
proof(*induct while* (*b*) *cx s rule*:*converse-rtranclp-induct2*)
case (*step s c'' s''*)
hence $c'' = cx;;$*while* (*b*) *cx* $\wedge\ s'' = s$ **by**(*auto elim*:*red.cases*)
with $\langle\langle c'',s''\rangle \rightarrow* \langle Skip,s'\rangle\rangle$ **show** *?case* **by**(*auto dest*:*Seq-reds*)
qed

lemma *While-True-or-False*:
$\langle$*while* (*b*) *com*,*s*$\rangle \rightarrow* \langle Skip,s'\rangle \Longrightarrow [\![b]\!]\ s = Some\ true \vee [\![b]\!]\ s = Some\ false$
by(*induct c*==*while* (*b*) *com s rule*:*converse-rtranclp-induct2*)(*auto elim*:*red.cases*)

inductive *red-n* :: *com* $\Rightarrow$ *state* $\Rightarrow$ *nat* $\Rightarrow$ *com* $\Rightarrow$ *state* $\Rightarrow$ *bool*
$(\langle((1\langle\text{-},/\text{-}\rangle) \rightarrow^{\text{-}} (1\langle\text{-},/\text{-}\rangle))\rangle\ [0,0,0,0,0]\ 81)$
where *red-n-Base*: $\langle c,s\rangle \rightarrow^0 \langle c,s\rangle$
| *red-n-Rec*: $[\![\langle c,s\rangle \rightarrow \langle c'',s''\rangle;\ \langle c'',s''\rangle \rightarrow^n \langle c',s'\rangle]\!] \Longrightarrow \langle c,s\rangle \rightarrow^{Suc\ n} \langle c',s'\rangle$

lemma *Seq-red-nE*: **assumes** $\langle c_1;;c_2,s\rangle \rightarrow^n \langle Skip,s'\rangle$

```
  obtains i j s'' where ⟨c1,s⟩ →^i ⟨Skip,s''⟩ and ⟨c2,s''⟩ →^j ⟨Skip,s'⟩
  and n = i + j + 1
proof −
  from ‹⟨c1;;c2,s⟩ →^n ⟨Skip,s'⟩›
  have ∃ i j s''. ⟨c1,s⟩ →^i ⟨Skip,s''⟩ ∧ ⟨c2,s''⟩ →^j ⟨Skip,s'⟩ ∧ n = i + j + 1
  proof(induct c1;;c2 s n Skip s' arbitrary:c1 rule:red-n.induct)
    case (red-n-Rec s c'' s'' n s')
    note IH = ‹⋀c1. c'' = c1;;c2
      ⟹ ∃ i j sx. ⟨c1,s''⟩ →^i ⟨Skip,sx⟩ ∧ ⟨c2,sx⟩ →^j ⟨Skip,s'⟩ ∧ n = i + j + 1›
    from ‹⟨c1;;c2,s⟩ → ⟨c'',s''⟩›
    have (c1 = Skip ∧ c'' = c2 ∧ s = s'') ∨
      (∃ c1'. c'' = c1';;c2 ∧ ⟨c1,s⟩ → ⟨c1',s''⟩)
      by(induct c1;;c2 - - - rule:red-induct) auto
    thus ?case
    proof
      assume c1 = Skip ∧ c'' = c2 ∧ s = s''
      hence c1 = Skip and c'' = c2 and s = s'' by simp-all
      from ‹c1 = Skip› have ⟨c1,s⟩ →^0 ⟨Skip,s⟩ by(fastforce intro:red-n-Base)
      with ‹⟨c'',s''⟩ →^n ⟨Skip,s'⟩› ‹c'' = c2› ‹s = s''›
      show ?thesis by(rule-tac x=0 in exI) auto
    next
      assume ∃ c1'. c'' = c1';;c2 ∧ ⟨c1,s⟩ → ⟨c1',s''⟩
      then obtain c1' where c'' = c1';;c2 and ⟨c1,s⟩ → ⟨c1',s''⟩ by blast
      from IH[OF ‹c'' = c1';;c2›] obtain i j sx
        where ⟨c1',s''⟩ →^i ⟨Skip,sx⟩ and ⟨c2,sx⟩ →^j ⟨Skip,s'⟩
        and n = i + j + 1 by blast
      from ‹⟨c1,s⟩ → ⟨c1',s''⟩› ‹⟨c1',s''⟩ →^i ⟨Skip,sx⟩›
      have ⟨c1,s⟩ →^Suc i ⟨Skip,sx⟩ by(rule red-n.red-n-Rec)
      with ‹⟨c2,sx⟩ →^j ⟨Skip,s'⟩› ‹n = i + j + 1› show ?thesis
        by(rule-tac x=Suc i in exI) auto
    qed
  qed
  with that show ?thesis by blast
qed


lemma while-red-nE:
  ⟨while (b) cx,s⟩ →^n ⟨Skip,s'⟩
  ⟹ (⟦b⟧ s = Some false ∧ s = s' ∧ n = 1) ∨
     (∃ i j s''. ⟦b⟧ s = Some true ∧ ⟨cx,s⟩ →^i ⟨Skip,s''⟩ ∧
                 ⟨while (b) cx,s''⟩ →^j ⟨Skip,s'⟩ ∧ n = i + j + 2)
proof(induct while (b) cx s n Skip s' rule:red-n.induct)
  case (red-n-Rec s c'' s'' n s')
  from ‹⟨while (b) cx,s⟩ → ⟨c'',s''⟩›
  have (⟦b⟧ s = Some false ∧ c'' = Skip ∧ s'' = s) ∨
    (⟦b⟧ s = Some true ∧ c'' = cx;;while (b) cx ∧ s'' = s)
    by(induct while (b) cx - - - rule:red-induct) auto
  thus ?case
  proof
```

```
    assume ⟦b⟧ s = Some false ∧ c'' = Skip ∧ s'' = s
    hence ⟦b⟧ s = Some false and c'' = Skip and s'' = s by simp-all
    with ‹⟨c'',s''⟩ →^n ⟨Skip,s'⟩› have s = s' and n = 0
      by(induct - - Skip - rule:red-n.induct,auto elim:red.cases)
    with ‹⟦b⟧ s = Some false› show ?thesis by fastforce
  next
    assume ⟦b⟧ s = Some true ∧ c'' = cx;;while (b) cx ∧ s'' = s
    hence ⟦b⟧ s = Some true and c'' = cx;;while (b) cx
      and s'' = s by simp-all
    with ‹⟨c'',s''⟩ →^n ⟨Skip,s'⟩›
    obtain i j sx where ⟨cx,s⟩ →^i ⟨Skip,sx⟩ and ⟨while (b) cx,sx⟩ →^j ⟨Skip,s'⟩
      and n = i + j + 1 by(fastforce elim:Seq-red-nE)
    with ‹⟦b⟧ s = Some true› show ?thesis by fastforce
  qed
qed


lemma while-red-n-induct [consumes 1, case-names false true]:
  assumes major: ⟨while (b) cx,s⟩ →^n ⟨Skip,s'⟩
  and IHfalse:⋀s. ⟦b⟧ s = Some false ⟹ P s s
  and IHtrue:⋀s i j s''. ⟦ ⟦b⟧ s = Some true; ⟨cx,s⟩ →^i ⟨Skip,s''⟩;
                           ⟨while (b) cx,s''⟩ →^j ⟨Skip,s'⟩; P s'' s'⟧ ⟹ P s s'
  shows P s s'
using major
proof(induct n arbitrary:s rule:nat-less-induct)
  fix n s
  assume IHall:∀ m<n. ∀ x. ⟨while (b) cx,x⟩ →^m ⟨Skip,s'⟩ ⟶ P x s'
    and ⟨while (b) cx,s⟩ →^n ⟨Skip,s'⟩
  from ‹⟨while (b) cx,s⟩ →^n ⟨Skip,s'⟩›
  have (⟦b⟧ s = Some false ∧ s = s' ∧ n = 1) ∨
     (∃ i j s''. ⟦b⟧ s = Some true ∧ ⟨cx,s⟩ →^i ⟨Skip,s''⟩ ∧
              ⟨while (b) cx,s''⟩ →^j ⟨Skip,s'⟩ ∧ n = i + j + 2)
    by(rule while-red-nE)
  thus P s s'
  proof
    assume ⟦b⟧ s = Some false ∧ s = s' ∧ n = 1
    hence ⟦b⟧ s = Some false and s = s' by auto
    from IHfalse[OF ‹⟦b⟧ s = Some false›] have P s s .
    with ‹s = s'› show ?thesis by simp
  next
    assume ∃ i j s''. ⟦b⟧ s = Some true ∧ ⟨cx,s⟩ →^i ⟨Skip,s''⟩ ∧
                    ⟨while (b) cx,s''⟩ →^j ⟨Skip,s'⟩ ∧ n = i + j + 2
    then obtain i j s'' where ⟦b⟧ s = Some true
      and ⟨cx,s⟩ →^i ⟨Skip,s''⟩ and ⟨while (b) cx,s''⟩ →^j ⟨Skip,s'⟩
      and n = i + j + 2 by blast
    with IHall have P s'' s'
      apply(erule-tac x=j in allE) apply clarsimp done
    from IHtrue[OF ‹⟦b⟧ s = Some true› ‹⟨cx,s⟩ →^i ⟨Skip,s''⟩›
      ‹⟨while (b) cx,s''⟩ →^j ⟨Skip,s'⟩› this] show ?thesis .
```

qed
qed

lemma *reds-to-red-n*:$\langle c,s\rangle \to* \langle c',s'\rangle \Longrightarrow \exists\, n.\ \langle c,s\rangle \to^n \langle c',s'\rangle$
by(*induct rule*:*converse-rtranclp-induct2*,*auto intro*:*red-n.intros*)

lemma *red-n-to-reds*:$\langle c,s\rangle \to^n \langle c',s'\rangle \Longrightarrow \langle c,s\rangle \to* \langle c',s'\rangle$
by(*induct rule*:*red-n.induct*,*auto intro*:*converse-rtranclp-into-rtranclp*)

lemma *while-reds-induct*[*consumes 1*, *case-names false true*]:
$[\![\langle$*while* (b) $cx,s\rangle \to* \langle$*Skip*,$s'\rangle$; $\bigwedge s.\ [\![b]\!]\ s =$ *Some false* $\Longrightarrow P\ s\ s$;
$\bigwedge s\ s''.\ [\![\ [\![b]\!]\ s =$ *Some true*; $\langle cx,s\rangle \to* \langle$*Skip*,$s''\rangle$;
$\langle$*while* (b) $cx,s''\rangle \to* \langle$*Skip*,$s'\rangle$; $P\ s''\ s']\!] \Longrightarrow P\ s\ s']\!]$
$\Longrightarrow P\ s\ s'$
apply(*drule reds-to-red-n*,*clarsimp*)
apply(*erule while-red-n-induct*,*clarsimp*)
by(*auto dest*:*red-n-to-reds*)

lemma *red-det*:
$[\![\langle c,s\rangle \to \langle c_1,s_1\rangle$; $\langle c,s\rangle \to \langle c_2,s_2\rangle]\!] \Longrightarrow c_1 = c_2 \wedge s_1 = s_2$
proof(*induct arbitrary*:c_2 *rule*:*red-induct*)
case (*SeqRed* $c_1\ s\ c_1'\ s'\ c_2'$)
note *IH* = $\langle\bigwedge c_2.\ \langle c_1,s\rangle \to \langle c_2,s_2\rangle \Longrightarrow c_1' = c_2 \wedge s' = s_2\rangle$
from $\langle\langle c_1;;c_2',s\rangle \to \langle c_2,s_2\rangle\rangle$ **have** $c_1 =$ *Skip* $\vee$ $(\exists\, cx.\ c_2 = cx;;c_2' \wedge \langle c_1,s\rangle \to \langle cx,s_2\rangle)$
by(*fastforce elim*:*red.cases*)
thus *?case*
proof
assume $c_1 =$ *Skip*
with $\langle\langle c_1,s\rangle \to \langle c_1',s'\rangle\rangle$ **have** *False* **by**(*fastforce elim*:*red.cases*)
thus *?thesis* **by** *simp*
next
assume $\exists\, cx.\ c_2 = cx;;c_2' \wedge \langle c_1,s\rangle \to \langle cx,s_2\rangle$
then obtain *cx* **where** $c_2 = cx;;c_2'$ **and** $\langle c_1,s\rangle \to \langle cx,s_2\rangle$ **by** *blast*
from *IH*[*OF* $\langle\langle c_1,s\rangle \to \langle cx,s_2\rangle\rangle$] **have** $c_1' = cx \wedge s' = s_2$ **.**
with $\langle c_2 = cx;;c_2'\rangle$ **show** *?thesis* **by** *simp*
qed
qed (*fastforce elim*:*red.cases*)+

theorem *reds-det*:
$[\![\langle c,s\rangle \to* \langle$*Skip*,$s_1\rangle$; $\langle c,s\rangle \to* \langle$*Skip*,$s_2\rangle]\!] \Longrightarrow s_1 = s_2$
proof(*induct rule*:*converse-rtranclp-induct2*)
case *refl*
from $\langle\langle$*Skip*,$s_1\rangle \to* \langle$*Skip*,$s_2\rangle\rangle$ **show** *?case*

by −(*erule converse-rtranclpE*,*auto elim*:*red.cases*)
next
case (*step c'' s'' c' s'*)
note *IH* = ‹⟨*c'*,*s'*⟩ →* ⟨*Skip*,s_2⟩ ⟹ $s_1 = s_2$›
from *step* **have** ⟨*c''*,*s''*⟩ → ⟨*c'*,*s'*⟩
by *simp*
from ‹⟨*c''*,*s''*⟩ →* ⟨*Skip*,s_2⟩› *this* **have** ⟨*c'*,*s'*⟩ →* ⟨*Skip*,s_2⟩
by −(*erule converse-rtranclpE*,*auto elim*:*red.cases dest*:*red-det*)
from *IH*[*OF this*] **show** *?thesis* **.**
qed

end
theory *secTypes*
imports *Semantics*
begin

2 Security types

2.1 Security definitions

datatype *secLevel* = *Low* | *High*

type-synonym *typeEnv* = *vname* ⇀ *secLevel*

inductive *secExprTyping* :: *typeEnv* ⇒ *expr* ⇒ *secLevel* ⇒ *bool* (‹- ⊢ - : ->)
where *typeVal*: Γ ⊢ *Val V* : *lev*

| *typeVar*: Γ *Vn* = *Some lev* ⟹ Γ ⊢ *Var Vn* : *lev*

| *typeBinOp1*: ⟦Γ ⊢ *e1* : *Low*; Γ ⊢ *e2* : *Low*⟧ ⟹ Γ ⊢ *e1* «*bop*» *e2* : *Low*

| *typeBinOp2*: ⟦Γ ⊢ *e1* : *High*; Γ ⊢ *e2* : *lev*⟧ ⟹ Γ ⊢ *e1* «*bop*» *e2* : *High*

| *typeBinOp3*: ⟦Γ ⊢ *e1* : *lev*; Γ ⊢ *e2* : *High*⟧ ⟹ Γ ⊢ *e1* «*bop*» *e2* : *High*

inductive *secComTyping* :: *typeEnv* ⇒ *secLevel* ⇒ *com* ⇒ *bool* (‹-,- ⊢ ->)
where *typeSkip*: Γ,*T* ⊢ *Skip*

| *typeAssH*: Γ *V* = *Some High* ⟹ Γ,*T* ⊢ *V* := *e*

| *typeAssL*: ⟦Γ ⊢ *e* : *Low*; Γ *V* = *Some Low*⟧ ⟹ Γ,*Low* ⊢ *V* := *e*

| *typeSeq*: ⟦Γ,*T* ⊢ *c1*; Γ,*T* ⊢ *c2*⟧ ⟹ Γ,*T* ⊢ *c1*;;*c2*

| *typeWhile*: ⟦Γ ⊢ *b* : *T*; Γ,*T* ⊢ *c*⟧ ⟹ Γ,*T* ⊢ *while* (*b*) *c*

| *typeIf*: $[\![\Gamma \vdash b : T; \Gamma,T \vdash c1; \Gamma,T \vdash c2]\!] \Longrightarrow \Gamma,T \vdash \mathit{if}\ (b)\ c1\ \mathit{else}\ c2$

| *typeConvert*: $\Gamma,\mathit{High} \vdash c \Longrightarrow \Gamma,\mathit{Low} \vdash c$

2.2 Lemmas concerning expressions

lemma *exprTypeable*:
assumes $fv\ e \subseteq dom\ \Gamma$ **obtains** T **where** $\Gamma \vdash e : T$
proof −
from ‹$fv\ e \subseteq dom\ \Gamma$› **have** $\exists\, T.\ \Gamma \vdash e : T$
proof(*induct e*)
case (*Val V*)
have $\Gamma \vdash \mathit{Val}\ V : \mathit{Low}$ **by**(*rule typeVal*)
thus *?case* **by** (*rule exI*)
next
case (*Var V*)
have $V \in fv\ (\mathit{Var}\ V)$ **by** *simp*
with ‹$fv\ (\mathit{Var}\ V) \subseteq dom\ \Gamma$› **have** $V \in dom\ \Gamma$ **by** *simp*
then obtain T **where** $\Gamma\ V = \mathit{Some}\ T$ **by** *auto*
hence $\Gamma \vdash \mathit{Var}\ V : T$ **by** (*rule typeVar*)
thus *?case* **by** (*rule exI*)
next
case (*BinOp e1 bop e2*)
note *IH1* = ‹$fv\ e1 \subseteq dom\ \Gamma \Longrightarrow \exists\, T.\ \Gamma \vdash e1 : T$›
note *IH2* = ‹$fv\ e2 \subseteq dom\ \Gamma \Longrightarrow \exists\, T.\ \Gamma \vdash e2 : T$›
from ‹$fv\ (e1\ «bop»\ e2) \subseteq dom\ \Gamma$›
have $fv\ e1 \subseteq dom\ \Gamma$ **and** $fv\ e2 \subseteq dom\ \Gamma$ **by** *auto*
from *IH1*[*OF* ‹$fv\ e1 \subseteq dom\ \Gamma$›] **obtain** *T1* **where** $\Gamma \vdash e1 : T1$ **by** *auto*
from *IH2*[*OF* ‹$fv\ e2 \subseteq dom\ \Gamma$›] **obtain** *T2* **where** $\Gamma \vdash e2 : T2$ **by** *auto*
show *?case*
proof (*cases T1*)
case *Low*
show *?thesis*
proof (*cases T2*)
case *Low*
with ‹$\Gamma \vdash e1 : T1$› ‹$\Gamma \vdash e2 : T2$› ‹$T1 = \mathit{Low}$›
have $\Gamma \vdash e1\ «bop»\ e2 : \mathit{Low}$ **by**(*simp add*:*typeBinOp1*)
thus *?thesis* **by**(*rule exI*)
next
case *High*
with ‹$\Gamma \vdash e1 : T1$› ‹$\Gamma \vdash e2 : T2$› ‹$T1 = \mathit{Low}$›
have $\Gamma \vdash e1\ «bop»\ e2 : \mathit{High}$ **by**(*simp add*:*typeBinOp3*)
thus *?thesis* **by**(*rule exI*)
qed
next
case *High*
with ‹$\Gamma \vdash e1 : T1$› ‹$\Gamma \vdash e2 : T2$›
have $\Gamma \vdash e1\ «bop»\ e2 : \mathit{High}$ **by** (*simp add*: *typeBinOp2*)
thus *?thesis* **by** (*rule exI*)

```
    qed
  qed
  with that show ?thesis by blast
qed


lemma exprBinopTypeable:
  assumes Γ ⊢ e1 «bop» e2 : T
  shows (∃ T1. Γ ⊢ e1 : T1) ∧ (∃ T2. Γ ⊢ e2 : T2)
using assms by(auto elim:secExprTyping.cases)



lemma exprTypingHigh:
  assumes Γ ⊢ e : T and x ∈ fv e and Γ x = Some High
  shows Γ ⊢ e : High
using assms
proof (induct e arbitrary:T)
  case (Val V) show ?case by (rule typeVal)
next
  case (Var V)
  from ‹x ∈ fv (Var V)› have x = V by simp
  with ‹Γ x = Some High› show ?case by(simp add:typeVar)
next
  case (BinOp e1 bop e2)
  note IH1 = ‹⋀T. ⟦Γ ⊢ e1 : T; x ∈ fv e1; Γ x = Some High⟧ ⟹ Γ ⊢ e1 : High›
  note IH2 = ‹⋀T. ⟦Γ ⊢ e2 : T; x ∈ fv e2; Γ x = Some High⟧ ⟹ Γ ⊢ e2 : High›
  from ‹Γ ⊢ e1 «bop» e2 : T›
 have T:(∃ T1. Γ ⊢ e1 : T1) ∧ (∃ T2. Γ ⊢ e2 : T2) by (auto intro!:exprBinopTypeable)
  then obtain T1 where Γ ⊢ e1 : T1 by auto
  from T obtain T2 where Γ ⊢ e2 : T2 by auto
  from ‹x ∈ fv (e1 «bop» e2)› have x ∈ (fv e1 ∪ fv e2) by simp
  hence x ∈ fv e1 ∨ x ∈ fv e2 by auto
  thus ?case
  proof
    assume x ∈ fv e1
    from IH1[OF ‹Γ ⊢ e1 : T1› this ‹Γ x = Some High›] have Γ ⊢ e1 : High .
    with ‹Γ ⊢ e2 : T2› show ?thesis by(simp add:typeBinOp2)
  next
    assume x ∈ fv e2
    from IH2[OF ‹Γ ⊢ e2 : T2› this ‹Γ x = Some High›] have Γ ⊢ e2 : High .
    with ‹Γ ⊢ e1 : T1› show ?thesis by(simp add:typeBinOp3)
  qed
qed


lemma exprTypingLow:
  assumes Γ ⊢ e : Low and x ∈ fv e shows Γ x = Some Low
using assms
```

```
proof (induct e)
  case (Val V)
  have fv (Val V) = {} by (rule FVc)
  with ‹x ∈ fv (Val V)› have False by auto
  thus ?thesis by simp
next
  case (Var V)
  from ‹x ∈ fv (Var V)› have xV: x = V by simp
 from ‹Γ ⊢ Var V : Low› have Γ V = Some Low by (auto elim:secExprTyping.cases)
  with xV show ?thesis by simp
next
  case (BinOp e1 bop e2)
  note IH1 = ‹⟦Γ ⊢ e1 : Low; x ∈ fv e1⟧ ⟹ Γ x = Some Low›
  note IH2=‹⟦Γ ⊢ e2 : Low; x ∈ fv e2⟧ ⟹ Γ x = Some Low›
  from ‹Γ ⊢ e1 «bop» e2 : Low› have Γ ⊢ e1 : Low and Γ ⊢ e2 : Low
    by(auto elim:secExprTyping.cases)
  from ‹x ∈ fv (e1 «bop» e2)› have x ∈ fv e1 ∪ fv e2 by (simp add:FVe)
  hence x ∈ fv e1 ∨ x ∈ fv e2 by auto
  thus ?case
  proof
    assume x ∈ fv e1
    with IH1[OF ‹Γ ⊢ e1 : Low›] show ?thesis by auto
  next
    assume x ∈ fv e2
    with IH2[OF ‹Γ ⊢ e2 : Low›] show ?thesis by auto
  qed
qed


lemma typeableFreevars:
  assumes Γ ⊢ e : T shows fv e ⊆ dom Γ
using assms
proof(induct e arbitrary:T)
  case (Val V)
  have fv (Val V) = {} by (rule FVc)
  thus ?case by simp
next
  case (Var V)
  show ?case
  proof
    fix x assume x ∈ fv (Var V)
    hence x = V by simp
   from ‹Γ ⊢ Var V : T› have Γ V = Some T by (auto elim:secExprTyping.cases)
    with ‹x = V› show x ∈ dom Γ by auto
  qed
next
  case (BinOp e1 bop e2)
  note IH1 = ‹⋀T. Γ ⊢ e1 : T ⟹ fv e1 ⊆ dom Γ›
  note IH2 = ‹⋀T. Γ ⊢ e2 : T ⟹ fv e2 ⊆ dom Γ›
```

```
  show ?case
  proof
    fix x assume x ∈ fv (e1 «bop» e2)
    from ‹Γ ⊢ e1 «bop» e2 : T›
    have Q:(∃ T1. Γ ⊢ e1 : T1) ∧ (∃ T2. Γ ⊢ e2 : T2)
      by(rule exprBinopTypeable)
    then obtain T1 where Γ ⊢ e1 : T1 by blast
    from Q obtain T2 where Γ ⊢ e2 : T2 by blast
    from IH1[OF ‹Γ ⊢ e1 : T1›] have fv e1 ⊆ dom Γ .
    moreover
    from IH2[OF ‹Γ ⊢ e2 : T2›] have fv e2 ⊆ dom Γ .
    ultimately have (fv e1) ∪ (fv e2) ⊆ dom Γ by auto
    hence fv (e1 «bop» e2) ⊆ dom Γ by(simp add:FVe)
    with ‹x ∈ fv (e1 «bop» e2)› show x ∈ dom Γ by auto
  qed
qed



lemma exprNotNone:
assumes Γ ⊢ e : T and fv e ⊆ dom s
shows ⟦e⟧ s ≠ None
using assms
proof (induct e arbitrary: Γ T s)
  case (Val v)
  show ?case by(simp add:Val)
next
  case (Var V)
  have ⟦Var V⟧ s = s V by (simp add:Var)
  have V ∈ fv (Var V) by (auto simp add:FVv)
  with ‹fv (Var V) ⊆ dom s› have V ∈ dom s by simp
  thus ?case by auto
next
  case (BinOp e1 bop e2)
  note IH1 = ‹⋀T. ⟦Γ ⊢ e1 : T; fv e1 ⊆ dom s ⟧ ⟹ ⟦e1⟧ s ≠ None›
  note IH2 = ‹⋀T. ⟦Γ ⊢ e2 : T; fv e2 ⊆ dom s ⟧ ⟹ ⟦e2⟧ s ≠ None›
  from ‹Γ ⊢ e1 «bop» e2 : T› have (∃ T1. Γ ⊢ e1 : T1) ∧ (∃ T2. Γ ⊢ e2 : T2)
    by(rule exprBinopTypeable)
  then obtain T1 T2 where Γ ⊢ e1 : T1 and Γ ⊢ e2 : T2 by blast
 from ‹fv (e1 «bop» e2) ⊆ dom s› have fv e1 ∪ fv e2 ⊆ dom s by(simp add:FVe)
  hence fv e1 ⊆ dom s and fv e2 ⊆ dom s by auto
  from IH1[OF ‹Γ ⊢ e1 : T1› ‹fv e1 ⊆ dom s›] have ⟦e1⟧s ≠ None .
  moreover from IH2[OF ‹Γ ⊢ e2 : T2› ‹fv e2 ⊆ dom s›] have ⟦e2⟧s ≠ None .
  ultimately show ?case
    apply(cases bop) apply auto
    apply(case-tac y,auto,case-tac ya,auto)+
    done
qed
```

2.3 Noninterference definitions

2.3.1 Low Equivalence

Low Equivalence is reflexive even if the involved states are undefined. But in non-reflexive situations low variables must be initialized (i.e. $\in$ *dom state*), otherwise the proof will not work. This is not a restriction, but a natural requirement, and could be formalized as part of a standard type system.

Low equivalence is also symmetric and transitiv (see lemmas) hence an equivalence relation.

definition *lowEquiv* :: *typeEnv* $\Rightarrow$ *state* $\Rightarrow$ *state* $\Rightarrow$ *bool* ($\langle$- $\vdash$ - $\approx_L$ -$\rangle$)
where $\Gamma \vdash s1 \approx_L s2 \equiv \forall v{\in}dom\ \Gamma.\ \Gamma\ v = Some\ Low \longrightarrow (s1\ v = s2\ v)$

lemma *lowEquivReflexive*: $\Gamma \vdash s1 \approx_L s1$
by(*simp add*:*lowEquiv-def*)

lemma *lowEquivSymmetric*:
$\Gamma \vdash s1 \approx_L s2 \Longrightarrow \Gamma \vdash s2 \approx_L s1$
by(*simp add*:*lowEquiv-def*)

lemma *lowEquivTransitive*:
$[\![\Gamma \vdash s1 \approx_L s2; \Gamma \vdash s2 \approx_L s3]\!] \Longrightarrow \Gamma \vdash s1 \approx_L s3$
by(*simp add*:*lowEquiv-def*)

2.3.2 Non Interference

definition *nonInterference* :: *typeEnv* $\Rightarrow$ *com* $\Rightarrow$ *bool*
where *nonInterference* $\Gamma\ c \equiv$
$(\forall s1\ s2\ s1'\ s2'.\ (\Gamma \vdash s1 \approx_L s2 \wedge \langle c,s1\rangle \rightarrow* \langle Skip,s1'\rangle \wedge \langle c,s2\rangle \rightarrow* \langle Skip,s2'\rangle)$
$\longrightarrow \Gamma \vdash s1' \approx_L s2')$

lemma *nonInterferenceI*:
$[\![\bigwedge s1\ s2\ s1'\ s2'.\ [\![\Gamma \vdash s1 \approx_L s2; \langle c,s1\rangle \rightarrow* \langle Skip,s1'\rangle; \langle c,s2\rangle \rightarrow* \langle Skip,s2'\rangle]\!]$
$\Longrightarrow \Gamma \vdash s1' \approx_L s2']\!] \Longrightarrow nonInterference\ \Gamma\ c$
by(*auto simp*:*nonInterference-def*)

lemma *interpretLow*:
assumes $\Gamma \vdash s1 \approx_L s2$ **and** *all*:$\forall\ V{\in}fv\ e.\ \Gamma\ V = Some\ Low$
shows $[\![e]\!]\ s1 = [\![e]\!]\ s2$
using *all*
proof (*induct e*)
case (*Val v*)
show *?case* **by** (*simp add*: *Val*)
next
case (*Var V*)
have $[\![Var\ V]\!]\ s1 = s1\ V$ **and** $[\![Var\ V]\!]\ s2 = s2\ V$ **by**(*auto simp*:*Var*)
have $V \in fv\ (Var\ V)$ **by**(*simp add*:*FVv*)

from ‹$V \in fv\ (Var\ V)$› ‹$\forall X{\in}fv\ (Var\ V).\ \Gamma\ X = Some\ Low$› **have** $\Gamma\ V = Some\ Low$ **by** *simp*
with *assms* **have** $s1\ V = s2\ V$ **by**(*auto simp add*:*lowEquiv-def*)
thus *?case* **by** *auto*
next
case (*BinOp e1 bop e2*)
note *IH1* = ‹$\forall V{\in}fv\ e1.\ \Gamma\ V = Some\ Low \Longrightarrow [\![e1]\!]s1 = [\![e1]\!]s2$›
note *IH2* = ‹$\forall V{\in}fv\ e2.\ \Gamma\ V = Some\ Low \Longrightarrow [\![e2]\!]s1 = [\![e2]\!]s2$›
from ‹$\forall V{\in}fv\ (e1\ «bop»\ e2).\ \Gamma\ V = Some\ Low$› **have** $\forall V{\in}fv\ e1.\ \Gamma\ V = Some\ Low$
and $\forall V{\in}fv\ e2.\ \Gamma\ V = Some\ Low$ **by** *auto*
from *IH1*[*OF* ‹$\forall V{\in}fv\ e1.\ \Gamma\ V = Some\ Low$›] **have** $[\![e1]\!]\ s1 = [\![e1]\!]\ s2$ **.**
moreover
from *IH2*[*OF* ‹$\forall V{\in}fv\ e2.\ \Gamma\ V = Some\ Low$›] **have** $[\![e2]\!]\ s1 = [\![e2]\!]\ s2$ **.**
ultimately show *?case* **by**(*cases* $[\![e1]\!]\ s2$,*auto*)
qed

lemma *interpretLow2*:
assumes $\Gamma \vdash e : Low$ **and** $\Gamma \vdash s1 \approx_L s2$ **shows** $[\![e]\!]\ s1 = [\![e]\!]\ s2$
proof −
from ‹$\Gamma \vdash e : Low$› **have** $fv\ e \subseteq dom\ \Gamma$ **by**(*auto dest*:*typeableFreevars*)
have $\forall x{\in}\ fv\ e.\ \Gamma\ x = Some\ Low$
proof
fix x **assume** $x \in fv\ e$
with ‹$\Gamma \vdash e : Low$› **show** $\Gamma\ x = Some\ Low$ **by**(*auto intro*:*exprTypingLow*)
qed
with ‹$\Gamma \vdash s1 \approx_L s2$› **show** *?thesis* **by**(*rule interpretLow*)
qed

lemma *assignNIhighlemma*:
assumes $\Gamma \vdash s1 \approx_L s2$ **and** $\Gamma\ V = Some\ High$ **and** $s1' = s1(V{:=}\ [\![e]\!]\ s1)$
and $s2' = s2(V{:=}\ [\![e]\!]\ s2)$
shows $\Gamma \vdash s1' \approx_L s2'$
proof −
{ fix V' **assume** $V' \in dom\ \Gamma$ **and** $\Gamma\ V' = Some\ Low$
from ‹$\Gamma \vdash s1 \approx_L s2$› ‹$\Gamma\ V' = Some\ Low$› **have** $s1\ V' = s2\ V'$
by(*auto simp add*:*lowEquiv-def*)
have $s1'\ V' = s2'\ V'$
proof(*cases* $V' = V$)
case *True*
with ‹$\Gamma\ V' = Some\ Low$› ‹$\Gamma\ V = Some\ High$› **have** *False* **by** *simp*
thus *?thesis* **by** *simp*
next
case *False*
with ‹$s1' = s1(V{:=}\ [\![e]\!]\ s1)$› ‹$s2' = s2(V{:=}\ [\![e]\!]\ s2)$›
have $s1\ V' = s1'\ V'$ **and** $s2\ V' = s2'\ V'$ **by** *auto*
with ‹$s1\ V' = s2\ V'$› **show** *?thesis* **by** *simp*

```
    qed
  }
  thus ?thesis by(auto simp add:lowEquiv-def)
qed



lemma assignNIlowlemma:
  assumes Γ ⊢ s1 ≈L s2 and Γ V = Some Low and Γ ⊢ e : Low
  and s1' = s1(V:= ⟦e⟧ s1) and s2' = s2(V:= ⟦e⟧ s2)
  shows Γ ⊢ s1' ≈L s2'
proof −
  { fix V' assume V' ∈ dom Γ and Γ V' = Some Low
    from ‹Γ ⊢ s1 ≈L s2› ‹Γ V' = Some Low›
    have s1 V' = s2 V' by(auto simp add:lowEquiv-def)
    have s1' V' = s2' V'
    proof(cases V' = V)
      case True
      with ‹s1' = s1(V:= ⟦e⟧ s1)› ‹s2' = s2(V:= ⟦e⟧ s2)›
      have s1' V' = ⟦e⟧ s1 and s2' V' = ⟦e⟧ s2 by auto
      from ‹Γ ⊢ e : Low› ‹Γ ⊢ s1 ≈L s2› have ⟦e⟧ s1 = ⟦e⟧ s2
        by(auto intro:interpretLow2)
      with ‹s1' V' = ⟦e⟧ s1› ‹s2' V' = ⟦e⟧ s2› show ?thesis by simp
    next
      case False
      with ‹s1' = s1(V:= ⟦e⟧ s1)› ‹s2' = s2(V:= ⟦e⟧ s2)›
      have s1' V' = s1 V' and s2' V' = s2 V' by auto
      with ‹s1 V' = s2 V'› have s1' V' = s2' V' by simp
      with False ‹s1' V' = s1 V'› ‹s2' V' = s2 V'›
      show ?thesis by auto
    qed
  }
  thus ?thesis by(simp add:lowEquiv-def)
qed
```

Sequential Compositionality is given the status of a theorem because compositionality is no longer valid in case of concurrency

```
theorem SeqCompositionality:
  assumes nonInterference Γ c1 and nonInterference Γ c2
  shows nonInterference Γ (c1;;c2)
proof(rule nonInterferenceI)
  fix s1 s2 s1' s2'
  assume Γ ⊢ s1 ≈L s2 and ⟨c1;;c2,s1⟩ →* ⟨Skip,s1'⟩
    and ⟨c1;;c2,s2⟩ →* ⟨Skip,s2'⟩
  from ‹⟨c1;;c2,s1⟩ →* ⟨Skip,s1'⟩› obtain s1'' where ⟨c1,s1⟩ →* ⟨Skip,s1''⟩
    and ⟨c2,s1''⟩ →* ⟨Skip,s1'⟩ by(auto dest:Seq-reds)
  from ‹⟨c1;;c2,s2⟩ →* ⟨Skip,s2'⟩› obtain s2'' where ⟨c1,s2⟩ →* ⟨Skip,s2''⟩
    and ⟨c2,s2''⟩ →* ⟨Skip,s2'⟩ by(auto dest:Seq-reds)
  from ‹Γ ⊢ s1 ≈L s2› ‹⟨c1,s1⟩ →* ⟨Skip,s1''⟩› ‹⟨c1,s2⟩ →* ⟨Skip,s2''⟩›
```

```
    ‹nonInterference Γ c1›
  have Γ ⊢ s1'' ≈L s2'' by(auto simp:nonInterference-def)
  with ‹⟨c2,s1''⟩ →* ⟨Skip,s1'⟩› ‹⟨c2,s2''⟩ →* ⟨Skip,s2'⟩› ‹nonInterference Γ c2›

  show Γ ⊢ s1' ≈L s2' by(auto simp:nonInterference-def)
qed



lemma WhileStepInduct:
  assumes while:⟨while (b) c,s1⟩ →* ⟨Skip,s2⟩
  and body:⋀s1 s2. ⟨c,s1⟩ →* ⟨Skip,s2⟩ ⟹ Γ ⊢ s1 ≈L s2 and Γ,High ⊢ c
  shows Γ ⊢ s1 ≈L s2
using while
proof (induct rule:while-reds-induct)
  case (false s) thus ?case by(auto simp add:lowEquiv-def)
next
  case (true s1 s3)
  from body[OF ‹⟨c,s1⟩ →* ⟨Skip,s3⟩›] have Γ ⊢ s1 ≈L s3 by simp
  with ‹Γ ⊢ s3 ≈L s2› show ?case by(auto intro:lowEquivTransitive)
qed
```

In case control conditions from if/while are high, the body of an if/while must not change low variables in order to prevent implicit flow. That is, start and end state of any if/while body must be low equivalent.

```
theorem highBodies:
  assumes Γ,High ⊢ c and ⟨c,s1⟩ →* ⟨Skip,s2⟩
  shows Γ ⊢ s1 ≈L s2
  — all intermediate states must be well formed, otherwise the proof does not work
for uninitialized variables. Thus it is propagated through the theorem conclusion
using assms
proof(induct c arbitrary:s1 s2 rule:com.induct)
  case Skip
  from ‹⟨Skip,s1⟩ →* ⟨Skip,s2⟩› have s1 = s2 by (auto dest:Skip-reds)
  thus ?case by(simp add:lowEquiv-def)
next
  case (LAss V e)
 from ‹Γ,High ⊢ V:=e› have Γ V = Some High by(auto elim:secComTyping.cases)
 from ‹⟨V:=e,s1⟩ →* ⟨Skip,s2⟩› have s2 = s1(V:= [[e]]s1) by (auto intro:LAss-reds)
 { fix V' assume V' ∈ dom Γ and Γ V' = Some Low
   have s1 V' = s2 V'
   proof(cases V' = V)
     case True
     with ‹Γ V' = Some Low› ‹Γ V = Some High› have False by simp
     thus ?thesis by simp
   next
     case False
     with ‹s2 = s1(V:= [[e]]s1)› show ?thesis by simp
   qed
```

```
  }
  thus ?case by(auto simp add:lowEquiv-def)
next
  case (Seq c1 c2)
  note IH1 = ‹⋀s1 s2. ⟦Γ,High ⊢ c1; ⟨c1,s1⟩ →* ⟨Skip,s2⟩⟧ ⟹ Γ ⊢ s1 ≈L s2›
  note IH2 = ‹⋀s1 s2. ⟦Γ,High ⊢ c2; ⟨c2,s1⟩ →* ⟨Skip,s2⟩⟧ ⟹ Γ ⊢ s1 ≈L s2›
  from ‹Γ,High ⊢ c1;;c2› have Γ,High ⊢ c1 and Γ,High ⊢ c2
    by(auto elim:secComTyping.cases)
  from ‹⟨c1;;c2,s1⟩ →* ⟨Skip,s2⟩›
  have ∃s3. ⟨c1,s1⟩ →* ⟨Skip,s3⟩ ∧ ⟨c2,s3⟩ →* ⟨Skip,s2⟩ by(auto intro:Seq-reds)
  then obtain s3 where ⟨c1,s1⟩ →* ⟨Skip,s3⟩ and ⟨c2,s3⟩ →* ⟨Skip,s2⟩ by auto
  from IH1[OF ‹Γ,High ⊢ c1› ‹⟨c1,s1⟩ →* ⟨Skip,s3⟩›]
  have Γ ⊢ s1 ≈L s3 by simp
  from IH2[OF ‹Γ,High ⊢ c2› ‹⟨c2,s3⟩ →* ⟨Skip,s2⟩›]
  have Γ ⊢ s3 ≈L s2 by simp
  from ‹Γ ⊢ s1 ≈L s3› ‹Γ ⊢ s3 ≈L s2› show ?case
    by(auto intro:lowEquivTransitive)
next
  case (Cond b c1 c2)
  note IH1 = ‹⋀s1 s2. ⟦Γ,High ⊢ c1; ⟨c1,s1⟩ →* ⟨Skip,s2⟩⟧ ⟹ Γ ⊢ s1 ≈L s2›
  note IH2 = ‹⋀s1 s2. ⟦Γ,High ⊢ c2; ⟨c2,s1⟩ →* ⟨Skip,s2⟩⟧ ⟹ Γ ⊢ s1 ≈L s2›
  from ‹Γ,High ⊢ if (b) c1 else c2› have Γ,High ⊢ c1 and Γ,High ⊢ c2
    by(auto elim:secComTyping.cases)
  from ‹⟨if (b) c1 else c2,s1⟩ →* ⟨Skip,s2⟩›
 have ⟦b⟧ s1 = Some true ∨ ⟦b⟧ s1 = Some false by(auto dest:Cond-True-or-False)
  thus ?case
  proof
    assume ⟦b⟧ s1 = Some true
    with ‹⟨if (b) c1 else c2,s1⟩ →* ⟨Skip,s2⟩› have ⟨c1,s1⟩ →* ⟨Skip,s2⟩
      by (auto intro:CondTrue-reds)
    from IH1[OF ‹Γ,High ⊢ c1› this] show ?thesis .
  next
    assume ⟦b⟧ s1 = Some false
    with ‹⟨if (b) c1 else c2,s1⟩ →* ⟨Skip,s2⟩› have ⟨c2,s1⟩ →* ⟨Skip,s2⟩
      by(auto intro:CondFalse-reds)
    from IH2[OF ‹Γ,High ⊢ c2› this] show ?thesis .
  qed
next
  case (While b c')
  note IH = ‹⋀s1 s2. ⟦Γ,High ⊢ c'; ⟨c',s1⟩ →* ⟨Skip,s2⟩⟧ ⟹ Γ ⊢ s1 ≈L s2›
 from ‹Γ,High ⊢ while (b) c'› have Γ,High ⊢ c' by(auto elim:secComTyping.cases)
  from IH[OF this]
  have ⋀s1 s2. ⟦⟨c',s1⟩ →* ⟨Skip,s2⟩⟧ ⟹ Γ ⊢ s1 ≈L s2 .
  with ‹⟨while (b) c',s1⟩ →* ⟨Skip,s2⟩› ‹Γ,High ⊢ c'›
  show ?case by(auto dest:WhileStepInduct)
qed
```

lemma *CondHighCompositionality*:
assumes Γ,*High* $\vdash$ *if* (*b*) *c1 else c2*
shows *nonInterference* Γ (*if* (*b*) *c1 else c2*)
proof(*rule nonInterferenceI*)
fix *s1 s2 s1' s2'*
assume $\Gamma \vdash$ *s1* $\approx_L$ *s2* **and** $\langle$*if* (*b*) *c1 else c2*,*s1*$\rangle \rightarrow* \langle$*Skip*,*s1'*$\rangle$
and $\langle$*if* (*b*) *c1 else c2*,*s2*$\rangle \rightarrow* \langle$*Skip*,*s2'*$\rangle$
show $\Gamma \vdash$ *s1'* $\approx_L$ *s2'*
proof −
from ‹Γ,*High* $\vdash$ *if* (*b*) *c1 else c2*› ‹$\langle$*if* (*b*) *c1 else c2*,*s1*$\rangle \rightarrow* \langle$*Skip*,*s1'*$\rangle$›
have $\Gamma \vdash$ *s1* $\approx_L$ *s1'* **by**(*auto dest*:*highBodies*)
from ‹Γ,*High* $\vdash$ *if* (*b*) *c1 else c2*› ‹$\langle$*if* (*b*) *c1 else c2*,*s2*$\rangle \rightarrow* \langle$*Skip*,*s2'*$\rangle$›
have $\Gamma \vdash$ *s2* $\approx_L$ *s2'* **by**(*auto dest*:*highBodies*)
with ‹$\Gamma \vdash$ *s1* $\approx_L$ *s2*› **have** $\Gamma \vdash$ *s1* $\approx_L$ *s2'* **by**(*auto intro*:*lowEquivTransitive*)
from ‹$\Gamma \vdash$ *s1* $\approx_L$ *s1'*› **have** $\Gamma \vdash$ *s1'* $\approx_L$ *s1* **by**(*auto intro*:*lowEquivSymmetric*)
with ‹$\Gamma \vdash$ *s1* $\approx_L$ *s2'*› **show** *?thesis* **by**(*auto intro*:*lowEquivTransitive*)
qed
qed

lemma *CondLowCompositionality*:
assumes *nonInterference* Γ *c1* **and** *nonInterference* Γ *c2* **and** $\Gamma \vdash b : Low$
shows *nonInterference* Γ (*if* (*b*) *c1 else c2*)
proof(*rule nonInterferenceI*)
fix *s1 s2 s1' s2'*
assume $\Gamma \vdash$ *s1* $\approx_L$ *s2* **and** $\langle$*if* (*b*) *c1 else c2*,*s1*$\rangle \rightarrow* \langle$*Skip*,*s1'*$\rangle$
and $\langle$*if* (*b*) *c1 else c2*,*s2*$\rangle \rightarrow* \langle$*Skip*,*s2'*$\rangle$
from ‹$\Gamma \vdash b : Low$› ‹$\Gamma \vdash$ *s1* $\approx_L$ *s2*› **have** $[\![b]\!]$ *s1* = $[\![b]\!]$ *s2*
by(*auto intro*:*interpretLow2*)
from ‹$\langle$*if* (*b*) *c1 else c2*,*s1*$\rangle \rightarrow* \langle$*Skip*,*s1'*$\rangle$›
have $[\![b]\!]$ *s1* = *Some true* $\vee$ $[\![b]\!]$ *s1* = *Some false* **by**(*auto dest*:*Cond-True-or-False*)
thus $\Gamma \vdash$ *s1'* $\approx_L$ *s2'*
proof
assume $[\![b]\!]$ *s1* = *Some true*
with ‹$[\![b]\!]$ *s1* = $[\![b]\!]$ *s2*› **have** $[\![b]\!]$ *s2* = *Some true* **by**(*auto intro*:*CondTrue-reds*)
from ‹$[\![b]\!]$ *s1* = *Some true*› ‹$\langle$*if* (*b*) *c1 else c2*,*s1*$\rangle \rightarrow* \langle$*Skip*,*s1'*$\rangle$›
have $\langle$*c1*,*s1*$\rangle \rightarrow* \langle$*Skip*,*s1'*$\rangle$ **by**(*auto intro*:*CondTrue-reds*)
from ‹$[\![b]\!]$ *s2* = *Some true*› ‹$\langle$*if* (*b*) *c1 else c2*,*s2*$\rangle \rightarrow* \langle$*Skip*,*s2'*$\rangle$›
have $\langle$*c1*,*s2*$\rangle \rightarrow* \langle$*Skip*,*s2'*$\rangle$ **by**(*auto intro*:*CondTrue-reds*)
with ‹$\langle$*c1*,*s1*$\rangle \rightarrow* \langle$*Skip*,*s1'*$\rangle$› ‹$\Gamma \vdash$ *s1* $\approx_L$ *s2*› ‹*nonInterference* Γ *c1*›
show *?thesis* **by**(*auto simp*:*nonInterference-def*)
next
assume $[\![b]\!]$ *s1* = *Some false*
with ‹$[\![b]\!]$ *s1* = $[\![b]\!]$ *s2*› **have** $[\![b]\!]$ *s2* = *Some false* **by**(*auto intro*:*CondTrue-reds*)
from ‹$[\![b]\!]$ *s1* = *Some false*› ‹$\langle$*if* (*b*) *c1 else c2*,*s1*$\rangle \rightarrow* \langle$*Skip*,*s1'*$\rangle$›
have $\langle$*c2*,*s1*$\rangle \rightarrow* \langle$*Skip*,*s1'*$\rangle$ **by**(*auto intro*:*CondFalse-reds*)
from ‹$[\![b]\!]$ *s2* = *Some false*› ‹$\langle$*if* (*b*) *c1 else c2*,*s2*$\rangle \rightarrow* \langle$*Skip*,*s2'*$\rangle$›
have $\langle$*c2*,*s2*$\rangle \rightarrow* \langle$*Skip*,*s2'*$\rangle$ **by**(*auto intro*:*CondFalse-reds*)

with ‹⟨*c2*,*s1*⟩ →* ⟨*Skip*,*s1'*⟩› ‹Γ ⊢ *s1* ≈$_L$ *s2*› ‹*nonInterference* Γ *c2*›
show *?thesis* **by**(*auto simp*:*nonInterference-def*)
qed
qed

lemma *WhileHighCompositionality*:
assumes Γ,*High* ⊢ *while* (*b*) *c'*
shows *nonInterference* Γ (*while* (*b*) *c'*)
proof(*rule nonInterferenceI*)
fix *s1 s2 s1' s2'*
assume Γ ⊢ *s1* ≈$_L$ *s2* **and** ⟨*while* (*b*) *c'*,*s1*⟩ →* ⟨*Skip*,*s1'*⟩
and ⟨*while* (*b*) *c'*,*s2*⟩ →* ⟨*Skip*,*s2'*⟩
show Γ ⊢ *s1'* ≈$_L$ *s2'*
proof −
from ‹Γ,*High* ⊢ *while* (*b*) *c'*› ‹⟨*while* (*b*) *c'*,*s1*⟩ →* ⟨*Skip*,*s1'*⟩›
have Γ ⊢ *s1* ≈$_L$ *s1'* **by**(*auto dest*:*highBodies*)
from ‹Γ,*High* ⊢ *while* (*b*) *c'*› ‹⟨*while* (*b*) *c'*,*s2*⟩ →* ⟨*Skip*,*s2'*⟩›
have Γ ⊢ *s2* ≈$_L$ *s2'* **by**(*auto dest*:*highBodies*)
with ‹Γ ⊢ *s1* ≈$_L$ *s2*› **have** Γ ⊢ *s1* ≈$_L$ *s2'* **by**(*auto intro*:*lowEquivTransitive*)
from ‹Γ ⊢ *s1* ≈$_L$ *s1'*› **have** Γ ⊢ *s1'* ≈$_L$ *s1* **by**(*auto intro*:*lowEquivSymmetric*)
with ‹Γ ⊢ *s1* ≈$_L$ *s2'*› **show** *?thesis* **by**(*auto intro*:*lowEquivTransitive*)
qed
qed

lemma *WhileLowStepInduct*:
assumes *while1*: ⟨*while* (*b*) *c'*,*s1*⟩ →* ⟨*Skip*,*s1'*⟩
and *while2*: ⟨*while* (*b*) *c'*,*s2*⟩→*⟨*Skip*,*s2'*⟩
and Γ ⊢ *b* : *Low*
and *body*:⋀*s1 s1' s2 s2'*. ⟦⟨*c'*,*s1*⟩ →* ⟨*Skip*,*s1'*⟩; ⟨*c'*,*s2*⟩ →* ⟨*Skip*,*s2'*⟩;
Γ ⊢ *s1* ≈$_L$ *s2*⟧ ⟹ Γ ⊢ *s1'* ≈$_L$ *s2'*
and *le*: Γ ⊢ *s1* ≈$_L$ *s2*
shows Γ ⊢ *s1'* ≈$_L$ *s2'*
using *while1 le while2*
proof (*induct arbitrary*:*s2 rule*:*while-reds-induct*)
case (*false s1*)
from ‹Γ ⊢ *b* : *Low*› ‹Γ ⊢ *s1* ≈$_L$ *s2*› **have** ⟦*b*⟧ *s1* = ⟦*b*⟧ *s2* **by**(*auto intro*:*interpretLow2*)
with ‹⟦*b*⟧ *s1* = *Some false*› **have** ⟦*b*⟧ *s2* = *Some false* **by** *simp*
with ‹⟨*while* (*b*) *c'*,*s2*⟩→*⟨*Skip*,*s2'*⟩› **have** *s2* = *s2'* **by**(*auto intro*:*WhileFalse-reds*)
with ‹Γ ⊢ *s1* ≈$_L$ *s2*› **show** *?case* **by** *auto*
next
case (*true s1 s1''*)
note *IH* = ‹⋀*s2''*. ⟦Γ ⊢ *s1''* ≈$_L$ *s2''*; ⟨*while* (*b*) *c'*,*s2''*⟩ →* ⟨*Skip*,*s2'*⟩⟧
⟹ Γ ⊢ *s1'* ≈$_L$ *s2'*›
from ‹Γ ⊢ *b* : *Low*› ‹Γ ⊢ *s1* ≈$_L$ *s2*› **have** ⟦*b*⟧ *s1* = ⟦*b*⟧ *s2* **by**(*auto intro*:*interpretLow2*)
with ‹⟦*b*⟧ *s1* = *Some true*› **have** ⟦*b*⟧ *s2* = *Some true* **by** *simp*
with ‹⟨*while* (*b*) *c'*,*s2*⟩ →* ⟨*Skip*,*s2'*⟩› **obtain** *s2''* **where** ⟨*c'*,*s2*⟩ →* ⟨*Skip*,*s2''*⟩

```
    and ⟨while (b) c',s2''⟩ →* ⟨Skip,s2'⟩ by(auto dest:WhileTrue-reds)
  from body[OF ‹⟨c',s1⟩ →* ⟨Skip,s1''⟩› ‹⟨c',s2⟩ →* ⟨Skip,s2''⟩› ‹Γ ⊢ s1 ≈L s2›]
  have Γ ⊢ s1'' ≈L s2'' .
  from IH[OF this ‹⟨while (b) c',s2''⟩ →* ⟨Skip,s2'⟩›] show ?case .
qed



lemma WhileLowCompositionality:
  assumes nonInterference Γ c' and Γ ⊢ b : Low and Γ,Low ⊢ c'
  shows nonInterference Γ (while (b) c')
proof(rule nonInterferenceI)
  fix s1 s2 s1' s2'
  assume Γ ⊢ s1 ≈L s2 and ⟨while (b) c',s1⟩ →* ⟨Skip,s1'⟩
    and ⟨while (b) c',s2⟩ →* ⟨Skip,s2'⟩
  { fix s1 s2 s1'' s2''
    assume ⟨c',s1⟩ →* ⟨Skip,s1''⟩ and ⟨c',s2⟩ →* ⟨Skip,s2''⟩
      and  Γ ⊢ s1 ≈L s2
    with ‹nonInterference Γ c'› have Γ ⊢ s1'' ≈L s2''
      by(auto simp:nonInterference-def)
  }
  hence ⋀s1 s1'' s2 s2''. ⟦⟨c',s1⟩ →* ⟨Skip,s1''⟩; ⟨c',s2⟩ →* ⟨Skip,s2''⟩;
                           Γ ⊢ s1 ≈L s2⟧ ⟹ Γ ⊢ s1'' ≈L s2'' by auto
  with ‹⟨while (b) c',s1⟩ →* ⟨Skip,s1'⟩› ‹⟨while (b) c',s2⟩ →* ⟨Skip,s2'⟩›
    ‹Γ ⊢ b : Low› ‹Γ ⊢ s1 ≈L s2› show Γ ⊢ s1' ≈L s2'
    by(auto intro:WhileLowStepInduct)
qed
```

and now: the main theorem:

```
theorem secTypeImpliesNonInterference:
  Γ,T ⊢ c ⟹ nonInterference Γ c
proof (induct c arbitrary:T rule:com.induct)
  case Skip
  show ?case
  proof(rule nonInterferenceI)
    fix s1 s2 s1' s2'
   assume Γ ⊢ s1 ≈L s2 and ⟨Skip,s1⟩ →* ⟨Skip,s1'⟩ and ⟨Skip,s2⟩ →* ⟨Skip,s2'⟩
    from ‹⟨Skip,s1⟩ →* ⟨Skip,s1'⟩› have s1 = s1' by(auto dest:Skip-reds)
    from ‹⟨Skip,s2⟩ →* ⟨Skip,s2'⟩› have s2 = s2' by(auto dest:Skip-reds)
    from ‹Γ ⊢ s1 ≈L s2› and ‹s1 = s1'› and ‹s2 = s2'›
    show Γ ⊢ s1' ≈L s2' by simp
  qed
next
  case (LAss V e)
  from ‹Γ,T ⊢ V:=e›
  have varprem:(Γ V = Some High) ∨ (Γ V = Some Low ∧ Γ ⊢ e : Low ∧ T =
Low)
    by (auto elim:secComTyping.cases)
  show ?case
```

```
  proof(rule nonInterferenceI)
    fix s1 s2 s1' s2'
     assume Γ ⊢ s1 ≈L s2 and ⟨V:=e,s1⟩ →* ⟨Skip,s1'⟩ and ⟨V:=e,s2⟩ →*
⟨Skip,s2'⟩
      from ‹⟨V:=e,s1⟩ →* ⟨Skip,s1'⟩› have s1' = s1(V:=⟦e⟧ s1) by(auto in-
tro:LAss-reds)
      from ‹⟨V:=e,s2⟩ →* ⟨Skip,s2'⟩› have s2' = s2(V:=⟦e⟧ s2) by(auto in-
tro:LAss-reds)
    from varprem show Γ ⊢ s1' ≈L s2'
    proof
      assume Γ V = Some High
      with ‹Γ ⊢ s1 ≈L s2› ‹s1' = s1(V:=⟦e⟧ s1)› ‹s2' = s2(V:=⟦e⟧ s2)›
      show ?thesis by(auto intro:assignNIhighlemma)
    next
      assume Γ V = Some Low ∧ Γ ⊢ e : Low ∧ T = Low
      with ‹Γ ⊢ s1 ≈L s2› ‹s1' = s1(V:=⟦e⟧ s1)› ‹s2' = s2(V:=⟦e⟧ s2)›
      show ?thesis by(auto intro:assignNIlowlemma)
    qed
  qed
next
  case (Seq c1 c2)
  note IH1 = ‹⋀T. Γ,T ⊢ c1 ⟹ nonInterference Γ c1›
  note IH2 = ‹⋀T. Γ,T ⊢ c2 ⟹ nonInterference Γ c2›
  show ?case
  proof (cases T)
    case High
    with ‹Γ,T ⊢ c1;;c2› have Γ,High ⊢ c1 and Γ,High ⊢ c2
      by(auto elim:secComTyping.cases)
    from IH1[OF ‹Γ,High ⊢ c1›] have nonInterference Γ c1 .
    moreover
    from IH2[OF ‹Γ,High ⊢ c2›] have nonInterference Γ c2 .
    ultimately show ?thesis by (auto intro:SeqCompositionality)
  next
    case Low
    with ‹Γ,T ⊢ c1;;c2›
    have (Γ,Low ⊢ c1 ∧ Γ,Low ⊢ c2) ∨ Γ,High ⊢ c1;;c2
      by(auto elim:secComTyping.cases)
    thus ?thesis
    proof
      assume Γ,Low ⊢ c1 ∧ Γ,Low ⊢ c2
      hence Γ,Low ⊢ c1 and Γ,Low ⊢ c2 by simp-all
      from IH1[OF ‹Γ,Low ⊢ c1›] have nonInterference Γ c1 .
      moreover
      from IH2[OF ‹Γ,Low ⊢ c2›] have nonInterference Γ c2 .
      ultimately show ?thesis by(auto intro:SeqCompositionality)
    next
      assume Γ,High ⊢ c1;;c2
      hence Γ,High ⊢ c1 and Γ,High ⊢ c2 by(auto elim:secComTyping.cases)
      from IH1[OF ‹Γ,High ⊢ c1›] have nonInterference Γ c1 .
```

```
      moreover
      from IH2[OF ‹Γ,High ⊢ c2›] have nonInterference Γ c2 .
      ultimately show ?thesis by(auto intro:SeqCompositionality)
    qed
  qed
next
  case (Cond b c1 c2)
  note IH1 = ‹⋀T. Γ,T ⊢ c1 ⟹ nonInterference Γ c1›
  note IH2 = ‹⋀T. Γ,T ⊢ c2 ⟹ nonInterference Γ c2›
  show ?case
  proof (cases T)
    case High
    with ‹Γ,T ⊢ if (b) c1 else c2› show ?thesis
      by(auto intro:CondHighCompositionality)
  next
    case Low
    with ‹Γ,T ⊢ if (b) c1 else c2›
    have (Γ ⊢ b : Low ∧ Γ,Low ⊢ c1 ∧Γ,Low ⊢ c2) ∨ Γ,High ⊢ if (b) c1 else c2
      by(auto elim:secComTyping.cases)
    thus ?thesis
    proof
      assume Γ ⊢ b : Low ∧ Γ,Low ⊢ c1 ∧ Γ,Low ⊢ c2
      hence Γ ⊢ b : Low and Γ,Low ⊢ c1 and Γ,Low ⊢ c2 by simp-all
      from IH1[OF ‹Γ,Low ⊢ c1›] have nonInterference Γ c1 .
      moreover
      from IH2[OF ‹Γ,Low ⊢ c2›] have nonInterference Γ c2 .
      ultimately show ?thesis using ‹Γ ⊢ b : Low›
        by(auto intro:CondLowCompositionality)
    next
      assume Γ,High ⊢ if (b) c1 else c2
      thus ?thesis by(auto intro:CondHighCompositionality)
    qed
  qed
next
  case (While b c')
  note IH = ‹⋀T. Γ,T ⊢ c' ⟹ nonInterference Γ c'›
  show ?case
  proof(cases T)
    case High
   with ‹Γ,T ⊢ while (b) c'› show ?thesis by(auto intro:WhileHighCompositionality)
  next
    case Low
    with ‹Γ,T ⊢ while (b) c'›
    have (Γ ⊢ b : Low ∧ Γ,Low ⊢ c') ∨ Γ,High ⊢ while (b) c'
      by(auto elim:secComTyping.cases)
    thus ?thesis
    proof
      assume Γ ⊢ b : Low ∧ Γ,Low ⊢ c'
      hence Γ ⊢ b : Low and Γ,Low ⊢ c' by simp-all
```

```
    moreover
    from IH[OF ‹Γ,Low ⊢ c'›] have nonInterference Γ c' .
    ultimately show ?thesis by(auto intro:WhileLowCompositionality)
   next
    assume Γ,High ⊢ while (b) c'
    thus ?thesis by(auto intro:WhileHighCompositionality)
   qed
  qed
qed

end


theory Execute
imports secTypes
begin
```

3 Executing the small step semantics

```
code-pred (modes: i => o => bool as exec-red, i => i * o => bool, i => o * i
=> bool, i => i => bool) red .
thm red.equation

definition [code]: one-step x = Predicate.the (exec-red x)

lemmas [code-pred-intro] = typeVal[where lev = Low] typeVal[where lev = High]
  typeVar
  typeBinOp1 typeBinOp2[where lev = Low] typeBinOp2[where lev = High] type-
BinOp3[where lev = Low]

code-pred (modes: i => i => o => bool as compute-secExprTyping,
  i => i => i => bool as check-secExprTyping) secExprTyping
proof −
  case secExprTyping
  from secExprTyping.prems show thesis
  proof
    fix Γ V lev assume x = Γ xa = Val V xb = lev
    from secExprTyping(1−2) this show thesis by (cases lev) auto
  next
    fix Γ Vn lev
    assume x = Γ xa = Var Vn xb = lev Γ Vn = Some lev
   from secExprTyping(3) this show thesis by (auto simp add: Predicate.eq-is-eq)
  next
    fix Γ e1 e2 bop
    assume x = Γ xa = e1«bop» e2 xb = Low
      Γ ⊢ e1 : Low Γ ⊢ e2 : Low
    from secExprTyping(4) this show thesis by auto
  next
    fix Γ e1 e2 lev bop
```

```
    assume x = Γ xa = e1 «bop» e2 xb = High
    Γ ⊢ e1 : High Γ ⊢ e2 : lev
    from secExprTyping(5−6) this show thesis by (cases lev) (auto)
  next
    fix Γ e1 e2 lev bop
    assume x = Γ xa = e1 «bop» e2 xb = High
    Γ ⊢ e1 : lev Γ ⊢ e2 : High
    from secExprTyping(6−7) this show thesis by (cases lev) (auto)
  qed
qed

lemmas [code-pred-intro] = typeSkip[where T=Low] typeSkip[where T=High]
  typeAssH[where T = Low] typeAssH[where T=High]
  typeAssL typeSeq typeWhile typeIf typeConvert

code-pred (modes: i => o => i => bool as compute-secComTyping,
  i => i => i => bool as check-secComTyping) secComTyping
proof −
  case secComTyping
  from secComTyping.prems show thesis
  proof
    fix Γ T assume x = Γ xa = T xb = Skip
    from secComTyping(1−2) this show thesis by (cases T) auto
  next
    fix Γ V T e assume x = Γ xa = T xb = V:=e Γ V = Some High
    from secComTyping(3−4) this show thesis by (cases T) (auto)
  next
    fix Γ e V
    assume x = Γ xa = Low xb = V:=e Γ ⊢ e : Low Γ V = Some Low
    from secComTyping(5) this show thesis by auto
  next
    fix Γ T c1 c2
    assume x = Γ xa = T xb = Seq c1 c2 Γ,T ⊢ c1 Γ,T ⊢ c2
    from secComTyping(6) this show thesis by auto
  next
    fix Γ b T c
    assume x = Γ xa = T xb = while (b) c Γ ⊢ b : T Γ,T ⊢ c
    from secComTyping(7) this show thesis by auto
  next
    fix Γ b T c1 c2
    assume x = Γ xa = T xb = if (b) c1 else c2 Γ ⊢ b : T Γ,T ⊢ c1 Γ,T ⊢ c2
    from secComTyping(8) this show thesis by blast
  next
    fix Γ c
    assume x = Γ xa = Low xb = c Γ,High ⊢ c
    from secComTyping(9) this show thesis by blast
  qed
qed
```

thm *secComTyping.equation*

3.1 An example taken from Volpano, Smith, Irvine

definition *com* = *if* (*Var* ''*x*'' «*Eq*» *Val* (*Intg 1*)) (''*y*'' := *Val* (*Intg 1*)) *else* (''*y*'' := *Val* (*Intg 0*))
definition *Env* = *map-of* [(''*x*'', *High*), (''*y*'', *High*)]

values {*T*. *Env* ⊢ (*Var* ''*x*'' «*Eq*» *Val* (*Intg 1*)): *T*}
value *Env*, *High* ⊢ *com*
value *Env*, *Low* ⊢ *com*
values *1* {*T*. *Env*, *T* ⊢ *com*}

definition *Env'* = *map-of* [(''*x*'', *Low*), (''*y*'', *High*)]

value *Env'*, *Low* ⊢ *com*
value *Env'*, *High* ⊢ *com*
values *1* {*T*. *Env*, *T* ⊢ *com*}

end

References

[1] Andrei Sabelfeld and Andrew C. Myers. Language-based information-flow security. *IEEE Journal on Selected Areas in Communications*, 21(1):5–19, 2003.

[2] Dennis Volpano, Cynthia Irvine, and Geoffrey Smith. A sound type system for secure flow analysis. *Journal of Computer Security*, 4(2-3):167–187, 1996.