VerifyThis 2018 - Polished Isabelle Solutions

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Abstract. VerifyThis 2018 http://www.pm.inf.ethz.ch/research/verifythis.html was a program verification competition associated with ETAPS 2018. It was the 7th event in the VerifyThis competition series. In this entry, we present polished and completed versions of our solutions that we created during the competition.
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Gap Buffer

1.1 Challenge

A gap buffer is a data structure for the implementation of text editors, which can efficiently move the cursor, as well add and delete characters. The idea is simple: the editor's content is represented as a character array $a$ of length $n$, which has a gap of unused entries $a[l], \ldots, a[r-1]$, with respect to two indices $l \leq r$. The data it represents is composed as $a[0], \ldots, a[l-1], a[r], \ldots, a[n-1]$. The current cursor position is at the left index $l$, and if we type a character, it is written to $a[l]$ and $l$ is increased. When the gap becomes empty, the array is enlarged and the data from $r$ is shifted to the right.

Implementation task. Implement the following four operations in the language of your tool: Procedures $\text{left()}$ and $\text{right()}$ move the cursor by one character; $\text{insert()}$ places a character at the beginning of the gap $a[l]$; $\text{delete()}$ removes the character at $a[l]$ from the range of text.

```plaintext
procedure left()
    if l != 0 then
        l := l - 1
        r := r - 1
        a[r] := a[l]
    end-if
end-procedure

procedure insert(x: char)
    if l == r then
        // see extended task
        grow()
    end-if
    a[l] := x
    l := l + 1
end-procedure

procedure right()
    // your task: similar to left()
    // but pay attention to the order of statements
end-procedure

procedure delete()
    if l != 0 then
        l := l - 1
    end-if
end-procedure
```

Verification task. Specify the intended behavior of the buffer in terms of a contiguous representation of the editor content. This can for example be based on strings, functional arrays, sequences, or lists. Verify that the gap buffer implementation satisfies this specification, and that every access to the array is within bounds.
Hint: For this task you may assume that insert() has the precondition $l < r$ and remove the call to grow(). Alternatively, assume a contract for grow() that ensures that this call does not change the abstract representation.

**Extended verification task.** Implement the operation grow(), specify its behavior in a way that lets you verify insert() in a modular way (i.e. not by referring to the implementation of grow()), and verify that grow() satisfies this specification.

*Hint:* You may assume that the allocation of the new buffer always succeeds. If your tool/language supports copying array ranges (such as System.arraycopy() in Java), consider using these primitives instead of the loops in the pseudo-code below.

**procedure grow()**

```plaintext
var b := new char[a.length + K]

// b[0..l] := a[0..l]
for i = 0 to l - 1 do
    b[i] := a[i]
end-for

// b[r+K..] := a[r..]
for i = r to a.length - 1 do
    b[i + K] := a[i]
end-for

r := r + K
a := b
end-procedure
```

**Resources**

1.2 Solution

theory Challenge1
imports lib/VTcomp
begin

Fully fledged specification of textbuffer ADT, and its implementation by a gap buffer.

1.2.1 Abstract Specification

Initially, we modelled the abstract text as a cursor position and a list. However, this gives you an invariant on the abstract level. An isomorphic but invariant free formulation is a pair of lists, representing the text before and after the cursor.

datatype 'a textbuffer = BUF 'a list 'a list

The primitive operations are the empty textbuffer, and to extract the text and the cursor position

definition empty :: 'a textbuffer where empty = BUF [] []
primrec get-text :: 'a textbuffer ⇒ 'a list where get-text (BUF a b) = a@b
primrec get-pos :: 'a textbuffer ⇒ nat where get-pos (BUF a b) = length a

These are the operations that were specified in the challenge

primrec move-left :: 'a textbuffer ⇒ 'a textbuffer where
  move-left (BUF a b) = (if a@[] then BUF (butlast a) (last a@b) else BUF a b)
primrec move-right :: 'a textbuffer ⇒ 'a textbuffer where
  move-right (BUF a b) = (if b@[] then BUF (a@[hd b]) (tl b) else BUF a b)
primrec insert :: 'a ⇒ 'a textbuffer ⇒ 'a textbuffer where
  insert x (BUF a b) = BUF (a@[x]) b
primrec delete :: 'a textbuffer ⇒ 'a textbuffer where
  delete (BUF a b) = BUF (butlast a) b
— Note that butlast [] = [] in Isabelle

We can also assign them a meaning wrt position and text

lemma empty-pos[simp]: get-pos empty = 0
  ⟨proof⟩
lemma empty-text[simp]: get-text empty = []
  ⟨proof⟩
lemma move-left-pos[simp]: get-pos (move-left b) = get-pos b − 1
— Note that 0 − 1 = 0 in Isabelle
  ⟨proof⟩
lemma move-left-text[simp]: get-text (move-left b) = get-text b
  ⟨proof⟩
lemma move-right-pos[simp]:
CHAPTER 1. GAP BUFFER

get-pos (move-right b) = min (get-pos b + 1) (length (get-text b))
⟨proof⟩

lemma move-right-text[simp]: get-text (move-right b) = get-text b
⟨proof⟩

lemma insert-pos[simp]: get-pos (insert x b) = get-pos b + 1
⟨proof⟩

lemma insert-text: get-text (insert x b)
= take (get-pos b) (get-text b)@x@drop (get-pos b) (get-text b)
⟨proof⟩

lemma delete-pos[simp]: get-pos (delete b) = get-pos b − 1
⟨proof⟩

lemma delete-text: get-text (delete b)
= take (get-pos b − 1) (get-text b)@drop (get-pos b) (get-text b)
⟨proof⟩

For the zero case, we can prove a simpler (equivalent) lemma

lemma delete-text0[simp]: get-pos b=0 ⇒ get-text (delete b) = get-text b
⟨proof⟩

To fully exploit the capabilities of our tool, we can (optionally) show that the operations of a text buffer are parametric in its content. Then, we can automatically refine the representation of the content.

definition [to-relAPP]:
textbuffer-rel A ≡ {
(BUF a b, BUF a' b') | a b a' b'.
(a,a')∈⟨A⟩list-rel ∧ (b,b')∈⟨A⟩list-rel}

lemma [param]: (BUF,BUF) ∈ ⟨A⟩list-rel → ⟨A⟩list-rel → ⟨A⟩textbuffer-rel
⟨proof⟩

lemma [param]: (rec-textbuffer,rec-textbuffer)
∈ ⟨⟨A⟩list-rel → ⟨A⟩list-rel→B⟩ → ⟨A⟩textbuffer-rel → B
⟨proof⟩

class

context

notes[simp] =
empty-def get-text-def get-pos-def move-left-def move-right-def
insert-def delete-def conv-to-is-nil

begin
sepref-decl-op (no-def) empty :: ⟨A⟩textbuffer-rel ⟨proof⟩
sepref-decl-op (no-def) get-text :: ⟨A⟩textbuffer-rel → ⟨A⟩list-rel ⟨proof⟩
sepref-decl-op (no-def) get-pos :: ⟨A⟩textbuffer-rel → nat-rel ⟨proof⟩
sepref-decl-op (no-def) move-left :: ⟨A⟩textbuffer-rel → ⟨A⟩textbuffer-rel ⟨proof⟩
sepref-decl-op (no-def) move-right :: ⟨A⟩textbuffer-rel → ⟨A⟩textbuffer-rel ⟨proof⟩
sepref-decl-op (no-def) insert :: A→⟨A⟩textbuffer-rel → ⟨A⟩textbuffer-rel ⟨proof⟩
sepref-decl-op (no-def) delete :: ⟨A⟩textbuffer-rel → ⟨A⟩textbuffer-rel ⟨proof⟩
end
1.2. Solution

1.2.2 Refinement 1: List with Gap

1.2.3 Implementation on List-Level

- type-synonym 'a gap-buffer = nat × nat × 'a list

Abstraction Relation

Also called coupling relation sometimes. Can be any relation, here we define it by an invariant and an abstraction function.

- definition gap-α ≡ λ(l,r,buf). BUF (take l buf) (drop r buf)
- definition gap-invar ≡ λ(l,r,buf). l ≤ r ∧ r ≤ length buf
- abbreviation gap-rel ≡ br gap-α gap-invar

Empty

- definition empty1 ≡ RETURN (0,0,[])
- lemma empty1-correct: (empty1, RETURN empty) ∈ ⟨gap-rel⟩ nres-rel

Left

- definition move-left1 ≡ λ(l,r,buf). doN {
  if l ≠ 0 then doN {
    ASSERT (r-1 < length buf ∧ l-1 < length buf);
    RETURN (l-1,r-1,buf[buf!l:=buf!r])
  } else RETURN (l,r,buf)
}
- lemma move-left1-correct: (move-left1, RETURN o move-left1) ∈ gap-rel → ⟨gap-rel⟩ nres-rel

Right

- definition move-right1 ≡ λ(l,r,buf). doN {
  if r < length buf then doN {
    ASSERT (l < length buf);
    RETURN (l+1,r+1,buf[l:=buf!r])
  } else RETURN (l,r,buf)
}
- lemma move-right1-correct: (move-right1, RETURN o move-right1) ∈ gap-rel → ⟨gap-rel⟩ nres-rel

Insert and Grow

- definition can-insert ≡ λ(l,r,buf). l < r
definition \textit{grow1} \( K \equiv \lambda (l, r, \text{buf}). \) doN \{
\begin{align*}
\text{let } b &= \text{op-array-replicate} (\text{length } \text{buf} + K) \text{ default}; \\
 b &\leftarrow \text{mop-list-blit} \text{ buf } 0 b \ 0 \ l; \\
 b &\leftarrow \text{mop-list-blit} \ r b \ (r + K) (\text{length } \text{buf} - r); \\
 \text{RETURN} \ (l, r + K, b)
\end{align*}
\}

lemma \textit{grow1-correct}[\text{THEN SPEC-trans, refine-vcg}]:
\begin{align*}
\text{assumes } &\text{gap-invar } gb \\
\text{shows } &\text{grow1 } K \ gb \leq (\text{SPEC } (\lambda gb'. \\
&\text{gap-invar } gb' \\
&\land \text{gap-} \alpha \ gb' = \text{gap-} \alpha \ gb \ \\
&\land (K > 0 \longrightarrow \text{can-insert } gb'))) \\
\langle \text{proof} \rangle
\end{align*}

definition \textit{insert1} \( x \equiv \lambda (l, r, \text{buf}). \) doN \{
\begin{align*}
(l, r, \text{buf}) &\leftarrow \\
&\text{if } (l = r) \text{ then } \text{grow1 } (\text{length } \text{buf} + 1) \ (l, r, \text{buf}) \text{ else } \text{RETURN } (l, r, \text{buf}); \\
\text{ASSERT } (&l < \text{length } \text{buf}); \\
\text{RETURN } (l + 1, r, \text{buf}[l := x])
\end{align*}
\}

lemma \textit{insert1-correct}:
\begin{align*}
(\text{insert1, RETURN oo insert}) \in \text{Id } \rightarrow \text{gap-rel } \rightarrow (\text{gap-rel})nres-rel \\
\langle \text{proof} \rangle
\end{align*}

\section*{1.2.4 Imperative Arrays and Executable Code}

abbreviation \textit{gap-impl-assn} \equiv \text{nat-assn} \times a \text{nat-assn} \times a \text{array-assn id-assn}

definition \textit{gap-assn } A \equiv \text{hr-comp } (\text{hr-comp gap-impl-assn gap-rel}) \ (\text{the-pure } A)\text{textbuffer-rel}

context
\begin{align*}
\text{notes } &\text{gap-assn-def [symmetric,fcomp-norm-unfold]} \\
\text{begin } \text{sepref-definition empty-impl} \\
&\text{is uncurry0 empty1 :: unit-assn}^k \rightarrow_a \text{gap-impl-assn} \\
\langle \text{proof} \rangle \\
\text{sepref-decl-impl empty-impl: empty-impl.refine[FCOMP empty1-correct]} \langle \text{proof} \rangle
\end{align*}
1.2. SOLUTION

sepref-definition move-left-impl
  is move-left1 :: gap-impl-assn \rightarrow_a gap-impl-assn
  ⟨proof⟩

sepref-decl-impl move-left-impl. move-left-impl.refine[FCOMP move-left1-correct] ⟨proof⟩

sepref-definition move-right-impl
  is move-right1 :: gap-impl-assn \rightarrow_a gap-impl-assn
  ⟨proof⟩

sepref-decl-impl move-right-impl. move-right-impl.refine[FCOMP move-right1-correct] ⟨proof⟩

sepref-definition insert-impl
  is uncurry insert1 :: id-assn \ast_a gap-impl-assn \rightarrow_a gap-impl-assn
  ⟨proof⟩

sepref-decl-impl insert-impl. insert-impl.refine[FCOMP insert1-correct] ⟨proof⟩

sepref-definition delete-impl
  is delete1 :: gap-impl-assn \rightarrow_a gap-impl-assn
  ⟨proof⟩

sepref-decl-impl delete-impl. delete-impl.refine[FCOMP delete1-correct] ⟨proof⟩

end

The above setup generated the following refinement theorems, connecting the implementations with our abstract specification:

\((\text{uncurry0 Challenge1.empty-impl, uncurry0 (RETURN Challenge1.empty)}) \in \text{unit-assn}\ast_a \text{gap-assn} \Rightarrow A\)

\((\text{move-left-impl, RETURN o move-left}) \in (\text{gap-assn} \Rightarrow A)^d \rightarrow_a \text{gap-assn} \Rightarrow A\)

\((\text{move-right-impl, RETURN o move-right}) \in (\text{gap-assn} \Rightarrow A)^d \rightarrow_a \text{gap-assn} \Rightarrow A\)

CONSTRAINT is-pure \Rightarrow A

\((\text{uncurry Challenge1.insert-impl, uncurry (RETURN o Challenge1.insert)}) \in \text{?A}\ast_a (\text{gap-assn} \Rightarrow A)^d \rightarrow_a \text{gap-assn} \Rightarrow A\)

\((\text{delete-impl, RETURN o delete}) \in (\text{gap-assn} \Rightarrow A)^d \rightarrow_a \text{gap-assn} \Rightarrow A\)

export-code move-left-impl move-right-impl insert-impl delete-impl
  in \text{SML-imp module-name Gap-Buffer}
  in \text{OCaml-imp module-name Gap-Buffer}
  in \text{Haskell module-name Gap-Buffer}
  in \text{Scala module-name Gap-Buffer}

1.2.5 Simple Client

definition client ≡ RETURN (fold (λf. f) [insert (1::int),
insert (2::int),
insert (3::int),
insert (5::int),
move-left,
insert (4::int),
insert (6::int),
insert (7::int),
insert (8::int),
insert (9::int)])
move-right, 
insert (6::int), 
delete 
| empty

lemma client \leq SPEC (\lambda r. \text{get-text } r=[1,2,3,4,5]) 
(proof)

sepref-definition client-impl
is uncurry0 client :: unit-assn^k \rightarrow a gap-assn id-assn 
(proof)

⟨ML⟩ 
end

1.3 Shorter Solution

theory Challenge1-short 
imports lib/VTcomp 
begin 
Small specification of textbuffer ADT, and its implementation by a gap buffer. 
Annotated and elaborated version of just the challenge requirements.

1.3.1 Abstract Specification 

datatype 'a textbuffer = BUF (pos: nat) (text: 'a list) 
— Note that we do not model the abstract invariant — pos in range — here, as it is not 
strictly required for the challenge spec.

These are the operations that were specified in the challenge. Note: Isabelle has 
type inference, so we do not need to specify types. Note: We exploit that, in Is- 
abelle, we have 0 − 1 = 0.

primrec move-left where move-left (BUF p t) = BUF (p−1) t 
primrec move-right where move-right (BUF p t) = BUF (min (length t) (p+1)) t 
primrec insert where insert x (BUF p t) = BUF (p+1) (take p t@x#drop p t) 
primrec delete where delete (BUF p t) = BUF (p−1) (take (p−1) t@drop p t)

1.3.2 Refinement 1: List with Gap
1.3.3 Implementation on List-Level 

type-synonym 'a gap-buffer = nat \times nat \times 'a list
1.3. SHORTER SOLUTION

Abstraction Relation

We define an invariant on the concrete gap-buffer, and its mapping to the abstract model. From these two, we define a relation $\text{gap-rel}$ between concrete and abstract buffers.

**Definition** $\text{gap-\alpha} \equiv \lambda(l, r, \text{buf}). \text{BUF} l \ (\text{take} \ l \ \text{buf} \ @ \ \text{drop} \ r \ \text{buf})$

**Definition** $\text{gap-invar} \equiv \lambda(l, r, \text{buf}). l \leq r \land r \leq \text{length buf}$

**Abbreviation** $\text{gap-rel} \equiv \text{br gap-\alpha gap-invar}$

Left

For the operations, we insert assertions. These are not required to prove the list-level specification correct (during the proof, they are inferred easily). However, they are required in the subsequent automatic refinement step to arrays, to give our tool the information that all indexes are, indeed, in bounds.

**Definition** $\text{move-left1} \equiv \lambda(l, r, \text{buf}). \text{doN} \{\text{if} \ l \neq 0 \ \text{then doN} \{\text{assert} (r - 1 < \text{length buf} \land l - 1 < \text{length buf}); \text{return} (l - 1, r - 1, \text{buf}[r - 1:= \text{buf}![l - 1]]) \} \ \text{else return} \ (l, r, \text{buf})\}$

**Lemma** $\text{move-left1-correct}: \ (\text{move-left1, \text{RETURN o move-left}) \in \text{gap-rel} \rightarrow (\text{gap-rel})\text{nres-rel} \ (proof)}$

Right

**Definition** $\text{move-right1} \equiv \lambda(l, r, \text{buf}). \text{doN} \{\text{if} \ r < \text{length buf} \ \text{then doN} \{\text{assert} (l < \text{length buf}); \text{return} (l + 1, r + 1, \text{buf}[l:= \text{buf}![r]]) \} \ \text{else return} \ (l, r, \text{buf})\}$

**Lemma** $\text{move-right1-correct}: \ (\text{move-right1, \text{RETURN o move-right}) \in \text{gap-rel} \rightarrow (\text{gap-rel})\text{nres-rel} \ (proof)}$

Insert and Grow

**Definition** $\text{can-insert} \equiv \lambda(l, r, \text{buf}). l < r$

**Definition** $\text{grow1 K} \equiv \lambda(l, r, \text{buf}). \text{doN} \{\text{let} \ b = \text{op-array-replicate} \ (\text{length buf} + K) \ \text{default}; \ b \leftarrow \text{mop-list-blit} \ \text{buf} \ 0 \ b \ 0 \ l; \ b \leftarrow \text{mop-list-blit} \ \text{buf} \ r \ b \ (r+K) \ (\text{length buf} - r); \ \text{return} \ (l, r+K, b)\}$
— Note: Most operations have also a variant prefixed with `mop`. These are defined
in the refinement monad and already contain the assertion of their precondition. The backside is
that they cannot be easily used in as part of expressions, e.g., in `buf'[l := buf ! r]'`, we would
have to explicitly bind each intermediate value: `mop-list-get buf r >> mop-list-set buf l`.

**lemma** `grow1-correct[THEN SPEC-trans, refine-vcg]`:
— Declares this as a rule to be used by the VCG

**assumes** `gap-invar gb`

**shows** `grow1 K gb ≤ (SPEC (λ gb'.
    gap-invar gb'.
    gap-α gb' = gap-α gb
    ∧ (K>0 ⇔ can-insert gb'))))`

**definition** `insert1 x ≡ λ (l,r,buf). doN {
    (l,r,buf) ←
    if (l=r) then grow1 (length buf+1) (l,r,buf) else RETURN (l,r,buf);
    ASSERT (l<length buf);
    RETURN (l+1,r,buf'[l:=x])
}

**lemma** `insert1-correct`:
`(insert1,RETURN oo insert) ∈ Id → gap-rel → ⟨gap-rel⟩nres-rel`

**Delete**

**definition** `delete1`  
≡ λ (l,r,buf). if l>0 then RETURN (l−1,r,buf) else RETURN (l,r,buf)

**lemma** `delete1-correct`:
`(delete1,RETURN o delete) ∈ gap-rel → ⟨gap-rel⟩nres-rel`

1.3.4 Imperative Arrays

The following indicates how we will further refine the gap-buffer: The list will
become an array, the indices and the content will not be refined (expressed by `nat-assn` and `id-assn`).

**abbreviation** `gap-impl-assn ≡ nat-assn × a nat-assn × a array-assn id-assn`

**sepref-definition** `move-left-impl`  
is `move-left1 :: gap-impl-assn d→ a gap-impl-assn`

**sepref-definition** `move-right-impl`  
is `move-right1 :: gap-impl-assn d→ a gap-impl-assn`
1.3. SHORTER SOLUTION

sepref-definition insert-impl
is uncurry insert1 :: \text{id-assn}^k \*_{a} \text{gap-impl-assn}^d \rightarrow_{a} \text{gap-impl-assn}
(proof)

sepref-definition delete-impl
is delete1 :: \text{gap-impl-assn}^d \rightarrow_{a} \text{gap-impl-assn}
(proof)

Finally, we combine the two refinement steps, to get overall correctness theorems

definition gap-assn \equiv \text{hr-comp} \text{gap-impl-assn} \text{gap-rel}
— \text{hr-comp} is composition of refinement relations

context notes gap-assn-def
\text{symmetric}, \text{fcomp-norm-unfold} \begin{proof}
lemmas move-left-impl-correct = move-left-impl.refine[FCOMP move-left1-correct]
and move-right-impl-correct = move-right-impl.refine[FCOMP move-right1-correct]
and insert-impl-correct = insert-impl.refine[FCOMP insert1-correct]
and delete-impl-correct = delete-impl.refine[FCOMP delete1-correct]

Proves:

(move-left-impl, RETURN o move-left) \in \text{gap-assn}^d \rightarrow_{a} \text{gap-assn}

(move-right-impl, RETURN o move-right) \in \text{gap-assn}^d \rightarrow_{a} \text{gap-assn}

(uncurry Challenge1-short.insert-impl, uncurry (RETURN o Challenge1-short.insert))
\in \text{id-assn}^k \*_{a} \text{gap-assn}^d \rightarrow_{a} \text{gap-assn}

(delete-impl, RETURN o delete) \in \text{gap-assn}^d \rightarrow_{a} \text{gap-assn}
\end{proof}

1.3.5 Executable Code

Isabelle/HOL can generate code in various target languages.

export-code move-left-impl move-right-impl insert-impl delete-impl
in SML-imp module-name Gap-Buffer
in OCaml-imp module-name Gap-Buffer
in Haskell module-name Gap-Buffer
in Scala module-name Gap-Buffer

end
Colored Tiles

2.1 Challenge

This problem is based on Project Euler problem #114. Alice and Bob are decorating their kitchen, and they want to add a single row of fifty tiles on the edge of the kitchen counter. Tiles can be either red or black, and for aesthetic reasons, Alice and Bob insist that red tiles come by blocks of at least three consecutive tiles. Before starting, they wish to know how many ways there are of doing this. They come up with the following algorithm:

```plaintext
var count[51] // count[i] is the number of valid rows of size i
count[0] := 1 // []
count[1] := 1 // [B] - cannot have a single red tile
count[2] := 1 // [BB] - cannot have one or two red tiles
for n = 4 to 50 do
    count[n] := count[n-1] // either the row starts with a black tile
    for k = 3 to n-1 do // or it starts with a block of k red tiles
        count[n] := count[n] + count[n-k-1] // followed by a black one
    end-for
    count[n] := count[n]+1 // or the entire row is red
end-for
```

Verification tasks. You should verify that at the end, count[50] will contain the right number.

Hint: Since the algorithm works by enumerating the valid colorings, we expect you to give a nice specification of a valid coloring and to prove the following properties:

1. Each coloring counted by the algorithm is valid.
2. No coloring is counted twice.
3. No valid coloring is missed.
2.2 Solution

theory Challenge2
imports lib/VTcomp
begin

The algorithm describes a dynamic programming scheme.
Instead of proving the 3 properties stated in the challenge separately, we approach
the problem by

1. Giving a natural specification of a valid tiling as a grammar
2. Deriving a recursion equation for the number of valid tilings
3. Verifying that the program returns the correct number (which obviously im-
   plies all three properties stated in the challenge)

2.2.1 Problem Specification

Colors

datatype color = R | B

Direct Natural Definition of a Valid Line

inductive valid where
valid [] |
valid xs \implies valid (B # xs) |
valid xs \implies n \geq 3 \implies valid (replicate n R @ xs)
definition lcount n = card \{ l. length l=n \land valid l \}

2.2.2 Derivation of Recursion Equations

This alternative variant helps us to prove the split lemma below.

inductive valid' where
valid' [] |
n \geq 3 \implies valid'' (replicate n R) |
valid' xs \implies valid' (B # xs) |
valid' xs \implies n \geq 3 \implies valid' (replicate n R @ B # xs)

lemma valid-valid':
valid l \implies valid' l
(proof)

lemmas valid-red = valid.intros(3)[OF valid.intros(1), simplified]
2.2. SOLUTION

**lemma** valid'-valid:
valid'\ l \implies valid \ l
⟨proof⟩

**lemma** valid-eq-valid':
valid' \ l = valid \ l
⟨proof⟩

**Additional Facts on Replicate**

**lemma** replicate-iff:
(\forall i < \text{length} \ l. \ l!i = R) \iff (∃ n. \ l = \text{replicate} \ n \ R)
⟨proof⟩

**lemma** replicate-iff2:
(\forall i < n. \ l!i = R) \iff (∃ l'. \ l = \text{replicate} \ n R @ l') \text{ if } n < \text{length} \ l
⟨proof⟩

**lemma** replicate-Cons-eq:
\text{replicate} \ n \ x = y \# ys \iff (∃ n'. n = \text{Suc} \ n' \land x = y \land \text{replicate} \ n' \ x = ys)
⟨proof⟩

**Main Case Analysis on @term valid**

**lemma** valid-split:
valid \ l \iff
l = [] \lor
(l!0 = B \land valid (tl l)) \lor
\text{length} \ l \geq 3 \land (∀ i < \text{length} \ l. \ l!i = R) \lor
(∃ j < \text{length} \ l. j \geq 3 \land (∀ i < j. \ l!i = R) \land l!j = B \land valid (\text{drop} (j + 1) l))
⟨proof⟩

**Base cases**

**lemma** lc0-aux:
\{l. l = [] \land valid l\} = \{[]\}
⟨proof⟩

**lemma** lc0: lcount 0 = 1
⟨proof⟩

**lemma** lc1aux: \{l. \text{length} l=1 \land valid l\} = \{[B]\}
⟨proof⟩

**lemma** lc2aux: \{l. \text{length} l=2 \land valid l\} = \{[B,B]\}
⟨proof⟩

**lemma** lc3-aux: \{l. \text{length} l=3 \land valid l\} = \{[B,B,B], [R,R,R]\}
⟨proof⟩
lemma \text{lc}counts-init: lcount 0 = 1 lcount 1 = 1 lcount 2 = 1 lcount 3 = 2
⟨proof⟩

The Recursion Case

lemma \text{finite-valid-length}:
finite \{l. length l = n ∧ valid l\} (is finite ?S)
⟨proof⟩

lemma valid-line-just-B:
valid (replicate n B)
⟨proof⟩

lemma valid-line-aux:
\{l. length l = n ∧ valid l\} \neq {} (is ?S \neq {})
⟨proof⟩

lemma replicate-unequal-aux:
replicate x R @ B # l \neq replicate y R @ B # l' (is ?l \neq ?r) if \langle x < y \rangle for \langle l = l' \rangle
⟨proof⟩

lemma valid-prepend-B-iff:
valid (B # xs) <-> valid xs
⟨proof⟩

lemma \text{lcrec}: \text{l}count n = \text{l}count (n-1) + 1 + (∑ i=3..<n. \text{l}count (n-i-1)) if \langle n>3 \rangle
⟨proof⟩

2.2.3 Verification of Program

Inner Loop: Summation

definition \text{sum-prog} \Phi l u f ≡
nfoldli [l..<u] (λ-. True) (λi s. doN {\nASSERT (\text{Φ \ i});
RETURN (s+f i)} ) 0

lemma \text{sum-spec[THEN SPEC-trans, refine-vcg]}:
assumes \langle i. l\leq i \implies i<u \implies \Phi \ i \rangle
shows \text{sum-prog} \Phi l u f \leq \text{SPEC} (λr. r=(∑ i=l..<u. f \ i))
⟨proof⟩

Main Program

definition \text{ic}count M ≡ doN {
ASSERT (M>2);
let c = op-array-replicate (M+1) 0;
let c = c[0:=1, 1:=1, 2:=1, 3:=2];

ASSERT (\forall i<4. c!i = lcount i);

c ← nfoldli [4..<M+1] (λ c. doN { (* let sum = (\sum i=3..<n. c!(n-i-1)); *)
  sum ← sum-prog (λ i. n−i−1 < length c) 3 n (λ i. c!(n−i−1));
  ASSERT (n−1<length c ∧ n<length c);
  RETURN (c[n := c!(n−1) + 1 + sum])
}) c;

ASSERT (\forall i\leq M. c!i = lcount i);

ASSERT (M < length c);
RETURN (c!M)

Abstract Correctness Statement

theorem icount-correct: M>2 \Rightarrow icount M \leq SPEC (\lambda r. r=lcount M)
(proof)

2.2.4 Refinement to Imperative Code

sepref-definition icount-impl is icount :: nat-assn \rightarrow a nat-assn
(proof)

Main Correctness Statement

As the main theorem, we prove the following Hoare triple, stating: starting from
the empty heap, our program will compute the correct result (lcount M).

theorem icount-impl-correct:
  M>2 \Rightarrow <emp> icount-impl M <\lambda r. ↑(r = lcount M)>,
(proof)

Code Export

export-code icount-impl in SML-imp module-name Tiling
export-code icount-impl in OCaml-imp module-name Tiling
export-code icount-impl in Haskell module-name Tiling
export-code icount-impl in Scala-imp module-name Tiling

2.2.5 Alternative Problem Specification

Alternative definition of a valid line that we used in the competition

context fixes l :: color list begin
inductive valid-point where
\[ [ \begin{align*}
& i + 2 < \text{length } l; \ l! = \text{R}; \ l!(i+1) = \text{R}; \ l!(i+2) = \text{R} \\
\end{align*} ] \implies \text{valid-point } i
\]
\[ [ \begin{align*}
& 1 \leq i + 1 < \text{length } l; \ l!(i-1) = \text{R}; \ l!(i) = \text{R}; \ l!(i+1) = \text{R} \\
\end{align*} ] \implies \text{valid-point } i
\]
\[ [ \begin{align*}
& 2 \leq i; \ i < \text{length } l; \ l!(i-2) = \text{R}; \ l!(i-1) = \text{R}; \ l!(i) = \text{R} \\
\end{align*} ] \implies \text{valid-point } i
\]
\[ [ i < \text{length } l; \ l! = \text{B} ] \implies \text{valid-point } i \]

definition valid-line = (\( \forall i < \text{length } l. \text{valid-point } i \))

end

lemma valid-line1:
\[ \text{assumes } \bigwedge i. \ i < \text{length } l \implies \text{valid-point } l \ i \]
\[ \text{shows } \text{valid-line } l \]
\[ \langle \text{proof} \rangle \]

lemma valid-B-first:
valid-point xs i \implies i < \text{length } xs \implies \text{valid-point } (\text{B } \# \ xs) (i + 1)
\[ \langle \text{proof} \rangle \]

lemma valid-line-prepend-B:
valid-line (\text{B } \# \ xs) if valid-line xs
\[ \langle \text{proof} \rangle \]

lemma valid-drop-B:
valid-point xs (i - 1) if valid-point (\text{B } \# \ xs) i i > 0
\[ \langle \text{proof} \rangle \]

lemma valid-line-drop-B:
valid-line xs if valid-line (\text{B } \# \ xs)
\[ \langle \text{proof} \rangle \]

lemma valid-line-prepend-B-iff:
valid-line (\text{B } \# \ xs) \longleftrightarrow \text{valid-line } xs
\[ \langle \text{proof} \rangle \]

lemma cases-valid-line:
\[ \text{assumes } \]
\[ l = \text{[]} \lor \]
\[ (\text{!}!0 = \text{B } \land \text{valid-line } (\text{tl } l)) \lor \]
\[ \text{length } l \geq 3 \land (\forall i < \text{length } l. \ l! i = \text{R}) \lor \]
\[ (\exists j < \text{length } l. \ j \geq 3 \land (\forall i < j. \ l! i = \text{R}) \land l! j = \text{B } \land \text{valid-line } (\text{drop } (j + 1) \ l)) \]
\[ (\text{is } ?a \lor \text{?b } \lor \text{?c } \lor \text{?d}) \]
\[ \text{shows } \text{valid-line } l \]
\[ \langle \text{proof} \rangle \]

lemma valid-line-cases:
\[ l = \text{[]} \lor \]
\[ (\text{!}!0 = \text{B } \land \text{valid-line } (\text{tl } l)) \lor \]
\[ \text{length } l \geq 3 \land (\forall i < \text{length } l. \ l! i = \text{R}) \lor \]
2.2. SOLUTION

$$\exists j < \text{length } l. j \geq 3 \land (\forall i < j. l \! i = R) \land l \! j = B \land \text{valid-line } (\text{drop } (j + 1) l))$$

if valid-line l

\langle proof \rangle

lemma valid-line-split:

valid-line l \iff

l = [] \lor

(l\!0 = B \land \text{valid-line } (\text{tl } l)) \lor

\text{length } l \geq 3 \land (\forall i < \text{length } l. l \! i = R) \lor

(\exists j < \text{length } l. j \geq 3 \land (\forall i < j. l \! i = R) \land l \! j = B \land \text{valid-line } (\text{drop } (j + 1) l))

\langle proof \rangle

Connection to the easier definition given above

lemma valid-valid-line:

valid l \iff \text{valid-line } l

\langle proof \rangle

end
Array-Based Queuing Lock

3.1 Challenge

Array-Based Queuing Lock (ABQL) is a variation of the Ticket Lock algorithm with a bounded number of concurrent threads and improved scalability due to better cache behaviour.

We assume that there are \( N \) threads and we allocate a shared Boolean array \( \text{pass}[\cdot] \) of length \( N \). We also allocate a shared integer value \( \text{next} \). In practice, \( \text{next} \) is an unsigned bounded integer that wraps to 0 on overflow, and we assume that the maximal value of \( \text{next} \) is of the form \( kN - 1 \). Finally, we assume at our disposal an atomic \text{fetch_and_add} \) instruction, such that \text{fetch_and_add}(\text{next},1) \) increments the value of \( \text{next} \) by 1 and returns the original value of \( \text{next} \).

The elements of \( \text{pass}[\cdot] \) are spinlocks, assigned individually to each thread in the waiting queue. Initially, each element of \( \text{pass}[\cdot] \) is set to \( \text{false} \), except \( \text{pass}[0] \) which is set to \( \text{true} \), allowing the first coming thread to acquire the lock. Variable \( \text{next} \) contains the number of the first available place in the waiting queue and is initialized to 0.

Here is an implementation of the locking algorithm in pseudocode:

```plaintext
procedure abql_init()
    for i = 1 to N - 1 do
        pass[i] := false
    end-for
    pass[0] := true
    next := 0
end-procedure

function abql_acquire()
    var my_ticket := fetch_and_add(next,1) mod N
    while not pass[my_ticket] do
        end-while
    return my_ticket
end-function

procedure abql_release(my_ticket)
    pass[my_ticket] := false
    pass[(my_ticket + 1) mod N] := true
end-procedure
```

Each thread that acquires the lock must eventually release it by calling \text{abql_release}(\text{my_ticket}).
where my_ticket is the return value of the earlier call of abql_acquire(). We assume that no thread tries to re-acquire the lock while already holding it, neither it attempts to release the lock which it does not possess. Notice that the first assignment in abql_release() can be moved at the end of abql_acquire().

Verification task 1. Verify the safety of ABQL under the given assumptions. Specifically, you should prove that no two threads can hold the lock at any given time.

Verification task 2. Verify the fairness, namely that the threads acquire the lock in order of request.

Verification task 3. Verify the liveness under a fair scheduler, namely that each thread requesting the lock will eventually acquire it.

You have liberty of adapting the implementation and specification of the concurrent setting as best suited for your verification tool. In particular, solutions with a fixed value of $N$ are acceptable. We expect, however, that the general idea of the algorithm and the non-deterministic behaviour of the scheduler shall be preserved.
3.2 Solution

theory Challenge3
imports lib/VTcomp lib/DF-System
begin

The Isabelle Refinement Framework does not support concurrency. However, Isabelle is a general purpose theorem prover, thus we can model the problem as a state machine, and prove properties over runs. For this polished solution, we make use of a small library for transition systems and simulations: VerifyThis2018.DF-System. Note, however, that our definitions are still quite ad-hoc, and there are lots of opportunities to define libraries that make similar proofs simpler and more canonical.

We approach the final ABQL with three refinement steps:

1. We model a ticket lock with unbounded counters, and prove safety, fairness, and liveness.
2. We bound the counters by $mod\ N$ and $mod\ (k\times\ N)$ respectively
3. We implement the current counter by an array, yielding exactly the algorithm described in the challenge.

With a simulation argument, we transfer the properties of the abstract system over the refinements.

The final theorems proving safety, fairness, and liveness can be found at the end of this chapter, in Subsection 3.2.6.

3.2.1 General Definitions

We fix a positive number $N$ of threads

```plaintext
consts N :: nat
specification (N) N-not0[simp, intro!]: N\neq0 [proof]
lemma N-gt0[simp, intro!]: 0<N [proof]
```

A thread’s state, representing the sequence points in the given algorithm. This will not change over the refinements.

```plaintext
datatype thread =
  INIT
  | is-WAIT: WAIT (ticket: nat)
  | is-HOLD: HOLD (ticket: nat)
  | is-REL: REL (ticket: nat)
```
3.2.2 Refinement 1: Ticket Lock with Unbounded Counters

System’s state: Current ticket, next ticket, thread states

**type-synonym**

\[ \text{astate} = \text{nat} \times \text{nat} \times (\text{nat} \Rightarrow \text{thread}) \]

**abbreviation**

\[ \text{cc} \equiv \text{fst} \]

\[ \text{nn s} \equiv \text{fst}(\text{snd s}) \]

\[ \text{tts s} \equiv \text{snd}(\text{snd s}) \]

The step relation of a single thread

**inductive** \( \text{astep-sng} \) where

- enter-wait: \( \text{astep-sng} (c, n, \text{INIT}) (c, (n+1), \text{WAIT n}) \)
- loop-wait: \( c \neq k \Longrightarrow \text{astep-sng} (c, n, \text{WAIT k}) (c, n, \text{WAIT k}) \)
- exit-wait: \( \text{astep-sng} (c, n, \text{WAIT c}) (c, n, \text{HOLD c}) \)
- start-release: \( \text{astep-sng} (c, n, \text{HOLD k}) (c, n, \text{REL k}) \)
- release: \( \text{astep-sng} (c, n, \text{REL k}) (k+1, n, \text{INIT}) \)

The step relation of the system

**inductive** \( \text{alstep} \) for \( t \) where

\[ \left[ t < N; \text{astep-sng} (c, n, ts t) (c', n', ts') \right] \]

\[ \Longrightarrow \text{alstep} t (c, n, ts) (c', n', ts(t:=s')) \]

Initial state of the system

**definition** \( \text{as}_0 \equiv (0, 0, \lambda\cdot \text{INIT}) \)

**interpretation** \( A: \text{system as}_0 \text{ alstep} \) (proof)

In our system, each thread can always perform a step

**lemma** never-blocked: \( A.\text{can-step l s} \leftrightarrow l < N \) (proof)

Thus, our system is in particular deadlock free

**interpretation** \( A: \text{df-system as}_0 \text{ alstep} \) (proof)

**Safety: Mutual Exclusion**

Predicates to express that a thread uses or holds a ticket

**definition** \( \text{has-ticket s k} \equiv s = \text{WAIT k} \lor s = \text{HOLD k} \lor s = \text{REL k} \)

**lemma** has-ticket-simps[simp]:

- \( \neg \text{has-ticket INIT k} \)
- \( \text{has-ticket (WAIT k) k'\leftrightarrow k'\equiv k} \)
- \( \text{has-ticket (HOLD k) k'\leftrightarrow k'\equiv k} \)
- \( \text{has-ticket (REL k) k'\leftrightarrow k'\equiv k} \) (proof)

**definition** \( \text{locks-ticket s k} \equiv s = \text{HOLD k} \lor s = \text{REL k} \)
3.2. SOLUTION

\textbf{lemma} locks-ticket-simps\{simp\}:
- \neg \text{locks-ticket} \text{ INIT} k
- \neg \text{locks-ticket} \ (\text{WAIT} k) k'
\text{locks-ticket} \ (\text{HOLD} k) k' \rightarrow k' = k
\text{locks-ticket} \ (\text{REL} k) k' \leftarrow k' = k

\textbf{proof}

\textbf{lemma} holds-imp-uses: locks-ticket s k \implies has-ticket s k

\textbf{proof}

We show the following invariant. Intuitively, it can be read as follows:

- Current lock is less than or equal next lock
- For all threads that use a ticket (i.e., are waiting, holding, or releasing):
  - The ticket is in between current and next
  - No other thread has the same ticket
  - Only the current ticket can be held (or released)

\textbf{definition} invar1 \equiv \lambda (c,n,\text{ts}).
\begin{align*}
c & \leq n \\
\land & (\forall k. t < N \land \text{has-ticket} \ (ts) \ k \rightarrow c \leq k \land k < n) \\
\land & (\forall k'. t' < N \land \text{has-ticket} \ (ts) \ k' \land t \neq t' \rightarrow k \neq k') \\
\land & (\forall k. k \neq c \rightarrow \neg \text{locks-ticket} \ (ts) \ k)
\end{align*}

\textbf{lemma} is-invar1: A.is-invar invar1

\textbf{proof}

From the above invariant, it’s straightforward to show mutual exclusion

\textbf{theorem} mutual-exclusion: \[ A.\text{reachable} \ s; t<N; t'<N; t \neq t'; \text{is-HOLD} \ (tts) \ s \ t; \text{is-HOLD} \ (tts) \ s \ t' \] \implies \text{False}

\textbf{proof}

\textbf{lemma} mutual-exclusion': \[ A.\text{reachable} \ s; t<N; t'<N; t \neq t'; \text{locks-ticket} \ (tts) \ s \ t k; \text{locks-ticket} \ (tts) \ s \ t' k' \] \implies \text{False}

\textbf{proof}

\textbf{Fairness: Ordered Lock Acquisition}

We first show an auxiliary lemma: Consider a segment of a run from \(i\) to \(j\). Every thread that waits for a ticket in between the current ticket at \(i\) and the current ticket at \(j\) will be granted the lock in between \(i\) and \(j\)
CHAPTER 3. ARRAY-BASED QUEUING LOCK

**lemma** fair-aux:
assumes R: A.is-run s
assumes A: i<j cc (s i) ≤ k k < cc (s j) t<N tts (s i) t=WAIT k
shows ∃l. i≤l ∧ l<j ∧ tts (s l) t = HOLD k
⟨proof⟩

**lemma** s-case-expand:
(case s of (c, n, ts) ⇒ P c n ts) = P (cc s) (nn s) (tts s)
⟨proof⟩

A version of the fairness lemma which is very detailed on the actual ticket numbers.
We will weaken this later.

**lemma** fair-aux2:
assumes RUN: A.is-run s
assumes ACQ: t<N tts (s i) t=INIT tts (s (Suc i)) t=WAIT k
assumes HOLD: i<j tts (s j) t = HOLD k
assumes WAIT: t'<N tts (s i) t' = WAIT k'
obtains l where i<l ∧ l<j ∧ tts (s l) t' = HOLD k'
⟨proof⟩

**lemma** find-hold-position:
assumes RUN: A.is-run s
assumes WAIT: t<N tts (s i) t = WAIT tk
assumes NWAIT: i<j tts (s j) t ≠ WAIT tk
obtains l where i<l ∧ l<j ∧ tts (s l) t = HOLD tk
⟨proof⟩

Finally we can show fairness, which we state as follows: Whenever a thread t gets a ticket, all other threads t′ waiting for the lock will be granted the lock before t.

**theorem** fair:
assumes RUN: A.is-run s
assumes ACQ: t<N tts (s i) t=INIT is-WAIT (tts (s (Suc i)) t)
— Thread t calls acquire in step i
assumes HOLD: i<j is-HOLD (tts (s j) t)
— Thread t holds lock in step j
assumes WAIT: t'<N is-WAIT (tts (s i) t')
— Thread t′ waits for lock at step i
obtains l where i<l ∧ is-HOLD (tts (s l) t')
— Then, t′ gets lock earlier
⟨proof⟩

**Liveness**

For all tickets in between the current and the next ticket, there is a thread that has this ticket

**definition** invar2
≡ λ(c,n,ts). ∀k. c≤k ∧ k<n → (∃t<N. has-ticket (ts t) k)
3.2. SOLUTION

lemma is-invar2: A.is-invar invar2
  ⟨proof⟩

If a thread \( t \) is waiting for a lock, the current lock is also used by a thread

corollary current-lock-used:
  assumes R: A.reachable (c,n,ts)
  assumes WAIT: t<N ts t = WAIT k
  obtains t' where t'<N has-ticket (ts t') c
  ⟨proof⟩

Used tickets are unique (Corollary from invariant 1)

lemma has-ticket-unique: [\[ A.reachable (c,n,ts); t<N; has-ticket (ts t) k; t'<N; has-ticket (ts t') k \] \implies t'=t]
  ⟨proof⟩

We define the thread that holds a specified ticket

definition tkt-thread ≡ \( \lambda ts k. \) THE \( t. t<N \land has-ticket (ts t) k \)
lemma tkt-thread-eq:
  assumes R: A.reachable (c,n,ts)
  assumes A: t<N has-ticket (ts t) k
  shows tkt-thread ts k = t
  ⟨proof⟩

lemma holds-only-current:
  assumes R: A.reachable (c,n,ts)
  assumes A: t<N locks-ticket (ts t) k
  shows k=c
  ⟨proof⟩

For the inductive argument, we will use this measure, that decreases as a single thread progresses through its phases.

definition tweight s ≡ case s of WAIT - \Rightarrow 3:nat | HOLD - \Rightarrow 2 | REL - \Rightarrow 1 | INIT \Rightarrow 0

We show progress: Every thread that waits for the lock will eventually hold the lock.

theorem progress:
  assumes FRUN: A.is-fair-run s
  assumes A: t<N is-WAIT (ts (s i) t)
  shows \( \exists j>i. \) is-HOLD (ts (s j) t)
  ⟨proof⟩

3.2.3 Refinement 2: Bounding the Counters

We fix the \( k \) from the task description, which must be positive

consts k::nat
CHAPTER 3. ARRAY-BASED QUEUING LOCK

specification \((k) \ k\text{-not0 simp}; \ k \neq 0 \langle \text{proof} \rangle\)

lemma \(k\text{-gt0 simp}; \ 0 < k \langle \text{proof} \rangle\)

System’s state: Current ticket, next ticket, thread states

type-synonym \(bstate = \text{nat} \times \text{nat} \times (\text{nat} \Rightarrow \text{thread})\)

The step relation of a single thread

inductive \(bstep\text{-sng}\) where

- enter-wait: \(bstep\text{-sng} (c, n, \text{INIT}) (c, (n + 1) \ mod (k * N), \text{WAIT} (n \ mod N))\)
- loop-wait: \(c \neq tk \implies bstep\text{-sng} (c, n, \text{WAIT} tk) (c, n, \text{WAIT} tk)\)
- exit-wait: \(bstep\text{-sng} (c, n, \text{WAIT} c) (c, n, \text{HOLD} c)\)
- start-release: \(bstep\text{-sng} (c, n, \text{HOLD} tk) (c, n, \text{REL} tk)\)
- release: \(bstep\text{-sng} (c, n, \text{REL} tk) ((tk + 1) \ mod N, n, \text{INIT})\)

The step relation of the system, labeled with the thread \(t\) that performs the step

inductive \(blstep\) for \(t\) where

- \(\lfloor t < N; bstep\text{-sng} (c, n, ts t) (c', n', ts') \rfloor \implies blstep t (c, n, ts) (c', n', ts(t := s'))\)

Initial state of the system

definition \(bs_0 \equiv (0, 0, \lambda - \text{INIT})\)

interpretation \(B: \text{system } bs_0 \ blstep \langle \text{proof} \rangle\)

lemma \(b\text{-never-blocked}: \text{B.can-step l s } \longleftrightarrow \text{l} < N \langle \text{proof} \rangle\)

interpretation \(B: \text{df-system } bs_0 \ blstep \langle \text{proof} \rangle\)

Simulation

We show that the abstract system simulates the concrete one.

A few lemmas to ease the automation further below

lemma \(\text{nat-sum-gtZ-iff simp}: \text{finite s } \implies \text{sum f s } \neq (0 :: \text{nat}) \longleftrightarrow (\exists x \in s. f x \neq 0) \langle \text{proof} \rangle\)

lemma \(\text{n-eq-Suc-sub1-conv simp}: n = \text{Suc} (n - \text{Suc} 0) \longleftrightarrow n \neq 0 \langle \text{proof} \rangle\)

lemma \(\text{mod-mult-mod-eq [mod-simps]}: x \ mod (k * N) \ mod N = x \ mod N \langle \text{proof} \rangle\)

lemma \(\text{mod-eq-imp-eq-aux}: b \ mod N = (a :: \text{nat}) \ mod N \implies a \leq b \implies b < a + N \implies b = a \langle \text{proof} \rangle\)
3.2. SOLUTION

lemma mod-eq-imp-eq:
\[ \begin{align*}
  & [ b \leq x; x < b + N; b \leq y; y < b + N; x \mod N = y \mod N ] \implies x = y \\
\end{align*} \]
\langle proof \rangle

Map the ticket of a thread

fun map-ticket \ where \\
map-ticket \ INIT = INIT \\
| map-ticket \ (WAIT \ tk) = WAIT (f \ tk) \\
| map-ticket \ (HOLD \ tk) = HOLD (f \ tk) \\
| map-ticket \ (REL \ tk) = REL (f \ tk) \\

lemma map-ticket-addsims[simp]: \\
map-ticket \ f \ t = INIT \iff t = INIT \\
| map-ticket \ f \ t = WAIT \ tk \iff (\exists \ tk'. tk = f \ tk' \land t = WAIT \ tk') \\
| map-ticket \ f \ t = HOLD \ tk \iff (\exists \ tk'. tk = f \ tk' \land t = HOLD \ tk') \\
| map-ticket \ f \ t = REL \ tk \iff (\exists \ tk'. tk = f \ tk' \land t = REL \ tk') \\
\langle proof \rangle

We define the number of threads that use a ticket

fun ni-weight \ :: \ thread \Rightarrow \ nat \ where \\
ni-weight \ INIT = 0 \ | \ ni-weight - = 1 \\

lemma ni-weight-le1[simp]: \ ni-weight \ s \leq \ Suc \ 0 \\
\langle proof \rangle

definition num-ni \ ts \equiv \ \sum_{i=0..<N} \ ni-weight \ (ts \ i) \\
lemma num-ni-init[simp]: \ num-ni \ (\lambda - INIT) = 0 \langle proof \rangle

lemma num-ni-upd: \\
\ t < N \implies num-ni \ (ts(t:=s)) = num-ni \ ts \ - \ ni-weight \ (ts \ t) \ + \ ni-weight \ s \\
\langle proof \rangle

lemma num-ni-nz-if[simp]: \ [t < N; ts \ t \neq INIT] \implies num-ni \ ts \neq \ 0 \\
\langle proof \rangle

lemma num-ni-leN: \ num-ni \ ts \ \leq \ N \\
\langle proof \rangle

We provide an additional invariant, considering the distance of \( c \) and \( n \). Although we could probably get this from the previous invariants, it is easy enough to prove directly.

definition invar3 \equiv \ \lambda (c,n,ts). n = c + num-ni \ ts \\

lemma is-invar3: A.is-invar invar3 \\
\langle proof \rangle

We establish a simulation relation: The concrete counters are the abstract ones, wrapped around.
CHAPTER 3. ARRAY-BASED QUEUING LOCK

**definition** sim-rel1 \( \equiv \lambda (c,n,ts) \ (ci,ni,tsi) \).

\( ci = c \mod N \)
\( \land \ ni = n \mod (k*N) \)
\( \land \ tsi = (\text{map-ticket} \ (\lambda t. \ t \mod N)) \ o \ ts \)

**lemma** sraux:

\[ \text{sim-rel1} (c,n,ts) (ci,ni,tsi) \implies ci = c \mod N \land ni = n \mod (k*N) \]

**proof**

**lemma** sraux2:

\[ \text{sim-rel1} (c,n,ts) (ci,ni,tsi) ; t < N \]

\[ \implies tsi t = \text{map-ticket} \ (\lambda x. \ x \mod N) \ (ts t) \]

**proof**

**interpretation** sim1: simulationI as \( as_0 \) alstep bs_0 blstep sim-rel1

**proof**

**Transfer of Properties**

We transfer a few properties over the simulation, which we need for the next refinement step.

**lemma** xfer-locks-ticket:

**assumes** locks-ticket (\( \text{map-ticket} \ (\lambda t. \ t \mod N) \ (ts t) \)) tki

**obtains** tk \ where tki = tk mod N locks-ticket (ts t) tk

**proof**

**lemma** b-holds-only-current:

\[ B.\text{reachable} \ (c, n, ts); t < N; \text{locks-ticket} \ (ts t) tk \]

\[ \implies \ tk = c \]

**proof**

**lemma** b-mutual-exclusion': \[ B.\text{reachable} s; \]

\[ t < N; t' < N; t \neq t'; \text{locks-ticket} \ (ts s t) tk; \text{locks-ticket} \ (ts s t') tk' \]

\[ \implies \ False \]

**proof**

**lemma** xfer-has-ticket:

**assumes** has-ticket (\( \text{map-ticket} \ (\lambda t. \ t \mod N) \ (ts t) \)) tki

**obtains** tk \ where tki = tk mod N has-ticket (ts t) tk

**proof**

**lemma** has-ticket-in-range:

**assumes** Ra: A.\text{reachable} (c,n,ts) \ and \ t < N \ and \ U: \text{has-ticket} \ (ts t) tk

**shows** c \leq tk \land tk < c+N

**proof**

**lemma** b-has-ticket-unique:

\[ B.\text{reachable} (ci,ni,tsi); \]

\[ t < N; \text{has-ticket} \ (tsi t) tki; \ t' < N; \text{has-ticket} \ (tsi t') tki \]
\[ t' = t \]

\[ \text{(proof)} \]

### 3.2.4 Refinement 3: Using an Array

Finally, we use an array instead of a counter, thus obtaining the exact data structures from the challenge assignment.

Note that we model the array by a list of Booleans here.

System’s state: Current ticket array, next ticket, thread states

\[
\text{type-synonym } \text{cstate} = \text{bool list } \times \text{nat } \times (\text{nat } \Rightarrow \text{thread})
\]

The step relation of a single thread

\[
\text{inductive } \text{cstep-sng} \text{ where}
\begin{align*}
\text{enter-wait: } & \neg p!tk \implies \text{cstep-sng} (p,n,\text{WAIT tk}) (p,n,\text{WAIT tk}) \\
\text{loop-wait: } & p!tk \implies \text{cstep-sng} (p,n,\text{WAIT tk}) (p,n,\text{WAIT tk}) \\
\text{exit-wait: } & p!tk \implies \text{cstep-sng} (p,n,\text{WAIT tk}) (p,n,\text{HOLD tk}) \\
\text{start-release: } & \text{cstep-sng} (p,n,\text{HOLD tk}) (p|tk:=\text{False},n,\text{REL tk}) \\
\text{release: } & \text{cstep-sng} (p,n,\text{REL tk}) (p|(tk+l) \mod N := \text{True},n,\text{INIT})
\end{align*}
\]

The step relation of the system, labeled with the thread \( t \) that performs the step

\[
\text{inductive } \text{clstep for } t \text{ where}
\begin{align*}
\ll [ t < N; \text{cstep-sng} (c,n,ts t) (c',n',s')] \rightarrow \text{clstep } t (c,n,ts) (c',n',ts(t := s'))
\end{align*}
\]

Initial state of the system

\[
\text{definition } c_0 \equiv ((\text{replicate } N \text{ False})[0 := \text{True}], 0, \lambda -. \text{INIT})
\]

\[
\text{interpretation } C: \text{system } c_0 \text{ clstep } \langle \text{proof} \rangle
\]

\[
\text{lemma c-never-blocked: } C.\text{can-step } l s \iff l < N
\]

\[
\text{(proof)}
\]

\[
\text{interpretation } C: \text{df-system } c_0 \text{ clstep } \langle \text{proof} \rangle
\]

We establish another invariant that states that the ticket numbers are bounded.

\[
\text{definition } \text{invar4} \equiv \lambda (c,n,ts). c < N \land (\forall t < N. \forall tk. \text{has-ticket } (ts t) tk \implies tk < N)
\]

\[
\text{lemma is-invar4: } B.\text{is-invar } \text{invar4}
\]

\[
\text{(proof)}
\]

We define a predicate that describes that a thread of the system is at the release sequence point — in this case, the array does not have a set bit, otherwise, the set bit corresponds to the current ticket.

\[
\text{definition } \text{is-REL-state} \equiv \lambda ts. \exists t < N. \exists tk. ts t = \text{REL tk}
\]
lemma is-REL-state-simps[simp]:
\[ t < N \implies \text{is-REL-state} (ts(t:=\text{REL } tk)) \]
\[ t < N \implies \neg \text{is-REL} (ts t) \implies \neg \text{is-REL} s' \]
\[ \implies \text{is-REL-state} (ts(t:=s')) \iff \text{is-REL-state} ts \]
\( \langle \text{proof} \rangle \)

lemma is-REL-state-aux1:
assumes R: B.reachable \((c,n,ts)\)
assumes REL: \text{is-REL-state} ts
assumes \(t < N\) and [simp]: \(ts t = \text{WAIT} tk\)
shows \(tk \neq c\)
\( \langle \text{proof} \rangle \)

lemma is-REL-state-aux2:
assumes R: B.reachable \((c,n,ts)\)
assumes A: \(t < N\) \(ts t = \text{REL} tk\)
shows \(\neg \text{is-REL-state} (ts(t:=\text{INIT}))\)
\( \langle \text{proof} \rangle \)

Simulation relation that implements current ticket by array

definition sim-rel2 \equiv \lambda (c, n, ts) (ci, ni, tsi).
\(\begin{align*}
\text{(if is-REL-state ts then} \\
\text{ci = replicate N False} \\
\text{else} \\
\text{ci = (replicate N False)[c:=True]}
\end{align*}\)
\(\land ni = n \\
\land tsi = ts\)

interpretation sim2: simulationI bs_0 blstep cs_0 clstep sim-rel2
\( \langle \text{proof} \rangle \)

3.2.5 Transfer Setup

We set up the final simulation relation, and the transfer of the concepts used in the correctness statements.

definition sim-rel \equiv sim-rel1 OO sim-rel2
interpretation sim: simulation as_0 alstep cs_0 clstep sim-rel
\( \langle \text{proof} \rangle \)

lemma xfer-holds:
assumes sim-rel s cs
shows \(\text{is-HOLD} (tts cs t) \iff \text{is-HOLD} (tts s t)\)
\( \langle \text{proof} \rangle \)
3.2. SOLUTION

lemma xfer-waits:
assumes sim-rel s cs
shows is-WAIT (tts cs t) ⟷ is-WAIT (tts s t)
(proof)

lemma xfer-init:
assumes sim-rel s cs
shows tts cs t = INIT ⟷ tts s t = INIT
(proof)

3.2.6 Main Theorems

Trusted Code Base

Note that the trusted code base for these theorems is only the formalization of the concrete system as defined in Section 3.2.4. The simulation setup and the abstract systems are only auxiliary constructions for the proof.

For completeness, we display the relevant definitions of reachability, runs, and fairness here:

\[ C.\text{step} \ s \ s' = (\exists l. \text{clstep} \ l \ s \ s') \]

\[ C.\text{reachable} \equiv C.\text{step}^{**} \ c s_0 \]
\[ C.\text{is-lrun} \ l \ s \equiv s_0 = c s_0 \land (\forall i. \text{clstep} (l i) (s i) (s (\text{Suc} \ i))) \]
\[ C.\text{is-run} \ s \equiv \exists l. C.\text{is-lrun} \ l \ s \]
\[ C.\text{is-lfair} \ ls \ ss \equiv \forall i. \exists j \geq i. \neg C.\text{can-step} \ l \ (ss j) \lor ls j = l \]
\[ C.\text{is-fair-run} \ s \equiv \exists l. C.\text{is-lrun} \ l \ s \land C.\text{is-lfair} \ l \ s \]

Safety

We show that there is no reachable state in which two different threads hold the lock.

\[
\text{theorem} \ \text{final-mutual-exclusion}: \ [C.\text{reachable} \ s; \ t < N; t' < N; t \neq t'; \text{is-HOLD} (tts s t); \text{is-HOLD} (tts s t') ] \implies \text{False} \\
\text{(proof)}
\]

Fairness

We show that, whenever a thread \( t \) draws a ticket, all other threads \( t' \) waiting for the lock will be granted the lock before \( t \).

\[
\text{theorem} \ \text{final-fair}: \\
\text{assumes} \ \text{RUN}: C.\text{is-run} \ s \\
\text{assumes} \ \text{ACQ}: t < N \ \text{and} \ tts (s i) t = \text{INIT} \ \text{and} \ \text{is-WAIT} (tts (s (\text{Suc} \ i)) t) \\
\text{— Thread} \ t \ \text{draws ticket in step} \ i
\]
assumes \textit{HOLD}: \(i < j\) and \textit{is-HOLD} (\(\text{tts} (s\ j) t\))

— Thread \(t\) holds lock in step \(j\)

assumes \textit{WAIT}: \(t' < N\) and \textit{is-WAIT} (\(\text{tts} (s\ i) t'\))

— Thread \(t'\) waits for lock at step \(i\)

obtains \(l\) where \(i < l\) and \(l < j\) and \textit{is-HOLD} (\(\text{tts} (s\ l) t'\))

— Then, \(t'\) gets lock earlier

\(\langle \text{proof} \rangle\)

\textbf{Liveness}

We show that, for a fair run, every thread that waits for the lock will eventually hold the lock.

\textbf{theorem} \textit{final-progress}:

\textbf{assumes} \textit{FRUN}: \(C.\text{is-fair-run}\ s\)

\textbf{assumes} \textit{WAIT}: \(t < N\) and \textit{is-WAIT} (\(\text{tts} (s\ i) t\))

\textbf{shows} \(\exists j > i.\ \textit{is-HOLD} (\text{tts} (s\ j) t)\)

\(\langle \text{proof} \rangle\)

\textbf{end}