Fundamentals of Unconstrained Optimization

Dustin Bryant

August 1, 2025

Abstract

As formal methods gain traction in machine learning and numerical analysis, the community needs computer-checked proofs of core optimization results. Existing Isabelle libraries still lack a foundational framework for unconstrained optimization. We close this gap with a comprehensive Isabelle/HOL development that formalizes:

(1) minimizers, strict and isolated local minimizers;

(2) first- and second-order optimality conditions for scalar functions $f : \mathbb{R} \to \mathbb{R}$;

(3) first-order optimality conditions for vector functions $g : \mathbb{R}^n \to \mathbb{R}$; and

(4) a worked example showing that the continuous function

$$h(x) = \begin{cases} x^4\big(\cos(1/x) + 2\big), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

has a *strict* but *non-isolated* local minimizer at $x = 0$.

The new session `Unconstrained_Optimization` provides sound, reusable foundations for future proof-checking tools and mechanized research in optimization, analysis, and algorithmic correctness.

Contents

1 Auxiliary Facts

theory *Auxilary-Facts*
  imports
    Sigmoid-Universal-Approximation.Limits-Higher-Order-Derivatives
begin

1.1 Differentiation Lemmas

lemma *has-derivative-imp*:
  fixes f :: *real* ⇒ *real*
  assumes (f *has-derivative* f') (*at* x)
  shows f *differentiable* (*at* x) ∧ *deriv* $f\ x = f'\ 1$
proof *safe*
  show f *differentiable at* x
    by (*meson assms differentiableI*)
  then show *deriv* $f\ x = f'\ 1$
    by (*metis DERIV-deriv-iff-real-differentiable assms has-derivative-unique*
      has-field-derivative-imp-has-derivative mult.comm-neutral)
qed

lemma *DERIV-inverse-func*:
  assumes $x \neq 0$
  shows *DERIV* ($\lambda w.\ 1\ /\ w$) x :> $-1\ /\ x\hat{}2$
proof −
  have *inverse* = (/) (*1*::′*a*)
    using *inverse-eq-divide* **by** *auto*
  then show *?thesis*
    by (*metis* (*no-types*) *DERIV-inverse assms divide-minus-left numeral-2-eq-2*
power-one-over)
qed

lemma *power-rule*:

fixes z :: *real* **and** n :: *nat*
shows *deriv* $(\lambda x.\ x \mathbin{\hat{}} n)\ z = (\mathit{if}\ n = 0\ \mathit{then}\ 0\ \mathit{else}\ \mathit{real}\ n * z \mathbin{\hat{}} (n - 1))$
by(*subst deriv-pow*, *simp-all*)

1.1.1 Transfer Lemmas

lemma *has-derivative-transfer-on-ball*:
fixes *f g* :: *real* ⇒ *real*
assumes *eps-gt0*: $0 < \varepsilon$
assumes *eq-on-ball*: $\forall y.\ y \in \mathit{ball}\ x\ \varepsilon \longrightarrow f\ y = g\ y$
assumes *f-has-deriv*: (*f has-derivative D*) (*at x*)
shows (*g has-derivative D*) (*at x*)
proof −
from *f-has-deriv*
have *lim*: $((\lambda y.\ (f\ y - f\ x - D\ (y - x))\ /\ |y - x|) \longrightarrow 0)\ (\mathit{at}\ x)$
unfolding *has-derivative-def*
by (*simp add*: *divide-inverse-commute*)

— Using $[\![(?f \longrightarrow ?l)\ (\mathit{at}\ ?a\ \mathit{within}\ ?T);\ \mathit{open}\ ?s;\ ?a \in ?s;\ \bigwedge x.\ [\![x \in ?s;\ x \neq ?a]\!] \Longrightarrow ?f\ x = ?g\ x]\!] \Longrightarrow (?g \longrightarrow ?l)\ (\mathit{at}\ ?a\ \mathit{within}\ ?T)$, we switch from$f$ to$g$ in the difference quotient.
from *assms*(*1*,*2*) *lim* **have** $((\lambda y.\ (g\ y - f\ x - D\ (y - x))\ /\ |y - x|) \longrightarrow 0)\ (\mathit{at}\ x)$
by (*subst Lim-transform-within-open*
[**where** $f = \lambda xa.\ (f\ xa - f\ x - D\ (xa - x))\ /\ |xa - x|$ **and** $s = \mathit{ball}\ x\ \varepsilon$], *simp-all*)
— Then we replace $f(x)$ by $g(x)$ using the assumption `eq_on_ball`.

then have $((\lambda y.\ (g\ y - g\ x - D\ (y - x))\ /\ |y - x|) \longrightarrow 0)\ (\mathit{at}\ x)$
by (*simp add*: *assms*(*1*) *eq-on-ball*)
thus *?thesis*
using *assms centre-in-ball has-derivative-transform-within-open* **by** *blast*
qed

corollary *field-differentiable-transfer-on-ball*:
fixes *f g* :: *real* ⇒ *real*
assumes $0 < \varepsilon$
assumes *eq-on-ball*: $\forall y.\ y \in \mathit{ball}\ x\ \varepsilon \longrightarrow f\ y = g\ y$
assumes *f-diff*: *f field-differentiable at x*
shows *g field-differentiable at x*
proof −
from *f-diff* **obtain** *d*
where *f-has-real-deriv*: (*f has-real-derivative d*) (*at x*)
by (*auto simp*: *field-differentiable-def*)

have (*g has-real-derivative d*) (*at x*)
by (*meson Elementary-Metric-Spaces.open-ball assms*(*1*,*2*) *centre-in-ball f-has-real-deriv has-field-derivative-transform-within-open*)
thus *?thesis*

unfolding *field-differentiable-def*
by *blast*
qed

1.2 Trigonometric Contraction

lemma *cos-contractive*:
fixes *x y* :: *real*
shows |*cos x* − *cos y*| ≤ |*x* − *y*|
proof −
have |*cos x* − *cos y*| = |−*2* ∗ *sin* ((*x* + *y*) / *2*) ∗ *sin* ((*x* − *y*) / *2*)|
by (*smt* (*verit*) *cos-diff-cos mult-minus-left*)
also have ... ≤ |*sin* ((*x* + *y*) / *2*)| ∗ (*2*∗ |*sin* ((*x* − *y*) / *2*)|)
by (*subst abs-mult*, *force*)
also have ... ≤ *2* ∗ |*sin* ((*x* − *y*) / *2*)|
proof −
have |*sin* ((*x* + *y*) / *2*)| ≤ *1*
using *abs-sin-le-one* **by** *blast*
then have |*sin* ((*x* + *y*) / *2*)| ∗ (*2*∗ |*sin* ((*x* − *y*) / *2*)|) ≤ *1* ∗ (*2*∗ |*sin* ((*x* − *y*) / *2*)|)
by(*rule mult-right-mono*, *simp*)
then show *?thesis*
by *linarith*
qed
also have ... ≤ *2* ∗ |(*x* − *y*) / *2*|
using *abs-sin-le-one* **by** (*smt* (*verit*, *del-insts*) *abs-sin-x-le-abs-x*)
also have ... = |*x* − *y*|
by *simp*
finally show *?thesis*.
qed

lemma *sin-contractive*:
fixes *x y* :: *real*
shows |*sin x* − *sin y*| ≤ |*x* − *y*|
proof −
have |*sin x* − *sin y*| = |*2* ∗ *cos* ((*x* + *y*) / *2*) ∗ *sin* ((*x* − *y*) / *2*)|
by (*metis* (*no-types*) *mult.assoc mult.commute sin-diff-sin*)
also have ... ≤ |*cos* ((*x* + *y*) / *2*)| ∗ (*2* ∗ |*sin* ((*x* − *y*) / *2*)|)
by (*subst abs-mult*, *force*)
also have ... ≤ *2* ∗ |*sin* ((*x* − *y*) / *2*)|
proof −
have |*cos* ((*x* + *y*) / *2*)| ≤ *1*
using *abs-cos-le-one* **by** *blast*
then have |*cos* ((*x* + *y*) / *2*)| ∗ (*2* ∗ |*sin* ((*x* − *y*) / *2*)|) ≤ *1* ∗ (*2* ∗ |*sin* ((*x* − *y*) / *2*)|)
by (*rule mult-right-mono*, *simp*)
then show *?thesis*
by *linarith*
qed

also have ... $\leq$ *2* $*$ $|$(*x* $-$ *y*) / *2*$|$
using *abs-sin-le-one* **by** (*smt* (*verit*, *del-insts*) *abs-sin-x-le-abs-x*)
also have ... = $|$*x* $-$ *y*$|$
by *simp*
finally show *?thesis*.
qed

1.3 Algebraic Factorizations

lemma *biquadrate-diff-biquadrate-factored*:
fixes *x y*::*real*
shows *y*^*4* $-$ *x*^*4* = (*y* $-$ *x*) $*$ (*y*^*3* + *y*^*2* $*$ *x* + *y* $*$ *x*^*2* + *x*^*3*)
proof $-$
have *y*^*4* $-$ *x*^*4* = (*y*^*2* $-$ *x*^*2*) $*$ (*y*^*2* + *x*^*2*)
by (*metis mult.commute numeral-Bit0 power-add square-diff-square-factored*)
also have ... = (*y* $-$ *x*) $*$ (*y* + *x*) $*$ (*y*^*2* + *x*^*2*)
by (*simp add*: *power2-eq-square square-diff-square-factored*)
also have ... = (*y* $-$ *x*) $*$ (*y*^*3* + *y*^*2* $*$ *x* + *y* $*$ *x*^*2* + *x*^*3*)
by (*simp add*: *distrib-left mult.commute power2-eq-square power3-eq-cube*)
finally show *?thesis*.
qed

1.4 Specific Trigonometric Values

lemma *sin-5pi-div-4*: *sin* (*5* $*$ *pi* / *4*) = $-$ (*sqrt 2* / *2*)
proof $-$
have *5* $*$ *pi* / *4* = *pi* + *pi* / *4*
by *simp*
moreover have *sin* (*pi* + *x*) = $-$ *sin x* **for** *x*
by (*simp add*: *sin-add*)
ultimately show *?thesis*
using *sin-45* **by** *presburger*
qed

lemma *cos-5pi-div-4*: *cos* (*5* $*$ *pi* / *4*) = $-$ (*sqrt 2* / *2*)
proof $-$
have *5* $*$ *pi* / *4* = *pi* + *pi* / *4*
by *simp*
moreover have *cos* (*pi* + *x*) = $-$ *cos x* **for** *x*
by (*simp add*: *cos-add*)
moreover have *cos* (*pi* / *4*) = *sqrt 2* / *2*
by (*simp add*: *real-div-sqrt cos-45*)
ultimately show *?thesis*
by *presburger*
qed

1.5 Local Sign Preservation of Continuous Functions

1.5.1 Local Positivity

lemma *cont-at-pos-imp-loc-pos*:
  fixes g :: *real* ⇒ *real* **and** x :: *real*
  assumes *continuous* (*at* x) g **and** $g\ x > 0$
  shows $\exists\,\delta > 0.\ \forall\,y.\ |y - x| < \delta \longrightarrow g\ y > 0$
proof –
  from *assms* **obtain** δ **where** δ*-pos*: $\delta > 0$
    and $\forall\,y.\ |y - x| < \delta \longrightarrow |g\ y - g\ x| < (g\ x)/2$
    using *continuous-at-eps-delta half-gt-zero* **by** *blast*
  then have $\forall\,y.\ |y - x| < \delta \longrightarrow g\ y > 0$
    by (*smt* (*verit*, *best*) *field-sum-of-halves*)
  then show *?thesis*
    using δ*-pos* **by** *blast*
qed

lemma *cont-at-pos-imp-loc-pos′*:
  fixes g :: *real* ⇒ *real* **and** x :: *real*
  assumes *continuous* (*at* x) g **and** $g\ x > 0$
  shows $\exists\,\Delta > 0.\ \forall\,\delta.\ 0 < \delta \wedge \delta \leq \Delta \longrightarrow (\forall\,y.\ |y - x| < \delta \longrightarrow g\ y > 0)$
proof –
  from *assms* **obtain** δ **where** δ*-pos*: $\delta > 0$ **and** H: $\forall\,y.\ |y - x| < \delta \longrightarrow g\ y > 0$
    using *cont-at-pos-imp-loc-pos* **by** *blast*
  have $\forall\,\delta' \leq \delta.\ \forall\,y.\ |y - x| < \delta' \longrightarrow g\ y > 0$
  proof *clarify*
    fix δ' y :: *real*
    assume $\delta' \leq \delta$ **and** $|y - x| < \delta'$
    thus $g\ y > 0$ **by** (*auto simp*: H)
  qed
  then show *?thesis*
    using δ*-pos* **by** *blast*
qed

1.5.2 Local Negativity

lemma *cont-at-neg-imp-loc-neg*:
  fixes g :: *real* ⇒ *real* **and** x :: *real*
  assumes *continuous* (*at* x) g **and** $g\ x < 0$
  shows $\exists\,\delta > 0.\ \forall\,y.\ |y - x| < \delta \longrightarrow g\ y < 0$
proof –
  from *assms* **obtain** δ **where** δ*-pos*: $\delta > 0$
    and $\forall\,y.\ |y - x| < \delta \longrightarrow |g\ y - g\ x| < -(g\ x)/2$
    by (*metis continuous-at-eps-delta half-gt-zero neg-0-less-iff-less*)
  then have $\forall\,y.\ |y - x| < \delta \longrightarrow -\ g\ y > 0$
    by (*smt* (*verit*, *best*) *field-sum-of-halves*)
  then show *?thesis*
    using δ*-pos neg-0-less-iff-less* **by** *blast*
qed

lemma *cont-at-neg-imp-loc-neg'*:
fixes g :: *real* ⇒ *real* **and** x :: *real*
assumes *continuous* (*at x*) g **and** $g\ x < 0$
shows $\exists\,\Delta > 0.\ \forall\,\delta.\ 0 < \delta \wedge \delta \leq \Delta \longrightarrow (\forall\,y.\ |y - x| < \delta \longrightarrow g\ y < 0)$
proof −
from *assms* **obtain** δ **where** *δ-pos*: $\delta > 0$
and H: $\forall\,y.\ |y - x| < \delta \longrightarrow -(g\ y) > 0$
by (*smt* (*verit*) *cont-at-neg-imp-loc-neg*)
have $\forall\,\delta' \leq \delta.\ \forall\,y.\ |y - x| < \delta' \longrightarrow -(g\ y) > 0$
proof *clarify*
fix δ' y :: *real*
assume $\delta' \leq \delta$ **and** $|y - x| < \delta'$
then show $-(g\ y) > 0$
using H **by** *auto*
qed
then show *?thesis*
using *δ-pos neg-0-less-iff-less* **by** *blast*
qed

end

2 Minimizers in Topological and Metric Spaces

theory *Minimizers-Definition*
imports *Auxilary-Facts*
begin

2.1 Abstract Topological Definitions

definition *global-minimizer* :: (*'a*::*topological-space* ⇒ *real*) ⇒ *'a* ⇒ *bool* **where**
global-minimizer f x-star ⟷ (∀ *x*. *f x-star* ≤ *f x*)

definition *local-minimizer-on* :: (*'a*::*topological-space* ⇒ *real*) ⇒ *'a* ⇒ *'a set* ⇒ *bool* **where**
local-minimizer-on f x-star U ⟷ (*open U* ∧ *x-star* ∈ *U* ∧ (∀ *x* ∈ *U*. *f x-star* ≤ *f x*))

definition *local-minimizer* :: (*'a*::*topological-space* ⇒ *real*) ⇒ *'a* ⇒ *bool* **where**
local-minimizer f x-star ⟷ (∃ *U*. *open U* ∧ *x-star* ∈ *U* ∧ (∀ *x* ∈ *U*. *f x-star* ≤ *f x*))

definition *isolated-local-minimizer-on* :: (*'a*::*topological-space* ⇒ *real*) ⇒ *'a* ⇒ *'a set* ⇒ *bool* **where**
isolated-local-minimizer-on f x-star U ⟷
(*local-minimizer-on f x-star U* ∧ ({*x* ∈ *U*. *local-minimizer f x*} = {*x-star*}))

definition *isolated-local-minimizer* :: (*'a*::*topological-space* ⇒ *real*) ⇒ *'a* ⇒ *bool* **where**

isolated-local-minimizer f x-star $\longleftrightarrow$
($\exists$ *U*. *local-minimizer-on f x-star U* $\wedge$ ({*x* $\in$ *U*. *local-minimizer f x*} = {*x-star*}))

definition *strict-local-minimizer-on* :: (*'a::topological-space* $\Rightarrow$ *real*) $\Rightarrow$ *'a* $\Rightarrow$ *'a set* $\Rightarrow$ *bool* **where**
strict-local-minimizer-on f x-star U $\longleftrightarrow$
(*open U* $\wedge$ *x-star* $\in$ *U* $\wedge$ ($\forall$ *x* $\in$ *U* − {*x-star*}. *f x-star* < *f x*))

definition *strict-local-minimizer* :: (*'a::topological-space* $\Rightarrow$ *real*) $\Rightarrow$ *'a* $\Rightarrow$ *bool* **where**
strict-local-minimizer f x-star $\longleftrightarrow$ ($\exists$ *U*. *strict-local-minimizer-on f x-star U*)

2.2 Metric Space Reformulations

lemma *local-minimizer-on-def2*:
fixes *f* :: *'a::metric-space* $\Rightarrow$ *real*
assumes *local-minimizer f x-star*
shows $\exists$ *N* > *0*. $\forall$ *x* $\in$ *ball x-star N*. *f x-star* $\leq$ *f x*
proof −
from *assms* **obtain** *U* **where**
open U x-star $\in$ *U* **and** *local-min*: $\forall$ *x* $\in$ *U*. *f x-star* $\leq$ *f x*
unfolding *local-minimizer-def* **by** *auto*
then obtain *N* **where** *N-pos*: *N* > *0* **and** *ball-in-U*: *ball x-star N* $\subseteq$ *U*
using *open-contains-ball* **by** *blast*
hence $\forall$ *x* $\in$ *ball x-star N*. *f x-star* $\leq$ *f x*
using *ball-in-U local-min* **by** *auto*
thus *?thesis*
using *N-pos* **by** *auto*
qed

lemma *local-minimizer-def2*:
fixes *f* :: *'a::metric-space* $\Rightarrow$ *real*
assumes *local-minimizer f x-star*
shows $\exists$ *N* > *0*. $\forall$ *x*. *dist x x-star* < *N* $\longrightarrow$ *f x-star* $\leq$ *f x*
proof −
from *assms* **obtain** *U* **where**
open U x-star $\in$ *U* **and** *local-min*: $\forall$ *x* $\in$ *U*. *f x-star* $\leq$ *f x*
unfolding *local-minimizer-def* **by** *auto*
then obtain *N* **where** *N-pos*: *N* > *0* **and** *ball-in-U*: *ball x-star N* $\subseteq$ *U*
using *open-contains-ball* **by** *blast*
hence $\forall$ *x*. *dist x x-star* < *N* $\longrightarrow$ *x* $\in$ *ball x-star N*
by (*subst mem-ball*, *simp add*: *dist-commute*)
hence $\forall$ *x*. *dist x x-star* < *N* $\longrightarrow$ *f x-star* $\leq$ *f x*
using *ball-in-U local-min* **by** *blast*
thus *?thesis*
using *N-pos* **by** *auto*
qed

lemma *isolated-local-minimizer-on-def2*:

```
  fixes f :: 'a::metric-space ⇒ real
  assumes isolated-local-minimizer-on f x-star U
  shows ∃ N > 0. ∀ x ∈ ball x-star N. (local-minimizer f x ⟶ x = x-star)
proof −
  from assms have
    local-minimizer-on f x-star U
    and unique-min: {x ∈ U. local-minimizer f x} = {x-star}
    unfolding isolated-local-minimizer-on-def by auto
  then obtain N where N-pos: N > 0 and ball-in-U: ball x-star N ⊆ U
    using open-contains-ball by (metis local-minimizer-on-def)
  have ∀ x ∈ ball x-star N. local-minimizer f x ⟶ x = x-star
  proof(clarify)
    fix x
    assume x ∈ ball x-star N
    then have x ∈ U using ball-in-U by auto
    moreover assume local-minimizer f x
    hence x ∈ {x ∈ U. local-minimizer f x} using ‹x ∈ U› by auto
    hence x ∈ {x-star} using unique-min by auto
    ultimately show x = x-star
      by simp
  qed
  thus ?thesis using N-pos by auto
qed

lemma isolated-local-minimizer-def2:
  fixes f :: 'a::metric-space ⇒ real
  assumes isolated-local-minimizer f x-star
  shows ∃ N > 0. ∀ x ∈ ball x-star N. (local-minimizer f x ⟶ x = x-star)
proof −
  from assms obtain U where
    local-minimizer-on f x-star U
    and unique-min: {x ∈ U. local-minimizer f x} = {x-star}
    unfolding isolated-local-minimizer-def by auto
  then obtain N where N-pos: N > 0 and ball-in-U: ball x-star N ⊆ U
    using open-contains-ball by (metis local-minimizer-on-def)
  have ∀ x ∈ ball x-star N. local-minimizer f x ⟶ x = x-star
  proof(clarify)
    fix x
    assume x ∈ ball x-star N
    then have x ∈ U using ball-in-U by auto
    moreover assume local-minimizer f x
    hence x ∈ {x ∈ U. local-minimizer f x} using ‹x ∈ U› by auto
    hence x ∈ {x-star} using unique-min by auto
    ultimately show x = x-star by simp
  qed
  thus ?thesis using N-pos by auto
qed

lemma strict-local-minimizer-on-def2:
```

```
  fixes f :: 'a::metric-space ⇒ real
  assumes strict-local-minimizer-on f x-star U
  shows ∃ N > 0. ∀ x ∈ ball x-star N − {x-star}. f x-star < f x
proof −
  from assms have
    open U x-star ∈ U and strict-min: ∀ x ∈ U − {x-star}. f x-star < f x
    unfolding strict-local-minimizer-on-def by auto
  then obtain N where N-pos: N > 0 and ball-in-U: ball x-star N ⊆ U
    using open-contains-ball by metis
  have ∀ x ∈ ball x-star N − {x-star}. f x-star < f x
  proof
    fix x
    assume x ∈ ball x-star N − {x-star}
    hence x ∈ U − {x-star} using ball-in-U by auto
    thus f x-star < f x
      using strict-min by auto
  qed
  thus ?thesis using N-pos by auto
qed

lemma strict-local-minimizer-def2:
  fixes f :: 'a::metric-space ⇒ real
  assumes strict-local-minimizer f x-star
  shows ∃ N > 0. ∀ x ∈ ball x-star N − {x-star}. f x-star < f x
proof −
  from assms obtain U where
    strict-local-minimizer-on f x-star U
    unfolding strict-local-minimizer-def by auto
  then have
    open U x-star ∈ U and strict-min: ∀ x ∈ U − {x-star}. f x-star < f x
    unfolding strict-local-minimizer-on-def by auto
  then obtain N where N-pos: N > 0 and ball-in-U: ball x-star N ⊆ U
    using open-contains-ball by metis
  have ∀ x ∈ ball x-star N − {x-star}. f x-star < f x
  proof
    fix x
    assume x ∈ ball x-star N − {x-star}
    hence x ∈ U − {x-star} using ball-in-U by auto
    thus f x-star < f x
      using strict-min by auto
  qed
  thus ?thesis using N-pos by auto
qed

lemma local-minimizer-neighborhood:
  fixes f :: real ⇒ real
  assumes loc-min: local-minimizer f x-min
  shows ∃ δ > 0. ∀ h. |h| < δ ⟶ f (x-min + h) ≥ f x-min
proof −
```

obtain *N* **where** *N-pos*: $N > 0$ **and** *N-prop*: $\forall x.$ *dist x x-min* $< N \longrightarrow$ *f x-min* $\leq$ *f x*
using *local-minimizer-def2*[*OF loc-min*] **by** *auto*
then have $\forall h.$ *abs h* $< N \longrightarrow f$ (*x-min* $+ h) \geq$ *f x-min*
by (*simp add*: *dist-real-def*)
then show *?thesis*
using *N-pos* **by** *blast*
qed

lemma *local-minimizer-from-neighborhood*:
fixes *f* :: *real* ⇒ *real* **and** *x-min* :: *real*
assumes $\exists \delta > 0. \forall x. |x -$ *x-min*$| < \delta \longrightarrow$ *f x-min* $\leq$ *f x*
shows *local-minimizer f x-min*
proof −
from *assms* **obtain** δ **where** δ*-pos*: $\delta > 0$ **and** *H*: $\forall x. |x -$ *x-min*$| < \delta \longrightarrow f$ *x-min* $\leq$ *f x*
by *auto*
obtain *U* **where** *U-def*: $U = \{x. |x -$ *x-min*$| < \delta\}$
by *simp*
then have *open U*
by (*smt* (*verit*) *dist-commute dist-real-def mem-Collect-eq metric-space-class.open-ball subsetI topological-space-class.openI*)
moreover have *x-min* $\in U$
using *U-def* δ*-pos* **by** *force*
moreover have $\forall x \in U.$ *f x-min* $\leq$ *f x*
using *H U-def* **by** *blast*
ultimately show *?thesis*
unfolding *local-minimizer-def* **by** *auto*
qed

end

3 Minimizer Implications

theory *First-Order-Conditions*
imports *Minimizers-Definition*
begin

notation *norm* (‖-‖)

3.1 Implications for a Given Minimizer Type

lemma *strict-local-minimizer-imp-local-minimizer*:
assumes *strict-local-minimizer f x-star*
shows *local-minimizer f x-star*
by (*smt* (*verit*) *Diff-iff assms local-minimizer-def singletonD strict-local-minimizer-def strict-local-minimizer-on-def*)

lemma *isolated-local-minimizer-imp-strict*:

assumes *isolated-local-minimizer f x-star*
shows *strict-local-minimizer f x-star*
proof –
— From isolated_local_minimizer we obtain an open set U such that $x^\star$ is the *only* local minimizer.
from *assms* **obtain** U **where** *iso-props*:
isolated-local-minimizer-on f x-star U
unfolding *isolated-local-minimizer-def*
using *isolated-local-minimizer-on-def* **by** *blast*

— Unpack `isolated_local_minimizer_on`: $x^\star$ is a `local_minimizer_on` U, and $x^\star$ is unique.

from *iso-props* **have** *lm-on*: *local-minimizer-on f x-star U*
unfolding *isolated-local-minimizer-on-def* **using** *local-minimizer-on-def* **by** *presburger*
moreover from *iso-props* **have** *unique-min*: $\{x \in U.$ *local-minimizer f x*$\} = \{$*x-star*$\}$
unfolding *isolated-local-minimizer-on-def* **by** *auto*

— From `local_minimizer_on`, we have: U open, $x^\star \in U$, and $\forall x \in U.\ f(x^\star) \leq f(x)$.

from *lm-on* **have** *open-U*: *open U* **and** *x-in-U*: *x-star* $\in U$ **and** *le-prop*: $\forall\, x \in U.$ *f x-star* $\leq$ *f x*
unfolding *local-minimizer-on-def* **by** *auto*

— Assume, for contradiction, that $x^\star$ is not a strict local minimizer. Then there exists $y \in U \setminus \{x^\star\}$ with $f(y) \leq f(x^\star)$.

show *strict-local-minimizer f x-star*
proof (*rule ccontr*)
assume $\neg$ *strict-local-minimizer f x-star*
then obtain y **where** *y-props*:
$y \in U - \{$*x-star*$\}$ **and** *f y* $\leq$ *f x-star*
unfolding *strict-local-minimizer-def strict-local-minimizer-on-def*
by (*smt* (*verit*, *ccfv-SIG*) *open-U x-in-U*)

from *y-props* **have** $y \in U$ **and** $y \neq$ *x-star*
by *auto*

— We already have $f(x^\star) \leq f(y)$ from $\forall\, x{\in}U.$ *f x-star* $\leq$ *f x* and $y \in U$. Together with $f(y) \leq f(x^\star)$, this yields $f(x^\star) = f(y)$.

from *le-prop* ‹$y \in U$› **have** *f x-star* $\leq$ *f y*
by *auto*
with ‹*f y* $\leq$ *f x-star*› **have** *f x-star* $=$ *f y*
by *auto*

— Now we show that y is also a local minimizer, contradicting the uniqueness of $x^\star$. To prove this, we must exhibit an open set V around y such that $f(y) \leq f(x)$ for all $x \in V$.

have *local-minimizer f y*
proof −
— Since U is open and $y \in U$, there exists an open set $V \subseteq U$ containing y.
obtain V **where** *open* V **and** $y \in V$ **and** $V \subseteq U$
using ‹*open* U› ‹$y \in U$› *open-subset* **by** *auto*

— On this subset, $f(y) = f(x^\star) \leq f(x)$ for all $x \in V$ (since $V \subseteq U$).

moreover from *le-prop* **and** ‹*f x-star* = *f y*› **have** $\forall x \in V.\ f\ y \leq f\ x$
using *calculation(3)* **by** *auto*

ultimately show *local-minimizer f y*
unfolding *local-minimizer-def local-minimizer-on-def* **by** *auto*
qed

— Since y is a local minimizer and $y \in U$, we have $y \in \{x \in U.$ `local_minimizer` $f\ x\}$. By uniqueness, $\{x \in U.$ `local_minimizer` $f\ x\} = \{x^\star\}$, hence $y = x^\star$, contradicting $y \neq x^\star$.

hence $y \in \{x \in U.$ *local-minimizer f x*$\}$
by (*simp add*: ‹$y \in U$›)
with *unique-min* **have** y = *x-star* **by** *auto*
thus *False* **using** ‹$y \neq$ *x-star*› **by** *contradiction*
qed
— Having reached a contradiction under the assumption that $x^\star$ is not a strict local minimizer, it follows that $x^\star$ must indeed be a strict local minimizer.
qed

3.2 Characterization of Non-Isolated Minimizers

lemma *not-isolated-minimizer-def*:
assumes *local-minimizer f x-star*
shows ($\exists$ *x-seq*::*nat* $\Rightarrow$ *real*. ($\forall n.$ *local-minimizer f* (*x-seq n*) $\wedge$ *x-seq n* $\neq$ *x-star*) $\wedge$ ((*x-seq* $\longrightarrow$ *x-star*) *at-top*)) = ($\neg$ *isolated-local-minimizer f x-star*)
proof(*safe*)
show $\bigwedge$*x-seq*. *isolated-local-minimizer f x-star* $\Longrightarrow$ $\forall n.$ *local-minimizer f* (*x-seq n*) $\wedge$ *x-seq n* $\neq$ *x-star* $\Longrightarrow$ *x-seq* $\longrightarrow$ *x-star* $\Longrightarrow$ *False*
proof −
fix *x-seq* :: *nat* $\Rightarrow$ *real*
assume *x-star-isolated-minimizer*: *isolated-local-minimizer f x-star*
assume *with-sequence-of-local-miniziers*: $\forall n.$ *local-minimizer f* (*x-seq n*) $\wedge$ *x-seq n* $\neq$ *x-star*
assume *converging-to-x-star*: *x-seq* $\longrightarrow$ *x-star*
have *open-ball-with-unique-min*: $\exists N > 0.\ \forall x \in$ *ball x-star N*. (*local-minimizer f x* $\longrightarrow$ x = *x-star*)

by (*simp add*: *isolated-local-minimizer-def2 x-star-isolated-minimizer*)
then obtain *N* **where** *N-pos*: $N > 0$ **and** *N-prop*: $\forall x \in$ *ball x-star N*. (*local-minimizer f x* $\longrightarrow$ *x* = *x-star*)
by *blast*
— Use convergence to show x_{seq} eventually lies in ball$(x^\star, N)$.
from *converging-to-x-star* **have** $\exists M. \forall n \geq M.$ *x-seq n* $\in$ *ball x-star N*
by (*metis LIMSEQ-iff-nz N-pos dist-commute mem-ball*)
then obtain *M* **where** *M-def*: $\forall n \geq M.$ *x-seq n* $\in$ *ball x-star N*
by *auto*
then show *False*
by (*meson N-prop linorder-not-le order-less-irrefl with-sequence-of-local-miniziers*)
qed
next
show ¬ *isolated-local-minimizer f x-star* $\Longrightarrow$ ∃ *x-seq*. (∀ *n*. *local-minimizer f* (*x-seq n*) ∧ *x-seq n* ≠ *x-star*) ∧ *x-seq* ⟶ *x-star*
proof(*rule ccontr*)
assume *not-isolated-minimizer*: ¬*isolated-local-minimizer f x-star*
assume *BWOC*: ∄ *x-seq*. (∀ *n*. *local-minimizer f* (*x-seq n*) ∧ *x-seq n* ≠ *x-star*) ∧ *x-seq* ⟶ *x-star*

have $\exists N > 0. \forall x.$ *dist x x-star* $< N \longrightarrow$ *f x-star* $\leq$ *f x*
by (*simp add*: *assms local-minimizer-def2*)
then obtain *N* **where** *N-pos*: (*N*::*nat*) > *0* **and** *x-star-min-on-N-ball*: ∀ *x*. *dist x x-star* < *1*/ *real N* ⟶ *f x-star* ≤ *f x*
by (*metis dual-order.strict-trans ex-inverse-of-nat-less inverse-eq-divide*)

obtain *S-n* :: *nat* ⇒ *real set* **where** *S-n-def*: *S-n* = (λ*n*. {*x*. *dist x x-star* < *1* / (*real n* + *N*) ∧ *x* ≠ *x-star* ∧ *local-minimizer f x*})
by *blast*

from *not-isolated-minimizer*
have *non-isolated*: ∀ *U*. *local-minimizer-on f x-star U* ⟶ (∃ *y* ∈ *U*. *y* ≠ *x-star* ∧ *local-minimizer f y*)
by (*smt* (*verit*, *best*) *Collect-cong assms isolated-local-minimizer-def local-minimizer-on-def singleton-conv2*)

have ∀ *n*::*nat*. ∃ *x*. *x* ∈ *S-n n*
proof (*intro allI*)
fix *n*::*nat*
have *pos-radius*: *1* / (*real n* + *N*) > *0*
using *N-pos* **by** *simp*

obtain *U* **where** *U-def*: *U* = *ball x-star* (*1* / (*real n* + *N*)) **and** *open-U*: *open U* **and** *U-contains-x-star*: *x-star* ∈ *U*
using *pos-radius* **by** *auto*

have *U-contained-in-Inverse-N-Ball*: ∀ *x* ∈ *U*. *dist x x-star* < *1* / *N*
proof(*safe*)
fix *x*:: *real*

```
        assume x-in-U: x ∈ U
        then have dist x x-star < (1 / (real n + N))
          by (simp add: U-def dist-commute)
        also have ... ≤ 1 / real N
          by (simp add: N-pos frac-le)
        finally show dist x x-star < 1 / real N.
      qed

      have ball-non-empty: ∃ y ∈ U. y ≠ x-star ∧ local-minimizer f y
      proof −
        have local-minimizer-on f x-star U
       by (simp add: U-contains-x-star U-contained-in-Inverse-N-Ball local-minimizer-on-def
open-U x-star-min-on-N-ball)
        then show ∃ y∈U. y ≠ x-star ∧ local-minimizer f y
          by (simp add: non-isolated)
      qed
      then obtain y where y-in-ball: y ∈ U and y ≠ x-star and local-minimizer
f y
        by blast
      then show ∃ x. x ∈ S-n n
        by (smt (verit, best) S-n-def U-def dist-commute mem-Collect-eq mem-ball)
    qed
    then obtain x-seq where x-seq-def: ∀ n. x-seq n ∈ S-n n
      by metis
    have x-seq-converges-to-x-star: x-seq ⟶ x-star
    proof (rule LIMSEQ-I)
      fix r :: real
      assume r-pos: 0 < r
      obtain n-min where n-min-def: 1 / (real n-min + N) < r
        using real-arch-inverse N-pos r-pos
       by (smt (verit, ccfv-SIG) frac-le inverse-eq-divide inverse-positive-iff-positive)
      show ∃ no. ∀ n≥no. norm (x-seq n − x-star) < r
      proof (intro exI allI impI)
        fix n
        assume n ≥ n-min
        then have n-large-enough: 1 / (real n + N) ≤ 1 / (real n-min + N)
          using N-pos by (subst frac-le, simp-all)
        have dist (x-seq n) x-star < 1 / (real n + N)
          using x-seq-def S-n-def by auto
        also have ... ≤ 1 / (real n-min + N)
          using n-large-enough by auto
        also have ... < r
          using n-min-def by auto
        finally show norm (x-seq n − x-star) < r
          by (simp add: dist-real-def)
      qed
    qed
     have ∃ x-seq. (∀ n. local-minimizer f (x-seq n) ∧ x-seq n ≠ x-star) ∧ x-seq
⟶ x-star
```

using *S-n-def x-seq-converges-to-x-star x-seq-def* **by** *blast*
then show *False*
using *BWOC* **by** *auto*
qed
qed

3.3 First-Order Condition

theorem *Fermat's-theorem-on-stationary-points*:
fixes *f* :: *real* ⇒ *real*
assumes (*f has-derivative f'*) (*at x-min*)
assumes *local-minimizer f x-min*
shows (*deriv f*) *x-min* = *0*
by (*metis assms has-derivative-imp differential-zero-maxmin local-minimizer-def*)

definition *stand-basis-vector* :: *'n::finite* ⇒ *real^'n* — the i-th standard basis vector
where *stand-basis-vector i* = (χ *j*. *if j* = *i then 1 else 0*)

lemma *stand-basis-vector-index*[*simp*]: (*stand-basis-vector i*) $ *j* = (*if j* = *i then* (*1*::*real*) *else 0*)
by (*simp add*: *stand-basis-vector-def*)

lemma *stand-basis-vector-nonzero*[*simp*]: *stand-basis-vector i* ≠ *0*
by (*smt* (*verit*, *del-insts*) *stand-basis-vector-index zero-index*)

lemma *norm-stand-basis-vector*[*simp*]: *norm* (*stand-basis-vector i*) = *1*
by (*smt* (*verit*, *best*) *axis-nth component-le-norm-cart norm-axis-1 norm-le-componentwise-cart real-norm-def stand-basis-vector-index*)

lemma *inner-stand-basis-vector*[*simp*]: *inner* (*stand-basis-vector i*) (*stand-basis-vector j*) = (*if i* = *j then 1 else 0*)
by (*metis axis-nth cart-eq-inner-axis norm-eq-1 norm-stand-basis-vector stand-basis-vector-index vector-eq*)

lemma *Basis-characterisation*:
stand-basis-vector i ∈ (*Basis* :: (*real^'n*) *set*) **and**
∀ *b*∈(*Basis*::(*real^'n*)*set*). ∃ *i*. *b* = *stand-basis-vector i*
by (*metis* (*no-types*, *lifting*) *Basis-real-def axis-in-Basis-iff cart-eq-inner-axis inner-stand-basis-vector insert-iff norm-axis-1 norm-eq-1 stand-basis-vector-index vector-eq*,
metis axis-index axis-nth cart-eq-inner-axis inner-stand-basis-vector stand-basis-vector-index vector-eq)

lemma *stand-basis-expansion*:
fixes *x* :: *real^'n*
shows *x* = (∑ *j*∈*UNIV*. (*x* $ *j*) $*_R$ *stand-basis-vector j*)
proof −
have (∑ *j*∈*UNIV*. (*x* $ *j*) $*_R$ *stand-basis-vector j*) $ *k* = *x* $ *k* **for** *k*

```
  proof -
    have (∑ j∈UNIV. (x $ j) *R stand-basis-vector j) $ k
        = (∑ j∈UNIV. (x $ j) * (stand-basis-vector j $ k))
      by simp
    also have ... = (∑ j∈UNIV. (x $ j) * (if j = k then 1 else 0))
      by (smt (verit, best) stand-basis-vector-index sum.cong)
    also have ... = (∑ j∈UNIV. (if j = k then x $ j else 0))
      by (smt (verit, best) mult-cancel-left1 mult-cancel-right1 sum.cong)
    also have ... = x $ k
      by (subst sum.delta, simp-all)
    finally show ?thesis.
  qed
  thus ?thesis
    by (simp add: vec-eq-iff)
qed

lemma has-derivative-affine:
  fixes a v :: 'a::real-normed-vector
  shows ((λt. a + t *R v) has-derivative (λh. h *R v)) (at x)
  unfolding has-derivative-def
proof safe
  have a + y *R v − (a + netlimit (at x) *R v) − (y − netlimit (at x)) *R v = 0
if y ≠ netlimit (at x) for y
    by (simp add: cross3-simps(32))
  then show (λy. (a + y *R v − (a + netlimit (at x) *R v) − (y − netlimit (at
x)) *R v) /R ‖y − netlimit (at x)‖) −x→ 0
    by (simp add: scaleR-left-diff-distrib)
  show bounded-linear (λh. h *R v)
    by (simp add: bounded-linearI' vector-space-assms(2))
qed

theorem Fermat's-theorem-on-stationary-points-mult:
  fixes f :: real ^ 'n ⇒ real
  assumes der-f: (f has-derivative f') (at x-min)
  assumes min-f: local-minimizer f x-min
  shows GDERIV f x-min :> 0
proof -
 — Show that f' kills every standard-basis vector.

  {
    fix i :: 'n
    — Define the 1D slice g_i(t) = f(x_min + t · e_i).
    let ?g = λt::real. f (x-min + t *R stand-basis-vector i)

    — Chain rule gives g'_i(0) = f'(e_i).
    from has-derivative-affine have g-der:
      ((λt. f (x-min + t *R stand-basis-vector i))
      has-derivative (λh. f' (h *R stand-basis-vector i))) (at 0)
     by (metis (no-types) arithmetic-simps(50) der-f has-derivative-compose scaleR-simps(1))
```

— 0 is a local minimizer of g_i because $x_{\min}$ is one for f.
have *g-min*: *local-minimizer ?g 0*
proof(*rule local-minimizer-from-neighborhood*)
obtain δ **where** *δ-pos*: δ > *0*
and *mono*: $\bigwedge x$. *dist x-min x* < δ $\Longrightarrow$ *f x* $\geq$ *f x-min*
by (*metis assms*(*2*) *dist-commute local-minimizer-def2*)

have $\forall x$. |*x* − *0*| < δ $\longrightarrow$ *f* (*x-min* + *0* $*_R$ *stand-basis-vector i*) $\leq$ *f* (*x-min* + *x* $*_R$ *stand-basis-vector i*)
using *mono* **by** (*simp add*: *dist-norm*)
then show $\exists \delta$>*0*. $\forall x$. |*x* − *0*| < δ $\longrightarrow$ *f* (*x-min* + *0* $*_R$ *stand-basis-vector i*) $\leq$ *f* (*x-min* + *x* $*_R$ *stand-basis-vector i*)
using *δ-pos* **by** *blast*
qed

— Apply the 1-D Fermat lemma to g_i.
from *Fermat's-theorem-on-stationary-points*
have *f'* (*stand-basis-vector i*) = *0*
using *g-der g-min* **by** (*metis has-derivative-imp scale-one*)
}

— Collecting the result for every i:
hence *zero-on-basis*: $\bigwedge i$. *f'* (*stand-basis-vector i*) = *0***.**

— Use linearity and the coordinate expansion to show $f' = 0$ everywhere.
{
fix *v* :: *real*^*'n*
— Expand $v = \sum_j v_j \cdot e_j$ and push f' through the finite sum.
have *f' v* = *0*
proof −
have *f' v* = *f'* ($\sum$*j*$\in$*UNIV*. (*v* $ *j*) $*_R$ *stand-basis-vector j*)
by (*metis stand-basis-expansion*)
also have ... = ($\sum$*j*$\in$*UNIV*. (*v* $ *j*) $*_R$ *f'* (*stand-basis-vector j*))
by (*smt* (*verit*) *assms differential-zero-maxmin local-minimizer-def scale-eq-0-iff sum.neutral*)
also have ... = *0*
using *zero-on-basis* **by** *simp*
finally show *?thesis***.**
qed
}
hence *f'-zero*: *f'* = (λ-. *0*)
by (*simp add*: *fun-eq-iff*)

— Translate $f' = 0$ into the gradient statement.
have (*f has-derivative* (λh. *0*)) (*at x-min*)
using *der-f f'-zero* **by** *simp*
hence *GDERIV f x-min* :> (*0*::*real*^*'n*)
by (*simp add*: *gderiv-def*)

  thus *?thesis*.
qed

end

4 Second-Order Conditions

theory *Second-Derivative-Test*
  imports *First-Order-Conditions*
begin

4.1 Necessary Condition

lemma *snd-derivative-nonneg-at-local-min-necessary*:
  fixes *f* :: *real* ⇒ *real*
  assumes *C2-cont-diff-at-xmin*: *C-k-on 2 f* (*U* :: *real set*)
  assumes *min-in-U*: (*x-min* :: *real*) ∈ *U*
  assumes *loc-min*: *local-minimizer f x-min*
  shows *deriv* (*deriv f*) *x-min* ≥ *0*
proof −
  have (∃ *ε*. *0* < *ε* ∧ {*x-min* − *ε* .. *x-min* + *ε*} ⊂ *U*)
  proof −
    have (∃ *ε*. *0* < *ε* ∧ *ball x-min ε* ⊂ *U*)
    by (*smt C2-cont-diff-at-xmin C-k-on-def assms*(*2*) *ball-subset-cball cball-eq-ball-iff*

        open-contains-cball-eq order-le-less-trans psubsetI)
    then show *?thesis*
        by (*metis Elementary-Metric-Spaces.open-ball cball-eq-atLeastAtMost centre-in-ball*
        open-contains-cball order-trans-rules(*21*))
  qed
  then obtain *ε* **where** *ε-pos*: *0* < *ε* **and** *ε-def*: {*x-min* − *ε* .. *x-min* + *ε*} ⊂ *U*
    by *blast*
  have *f-diff*: (∀ *y* ∈ *U*. (*f has-real-derivative* (*deriv f*) *y*) (*at y*))
    using *C2-cont-diff C2-cont-diff-at-xmin* **by** *blast*
  have *f′-diff*: (∀ *y* ∈ *U*. (*deriv f has-real-derivative* (*deriv* (*deriv f*)) *y*) (*at y*))
    using *C2-cont-diff C2-cont-diff-at-xmin* **by** *blast*
  have *f′′-contin*: *continuous-on U* (*deriv* (*deriv f*))
    using *C2-cont-diff assms*(*1*) **by** *blast*

  have *f′-0*: (*deriv f*) *x-min* = *0*
    using *Fermat′s-theorem-on-stationary-points*
    by (*meson assms*(*2*,*3*) *f-diff has-field-derivative-imp-has-derivative*)

 — By local minimality at $x_{\min}$, there is a $\delta > 0$ such that for all h with $|h| < \delta$, we have $f(x_{\min} + h) \geq f(x_{\min})$.
  obtain *δ* **where** *δ-pos*: *δ* > *0* **and** *δ-prop*: ∀ *h*. |*h*| < *δ* ⟶ *f* (*x-min* + *h*) ≥ *f x-min*
    by (*meson assms*(*3*) *local-minimizer-neighborhood*)

from f'-0 **have** *second-deriv-limit-at-x-min*:
$((\lambda h.\ (deriv\ f\ (x\text{-}min + h))\ /\ h) \longrightarrow deriv\ (deriv\ f)\ x\text{-}min)\ (at\ 0)$
by (*smt* (*verit*, *best*) *DERIV-def Lim-cong-within assms*(*2*) f'-*diff*)

show *?thesis*
proof(*rule ccontr*)
assume $\neg\ 0 \leq deriv\ (deriv\ f)\ x\text{-}min$
then have *BWOC*: $0 > deriv\ (deriv\ f)\ x\text{-}min$
by *auto*
then obtain Δ **where** Δ-*pos*: $\Delta > 0$ **and**
Δ-*def*: $\forall\delta.\ 0 < \delta \wedge \delta \leq \Delta \longrightarrow (\forall y.\ |y - x\text{-}min| < \delta \longrightarrow deriv\ (deriv\ f)\ y < 0)$
by (*metis C2-cont-diff-at-xmin C-k-on-def min-in-U at-within-open cont-at-neg-imp-loc-neg'*
continuous-on-eq-continuous-within f''-*contin*)

— Choose h with $0 < h < \min\{\delta, \Delta\}$ so that $x_{\min} + h \in U$.
obtain h **where** *h-def*: $h = min\ \varepsilon\ (min\ (\delta/2)\ \Delta)$ **and** *h-pos*: $0 < h$
using ε-*pos* δ-*pos* Δ-*pos* **by** *fastforce*
have *h-lt*: $h \leq \varepsilon \wedge h < \delta \wedge h \leq \Delta$
using δ-*pos h-def* **by** *linarith*
have *neigh-in-U*: $x\text{-}min + h \in \{x\text{-}min - \varepsilon\ ..\ x\text{-}min + \varepsilon\}$
using *h-def h-pos* **by** *fastforce*

have $f\ (x\text{-}min + h) < f\ x\text{-}min$
proof(*rule DERIV-neg-imp-decreasing-open*[**where** $a = x\text{-}min$ **and** $f = f$ **and** $b = x\text{-}min + h$])
show $x\text{-}min < x\text{-}min + h$
using *h-pos* **by** *simp*
next
have $\{x\text{-}min..x\text{-}min + h\} \subset U$
using ε-*def dual-order.strict-trans2 neigh-in-U* **by** *auto*
then show *continuous-on* $\{x\text{-}min..x\text{-}min + h\}\ f$
by (*meson C2-cont-diff C2-cont-diff-at-xmin continuous-on-subset*
differentiable-imp-continuous-on le-less)
next
show $\bigwedge x.\ [\![x\text{-}min < x;\ x < x\text{-}min + h]\!] \Longrightarrow \exists y.\ (f\ has\text{-}real\text{-}derivative\ y)\ (at\ x) \wedge y < 0$
proof −
fix $x :: real$
assume *x-min-lt-x*: $x\text{-}min < x$
assume *x-lt-xmin-pls-h*: $x < x\text{-}min + h$

have *xmin-x-subset*: $\{x\text{-}min\ ..\ x\} \subseteq \{x\text{-}min - \varepsilon\ ..\ x\text{-}min + \varepsilon\}$
using *neigh-in-U x-lt-xmin-pls-h* **by** *auto*

— By the Mean Value Theorem applied to f' on $[x_{\min}, x]$, there exists some c with $x_{\min} < c < x$ such that:
have $\exists z > x\text{-}min.\ z < x \wedge deriv\ f\ (x) - deriv\ f\ x\text{-}min = (x - x\text{-}min) *$

```
deriv(deriv f) z
        proof(rule MVT2)
          show x-min < x
            using x-min-lt-x by auto
        next
          fix y :: real
          assume x-min-leq-y: x-min ≤ y
          assume y-leq-x: y ≤ x

          from xmin-x-subset have y ∈ U
            using ε-def atLeastAtMost-iff x-min-leq-y y-leq-x by blast
          then show (deriv f has-real-derivative deriv (deriv f) y) (at y)
            using f′-diff by blast
        qed
        then obtain z where z-gt-x-min: z > x-min and
                            z-lt-x: z < x and
                             z-def: deriv f (x) −  deriv f x-min = (x − x-min) ∗ deriv
(deriv f) z
          by blast
        then have mvt-f′: deriv f (x)  = (x − x-min) ∗ deriv (deriv f) z
          by (simp add: f′-0)

        then have x-diff-xmin-pos: x − x-min > 0
           using ‹x-min < x› by simp
        then have left-bound-satisfied: |z − x-min| < x − x-min
           using ‹x-min < z› ‹z < x› by auto
        then have x − x-min < h
           using ‹x < x-min + h› by simp
        then  have  |z − x-min| < h
          using left-bound-satisfied by fastforce
        then have deriv (deriv f) z < 0
           using Δ-def h-lt h-pos by blast
        then  have deriv f x < 0
          by (metis x-diff-xmin-pos mvt-f′ mult-pos-neg)
         moreover have x ∈ U
           using xmin-x-subset
           by (meson ε-def atLeastAtMost-iff dual-order.strict-iff-not
               subset-eq verit-comp-simplify(2) x-min-lt-x)
         ultimately show ∃ y. (f has-real-derivative y) (at x) ∧ y < 0
           using f-diff by blast
       qed
     qed
    then show False
      by (smt (verit, best) δ-prop h-lt h-pos)
  qed
qed
```

4.2 Sufficient Condition

```
lemma second-derivative-test:
  fixes f :: real ⇒ real and a :: real and b :: real and x-min :: real
  assumes valid-interval: a < b
  assumes twice-continuously-differentiable: C-k-on 2 f {a <..< b}
  assumes min-exists: x-min ∈ {a <..< b}
  assumes fst-deriv-req: (deriv f) x-min = 0
  assumes snd-deriv-req: deriv (deriv f) x-min > 0
  shows loc-min: local-minimizer f x-min
proof −
  from twice-continuously-differentiable
  have f''-cont: continuous-on {a <..< b} (deriv (deriv f))
    by (metis C-k-on-def Suc-1 lessI nat.simps(2) second-derivative-alt-def)
  then obtain Δ where Δ-pos: Δ > 0
    and Δ-prop: ∀δ. 0 < δ ∧ δ ≤ Δ ⟶ (∀y. |y − x-min| < δ ⟶ deriv (deriv f)
y > 0)
   by (metis assms(3,5) at-within-open cont-at-pos-imp-loc-pos' continuous-on-eq-continuous-within

      open-real-greaterThanLessThan)

  obtain δ where δ-min: δ = min Δ (min ((x-min − a) / 2) ((b − x-min) / 2))
    by blast

  have δ-pos: δ > 0
  proof (cases δ = Δ)
    show δ = Δ ⟹ 0 < δ
      by (simp add: Δ-pos)
  next
    assume δ ≠ Δ
    then have δ = min ((x-min − a) / 2) ((b − x-min) / 2)
      using δ-min by linarith
    then show 0 < δ
      using min-exists by force
  qed

  have neigh-of-x-min-contained-in-ab: a < x-min − δ ∧ x-min + δ < b
    by (smt (z3) δ-min δ-pos field-sum-of-halves)

  have local-min: ∀x. |x − x-min| < δ ⟶ f x ≥ f x-min
  proof clarify
    fix x
    assume A: |x − x-min| < δ
    consider (eq) x = x-min | (lt) x < x-min | (gt) x > x-min
      by linarith
    then show f x ≥ f x-min
    proof cases
      case eq
      then show ?thesis
        by simp
```

```
  next
    case lt
    have a-lt-x-and-xmin-lt-b: a < x ∧ x-min < b
      using A neigh-of-x-min-contained-in-ab by linarith
    have f x > f x-min
    proof (rule DERIV-neg-imp-decreasing-open[where a = x])
      show x < x-min
        by (simp add: lt)
    next
      fix y :: real
      assume x-lt-y: x < y
      assume y-lt-x-min: y < x-min
      — For x < x_min, apply the Mean Value Theorem to f on [x, x_min].
      have ∃ z > y. z < x-min ∧ deriv f x-min − deriv f y = (x-min − y) ∗ deriv
(deriv f) z
        proof (rule MVT2[where a = y and b = x-min and f = deriv f and f′
= deriv (deriv f)])
        show y < x-min
          by (simp add: y-lt-x-min)
      next
        fix z :: real
        assume y-lt-z: y ≤ z
        assume z-lt-x-min: z ≤ x-min
        show (deriv f has-real-derivative (deriv (deriv f)) z) (at z)
        proof (subst C2-cont-diff[where f = f, where U = {a <..< b}])
          show C-k-on 2 f {a<..<b}
            by (simp add: assms(2))
          show z ∈ {a<..<b} and True
            using a-lt-x-and-xmin-lt-b x-lt-y y-lt-z z-lt-x-min by auto
        qed
      qed
      then obtain z where
        z-props: y < z z < x-min and
        eq: deriv f x-min − deriv f y = (x-min − y) ∗ deriv (deriv f) z
        by blast
      have deriv f x-min = 0
        using fst-deriv-req by simp
      hence deriv f y = − (x-min − y) ∗ deriv (deriv f) z
        using eq by linarith
      moreover have x-min − x > 0
        using lt by simp
      have deriv (deriv f) z > 0
        by (smt (verit) A Δ-prop δ-min x-lt-y z-props)
      ultimately have deriv f y < 0
        by (simp add: mult-less-0-iff y-lt-x-min)
      then show ∃ z. (f has-real-derivative z) (at y) ∧ z < 0
      by (meson C2-cont-diff a-lt-x-and-xmin-lt-b assms(2) dual-order.strict-trans
              greaterThanLessThan-iff x-lt-y y-lt-x-min)
    next
```

```
      have continuous-on {a <..< b} f
        by (simp add: C2-cont-diff assms(2) differentiable-imp-continuous-on)
      then show continuous-on {x..x-min} f
        by (smt (verit, del-insts) a-lt-x-and-xmin-lt-b atLeastAtMost-iff
              continuous-on-subset greaterThanLessThan-iff subsetI)
    qed
    then show f x-min ≤ f x
      by simp
  next
    case gt
    have a-lt-xmin-and-x-lt-b: a < x-min ∧ x < b
      using A ‹a < x-min − δ ∧ x-min + δ < b› by linarith
    have f x > f x-min
    proof (rule DERIV-pos-imp-increasing-open[where a = x-min])
      show x-min < x
        by (simp add: gt)
    next
      fix y :: real
      assume y-gt-xmin: x-min < y
      assume y-lt-x: y < x
```
— For $x_{\min} < y$, apply the Mean Value Theorem to f' on $[x_{\min}, y]$.
```
      have ∃ z > x-min. z < y ∧ deriv f y − deriv f x-min = (y − x-min) ∗ deriv
(deriv f) z
      proof (rule MVT2[where a = x-min and b = y and f = deriv f and f′
= deriv (deriv f)])
        show x-min < y
          by (simp add: y-gt-xmin)
      next
        fix z :: real
        assume z-ge-xmin: x-min ≤ z
        assume z-le-y: z ≤ y
        show (deriv f has-real-derivative (deriv (deriv f)) z) (at z)
        proof (subst C2-cont-diff[where f = f and U = {a<..<b}])
          show C-k-on 2 f {a<..<b}
            by (simp add: assms(2))
          show z ∈ {a<..<b} and True
            using a-lt-xmin-and-x-lt-b y-lt-x z-ge-xmin z-le-y by auto
        qed
      qed
      then obtain z where
        z-props: x-min < z z < y
        and eq: deriv f y − deriv f x-min = (y − x-min) ∗ deriv (deriv f) z
        by blast
      have deriv f x-min = 0
        using fst-deriv-req by simp
      hence deriv f y = (y − x-min) ∗ deriv (deriv f) z
        using eq by simp
      moreover have y − x-min > 0
        using y-gt-xmin by simp
```

```
      moreover have deriv (deriv f) z > 0
        by (smt (verit, best) A Δ-prop δ-min y-lt-x z-props(1,2))
      ultimately have deriv f y > 0
        by auto
      then show ∃ d. (f has-real-derivative d) (at y) ∧ d > 0
      by (meson C2-cont-diff a-lt-xmin-and-x-lt-b assms(2) dual-order.strict-trans
            greaterThanLessThan-iff y-lt-x y-gt-xmin)
    next
      have continuous-on {a <..< b} f
        by (simp add: C2-cont-diff assms(2) differentiable-imp-continuous-on)
      then show continuous-on {x-min..x} f
        by (smt (verit, del-insts) a-lt-xmin-and-x-lt-b atLeastAtMost-iff
            continuous-on-subset greaterThanLessThan-iff subsetI)
    qed
    then show ?thesis
      by simp
   qed
  qed
  show ?thesis
   by (rule local-minimizer-from-neighborhood, smt δ-pos local-min)
qed

end
```

5 Pathological Example: Non-Isolated Strict Local Minima

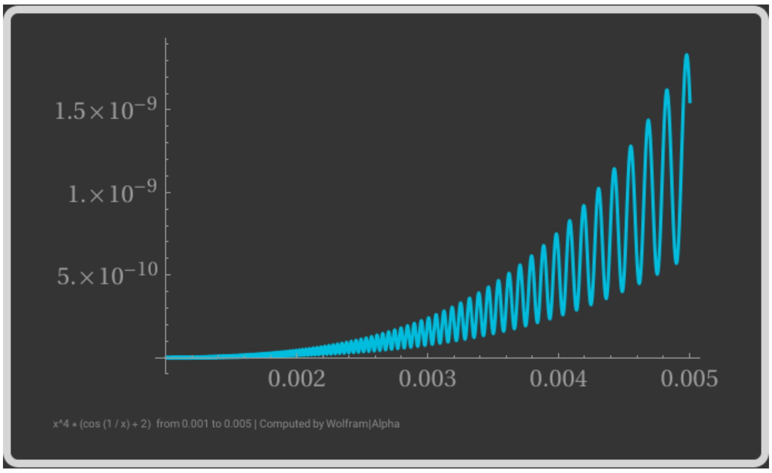
```
theory Cont-Nonisolated-Strict-Local-Minimizer-Exists
  imports Second-Derivative-Test HOL−Library.Quadratic-Discriminant
begin
```

Idea of the example. We construct a continuous function

$$f(x) = \begin{cases} x^4\big(\cos(1/x) + 2\big), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

whose oscillations *speed up* as $x \to 0$ because of the $\cos(1/x)$ term. Multiplying by x^4 makes the function and its first derivative vanish at the origin, ensuring that $x = 0$ is a strict local minimizer, while the shifted cosine creates infinitely many additional strict local minimizers that accumulate at 0. Hence the minimizer at 0 is *strict* but *not isolated*.

```
theorem Exists-Continuous-Func-with-non-isolated-strict-local-minimizer:
  ∃ f::real ⇒ real. continuous-on ℝ f ∧
    (∃ x-star. strict-local-minimizer f x-star ∧ ¬ isolated-local-minimizer f x-star)
proof −
```



obtain *f* **where** *f-def*: *f* = (λ(*x*::*real*). *if x* ≠ *0 then x*^*4* ∗ (*cos* (*1* / *x*) + *2*) *else 0*)
  by *simp*

have *deriv-f*: ⋀*x*::*real*. *deriv f x* = (*if x* = *0 then 0 else* x^2 ∗ *sin* (*1* / *x*) +
    4 ∗ *x*^*3* ∗ *cos* (*1* / *x*) + *8* ∗ *x*^*3*)
  ∧ (λ*x*. *f x*) *differentiable-on UNIV*
  ∧ *deriv* (*deriv f*) *x* = (*if x* = *0 then 0 else 6*∗*x* ∗ *sin* (*1* / *x*) +
    (*12*∗x^2 − *1*)∗ *cos* (*1* / *x*) + *24*∗x^2)
  ∧ (*deriv f*) *differentiable-on UNIV*
proof (*safe*)
  — First we compute the derivative away from 0, then we compute it at 0.
  have *deriv-f-at-nonzero*:
   ⋀*x*. *x* ≠ *0* ⟶ *deriv f x* = (x^2 ∗ *sin* (*1* / *x*) + *4*∗*x*^*3* ∗ *cos* (*1* / *x*) + *8*∗*x*^*3*)
     ∧ *f field-differentiable at x*
  proof (*safe*)
    fix *x* :: *real*
    assume *x-type*: *x* ≠ *0*

    have *cos-inverse-diff*: (λ*w*. *cos* (*1* / *w*)) *field-differentiable at x*
    proof −
      have *f1*: (λ*w*. *1* / *w*) *field-differentiable at x*
        by (*simp add*: *field-differentiable-divide x-type*)
      have (λ*z*. *cos z*) *field-differentiable at* (*1* / *x*)
        by (*simp add*: *field-differentiable-within-cos*)
      then show *?thesis*
        by (*metis DERIV-chain2 f1 field-differentiable-def*)
    qed
    then have (λ*x*. *cos* (*1* / *x*) + *2*) *field-differentiable at x*
      by (*simp add*: *Derivative.field-differentiable-add*)
    then have *f2*: (λ*x*. *x*^*4* ∗ (*cos* (*1* / *x*) + *2*)) *field-differentiable at x*
        by (*subst field-differentiable-mult*, *simp add*: *field-differentiable-power*, *simp-all*)

```
      have deriv-2nd-part: deriv (λw. (λx. cos (1 / x) + 2) w) x = (sin (1 / x))
/ x^2
      proof −
        have deriv (λw. (λx. cos (1 / x) + 2) w) x =
              (deriv (λw. (λx. cos (1 / x)) w) x + deriv (λw. (λx. 2) w) x)
          by (rule deriv-add, simp add: cos-inverse-diff, simp)
        also have ... = (sin (1 / x)) / x^2
        proof −
          have f1: DERIV (λz. cos z) (1 / x) :> −sin (1 / x)
            by simp
          have f2: DERIV (λw. 1 / w) x :> −1 / x^2
            using DERIV-inverse-func x-type by blast
          from f1 f2 have DERIV ((λz. cos z) ∘ (λw. 1 / w)) x :> (−sin (1 / x))
∗ (−1 / x^2)
              by (rule DERIV-chain)
          then show ?thesis
            by (simp add: DERIV-imp-deriv o-def)
        qed
        finally show ?thesis.
      qed

      show deriv f x = x^2 ∗ sin (1 / x) + 4∗x^3 ∗ cos (1 / x) + 8∗x^3
      proof −
        have deriv f x = deriv (λx. x^4 ∗ (cos (1 / x) + 2)) x
          by (metis (no-types, lifting) f-def mult-eq-0-iff power-zero-numeral)
        also have ... = x^4 ∗ deriv (λx. cos (1 / x) + 2) x +
                        deriv (λx. x^4) x ∗ (cos (1 / x) + 2)
          by (rule deriv-mult, simp add: field-differentiable-power,
              simp add: Derivative.field-differentiable-add cos-inverse-diff)
        also have ... = x^4 ∗ (sin (1 / x)) / x^2 +
                        deriv (λx. x^4) x ∗ (cos (1 / x) + 2)
          by (simp add: deriv-2nd-part)
        also have ... = x^4 ∗ (sin (1 / x)) / x^2 + (4∗x^3) ∗ (cos (1 / x) + 2)
          by (subst power-rule, simp)
        also have ... = x^2 ∗ (sin (1 / x)) + (4∗x^3) ∗ (cos (1 / x) + 2)
          by (simp add: power2-eq-square power4-eq-xxxx)
        also have ... = x^2 ∗ sin (1 / x) + 4∗x^3 ∗ cos (1 / x) + 8∗x^3
          by (simp add: Rings.ring-distribs(2) mult.commute)
        finally show ?thesis.
      qed
      from x-type f-def f2 show f field-differentiable at x
        by (subst field-differentiable-transfer-on-ball[where f = λ x. (x^4 ∗ (cos (1
/ x) + 2))
            and ε = |x|], simp-all)
    qed

    have deriv-f-at-0: deriv f 0 = 0 ∧ f field-differentiable at 0
    proof −
```

— By the definition of deriv, we need to show the limit of the difference quotient is 0.

```
  have dq-limit: ((λh. (f (0 + h) − f 0) / h) ⟶ 0) (at 0)
  proof
    fix ε :: real
    assume ε-pos: 0 < ε
    — Choose δ > 0 to make |difference quotient| < ε.
    obtain δ where δ-def: δ = (ε / 3) powr (1 / 3)
      by simp
    — A reasonable δ based on the growth of |h^3|.
    have δ-pos: δ > 0
      using ε-pos by (simp add: δ-def)
    have ∃δ>0. ∀h. 0 < |h| ∧ |h| < δ ⟶ |(f (0 + h) − f 0) / h − 0| < ε
    proof (intro exI[where x=δ], intro conjI insert δ-pos, clarify)
      fix h :: real
      assume h-pos: 0 < |h|
      assume h-lt-δ: |h| < δ

      have |(f (0 + h) − f 0) / h − 0| = |f h / h|
        by (simp add: f-def)
      also have ... = |h^4 ∗ (cos (1 / h) + 2) / h|
        using f-def by presburger
      also have ... = |h^3 ∗ (cos (1 / h) + 2)|
     by (simp add: power3-eq-cube power4-eq-xxxx vector-space-over-itself.scale-scale)
      also have ... ≤ |h^3| ∗ |cos (1 / h) + 2|
        by (metis abs-mult order.refl)
      also have ... ≤ |h^3| ∗ (|cos (1 / h)| + |2|)
        by (simp add: mult-left-mono)
      also have ... ≤ |h^3| ∗ (1 + 2)
        by (simp add: mult-left-mono)
      also have ... = 3 ∗ |h^3|
        by simp
      also have ... < 3 ∗ δ^3
        using power-strict-mono[of |h| δ 3] by (simp add: h-lt-δ power-abs)
      also have ... = 3 ∗ (ε / 3)
        by (metis δ-def ε-pos div-self less-le more-arith-simps(5)
                mult-eq-0-iff pos-le-divide-eq powr-numeral powr-one-gt-zero-iff
                        powr-powr times-divide-eq-left verit-comp-simplify(19)
zero-neq-numeral)
      also have ... = ε
        by simp
      finally show |(f (0 + h) − f 0) / h − 0| < ε.
    qed
    then show ∃d>0.∀x∈UNIV. 0 < dist x 0 ∧ dist x 0 < d ⟶ dist ((f (0
+ x) − f 0) / x) 0 ≤ ε
      by (metis arithmetic-simps(57) dist-real-def less-le)
  qed
  then show ?thesis
    using DERIV-def DERIV-imp-deriv field-differentiable-def by blast
```

qed

show *deriv-f*: $\bigwedge x.$ *deriv f x* =
($if\ x = 0\ then\ 0\ else\ x^2 * sin\ (1\ /\ x) + 4 * x\hat{}3 * cos\ (1\ /\ x) + 8 * x\hat{}3$)
using *deriv-f-at-0 deriv-f-at-nonzero* **by** *presburger*

show *f-is-differentiable*: ($\lambda x.$ *f x*) *differentiable-on UNIV*
by (*metis deriv-f-at-0 deriv-f-at-nonzero differentiable-on-def*
field-differentiable-imp-differentiable)

have *snd-deriv-f-at-nonzero*:
$\bigwedge x.\ x \neq 0 \longrightarrow deriv\ (deriv\ f)\ x = (6{*}x * sin\ (1\ /\ x) + (12{*}x^2 - 1){*}\ cos\ (1\ /\ x) + 24{*}x^2)$
$\wedge$ (*deriv f*) *field-differentiable at x*
proof (*safe*)
fix *x* :: *real*
assume *x-type*: $x \neq 0$

have *fst-term-diff*: ($\lambda w.\ w^2 * sin\ (1\ /\ w)$) *field-differentiable at x*
proof −
have *f1*: ($\lambda w.\ w^2$) *field-differentiable at x*
by (*simp add*: *field-differentiable-power*)
have ($\lambda w.$ *sin* (*1* / *w*)) *field-differentiable at x*
by (*metis DERIV-chain2 DERIV-inverse-func field-differentiable-at-sin*
field-differentiable-def x-type)
then show *?thesis*
by (*simp add*: *f1 field-differentiable-mult*)
qed

have *fst-term-deriv*: $deriv\ (\lambda w.\ w\hat{}2 * sin\ (1\ /\ w))\ x = 2 * x * sin\ (1\ /\ x) - cos\ (1\ /\ x)$
proof −
have $deriv\ (\lambda x.\ x\hat{}2 * sin\ (1\ /\ x))\ x =$
$x\hat{}2 * deriv\ (\lambda x.\ sin\ (1\ /\ x))\ x + deriv\ (\lambda x.\ x\hat{}2)\ x * sin\ (1\ /\ x)$
by (*rule deriv-mult*, *simp add*: *field-differentiable-power*,
metis DERIV-chain2 DERIV-inverse-func field-differentiable-at-sin
field-differentiable-def x-type)
moreover have $deriv\ (\lambda x.\ x\hat{}2)\ x = 2 * x$
using *power-rule* **by** *auto*
moreover have $deriv\ (\lambda x.\ sin\ (1\ /\ x))\ x = -cos\ (1\ /\ x)\ /\ x\hat{}2$
proof −
have *f1*: $DERIV\ (\lambda z.\ sin\ z)\ (1\ /\ x) :> cos\ (1\ /\ x)$
by *simp*
have *f2*: $DERIV\ (\lambda x.\ 1\ /\ x)\ x :> -1\ /\ x\hat{}2$
using *DERIV-inverse-func x-type* **by** *blast*
from *f1 f2* **have** $DERIV\ ((\lambda z.\ sin\ z) \circ (\lambda x.\ 1\ /\ x))\ x :> cos\ (1\ /\ x) * (-1\ /\ x\hat{}2)$
by (*rule DERIV-chain*)
then show *?thesis*

```
        by (simp add: DERIV-imp-deriv o-def)
      qed
      ultimately show ?thesis
        by (simp add: x-type)
    qed

    have snd-term-diff: (λx. 4 * x^3 * cos (1 / x)) field-differentiable at x
    proof -
      have t1: (λx. 4 * x^3) field-differentiable at x
        by (simp add: field-differentiable-power field-differentiable-mult)
      have t2: (λx. cos (1 / x)) field-differentiable at x
        by (metis DERIV-chain2 DERIV-inverse-func field-differentiable-at-cos
                field-differentiable-def x-type)
      show ?thesis
        by (simp add: t1 t2 field-differentiable-mult)
    qed
   have snd-term-diff': (λw. 4 * w ^ 3 * cos (1 / w) + 8 * w ^ 3) field-differentiable
at x
    proof -
      have t3: (λx. 8 * x^3) field-differentiable at x
        by (simp add: field-differentiable-mult field-differentiable-power)
      show ?thesis
        by (simp add: Derivative.field-differentiable-add t3 snd-term-diff)
    qed

    have snd-term-deriv:
      deriv (λx. 4 * x^3 * cos (1 / x) + 8 * x^3) x =
       12 * x^2 * cos (1 / x) + 4 * x * sin (1 / x) + 24 * x^2
    proof -
      have deriv (λx. 4 * x^3 * cos (1 / x) + 8 * x^3) x =
            deriv (λx. 4 * x^3 * cos (1 / x)) x + deriv (λx. 8 * x^3) x
        by (rule deriv-add, simp add: snd-term-diff,
            simp add: field-differentiable-mult field-differentiable-power)
      also have ... = (4*x^3) * (deriv (λx. cos (1 / x)) x) +
                      ((12 * x^2) * (cos (1 / x))) + deriv (λx. 8 * x^3) x
      proof -
        have deriv (λx. 4 * x^3 * cos (1 / x)) x =
              (4*x^3) * (deriv (λx. cos (1 / x)) x) +
              (deriv (λx. 4 * x^3) x) * (cos (1 / x))
         by (rule deriv-mult, simp add: field-differentiable-mult field-differentiable-power,
                metis DERIV-fun-cos DERIV-inverse-func field-differentiable-def
x-type)
        then have deriv (λx. 4 * x^3) x = 12 * x^2
        proof -
          have deriv (λx. 4 * x^3) x = 4 * deriv (λx. x^3) x
            by (rule deriv-cmult, simp add: field-differentiable-power)
          then show ?thesis
            by (simp add: power-rule)
        qed
```

```
      then show ?thesis
        using ‹deriv (λx. 4 * x^3 * cos (1 / x)) x = (4*x^3) * (deriv (λx. cos
(1 / x)) x) +
                                          (deriv (λx. 4 * x^3) x) * (cos (1 / x))›
        by auto
    qed
    also have ... = (4*x^3) * (deriv (λx. cos (1 / x)) x) +
                 ((12 * x^2) * (cos (1 / x))) + 24 * x^2
    proof −
      have deriv (λx. 8 * x^3) x = 24 * x^2
      proof −
        have deriv (λx. 8 * x^3) x = 8 * deriv (λx. x^3) x
          by (rule deriv-cmult, simp add: field-differentiable-power)
        then show ?thesis
          by (simp add: power-rule)
      qed
      then show ?thesis
        by auto
    qed
    also have ... = (4*x^3) * sin (1 / x) / x^2 + ((12 * x^2) * (cos (1 / x)))
+ 24 * x^2
    proof −
      have deriv (λx. cos (1 / x)) x = sin (1 / x) / x^2
      proof −
        have f1: DERIV (λz. cos z) (1 / x) :> −sin (1 / x)
          by simp
        have f2: DERIV (λx. 1 / x) x :> −1 / x^2
          using DERIV-inverse-func x-type by blast
        from f1 f2 have DERIV ((λz. cos z) ∘ (λx. 1 / x)) x :> (−sin (1 / x))
* (−1 / x^2)
          by (rule DERIV-chain)
        then show ?thesis
          by (simp add: DERIV-imp-deriv o-def)
      qed
      then show ?thesis
        by auto
    qed
    also have ... = ((12 * x^2) * (cos (1 / x))) + (4*x^3) * sin (1 / x) / x^2
+ 24 * x^2
      by linarith
    also have ... = (12 * x^2) * (cos (1 / x)) + 4*x * sin (1 / x) + 24 * x^2
    proof −
      have (4*x^3) * sin (1 / x) / x^2 = 4*x * sin (1 / x)
        by (simp add: power2-eq-square power3-eq-cube)
      then show ?thesis
        by presburger
    qed
    finally show ?thesis.
  qed
```

show *deriv* (*deriv f*) x = ($6*x * sin$ (1 / x) + ($12*x^2 - 1$)* *cos* (1 / x) + $24*x^2$)
proof −
have *deriv* (*deriv f*) x = *deriv* ($\lambda x.\ x^2 * sin\ (1 / x) + 4 * x \hat{} 3 * cos\ (1 / x) + 8 * x \hat{} 3$) x
by (*metis* (*no-types*, *opaque-lifting*) *deriv-f mult-cancel-left2 mult-cancel-right2*

power-zero-numeral pth-7(*2*))
also have ... = *deriv* ($\lambda x.\ x^2 * sin\ (1 / x) + (4 * x \hat{} 3 * cos\ (1 / x) + 8 * x \hat{} 3)$) x
by (*meson Groups.add-ac*(*1*))
also have ... = *deriv* ($\lambda x.\ x \hat{} 2 * sin\ (1 / x)$) x +
deriv ($\lambda x.\ 4 * x \hat{} 3 * cos\ (1 / x) + 8 * x \hat{} 3$) x
by (*rule deriv-add*, *simp add*: *fst-term-diff*, *simp add*: *snd-term-diff′*)
also have ... = $2 * x * sin\ (1 / x) - cos\ (1 / x)$ +
deriv ($\lambda x.\ 4 * x \hat{} 3 * cos\ (1 / x) + 8 * x \hat{} 3$) x
by (*simp add*: *fst-term-deriv*)
also have ... = $2 * x * sin\ (1 / x) - cos\ (1 / x)$ +
$12 * x \hat{} 2 * cos\ (1 / x) + 4 * x * sin\ (1 / x) + 24 * x \hat{} 2$
by (*simp add*: *snd-term-deriv*)
also have ... = $2 * x * sin\ (1 / x) + 4 * x * sin\ (1 / x)$ +
$12 * x \hat{} 2 * cos\ (1 / x) - cos\ (1 / x) + 24 * x \hat{} 2$
by *simp*
also have ... = ($6*x * sin\ (1 / x) + (12*x^2 - 1)* cos\ (1 / x) + 24*x^2$)
by (*smt* (*verit*, *best*) *cos-add cos-zero mult-diff-mult sin-zero*)
finally show *?thesis*.
qed

show (*deriv f*) *field-differentiable at x*
proof (*rule field-differentiable-transfer-on-ball*
[**where** $f = \lambda\ x.\ (x^2 * sin\ (1 / x) + 4 * x \hat{} 3 * cos\ (1 / x) + 8 * x \hat{} 3)$ **and** $\varepsilon = |x|$])
show $0 < |x|$
by (*simp add*: *x-type*)
show $\forall\, y.\ y \in ball\ x\ |x| \longrightarrow y^2 * sin\ (1 / y) + 4 * y \hat{}\ 3 * cos\ (1 / y) + 8 * y \hat{}\ 3 =$
deriv f y
by (*simp add*: *deriv-f*)
show ($\lambda x.\ x^2 * sin\ (1 / x) + 4 * x \hat{}\ 3 * cos\ (1 / x) + 8 * x \hat{}\ 3$)*field-differentiable at x*
by (*simp add*: *Derivative.field-differentiable-add fst-term-diff is-num-normalize*(*1*)

snd-term-diff′)
qed
qed

have *deriv2-f-at-0*:
deriv (*deriv f*) $0 = 0$ ∧ (*deriv f*) *field-differentiable at 0*

```
  proof -
    — By the definition of deriv, we need to show the limit of the difference quotient of f′ is 0.
    have dq-limit: ((λh. (deriv f (0 + h) - deriv f 0) / h) ⟶ 0) (at 0)
    proof
      fix ε :: real
      assume ε-pos: 0 < ε
      have ∃δ>0. ∀h. 0 < |h| ∧ |h| < δ ⟶ |(deriv f (0 + h) - deriv f 0) / h - 0| < ε
      proof (cases ε < 1/6)
        assume eps-lt-inv6: ε < 1/6
        — Choose δ > 0 to ensure |difference quotient| < ε.
        obtain δ where δ-def: δ = ε / 2
          by blast
        have δ-pos: δ > 0
          using ε-pos by (simp add: δ-def)
        show ∃δ>0. ∀h. 0 < |h| ∧ |h| < δ ⟶ |(deriv f (0 + h) - deriv f 0) / h - 0| < ε
        proof (intro exI[where x=δ], intro conjI insert δ-pos, clarify)
          fix h :: real
          assume h-pos: 0 < |h|
          assume h-lt-δ: |h| < δ

          have h-bound1: |h| < ε / 2
            using h-lt-δ by (simp add: δ-def)
          have h-bound2: 12 * |h^2| < ε / 2
          proof -
            have |h| < ε / 2 using h-bound1 by blast
            then have |h^2| < (ε / 2)^2
                by (metis abs-ge-zero abs-power2 power2-abs power-strict-mono zero-less-numeral)
            then have 12 * |h^2| < 12 * (ε / 2)^2
              by (simp add: mult-strict-left-mono)
            also have ... = 12 * (ε^2 / 4)
              by (simp add: power2-eq-square)
            also have ... = 3 * ε^2
              by simp
            also have ... < ε/2
            proof -
              have ε * 6 < 1
                by (meson eps-lt-inv6 less-divide-eq-numeral1(1))
              then show ?thesis
                by (simp add: ε-pos power2-eq-square)
            qed
            finally show ?thesis.
          qed
          have |(deriv f (0 + h) - deriv f 0) / h - 0| = |deriv f h / h|
            by (simp add: deriv-f-at-0)
         also have ... = |(h² * sin (1 / h) + 4*h^3 * cos (1 / h) + 8*h^3) / h|
```

```
        using deriv-f by presburger
       also have ... = |(h^2 * sin (1 / h) / h) + (4*h^3 * cos (1 / h)) / h +
(8*h^3) / h|
        by (simp add: add-divide-distrib)
      also have ... = |h * sin (1 / h) + (4*h^2 * cos (1 / h)) + 8 * h^2|
        by (simp add: more-arith-simps(11) power2-eq-square power3-eq-cube)
      also have ... ≤ |h * sin (1 / h)| + |4*h^2 * cos (1 / h)| + |8 * h^2|
        by linarith
     also have ... ≤ |h| * |sin (1 / h)| + 4 * |h^2| * |cos (1 / h)| + 8 * |h^2|
        by (simp add: abs-mult)
      also have ... ≤ |h| + 4 * |h^2| + 8 * |h^2|
      proof −
        have i1: |h| * |sin (1 / h)| ≤ |h|
          using h-pos by fastforce
        have |h| * |cos (1 / h)| ≤ |h|
          by (simp add: mult-left-le)
        then show ?thesis
          by (smt (verit) cos-ge-minus-one cos-le-one i1 mult-left-le)
      qed
      also have ... = |h| + 12 * |h^2|
        by simp
      also have ... < ε
        using h-bound1 h-bound2 by auto
      finally show |(deriv f (0 + h) − deriv f 0) / h − 0| < ε.
    qed
  next
    assume ¬ ε < 1/6
    then have ε ≥ 1/6 by linarith
    then have eps-half: ε / 2 ≥ 1/12 by linarith
    obtain δ where δ-def: δ = (1::real)/12 by blast
    have δ-pos: δ > 0 using ε-pos by (simp add: δ-def)
    show ∃δ>0. ∀h. 0 < |h| ∧ |h| < δ ⟶ |(deriv f (0 + h) − deriv f 0) /
h − 0| < ε
    proof (intro exI[where x=δ], intro conjI insert δ-pos, clarify)
      fix h :: real
      assume h-pos: 0 < |h|
      assume h-lt-δ: |h| < δ
      have h-bound1: |h| < ε / 2
      proof −
        have |h| < δ using h-lt-δ by blast
        also have ... = (1::real)/12 by (simp add: δ-def)
        also have ... ≤ ε / 2 using eps-half by blast
        finally show ?thesis.
      qed
      have h-bound2: 12 * |h|^2 < ε / 2
      proof −
        from h-bound1 have |h|^2 < (1/12)^2
        by (metis δ-def abs-ge-zero h-lt-δ power-strict-mono zero-less-numeral)
        hence 12 * |h|^2 < 12 * (1/12)^2
```

```
        by (rule mult-strict-left-mono, simp-all)
      also have ... = 1/12 by (simp add: power-one-over)
      also have ... ≤ ε / 2 using eps-half by blast
      finally show ?thesis.
    qed
    have |(deriv f (0 + h) − deriv f 0) / h − 0| = |deriv f h / h|
      by (simp add: deriv-f-at-0)
   also have ... = |(h² ∗ sin (1 / h) + 4∗h^3 ∗ cos (1 / h) + 8∗h^3) / h|
      using deriv-f by presburger
    also have ... = |(h² ∗ sin (1 / h) / h) + (4∗h^3 ∗ cos (1 / h)) / h +
(8∗h^3) / h|
      by (simp add: add-divide-distrib)
    also have ... = |h ∗ sin (1 / h) + (4∗h^2 ∗ cos (1 / h)) + 8 ∗ h^2|
      by (simp add: more-arith-simps(11) power2-eq-square power3-eq-cube)
    also have ... ≤ |h ∗ sin (1 / h)| + |4∗h^2 ∗ cos (1 / h)| + |8 ∗ h^2|
      by linarith
   also have ... ≤ |h| ∗ |sin (1 / h)| + 4 ∗ |h^2| ∗ |cos (1 / h)| + 8 ∗ |h^2|
      by (simp add: abs-mult)
    also have ... ≤ |h| + 4 ∗ |h^2| + 8 ∗ |h^2|
    proof −
      have i1: |h| ∗ |sin (1 / h)| ≤ |h|
        using h-pos by fastforce
      have |h| ∗ |cos (1 / h)| ≤ |h|
        by (simp add: mult-left-le)
      then show ?thesis
        by (smt (verit) cos-ge-minus-one cos-le-one i1 mult-left-le)
    qed
    also have ... = |h| + 12 ∗ |h^2|
      by simp
    also have ... < ε
      using h-bound1 h-bound2 by auto
    finally show |(deriv f (0 + h) − deriv f 0) / h − 0| < ε.
  qed
 qed
 then show ∃ d>0. ∀ x∈UNIV. 0 < dist x 0 ∧ dist x 0 < d ⟶
          dist ((deriv f (0 + x) − deriv f 0) / x) 0 ≤ ε
   by (metis cancel-comm-monoid-add-class.diff-zero dist-real-def le-less)
qed
then show ?thesis
  using DERIV-def DERIV-imp-deriv field-differentiable-def by blast
qed

show ⋀x. deriv (deriv f) x = (if x = 0 then 0 else 6 ∗ x ∗ sin (1 / x)
                             + (12 ∗ x² − 1) ∗ cos (1 / x)
                             + 24 ∗ x²)
  using snd-deriv-f-at-nonzero deriv2-f-at-0 by presburger

show (deriv f) differentiable-on UNIV
  by (metis deriv2-f-at-0 differentiable-on-def
```

```
              field-differentiable-imp-differentiable snd-deriv-f-at-nonzero)
qed
then have f-cont: continuous-on ℝ f
 by (meson continuous-on-subset differentiable-imp-continuous-on top.extremum)
have f'-cont: continuous-on ℝ (deriv f)
 by (meson continuous-on-subset deriv-f differentiable-imp-continuous-on top.extremum)

obtain U where U-def: U = {x :: real. −1 < x ∧ x < 1}
  by blast
then have open-neighborhood-of-zero: open U ∧ 0 ∈ U
  using lemma-interval-lt by (subst open-dist, subst dist-real-def,fastforce)

have strict-local-minimizer-at-0: strict-local-minimizer f 0
  unfolding strict-local-minimizer-def  strict-local-minimizer-on-def
proof (intro exI[where x=U],(subst sym[OF conj-assoc],rule conjI), rule open-neighborhood-of-zero)
  show ∀ x ∈ U − {0}. f 0 < f x
  proof
    fix x
    assume x-type: x ∈ U − {0}
    then have x-nonzero: x ≠ 0
      by blast
    have cos(1/x) + 2 ≥ 1
      by (smt (verit) cos-ge-minus-one)
    then have x^4 ∗ (cos(1/x) + 2) ≥ x^4 ∗ 1
      by (rule mult-left-mono, force)
    then have f x ≥ x^4
      by (simp add: f-def x-nonzero)
    then have f x > 0
    by (smt (verit, del-insts) mult-le-0-iff power4-eq-xxxx x-nonzero zero-le-mult-iff)
    then show f 0 < f x
      using f-def by force
  qed
qed
then have zero-min: local-minimizer f 0
  by (simp add: strict-local-minimizer-imp-local-minimizer)
have (∃ x-seq::nat ⇒ real. (∀ n. local-minimizer f (x-seq n) ∧ x-seq n ≠ 0) ∧
                                        ((x-seq ⟶ 0) at-top))
proof −
  obtain left-seq :: nat ⇒ real where left-seq-def: ∀ n ∈ ℕ. n ≠ 0 ⟶
        left-seq n = inverse ((5 ∗ pi / 4) + 2 ∗ real n ∗ pi)
    by force
  obtain right-seq :: nat ⇒ real where right-seq-def: ∀ n ∈ ℕ. n ≠ 0 ⟶
        right-seq n = inverse  (pi + 2 ∗ real n ∗ pi)
    by force

  have zero-lt-left-seq-lt-right-seq-both-pos: ∀ n ∈ ℕ. n ≠ 0 ⟶
                                        0 < left-seq n ∧ left-seq n < right-seq n
  proof clarify
    fix n::nat
```

```
    assume n-pos: 0 < n
    then have inv-left: inverse (left-seq n) = (5 * pi / 4) + 2 * real n * pi
      by (metis bot-nat-0.not-eq-extremum id-apply inverse-inverse-eq left-seq-def
of-nat-eq-id
          of-nat-in-Nats)

    have inv-right: inverse (right-seq n) = pi + 2 * real n * pi
          by (metis bot-nat-0.not-eq-extremum id-apply inverse-inverse-eq n-pos
of-nat-eq-id
          of-nat-in-Nats right-seq-def)

    have denom-ineq: (pi + 2 * real n * pi) < ((5 * pi / 4) + 2 * real n * pi)
    proof -
      have (5 * pi / 4) + 2 * real n * pi = 2 * real n * pi + (5 * pi / 4)
        by simp
      have ((5 * pi / 4) + 2 * real n * pi) - (pi + 2 * real n * pi) =
              (5 * pi / 4) + 2 * real n * pi - pi - 2 * real n * pi
        by simp
      also have ... = (5 * pi / 4) - pi
        by simp
      also have ... = (5 * pi / 4) - (4 * pi / 4)
        by simp
      also have ... = (5 - 4) * pi / 4
        by simp
      also have ... = pi / 4
        by simp
      then show ?thesis
        by simp
    qed
    then have left-seq n < right-seq n
     by (smt (verit) inv-left inv-right inverse-positive-iff-positive le-imp-inverse-le

          mult-nonneg-nonneg of-nat-less-0-iff pi-gt3)
    then show 0 < left-seq n ∧ left-seq n < right-seq n
    by (smt (verit, best) denom-ineq inv-left inverse-positive-iff-positive mult-nonneg-nonneg

          of-nat-less-0-iff pi-gt3)
  qed
  have first-and-second-order-conditions: ∀ n. n ≠ 0 ⟶
 (∃ y ∈ {left-seq n .. right-seq n}. (y^2 * sin (1 / y) + 4 * y^3 * cos (1 / y) +
8 * y^3) = 0 ∧
 (6*y * sin (1 / y) + (12*y² - 1)* cos (1 / y) + 24*y²) > 0) ∧
 ((left-seq n)^2 * sin (1 /(left-seq n)) + 4 * (left-seq n)^3 * cos (1 / (left-seq n))
+
                          8 * (left-seq n)^3) < 0 ∧
 ((right-seq n)^2 * sin (1 /(right-seq n)) + 4 * (right-seq n)^3 * cos (1 / (right-seq
n))
                         + 8 * (right-seq n)^3) > 0
  proof(clarify)
```

fix *n*:: *nat*
assume *n-pos*: *0* < *n*
then have *n-ge-1*: *1* ≤ *n*
by *simp*
show (∃ *y*∈{*left-seq n*..*right-seq n*}. y^2 ∗ *sin* (*1* / *y*) + *4* ∗ *y* ^ *3* ∗ *cos* (*1* / *y*) + *8* ∗ *y* ^ *3* = *0* ∧ *0* < *6* ∗ *y* ∗ *sin* (*1* / *y*) + (*12* ∗ y^2 − *1*) ∗ *cos* (*1* / *y*) + *24* ∗ y^2) ∧
(*left-seq n*)^2 ∗ *sin* (*1* / *left-seq n*) + *4* ∗ *left-seq n* ^ *3* ∗ *cos* (*1* / *left-seq n*) + *8* ∗ *left-seq n* ^ *3* < *0* ∧
0 < (*right-seq n*)^2 ∗ *sin* (*1* / *right-seq n*) + *4* ∗ *right-seq n* ^ *3* ∗ *cos* (*1* / *right-seq n*) + *8* ∗ *right-seq n* ^ *3*
proof *safe*
show *left-seq-less-zero*: (λ*x*. *x*^*2* ∗ *sin* (*1* / *x*) + *4* ∗ *x*^*3* ∗ *cos* (*1* / *x*) + *8* ∗ *x*^*3*) (*left-seq n*) < *0*
proof −
obtain *x* **where** *x-def*: *x* = *left-seq n*
by *blast*
— Rewrite $1/x$ in terms of $\frac{5\pi}{4} + 2n\pi$.

then have *inv-x-eqs*: *inverse x* = *inverse* (*inverse* ((*5* ∗ *pi* / *4*) + *2* ∗ *real n* ∗ *pi*))
by (*metis bot-nat-0.not-eq-extremum id-apply left-seq-def n-pos of-nat-eq-id of-nat-in-Nats*)
then have *x-inv*: *1*/*x* = (*5* ∗ *pi* / *4*) + *2* ∗ *real n* ∗ *pi*
by (*simp add*: *inverse-eq-divide*)

— Evaluate $\sin(1/x)$ and $\cos(1/x)$.
have *sin-inv-x*: *sin* (*1* / *x*) = − (*sqrt 2* / *2*)
proof −
have *sin* (*1* / *x*) = *sin* ((*5* ∗ *pi* / *4*) + *2* ∗ *real n* ∗ *pi*)
using *x-inv* **by** *presburger*
also have ... = *sin* (*5* ∗ *pi* / *4*)
by (*simp add*: *sin-add*)
also have ... = − (*sqrt 2* / *2*)
using *sin-5pi-div-4* **by** *blast*
finally show *sin* (*1* / *x*) = − (*sqrt 2* / *2*).
qed

have *cos-inv-x*: *cos* (*1* / *x*) = − (*sqrt 2* / *2*)
proof −
have *cos-val*: *cos* (*1* / *x*) = *cos* ((*5* ∗ *pi* / *4*) + *2* ∗ *real n* ∗ *pi*)
using *x-inv* **by** *presburger*
also have ... = *cos* (*5* ∗ *pi* / *4*)
by (*simp add*: *cos-add*)
also have ... = − (*sqrt 2* / *2*)
using *cos-5pi-div-4* **by** *linarith*
finally show *cos* (*1* / *x*) = − (*sqrt 2* / *2*).
qed

```
— Substitute these into the expression.
have expr: x^2 * sin (1 / x) + 4 * x^3 * cos (1 / x) + 8 * x^3
        = − (sqrt 2 / 2) * x^2 + (8 − 2 * sqrt 2) * x^3
proof −
  have x^2 * sin (1 / x) + 4 * x^3 * cos (1 / x) + 8 * x^3
       = (x^2 * − (sqrt 2 / 2)) + 4 * x^3 * (−(sqrt 2 / 2)) + 8 * x^3
    by (simp add: cos-inv-x sin-inv-x)

  also have ... = x^2 * − (sqrt 2 / 2) + (−2 * sqrt 2) * x^3 + 8 * x^3
    by simp
  also have ... =  − (sqrt 2 / 2) * x^2 + (8 − 2 * sqrt 2) * x^3
  proof −
    have − (sqrt 2 / 2) + (x ^ 3 * (sqrt 2 * − 2) + x ^ 3 * 8) =
        − (sqrt 2 / 2) + x ^ 3 * (sqrt 2 * − 2 + 8)
      by (metis (no-types) nat-distrib(2))
    then show ?thesis
      by (simp add: Groups.mult-ac(2))
  qed
  finally show rewrite-expr:
    x^2 * sin (1 / x) + 4 * x^3 * cos (1 / x) + 8 * x^3
     = − (sqrt 2 / 2) * x^2 + (8 − 2 * sqrt 2) * x^3.
qed
```

— Factor out x^3, and rewrite x^3 as $\left(\frac{5\pi}{4} + 2n\pi\right)^{-1}$.

```
have deriv-right-seq-eval: sin (1 / x) * x^2 + 4 * x^3 * cos (1 / x) + 8
* x^3 =
    (− (sqrt 2 / 2)*((5 * pi / 4) + 2 * real n * pi) + (8 − 2 * sqrt 2))
* x^3
proof −
  have sin (1 / x) * x^2 + 4 * x^3 * cos (1 / x) + 8 * x^3 =
    − (sqrt 2 / 2)*inverse x * x^3 + (8 − 2 * sqrt 2) * x^3
      by (smt (verit, del-insts) Groups.mult-ac(2) cos-inv-x cos-zero di-
vide-eq-0-iff expr
          left-inverse more-arith-simps(11) one-power2 power2-eq-square
power3-eq-cube
        power-minus sin-inv-x sin-zero)
  also have ... = (− (sqrt 2 / 2)*inverse x + (8 − 2 * sqrt 2)) * x^3
    by (metis (no-types) distrib-right)
  also have ... = (− (sqrt 2 / 2)*((5 * pi / 4) + 2 * real n * pi) +
                                       (8 − 2 * sqrt 2)) * x^3
    by (simp add: inv-x-eqs)
  finally show ?thesis.
qed
```

— Combine into a single fraction and show negativity.

```
have first-term-eval: x^3 > 0
by (smt (verit) mult-nonneg-nonneg of-nat-0-le-iff pi-gt3 x-inv zero-compare-simps(7)
```

```
          zero-less-power)
       have neg-term: (−(sqrt 2 / 2)∗((5 ∗ pi / 4) + 2 ∗ real n ∗ pi) + (8 −
2 ∗ sqrt 2)) < 0
        proof −
          have n-ge1: n ≥ 1
            using n-ge-1 by auto
          have lower-bound: 2 ∗ real n ∗ pi ≥ 2 ∗ pi
            using n-ge1 by (simp add: mult-left-mono)
          then have mult-bound: − (sqrt 2 / 2) ∗ ((5 ∗ pi / 4) + 2 ∗ real n ∗
pi)
                          ≤ − (sqrt 2 / 2) ∗ (5 ∗ pi / 4 + 2 ∗ pi)
            by (simp add: mult-left-mono)
          moreover have (− (sqrt 2 / 2) ∗ (5 ∗ pi / 4 + 2 ∗ pi) + (8 − 2 ∗
sqrt 2)) < 0
          proof −
            have 5 ∗ pi / 4 + 2 ∗ pi = 13 ∗ pi / 4
              by simp
            then have simpification: (− (sqrt 2 / 2) ∗ (5 ∗ pi / 4 + 2 ∗ pi) +
(8 − 2 ∗ sqrt 2))
                =    (64 − 16 ∗ sqrt 2 − 13 ∗ pi ∗ sqrt 2) / 8
              by (simp add: field-simps)
             have sufficies-to-show-numerator-neg:((64 − 16 ∗ sqrt 2 − 13 ∗ pi
∗ sqrt 2) / 8 < 0)
                = (64 − 16 ∗ sqrt 2 − 13 ∗ pi ∗ sqrt 2 < 0)
              by simp
            have sqrt 2 ∗ (16 + 13 ∗ pi) > 64
            proof −
              have pi-gt-3: pi > 3
                by (simp add: pi-gt3)
              hence 16 + 13 ∗ pi > 16 + 13 ∗ 3
                by (simp add: mult-strict-left-mono)
              hence 16 + 13 ∗ pi > 55
                by simp
              then have sqrt 2 ∗ (16 + 13 ∗ pi) > sqrt 2 ∗ 55
                by (simp add: mult-strict-left-mono)
              moreover have sqrt 2 ∗ 55 > 64
              proof −
                have (sqrt 2 ∗ 55)^2 = 2 ∗ 55^2
                  by (simp add: power-mult-distrib)
                also have ... = 2 ∗ (55∗55)
                  by auto
                also have ... = 6050
                  by simp
                also have ... > 64∗64
                  by eval
                moreover have sqrt 2 ∗ 55 > 0
                  by simp
                ultimately show sqrt 2 ∗ 55 > 64
                  using power-mono-iff
```

```
            by (metis less-le power2-eq-square zero-less-numeral)
          qed
          ultimately show ?thesis
            by linarith
        qed
        then have 64 − 16 ∗ sqrt 2 − 13 ∗ pi ∗ sqrt 2 < 0
          by (simp add: Groups.mult-ac(2) distrib-left)
        then show ?thesis
          using simpification sufficies-to-show-numerator-neg by presburger
      qed
      then show ?thesis
        using mult-bound by linarith
    qed
    then show (left-seq n)^2 ∗ sin (1 / left-seq n) +
          4 ∗ left-seq n ^ 3 ∗ cos (1 / left-seq n) + 8 ∗ left-seq n ^ 3 < 0
      by (metis deriv-right-seq-eval first-term-eval mult.commute x-def
          zero-compare-simps(10))
  qed
  show right-seq-greater-zero:(λx. x^2 ∗ sin (1 / x) + 4 ∗ x^3 ∗ cos (1 /
x) + 8 ∗ x^3)
                                                              (right-seq n) > 0
proof −
  obtain x where x-def: x = right-seq n
    by blast
  then have inv-x-eqs: inverse x = inverse (inverse (pi + 2 ∗ real n ∗ pi))
      by (metis id-apply n-pos of-nat-eq-id of-nat-in-Nats of-nat-less-0-iff
right-seq-def)
  have x-inv: 1 / x = pi + 2 ∗ real n ∗ pi
      unfolding right-seq-def by (metis inv-x-eqs inverse-eq-divide in-
verse-inverse-eq)

  have sin-inv-x: sin (1 / x) = 0
    by (metis add.inverse-neutral sin-2npi sin-periodic-pi2 x-inv)

  have cos-inv-x: cos (1 / x) = −1
    using cos-2npi cos-periodic-pi2 x-inv by presburger

  have f-x: x^2 ∗ sin (1 / x) + 4 ∗ x^3 ∗ cos (1 / x) + 8 ∗ x^3 = 4 ∗ x^3
    by (simp add: cos-inv-x sin-inv-x)

  have x-pos: x > 0
    unfolding right-seq-def
      by (smt (verit) mult-nonneg-nonneg of-nat-less-0-iff pi-gt-zero x-inv
zero-less-divide-iff)
  then show 0 < (right-seq n)^2 ∗ sin (1 / right-seq n) + 4 ∗ right-seq n ^
3 ∗ cos (1 / right-seq n) + 8 ∗ right-seq n ^ 3
    using cos-inv-x sin-inv-x x-def by fastforce
qed
```

show $\exists y \in \{$*left-seq n..right-seq n*$\}$. $y^2 * sin\ (1\ /\ y) + 4 * y\ \hat{}\ 3 * cos\ (1\ /\ y) + 8 * y\ \hat{}\ 3 =$
$0 \wedge 0 < 6 * y * sin\ (1\ /\ y) + (12 * y^2 - 1) * cos\ (1\ /\ y) + 24 * y^2$
proof −
have *existence-of-minimizing-sequence*: $\exists y \in \{$*left-seq n..right-seq n*$\}$. $y^2 * sin\ (1\ /\ y) + 4 * y\ \hat{}\ 3 * cos\ (1\ /\ y) + 8 * y\ \hat{}\ 3 = 0$
proof −
have $\exists x \geq$ *left-seq n*. $x \leq$ *right-seq n* $\wedge$ $(\lambda x.\ x\hat{}2 * sin\ (1\ /\ x) + 4 * x\hat{}3 * cos\ (1\ /\ x) + 8 * x\hat{}3)\ x = 0$
proof(*rule IVT′*)
show (*left-seq n*)^2 $*$ *sin* (1 / *left-seq n*) + 4 $*$ *left-seq n* ^ 3 $*$ *cos* (1 / *left-seq n*) + 8 $*$ *left-seq n* ^ 3 ≤ 0
using *left-seq-less-zero* **by** *auto*
show $0 \leq$ (*right-seq n*)^2 $*$ *sin* (1 / *right-seq n*) + 4 $*$ *right-seq n* ^ 3 $*$ *cos* (1 / *right-seq n*) + 8 $*$ *right-seq n* ^ 3
using *right-seq-greater-zero* **by** *linarith*
show *left-seq n* $\leq$ *right-seq n*
by (*metis id-apply leD linorder-linear n-pos of-nat-eq-id of-nat-in-Nats zero-lt-left-seq-lt-right-seq-both-pos*)
show *continuous-on* {*left-seq n..right-seq n*} $(\lambda x.\ x^2 * sin\ (1\ /\ x) + 4 * x\ \hat{}\ 3 * cos\ (1\ /\ x) + 8 * x\ \hat{}\ 3)$
proof − — We prove continuity by establishing it is differentiable.
— First, note that $left\_seq_n$ is positive, so the interval does not contain 0.
have *left-seq-pos*: *left-seq n* > 0
by (*metis bot-nat-0.extremum-strict id-apply n-pos of-nat-eq-id of-nat-in-Nats zero-lt-left-seq-lt-right-seq-both-pos*)

— Transfer global differentiability to local differentiability of deriv f.

have $\bigwedge x.\ x \in$ {*left-seq n..right-seq n*} $\longrightarrow$ $(\lambda x.\ x^2 * sin\ (1\ /\ x) + 4 * x\ \hat{}\ 3 * cos\ (1\ /\ x) + 8 * x\ \hat{}\ 3)$ *field-differentiable at x*
proof *clarify*
fix *x*::*real*
assume *x-type*: $x \in$ {*left-seq n..right-seq n*}
show $(\lambda x.\ x^2 * sin\ (1\ /\ x) + 4 * x\ \hat{}\ 3 * cos\ (1\ /\ x) + 8 * x\ \hat{}\ 3)$ *field-differentiable at x*
proof(*rule field-differentiable-transfer-on-ball*[**where** *f* = *deriv f* **and** $\varepsilon = x$])
show $0 < x$
using *left-seq-pos x-type* **by** *auto*
show $\forall y.\ y \in$ *ball x x* $\longrightarrow$ *deriv f y* $= y^2 * sin\ (1\ /\ y) + 4 * y\ \hat{}\ 3 * cos\ (1\ /\ y) + 8 * y\ \hat{}\ 3$
by (*simp add*: *deriv-f*)
show *deriv f field-differentiable at x*
by (*meson UNIV-I deriv-f differentiable-on-def field-differentiable-def real-differentiableE*)
qed

```
          qed
          then have (λx. x^2 * sin (1 / x) + 4 * x ^ 3 * cos (1 / x) + 8 * x
^ 3) differentiable-on {left-seq n..right-seq n}
        by (meson differentiable-at-imp-differentiable-on field-differentiable-imp-differentiable)
          then show ?thesis
            using differentiable-imp-continuous-on by blast
        qed
      qed
      then show ∃y∈{left-seq n..right-seq n}. y^2 * sin (1 / y) + 4 * y ^ 3 *
cos (1 / y) + 8 * y ^ 3 = 0
        by presburger
    qed
    then obtain min-n where min-n-def: min-n ∈{left-seq n..right-seq n} ∧
min-n^2 * sin (1 / min-n) + 4 * min-n ^ 3 * cos (1 / min-n) + 8 * min-n ^ 3 =
0
      by blast
     have ⋀y. y ∈ {left-seq n .. right-seq n} ⟶ 0 < 6 * y * sin (1 / y) +
(12 * y^2 − 1) * cos (1 / y) + 24 * y^2
    proof (clarify)
      fix y :: real
      assume y-int: y ∈ {left-seq n .. right-seq n}
      — Since left_seq_n > 0, every y in the interval is positive.
      then have y-pos: y > 0
     by (metis atLeastAtMost-iff bot-nat-0.extremum id-apply linorder-not-less
n-pos
       of-nat-eq-id of-nat-in-Nats order-less-le-trans zero-lt-left-seq-lt-right-seq-both-pos)

      have ∃ x-nc :: real ⇒ real. ∀ c ∈ {0..pi/4}. x-nc c = inverse (pi + c
+ 2*pi*real n)
        by auto
      then obtain x-nc :: real ⇒ real where x-nc-def: ∀ c ∈ {0..pi/4}. x-nc
c = inverse (pi + c + 2*pi*real n)
        by auto
       have ∃ x-nc :: real ⇒ real. ∀ c ∈ {0..pi/4}. x-nc c = inverse (pi + c
+ 2*pi*real n)
        by auto
      then obtain x-nc :: real ⇒ real where x-nc-def: ∀ c ∈ {0..pi/4}. x-nc
c = inverse (pi + c + 2*pi*real n)
        by auto
      have continuous-on-subinterval: continuous-on {0..pi/4} x-nc
      proof −
       have cont-denom: continuous-on {0..pi/4} (λc. pi + c + 2*pi*real n)
        proof −
          have continuous-on {0..pi/4} (λc. c)
            using continuous-on-id by blast
          moreover have continuous-on {0..pi/4} (λc. pi + 2*pi*real n)
            using continuous-on-const by blast
          ultimately show ?thesis
            by (simp add: continuous-on-add)
```

```
      qed
        then have continuous-on {0..pi/4} (λx. inverse ((λc. pi + c +
2∗pi∗real n) x))
          by(rule continuous-on-inverse,
            smt (verit) add-mono-thms-linordered-field(4) atLeastAtMost-iff
            of-nat-less-0-iff pi-neq-zero pi-not-less-zero zero-compare-simps(4))

      then show ?thesis
        using continuous-on-cong x-nc-def by fastforce
    qed

    have minimizer-dom: ∃x. 0 ≤ x ∧ x ≤ pi/4 ∧ x-nc x = y
    proof(rule IVT2')
      show x-nc (pi / 4) ≤ y
      proof −
        have x-nc (pi / 4) = inverse ( pi + pi / 4 + 2 ∗ real n ∗ pi)
         by (metis (no-types, opaque-lifting) atLeastAtMost-iff divide-eq-imp
          divide-real-def linorder-not-less mult.left-commute mult.right-neutral

            mult-le-0-iff nle-le of-nat-0-le-iff of-nat-numeral pi-gt-zero x-nc-def

              zero-neq-numeral)
        also have ... = inverse ((5 ∗ pi / 4) + 2 ∗ real n ∗ pi)
          by simp
        also have ... = left-seq n
              by (metis bot-nat-0.not-eq-extremum id-apply left-seq-def n-pos
of-nat-eq-id of-nat-in-Nats)
        also have ... ≤ y
          using y-int by presburger
        finally show ?thesis.
      qed
      show y ≤ x-nc 0
      proof −
        have y ≤ right-seq n
          using y-int by presburger
        also have ... = inverse (pi + 2 ∗ real n ∗ pi)
             by (metis bot-nat-0.not-eq-extremum id-apply n-pos of-nat-eq-id
of-nat-in-Nats right-seq-def)
        also have ... = x-nc 0
          using x-nc-def by auto
        finally show ?thesis.
      qed
      show 0 ≤ pi / 4
        by simp
      show continuous-on {0..pi / 4} x-nc
        using continuous-on-subinterval by simp
    qed
    then have minimizer-dom': ∃c ∈ {0..pi/4}. y = x-nc c
      using atLeastAtMost-iff by blast
```

— We will show that $f''(x_{nc}(c)) > 0$ for all $c \in [0, 1]$, then use the fact that $left\_seq_n \leq x_{nc}(c) \leq right\_seq_n$ together with the IVT to establish the existence of $c \in [0, \frac{\pi}{4}]$ such that $x_{nc}(c) = y$, and then conclude that $f''(y) > 0$.

have *snd-deriv-positive-in-neighborhood*: ∀ *c* ∈ {*0..pi/4*}. *left-seq n* ≤ *x-nc c* ∧ *x-nc c* ≤ *right-seq n* ∧ *deriv* (*deriv f*) (*x-nc c*) > *0*
proof (*safe*)
fix *c* :: *real*
assume *c-type*: *c* ∈ {*0..pi/4*}
then have *c-bounds*: *0* ≤ *c* ∧ *c* ≤ *pi/4*
by *simp*

have *x-nc-eqs*: *x-nc c* = *inverse* (*pi* + *c* + *2∗pi∗real n*)
using *c-bounds inverse-eq-divide pi-half-le-two x-nc-def* **by** *auto*
show *left-seq n* ≤ *x-nc c*
proof −
have *f1*: *left-seq n* = *inverse* ((*5* ∗ *pi* / *4*) + *2* ∗ *real n* ∗ *pi*)
by (*metis bot-nat-0.not-eq-extremum id-apply left-seq-def n-pos of-nat-eq-id of-nat-in-Nats*)
from *c-bounds* **have** *1*/ ((*5* ∗ *pi* / *4*) + *2* ∗ *real n* ∗ *pi*) ≤ *1*/ (*pi* + *c* + *2∗pi∗real n*)
by(*subst frac-le*, *simp-all*, *simp add*: *add-sign-intros*(*1*))
then show *?thesis*
by (*simp add*: *f1 x-nc-eqs inverse-eq-divide*)
qed

then have *x-nc-pos*: *x-nc c* > *0*
by (*metis id-apply n-pos of-nat-eq-id of-nat-in-Nats order-less-le-trans zero-lt-left-seq-lt-right-seq-both-pos zero-order*(*5*))

show *x-nc c* ≤ *right-seq n*
proof −
have *f1*: *right-seq n* = *inverse* (*pi* + *2* ∗ *real n* ∗ *pi*)
by (*metis bot-nat-0.not-eq-extremum id-apply n-pos of-nat-eq-id of-nat-in-Nats right-seq-def*)
from *c-bounds* **have** *1*/ (*pi* + *c* + *2∗pi∗real n*) ≤ *1* /(*pi* + *2* ∗ *real n* ∗ *pi*)
by(*subst frac-le*, *simp-all*, *smt* (*verit*, *del-insts*) *m2pi-less-pi mult-sign-intros*(*1*) *of-nat-less-0-iff*)
then show *?thesis*
by (*simp add*: *f1 x-nc-eqs inverse-eq-divide*)
qed

— Bounds on sin(c) and cos(c).
have *pi* + *c* + *2∗pi∗real n* ≥ *3∗pi*
proof −
have *pi* + *c* + *2∗pi∗real n* ≥ *pi* + *0* + *2∗pi∗real 1*
by (*smt* (*verit*, *best*) *Num.of-nat-simps*(*2*) *c-bounds mult-left-mono*

```
n-ge-1
          pi-not-less-zero real-of-nat-ge-one-iff)
      then show ?thesis
        by linarith
    qed
    then have x-nc-bound: x-nc c ≤ inverse(3*pi)
      by (smt (verit) le-imp-inverse-le pi-gt-zero x-nc-eqs)
        then have cos-coef-bound: (1− 12 ∗ (x-nc c)^2) ≥ (1− 12 ∗
(inverse(3*pi))^2)
      using x-nc-pos by force

    have sin-bound: 0 ≤ sin c ∧ sin c ≤ sqrt(2)/2
    proof safe
      show 0 ≤ sin c
        using c-bounds sin-ge-zero by auto
      show sin c ≤ sqrt(2)/2
          by (smt (verit, best) c-bounds frac-le pi-not-less-zero sin-45
sin-mono-less-eq)
    qed
    have cos-bound: sqrt(2)/2 ≤ cos c ∧ cos c ≤ 1
    proof safe
      show sqrt 2 / 2 ≤ cos c
          by (smt (verit) c-bounds cos-45 cos-monotone-0-pi-le machin
pi-machin)
      show cos c ≤ 1
        by simp
    qed

    show 0 < deriv (deriv f) (x-nc c)
    proof −
      — Lower bound of f''(x_nc).
        have snd-deriv-at-x-nc: deriv (deriv f) (x-nc c) = (1− 12 ∗ (x-nc
c)^2) ∗ cos c − 6 ∗ (x-nc c) ∗ sin c + 24 ∗ (x-nc c)^2
      proof−
        have f1: sin (1 / (x-nc c)) = −sin c
        proof −
          have sin (1 / (x-nc c)) = sin (pi + c + 2*pi*real n)
            by (simp add: inverse-eq-divide x-nc-eqs)
          also have ... = sin (pi + c)
          by (metis Groups.mult-ac(2) id-apply of-real-eq-id sin.plus-of-nat)
          also have ... = −sin c
            by simp
          finally show ?thesis.
        qed
        have f2: cos (1 / (x-nc c)) = −cos c
        proof −
          have cos (1 / (x-nc c)) = cos (pi + c + 2*pi*real n)
            by (simp add: inverse-eq-divide x-nc-eqs)
          also have ... = cos (pi + c)
```

by (*metis Groups.mult-ac*(*2*) *id-apply of-real-eq-id cos.plus-of-nat*)
also have ... = $-$*cos c*
by *simp*
finally show *?thesis*.
qed

have *deriv* (*deriv f*) (*x-nc c*) = (*12*$*$(*x-nc c*)^2 $-$ *1*)$*$ *cos* (*1* / (*x-nc c*)) + *6*$*$(*x-nc c*) $*$ *sin* (*1* / (*x-nc c*)) + *24*$*$(*x-nc c*)^2
using *deriv-f x-nc-pos* **by** *auto*
also have ... = (*1*$-$ *12* $*$ (*x-nc c*)^2) $*$ *cos c* $-$ *6* $*$ (*x-nc c*) $*$ *sin c* + *24* $*$ (*x-nc c*)^2
by (*smt* (*verit*) *f1 f2 minus-mult-commute more-arith-simps*(*8*))
finally show *?thesis*.
qed
have *snd-deriv-bound*: *deriv* (*deriv f*) (*x-nc c*) $\geq$ (*1* $-$ *12* $*$ (*x-nc c*)^2 $-$ *6* $*$ (*x-nc c*)) $*$ (*sqrt 2* / *2*)
proof $-$
have *deriv* (*deriv f*) (*x-nc c*) $\geq$ (*1*$-$ *12* $*$ (*x-nc c*)^2) $*$ *cos c* $-$ *6* $*$ (*x-nc c*) $*$ (*sqrt*(*2*)/*2*) + *24* $*$ (*x-nc c*)^2
using *snd-deriv-at-x-nc sin-bound x-nc-pos* **by** *auto*
also have ... $\geq$ (*1* $-$ *12* $*$ (*x-nc c*)^2 $-$ *6* $*$ (*x-nc c*)) $*$ (*sqrt 2* / *2*)
by (*smt* (*verit*, *best*) *calculation cos-bound divide-pos-pos one-power2 real-le-rsqrt right-diff-distrib′ sum-le-prod1 vector-space-over-itself.scale-left-diff-distrib zero-compare-simps*(*12*))
then show *?thesis*.
qed

show *0* < *deriv* (*deriv f*) (*x-nc c*)
proof $-$
obtain *h* :: *real* ⇒ *real* **where** *h-def*: *h* = (λ*x*. $-$ *12* $*$ *x*^{2} $-$ *6* $*$ *x* + *1*)
by *auto*
have *diff-h*: ∀ *x*. *h field-differentiable at x*
unfolding *h-def*
proof *clarify*
fix *x*::*real*
have *d1*: (λ*x*. $-$ *12* $*$ *x*^{2}) *field-differentiable at x*
by(*rule field-differentiable-mult*, *simp*, *simp add*: *field-differentiable-power*)
have *d2*: (λ*x*. $-$ *6* $*$ *x*) *field-differentiable at x*
by(*rule field-differentiable-mult*, *simp*, *simp add*: *field-differentiable-power*)
from *d1 d2* **show** (λ*x*. $-$ *12* $*$ *x*^{2} $-$ *6* $*$ *x* + *1*) *field-differentiable at x*
by(*subst field-differentiable-add*, *simp add*: *Derivative.field-differentiable-diff*, *simp-all*)
qed

have *h-roots*: ∀ *x*. *h x* = *0* ⟷ *x* = ($-$*6* + *sqrt 84*) / *24* ∨ *x* = ($-$*6* $-$ *sqrt 84*) / *24*
proof(*clarify*)

```
          fix x ::real
          have roots: (12 * x^2 + 6 * x + (−1) = 0) = (x = (− 6 + sqrt
(6^2 − 4 * 12 * (−1))) / (2 * 12) ∨ x = (− 6 − sqrt (6^2 − 4 * 12 * (−1))) /
(2 * 12))
                  using discrim-def by(subst discriminant-iff, eval, force)

          then show (h x = 0) = (x = (− 6 + sqrt 84) / 24 ∨ x = (− 6
− sqrt 84) / 24)
              using h-def by auto
        qed

        have right-root-positive: (− 6 + sqrt 84) / 24 > 0
        proof −
          have − 6 + sqrt 84 > − 6 + sqrt 64
            by (smt (verit) real-sqrt-less-mono)
          then show (− 6 + sqrt 84) / 24 > 0
            by simp
        qed
        then have left-root-neg: 0 > (− 6 − sqrt 84) / 24
          by fastforce
        have h-pos-on-interval: ∀ x ∈ {0..<(−6 + sqrt 84) / 24}. h x > 0
        proof(rule ccontr)
          assume ¬ (∀x∈{0..<(− 6 + sqrt 84) / 24}. 0 < h x)
          then obtain z where z-def: z ∈ {0..<(− 6 + sqrt 84) / 24} ∧
0 ≥ h z
            by fastforce
          then have z-not-root: z ≠ (− 6 + sqrt 84) / 24 ∧ z ≠ (− 6 −
sqrt 84) / 24
            using z-def by force
          show False
          proof(cases h z = 0)
            show h z = 0 ⟹ False
              using h-roots z-not-root by blast
          next
            assume h z ≠ 0
            then have hz-neg: h z < 0
              using z-def by auto
            have ∃x. 0 ≤ x ∧ x ≤ z ∧ h x = 0
            proof(rule IVT2')
              show h z ≤ 0
                by (simp add: z-def)
              show 0 ≤ h 0
                by (simp add: h-def)
              show 0 ≤ z
                using z-def by fastforce
              show continuous-on {0..z} h
                      by (meson continuous-at-imp-continuous-on diff-h
field-differentiable-imp-continuous-at)
            qed
```

```
    then show False
      by (metis atLeastLessThan-iff h-roots left-root-neg not-less z-def)
  qed
qed

have (−6 + sqrt 84) / 24 > 1 / (3 ∗ pi)
proof −
  have i1: 64 / pi^2 < 8
  proof −
    have pi∗pi > 3∗3
by (meson pi-gt3 mult-strict-mono pi-gt-zero verit-comp-simplify(7))
    then have pi^2 > 9
      by (simp add: power2-eq-square)
    then have 64/pi^2 < 64/8
      by (smt (verit) frac-less2)
    also have ... = 8
      by eval
    finally show ?thesis.
  qed

  have i2: 96/pi < 32
  proof −
    have 96/pi < 96/3
      by (meson frac-less2 order.refl pi-gt3 verit-comp-simplify(19))
    also have ... = 32
      by eval
    finally show ?thesis.
  qed

  have (8/pi + 6)^2 < 84
  proof −
    have ((8::real)/pi + 6)^2 = (8/pi)^2 + 2∗(8/pi)∗6 + 6^2
      by (simp add: power2-sum)
    also have ... = 8^2/pi^2 + 2∗(8/pi)∗6 + 6^2
      by (simp add: power-divide)
    also have ... = 64/pi^2 +96/pi + 36
      by simp
    also have ... < 84
      using i1 i2 by linarith
    finally show ?thesis.
  qed
  then have lt-sqrt84: 8/pi + 6 < sqrt(84)
    using real-less-rsqrt by presburger
  have lt-3pi-sqrt84: 24 + 18∗pi < 3 ∗ pi ∗ sqrt (84)
  proof −
    have 24 + 18∗pi = 3∗8 + 3∗6∗pi
      by simp
    also have ... = 3∗pi∗(8/pi) + 3∗pi∗6
      by simp
```

```
                also have ... = 3*pi*((8/pi)+6)
                  by (simp add: distrib-left)
                also have ... < 3 * pi * sqrt(84)
                  by (simp add: lt-sqrt84)
                finally show ?thesis.
              qed
              have (−6+sqrt(84))*(3*pi) > 24
              proof −
                have (−6+sqrt(84))*(3*pi) = −6*(3*pi) + sqrt(84)*(3*pi)
                  by (meson ring-class.ring-distribs(2))
                also have ... = −18*pi + 3*pi * sqrt(84)
                  by simp
                also have ... > 24
                  using lt-3pi-sqrt84 by auto
                finally show ?thesis.
              qed
              then have (−6+sqrt(84))*(3*pi) / 24 > 1
                by simp
              then show (−6+sqrt(84)) / 24 > 1 / (3*pi)
                       by (metis pi-gt-zero pos-divide-less-eq times-divide-eq-left
zero-compare-simps(6) zero-less-numeral)
            qed
            then have x-nc c < (−6+sqrt(84)) / 24
              by (metis dual-order.strict-trans2 inverse-eq-divide x-nc-bound)
            then have h-x-nc-pos: h (x-nc c) > 0
              by (simp add: h-pos-on-interval less-eq-real-def x-nc-pos)

            have deriv (deriv f) (x-nc c) ≥ (sqrt(2)/2) * h (x-nc c)
                 by (metis Groups.mult-ac(2) snd-deriv-bound diff-add-eq h-def
mult-minus-left uminus-add-conv-diff)
            then show ?thesis
          by (smt (verit) h-x-nc-pos half-gt-zero-iff mult-pos-pos real-sqrt-gt-0-iff)
          qed
        qed
      qed

       then show 0 < 6 * y * sin (1 / y) + (12 * y^2 − 1) * cos (1 / y) +
24 * y^2
         by (smt (verit, best) deriv-f minimizer-dom')
     qed
       then show ∃y∈{left-seq n..right-seq n}. y^2 * sin (1 / y) + 4 * y ^ 3 *
cos (1 / y) + 8 * y ^ 3 = 0 ∧ 0 < 6 * y * sin (1 / y) + (12 * y^2 − 1) * cos (1
/ y) + 24 * y^2
         using min-n-def by blast
   qed
  qed
qed

have optimality-conditions: ∀n. n ≠ 0 ⟶ (∃ y ∈ {left-seq n .. right-seq n}.
```

(*deriv f*) *y* = *0* ∧ *deriv* (*deriv f*) *y* > *0*)
  proof *clarify*
    fix *n*::*nat*
    assume *0* < *n*
    then obtain *min-n* **where** *min-n-def*: *min-n* ∈{*left-seq n*..*right-seq n*}
      ∧ $min\text{-}n^2$ ∗ *sin* (*1* / *min-n*) + *4* ∗ *min-n* ^ *3* ∗ *cos* (*1* / *min-n*) + *8* ∗ *min-n* ^ *3* = *0*
      ∧ *0* < *6* ∗ *min-n* ∗ *sin* (*1* / *min-n*) + (*12* ∗ $min\text{-}n^2$ − *1*) ∗ *cos* (*1* / *min-n*) + *24* ∗ $min\text{-}n^2$
      using *first-and-second-order-conditions bot-nat-0.not-eq-extremum* **by** *presburger*
    have *fst-order-condition*: *deriv f min-n* = *0*
      using *deriv-f min-n-def* **by** *presburger*
    have *snd-order-condition*: *deriv* (*deriv f*) *min-n* > *0*
      using *deriv-f min-n-def* **by** *fastforce*
    show ∃ *y*∈{*left-seq n*..*right-seq n*}. *deriv f y* = *0* ∧ *0* < *deriv* (*deriv f*) *y*
      using *fst-order-condition min-n-def snd-order-condition* **by** *blast*
  qed

  have *seq-of-local-minizers-exists*: ∀ *n*. *n* ≠ *0* ⟶ (∃ *y* ∈ {*left-seq n* .. *right-seq n*}. *local-minimizer f y*)
  proof(*clarify*)
    fix *n*::*nat*
    assume *n-pos*: *0* < *n*
    then obtain *y* **where** *y-def*: (*y* ∈ {*left-seq n* .. *right-seq n*} ∧ (*deriv f*) *y* = *0* ∧ *deriv* (*deriv f*) *y* > *0*)
      using *gr-implies-not0 optimality-conditions* **by** *presburger*
    have *right-seq-def2*: *right-seq n* = *inverse* (*pi* + *2* ∗ *real n* ∗ *pi*)
     by (*metis id-apply less-not-refl n-pos of-nat-eq-id of-nat-in-Nats right-seq-def*)

    have *y* ∈ {*left-seq n*..*right-seq n*} ∧ *local-minimizer f y*
    proof(*subst second-derivative-test*[**where** *a* = *left-seq n*, **where** *b* = *right-seq n*])
      show *proper-interval*: *left-seq n* < *right-seq n*
      by (*metis* (*no-types*) *id-apply n-pos of-nat-eq-id of-nat-in-Nats rel-simps*(*70*) *zero-lt-left-seq-lt-right-seq-both-pos*)
      show *C-k-on 2 f* {*left-seq n* <..<*right-seq n*}
      proof(*rule C-k-on-subset*[**where** *U* = {*0*<..<(*1*::*real*)}])
        show *f-contin-diff-on-right*: *C-k-on 2 f* {*0*<..<(*1*::*real*)}
        proof(*rule C2-on-open-U-def2*)
          show *open* {*0*<..<(*1*::*real*)}
           using *lemma-interval* **by**(*subst open-dist*, *subst dist-real-def*, *simp add*: *abs-minus-commute lemma-interval-lt*)
          show *f differentiable-on* {*0*<..<(*1*::*real*)}
            by (*meson deriv-f differentiable-on-subset top.extremum*)
          show *deriv f differentiable-on* {*0*<..<(*1*::*real*)}
            by (*meson deriv-f differentiable-on-subset top.extremum*)
          show *continuous-on* {*0*<..<(*1*::*real*)} (*deriv* (*deriv f*))
          proof −

```
            have ∀x ∈ {0<..<1}. deriv (deriv f) x = 6*x * sin(1/x) + (12*x^2 − 1)*cos(1/x) + 24*x^2
              by (simp add: deriv-f)
            moreover have continuous-on {0<..<(1::real)} (λx. 6*x * sin(1/x) + (12*x^2 − 1)*cos(1/x) + 24*x^2)
            proof −
              have {0<..<(1::real)} ⊆ {x :: real. x>0}
                by fastforce
              moreover have continuous-on {x :: real. x>0} (λx. 6*x * sin(1/x) + (12*x^2 − 1)*cos(1/x) + 24*x^2)
                by (auto intro!: continuous-intros)
              ultimately show ?thesis
                using  continuous-on-subset by blast
            qed
            ultimately show continuous-on {0<..<1} (deriv (deriv f))
              using continuous-on-cong by fastforce
          qed
        qed

          show open {left-seq n<..<right-seq n} ∧ {left-seq n<..<right-seq n} ⊂ {0<..<1}
        proof −
          have 0 < left-seq n
              by (metis id-apply n-pos of-nat-eq-id of-nat-in-Nats order.asym zero-lt-left-seq-lt-right-seq-both-pos)
          moreover have right-seq n < 1
            using right-seq-def2
          by (smt (verit, ccfv-SIG) inverse-1 inverse-le-imp-le mult-sign-intros(5) n-pos of-nat-0-less-iff pi-gt3)
          ultimately show ?thesis
            using proper-interval by fastforce
        qed
      qed

      show y ∈ {left-seq n<..<right-seq n}
      proof −
        have y ∈ {left-seq n..right-seq n}
          using y-def by blast
        moreover have y ≠ left-seq n
        proof(rule ccontr)
          assume ¬ y ≠ left-seq n
          then have deriv f y ≠ 0
            using deriv-f first-and-second-order-conditions
            by (metis n-pos rel-simps(70) y-def)
          then show False
            by (simp add: y-def)
        qed
        moreover have y ≠ right-seq n
        proof(rule ccontr)
```

assume ¬ *y* ≠ *right-seq n*
then have *deriv f y* ≠ *0*
using *deriv-f first-and-second-order-conditions*
by (*metis n-pos rel-simps*(*70*) *y-def*)
then show *False*
by (*simp add*: *y-def*)
qed
ultimately show *y* ∈ {*left-seq n*<..<*right-seq n*}
by *auto*
qed
show *deriv f y* = *0* **and** *0* < *deriv* (*deriv f*) *y*
using *y-def* **by** *auto*
show *y* ∈ {*left-seq n*..*right-seq n*} ∧ *True*
using *y-def* **by** *blast*
qed
then show ∃ *y*∈{*left-seq n*..*right-seq n*}. *local-minimizer f y*
by *blast*
qed
show ∃ *x-seq*. (∀ *n*. *local-minimizer f* (*x-seq n*) ∧ *x-seq n* ≠ *0*) ∧ *x-seq* ⟶ *0*
proof −
define *x-seq* **where**
x-seq n = (*SOME y*. *y* ∈ {*left-seq* (*n*+*1*)..*right-seq* (*n*+*1*)} ∧ *local-minimizer f y*) **for** *n*
have *x-seq-prop*: ∀ *n*. *x-seq n* ∈ {*left-seq* (*n*+*1*)..*right-seq* (*n*+*1*)} ∧ *local-minimizer f* (*x-seq n*)
by (*metis* (*mono-tags*, *lifting*) *seq-of-local-minizers-exists someI-ex verit-eq-simplify*(*7*) *x-seq-def zero-eq-add-iff-both-eq-0*)

from *x-seq-prop* **have** *bounds*: ∀ *n*. *left-seq* (*n*+*1*) ≤ *x-seq n* ∧ *x-seq n* ≤ *right-seq* (*n*+*1*)
by *auto*

have *nonzero*: ∀ *n*. *x-seq n* ≠ *0*
by (*metis Suc-eq-plus1 bounds id-apply nat.simps*(*3*) *not-less of-nat-eq-id of-nat-in-Nats zero-lt-left-seq-lt-right-seq-both-pos*)

have *left-seq-converges*: *left-seq* ⟶ *0*
proof (*rule LIMSEQ-I*)
fix *ε* :: *real*
assume *ε-pos*: *0* < *ε*
then obtain *N* **where** *N-def*: (*N*::*nat*) = ⌈*1* / (*2* ∗ *pi* ∗ *ε*)⌉ + *1*
by (*metis add-mono-thms-linordered-field*(*5*) *arithmetic-simps*(*50*) *divide-pos-pos mult-sign-intros*(*5*) *pi-gt-zero pos-int-cases semiring-norm*(*172*) *zero-less-ceiling zero-less-numeral*)
then have *N-gt-0*: *N* > *0*
by (*smt* (*verit*) *ε-pos divide-pos-pos gr0I int-ops*(*1*) *m2pi-less-pi mult-sign-intros*(*5*) *zero-less-ceiling*)

```
have ∀ n ≥ N. |left-seq n| < ε
proof clarify
  fix n :: nat
  assume n-ge: n ≥ N
  have left-seq-eqs: left-seq n = inverse ((5 ∗ pi / 4) + 2 ∗ pi ∗ real n)
    unfolding left-seq-def
   by (metis N-gt-0 id-apply left-seq-def linorder-not-less mult.commute n-ge
of-nat-eq-id of-nat-in-Nats vector-space-over-itself.scale-scale)
  show |left-seq n| < ε
  proof −
    have |left-seq n| = 1 / ((5 ∗ pi / 4) + 2 ∗ pi ∗ real n)
      by (simp add: left-seq-eqs inverse-eq-divide)
    also have ... ≤  1 / (2 ∗ pi ∗ real N)
       by (smt (verit, best) N-gt-0 divide-nonneg-nonneg frac-le m2pi-less-pi
mult-left-mono mult-sign-intros(5) n-ge of-nat-0-less-iff of-nat-mono)
    also have ... <  1 / (2 ∗ pi ∗ (⌈1 / (2 ∗ pi ∗ ε)⌉))
     by (smt (verit, best) N-def ε-pos ceiling-correct divide-pos-pos frac-less2
m2pi-less-pi mult-less-cancel-left-pos mult-sign-intros(5) of-int-1 of-int-add of-int-of-nat-eq)
    also have ... ≤ 1 / (2 ∗ pi ∗ (1 / (2 ∗ pi ∗ ε)))
        by (smt (verit, ccfv-SIG) ε-pos ceiling-correct frac-le mult-left-mono
mult-sign-intros(5) pi-gt-zero zero-less-divide-iff)
    also have ... = ε
      by simp
    finally show ?thesis.
  qed
qed
then show ∃ N. ∀ n≥N. ‖left-seq n − 0‖ < ε
  by (metis cancel-comm-monoid-add-class.diff-zero real-norm-def)
qed
have right-seq-converges: right-seq ⟶ 0
proof (rule LIMSEQ-I)
  fix ε::real
  assume eps-pos: 0 < ε
  then obtain N where N-def: (N::nat) = ⌈1 / (2 ∗ pi ∗ ε)⌉ + 1
      by (metis add-mono-thms-linordered-field(5) arithmetic-simps(50) di-
vide-pos-pos
            mult-sign-intros(5) pi-gt-zero pos-int-cases semiring-norm(172)
            zero-less-ceiling zero-less-numeral)
  hence N-gt-0: N > 0
        by (smt (verit) eps-pos divide-pos-pos gr0I int-ops(1) m2pi-less-pi
mult-sign-intros(5)
            zero-less-ceiling)

  have ∀ n≥N. |right-seq n| < ε
  proof clarify
    fix n :: nat
    assume n-ge: n ≥ N
    have right-seq-eqs: right-seq n = inverse (pi + 2 ∗ pi ∗ real n)
      unfolding right-seq-def
```

by (*metis N-gt-0 id-apply linorder-not-less mult.commute mult.left-commute n-ge of-nat-eq-id of-nat-in-Nats right-seq-def*)
show $|$*right-seq n*$| < \varepsilon$
proof −
have $|$*right-seq n*$| = 1 / (pi + 2 * pi * real\ n)$
by (*simp add*: *right-seq-eqs inverse-eq-divide*)
also have ... $\leq 1 / (2 * pi * real\ N)$
by (*smt* (*verit*, *best*) *N-gt-0 divide-nonneg-nonneg frac-le m2pi-less-pi mult-left-mono mult-sign-intros*(*5*) *n-ge of-nat-0-less-iff of-nat-mono*)
also have ... $< 1 / (2 * pi * (\lceil 1 / (2 * pi * \varepsilon)\rceil))$
by (*smt* (*verit*, *best*) *N-def eps-pos ceiling-correct divide-pos-pos frac-less2 m2pi-less-pi mult-less-cancel-left-pos mult-sign-intros*(*5*) *of-int-1 of-int-add of-int-of-nat-eq*)
also have ... $\leq 1 / (2 * pi * (1 / (2 * pi * \varepsilon)))$
by (*smt* (*verit*, *ccfv-SIG*) *eps-pos ceiling-correct frac-le mult-left-mono mult-sign-intros*(*5*) *pi-gt-zero zero-less-divide-iff*)
also have ... $= \varepsilon$
by *simp*
finally show *?thesis*
by *blast*
qed
qed
then show $\exists$ *no*. $\forall$ *n*$\geq$*no*. $\|$*right-seq n* $- 0\| < \varepsilon$
by (*metis cancel-comm-monoid-add-class.diff-zero real-norm-def*)
qed
have *x-seq-converges*: *x-seq* $\longrightarrow 0$
proof (*rule LIMSEQ-I*)
fix ε :: *real*
assume ε*-pos*: $0 < \varepsilon$

obtain N_0 **where** N_0: $\forall n \geq N_0$. $\|$*left-seq* $(n+1) - 0\| < \varepsilon$
using *left-seq-converges*
by (*meson LIMSEQ-iff* ε*-pos le-diff-conv*)

obtain N_1 **where** N_1: $\forall n \geq N_1$. $\|$*right-seq* $(n+1) - 0\| < \varepsilon$
using *right-seq-converges*
by (*meson LIMSEQ-iff* ε*-pos le-diff-conv*)

obtain N **where** $N = max\ N_0\ N_1$
by *simp*
hence *N-def*: $N \geq N_0 \wedge N \geq N_1$
by *simp*

show $\exists N. \forall n \geq N.$ $\|$*x-seq n* $- 0\| < \varepsilon$
proof (*intro exI*[**where** $x=N$] *exI allI impI*)
fix *n* :: *nat*
assume *N-leq-n*: $N \leq n$

from *bounds* **have** *left-seq* $(n+1) \leq$ *x-seq* $n \wedge$ *x-seq* $n \leq$ *right-seq* $(n+1)$
by *auto*
hence $\|$*x-seq* $n\| \leq \|$*left-seq* $(n+1)\| \vee \|$*x-seq* $n\| \leq \|$*right-seq* $(n+1)\|$
by *force*
moreover have $\|$*left-seq* $(n+1)\| < \varepsilon \wedge \|$*right-seq* $(n+1)\| < \varepsilon$
using N_0 N_1 *N-leq-n N-def* **by** *auto*
ultimately show $\|$*x-seq* $n - 0\| < \varepsilon$
by *auto*
qed
qed
then show *?thesis*
using *nonzero x-seq-prop* **by** *blast*
qed
qed
then show *?thesis*
using *zero-min f-cont not-isolated-minimizer-def strict-local-minimizer-at-0* **by** *auto*
qed

end
theory *Unconstrained-Optimization*
imports *Auxilary-Facts*
Minimizers-Definition
First-Order-Conditions
Second-Derivative-Test
Cont-Nonisolated-Strict-Local-Minimizer-Exists
begin

end

References

[1] T. Nipkow, M. Wenzel, and L. C. Paulson. *Isabelle/HOL: A Proof Assistant for Higher-Order Logic*, volume 2283 of *Lecture Notes in Computer Science*. Springer, Berlin, Heidelberg, 2002.

[2] J. Nocedal and S. J. Wright. *Numerical Optimization*. Springer Series in Operations Research and Financial Engineering. Springer, New York, second edition, 2006.