

Uncertainty Principle

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Abstract

This is a formal proof of the uncertainty principle known from quantum mechanics. It is based upon work on complex vector spaces contained in the QHLProver session[1]. The formalization follows the proof outlined in the book "Quantum computation and quantum information" by Nielsen and Chuang[2].

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```

theory Uncertainty-Principle
  imports QHLProver.Complex-Matrix
begin

```

1 Setup

```

abbreviation bra-ket ( $\langle\langle\cdot|\cdot\rangle\rangle$ )
  where  $\langle u|v\rangle \equiv \text{inner-prod } u \ v$ 

```

Fix an n-dimensional normalized quantum state ψ .

```

locale quantum-state =
  fixes  $n:: \text{nat}$ 
  and  $\psi:: \text{complex Matrix.vec}$ 
  assumes  $\text{dim}[simp]: \psi \in \text{carrier-vec } n$ 
  and  $\text{normalized}[simp]: \langle\psi|\psi\rangle = 1$ 

```

```

begin

```

Observables on ψ are hermitian matrices of appropriate dimensions.

```

abbreviation observable::  $\text{complex Matrix.mat} \Rightarrow \text{bool}$  where
  observable  $A \equiv A \in \text{carrier-mat } n \ n \wedge \text{hermitian } A$ 

```

The mean value of an observable A is defined as $\langle\psi|A|\psi\rangle$. It is useful to have a scalar matrix of appropriate dimension containing this value. On paper, this is usually implicit.

```

abbreviation mean-mat ::  $\text{complex Matrix.mat} \Rightarrow \text{complex Matrix.mat}$  ( $\langle\langle\cdot\rangle\rangle$ )
  where  $\langle\langle A\rangle\rangle \equiv \langle\psi| A *_v \psi\rangle \cdot_m 1_m \ n$ 

```

The standard deviation of an observable $A = \sqrt{\langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2}$. Since the standard deviation is real (see lemma `std-dev-real`), we can define it as being of type `real` using `norm`. This simultaneously restricts it to positive values. (powers of two are expanded for simplicity)

```

abbreviation std-dev ::  $\text{complex Matrix.mat} \Rightarrow \text{real}$  ( $\Delta$ )
  where  $\Delta \ A \equiv \text{norm } (\text{csqrt } (\langle\psi| (A * A *_v \psi)\rangle - \langle\psi| A *_v \psi\rangle * \langle\psi| A *_v \psi\rangle))$ 

```

```

end

```

```

abbreviation commutator ::  $\text{complex Matrix.mat} \Rightarrow \text{complex Matrix.mat} \Rightarrow \text{complex Matrix.mat}$  ( $\llbracket\cdot, \cdot\rrbracket$ )
  where  $\text{commutator } A \ B \equiv (A * B - B * A)$ 

```

```

abbreviation anticommutator ::  $\text{complex Matrix.mat} \Rightarrow \text{complex Matrix.mat} \Rightarrow \text{complex Matrix.mat}$  ( $\{\!\{ \cdot, \cdot \}\!\}$ )
  where  $\text{anticommutator } A \ B \equiv (A * B + B * A)$ 

```

2 Auxiliary Lemmas

```

lemma inner-prod-distrib-add-mat:

```

fixes $u\ v :: \text{complex vec}$
assumes
 $u \in \text{carrier-vec } n$
 $v \in \text{carrier-vec } m$
 $A \in \text{carrier-mat } n\ m$
 $B \in \text{carrier-mat } n\ m$
shows $\langle u | (A + B) *_v v \rangle = \langle u | A *_v v \rangle + \langle u | B *_v v \rangle$
 $\langle \text{proof} \rangle$

lemma *inner-prod-distrib-minus-mat*:
fixes $u\ v :: \text{complex vec}$
assumes
 $u \in \text{carrier-vec } n$
 $v \in \text{carrier-vec } m$
 $A \in \text{carrier-mat } n\ m$
 $B \in \text{carrier-mat } n\ m$
shows $\langle u | (A - B) *_v v \rangle = \langle u | A *_v v \rangle - \langle u | B *_v v \rangle$
 $\langle \text{proof} \rangle$

Proving the usual Cauchy-Schwarz inequality using its formulation for complex vector spaces.

lemma *Cauchy-Schwarz*:
assumes $v \in \text{carrier-vec } n\ u \in \text{carrier-vec } n$
shows $\text{norm } (\langle u | v \rangle)^2 \leq \text{Re } (\langle u | u \rangle * \langle v | v \rangle)$
 $\langle \text{proof} \rangle$

context *quantum-state*
begin

Show that the the standard deviation yields a real value. This justifies our definition in terms of the norm.

lemma *std-dev-real*:
assumes *observable* A
shows $\text{csqrt } (\langle \psi | (A * A *_v \psi) \rangle - \langle \psi | A *_v \psi \rangle * \langle \psi | A *_v \psi \rangle) \in \mathbf{R}$
 $\langle \text{proof} \rangle$

This is an alternative way of formulating the standard deviation.

lemma *std-dev-alt*:
assumes *observable* A
shows $\Delta A = \text{norm } (\text{csqrt } (\langle \psi | (A - \langle A \rangle) * (A - \langle A \rangle) *_v \psi \rangle))$
 $\langle \text{proof} \rangle$

3 Main Proof

Note that when swapping two observables inside an inner product, it is the same as conjugating the result.

lemma *cnj-observables*:
assumes *observable* A *observable* B

shows *cnj* $\langle \psi | (A * B) *_v \psi \rangle = \langle \psi | (B * A) *_v \psi \rangle$
 $\langle proof \rangle$

With the above lemma we can make two observations about the behaviour of the commutator/ anticommutator inside an inner product.

lemma *commutator-im*:

assumes *observable A observable B*
shows $\langle \psi | \llbracket A, B \rrbracket *_v \psi \rangle = 2 * i * Im(\langle \psi | A * B *_v \psi \rangle)$
 $\langle proof \rangle$

lemma *anticommutator-re*:

assumes *observable A observable B*
shows $\langle \psi | \{\!| A, B \!\!\} *_v \psi \rangle = 2 * Re(\langle \psi | A * B *_v \psi \rangle)$
 $\langle proof \rangle$

This intermediate step already looks similar to the uncertainty principle. The LHS will play the role of the lower bound in the uncertainty principle. The RHS will turn into the standard deviation of our observables under a certain substitution.

lemma *commutator-ineq*:

assumes *observable A observable B*
shows $(norm \langle \psi | \llbracket A, B \rrbracket *_v \psi \rangle)^2 \leq 4 * Re(\langle \psi | A * A *_v \psi \rangle * \langle \psi | B * B *_v \psi \rangle)$
 $\langle proof \rangle$

This is part of the substitution we need in the final proof. This lemma shows that the commutator simplifies nicely under that substitution.

lemma *commutator-sub-mean[simp]*:

assumes $A \in carrier_mat\ n\ n\ B \in carrier_mat\ n\ n$
shows $\llbracket A - \langle A \rangle, B - \langle B \rangle \rrbracket = \llbracket A, B \rrbracket$
 $\langle proof \rangle$

theorem *uncertainty-principle*:

assumes *observable C observable D*
shows $\Delta C * \Delta D \geq norm \langle \psi | \llbracket C, D \rrbracket *_v \psi \rangle / 2$
 $\langle proof \rangle$

end

end

References

- [1] J. Liu, B. Zhan, S. Wang, S. Ying, T. Liu, Y. Li, M. Ying, and N. Zhan. Quantum hoare logic. *Archive of Formal Proofs*, March 2019. <https://isa-afp.org/entries/QHLProver.html>, Formal proof development.

- [2] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2010.