Isabelle/UTP: Mechanised Theory Engineering for Unifying Theories of Programming

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Abstract

Isabelle/UTP is a mechanised theory engineering toolkit based on Hoare and He's Unifying Theories of Programming (UTP). UTP enables the creation of denotational, algebraic, and operational semantics for different programming languages using an alphabetised relational calculus. We provide a semantic embedding of the alphabetised relational calculus in Isabelle/HOL, including new type definitions, relational constructors, automated proof tactics, and accompanying algebraic laws. Isabelle/UTP can be used to both capture laws of programming for different languages, and put these fundamental theorems to work in the creation of associated verification tools, using calculi like Hoare logics. This document describes the relational core of the UTP in Isabelle/HOL.

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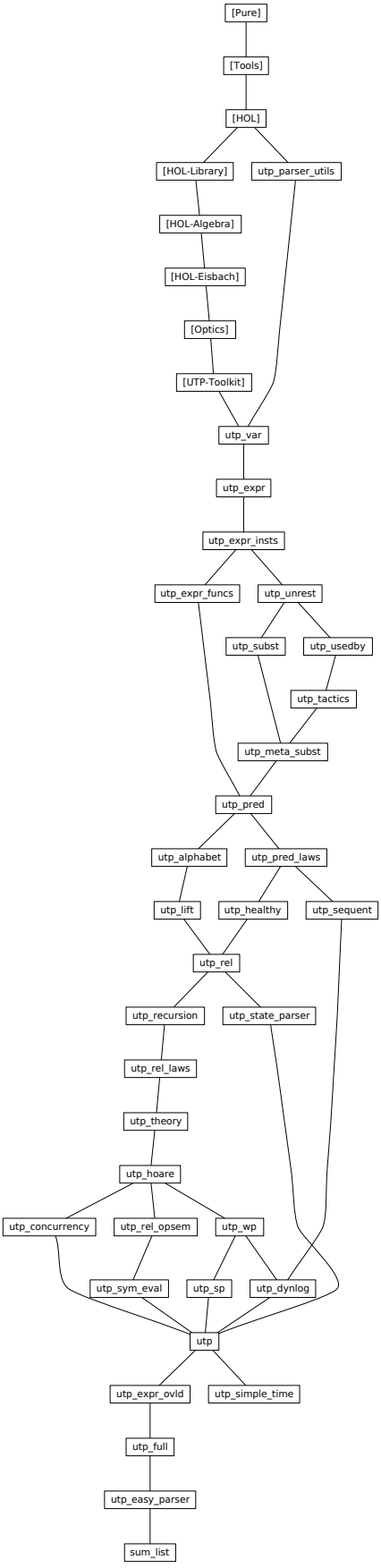
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1 Introduction

This document contains the description of our mechanisation of Hoare and He's *Unifying Theories of Programming* [22, 7] (UTP) in Isabelle/HOL. UTP uses the "programs-as-predicates" approach, pioneered by Hehner [20, 18, 19], to encode denotational semantics and facilitate reasoning about programs. It uses the alphabetised relational calculus, which combines predicate calculus and relation algebra, to denote programs as relations between initial variables (x) and their subsequent values (x'). Isabelle/UTP[1] [16, 28, 15] semantically embeds this relational calculus into Isabelle/HOL, which enables application of the latter's proof facilities to program verification. For an introduction to UTP, we recommend two tutorials [6, 7], and also the UTP book [22].

The Isabelle/UTP core mechanises most of definitions and theorems from chapters 1, 2, 4, and 7 of [22], and some material contained in chapters 5 and 10. This essentially amounts to alphabetised predicate calculus, its core laws, the UTP theory infrastructure, and also parallel-by-merge [22, chapter 5], which adds concurrency primitives. The Isabelle/UTP core does not contain the theory of designs [6] and CSP [7], which are both represented in their own theory developments.

A large part of the mechanisation, however, is foundations that enable these core UTP theories. In particular, Isabelle/UTP builds on our implementation of lenses [16, 14], which gives a formal semantics to state spaces and variables. This, in turn, builds on a previous version of Isabelle/UTP [9, 10], which provided a shallow embedding of UTP by using Isabelle record types to represent alphabets. We follow this approach and, additionally, use the lens laws [11, 16] to characterise well-behaved variables. We also add meta-logical infrastructure for dealing with free variables and substitution. All this, we believe, adds an additional layer rigour to the UTP.

The alphabets-as-types approach does impose a number of theoretical limitations. For example, alphabets can only be extended when an injection into a larger state-space type can be exhibited. It is therefore not possible to arbitrarily augment an alphabet with additional variables, but new types must be created to do this. This is largely because as in previous work [9, 10], we actually encode state spaces rather than alphabets, the latter being implicit. Namely, a relation is typed by the state space type that it manipulates, and the alphabet is represented by collection of lenses into this state space. This aspect of our mechanisation is actually much closer to the relational program model in Back's refinement calculus [3].

The pay-off is that the Isabelle/HOL type checker can be directly applied to relational constructions, which makes proof much more automated and efficient. Moreover, our use of lenses mitigates the limitations by providing meta-logical style operators, such as equality on variables, and alphabet membership [16]. Isabelle/UTP can therefore directly harness proof automation from Isabelle/HOL, which allows its use in building efficient verification tools [13, 12]. For a detailed discussion of semantic embedding approaches, please see [28].

In addition to formalising variables, we also make a number of generalisations to UTP laws. Notably, our lens-based representation of state leads us to adopt Back's approach to both assignment and local variables [3]. Assignment becomes a point-free operator that acts on state-space update functions, which provides a rich set of algebraic theorems. Local variables are represented using stacks, unlike in the UTP book where they utilise alphabet extension.

[1] Isabelle/UTP website: https://www.cs.york.ac.uk/circus/isabelle-utp/

We give a summary of the main contributions within the Isabelle/UTP core, which can all be seen in the table of contents.

1. Formalisation of variables and state-spaces using lenses [16];
2. an expression model, together with lifted operators from HOL;
3. the meta-logical operators of unrestriction, used-by, substitution, alphabet extrusion, and alphabet restriction;
4. the alphabetised predicate calculus and associated algebraic laws;
5. the alphabetised relational calculus and associated algebraic laws;
6. proof tactics for the above based on interpretation [23];
7. a formalisation of UTP theories using locales [4] and building on HOL-Algebra [5];
8. Hoare logic [21] and dynamic logic [17];
9. weakest precondition and strongest postcondition calculi [8];
10. concurrent programming with parallel-by-merge;
11. relational operational semantics.

2 UTP Variables

theory *utp-var*
imports
UTP−Toolkit.utp-toolkit
utp-parser-utils
begin

In this first UTP theory we set up variables, which are are built on lenses [11, 16]. A large part of this theory is setting up the parser for UTP variable syntax.

2.1 Initial syntax setup

We will overload the square order relation with refinement and also the lattice operators so we will turn off these notations.

purge-notation
Order.le (**infixl** ‹⊑₁› *50*) **and**
Lattice.sup (‹⨆₁-› [*90*] *90*) **and**
Lattice.inf (‹⨅₁-› [*90*] *90*) **and**
Lattice.join (**infixl** ‹⊔₁› *65*) **and**
Lattice.meet (**infixl** ‹⊓₁› *70*) **and**
Set.member (‹'(:')›) **and**
Set.member (‹(‹*notation*=‹*infix* :››-/ : -)› [*51*, *51*] *50*) **and**
disj (**infixr** ‹|› *30*) **and**
conj (**infixr** ‹&› *35*)

declare *fst-vwb-lens* [*simp*]
declare *snd-vwb-lens* [*simp*]
declare *comp-vwb-lens* [*simp*]

declare *lens-indep-left-ext* [*simp*]
declare *lens-indep-right-ext* [*simp*]
declare *lens-comp-quotient* [*simp*]
declare *plus-lens-distr* [*THEN sym*, *simp*]

2.2 Variable foundations

This theory describes the foundational structure of UTP variables, upon which the rest of our model rests. We start by defining alphabets, which following [9, 10] in this shallow model are simply represented as types $'\alpha$, though by convention usually a record type where each field corresponds to a variable. UTP variables in this frame are simply modelled as lenses $'a \Longrightarrow '\alpha$, where the view type $'a$ is the variable type, and the source type $'\alpha$ is the alphabet or state-space type.

We define some lifting functions for variables to create input and output variables. These simply lift the alphabet to a tuple type since relations will ultimately be defined by a tuple alphabet.

definition *in-var* :: $('a \Longrightarrow '\alpha) \Rightarrow ('a \Longrightarrow '\alpha \times '\beta)$ **where**
[*lens-defs*]: *in-var* $x = x ;_L fst_L$

definition *out-var* :: $('a \Longrightarrow '\beta) \Rightarrow ('a \Longrightarrow '\alpha \times '\beta)$ **where**
[*lens-defs*]: *out-var* $x = x ;_L snd_L$

Variables can also be used to effectively define sets of variables. Here we define the the universal alphabet (Σ) to be the bijective lens 1_L. This characterises the whole of the source type, and thus is effectively the set of all alphabet variables.

abbreviation (*input*) *univ-alpha* :: $('\alpha \Longrightarrow '\alpha)$ $(\langle\Sigma\rangle)$ **where**
univ-alpha $\equiv 1_L$

The next construct is vacuous and simply exists to help the parser distinguish predicate variables from input and output variables.

definition *pr-var* :: $('a \Longrightarrow '\beta) \Rightarrow ('a \Longrightarrow '\beta)$ **where**
[*lens-defs*]: *pr-var* $x = x$

2.3 Variable lens properties

We can now easily show that our UTP variable construction are various classes of well-behaved lens .

lemma *in-var-weak-lens* [*simp*]:
weak-lens $x \Longrightarrow$ *weak-lens* (*in-var* x)
⟨*proof*⟩

lemma *in-var-semi-uvar* [*simp*]:
mwb-lens $x \Longrightarrow$ *mwb-lens* (*in-var* x)
⟨*proof*⟩

lemma *pr-var-weak-lens* [*simp*]:
weak-lens $x \Longrightarrow$ *weak-lens* (*pr-var* x)
⟨*proof*⟩

lemma *pr-var-mwb-lens* [*simp*]:
mwb-lens $x \Longrightarrow$ *mwb-lens* (*pr-var* x)
⟨*proof*⟩

lemma *pr-var-vwb-lens* [*simp*]:
vwb-lens x $\Longrightarrow$ *vwb-lens* (*pr-var x*)
⟨*proof*⟩

lemma *in-var-uvar* [*simp*]:
vwb-lens x $\Longrightarrow$ *vwb-lens* (*in-var x*)
⟨*proof*⟩

lemma *out-var-weak-lens* [*simp*]:
weak-lens x $\Longrightarrow$ *weak-lens* (*out-var x*)
⟨*proof*⟩

lemma *out-var-semi-uvar* [*simp*]:
mwb-lens x $\Longrightarrow$ *mwb-lens* (*out-var x*)
⟨*proof*⟩

lemma *out-var-uvar* [*simp*]:
vwb-lens x $\Longrightarrow$ *vwb-lens* (*out-var x*)
⟨*proof*⟩

Moreover, we can show that input and output variables are independent, since they refer to different sections of the alphabet.

lemma *in-out-indep* [*simp*]:
in-var x $\bowtie$ *out-var y*
⟨*proof*⟩

lemma *out-in-indep* [*simp*]:
out-var x $\bowtie$ *in-var y*
⟨*proof*⟩

lemma *in-var-indep* [*simp*]:
x $\bowtie$ *y* $\Longrightarrow$ *in-var x* $\bowtie$ *in-var y*
⟨*proof*⟩

lemma *out-var-indep* [*simp*]:
x $\bowtie$ *y* $\Longrightarrow$ *out-var x* $\bowtie$ *out-var y*
⟨*proof*⟩

lemma *pr-var-indeps* [*simp*]:
x $\bowtie$ *y* $\Longrightarrow$ *pr-var x* $\bowtie$ *y*
x $\bowtie$ *y* $\Longrightarrow$ *x* $\bowtie$ *pr-var y*
⟨*proof*⟩

lemma *prod-lens-indep-in-var* [*simp*]:
a $\bowtie$ *x* $\Longrightarrow$ *a* $\times_L$ *b* $\bowtie$ *in-var x*
⟨*proof*⟩

lemma *prod-lens-indep-out-var* [*simp*]:
b $\bowtie$ *x* $\Longrightarrow$ *a* $\times_L$ *b* $\bowtie$ *out-var x*
⟨*proof*⟩

lemma *in-var-pr-var* [*simp*]:
in-var (*pr-var x*) = *in-var x*
⟨*proof*⟩

lemma *out-var-pr-var* [*simp*]:
out-var (*pr-var x*) = *out-var x*
⟨*proof*⟩

lemma *pr-var-idem* [*simp*]:
pr-var (*pr-var x*) = *pr-var x*
⟨*proof*⟩

lemma *pr-var-lens-plus* [*simp*]:
pr-var (*x* +$_L$ *y*) = (*x* +$_L$ *y*)
⟨*proof*⟩

lemma *pr-var-lens-comp-1* [*simp*]:
pr-var x ;$_L$ *y* = *pr-var* (*x* ;$_L$ *y*)
⟨*proof*⟩

lemma *in-var-plus* [*simp*]: *in-var* (*x* +$_L$ *y*) = *in-var x* +$_L$ *in-var y*
⟨*proof*⟩

lemma *out-var-plus* [*simp*]: *out-var* (*x* +$_L$ *y*) = *out-var x* +$_L$ *out-var y*
⟨*proof*⟩

Similar properties follow for sublens

lemma *in-var-sublens* [*simp*]:
y ⊆$_L$ *x* ⟹ *in-var y* ⊆$_L$ *in-var x*
⟨*proof*⟩

lemma *out-var-sublens* [*simp*]:
y ⊆$_L$ *x* ⟹ *out-var y* ⊆$_L$ *out-var x*
⟨*proof*⟩

lemma *pr-var-sublens* [*simp*]:
y ⊆$_L$ *x* ⟹ *pr-var y* ⊆$_L$ *pr-var x*
⟨*proof*⟩

2.4 Lens simplifications

We also define some lookup abstraction simplifications.

lemma *var-lookup-in* [*simp*]: *lens-get* (*in-var x*) (*A*, *A′*) = *lens-get x A*
⟨*proof*⟩

lemma *var-lookup-out* [*simp*]: *lens-get* (*out-var x*) (*A*, *A′*) = *lens-get x A′*
⟨*proof*⟩

lemma *var-update-in* [*simp*]: *lens-put* (*in-var x*) (*A*, *A′*) *v* = (*lens-put x A v*, *A′*)
⟨*proof*⟩

lemma *var-update-out* [*simp*]: *lens-put* (*out-var x*) (*A*, *A′*) *v* = (*A*, *lens-put x A′ v*)
⟨*proof*⟩

lemma *get-lens-plus* [*simp*]: $get_{x +_L y}\ s = (get_x\ s, get_y\ s)$
⟨*proof*⟩

2.5 Syntax translations

In order to support nice syntax for variables, we here set up some translations. The first step is to introduce a collection of non-terminals.

nonterminal *svid* **and** *svids* **and** *svar* **and** *svars* **and** *salpha*

These non-terminals correspond to the following syntactic entities. Non-terminal *svid* is an atomic variable identifier, and *svids* is a list of identifier. *svar* is a decorated variable, such as an input or output variable, and *svars* is a list of decorated variables. *salpha* is an alphabet or set of variables. Such sets can be constructed only through lens composition due to typing restrictions. Next we introduce some syntax constructors.

syntax — Identifiers
-svid :: *id* ⇒ *svid* (‹-› [*999*] *999*)
-svid-unit :: *svid* ⇒ *svids* (‹-›)
-svid-list :: *svid* ⇒ *svids* ⇒ *svids* (‹-,/ -›)
-svid-alpha :: *svid* (‹**v**›)
-svid-dot :: *svid* ⇒ *svid* ⇒ *svid* (‹-:-› [*998*,*999*] *998*)
-mk-svid-list :: *svids* ⇒ *logic* — Helper function for summing a list of identifiers

A variable identifier can either be a HOL identifier, the complete set of variables in the alphabet **v**, or a composite identifier separated by colons, which corresponds to a sort of qualification. The final option is effectively a lens composition.

syntax — Decorations
-spvar :: *svid* ⇒ *svar* (‹&-› [*990*] *990*)
-sinvar :: *svid* ⇒ *svar* (‹$-› [*990*] *990*)
-soutvar :: *svid* ⇒ *svar* (‹$-´› [*990*] *990*)

A variable can be decorated with an ampersand, to indicate it is a predicate variable, with a dollar to indicate its an unprimed relational variable, or a dollar and "acute" symbol to indicate its a primed relational variable. Isabelle's parser is extensible so additional decorations can be and are added later.

syntax — Variable sets
-salphaid :: *svid* ⇒ *salpha* (‹-› [*990*] *990*)
-salphavar :: *svar* ⇒ *salpha* (‹-› [*990*] *990*)
-salphaparen :: *salpha* ⇒ *salpha* (‹′(-′)›)
-salphacomp :: *salpha* ⇒ *salpha* ⇒ *salpha* (**infixr** ‹;› *75*)
-salphaprod :: *salpha* ⇒ *salpha* ⇒ *salpha* (**infixr** ‹×› *85*)
-salpha-all :: *salpha* (‹Σ›)
-salpha-none :: *salpha* (‹∅›)
-svar-nil :: *svar* ⇒ *svars* (‹-›)
-svar-cons :: *svar* ⇒ *svars* ⇒ *svars* (‹-,/ -›)
-salphaset :: *svars* ⇒ *salpha* (‹{-}›)
-salphamk :: *logic* ⇒ *salpha*

The terminals of an alphabet are either HOL identifiers or UTP variable identifiers. We support two ways of constructing alphabets; by composition of smaller alphabets using a semi-colon or by a set-style construction $\{a, b, c\}$ with a list of UTP variables.

syntax — Quotations
-ualpha-set :: *svars* ⇒ *logic* (‹{-}$_\alpha$›)
-svar :: *svar* ⇒ *logic* (‹′(-′)$_v$›)

For various reasons, the syntax constructors above all yield specific grammar categories and will not parser at the HOL top level (basically this is to do with us wanting to reuse the syntax for expressions). As a result we provide some quotation constructors above.

Next we need to construct the syntax translations rules. First we need a few polymorphic constants.

consts
svar :: $'v \Rightarrow 'e$
ivar :: $'v \Rightarrow 'e$
ovar :: $'v \Rightarrow 'e$

adhoc-overloading
svar $\rightleftharpoons$ *pr-var* **and** *ivar* $\rightleftharpoons$ *in-var* **and** *ovar* $\rightleftharpoons$ *out-var*

The functions above turn a representation of a variable (type $'v$), including its name and type, into some lens type $'e$. *svar* constructs a predicate variable, *ivar* and input variables, and *ovar* and output variable. The functions bridge between the model and encoding of the variable and its interpretation as a lens in order to integrate it into the general lens-based framework. Overriding these functions is then all we need to make use of any kind of variables in terms of interfacing it with the system. Although in core UTP variables are always modelled using record field, we can overload these constants to allow other kinds of variables, such as deep variables with explicit syntax and type information.

Finally, we set up the translations rules.

translations
— Identifiers
-svid x $\rightharpoonup$ *x*
-svid-alpha $\rightleftharpoons$ Σ
-svid-dot x y $\rightharpoonup$ *y* $;_L$ *x*
-mk-svid-list (*-svid-unit x*) $\rightharpoonup$ *x*
-mk-svid-list (*-svid-list x xs*) $\rightharpoonup$ *x* $+_L$ *-mk-svid-list xs*

— Decorations
-spvar Σ $\leftharpoondown$ *CONST svar CONST id-lens*
-sinvar Σ $\leftharpoondown$ *CONST ivar* 1_L
-soutvar Σ $\leftharpoondown$ *CONST ovar* 1_L
-spvar (*-svid-dot x y*) $\leftharpoondown$ *CONST svar* (*CONST lens-comp y x*)
-sinvar (*-svid-dot x y*) $\leftharpoondown$ *CONST ivar* (*CONST lens-comp y x*)
-soutvar (*-svid-dot x y*) $\leftharpoondown$ *CONST ovar* (*CONST lens-comp y x*)
-svid-dot (*-svid-dot x y*) *z* $\leftharpoondown$ *-svid-dot* (*CONST lens-comp y x*) *z*

-spvar x $\rightleftharpoons$ *CONST svar x*
-sinvar x $\rightleftharpoons$ *CONST ivar x*
-soutvar x $\rightleftharpoons$ *CONST ovar x*

— Alphabets
-salphaparen a $\rightharpoonup$ *a*
-salphaid x $\rightharpoonup$ *x*
-salphacomp x y $\rightharpoonup$ *x* $+_L$ *y*
-salphaprod a b $\rightleftharpoons$ *a* $\times_L$ *b*
-salphavar x $\rightharpoonup$ *x*
-svar-nil x $\rightharpoonup$ *x*
-svar-cons x xs $\rightharpoonup$ *x* $+_L$ *xs*
-salphaset A $\rightharpoonup$ *A*
(*-svar-cons x* (*-salphamk y*)) $\leftharpoondown$ *-salphamk* (*x* $+_L$ *y*)
x $\leftharpoondown$ *-salphamk x*
-salpha-all $\rightleftharpoons$ 1_L
-salpha-none $\rightleftharpoons$ 0_L

— Quotations
-ualpha-set A ⇀ A
-svar x ⇀ x

The translation rules mainly convert syntax into lens constructions, using a mixture of lens operators and the bespoke variable definitions. Notably, a colon variable identifier qualification becomes a lens composition, and variable sets are constructed using len sum. The translation rules are carefully crafted to ensure both parsing and pretty printing.

Finally we create the following useful utility translation function that allows us to construct a UTP variable (lens) type given a return and alphabet type.

syntax
-uvar-ty :: *type* ⇒ *type* ⇒ *type*

⟨*ML*⟩

end

3 UTP Expressions

theory *utp-expr*
imports
utp-var
begin

3.1 Expression type

purge-notation *BNF-Def.convol* (‹(‹*indent=1 notation=‹mixfix convol›*›⟨-,/ -⟩)›)

Before building the predicate model, we will build a model of expressions that generalise alphabetised predicates. Expressions are represented semantically as mapping from the alphabet $'\alpha$ to the expression's type $'a$. This general model will allow us to unify all constructions under one type. The majority definitions in the file are given using the *lifting* package [23], which allows us to reuse much of the existing library of HOL functions.

typedef ('*t*, 'α) *uexpr* = *UNIV* :: ('α ⇒ '*t*) *set* ⟨*proof*⟩

setup-lifting *type-definition-uexpr*

notation *Rep-uexpr* (‹⟦-⟧$_e$›)
notation *Abs-uexpr* (‹*mk*$_e$›)

lemma *uexpr-eq-iff*:
e = *f* ⟷ (∀ *b*. ⟦*e*⟧$_e$ *b* = ⟦*f*⟧$_e$ *b*)
⟨*proof*⟩

The term ⟦*e*⟧$_e$ *b* effectively refers to the semantic interpretation of the expression under the state-space valuation (or variables binding) *b*. It can be used, in concert with the lifting package, to interpret UTP constructs to their HOL equivalents. We create some theorem sets to store such transfer theorems.

named-theorems *uexpr-defs* **and** *ueval* **and** *lit-simps* **and** *lit-norm*

3.2 Core expression constructs

A variable expression corresponds to the lens *get* function associated with a variable. Specifically, given a lens the expression always returns that portion of the state-space referred to by the lens.

lift-definition *var* :: $('t \Longrightarrow 'a) \Rightarrow ('t, 'a)$ *uexpr* **is** *lens-get* ⟨*proof*⟩

A literal is simply a constant function expression, always returning the same value for any binding.

lift-definition *lit* :: $'t \Rightarrow ('t, 'a)$ *uexpr* (‹«-»›) **is** $\lambda\ v\ b.\ v$ ⟨*proof*⟩

We define lifting for unary, binary, ternary, and quaternary expression constructs, that simply take a HOL function with correct number of arguments and apply it function to all possible results of the expressions.

lift-definition *uop* :: $('a \Rightarrow 'b) \Rightarrow ('a, 'a)$ *uexpr* $\Rightarrow ('b, 'a)$ *uexpr*
is $\lambda\ f\ e\ b.\ f\ (e\ b)$ ⟨*proof*⟩
lift-definition *bop* ::
$('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('a, 'a)$ *uexpr* $\Rightarrow ('b, 'a)$ *uexpr* $\Rightarrow ('c, 'a)$ *uexpr*
is $\lambda\ f\ u\ v\ b.\ f\ (u\ b)\ (v\ b)$ ⟨*proof*⟩
lift-definition *trop* ::
$('a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd) \Rightarrow ('a, 'a)$ *uexpr* $\Rightarrow ('b, 'a)$ *uexpr* $\Rightarrow ('c, 'a)$ *uexpr* $\Rightarrow ('d, 'a)$ *uexpr*
is $\lambda\ f\ u\ v\ w\ b.\ f\ (u\ b)\ (v\ b)\ (w\ b)$ ⟨*proof*⟩
lift-definition *qtop* ::
$('a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow 'e) \Rightarrow$
$('a, 'a)$ *uexpr* $\Rightarrow ('b, 'a)$ *uexpr* $\Rightarrow ('c, 'a)$ *uexpr* $\Rightarrow ('d, 'a)$ *uexpr* $\Rightarrow$
$('e, 'a)$ *uexpr*
is $\lambda\ f\ u\ v\ w\ x\ b.\ f\ (u\ b)\ (v\ b)\ (w\ b)\ (x\ b)$ ⟨*proof*⟩

We also define a UTP expression version of function (λ) abstraction, that takes a function producing an expression and produces an expression producing a function.

lift-definition *ulambda* :: $('a \Rightarrow ('b, 'a)$ *uexpr*$) \Rightarrow ('a \Rightarrow 'b, 'a)$ *uexpr*
is $\lambda\ f\ A\ x.\ f\ x\ A$ ⟨*proof*⟩

We set up syntax for the conditional. This is effectively an infix version of if-then-else where the condition is in the middle.

definition *uIf* :: *bool* $\Rightarrow 'a \Rightarrow 'a \Rightarrow 'a$ **where**
[*uexpr-defs*]: *uIf* = *If*

abbreviation *cond* ::
$('a,'\alpha)$ *uexpr* $\Rightarrow$ (*bool*, $'\alpha$) *uexpr* $\Rightarrow ('a,'\alpha)$ *uexpr* $\Rightarrow ('a,'\alpha)$ *uexpr*
(‹(*3*- ◁ - ▷/ -)› [*52*,*0*,*53*] *52*)
where $P \lhd b \rhd Q \equiv$ *trop uIf b P Q*

UTP expression is equality is simply HOL equality lifted using the *bop* binary expression constructor.

definition *eq-upred* :: $('a, 'a)$ *uexpr* $\Rightarrow ('a, 'a)$ *uexpr* $\Rightarrow$ (*bool*, $'a$) *uexpr* (**infixl** $‹=_u›$ *50*)
where [*uexpr-defs*]: *eq-upred x y* = *bop HOL.eq x y*

A literal is the expression «*v*», where *v* is any HOL term. Actually, the literal construct is very versatile and also allows us to refer to HOL variables within UTP expressions, and has a variety of other uses. It can therefore also be considered as a kind of quotation mechanism.

We also set up syntax for UTP variable expressions.

syntax
-uuvar :: *svar* ⇒ *logic* (‹-›)

syntax-consts
-uuvar == *var*

translations
-uuvar x == *CONST var x*

Since we already have a parser for variables, we can directly reuse it and simply apply the *var* expression construct to lift the resulting variable to an expression.

3.3 Type class instantiations

Isabelle/HOL of course provides a large hierarchy of type classes that provide constructs such as numerals and the arithmetic operators. Fortunately we can directly make use of these for UTP expressions, and thus we now perform a long list of appropriate instantiations. We first lift the core arithemtic constants and operators using a mixture of literals, unary, and binary expression constructors.

instantiation *uexpr* :: (*zero*, *type*) *zero*
begin
definition *zero-uexpr-def* [*uexpr-defs*]: *0* = *lit 0*
instance ⟨*proof*⟩
end

instantiation *uexpr* :: (*one*, *type*) *one*
begin
definition *one-uexpr-def* [*uexpr-defs*]: *1* = *lit 1*
instance ⟨*proof*⟩

end

instantiation *uexpr* :: (*plus*, *type*) *plus*
begin
definition *plus-uexpr-def* [*uexpr-defs*]: *u* + *v* = *bop* (+) *u v*
instance ⟨*proof*⟩
end

instance *uexpr* :: (*semigroup-add*, *type*) *semigroup-add*
⟨*proof*⟩

The following instantiation sets up numerals. This will allow us to have Isabelle number representations (i.e. 3,7,42,198 etc.) to UTP expressions directly.

instance *uexpr* :: (*numeral*, *type*) *numeral*
⟨*proof*⟩

We can also define the order relation on expressions. Now, unlike the previous group and ring constructs, the order relations (≤) and (≤) return a *bool* type. This order is not therefore the lifted order which allows us to compare the valuation of two expressions, but rather the order on expressions themselves. Notably, this instantiation will later allow us to talk about predicate refinements and complete lattices.

instantiation *uexpr* :: (*ord*, *type*) *ord*
begin

lift-definition *less-eq-uexpr* :: ('*a*, '*b*) *uexpr* ⇒ ('*a*, '*b*) *uexpr* ⇒ *bool*
is λ *P Q*. (∀ *A*. *P A* ≤ *Q A*) ⟨*proof*⟩
definition *less-uexpr* :: ('*a*, '*b*) *uexpr* ⇒ ('*a*, '*b*) *uexpr* ⇒ *bool*
where [*uexpr-defs*]: *less-uexpr P Q* = (*P* ≤ *Q* ∧ ¬ *Q* ≤ *P*)
instance ⟨*proof*⟩
end

UTP expressions whose return type is a partial ordered type, are also partially ordered as the following instantiation demonstrates.

instance *uexpr* :: (*order*, *type*) *order*
⟨*proof*⟩

3.4 Syntax translations

The follows a large number of translations that lift HOL functions to UTP expressions using the various expression constructors defined above. Much of the time we try to keep the HOL syntax but add a "u" subscript.

This operator allows us to get the characteristic set of a type. Essentially this is *UNIV*, but it retains the type syntactically for pretty printing.

definition *set-of* :: '*a itself* ⇒ '*a set* **where**
[*uexpr-defs*]: *set-of t* = *UNIV*

We add new non-terminals for UTP tuples and maplets.

nonterminal *utuple-args* **and** *umaplet* **and** *umaplets*

syntax — Core expression constructs
-ucoerce :: *logic* ⇒ *type* ⇒ *logic* (**infix** ‹:$_u$› *50*)
-ulambda :: *pttrn* ⇒ *logic* ⇒ *logic* (‹λ - • -› [*0*, *10*] *10*)
-ulens-ovrd :: *logic* ⇒ *logic* ⇒ *salpha* ⇒ *logic* (‹- ⊕ - *on* -› [*85*, *0*, *86*] *86*)
-ulens-get :: *logic* ⇒ *svar* ⇒ *logic* (‹-:-› [*900*,*901*] *901*)
-umem :: ('*a*, '*α*) *uexpr* ⇒ ('*a set*, '*α*) *uexpr* ⇒ (*bool*, '*α*) *uexpr* (**infix** ‹∈$_u$› *50*)

translations
λ *x* • *p* == *CONST ulambda* (λ *x*. *p*)
x :$_u$ '*a* == *x* :: ('*a*, -) *uexpr*
-ulens-ovrd f g a => *CONST bop* (*CONST lens-override a*) *f g*
-ulens-ovrd f g a <= *CONST bop* (λ*x y*. *CONST lens-override x1 y1 a*) *f g*
-ulens-get x y == *CONST uop* (*CONST lens-get y*) *x*
x ∈$_u$ *A* == *CONST bop* (∈) *x A*

syntax — Tuples
-utuple :: ('*a*, '*α*) *uexpr* ⇒ *utuple-args* ⇒ ('*a* ∗ '*b*, '*α*) *uexpr* (‹(*1* '(-,/ -')$_u$)›)
-utuple-arg :: ('*a*, '*α*) *uexpr* ⇒ *utuple-args* (‹-›)
-utuple-args :: ('*a*, '*α*) *uexpr* => *utuple-args* ⇒ *utuple-args* (‹-,/ -›)
-uunit :: ('*a*, '*α*) *uexpr* (‹'(')$_u$›)
-ufst :: ('*a* × '*b*, '*α*) *uexpr* ⇒ ('*a*, '*α*) *uexpr* (‹π_1'(-')›)
-usnd :: ('*a* × '*b*, '*α*) *uexpr* ⇒ ('*b*, '*α*) *uexpr* (‹π_2'(-')›)

translations
()$_u$ == «()»
(*x*, *y*)$_u$ == *CONST bop* (*CONST Pair*) *x y*
-utuple x (*-utuple-args y z*) == *-utuple x* (*-utuple-arg* (*-utuple y z*))
π_1(*x*) == *CONST uop CONST fst x*

$\pi_2(x)$ == *CONST uop CONST snd x*

syntax — Orders
-uless :: *logic* ⇒ *logic* ⇒ *logic* (**infix** ‹$<_u$› *50*)
-uleq :: *logic* ⇒ *logic* ⇒ *logic* (**infix** ‹$\leq_u$› *50*)
-ugreat :: *logic* ⇒ *logic* ⇒ *logic* (**infix** ‹$>_u$› *50*)
-ugeq :: *logic* ⇒ *logic* ⇒ *logic* (**infix** ‹$\geq_u$› *50*)

translations
$x <_u y$ == *CONST bop* (<) *x y*
$x \leq_u y$ == *CONST bop* (≤) *x y*
$x >_u y$ => $y <_u x$
$x \geq_u y$ => $y \leq_u x$

3.5 Evaluation laws for expressions

The following laws show how to evaluate the core expressions constructs in terms of which the above definitions are defined. Thus, using these theorems together, we can convert any UTP expression into a pure HOL expression. All these theorems are marked as *ueval* theorems which can be used for evaluation.

lemma *lit-ueval* [*ueval*]: $[\![«x»]\!]_e b = x$
⟨*proof*⟩

lemma *var-ueval* [*ueval*]: $[\![var\ x]\!]_e b = get_x\ b$
⟨*proof*⟩

lemma *uop-ueval* [*ueval*]: $[\![uop\ f\ x]\!]_e b = f\ ([\![x]\!]_e b)$
⟨*proof*⟩

lemma *bop-ueval* [*ueval*]: $[\![bop\ f\ x\ y]\!]_e b = f\ ([\![x]\!]_e b)\ ([\![y]\!]_e b)$
⟨*proof*⟩

lemma *trop-ueval* [*ueval*]: $[\![trop\ f\ x\ y\ z]\!]_e b = f\ ([\![x]\!]_e b)\ ([\![y]\!]_e b)\ ([\![z]\!]_e b)$
⟨*proof*⟩

lemma *qtop-ueval* [*ueval*]: $[\![qtop\ f\ x\ y\ z\ w]\!]_e b = f\ ([\![x]\!]_e b)\ ([\![y]\!]_e b)\ ([\![z]\!]_e b)\ ([\![w]\!]_e b)$
⟨*proof*⟩

3.6 Misc laws

We also prove a few useful algebraic and expansion laws for expressions.

lemma *uop-const* [*simp*]: *uop id u* = *u*
⟨*proof*⟩

lemma *bop-const-1* [*simp*]: *bop* (λ*x y*. *y*) *u v* = *v*
⟨*proof*⟩

lemma *bop-const-2* [*simp*]: *bop* (λ*x y*. *x*) *u v* = *u*
⟨*proof*⟩

lemma *uexpr-fst* [*simp*]: $\pi_1((e, f)_u) = e$
⟨*proof*⟩

lemma *uexpr-snd* [*simp*]: $\pi_2((e, f)_u) = f$

⟨*proof*⟩

3.7 Literalise tactics

The following tactic converts literal HOL expressions to UTP expressions and vice-versa via a collection of simplification rules. The two tactics are called "literalise", which converts UTP to expressions to HOL expressions – i.e. it pushes them into literals – and unliteralise that reverses this. We collect the equations in a theorem attribute called "lit_simps".

lemma *lit-fun-simps* [*lit-simps*]:
«*i x y z u*» = *qtop i* «*x*» «*y*» «*z*» «*u*»
«*h x y z*» = *trop h* «*x*» «*y*» «*z*»
«*g x y*» = *bop g* «*x*» «*y*»
«*f x*» = *uop f* «*x*»
⟨*proof*⟩

The following two theorems also set up interpretation of numerals, meaning a UTP numeral can always be converted to a HOL numeral.

lemma *numeral-uexpr-rep-eq* [*ueval*]: $[\![\mathit{numeral}\ x]\!]_e\ b = \mathit{numeral}\ x$
⟨*proof*⟩

lemma *numeral-uexpr-simp*: *numeral x* = «*numeral x*»
⟨*proof*⟩

lemma *lit-zero* [*lit-simps*]: «*0*» = *0* ⟨*proof*⟩
lemma *lit-one* [*lit-simps*]: «*1*» = *1* ⟨*proof*⟩
lemma *lit-plus* [*lit-simps*]: «*x* + *y*» = «*x*» + «*y*» ⟨*proof*⟩
lemma *lit-numeral* [*lit-simps*]: «*numeral n*» = *numeral n* ⟨*proof*⟩

In general unliteralising converts function applications to corresponding expression liftings. Since some operators, like + and *, have specific operators we also have to use *uIf* = *If*
$(?x =_u ?y) = \mathit{bop}\ (=)\ ?x\ ?y$
$0 = «0»$
$1 = «1»$
$?u + ?v = \mathit{bop}\ (+)\ ?u\ ?v$
$(?P < ?Q) = (?P \leq ?Q \wedge \neg\ ?Q \leq ?P)$
set-of ?t = *UNIV* in reverse to correctly interpret these. Moreover, numerals must be handled separately by first simplifying them and then converting them into UTP expression numerals; hence the following two simplification rules.

lemma *lit-numeral-1*: *uop numeral x* = *Abs-uexpr* ($\lambda b.$ *numeral* ($[\![x]\!]_e$ *b*))
⟨*proof*⟩

lemma *lit-numeral-2*: *Abs-uexpr* (λ *b. numeral v*) = *numeral v*
⟨*proof*⟩

method *literalise* = (*unfold lit-simps*[*THEN sym*])
method *unliteralise* = (*unfold lit-simps uexpr-defs*[*THEN sym*];
(*unfold lit-numeral-1* ; (*unfold uexpr-defs ueval*); (*unfold lit-numeral-2*))*?*)+

The following tactic can be used to evaluate literal expressions. It first literalises UTP expressions, that is pushes as many operators into literals as possible. Then it tries to simplify, and final unliteralises at the end.

method *uexpr-simp* **uses** *simps* = ((*literalise*)*?*, *simp add*: *lit-norm simps*, (*unliteralise*)*?*)

lemma (*1*::(*int*, *'α*) *uexpr*) + «*2*» = *4* ⟷ «*3*» = *4*
⟨*proof*⟩

end

4 Expression Type Class Instantiations

theory *utp-expr-insts*
imports *utp-expr*
begin

It should be noted that instantiating the unary minus class, *uminus*, will also provide negation UTP predicates later.

instantiation *uexpr* :: (*uminus*, *type*) *uminus*
begin
definition *uminus-uexpr-def* [*uexpr-defs*]: − *u* = *uop uminus u*
instance ⟨*proof*⟩
end

instantiation *uexpr* :: (*minus*, *type*) *minus*
begin
definition *minus-uexpr-def* [*uexpr-defs*]: *u* − *v* = *bop* (−) *u v*
instance ⟨*proof*⟩
end

instantiation *uexpr* :: (*times*, *type*) *times*
begin
definition *times-uexpr-def* [*uexpr-defs*]: *u* ∗ *v* = *bop times u v*
instance ⟨*proof*⟩
end

instance *uexpr* :: (*Rings.dvd*, *type*) *Rings.dvd* ⟨*proof*⟩

instantiation *uexpr* :: (*divide*, *type*) *divide*
begin
definition *divide-uexpr* :: (*'a*, *'b*) *uexpr* ⇒ (*'a*, *'b*) *uexpr* ⇒ (*'a*, *'b*) *uexpr* **where**
[*uexpr-defs*]: *divide-uexpr u v* = *bop divide u v*
instance ⟨*proof*⟩
end

instantiation *uexpr* :: (*inverse*, *type*) *inverse*
begin
definition *inverse-uexpr* :: (*'a*, *'b*) *uexpr* ⇒ (*'a*, *'b*) *uexpr*
where [*uexpr-defs*]: *inverse-uexpr u* = *uop inverse u*
instance ⟨*proof*⟩
end

instantiation *uexpr* :: (*modulo*, *type*) *modulo*
begin
definition *mod-uexpr-def* [*uexpr-defs*]: *u mod v* = *bop* (*mod*) *u v*
instance ⟨*proof*⟩

end

instantiation *uexpr* :: (*sgn*, *type*) *sgn*
begin
definition *sgn-uexpr-def* [*uexpr-defs*]: *sgn u* = *uop sgn u*
instance ⟨*proof*⟩
end

instantiation *uexpr* :: (*abs*, *type*) *abs*
begin
definition *abs-uexpr-def* [*uexpr-defs*]: *abs u* = *uop abs u*
instance ⟨*proof*⟩
end

Once we've set up all the core constructs for arithmetic, we can also instantiate the type classes for various algebras, including groups and rings. The proofs are done by definitional expansion, the *transfer* tactic, and then finally the theorems of the underlying HOL operators. This is mainly routine, so we don't comment further.

instance *uexpr* :: (*semigroup-mult*, *type*) *semigroup-mult*
⟨*proof*⟩

instance *uexpr* :: (*monoid-mult*, *type*) *monoid-mult*
⟨*proof*⟩

instance *uexpr* :: (*monoid-add*, *type*) *monoid-add*
⟨*proof*⟩

instance *uexpr* :: (*ab-semigroup-add*, *type*) *ab-semigroup-add*
⟨*proof*⟩

instance *uexpr* :: (*cancel-semigroup-add*, *type*) *cancel-semigroup-add*
⟨*proof*⟩

instance *uexpr* :: (*cancel-ab-semigroup-add*, *type*) *cancel-ab-semigroup-add*
⟨*proof*⟩

instance *uexpr* :: (*group-add*, *type*) *group-add*
⟨*proof*⟩

instance *uexpr* :: (*ab-group-add*, *type*) *ab-group-add*
⟨*proof*⟩

instance *uexpr* :: (*semiring*, *type*) *semiring*
⟨*proof*⟩

instance *uexpr* :: (*ring-1*, *type*) *ring-1*
⟨*proof*⟩

We also lift the properties from certain ordered groups.

instance *uexpr* :: (*ordered-ab-group-add*, *type*) *ordered-ab-group-add*
⟨*proof*⟩

instance *uexpr* :: (*ordered-ab-group-add-abs*, *type*) *ordered-ab-group-add-abs*
⟨*proof*⟩

The next theorem lifts powers.

lemma *power-rep-eq* [*ueval*]: $[\![P \mathbin{\hat{}} n]\!]_e = (\lambda\ b.\ [\![P]\!]_e\ b \mathbin{\hat{}} n)$
⟨*proof*⟩

lemma *of-nat-uexpr-rep-eq* [*ueval*]: $[\![of\text{-}nat\ x]\!]_e\ b = of\text{-}nat\ x$
⟨*proof*⟩

lemma *lit-uminus* [*lit-simps*]: «$-\ x$» = $-$ «x» ⟨*proof*⟩
lemma *lit-minus* [*lit-simps*]: «$x - y$» = «x» $-$ «y» ⟨*proof*⟩
lemma *lit-times* [*lit-simps*]: «$x * y$» = «x» $*$ «y» ⟨*proof*⟩
lemma *lit-divide* [*lit-simps*]: «$x\ /\ y$» = «x» $/$ «y» ⟨*proof*⟩
lemma *lit-div* [*lit-simps*]: «$x\ div\ y$» = «x» *div* «y» ⟨*proof*⟩
lemma *lit-power* [*lit-simps*]: «$x \mathbin{\hat{}} n$» = «x» $\mathbin{\hat{}} n$ ⟨*proof*⟩

4.1 Expression construction from HOL terms

Sometimes it is convenient to cast HOL terms to UTP expressions, and these simplifications automate this process.

named-theorems *mkuexpr*

lemma *mkuexpr-lens-get* [*mkuexpr*]: $mk_e\ get_x = \&x$
⟨*proof*⟩

lemma *mkuexpr-zero* [*mkuexpr*]: $mk_e\ (\lambda\ s.\ 0) = 0$
⟨*proof*⟩

lemma *mkuexpr-one* [*mkuexpr*]: $mk_e\ (\lambda\ s.\ 1) = 1$
⟨*proof*⟩

lemma *mkuexpr-numeral* [*mkuexpr*]: $mk_e\ (\lambda\ s.\ numeral\ n) = numeral\ n$
⟨*proof*⟩

lemma *mkuexpr-lit* [*mkuexpr*]: $mk_e\ (\lambda\ s.\ k) =$ «k»
⟨*proof*⟩

lemma *mkuexpr-pair* [*mkuexpr*]: $mk_e\ (\lambda s.\ (f\ s,\ g\ s)) = (mk_e\ f,\ mk_e\ g)_u$
⟨*proof*⟩

lemma *mkuexpr-plus* [*mkuexpr*]: $mk_e\ (\lambda\ s.\ f\ s + g\ s) = mk_e\ f + mk_e\ g$
⟨*proof*⟩

lemma *mkuexpr-uminus* [*mkuexpr*]: $mk_e\ (\lambda\ s.\ -\ f\ s) = -\ mk_e\ f$
⟨*proof*⟩

lemma *mkuexpr-minus* [*mkuexpr*]: $mk_e\ (\lambda\ s.\ f\ s - g\ s) = mk_e\ f - mk_e\ g$
⟨*proof*⟩

lemma *mkuexpr-times* [*mkuexpr*]: $mk_e\ (\lambda\ s.\ f\ s * g\ s) = mk_e\ f * mk_e\ g$
⟨*proof*⟩

lemma *mkuexpr-divide* [*mkuexpr*]: $mk_e\ (\lambda\ s.\ f\ s\ /\ g\ s) = mk_e\ f\ /\ mk_e\ g$
⟨*proof*⟩

end

theory *utp-expr-funcs*
imports *utp-expr-insts*
begin

syntax — Polymorphic constructs
-uceil :: *logic* ⇒ *logic* (‹⌈-⌉$_u$›)
-ufloor :: *logic* ⇒ *logic* (‹⌊-⌋$_u$›)
-umin :: *logic* ⇒ *logic* ⇒ *logic* (‹min_u'(-, -')›)
-umax :: *logic* ⇒ *logic* ⇒ *logic* (‹max_u'(-, -')›)
-ugcd :: *logic* ⇒ *logic* ⇒ *logic* (‹gcd_u'(-, -')›)

translations
— Type-class polymorphic constructs
$min_u(x, y)$ == *CONST bop* (*CONST min*) *x y*
$max_u(x, y)$ == *CONST bop* (*CONST max*) *x y*
$gcd_u(x, y)$ == *CONST bop* (*CONST gcd*) *x y*
$\lceil x \rceil_u$ == *CONST uop CONST ceiling x*
$\lfloor x \rfloor_u$ == *CONST uop CONST floor x*

syntax — Lists / Sequences
-ucons :: *logic* ⇒ *logic* ⇒ *logic* (**infixr** ‹$\#_u$› *65*)
-unil :: ('*a list*, 'α) *uexpr* (‹⟨⟩›)
-ulist :: *args* => ('*a list*, 'α) *uexpr* (‹⟨(-)⟩›)
-uappend :: ('*a list*, 'α) *uexpr* ⇒ ('*a list*, 'α) *uexpr* ⇒ ('*a list*, 'α) *uexpr* (**infixr** ‹$\frown_u$› *80*)
-udconcat :: *logic* ⇒ *logic* ⇒ *logic* (**infixr** ‹$\frown_u$› *90*)
-ulast :: ('*a list*, 'α) *uexpr* ⇒ ('*a*, 'α) *uexpr* (‹$last_u$'(-')›)
-ufront :: ('*a list*, 'α) *uexpr* ⇒ ('*a list*, 'α) *uexpr* (‹$front_u$'(-')›)
-uhead :: ('*a list*, 'α) *uexpr* ⇒ ('*a*, 'α) *uexpr* (‹$head_u$'(-')›)
-utail :: ('*a list*, 'α) *uexpr* ⇒ ('*a list*, 'α) *uexpr* (‹$tail_u$'(-')›)
-utake :: (*nat*, 'α) *uexpr* ⇒ ('*a list*, 'α) *uexpr* ⇒ ('*a list*, 'α) *uexpr* (‹$take_u$'(-,/ -')›)
-udrop :: (*nat*, 'α) *uexpr* ⇒ ('*a list*, 'α) *uexpr* ⇒ ('*a list*, 'α) *uexpr* (‹$drop_u$'(-,/ -')›)
-ufilter :: ('*a list*, 'α) *uexpr* ⇒ ('*a set*, 'α) *uexpr* ⇒ ('*a list*, 'α) *uexpr* (**infixl** ‹$\restriction_u$› *75*)
-uextract :: ('*a set*, 'α) *uexpr* ⇒ ('*a list*, 'α) *uexpr* ⇒ ('*a list*, 'α) *uexpr* (**infixl** ‹$\upharpoonright_u$› *75*)
-uelems :: ('*a list*, 'α) *uexpr* ⇒ ('*a set*, 'α) *uexpr* (‹$elems_u$'(-')›)
-usorted :: ('*a list*, 'α) *uexpr* ⇒ (*bool*, 'α) *uexpr* (‹$sorted_u$'(-')›)
-udistinct :: ('*a list*, 'α) *uexpr* ⇒ (*bool*, 'α) *uexpr* (‹$distinct_u$'(-')›)
-uupto :: *logic* ⇒ *logic* ⇒ *logic* (‹⟨-..-⟩›)
-uupt :: *logic* ⇒ *logic* ⇒ *logic* (‹⟨-..<-⟩›)
-umap :: *logic* ⇒ *logic* ⇒ *logic* (‹map_u›)
-uzip :: *logic* ⇒ *logic* ⇒ *logic* (‹zip_u›)

translations
x $\#_u$ *ys* == *CONST bop* (#) *x ys*
⟨⟩ == «[]»
⟨*x*, *xs*⟩ == *x* $\#_u$ ⟨*xs*⟩
⟨*x*⟩ == *x* $\#_u$ «[]»
x $\frown_u$ *y* == *CONST bop* (@) *x y*
A $\frown_u$ *B* == *CONST bop* ($\frown$) *A B*
$last_u(xs)$ == *CONST uop CONST last xs*
$front_u(xs)$ == *CONST uop CONST butlast xs*
$head_u(xs)$ == *CONST uop CONST hd xs*
$tail_u(xs)$ == *CONST uop CONST tl xs*
$drop_u(n,xs)$ == *CONST bop CONST drop n xs*
$take_u(n,xs)$ == *CONST bop CONST take n xs*
$elems_u(xs)$ == *CONST uop CONST set xs*

$sorted_u(xs) == CONST\ uop\ CONST\ sorted\ xs$
$distinct_u(xs) == CONST\ uop\ CONST\ distinct\ xs$
$xs \restriction_u A == CONST\ bop\ CONST\ seq\text{-}filter\ xs\ A$
$A \restriction_u xs == CONST\ bop\ (\restriction_l)\ A\ xs$
$\langle n..k\rangle == CONST\ bop\ CONST\ upto\ n\ k$
$\langle n..<k\rangle == CONST\ bop\ CONST\ upt\ n\ k$
$map_u\ f\ xs == CONST\ bop\ CONST\ map\ f\ xs$
$zip_u\ xs\ ys == CONST\ bop\ CONST\ zip\ xs\ ys$

syntax — Sets
$\text{-}ufinite :: logic \Rightarrow logic\ (\langle finite_u'(\text{-}')\rangle)$
$\text{-}uempset :: ('a\ set, 'a)\ uexpr\ (\langle\{\}_u\rangle)$
$\text{-}uset :: args => ('a\ set, 'a)\ uexpr\ (\langle\{(\text{-})\}_u\rangle)$
$\text{-}uunion :: ('a\ set, 'a)\ uexpr \Rightarrow ('a\ set, 'a)\ uexpr \Rightarrow ('a\ set, 'a)\ uexpr\ (\textbf{infixl}\ \langle\cup_u\rangle\ 65)$
$\text{-}uinter :: ('a\ set, 'a)\ uexpr \Rightarrow ('a\ set, 'a)\ uexpr \Rightarrow ('a\ set, 'a)\ uexpr\ (\textbf{infixl}\ \langle\cap_u\rangle\ 70)$
$\text{-}uinsert :: logic \Rightarrow logic \Rightarrow logic\ (\langle insert_u\rangle)$
$\text{-}uimage :: logic \Rightarrow logic \Rightarrow logic\ (\langle\text{-}(\!|\text{-}|\!)_u\rangle\ [10,0]\ 10)$
$\text{-}usubset :: ('a\ set, 'a)\ uexpr \Rightarrow ('a\ set, 'a)\ uexpr \Rightarrow (bool, 'a)\ uexpr\ (\textbf{infix}\ \langle\subset_u\rangle\ 50)$
$\text{-}usubseteq :: ('a\ set, 'a)\ uexpr \Rightarrow ('a\ set, 'a)\ uexpr \Rightarrow (bool, 'a)\ uexpr\ (\textbf{infix}\ \langle\subseteq_u\rangle\ 50)$
$\text{-}uconverse :: logic \Rightarrow logic\ (\langle(\text{-}^\sim)\rangle\ [1000]\ 999)$
$\text{-}ucarrier :: type \Rightarrow logic\ (\langle[\text{-}]_T\rangle)$
$\text{-}uid :: type \Rightarrow logic\ (\langle id[\text{-}]\rangle)$
$\text{-}uproduct :: logic \Rightarrow logic \Rightarrow logic\ (\textbf{infixr}\ \langle\times_u\rangle\ 80)$
$\text{-}urelcomp :: logic \Rightarrow logic \Rightarrow logic\ (\textbf{infixr}\ \langle;_u\rangle\ 75)$

translations
$finite_u(x) == CONST\ uop\ (CONST\ finite)\ x$
$\{\}_u == «\{\}»$
$insert_u\ x\ xs == CONST\ bop\ CONST\ insert\ x\ xs$
$\{x, xs\}_u == insert_u\ x\ \{xs\}_u$
$\{x\}_u == insert_u\ x\ «\{\}»$
$A \cup_u B == CONST\ bop\ (\cup)\ A\ B$
$A \cap_u B == CONST\ bop\ (\cap)\ A\ B$
$f(\!|A|\!)_u == CONST\ bop\ CONST\ image\ f\ A$
$A \subset_u B == CONST\ bop\ (\subset)\ A\ B$
$f \subset_u g <= CONST\ bop\ (\subset_p)\ f\ g$
$f \subset_u g <= CONST\ bop\ (\subset_f)\ f\ g$
$A \subseteq_u B == CONST\ bop\ (\subseteq)\ A\ B$
$f \subseteq_u g <= CONST\ bop\ (\subseteq_p)\ f\ g$
$f \subseteq_u g <= CONST\ bop\ (\subseteq_f)\ f\ g$
$P^\sim == CONST\ uop\ CONST\ converse\ P$
$['a]_T == «CONST\ set\text{-}of\ TYPE('a)»$
$id['a] == «CONST\ Id\text{-}on\ (CONST\ set\text{-}of\ TYPE('a))»$
$A \times_u B == CONST\ bop\ CONST\ Product\text{-}Type.Times\ A\ B$
$A ;_u B == CONST\ bop\ CONST\ relcomp\ A\ B$

syntax — Partial functions
$\text{-}umap\text{-}plus :: logic \Rightarrow logic \Rightarrow logic\ (\textbf{infixl}\ \langle\oplus_u\rangle\ 85)$
$\text{-}umap\text{-}minus :: logic \Rightarrow logic \Rightarrow logic\ (\textbf{infixl}\ \langle\ominus_u\rangle\ 85)$

translations
$f \oplus_u g => (f :: ((\text{-}, \text{-})\ pfun, \text{-})\ uexpr) + g$
$f \ominus_u g => (f :: ((\text{-}, \text{-})\ pfun, \text{-})\ uexpr) - g$

syntax — Sum types

-uinl :: *logic* ⇒ *logic* (‹inl_u'(-')›)
-uinr :: *logic* ⇒ *logic* (‹inr_u'(-')›)

translations
$inl_u(x)$ == *CONST uop CONST Inl x*
$inr_u(x)$ == *CONST uop CONST Inr x*

4.2 Lifting set collectors

We provide syntax for various types of set collectors, including intervals and the Z-style set comprehension which is purpose built as a new lifted definition.

syntax
-uset-atLeastAtMost :: ('a, 'α) *uexpr* ⇒ ('a, 'α) *uexpr* ⇒ ('a *set*, 'α) *uexpr* (‹(1{-..-}$_u$)›)
-uset-atLeastLessThan :: ('a, 'α) *uexpr* ⇒ ('a, 'α) *uexpr* ⇒ ('a *set*, 'α) *uexpr* (‹(1{-..<-}$_u$)›)
-uset-compr :: *pttrn* ⇒ ('a *set*, 'α) *uexpr* ⇒ (*bool*, 'α) *uexpr* ⇒ ('b, 'α) *uexpr* ⇒ ('b *set*, 'α) *uexpr* (‹(1{- :/ - |/ - •/ -}$_u$)›)
-uset-compr-nset :: *pttrn* ⇒ (*bool*, 'α) *uexpr* ⇒ ('b, 'α) *uexpr* ⇒ ('b *set*, 'α) *uexpr* (‹(1{- |/ - •/ -}$_u$)›)

lift-definition *ZedSetCompr* ::
('a *set*, 'α) *uexpr* ⇒ ('a ⇒ (*bool*, 'α) *uexpr* × ('b, 'α) *uexpr*) ⇒ ('b *set*, 'α) *uexpr*
is λ *A PF b*. { *snd* (*PF x*) *b* | *x*. *x* ∈ *A b* ∧ *fst* (*PF x*) *b*} ⟨*proof*⟩

translations
{*x*..*y*}$_u$ == *CONST bop CONST atLeastAtMost x y*
{*x*..<*y*}$_u$ == *CONST bop CONST atLeastLessThan x y*
{*x* | *P* • *F*}$_u$ == *CONST ZedSetCompr* (*CONST lit CONST UNIV*) (λ *x*. (*P*, *F*))
{*x* : *A* | *P* • *F*}$_u$ == *CONST ZedSetCompr A* (λ *x*. (*P*, *F*))

4.3 Lifting limits

We also lift the following functions on topological spaces for taking function limits, and describing continuity.

definition *ulim-left* :: 'a::*order-topology* ⇒ ('a ⇒ 'b) ⇒ 'b::*t2-space* **where**
[*uexpr-defs*]: *ulim-left* = (λ *p f*. *Lim* (*at-left p*) *f*)

definition *ulim-right* :: 'a::*order-topology* ⇒ ('a ⇒ 'b) ⇒ 'b::*t2-space* **where**
[*uexpr-defs*]: *ulim-right* = (λ *p f*. *Lim* (*at-right p*) *f*)

definition *ucont-on* :: ('a::*topological-space* ⇒ 'b::*topological-space*) ⇒ 'a *set* ⇒ *bool* **where**
[*uexpr-defs*]: *ucont-on* = (λ *f A*. *continuous-on A f*)

syntax
-ulim-left :: *id* ⇒ *logic* ⇒ *logic* ⇒ *logic* (‹lim_u'(- → -$^-$')'(-')›)
-ulim-right :: *id* ⇒ *logic* ⇒ *logic* ⇒ *logic* (‹lim_u'(- → -$^+$')'(-')›)
-ucont-on :: *logic* ⇒ *logic* ⇒ *logic* (**infix** ‹*cont*−*on*$_u$› *90*)

translations
$lim_u(x \to p^-)(e)$ == *CONST bop CONST ulim-left p* (λ *x* • *e*)
$lim_u(x \to p^+)(e)$ == *CONST bop CONST ulim-right p* (λ *x* • *e*)
f cont−*on*$_u$ *A* == *CONST bop CONST continuous-on A f*

lemma *uset-minus-empty* [*simp*]: $x - \{\}_u = x$
⟨*proof*⟩

lemma *uinter-empty-1* [*simp*]: $x \cap_u \{\}_u = \{\}_u$
⟨*proof*⟩

lemma *uinter-empty-2* [*simp*]: $\{\}_u \cap_u x = \{\}_u$
⟨*proof*⟩

lemma *uunion-empty-1* [*simp*]: $\{\}_u \cup_u x = x$
⟨*proof*⟩

lemma *uunion-insert* [*simp*]: $(bop\ insert\ x\ A) \cup_u B = bop\ insert\ x\ (A \cup_u B)$
⟨*proof*⟩

lemma *ulist-filter-empty* [*simp*]: $x \upharpoonright_u \{\}_u = \langle\rangle$
⟨*proof*⟩

lemma *tail-cons* [*simp*]: $tail_u(\langle x\rangle \mathbin{\hat{}_u} xs) = xs$
⟨*proof*⟩

lemma *uconcat-units* [*simp*]: $\langle\rangle \mathbin{\hat{}_u} xs = xs\ xs \mathbin{\hat{}_u} \langle\rangle = xs$
⟨*proof*⟩

end

5 Unrestriction

theory *utp-unrest*
imports *utp-expr-insts*
begin

5.1 Definitions and Core Syntax

Unrestriction is an encoding of semantic freshness that allows us to reason about the presence of variables in predicates without being concerned with abstract syntax trees. An expression p is unrestricted by lens x, written $x \sharp p$, if altering the value of x has no effect on the valuation of p. This is a sufficient notion to prove many laws that would ordinarily rely on an *fv* function.

Unrestriction was first defined in the work of Marcel Oliveira [27, 26] in his UTP mechanisation in *ProofPowerZ*. Our definition modifies his in that our variables are semantically characterised as lenses, and supported by the lens laws, rather than named syntactic entities. We effectively fuse the ideas from both Feliachi [9] and Oliveira's [26] mechanisations of the UTP, the former being also purely semantic in nature.

We first set up overloaded syntax for unrestriction, as several concepts will have this defined.

consts
unrest :: $'a \Rightarrow 'b \Rightarrow bool$

syntax
-unrest :: *salpha* ⇒ *logic* ⇒ *logic* ⇒ *logic* (**infix** ‹♯› *20*)

syntax-consts
-unrest == *unrest*

translations
-unrest x p == *CONST unrest x p*
-unrest (*-salphaset* (*-salphamk* ($x +_L y$))) *P* <= *-unrest* ($x +_L y$) *P*

Our syntax translations support both variables and variable sets such that we can write down predicates like $\&x \sharp P$ and also $\{\&x, \&y, \&z\} \sharp P$.

We set up a simple tactic for discharging unrestriction conjectures using a simplification set.

named-theorems *unrest*
method *unrest-tac* = (*simp add*: *unrest*)*?*

Unrestriction for expressions is defined as a lifted construct using the underlying lens operations. It states that lens x is unrestricted by expression e provided that, for any state-space binding b and variable valuation v, the value which the expression evaluates to is unaltered if we set x to v in b. In other words, we cannot effect the behaviour of e by changing x. Thus e does not observe the portion of state-space characterised by x. We add this definition to our overloaded constant.

lift-definition *unrest-uexpr* :: $('a \Longrightarrow '\alpha) \Rightarrow ('b, '\alpha)$ *uexpr* $\Rightarrow$ *bool*
is $\lambda\ x\ e.\ \forall\ b\ v.\ e\ (put_x\ b\ v) = e\ b\ \langle proof \rangle$

adhoc-overloading
unrest $\rightleftharpoons$ *unrest-uexpr*

lemma *unrest-expr-alt-def*:
weak-lens x $\Longrightarrow (x \sharp P) = (\forall\ b\ b'.\ [\![P]\!]_e\ (b \oplus_L b'\ on\ x) = [\![P]\!]_e\ b)$
$\langle proof \rangle$

5.2 Unrestriction laws

We now prove unrestriction laws for the key constructs of our expression model. Many of these depend on lens properties and so variously employ the assumptions *mwb-lens* and *vwb-lens*, depending on the number of assumptions from the lenses theory is required.

Firstly, we prove a general property – if x and y are both unrestricted in P, then their composition is also unrestricted in P. One can interpret the composition here as a union – if the two sets of variables x and y are unrestricted, then so is their union.

lemma *unrest-var-comp* [*unrest*]:
$[\![x \sharp P;\ y \sharp P]\!] \Longrightarrow x;y \sharp P$
$\langle proof \rangle$

lemma *unrest-svar* [*unrest*]: $(\&x \sharp P) \longleftrightarrow (x \sharp P)$
$\langle proof \rangle$

No lens is restricted by a literal, since it returns the same value for any state binding.

lemma *unrest-lit* [*unrest*]: $x \sharp \text{«}v\text{»}$
$\langle proof \rangle$

If one lens is smaller than another, then any unrestriction on the larger lens implies unrestriction on the smaller.

lemma *unrest-sublens*:
fixes P :: $('a, '\alpha)$ *uexpr*
assumes $x \sharp P\ y \subseteq_L x$
shows $y \sharp P$
$\langle proof \rangle$

If two lenses are equivalent, and thus they characterise the same state-space regions, then clearly unrestrictions over them are equivalent.

lemma *unrest-equiv*:
fixes $P :: ('a, 'α)\ uexpr$
assumes $mwb\text{-}lens\ y\ x \approx_L y\ x \sharp P$
shows $y \sharp P$
⟨*proof*⟩

If we can show that an expression is unrestricted on a bijective lens, then is unrestricted on the entire state-space.

lemma *bij-lens-unrest-all*:
fixes $P :: ('a, 'α)\ uexpr$
assumes $bij\text{-}lens\ X\ X \sharp P$
shows $\Sigma \sharp P$
⟨*proof*⟩

lemma *bij-lens-unrest-all-eq*:
fixes $P :: ('a, 'α)\ uexpr$
assumes $bij\text{-}lens\ X$
shows $(\Sigma \sharp P) \longleftrightarrow (X \sharp P)$
⟨*proof*⟩

If an expression is unrestricted by all variables, then it is unrestricted by any variable

lemma *unrest-all-var*:
fixes $e :: ('a, 'α)\ uexpr$
assumes $\Sigma \sharp e$
shows $x \sharp e$
⟨*proof*⟩

We can split an unrestriction composed by lens plus

lemma *unrest-plus-split*:
fixes $P :: ('a, 'α)\ uexpr$
assumes $x \bowtie y\ vwb\text{-}lens\ x\ vwb\text{-}lens\ y$
shows $unrest\ (x +_L y)\ P \longleftrightarrow (x \sharp P) \wedge (y \sharp P)$
⟨*proof*⟩

The following laws demonstrate the primary motivation for lens independence: a variable expression is unrestricted by another variable only when the two variables are independent. Lens independence thus effectively allows us to semantically characterise when two variables, or sets of variables, are different.

lemma *unrest-var* [*unrest*]: $[\![mwb\text{-}lens\ x; x \bowtie y]\!] \Longrightarrow y \sharp var\ x$
⟨*proof*⟩

lemma *unrest-iuvar* [*unrest*]: $[\![mwb\text{-}lens\ x; x \bowtie y]\!] \Longrightarrow \$y \sharp \$x$
⟨*proof*⟩

lemma *unrest-ouvar* [*unrest*]: $[\![mwb\text{-}lens\ x; x \bowtie y]\!] \Longrightarrow \$y\acute{} \sharp \$x\acute{}$
⟨*proof*⟩

The following laws follow automatically from independence of input and output variables.

lemma *unrest-iuvar-ouvar* [*unrest*]:
fixes $x :: ('a \Longrightarrow 'α)$
assumes $mwb\text{-}lens\ y$
shows $\$x \sharp \$y\acute{}$
⟨*proof*⟩

lemma *unrest-ouvar-iuvar* [*unrest*]:
fixes $x :: ('a \Longrightarrow 'a)$
assumes *mwb-lens y*
shows $\$x\acute{} \sharp \$y$
⟨*proof*⟩

Unrestriction distributes through the various function lifting expression constructs; this allows us to prove unrestrictions for the majority of the expression language.

lemma *unrest-uop* [*unrest*]: $x \sharp e \Longrightarrow x \sharp uop\ f\ e$
⟨*proof*⟩

lemma *unrest-bop* [*unrest*]: $[\![x \sharp u; x \sharp v]\!] \Longrightarrow x \sharp bop\ f\ u\ v$
⟨*proof*⟩

lemma *unrest-trop* [*unrest*]: $[\![x \sharp u; x \sharp v; x \sharp w]\!] \Longrightarrow x \sharp trop\ f\ u\ v\ w$
⟨*proof*⟩

lemma *unrest-qtop* [*unrest*]: $[\![x \sharp u; x \sharp v; x \sharp w; x \sharp y]\!] \Longrightarrow x \sharp qtop\ f\ u\ v\ w\ y$
⟨*proof*⟩

For convenience, we also prove unrestriction rules for the bespoke operators on equality, numbers, arithmetic etc.

lemma *unrest-eq* [*unrest*]: $[\![x \sharp u; x \sharp v]\!] \Longrightarrow x \sharp u =_u v$
⟨*proof*⟩

lemma *unrest-zero* [*unrest*]: $x \sharp 0$
⟨*proof*⟩

lemma *unrest-one* [*unrest*]: $x \sharp 1$
⟨*proof*⟩

lemma *unrest-numeral* [*unrest*]: $x \sharp (numeral\ n)$
⟨*proof*⟩

lemma *unrest-sgn* [*unrest*]: $x \sharp u \Longrightarrow x \sharp sgn\ u$
⟨*proof*⟩

lemma *unrest-abs* [*unrest*]: $x \sharp u \Longrightarrow x \sharp abs\ u$
⟨*proof*⟩

lemma *unrest-plus* [*unrest*]: $[\![x \sharp u; x \sharp v]\!] \Longrightarrow x \sharp u + v$
⟨*proof*⟩

lemma *unrest-uminus* [*unrest*]: $x \sharp u \Longrightarrow x \sharp - u$
⟨*proof*⟩

lemma *unrest-minus* [*unrest*]: $[\![x \sharp u; x \sharp v]\!] \Longrightarrow x \sharp u - v$
⟨*proof*⟩

lemma *unrest-times* [*unrest*]: $[\![x \sharp u; x \sharp v]\!] \Longrightarrow x \sharp u * v$
⟨*proof*⟩

lemma *unrest-divide* [*unrest*]: $[\![x \sharp u; x \sharp v]\!] \Longrightarrow x \sharp u\ /\ v$
⟨*proof*⟩

lemma *unrest-case-prod* [*unrest*]: $[\![\bigwedge i\ j.\ x \sharp P\ i\ j\ ]\!] \Longrightarrow x \sharp \mathit{case\text{-}prod}\ P\ v$
⟨*proof*⟩

For a λ-term we need to show that the characteristic function expression does not restrict v for any input value x.

lemma *unrest-ulambda* [*unrest*]:
$[\![\bigwedge x.\ v \sharp F\ x\ ]\!] \Longrightarrow v \sharp (\lambda\ x \cdot F\ x)$
⟨*proof*⟩

end

6 Used-by

theory *utp-usedby*
imports *utp-unrest*
begin

The used-by predicate is the dual of unrestriction. It states that the given lens is an upper-bound on the size of state space the given expression depends on. It is similar to stating that the lens is a valid alphabet for the predicate. For convenience, and because the predicate uses a similar form, we will reuse much of unrestriction's infrastructure.

consts
$\mathit{usedBy} :: 'a \Rightarrow 'b \Rightarrow \mathit{bool}$

syntax
$\mathit{-usedBy} :: \mathit{salpha} \Rightarrow \mathit{logic} \Rightarrow \mathit{logic} \Rightarrow \mathit{logic}$ (**infix** ‹♮› *20*)

syntax-consts
$\mathit{-usedBy} == \mathit{usedBy}$

translations
$\mathit{-usedBy}\ x\ p == \mathit{CONST\ usedBy}\ x\ p$
$\mathit{-usedBy}\ (\mathit{-salphaset}\ (\mathit{-salphamk}\ (x +_L y)))\ P <= \mathit{-usedBy}\ (x +_L y)\ P$

lift-definition $\mathit{usedBy\text{-}uexpr} :: ('b \Longrightarrow 'a) \Rightarrow ('a, 'a)\ \mathit{uexpr} \Rightarrow \mathit{bool}$
is $\lambda\ x\ e.\ (\forall\ b\ b'.\ e\ (b' \oplus_L b\ \mathit{on}\ x) = e\ b)$ ⟨*proof*⟩

adhoc-overloading $\mathit{usedBy} \rightleftharpoons \mathit{usedBy\text{-}uexpr}$

lemma *usedBy-lit* [*unrest*]: $x \natural \ll v \gg$
⟨*proof*⟩

lemma *usedBy-sublens*:
fixes $P :: ('a, 'a)\ \mathit{uexpr}$
assumes $x \natural P\ x \subseteq_L y$ *vwb-lens* y
shows $y \natural P$
⟨*proof*⟩

lemma *usedBy-svar* [*unrest*]: $x \natural P \Longrightarrow \&x \natural P$
⟨*proof*⟩

lemma *usedBy-lens-plus-1* [*unrest*]: $x \natural P \Longrightarrow x;y \natural P$
⟨*proof*⟩

lemma *usedBy-lens-plus-2* [*unrest*]: ⟦ $x \bowtie y$; $y \natural P$ ⟧ ⟹ $x;y \natural P$
⟨*proof*⟩

Linking used-by to unrestriction: if x is used-by P, and x is independent of y, then P cannot depend on any variable in y.

lemma *usedBy-indep-uses*:
fixes $P :: ('a, '\alpha)$ *uexpr*
assumes $x \natural P$ $x \bowtie y$
shows $y \sharp P$
⟨*proof*⟩

lemma *usedBy-var* [*unrest*]:
assumes *vwb-lens* x $y \subseteq_L x$
shows $x \natural$ *var* y
⟨*proof*⟩

lemma *usedBy-uop* [*unrest*]: $x \natural e \Longrightarrow x \natural$ *uop* f e
⟨*proof*⟩

lemma *usedBy-bop* [*unrest*]: ⟦ $x \natural u$; $x \natural v$ ⟧ ⟹ $x \natural$ *bop* f u v
⟨*proof*⟩

lemma *usedBy-trop* [*unrest*]: ⟦ $x \natural u$; $x \natural v$; $x \natural w$ ⟧ ⟹ $x \natural$ *trop* f u v w
⟨*proof*⟩

lemma *usedBy-qtop* [*unrest*]: ⟦ $x \natural u$; $x \natural v$; $x \natural w$; $x \natural y$ ⟧ ⟹ $x \natural$ *qtop* f u v w y
⟨*proof*⟩

For convenience, we also prove used-by rules for the bespoke operators on equality, numbers, arithmetic etc.

lemma *usedBy-eq* [*unrest*]: ⟦ $x \natural u$; $x \natural v$ ⟧ ⟹ $x \natural u =_u v$
⟨*proof*⟩

lemma *usedBy-zero* [*unrest*]: $x \natural 0$
⟨*proof*⟩

lemma *usedBy-one* [*unrest*]: $x \natural 1$
⟨*proof*⟩

lemma *usedBy-numeral* [*unrest*]: $x \natural$ (*numeral* n)
⟨*proof*⟩

lemma *usedBy-sgn* [*unrest*]: $x \natural u \Longrightarrow x \natural$ *sgn* u
⟨*proof*⟩

lemma *usedBy-abs* [*unrest*]: $x \natural u \Longrightarrow x \natural$ *abs* u
⟨*proof*⟩

lemma *usedBy-plus* [*unrest*]: ⟦ $x \natural u$; $x \natural v$ ⟧ ⟹ $x \natural u + v$
⟨*proof*⟩

lemma *usedBy-uminus* [*unrest*]: $x \natural u \Longrightarrow x \natural - u$
⟨*proof*⟩

lemma *usedBy-minus* [*unrest*]: ⟦ $x \natural u$; $x \natural v$ ⟧ ⟹ $x \natural u - v$

⟨*proof*⟩

lemma *usedBy-times* [*unrest*]: ⟦ $x \natural u$; $x \natural v$ ⟧ ⟹ $x \natural u * v$
⟨*proof*⟩

lemma *usedBy-divide* [*unrest*]: ⟦ $x \natural u$; $x \natural v$ ⟧ ⟹ $x \natural u \,/\, v$
⟨*proof*⟩

lemma *usedBy-ulambda* [*unrest*]:
⟦ ⋀ x. $v \natural F\ x$ ⟧ ⟹ $v \natural (\lambda\ x \cdot F\ x)$
⟨*proof*⟩

lemma *unrest-var-sep* [*unrest*]:
vwb-lens x ⟹ $x \natural \&x{:}y$
⟨*proof*⟩

end

7 Substitution

theory *utp-subst*
imports
utp-expr
utp-unrest
begin

7.1 Substitution definitions

Variable substitution, like unrestriction, will be characterised semantically using lenses and state-spaces. Effectively a substitution σ is simply a function on the state-space which can be applied to an expression e using the syntax $\sigma \dagger e$. We introduce a polymorphic constant that will be used to represent application of a substitution, and also a set of theorems to represent laws.

consts
usubst :: $'s \Rightarrow 'a \Rightarrow 'b$ (**infixr** ‹†› *80*)

named-theorems *usubst*

A substitution is simply a transformation on the alphabet; it shows how variables should be mapped to different values. Most of the time these will be homogeneous functions but for flexibility we also allow some operations to be heterogeneous.

type-synonym $('\alpha,'\beta)$ *psubst* = $'\alpha \Rightarrow '\beta$
type-synonym $'\alpha$ *usubst* = $'\alpha \Rightarrow '\alpha$

Application of a substitution simply applies the function σ to the state binding b before it is handed to e as an input. This effectively ensures all variables are updated in e.

lift-definition *subst* :: $('\alpha,\ '\beta)$ *psubst* ⇒ $('a,\ '\beta)$ *uexpr* ⇒ $('a,\ '\alpha)$ *uexpr* **is**
$\lambda\ \sigma\ e\ b.\ e\ (\sigma\ b)$ ⟨*proof*⟩

adhoc-overloading
usubst ⇌ *subst*

Substitutions can be updated by associating variables with expressions. We thus create an additional polymorphic constant to represent updating the value of a variable to an expression

in a substitution, where the variable is modelled by type $'v$. This again allows us to support different notions of variables, such as deep variables, later.

consts *subst-upd* :: $('\alpha,'\beta)$ *psubst* $\Rightarrow$ $'v \Rightarrow ('a, '\alpha)$ *uexpr* $\Rightarrow ('\alpha,'\beta)$ *psubst*

The following function takes a substitution form state-space $'\alpha$ to $'\beta$, a lens with source $'\beta$ and view "'a", and an expression over $'\alpha$ and returning a value of type "$'a$, and produces an updated substitution. It does this by constructing a substitution function that takes state binding b, and updates the state first by applying the original substitution σ, and then updating the part of the state associated with lens x with expression evaluated in the context of b. This effectively means that x is now associated with expression v. We add this definition to our overloaded constant.

definition *subst-upd-uvar* :: $('\alpha,'\beta)$ *psubst* $\Rightarrow$ $('a \Longrightarrow '\beta) \Rightarrow ('a, '\alpha)$ *uexpr* $\Rightarrow ('\alpha,'\beta)$ *psubst* **where**
subst-upd-uvar σ x v = $(\lambda\ b.\ put_x\ (\sigma\ b)\ (\llbracket v \rrbracket_e\, b))$

adhoc-overloading
subst-upd $\rightleftharpoons$ *subst-upd-uvar*

The next function looks up the expression associated with a variable in a substitution by use of the *get* lens function.

lift-definition *usubst-lookup* :: $('\alpha,'\beta)$ *psubst* $\Rightarrow$ $('a \Longrightarrow '\beta) \Rightarrow ('a, '\alpha)$ *uexpr* $(\langle\langle\text{-}\rangle_s\rangle)$
is $\lambda\ \sigma\ x\ b.\ get_x\ (\sigma\ b)\ \langle proof \rangle$

Substitutions also exhibit a natural notion of unrestriction which states that σ does not restrict x if application of σ to an arbitrary state ρ will not effect the valuation of x. Put another way, it requires that *put* and the substitution commute.

definition *unrest-usubst* :: $('a \Longrightarrow '\alpha) \Rightarrow '\alpha$ *usubst* $\Rightarrow$ *bool*
where *unrest-usubst* x σ = $(\forall\ \varrho\ v.\ \sigma\ (put_x\ \varrho\ v) = put_x\ (\sigma\ \varrho)\ v)$

adhoc-overloading
unrest $\rightleftharpoons$ *unrest-usubst*

A conditional substitution deterministically picks one of the two substitutions based on a Booolean expression which is evaluated on the present state-space. It is analogous to a functional if-then-else.

definition *cond-subst* :: $'\alpha$ *usubst* $\Rightarrow$ (*bool*, $'\alpha$) *uexpr* $\Rightarrow$ $'\alpha$ *usubst* $\Rightarrow$ $'\alpha$ *usubst* $(\langle\langle 3\text{-} \lhd \text{-} \rhd_s / \text{-}\rangle\rangle\ [52,0,53]\ 52)$ **where**
cond-subst σ b ϱ = $(\lambda\ s.\ if\ \llbracket b \rrbracket_e\ s\ then\ \sigma(s)\ else\ \varrho(s))$

Parallel substitutions allow us to divide the state space into three segments using two lens, A and B. They correspond to the part of the state that should be updated by the respective substitution. The two lenses should be independent. If any part of the state is not covered by either lenses then this area is left unchanged (framed).

definition *par-subst* :: $'\alpha$ *usubst* $\Rightarrow$ $('a \Longrightarrow '\alpha) \Rightarrow ('b \Longrightarrow '\alpha) \Rightarrow '\alpha$ *usubst* $\Rightarrow$ $'\alpha$ *usubst* **where**
par-subst σ_1 A B σ_2 = $(\lambda\ s.\ (s \oplus_L (\sigma_1\ s)\ on\ A) \oplus_L (\sigma_2\ s)\ on\ B)$

7.2 Syntax translations

We support two kinds of syntax for substitutions, one where we construct a substitution using a maplet-style syntax, with variables mapping to expressions. Such a constructed substitution can be applied to an expression. Alternatively, we support the more traditional notation, $P\llbracket v/x \rrbracket$, which also support multiple simultaneous substitutions. We have to use double square brackets as the single ones are already well used.

We set up non-terminals to represent a single substitution maplet, a sequence of maplets, a list of expressions, and a list of alphabets. The parser effectively uses *subst-upd* to construct substitutions from multiple variables.

nonterminal *smaplet* **and** *smaplets* **and** *uexp* **and** *uexprs* **and** *salphas*

syntax
-smaplet :: [*salpha*, *'a*] => *smaplet* (‹- /↦$_s$/ -›)
:: *smaplet* => *smaplets* (‹-›)
-SMaplets :: [*smaplet*, *smaplets*] => *smaplets* (‹-,/ -›)
-SubstUpd :: [*'m usubst*, *smaplets*] => *'m usubst* (‹-/'(-')› [*900*,*0*] *900*)
-Subst :: *smaplets* => *'a* ⇀ *'b* (‹(*1*[-])›)
-psubst :: [*logic*, *svars*, *uexprs*] ⇒ *logic*
-subst :: *logic* ⇒ *uexprs* ⇒ *salphas* ⇒ *logic* (‹(-[![-'/-]!])› [*990*,*0*,*0*] *991*)
-uexp-l :: *logic* ⇒ *uexp* (‹-› [*64*] *64*)
-uexprs :: [*uexp*, *uexprs*] => *uexprs* (‹-,/ -›)
:: *uexp* => *uexprs* (‹-›)
-salphas :: [*salpha*, *salphas*] => *salphas* (‹-,/ -›)
:: *salpha* => *salphas* (‹-›)
-par-subst :: *logic* ⇒ *salpha* ⇒ *salpha* ⇒ *logic* ⇒ *logic* (‹- [-|-]$_s$ -› [*100*,*0*,*0*,*101*] *101*)

translations
-SubstUpd m (*-SMaplets xy ms*) == *-SubstUpd* (*-SubstUpd m xy*) *ms*
-SubstUpd m (*-smaplet x y*) == *CONST subst-upd m x y*
-Subst ms == *-SubstUpd* (*CONST id*) *ms*
-Subst (*-SMaplets ms1 ms2*) <= *-SubstUpd* (*-Subst ms1*) *ms2*
-SMaplets ms1 (*-SMaplets ms2 ms3*) <= *-SMaplets* (*-SMaplets ms1 ms2*) *ms3*
-subst P es vs => *CONST subst* (*-psubst* (*CONST id*) *vs es*) *P*
-psubst m (*-salphas x xs*) (*-uexprs v vs*) => *-psubst* (*-psubst m x v*) *xs vs*
-psubst m x v => *CONST subst-upd m x v*
-subst P v x <= *CONST usubst* (*CONST subst-upd* (*CONST id*) *x v*) *P*
-subst P v x <= *-subst P* (*-spvar x*) *v*
-par-subst σ_1 *A B* σ_2 == *CONST par-subst* σ_1 *A B* σ_2
-uexp-l e => *e*

Thus we can write things like $\sigma(x \mapsto_s v)$ to update a variable x in σ with expression v, $[x \mapsto_s e,\ y \mapsto_s f]$ to construct a substitution with two variables, and finally $P[\![v/x]\!]$, the traditional syntax.

We can now express deletion of a substitution maplet.

definition *subst-del* :: *'α usubst* ⇒ (*'a* ⟹ *'α*) ⇒ *'α usubst* (**infix** ‹−$_s$› *85*) **where**
subst-del σ *x* = $\sigma(x \mapsto_s \&x)$

7.3 Substitution Application Laws

We set up a simple substitution tactic that applies substitution and unrestriction laws

method *subst-tac* = (*simp add*: *usubst unrest*)*?*

Evaluation of a substitution expression involves application of the substitution to different variables. Thus we first prove laws for these cases. The simplest substitution, *id*, when applied to any variable x simply returns the variable expression, since *id* has no effect.

lemma *usubst-lookup-id* [*usubst*]: ⟨*id*⟩$_s$ *x* = *var x*
⟨*proof*⟩

lemma *subst-upd-id-lam* [*usubst*]: *subst-upd* (λ *x*. *x*) *x v* = *subst-upd id x v*

⟨*proof*⟩

A substitution update naturally yields the given expression.

lemma *usubst-lookup-upd* [*usubst*]:
assumes *weak-lens x*
shows $\langle\sigma(x \mapsto_s v)\rangle_s\ x = v$
⟨*proof*⟩

lemma *usubst-lookup-upd-pr-var* [*usubst*]:
assumes *weak-lens x*
shows $\langle\sigma(x \mapsto_s v)\rangle_s\ (pr\text{-}var\ x) = v$
⟨*proof*⟩

Substitution update is idempotent.

lemma *usubst-upd-idem* [*usubst*]:
assumes *mwb-lens x*
shows $\sigma(x \mapsto_s u,\ x \mapsto_s v) = \sigma(x \mapsto_s v)$
⟨*proof*⟩

lemma *usubst-upd-idem-sub* [*usubst*]:
assumes $x \subseteq_L y$ *mwb-lens y*
shows $\sigma(x \mapsto_s u,\ y \mapsto_s v) = \sigma(y \mapsto_s v)$
⟨*proof*⟩

Substitution updates commute when the lenses are independent.

lemma *usubst-upd-comm*:
assumes $x \bowtie y$
shows $\sigma(x \mapsto_s u,\ y \mapsto_s v) = \sigma(y \mapsto_s v,\ x \mapsto_s u)$
⟨*proof*⟩

lemma *usubst-upd-comm2*:
assumes $z \bowtie y$
shows $\sigma(x \mapsto_s u,\ y \mapsto_s v,\ z \mapsto_s s) = \sigma(x \mapsto_s u,\ z \mapsto_s s,\ y \mapsto_s v)$
⟨*proof*⟩

lemma *subst-upd-pr-var*: $s(\&x \mapsto_s v) = s(x \mapsto_s v)$
⟨*proof*⟩

A substitution which swaps two independent variables is an injective function.

lemma *swap-usubst-inj*:
fixes $x\ y :: ('a \Longrightarrow 'a)$
assumes *vwb-lens x vwb-lens y* $x \bowtie y$
shows *inj* $[x \mapsto_s \&y,\ y \mapsto_s \&x]$
⟨*proof*⟩

lemma *usubst-upd-var-id* [*usubst*]:
vwb-lens x $\Longrightarrow [x \mapsto_s var\ x] = id$
⟨*proof*⟩

lemma *usubst-upd-pr-var-id* [*usubst*]:
vwb-lens x $\Longrightarrow [x \mapsto_s var\ (pr\text{-}var\ x)] = id$
⟨*proof*⟩

lemma *usubst-upd-comm-dash* [*usubst*]:
fixes $x :: ('a \Longrightarrow 'a)$

shows $\sigma(\$x\acute{} \mapsto_s v, \$x \mapsto_s u) = \sigma(\$x \mapsto_s u, \$x\acute{} \mapsto_s v)$
⟨*proof*⟩

lemma *subst-upd-lens-plus* [*usubst*]:
subst-upd σ $(x +_L y)$ «(u,v)» = $\sigma(y \mapsto_s$ «v», $x \mapsto_s$ «u»)
⟨*proof*⟩

lemma *subst-upd-in-lens-plus* [*usubst*]:
subst-upd σ (*ivar* $(x +_L y)$) «(u,v)» = $\sigma(\$y \mapsto_s$ «v», $\$x \mapsto_s$ «u»)
⟨*proof*⟩

lemma *subst-upd-out-lens-plus* [*usubst*]:
subst-upd σ (*ovar* $(x +_L y)$) «(u,v)» = $\sigma(\$y\acute{} \mapsto_s$ «v», $\$x\acute{} \mapsto_s$ «u»)
⟨*proof*⟩

lemma *usubst-lookup-upd-indep* [*usubst*]:
assumes *mwb-lens* x $x \bowtie y$
shows $\langle\sigma(y \mapsto_s v)\rangle_s\ x = \langle\sigma\rangle_s\ x$
⟨*proof*⟩

lemma *subst-upd-plus* [*usubst*]:
$x \bowtie y \Longrightarrow$ *subst-upd* s $(x +_L y)$ $e = s(x \mapsto_s \pi_1(e), y \mapsto_s \pi_2(e))$
⟨*proof*⟩

If a variable is unrestricted in a substitution then it's application has no effect.

lemma *usubst-apply-unrest* [*usubst*]:
⟦ *vwb-lens* x; $x \sharp \sigma$ ⟧ $\Longrightarrow \langle\sigma\rangle_s\ x = var\ x$
⟨*proof*⟩

There follows various laws about deleting variables from a substitution.

lemma *subst-del-id* [*usubst*]:
vwb-lens $x \Longrightarrow id -_s x = id$
⟨*proof*⟩

lemma *subst-del-upd-same* [*usubst*]:
mwb-lens $x \Longrightarrow \sigma(x \mapsto_s v) -_s x = \sigma -_s x$
⟨*proof*⟩

lemma *subst-del-upd-diff* [*usubst*]:
$x \bowtie y \Longrightarrow \sigma(y \mapsto_s v) -_s x = (\sigma -_s x)(y \mapsto_s v)$
⟨*proof*⟩

If a variable is unrestricted in an expression, then any substitution of that variable has no effect on the expression .

lemma *subst-unrest* [*usubst*]: $x \sharp P \Longrightarrow \sigma(x \mapsto_s v) \dagger P = \sigma \dagger P$
⟨*proof*⟩

lemma *subst-unrest-2* [*usubst*]:
fixes $P :: ('a, 'o)$ *uexpr*
assumes $x \sharp P\ x \bowtie y$
shows $\sigma(x \mapsto_s u, y \mapsto_s v) \dagger P = \sigma(y \mapsto_s v) \dagger P$
⟨*proof*⟩

lemma *subst-unrest-3* [*usubst*]:
fixes $P :: ('a, 'o)$ *uexpr*

assumes $x \sharp P\ x \bowtie y\ x \bowtie z$
shows $\sigma(x \mapsto_s u,\ y \mapsto_s v,\ z \mapsto_s w) \dagger P = \sigma(y \mapsto_s v,\ z \mapsto_s w) \dagger P$
⟨*proof*⟩

lemma *subst-unrest-4* [*usubst*]:
fixes $P :: ('a,\ 'α)\ uexpr$
assumes $x \sharp P\ x \bowtie y\ x \bowtie z\ x \bowtie u$
shows $\sigma(x \mapsto_s e,\ y \mapsto_s f,\ z \mapsto_s g,\ u \mapsto_s h) \dagger P = \sigma(y \mapsto_s f,\ z \mapsto_s g,\ u \mapsto_s h) \dagger P$
⟨*proof*⟩

lemma *subst-unrest-5* [*usubst*]:
fixes $P :: ('a,\ 'α)\ uexpr$
assumes $x \sharp P\ x \bowtie y\ x \bowtie z\ x \bowtie u\ x \bowtie v$
shows $\sigma(x \mapsto_s e,\ y \mapsto_s f,\ z \mapsto_s g,\ u \mapsto_s h,\ v \mapsto_s i) \dagger P = \sigma(y \mapsto_s f,\ z \mapsto_s g,\ u \mapsto_s h,\ v \mapsto_s i) \dagger P$
⟨*proof*⟩

lemma *subst-compose-upd* [*usubst*]: $x \sharp \sigma \Longrightarrow \sigma \circ \varrho(x \mapsto_s v) = (\sigma \circ \varrho)(x \mapsto_s v)$
⟨*proof*⟩

Any substitution is a monotonic function.

lemma *subst-mono*: *mono* (*subst* σ)
⟨*proof*⟩

7.4 Substitution laws

We now prove the key laws that show how a substitution should be performed for every expression operator, including the core function operators, literals, variables, and the arithmetic operators. They are all added to the *usubst* theorem attribute so that we can apply them using the substitution tactic.

lemma *id-subst* [*usubst*]: $id \dagger v = v$
⟨*proof*⟩

lemma *subst-lit* [*usubst*]: $\sigma \dagger «v» = «v»$
⟨*proof*⟩

lemma *subst-var* [*usubst*]: $\sigma \dagger var\ x = \langle\sigma\rangle_s\ x$
⟨*proof*⟩

lemma *usubst-ulambda* [*usubst*]: $\sigma \dagger (\lambda\ x \cdot P(x)) = (\lambda\ x \cdot \sigma \dagger P(x))$
⟨*proof*⟩

lemma *unrest-usubst-del* [*unrest*]: $[\![\ vwb\text{-}lens\ x;\ x \sharp (\langle\sigma\rangle_s\ x);\ x \sharp \sigma -_s x\ ]\!] \Longrightarrow\ x \sharp (\sigma \dagger P)$
⟨*proof*⟩

We add the symmetric definition of input and output variables to substitution laws so that the variables are correctly normalised after substitution.

lemma *subst-uop* [*usubst*]: $\sigma \dagger uop\ f\ v = uop\ f\ (\sigma \dagger v)$
⟨*proof*⟩

lemma *subst-bop* [*usubst*]: $\sigma \dagger bop\ f\ u\ v = bop\ f\ (\sigma \dagger u)\ (\sigma \dagger v)$
⟨*proof*⟩

lemma *subst-trop* [*usubst*]: $\sigma \dagger trop\ f\ u\ v\ w = trop\ f\ (\sigma \dagger u)\ (\sigma \dagger v)\ (\sigma \dagger w)$
⟨*proof*⟩

lemma *subst-qtop* [*usubst*]: $\sigma \dagger qtop\ f\ u\ v\ w\ x = qtop\ f\ (\sigma \dagger u)\ (\sigma \dagger v)\ (\sigma \dagger w)\ (\sigma \dagger x)$
⟨*proof*⟩

lemma *subst-case-prod* [*usubst*]:
fixes $P :: 'i \Rightarrow 'j \Rightarrow ('a, 'a)\ uexpr$
shows $\sigma \dagger case\text{-}prod\ (\lambda\ x\ y.\ P\ x\ y)\ v = case\text{-}prod\ (\lambda\ x\ y.\ \sigma \dagger P\ x\ y)\ v$
⟨*proof*⟩

lemma *subst-plus* [*usubst*]: $\sigma \dagger (x + y) = \sigma \dagger x + \sigma \dagger y$
⟨*proof*⟩

lemma *subst-times* [*usubst*]: $\sigma \dagger (x * y) = \sigma \dagger x * \sigma \dagger y$
⟨*proof*⟩

lemma *subst-mod* [*usubst*]: $\sigma \dagger (x\ mod\ y) = \sigma \dagger x\ mod\ \sigma \dagger y$
⟨*proof*⟩

lemma *subst-div* [*usubst*]: $\sigma \dagger (x\ div\ y) = \sigma \dagger x\ div\ \sigma \dagger y$
⟨*proof*⟩

lemma *subst-minus* [*usubst*]: $\sigma \dagger (x - y) = \sigma \dagger x - \sigma \dagger y$
⟨*proof*⟩

lemma *subst-uminus* [*usubst*]: $\sigma \dagger (- x) = - (\sigma \dagger x)$
⟨*proof*⟩

lemma *usubst-sgn* [*usubst*]: $\sigma \dagger sgn\ x = sgn\ (\sigma \dagger x)$
⟨*proof*⟩

lemma *usubst-abs* [*usubst*]: $\sigma \dagger abs\ x = abs\ (\sigma \dagger x)$
⟨*proof*⟩

lemma *subst-zero* [*usubst*]: $\sigma \dagger 0 = 0$
⟨*proof*⟩

lemma *subst-one* [*usubst*]: $\sigma \dagger 1 = 1$
⟨*proof*⟩

lemma *subst-eq-upred* [*usubst*]: $\sigma \dagger (x =_u y) = (\sigma \dagger x =_u \sigma \dagger y)$
⟨*proof*⟩

This laws shows the effect of applying one substitution after another – we simply use function composition to compose them.

lemma *subst-subst* [*usubst*]: $\sigma \dagger \varrho \dagger e = (\varrho \circ \sigma) \dagger e$
⟨*proof*⟩

The next law is similar, but shows how such a substitution is to be applied to every updated variable additionally.

lemma *subst-upd-comp* [*usubst*]:
fixes $x :: ('a \Longrightarrow 'a)$
shows $\varrho(x \mapsto_s v) \circ \sigma = (\varrho \circ \sigma)(x \mapsto_s \sigma \dagger v)$
⟨*proof*⟩

lemma *subst-singleton*:

fixes $x :: ('a \Longrightarrow 'a)$
assumes $x \sharp \sigma$
shows $\sigma(x \mapsto_s v) \dagger P = (\sigma \dagger P)[\![v/x]\!]$
⟨*proof*⟩

lemmas *subst-to-singleton* = *subst-singleton id-subst*

7.5 Ordering substitutions

A simplification procedure to reorder substitutions maplets lexicographically by variable syntax

⟨*ML*⟩

7.6 Unrestriction laws

These are the key unrestriction theorems for substitutions and expressions involving substitutions.

lemma *unrest-usubst-single* [*unrest*]:
$[\![\textit{mwb-lens } x;\ x \sharp v]\!] \Longrightarrow x \sharp P[\![v/x]\!]$
⟨*proof*⟩

lemma *unrest-usubst-id* [*unrest*]:
mwb-lens $x \Longrightarrow x \sharp \textit{id}$
⟨*proof*⟩

lemma *unrest-usubst-upd* [*unrest*]:
$[\![x \bowtie y;\ x \sharp \sigma;\ x \sharp v]\!] \Longrightarrow x \sharp \sigma(y \mapsto_s v)$
⟨*proof*⟩

lemma *unrest-subst* [*unrest*]:
$[\![x \sharp P;\ x \sharp \sigma]\!] \Longrightarrow x \sharp (\sigma \dagger P)$
⟨*proof*⟩

7.7 Conditional Substitution Laws

lemma *usubst-cond-upd-1* [*usubst*]:
$\sigma(x \mapsto_s u) \lhd b \rhd_s \varrho(x \mapsto_s v) = (\sigma \lhd b \rhd_s \varrho)(x \mapsto_s u \lhd b \rhd v)$
⟨*proof*⟩

lemma *usubst-cond-upd-2* [*usubst*]:
$[\![\textit{vwb-lens } x;\ x \sharp \varrho]\!] \Longrightarrow \sigma(x \mapsto_s u) \lhd b \rhd_s \varrho = (\sigma \lhd b \rhd_s \varrho)(x \mapsto_s u \lhd b \rhd \&x)$
⟨*proof*⟩

lemma *usubst-cond-upd-3* [*usubst*]:
$[\![\textit{vwb-lens } x;\ x \sharp \sigma]\!] \Longrightarrow \sigma \lhd b \rhd_s \varrho(x \mapsto_s v) = (\sigma \lhd b \rhd_s \varrho)(x \mapsto_s \&x \lhd b \rhd v)$
⟨*proof*⟩

lemma *usubst-cond-id* [*usubst*]:
$\sigma \lhd b \rhd_s \sigma = \sigma$
⟨*proof*⟩

7.8 Parallel Substitution Laws

lemma *par-subst-id* [*usubst*]:
$[\![\textit{vwb-lens } A;\ \textit{vwb-lens } B]\!] \Longrightarrow \textit{id}\ [A|B]_s\ \textit{id} = \textit{id}$

⟨*proof*⟩

lemma *par-subst-left-empty* [*usubst*]:
⟦ *vwb-lens A* ⟧ ⟹ $\sigma\ [\emptyset|A]_s\ \varrho = id\ [\emptyset|A]_s\ \varrho$
⟨*proof*⟩

lemma *par-subst-right-empty* [*usubst*]:
⟦ *vwb-lens A* ⟧ ⟹ $\sigma\ [A|\emptyset]_s\ \varrho = \sigma\ [A|\emptyset]_s\ id$
⟨*proof*⟩

lemma *par-subst-comm*:
⟦ $A \bowtie B$ ⟧ ⟹ $\sigma\ [A|B]_s\ \varrho = \varrho\ [B|A]_s\ \sigma$
⟨*proof*⟩

lemma *par-subst-upd-left-in* [*usubst*]:
⟦ *vwb-lens A*; $A \bowtie B$; $x \subseteq_L A$ ⟧ ⟹ $\sigma(x \mapsto_s v)\ [A|B]_s\ \varrho = (\sigma\ [A|B]_s\ \varrho)(x \mapsto_s v)$
⟨*proof*⟩

lemma *par-subst-upd-left-out* [*usubst*]:
⟦ *vwb-lens A*; $x \bowtie A$ ⟧ ⟹ $\sigma(x \mapsto_s v)\ [A|B]_s\ \varrho = (\sigma\ [A|B]_s\ \varrho)$
⟨*proof*⟩

lemma *par-subst-upd-right-in* [*usubst*]:
⟦ *vwb-lens B*; $A \bowtie B$; $x \subseteq_L B$ ⟧ ⟹ $\sigma\ [A|B]_s\ \varrho(x \mapsto_s v) = (\sigma\ [A|B]_s\ \varrho)(x \mapsto_s v)$
⟨*proof*⟩

lemma *par-subst-upd-right-out* [*usubst*]:
⟦ *vwb-lens B*; $A \bowtie B$; $x \bowtie B$ ⟧ ⟹ $\sigma\ [A|B]_s\ \varrho(x \mapsto_s v) = (\sigma\ [A|B]_s\ \varrho)$
⟨*proof*⟩

end

8 UTP Tactics

theory *utp-tactics*
imports
utp-expr utp-unrest utp-usedby
keywords *update-uexpr-rep-eq-thms* :: *thy-decl*
begin

declare *image-comp* [*simp*]

In this theory, we define several automatic proof tactics that use transfer techniques to re-interpret proof goals about UTP predicates and relations in terms of pure HOL conjectures. The fundamental tactics to achieve this are *pred-simp* and *rel-simp*; a more detailed explanation of their behaviour is given below. The tactics can be given optional arguments to fine-tune their behaviour. By default, they use a weaker but faster form of transfer using rewriting; the option *robust*, however, forces them to use the slower but more powerful transfer of Isabelle's lifting package. A second option *no-interp* suppresses the re-interpretation of state spaces in order to eradicate record for tuple types prior to automatic proof.

In addition to *pred-simp* and *rel-simp*, we also provide the tactics *pred-auto* and *rel-auto*, as well as *pred-blast* and *rel-blast*; they, in essence, sequence the simplification tactics with the methods *auto* and *blast*, respectively.

8.1 Theorem Attributes

The following named attributes have to be introduced already here since our tactics must be able to see them. Note that we do not want to import the theories *utp-pred* and *utp-rel* here, so that both can potentially already make use of the tactics we define in this theory.

named-theorems *upred-defs upred definitional theorems*
named-theorems *urel-defs urel definitional theorems*

8.2 Generic Methods

We set up several automatic tactics that recast theorems on UTP predicates into equivalent HOL predicates, eliminating artefacts of the mechanisation as much as this is possible. Our approach is first to unfold all relevant definition of the UTP predicate model, then perform a transfer, and finally simplify by using lens and variable definitions, the split laws of alphabet records, and interpretation laws to convert record-based state spaces into products. The definition of the respective methods is facilitated by the Eisbach tool: we define generic methods that are parametrised by the tactics used for transfer, interpretation and subsequent automatic proof. Note that the tactics only apply to the head goal.

Generic Predicate Tactics

method *gen-pred-tac* **methods** *transfer-tac interp-tac prove-tac* = (
((*unfold upred-defs*) [*1*])*?*;
(*transfer-tac*),
(*simp add*: *fun-eq-iff*
lens-defs upred-defs alpha-splits Product-Type.split-beta)*?*,
(*interp-tac*)*?*);
(*prove-tac*)

Generic Relational Tactics

method *gen-rel-tac* **methods** *transfer-tac interp-tac prove-tac* = (
((*unfold upred-defs urel-defs*) [*1*])*?*;
(*transfer-tac*),
(*simp add*: *fun-eq-iff relcomp-unfold OO-def*
lens-defs upred-defs alpha-splits Product-Type.split-beta)*?*,
(*interp-tac*)*?*);
(*prove-tac*)

8.3 Transfer Tactics

Next, we define the component tactics used for transfer.

8.3.1 Robust Transfer

Robust transfer uses the transfer method of the lifting package.

method *slow-uexpr-transfer* = (*transfer*)

8.3.2 Faster Transfer

Fast transfer side-steps the use of the (*transfer*) method in favour of plain rewriting with the underlying *rep-eq-...* laws of lifted definitions. For moderately complex terms, surprisingly, the transfer step turned out to be a bottle-neck in some proofs; we observed that faster transfer resulted in a speed-up of approximately 30% when building the UTP theory heaps. On the downside, tactics using faster transfer do not always work but merely in about 95% of the cases. The approach typically works well when proving predicate equalities and refinements conjectures.

A known limitation is that the faster tactic, unlike lifting transfer, does not turn free variables into meta-quantified ones. This can, in some cases, interfere with the interpretation step and cause subsequent application of automatic proof tactics to fail. A fix is in progress [TODO].

Attribute Setup We first configure a dynamic attribute *uexpr-rep-eq-thms* to automatically collect all *rep-eq-* laws of lifted definitions on the *uexpr* type.

⟨*ML*⟩

We next configure a command **update-uexpr-rep-eq-thms** in order to update the content of the *uexpr-rep-eq-thms* attribute. Although the relevant theorems are collected automatically, for efficiency reasons, the user has to manually trigger the update process. The command must hence be executed whenever new lifted definitions for type *uexpr* are created. The updating mechanism uses **find-theorems** under the hood.

⟨*ML*⟩

update-uexpr-rep-eq-thms — Read *uexpr-rep-eq-thms* here.

Lastly, we require several named-theorem attributes to record the manual transfer laws and extra simplifications, so that the user can dynamically extend them in child theories.

named-theorems *uexpr-transfer-laws uexpr transfer laws*

declare *uexpr-eq-iff* [*uexpr-transfer-laws*]
named-theorems *uexpr-transfer-extra extra simplifications for uexpr transfer*

declare *unrest-uexpr.rep-eq* [*uexpr-transfer-extra*]
usedBy-uexpr.rep-eq [*uexpr-transfer-extra*]
utp-expr.numeral-uexpr-rep-eq [*uexpr-transfer-extra*]
utp-expr.less-eq-uexpr.rep-eq [*uexpr-transfer-extra*]
Abs-uexpr-inverse [*simplified*, *uexpr-transfer-extra*]
Rep-uexpr-inverse [*uexpr-transfer-extra*]

Tactic Definition We have all ingredients now to define the fast transfer tactic as a single simplification step.

method *fast-uexpr-transfer* =
(*simp add*: *uexpr-transfer-laws uexpr-rep-eq-thms uexpr-transfer-extra*)

8.4 Interpretation

The interpretation of record state spaces as products is done using the laws provided by the utility theory *Interp*. Note that this step can be suppressed by using the *no-interp* option.

method *uexpr-interp-tac* = (*simp add*: *lens-interp-laws*)*?*

8.5 User Tactics

In this section, we finally set-up the six user tactics: *pred-simp*, *rel-simp*, *pred-auto*, *rel-auto*, *pred-blast* and *rel-blast*. For this, we first define the proof strategies that are to be applied *after* the transfer steps.

method *utp-simp-tac* = (*clarsimp*)*?*
method *utp-auto-tac* = ((*clarsimp*)*?*; *auto*)
method *utp-blast-tac* = ((*clarsimp*)*?*; *blast*)

The ML file below provides ML constructor functions for tactics that process arguments suitable and invoke the generic methods *gen-pred-tac* and *gen-rel-tac* with suitable arguments.

⟨*ML*⟩

Finally, we execute the relevant outer commands for method setup. Sadly, this cannot be done at the level of Eisbach since the latter does not provide a convenient mechanism to process symbolic flags as arguments. It may be worth to put in a feature request with the developers of the Eisbach tool.

⟨*ML*⟩

Simpler, one-shot versions of the above tactics, but without the possibility of dynamic arguments.

method *rel-simp′*
uses *simp*
= (*simp add*: *upred-defs urel-defs lens-defs prod.case-eq-if relcomp-unfold uexpr-transfer-laws uexpr-transfer-extra uexpr-rep-eq-thms simp*)

method *rel-auto′*
uses *simp intro elim dest*
= (*auto intro*: *intro elim*: *elim dest*: *dest simp add*: *upred-defs urel-defs lens-defs relcomp-unfold uexpr-transfer-laws uexpr-transfer-extra uexpr-rep-eq-thms simp*)

method *rel-blast′*
uses *simp intro elim dest*
= (*rel-simp′ simp*: *simp*, *blast intro*: *intro elim*: *elim dest*: *dest*)

end

9 Meta-level Substitution

theory *utp-meta-subst*
imports *utp-subst utp-tactics*
begin

Meta substitution substitutes a HOL variable in a UTP expression for another UTP expression. It is analogous to UTP substitution, but acts on functions.

lift-definition *msubst* :: $('b \Rightarrow ('a, '\alpha)\ uexpr) \Rightarrow ('b, '\alpha)\ uexpr \Rightarrow ('a, '\alpha)\ uexpr$
is $\lambda\ F\ v\ b.\ F\ (v\ b)\ b$ ⟨*proof*⟩

update-uexpr-rep-eq-thms — Reread *rep-eq* theorems.

syntax
-msubst :: *logic* ⇒ *pttrn* ⇒ *logic* ⇒ *logic* (‹(-⟦-→-⟧)› [*990*,*0*,*0*] *991*)

syntax-consts
-msubst == *msubst*

translations
-msubst P x v == *CONST msubst* $(\lambda\ x.\ P)\ v$

lemma *msubst-lit* [*usubst*]: $«x»⟦x{\rightarrow}v⟧ = v$
⟨*proof*⟩

lemma *msubst-const* [*usubst*]: $P⟦x{\rightarrow}v⟧ = P$
⟨*proof*⟩

lemma *msubst-pair* [*usubst*]: $(P\ x\ y)⟦(x,\ y) \rightarrow (e,\ f)_u⟧ = (P\ x\ y)⟦x \rightarrow e⟧⟦y \rightarrow f⟧$
⟨*proof*⟩

lemma *msubst-lit-2-1* [*usubst*]: $«x»⟦(x,y){\rightarrow}(u,v)_u⟧ = u$
⟨*proof*⟩

lemma *msubst-lit-2-2* [*usubst*]: $«y»⟦(x,y){\rightarrow}(u,v)_u⟧ = v$
⟨*proof*⟩

lemma *msubst-lit'* [*usubst*]: $«y»⟦x{\rightarrow}v⟧ = «y»$
⟨*proof*⟩

lemma *msubst-lit'-2* [*usubst*]: $«z»⟦(x,y){\rightarrow}v⟧ = «z»$
⟨*proof*⟩

lemma *msubst-uop* [*usubst*]: $(uop\ f\ (v\ x))⟦x{\rightarrow}u⟧ = uop\ f\ ((v\ x)⟦x{\rightarrow}u⟧)$
⟨*proof*⟩

lemma *msubst-uop-2* [*usubst*]: $(uop\ f\ (v\ x\ y))⟦(x,y){\rightarrow}u⟧ = uop\ f\ ((v\ x\ y)⟦(x,y){\rightarrow}u⟧)$
⟨*proof*⟩

lemma *msubst-bop* [*usubst*]: $(bop\ f\ (v\ x)\ (w\ x))⟦x{\rightarrow}u⟧ = bop\ f\ ((v\ x)⟦x{\rightarrow}u⟧)\ ((w\ x)⟦x{\rightarrow}u⟧)$
⟨*proof*⟩

lemma *msubst-bop-2* [*usubst*]: $(bop\ f\ (v\ x\ y)\ (w\ x\ y))[\![(x,y)\rightarrow u]\!] = bop\ f\ ((v\ x\ y)[\![(x,y)\rightarrow u]\!])\ ((w\ x\ y)[\![(x,y)\rightarrow u]\!])$
⟨*proof*⟩

lemma *msubst-var* [*usubst*]:
$(utp\text{-}expr.var\ x)[\![y\rightarrow u]\!] = utp\text{-}expr.var\ x$
⟨*proof*⟩

lemma *msubst-var-2* [*usubst*]:
$(utp\text{-}expr.var\ x)[\![(y,z)\rightarrow u]\!] = utp\text{-}expr.var\ x$
⟨*proof*⟩

lemma *msubst-unrest* [*unrest*]: $[\![\bigwedge v.\ x \sharp P(v);\ x \sharp k\ ]\!] \Longrightarrow x \sharp P(v)[\![v\rightarrow k]\!]$
⟨*proof*⟩

end

10 Alphabetised Predicates

theory *utp-pred*
imports
utp-expr-funcs
utp-subst
utp-meta-subst
utp-tactics
begin

In this theory we begin to create an Isabelle version of the alphabetised predicate calculus that is described in Chapter 1 of the UTP book [22].

10.1 Predicate type and syntax

An alphabetised predicate is a simply a boolean valued expression.

type-synonym $'\alpha\ upred = (bool, '\alpha)\ uexpr$

translations
$(type)\ '\alpha\ upred <= (type)\ (bool, '\alpha)\ uexpr$

We want to remain as close as possible to the mathematical UTP syntax, but also want to be conservative with HOL. For this reason we chose not to steal syntax from HOL, but where possible use polymorphism to allow selection of the appropriate operator (UTP vs. HOL). Thus we will first remove the standard syntax for conjunction, disjunction, and negation, and replace these with adhoc overloaded definitions. We similarly use polymorphic constants for the other predicate calculus operators.

purge-notation
conj (**infixr** ‹∧› *35*) **and**
disj (**infixr** ‹∨› *30*) **and**
Not (‹(‹*open-block notation*=‹*prefix* ¬››¬ -)› [*40*] *40*)

consts
utrue :: $'a$ (‹*true*›)
ufalse :: $'a$ (‹*false*›)
uconj :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** ‹∧› *35*)

udisj :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** ‹$\vee$› *30*)
uimpl :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** ‹$\Rightarrow$› *25*)
uiff :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** ‹$\Leftrightarrow$› *25*)
unot :: $'a \Rightarrow 'a$ (‹$\neg$ -› [*40*] *40*)
uex :: $('a \Longrightarrow 'a) \Rightarrow 'p \Rightarrow 'p$
uall :: $('a \Longrightarrow 'a) \Rightarrow 'p \Rightarrow 'p$
ushEx :: $['a \Rightarrow 'p] \Rightarrow 'p$
ushAll :: $['a \Rightarrow 'p] \Rightarrow 'p$

adhoc-overloading
uconj $\rightleftharpoons$ *conj* **and**
udisj $\rightleftharpoons$ *disj* **and**
unot $\rightleftharpoons$ *Not*

We set up two versions of each of the quantifiers: *uex* / *uall* and *ushEx* / *ushAll*. The former pair allows quantification of UTP variables, whilst the latter allows quantification of HOL variables in concert with the literal expression constructor «*x*». Both varieties will be needed at various points. Syntactically they are distinguished by a boldface quantifier for the HOL versions (achieved by the "bold" escape in Isabelle).

nonterminal *idt-list*

syntax
-idt-el :: *idt* $\Rightarrow$ *idt-list* (‹-›)
-idt-list :: *idt* $\Rightarrow$ *idt-list* $\Rightarrow$ *idt-list* (‹(-,/ -)› [*0*, *1*])
-uex :: *salpha* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (‹$\exists$ - $\cdot$ -› [*0*, *10*] *10*)
-uall :: *salpha* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (‹$\forall$ - $\cdot$ -› [*0*, *10*] *10*)
-ushEx :: *pttrn* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (‹$\exists$ - $\cdot$ -› [*0*, *10*] *10*)
-ushAll :: *pttrn* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (‹$\forall$ - $\cdot$ -› [*0*, *10*] *10*)
-ushBEx :: *pttrn* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (‹$\exists$ - $\in$ - $\cdot$ -› [*0*, *0*, *10*] *10*)
-ushBAll :: *pttrn* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (‹$\forall$ - $\in$ - $\cdot$ -› [*0*, *0*, *10*] *10*)
-ushGAll :: *pttrn* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (‹$\forall$ - | - $\cdot$ -› [*0*, *0*, *10*] *10*)
-ushGtAll :: *idt* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (‹$\forall$ - > - $\cdot$ -› [*0*, *0*, *10*] *10*)
-ushLtAll :: *idt* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (‹$\forall$ - < - $\cdot$ -› [*0*, *0*, *10*] *10*)
-uvar-res :: *logic* $\Rightarrow$ *salpha* $\Rightarrow$ *logic* (**infixl** ‹$\restriction_v$› *90*)

translations
-uex x P == *CONST uex x P*
-uex (*-salphaset* (*-salphamk* (x $+_L$ y))) P <= *-uex* (x $+_L$ y) P
-uall x P == *CONST uall x P*
-uall (*-salphaset* (*-salphamk* (x $+_L$ y))) P <= *-uall* (x $+_L$ y) P
-ushEx x P == *CONST ushEx* (λ x. P)
$\exists$ $x \in A \cdot P$ => $\exists$ $x \cdot$ «x» $\in_u$ $A \wedge P$
-ushAll x P == *CONST ushAll* (λ x. P)
$\forall$ $x \in A \cdot P$ => $\forall$ $x \cdot$ «x» $\in_u$ $A \Rightarrow P$
$\forall$ $x \mid P \cdot Q$ => $\forall$ $x \cdot P \Rightarrow Q$
$\forall$ $x > y \cdot P$ => $\forall$ $x \cdot$ «x» $>_u$ $y \Rightarrow P$
$\forall$ $x < y \cdot P$ => $\forall$ $x \cdot$ «x» $<_u$ $y \Rightarrow P$

10.2 Predicate operators

We chose to maximally reuse definitions and laws built into HOL. For this reason, when introducing the core operators we proceed by lifting operators from the polymorphic algebraic hierarchy of HOL. Thus the initial definitions take place in the context of type class instantiations. We first introduce our own class called *refine* that will add the refinement operator

syntax to the HOL partial order class.

class *refine* = *order*

abbreviation *refineBy* :: *'a*::*refine* ⇒ *'a* ⇒ *bool* (**infix** ‹⊑› *50*) **where**
P ⊑ *Q* ≡ *less-eq Q P*

Since, on the whole, lattices in UTP are the opposite way up to the standard definitions in HOL, we syntactically invert the lattice operators. This is the one exception where we do steal HOL syntax, but I think it makes sense for UTP. Indeed we make this inversion for all of the lattice operators.

purge-notation *Lattices.inf* (**infixl** ‹⊓› *70*)
notation *Lattices.inf* (**infixl** ‹⊔› *70*)
purge-notation *Lattices.sup* (**infixl** ‹⊔› *65*)
notation *Lattices.sup* (**infixl** ‹⊓› *65*)

purge-notation *Inf* (‹(‹*open-block notation*=‹*prefix* ⨅››⨅ -)› [*900*] *900*)
notation *Inf* (‹⨆-› [*900*] *900*)
purge-notation *Sup* (‹(‹*open-block notation*=‹*prefix* ⨆››⨆ -)› [*900*] *900*)
notation *Sup* (‹⨅-› [*900*] *900*)

purge-notation *Orderings.bot* (‹⊥›)
notation *Orderings.bot* (‹⊤›)
purge-notation *Orderings.top* (‹⊤›)
notation *Orderings.top* (‹⊥›)

purge-syntax
-INF1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨅››⨅-./ -)› [*0*, *10*] *10*)
-INF :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨅››⨅-∈-./ -)› [*0*, *0*, *10*] *10*)
-SUP1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨆››⨆-./ -)› [*0*, *10*] *10*)
-SUP :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨆››⨆-∈-./ -)› [*0*, *0*, *10*] *10*)

syntax
-INF1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨆››⨆-./ -)› [*0*, *10*] *10*)
-INF :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨆››⨆-∈-./ -)› [*0*, *0*, *10*] *10*)
-SUP1 :: *pttrns* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨅››⨅-./ -)› [*0*, *10*] *10*)
-SUP :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (‹(‹*indent*=*3 notation*=‹*binder* ⨅››⨅-∈-./ -)› [*0*, *0*, *10*] *10*)

We trivially instantiate our refinement class

instance *uexpr* :: (*order*, *type*) *refine* ⟨*proof*⟩

theorem *upred-ref-iff* [*uexpr-transfer-laws*]:
(*P* ⊑ *Q*) = (∀ *b*. ⟦*Q*⟧*e* *b* ⟶ ⟦*P*⟧*e* *b*)
⟨*proof*⟩

Next we introduce the lattice operators, which is again done by lifting.

instantiation *uexpr* :: (*lattice*, *type*) *lattice*
begin
lift-definition *sup-uexpr* :: (*'a*, *'b*) *uexpr* ⇒ (*'a*, *'b*) *uexpr* ⇒ (*'a*, *'b*) *uexpr*
is *λP Q A. Lattices.sup* (*P A*) (*Q A*) ⟨*proof*⟩
lift-definition *inf-uexpr* :: (*'a*, *'b*) *uexpr* ⇒ (*'a*, *'b*) *uexpr* ⇒ (*'a*, *'b*) *uexpr*
is *λP Q A. Lattices.inf* (*P A*) (*Q A*) ⟨*proof*⟩
instance
⟨*proof*⟩
end

instantiation *uexpr* :: (*bounded-lattice*, *type*) *bounded-lattice*
begin
lift-definition *bot-uexpr* :: (*'a*, *'b*) *uexpr* **is** λ *A*. *Orderings.bot* ⟨*proof*⟩
lift-definition *top-uexpr* :: (*'a*, *'b*) *uexpr* **is** λ *A*. *Orderings.top* ⟨*proof*⟩
instance
⟨*proof*⟩
end

lemma *top-uexpr-rep-eq* [*simp*]:
$[\![Orderings.bot]\!]_e\ b = False$
⟨*proof*⟩

lemma *bot-uexpr-rep-eq* [*simp*]:
$[\![Orderings.top]\!]_e\ b = True$
⟨*proof*⟩

instance *uexpr* :: (*distrib-lattice*, *type*) *distrib-lattice*
⟨*proof*⟩

Finally we show that predicates form a Boolean algebra (under the lattice operators), a complete lattice, a completely distribute lattice, and a complete boolean algebra. This equip us with a very complete theory for basic logical propositions.

instance *uexpr* :: (*boolean-algebra*, *type*) *boolean-algebra*
⟨*proof*⟩

instantiation *uexpr* :: (*complete-lattice*, *type*) *complete-lattice*
begin
lift-definition *Inf-uexpr* :: (*'a*, *'b*) *uexpr set* ⇒ (*'a*, *'b*) *uexpr*
is λ *PS A*. *INF P∈PS*. *P*(*A*) ⟨*proof*⟩
lift-definition *Sup-uexpr* :: (*'a*, *'b*) *uexpr set* ⇒ (*'a*, *'b*) *uexpr*
is λ *PS A*. *SUP P∈PS*. *P*(*A*) ⟨*proof*⟩
instance
⟨*proof*⟩
end

instance *uexpr* :: (*complete-distrib-lattice*, *type*) *complete-distrib-lattice*
⟨*proof*⟩

instance *uexpr* :: (*complete-boolean-algebra*, *type*) *complete-boolean-algebra* ⟨*proof*⟩

From the complete lattice, we can also define and give syntax for the fixed-point operators. Like the lattice operators, these are reversed in UTP.

syntax
-mu :: *pttrn* ⇒ *logic* ⇒ *logic* (‹μ - • -› [*0*, *10*] *10*)
-nu :: *pttrn* ⇒ *logic* ⇒ *logic* (‹ν - • -› [*0*, *10*] *10*)

syntax-consts
-mu == *lfp* **and**
-nu == *gfp*

notation *gfp* (‹μ›)
notation *lfp* (‹ν›)

translations

$\nu\ X \cdot P == CONST\ lfp\ (\lambda\ X.\ P)$
$\mu\ X \cdot P == CONST\ gfp\ (\lambda\ X.\ P)$

With the lattice operators defined, we can proceed to give definitions for the standard predicate operators in terms of them.

definition *true-upred* $= (Orderings.top :: 'a\ upred)$
definition *false-upred* $= (Orderings.bot :: 'a\ upred)$
definition *conj-upred* $= (Lattices.inf :: 'a\ upred \Rightarrow 'a\ upred \Rightarrow 'a\ upred)$
definition *disj-upred* $= (Lattices.sup :: 'a\ upred \Rightarrow 'a\ upred \Rightarrow 'a\ upred)$
definition *not-upred* $= (uminus :: 'a\ upred \Rightarrow 'a\ upred)$
definition *diff-upred* $= (minus :: 'a\ upred \Rightarrow 'a\ upred \Rightarrow 'a\ upred)$

abbreviation *Conj-upred* $:: 'a\ upred\ set \Rightarrow 'a\ upred\ (\langle\bigwedge\text{-}\rangle\ [900]\ 900)$ **where**
$\bigwedge\ A \equiv \bigsqcup\ A$

abbreviation *Disj-upred* $:: 'a\ upred\ set \Rightarrow 'a\ upred\ (\langle\bigvee\text{-}\rangle\ [900]\ 900)$ **where**
$\bigvee\ A \equiv \bigsqcap\ A$

notation
conj-upred (**infixr** $\langle\wedge_p\rangle$ *35*) **and**
disj-upred (**infixr** $\langle\vee_p\rangle$ *30*)

Perhaps slightly confusingly, the UTP infimum is the HOL supremum and vice-versa. This is because, again, in UTP the lattice is inverted due to the definition of refinement and a desire to have miracle at the top, and abort at the bottom.

lift-definition *UINF* $:: ('a \Rightarrow 'a\ upred) \Rightarrow ('a \Rightarrow ('b{::}complete\text{-}lattice, 'a)\ uexpr) \Rightarrow ('b, 'a)\ uexpr$
is $\lambda\ P\ F\ b.\ Sup\ \{[\![F\ x]\!]_e b \mid x.\ [\![P\ x]\!]_e b\}$ ⟨*proof*⟩

lift-definition *USUP* $:: ('a \Rightarrow 'a\ upred) \Rightarrow ('a \Rightarrow ('b{::}complete\text{-}lattice, 'a)\ uexpr) \Rightarrow ('b, 'a)\ uexpr$
is $\lambda\ P\ F\ b.\ Inf\ \{[\![F\ x]\!]_e b \mid x.\ [\![P\ x]\!]_e b\}$ ⟨*proof*⟩

syntax
-USup :: *pttrn* ⇒ *logic* ⇒ *logic* $(\langle\bigwedge\ \text{-} \cdot \text{-}\rangle\ [0, 10]\ 10)$
-USup :: *pttrn* ⇒ *logic* ⇒ *logic* $(\langle\bigsqcup\ \text{-} \cdot \text{-}\rangle\ [0, 10]\ 10)$
-USup-mem :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* $(\langle\bigwedge\ \text{-} \in \text{-} \cdot \text{-}\rangle\ [0, 10]\ 10)$
-USup-mem :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* $(\langle\bigsqcup\ \text{-} \in \text{-} \cdot \text{-}\rangle\ [0, 10]\ 10)$
-USUP :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* $(\langle\bigwedge\ \text{-} \mid \text{-} \cdot \text{-}\rangle\ [0, 0, 10]\ 10)$
-USUP :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* $(\langle\bigsqcup\ \text{-} \mid \text{-} \cdot \text{-}\rangle\ [0, 0, 10]\ 10)$
-UInf :: *pttrn* ⇒ *logic* ⇒ *logic* $(\langle\bigvee\ \text{-} \cdot \text{-}\rangle\ [0, 10]\ 10)$
-UInf :: *pttrn* ⇒ *logic* ⇒ *logic* $(\langle\bigsqcap\ \text{-} \cdot \text{-}\rangle\ [0, 10]\ 10)$
-UInf-mem :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* $(\langle\bigvee\ \text{-} \in \text{-} \cdot \text{-}\rangle\ [0, 10]\ 10)$
-UInf-mem :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* $(\langle\bigsqcap\ \text{-} \in \text{-} \cdot \text{-}\rangle\ [0, 10]\ 10)$
-UINF :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* $(\langle\bigvee\ \text{-} \mid \text{-} \cdot \text{-}\rangle\ [0, 10]\ 10)$
-UINF :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* $(\langle\bigsqcap\ \text{-} \mid \text{-} \cdot \text{-}\rangle\ [0, 10]\ 10)$

translations
$\bigsqcap\ x \mid P \cdot F => CONST\ UINF\ (\lambda\ x.\ P)\ (\lambda\ x.\ F)$
$\bigsqcap\ x \cdot F == \bigsqcap\ x \mid true \cdot F$
$\bigsqcap\ x \cdot F == \bigsqcap\ x \mid true \cdot F$
$\bigsqcap\ x \in A \cdot F => \bigsqcap\ x \mid «x» \in_u «A» \cdot F$
$\bigsqcap\ x \in A \cdot F <= \bigsqcap\ x \mid «y» \in_u «A» \cdot F$
$\bigsqcap\ x \mid P \cdot F <= CONST\ UINF\ (\lambda\ y.\ P)\ (\lambda\ x.\ F)$
$\bigsqcap\ x \mid P \cdot F(x) <= CONST\ UINF\ (\lambda\ x.\ P)\ F$
$\bigsqcup\ x \mid P \cdot F => CONST\ USUP\ (\lambda\ x.\ P)\ (\lambda\ x.\ F)$
$\bigsqcup\ x \cdot F == \bigsqcup\ x \mid true \cdot F$

$\bigsqcup\ x \in A \cdot F => \bigsqcup\ x \mid «x» \in_u «A» \cdot F$
$\bigsqcup\ x \in A \cdot F <= \bigsqcup\ x \mid «y» \in_u «A» \cdot F$
$\bigsqcup\ x \mid P \cdot F <= CONST\ USUP\ (\lambda\ y.\ P)\ (\lambda\ x.\ F)$
$\bigsqcup\ x \mid P \cdot F(x) <= CONST\ USUP\ (\lambda\ x.\ P)\ F$

We also define the other predicate operators

lift-definition *impl*::$'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *upred* **is**
$\lambda\ P\ Q\ A.\ P\ A \longrightarrow Q\ A\ \langle proof \rangle$

lift-definition *iff-upred* ::$'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *upred* **is**
$\lambda\ P\ Q\ A.\ P\ A \longleftrightarrow Q\ A\ \langle proof \rangle$

lift-definition *ex* :: $('a \Longrightarrow '\alpha) \Rightarrow$ $'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *upred* **is**
$\lambda\ x\ P\ b.\ (\exists\ v.\ P(put_x\ b\ v))\ \langle proof \rangle$

lift-definition *shEx* ::$['\beta \Rightarrow '\alpha$ *upred*$] \Rightarrow$ $'\alpha$ *upred* **is**
$\lambda\ P\ A.\ \exists\ x.\ (P\ x)\ A\ \langle proof \rangle$

lift-definition *all* :: $('a \Longrightarrow '\alpha) \Rightarrow$ $'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *upred* **is**
$\lambda\ x\ P\ b.\ (\forall\ v.\ P(put_x\ b\ v))\ \langle proof \rangle$

lift-definition *shAll* ::$['\beta \Rightarrow '\alpha$ *upred*$] \Rightarrow$ $'\alpha$ *upred* **is**
$\lambda\ P\ A.\ \forall\ x.\ (P\ x)\ A\ \langle proof \rangle$

We define the following operator which is dual of existential quantification. It hides the valuation of variables other than x through existential quantification.

lift-definition *var-res* :: $'\alpha$ *upred* $\Rightarrow$ $('a \Longrightarrow '\alpha) \Rightarrow$ $'\alpha$ *upred* **is**
$\lambda\ P\ x\ b.\ \exists\ b'.\ P\ (b' \oplus_L b\ on\ x)\ \langle proof \rangle$

syntax-consts
-uvar-res $\rightleftharpoons$ *var-res*
translations
-uvar-res P a $\rightleftharpoons$ *CONST var-res P a*

We have to add a u subscript to the closure operator as I don't want to override the syntax for HOL lists (we'll be using them later).

lift-definition *closure*::$'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *upred* $(\langle[\text{-}]_u\rangle)$ **is**
$\lambda\ P\ A.\ \forall\ A'.\ P\ A'\ \langle proof \rangle$

lift-definition *taut* :: $'\alpha$ *upred* $\Rightarrow$ *bool* $(\langle\text{`-`}\rangle)$
is $\lambda\ P.\ \forall\ A.\ P\ A\ \langle proof \rangle$

Configuration for UTP tactics

update-uexpr-rep-eq-thms — Reread *rep-eq* theorems.

declare *utp-pred.taut.rep-eq* [*upred-defs*]

adhoc-overloading
utrue $\rightleftharpoons$ *true-upred* **and**
ufalse $\rightleftharpoons$ *false-upred* **and**
unot $\rightleftharpoons$ *not-upred* **and**
uconj $\rightleftharpoons$ *conj-upred* **and**
udisj $\rightleftharpoons$ *disj-upred* **and**
uimpl $\rightleftharpoons$ *impl* **and**

uiff $\rightleftharpoons$ *iff-upred* **and**
uex $\rightleftharpoons$ *ex* **and**
uall $\rightleftharpoons$ *all* **and**
ushEx $\rightleftharpoons$ *shEx* **and**
ushAll $\rightleftharpoons$ *shAll*

syntax
-uneq :: *logic* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (**infixl** $\langle\neq_u\rangle$ *50*)
-unmem :: ('*a*, 'α) *uexpr* $\Rightarrow$ ('*a set*, 'α) *uexpr* $\Rightarrow$ (*bool*, 'α) *uexpr* (**infix** $\langle\notin_u\rangle$ *50*)

syntax-consts
-uneq -unmem == *unot*

translations
$x \neq_u y$ == *CONST unot* ($x =_u y$)
$x \notin_u A$ == *CONST unot* (*CONST bop* ($\in$) *x A*)

declare *true-upred-def* [*upred-defs*]
declare *false-upred-def* [*upred-defs*]
declare *conj-upred-def* [*upred-defs*]
declare *disj-upred-def* [*upred-defs*]
declare *not-upred-def* [*upred-defs*]
declare *diff-upred-def* [*upred-defs*]
declare *subst-upd-uvar-def* [*upred-defs*]
declare *cond-subst-def* [*upred-defs*]
declare *par-subst-def* [*upred-defs*]
declare *subst-del-def* [*upred-defs*]
declare *unrest-usubst-def* [*upred-defs*]
declare *uexpr-defs* [*upred-defs*]

lemma *true-alt-def*: *true* = «*True*»
⟨*proof*⟩

lemma *false-alt-def*: *false* = «*False*»
⟨*proof*⟩

declare *true-alt-def*[*THEN sym*,*simp*]
declare *false-alt-def*[*THEN sym*,*simp*]

10.3 Unrestriction Laws

lemma *unrest-allE*:
⟦ $\Sigma \sharp P$; $P = true \Longrightarrow Q$; $P = false \Longrightarrow Q$ ⟧ $\Longrightarrow Q$
⟨*proof*⟩

lemma *unrest-true* [*unrest*]: $x \sharp true$
⟨*proof*⟩

lemma *unrest-false* [*unrest*]: $x \sharp false$
⟨*proof*⟩

lemma *unrest-conj* [*unrest*]: ⟦ $x \sharp (P :: '\alpha\ upred)$; $x \sharp Q$ ⟧ $\Longrightarrow x \sharp P \wedge Q$
⟨*proof*⟩

lemma *unrest-disj* [*unrest*]: ⟦ $x \sharp (P :: '\alpha\ upred)$; $x \sharp Q$ ⟧ $\Longrightarrow x \sharp P \vee Q$
⟨*proof*⟩

lemma *unrest-UINF* [*unrest*]:
$[\![(\bigwedge i.\ x \sharp P(i)); (\bigwedge i.\ x \sharp Q(i))]\!] \Longrightarrow x \sharp (\bigsqcap i \mid P(i) \cdot Q(i))$
⟨*proof*⟩

lemma *unrest-USUP* [*unrest*]:
$[\![(\bigwedge i.\ x \sharp P(i)); (\bigwedge i.\ x \sharp Q(i))]\!] \Longrightarrow x \sharp (\bigsqcup i \mid P(i) \cdot Q(i))$
⟨*proof*⟩

lemma *unrest-UINF-mem* [*unrest*]:
$[\![(\bigwedge i.\ i \in A \Longrightarrow x \sharp P(i))]\!] \Longrightarrow x \sharp (\bigsqcap i{\in}A \cdot P(i))$
⟨*proof*⟩

lemma *unrest-USUP-mem* [*unrest*]:
$[\![(\bigwedge i.\ i \in A \Longrightarrow x \sharp P(i))]\!] \Longrightarrow x \sharp (\bigsqcup i{\in}A \cdot P(i))$
⟨*proof*⟩

lemma *unrest-impl* [*unrest*]: $[\![x \sharp P; x \sharp Q]\!] \Longrightarrow x \sharp P \Rightarrow Q$
⟨*proof*⟩

lemma *unrest-iff* [*unrest*]: $[\![x \sharp P; x \sharp Q]\!] \Longrightarrow x \sharp P \Leftrightarrow Q$
⟨*proof*⟩

lemma *unrest-not* [*unrest*]: $x \sharp (P :: {'\alpha}\ upred) \Longrightarrow x \sharp (\neg P)$
⟨*proof*⟩

The sublens proviso can be thought of as membership below.

lemma *unrest-ex-in* [*unrest*]:
$[\![mwb\text{-}lens\ y; x \subseteq_L y]\!] \Longrightarrow x \sharp (\exists\ y \cdot P)$
⟨*proof*⟩

declare *sublens-refl* [*simp*]
declare *lens-plus-ub* [*simp*]
declare *lens-plus-right-sublens* [*simp*]
declare *comp-wb-lens* [*simp*]
declare *comp-mwb-lens* [*simp*]
declare *plus-mwb-lens* [*simp*]

lemma *unrest-ex-diff* [*unrest*]:
assumes $x \bowtie y$ $y \sharp P$
shows $y \sharp (\exists\ x \cdot P)$
⟨*proof*⟩

lemma *unrest-all-in* [*unrest*]:
$[\![mwb\text{-}lens\ y; x \subseteq_L y]\!] \Longrightarrow x \sharp (\forall\ y \cdot P)$
⟨*proof*⟩

lemma *unrest-all-diff* [*unrest*]:
assumes $x \bowtie y$ $y \sharp P$
shows $y \sharp (\forall\ x \cdot P)$
⟨*proof*⟩

lemma *unrest-var-res-diff* [*unrest*]:
assumes $x \bowtie y$
shows $y \sharp (P \upharpoonright_v x)$

⟨*proof*⟩

lemma *unrest-var-res-in* [*unrest*]:
assumes *mwb-lens* x y $\subseteq_L$ x y $\sharp$ P
shows $y \sharp (P \upharpoonright_v x)$
⟨*proof*⟩

lemma *unrest-shEx* [*unrest*]:
assumes $\bigwedge y.\ x \sharp P(y)$
shows $x \sharp (\exists\ y \cdot P(y))$
⟨*proof*⟩

lemma *unrest-shAll* [*unrest*]:
assumes $\bigwedge y.\ x \sharp P(y)$
shows $x \sharp (\forall\ y \cdot P(y))$
⟨*proof*⟩

lemma *unrest-closure* [*unrest*]:
$x \sharp [P]_u$
⟨*proof*⟩

10.4 Used-by laws

lemma *usedBy-not* [*unrest*]:
$[\![\ x \natural P\ ]\!] \Longrightarrow x \natural (\neg P)$
⟨*proof*⟩

lemma *usedBy-conj* [*unrest*]:
$[\![\ x \natural P;\ x \natural Q\ ]\!] \Longrightarrow x \natural (P \wedge Q)$
⟨*proof*⟩

lemma *usedBy-disj* [*unrest*]:
$[\![\ x \natural P;\ x \natural Q\ ]\!] \Longrightarrow x \natural (P \vee Q)$
⟨*proof*⟩

lemma *usedBy-impl* [*unrest*]:
$[\![\ x \natural P;\ x \natural Q\ ]\!] \Longrightarrow x \natural (P \Rightarrow Q)$
⟨*proof*⟩

lemma *usedBy-iff* [*unrest*]:
$[\![\ x \natural P;\ x \natural Q\ ]\!] \Longrightarrow x \natural (P \Leftrightarrow Q)$
⟨*proof*⟩

10.5 Substitution Laws

Substitution is monotone

lemma *subst-mono*: $P \sqsubseteq Q \Longrightarrow (\sigma \dagger P) \sqsubseteq (\sigma \dagger Q)$
⟨*proof*⟩

lemma *subst-true* [*usubst*]: $\sigma \dagger true = true$
⟨*proof*⟩

lemma *subst-false* [*usubst*]: $\sigma \dagger false = false$
⟨*proof*⟩

lemma *subst-not* [*usubst*]: $\sigma \dagger (\neg\ P) = (\neg\ \sigma \dagger P)$
⟨*proof*⟩

lemma *subst-impl* [*usubst*]: $\sigma \dagger (P \Rightarrow Q) = (\sigma \dagger P \Rightarrow \sigma \dagger Q)$
⟨*proof*⟩

lemma *subst-iff* [*usubst*]: $\sigma \dagger (P \Leftrightarrow Q) = (\sigma \dagger P \Leftrightarrow \sigma \dagger Q)$
⟨*proof*⟩

lemma *subst-disj* [*usubst*]: $\sigma \dagger (P \vee Q) = (\sigma \dagger P \vee \sigma \dagger Q)$
⟨*proof*⟩

lemma *subst-conj* [*usubst*]: $\sigma \dagger (P \wedge Q) = (\sigma \dagger P \wedge \sigma \dagger Q)$
⟨*proof*⟩

lemma *subst-sup* [*usubst*]: $\sigma \dagger (P \sqcap Q) = (\sigma \dagger P \sqcap \sigma \dagger Q)$
⟨*proof*⟩

lemma *subst-inf* [*usubst*]: $\sigma \dagger (P \sqcup Q) = (\sigma \dagger P \sqcup \sigma \dagger Q)$
⟨*proof*⟩

lemma *subst-UINF* [*usubst*]: $\sigma \dagger (\bigsqcap\ i \mid P(i) \cdot Q(i)) = (\bigsqcap\ i \mid (\sigma \dagger P(i)) \cdot (\sigma \dagger Q(i)))$
⟨*proof*⟩

lemma *subst-USUP* [*usubst*]: $\sigma \dagger (\bigsqcup\ i \mid P(i) \cdot Q(i)) = (\bigsqcup\ i \mid (\sigma \dagger P(i)) \cdot (\sigma \dagger Q(i)))$
⟨*proof*⟩

lemma *subst-closure* [*usubst*]: $\sigma \dagger [P]_u = [P]_u$
⟨*proof*⟩

lemma *subst-shEx* [*usubst*]: $\sigma \dagger (\exists\ x \cdot P(x)) = (\exists\ x \cdot \sigma \dagger P(x))$
⟨*proof*⟩

lemma *subst-shAll* [*usubst*]: $\sigma \dagger (\forall\ x \cdot P(x)) = (\forall\ x \cdot \sigma \dagger P(x))$
⟨*proof*⟩

TODO: Generalise the quantifier substitution laws to n-ary substitutions

lemma *subst-ex-same* [*usubst*]:
$mwb\text{-}lens\ x \Longrightarrow \sigma(x \mapsto_s v) \dagger (\exists\ x \cdot P) = \sigma \dagger (\exists\ x \cdot P)$
⟨*proof*⟩

lemma *subst-ex-same′* [*usubst*]:
$mwb\text{-}lens\ x \Longrightarrow \sigma(x \mapsto_s v) \dagger (\exists\ \&x \cdot P) = \sigma \dagger (\exists\ \&x \cdot P)$
⟨*proof*⟩

lemma *subst-ex-indep* [*usubst*]:
assumes $x \bowtie y$ $y \sharp v$
shows $(\exists\ y \cdot P)[\![v/x]\!] = (\exists\ y \cdot P[\![v/x]\!])$
⟨*proof*⟩

lemma *subst-ex-unrest* [*usubst*]:
$x \sharp \sigma \Longrightarrow \sigma \dagger (\exists\ x \cdot P) = (\exists\ x \cdot \sigma \dagger P)$
⟨*proof*⟩

lemma *subst-all-same* [*usubst*]:

mwb-lens $x \Longrightarrow \sigma(x \mapsto_s v) \dagger (\forall\ x \cdot P) = \sigma \dagger (\forall\ x \cdot P)$
⟨*proof*⟩

lemma *subst-all-indep* [*usubst*]:
assumes $x \bowtie y$ $y \sharp v$
shows $(\forall\ y \cdot P)[\![v/x]\!] = (\forall\ y \cdot P[\![v/x]\!])$
⟨*proof*⟩

lemma *msubst-true* [*usubst*]: $true[\![x{\rightarrow}v]\!] = true$
⟨*proof*⟩

lemma *msubst-false* [*usubst*]: $false[\![x{\rightarrow}v]\!] = false$
⟨*proof*⟩
lemma *msubst-not* [*usubst*]: $(\neg\ P(x))[\![x{\rightarrow}v]\!] = (\neg\ ((P\ x)[\![x{\rightarrow}v]\!]))$
⟨*proof*⟩

lemma *msubst-not-2* [*usubst*]: $(\neg\ P\ x\ y)[\![(x,y){\rightarrow}v]\!] = (\neg\ ((P\ x\ y)[\![(x,y){\rightarrow}v]\!]))$
⟨*proof*⟩

lemma *msubst-disj* [*usubst*]: $(P(x) \vee Q(x))[\![x{\rightarrow}v]\!] = ((P(x))[\![x{\rightarrow}v]\!] \vee (Q(x))[\![x{\rightarrow}v]\!])$
⟨*proof*⟩

lemma *msubst-disj-2* [*usubst*]: $(P\ x\ y \vee Q\ x\ y)[\![(x,y){\rightarrow}v]\!] = ((P\ x\ y)[\![(x,y){\rightarrow}v]\!] \vee (Q\ x\ y)[\![(x,y){\rightarrow}v]\!])$
⟨*proof*⟩

lemma *msubst-conj* [*usubst*]: $(P(x) \wedge Q(x))[\![x{\rightarrow}v]\!] = ((P(x))[\![x{\rightarrow}v]\!] \wedge (Q(x))[\![x{\rightarrow}v]\!])$
⟨*proof*⟩

lemma *msubst-conj-2* [*usubst*]: $(P\ x\ y \wedge Q\ x\ y)[\![(x,y){\rightarrow}v]\!] = ((P\ x\ y)[\![(x,y){\rightarrow}v]\!] \wedge (Q\ x\ y)[\![(x,y){\rightarrow}v]\!])$
⟨*proof*⟩

lemma *msubst-implies* [*usubst*]:
$(P\ x \Rightarrow Q\ x)[\![x{\rightarrow}v]\!] = ((P\ x)[\![x{\rightarrow}v]\!] \Rightarrow (Q\ x)[\![x{\rightarrow}v]\!])$
⟨*proof*⟩

lemma *msubst-implies-2* [*usubst*]:
$(P\ x\ y \Rightarrow Q\ x\ y)[\![(x,y){\rightarrow}v]\!] = ((P\ x\ y)[\![(x,y){\rightarrow}v]\!] \Rightarrow (Q\ x\ y)[\![(x,y){\rightarrow}v]\!])$
⟨*proof*⟩

lemma *msubst-shAll* [*usubst*]:
$(\forall\ x \cdot P\ x\ y)[\![y{\rightarrow}v]\!] = (\forall\ x \cdot (P\ x\ y)[\![y{\rightarrow}v]\!])$
⟨*proof*⟩

lemma *msubst-shAll-2* [*usubst*]:
$(\forall\ x \cdot P\ x\ y\ z)[\![(y,z){\rightarrow}v]\!] = (\forall\ x \cdot (P\ x\ y\ z)[\![(y,z){\rightarrow}v]\!])$
⟨*proof*⟩

10.6 Sandbox for conjectures

definition *utp-sandbox* :: $'\alpha$ *upred* ⇒ *bool* (‹*TRY*'(-')›) **where**
$TRY(P) = (P = undefined)$

translations
$P <= CONST\ utp\text{-}sandbox\ P$

end

11 Alphabet Manipulation

theory *utp-alphabet*
imports
utp-pred utp-usedby
begin

11.1 Preliminaries

Alphabets are simply types that characterise the state-space of an expression. Thus the Isabelle type system ensures that predicates cannot refer to variables not in the alphabet as this would be a type error. Often one would like to add or remove additional variables, for example if we wish to have a predicate which ranges only a smaller state-space, and then lift it into a predicate over a larger one. This is useful, for example, when dealing with relations which refer only to undashed variables (conditions) since we can use the type system to ensure well-formedness.

In this theory we will set up operators for extending and contracting and alphabet. We first set up a theorem attribute for alphabet laws and a tactic.

named-theorems *alpha*

method *alpha-tac* = (*simp add*: *alpha unrest*)*?*

11.2 Alphabet Extrusion

Alter an alphabet by application of a lens that demonstrates how the smaller alphabet (β) injects into the larger alphabet (α). This changes the type of the expression so it is parametrised over the large alphabet. We do this by using the lens *get* function to extract the smaller state binding, and then apply this to the expression.

We call this "extrusion" rather than "extension" because if the extension lens is bijective then it does not extend the alphabet. Nevertheless, it does have an effect because the type will be different which can be useful when converting predicates with equivalent alphabets.

lift-definition *aext* :: ('*a*, 'β) *uexpr* ⇒ ('β, 'α) *lens* ⇒ ('*a*, 'α) *uexpr* (**infixr** ‹$\oplus_p$› *95*)
is λ *P x b*. *P* (*get*$_x$ *b*) ⟨*proof*⟩

update-uexpr-rep-eq-thms

Next we prove some of the key laws. Extending an alphabet twice is equivalent to extending by the composition of the two lenses.

lemma *aext-twice*: (*P* $\oplus_p$ *a*) $\oplus_p$ *b* = *P* $\oplus_p$ (*a* ;$_L$ *b*)
⟨*proof*⟩

The bijective Σ lens identifies the source and view types. Thus an alphabet extension using this has no effect.

lemma *aext-id* [*simp*]: *P* $\oplus_p$ 1$_L$ = *P*
⟨*proof*⟩

Literals do not depend on any variables, and thus applying an alphabet extension only alters the predicate's type, and not its valuation .

lemma *aext-lit* [*simp*]: «*v*» $\oplus_p$ *a* = «*v*»
⟨*proof*⟩

lemma *aext-zero* [*simp*]: *0* $\oplus_p$ *a* = *0*

⟨*proof*⟩

lemma *aext-one* [*simp*]: $1 \oplus_p a = 1$
⟨*proof*⟩

lemma *aext-numeral* [*simp*]: $numeral\ n \oplus_p a = numeral\ n$
⟨*proof*⟩

lemma *aext-true* [*simp*]: $true \oplus_p a = true$
⟨*proof*⟩

lemma *aext-false* [*simp*]: $false \oplus_p a = false$
⟨*proof*⟩

lemma *aext-not* [*alpha*]: $(\neg\ P) \oplus_p x = (\neg\ (P \oplus_p x))$
⟨*proof*⟩

lemma *aext-and* [*alpha*]: $(P \wedge Q) \oplus_p x = (P \oplus_p x \wedge Q \oplus_p x)$
⟨*proof*⟩

lemma *aext-or* [*alpha*]: $(P \vee Q) \oplus_p x = (P \oplus_p x \vee Q \oplus_p x)$
⟨*proof*⟩

lemma *aext-imp* [*alpha*]: $(P \Rightarrow Q) \oplus_p x = (P \oplus_p x \Rightarrow Q \oplus_p x)$
⟨*proof*⟩

lemma *aext-iff* [*alpha*]: $(P \Leftrightarrow Q) \oplus_p x = (P \oplus_p x \Leftrightarrow Q \oplus_p x)$
⟨*proof*⟩

lemma *aext-shAll* [*alpha*]: $(\forall\ x \cdot P(x)) \oplus_p a = (\forall\ x \cdot P(x) \oplus_p a)$
⟨*proof*⟩

lemma *aext-UINF-ind* [*alpha*]: $(\bigsqcap\ x \cdot P\ x) \oplus_p a = (\bigsqcap\ x \cdot (P\ x \oplus_p a))$
⟨*proof*⟩

lemma *aext-UINF-mem* [*alpha*]: $(\bigsqcap\ x{\in}A \cdot P\ x) \oplus_p a = (\bigsqcap\ x{\in}A \cdot (P\ x \oplus_p a))$
⟨*proof*⟩

Alphabet extension distributes through the function liftings.

lemma *aext-uop* [*alpha*]: $uop\ f\ u \oplus_p a = uop\ f\ (u \oplus_p a)$
⟨*proof*⟩

lemma *aext-bop* [*alpha*]: $bop\ f\ u\ v \oplus_p a = bop\ f\ (u \oplus_p a)\ (v \oplus_p a)$
⟨*proof*⟩

lemma *aext-trop* [*alpha*]: $trop\ f\ u\ v\ w \oplus_p a = trop\ f\ (u \oplus_p a)\ (v \oplus_p a)\ (w \oplus_p a)$
⟨*proof*⟩

lemma *aext-qtop* [*alpha*]: $qtop\ f\ u\ v\ w\ x \oplus_p a = qtop\ f\ (u \oplus_p a)\ (v \oplus_p a)\ (w \oplus_p a)\ (x \oplus_p a)$
⟨*proof*⟩

lemma *aext-plus* [*alpha*]:
$(x + y) \oplus_p a = (x \oplus_p a) + (y \oplus_p a)$
⟨*proof*⟩

lemma *aext-minus* [*alpha*]:
$(x - y) \oplus_p a = (x \oplus_p a) - (y \oplus_p a)$
⟨*proof*⟩

lemma *aext-uminus* [*simp*]:
$(- x) \oplus_p a = - (x \oplus_p a)$
⟨*proof*⟩

lemma *aext-times* [*alpha*]:
$(x * y) \oplus_p a = (x \oplus_p a) * (y \oplus_p a)$
⟨*proof*⟩

lemma *aext-divide* [*alpha*]:
$(x \mathbin{/} y) \oplus_p a = (x \oplus_p a) \mathbin{/} (y \oplus_p a)$
⟨*proof*⟩

Extending a variable expression over x is equivalent to composing x with the alphabet, thus effectively yielding a variable whose source is the large alphabet.

lemma *aext-var* [*alpha*]:
$var\ x \oplus_p a = var\ (x ;_L a)$
⟨*proof*⟩

lemma *aext-ulambda* [*alpha*]: $((\lambda\ x \cdot P(x)) \oplus_p a) = (\lambda\ x \cdot P(x) \oplus_p a)$
⟨*proof*⟩

Alphabet extension is monotonic and continuous.

lemma *aext-mono*: $P \sqsubseteq Q \Longrightarrow P \oplus_p a \sqsubseteq Q \oplus_p a$
⟨*proof*⟩

lemma *aext-cont* [*alpha*]: *vwb-lens* $a \Longrightarrow (\bigsqcap A) \oplus_p a = (\bigsqcap P{\in}A.\ \ P \oplus_p a)$
⟨*proof*⟩

If a variable is unrestricted in a predicate, then the extended variable is unrestricted in the predicate with an alphabet extension.

lemma *unrest-aext* [*unrest*]:
$[\![mwb\text{-}lens\ a; x \sharp p]\!] \Longrightarrow unrest\ (x ;_L a)\ (p \oplus_p a)$
⟨*proof*⟩

If a given variable (or alphabet) b is independent of the extension lens a, that is, it is outside the original state-space of p, then it follows that once p is extended by a then b cannot be restricted.

lemma *unrest-aext-indep* [*unrest*]:
$a \bowtie b \Longrightarrow b \sharp (p \oplus_p a)$
⟨*proof*⟩

11.3 Expression Alphabet Restriction

Restrict an alphabet by application of a lens that demonstrates how the smaller alphabet (β) injects into the larger alphabet (α). Unlike extension, this operation can lose information if the expressions refers to variables in the larger alphabet.

lift-definition *arestr* :: $('a, '\alpha)$ *uexpr* $\Rightarrow$ $('\beta, '\alpha)$ *lens* $\Rightarrow$ $('a, '\beta)$ *uexpr* (**infixr** $\leftharpoondown_e$ *90*)
is $\lambda\ P\ x\ b.\ P\ (create_x\ b)$ ⟨*proof*⟩

update-uexpr-rep-eq-thms

lemma *arestr-id* [*simp*]: $P \upharpoonright_e 1_L = P$
⟨*proof*⟩

lemma *arestr-aext* [*simp*]: *mwb-lens a* $\Longrightarrow (P \oplus_p a) \upharpoonright_e a = P$
⟨*proof*⟩

If an expression's alphabet can be divided into two disjoint sections and the expression does not depend on the second half then restricting the expression to the first half is loss-less.

lemma *aext-arestr* [*alpha*]:
assumes *mwb-lens a bij-lens* $(a +_L b)$ $a \bowtie b$ $b \sharp P$
shows $(P \upharpoonright_e a) \oplus_p a = P$
⟨*proof*⟩

Alternative formulation of the above law using used-by instead of unrestriction.

lemma *aext-arestr′* [*alpha*]:
assumes $a \natural P$
shows $(P \upharpoonright_e a) \oplus_p a = P$
⟨*proof*⟩

lemma *arestr-lit* [*simp*]: $\langle\!\langle v \rangle\!\rangle \upharpoonright_e a = \langle\!\langle v \rangle\!\rangle$
⟨*proof*⟩

lemma *arestr-zero* [*simp*]: $0 \upharpoonright_e a = 0$
⟨*proof*⟩

lemma *arestr-one* [*simp*]: $1 \upharpoonright_e a = 1$
⟨*proof*⟩

lemma *arestr-numeral* [*simp*]: *numeral n* $\upharpoonright_e a =$ *numeral n*
⟨*proof*⟩

lemma *arestr-var* [*alpha*]:
var x $\upharpoonright_e a =$ *var* $(x /_L a)$
⟨*proof*⟩

lemma *arestr-true* [*simp*]: *true* $\upharpoonright_e a =$ *true*
⟨*proof*⟩

lemma *arestr-false* [*simp*]: *false* $\upharpoonright_e a =$ *false*
⟨*proof*⟩

lemma *arestr-not* [*alpha*]: $(\neg P)\upharpoonright_e a = (\neg (P \upharpoonright_e a))$
⟨*proof*⟩

lemma *arestr-and* [*alpha*]: $(P \wedge Q)\upharpoonright_e x = (P\upharpoonright_e x \wedge Q\upharpoonright_e x)$
⟨*proof*⟩

lemma *arestr-or* [*alpha*]: $(P \vee Q)\upharpoonright_e x = (P\upharpoonright_e x \vee Q\upharpoonright_e x)$
⟨*proof*⟩

lemma *arestr-imp* [*alpha*]: $(P \Rightarrow Q)\upharpoonright_e x = (P\upharpoonright_e x \Rightarrow Q\upharpoonright_e x)$
⟨*proof*⟩

11.4 Predicate Alphabet Restriction

In order to restrict the variables of a predicate, we also need to existentially quantify away the other variables. We can't do this at the level of expressions, as quantifiers are not applicable here. Consequently, we need a specialised version of alphabet restriction for predicates. It both restricts the variables using quantification and then removes them from the alphabet type using expression restriction.

definition *upred-ares* :: $'\alpha$ *upred* $\Rightarrow$ $('\beta \Longrightarrow '\alpha) \Rightarrow '\beta$ *upred*
where [*upred-defs*]: *upred-ares* P a = $(P \restriction_v a) \restriction_e a$

syntax
-upred-ares :: *logic* $\Rightarrow$ *salpha* $\Rightarrow$ *logic* (**infixl** $\langle\restriction_p\rangle$ *90*)

syntax-consts
-upred-ares == *upred-ares*

translations
-upred-ares P a == *CONST upred-ares* P a

lemma *upred-aext-ares* [*alpha*]:
vwb-lens $a \Longrightarrow P \oplus_p a \restriction_p a = P$
⟨*proof*⟩

lemma *upred-ares-aext* [*alpha*]:
$a \natural P \Longrightarrow (P \restriction_p a) \oplus_p a = P$
⟨*proof*⟩

lemma *upred-arestr-lit* [*simp*]: $\ll v \gg \restriction_p a = \ll v \gg$
⟨*proof*⟩

lemma *upred-arestr-true* [*simp*]: *true* $\restriction_p a$ = *true*
⟨*proof*⟩

lemma *upred-arestr-false* [*simp*]: *false* $\restriction_p a$ = *false*
⟨*proof*⟩

lemma *upred-arestr-or* [*alpha*]: $(P \vee Q)\restriction_p x = (P\restriction_p x \vee Q\restriction_p x)$
⟨*proof*⟩

11.5 Alphabet Lens Laws

lemma *alpha-in-var* [*alpha*]: $x ;_L fst_L$ = *in-var* x
⟨*proof*⟩

lemma *alpha-out-var* [*alpha*]: $x ;_L snd_L$ = *out-var* x
⟨*proof*⟩

lemma *in-var-prod-lens* [*alpha*]:
wb-lens $Y \Longrightarrow$ *in-var* $x ;_L (X \times_L Y)$ = *in-var* $(x ;_L X)$
⟨*proof*⟩

lemma *out-var-prod-lens* [*alpha*]:
wb-lens $X \Longrightarrow$ *out-var* $x ;_L (X \times_L Y)$ = *out-var* $(x ;_L Y)$
⟨*proof*⟩

11.6 Substitution Alphabet Extension

This allows us to extend the alphabet of a substitution, in a similar way to expressions.

definition *subst-ext* :: $'\alpha$ *usubst* $\Rightarrow$ $('\alpha \Longrightarrow '\beta) \Rightarrow '\beta$ *usubst* (**infix** $\langle\oplus_s\rangle$ *65*) **where**
[*upred-defs*]: $\sigma \oplus_s x = (\lambda\ s.\ put_x\ s\ (\sigma\ (get_x\ s)))$

lemma *id-subst-ext* [*usubst*]:
wb-lens $x \Longrightarrow id \oplus_s x = id$
⟨*proof*⟩

lemma *upd-subst-ext* [*alpha*]:
vwb-lens $x \Longrightarrow \sigma(y \mapsto_s v) \oplus_s x = (\sigma \oplus_s x)(\&x{:}y \mapsto_s v \oplus_p x)$
⟨*proof*⟩

lemma *apply-subst-ext* [*alpha*]:
vwb-lens $x \Longrightarrow (\sigma \dagger e) \oplus_p x = (\sigma \oplus_s x) \dagger (e \oplus_p x)$
⟨*proof*⟩

lemma *aext-upred-eq* [*alpha*]:
$((e =_u f) \oplus_p a) = ((e \oplus_p a) =_u (f \oplus_p a))$
⟨*proof*⟩

lemma *subst-aext-comp* [*usubst*]:
vwb-lens $a \Longrightarrow (\sigma \oplus_s a) \circ (\varrho \oplus_s a) = (\sigma \circ \varrho) \oplus_s a$
⟨*proof*⟩

11.7 Substitution Alphabet Restriction

This allows us to reduce the alphabet of a substitution, in a similar way to expressions.

definition *subst-res* :: $'\alpha$ *usubst* $\Rightarrow$ $('\beta \Longrightarrow '\alpha) \Rightarrow '\beta$ *usubst* (**infix** $\langle\restriction_s\rangle$ *65*) **where**
[*upred-defs*]: $\sigma \restriction_s x = (\lambda\ s.\ get_x\ (\sigma\ (create_x\ s)))$

lemma *id-subst-res* [*usubst*]:
mwb-lens $x \Longrightarrow id \restriction_s x = id$
⟨*proof*⟩

lemma *upd-subst-res* [*alpha*]:
mwb-lens $x \Longrightarrow \sigma(\&x{:}y \mapsto_s v) \restriction_s x = (\sigma \restriction_s x)(\&y \mapsto_s v \restriction_e x)$
⟨*proof*⟩

lemma *subst-ext-res* [*usubst*]:
mwb-lens $x \Longrightarrow (\sigma \oplus_s x) \restriction_s x = \sigma$
⟨*proof*⟩

lemma *unrest-subst-alpha-ext* [*unrest*]:
$x \bowtie y \Longrightarrow x \sharp (P \oplus_s y)$
⟨*proof*⟩
end

12 Lifting Expressions

theory *utp-lift*
imports
utp-alphabet

begin

12.1 Lifting definitions

We define operators for converting an expression to and from a relational state space with the help of alphabet extrusion and restriction. In general throughout Isabelle/UTP we adopt the notation $\lceil P \rceil$ with some subscript to denote lifting an expression into a larger alphabet, and $\lfloor P \rfloor$ for dropping into a smaller alphabet.

The following two functions lift and drop an expression, respectively, whose alphabet is $'\alpha$, into a product alphabet $'\alpha \times '\beta$. This allows us to deal with expressions which refer only to undashed variables, and use the type-system to ensure this.

abbreviation *lift-pre* :: $('a, '\alpha)$ *uexpr* $\Rightarrow$ $('a, '\alpha \times '\beta)$ *uexpr* $(\langle \lceil \text{-} \rceil_< \rangle)$
where $\lceil P \rceil_< \equiv P \oplus_p fst_L$

abbreviation *drop-pre* :: $('a, '\alpha \times '\beta)$ *uexpr* $\Rightarrow$ $('a, '\alpha)$ *uexpr* $(\langle \lfloor \text{-} \rfloor_< \rangle)$
where $\lfloor P \rfloor_< \equiv P \upharpoonright_e fst_L$

The following two functions lift and drop an expression, respectively, whose alphabet is $'\beta$, into a product alphabet $'\alpha \times '\beta$. This allows us to deal with expressions which refer only to dashed variables.

abbreviation *lift-post* :: $('a, '\beta)$ *uexpr* $\Rightarrow$ $('a, '\alpha \times '\beta)$ *uexpr* $(\langle \lceil \text{-} \rceil_> \rangle)$
where $\lceil P \rceil_> \equiv P \oplus_p snd_L$

abbreviation *drop-post* :: $('a, '\alpha \times '\beta)$ *uexpr* $\Rightarrow$ $('a, '\beta)$ *uexpr* $(\langle \lfloor \text{-} \rfloor_> \rangle)$
where $\lfloor P \rfloor_> \equiv P \upharpoonright_e snd_L$

12.2 Lifting Laws

With the help of our alphabet laws, we can prove some intuitive laws about alphabet lifting. For example, lifting variables yields an unprimed or primed relational variable expression, respectively.

lemma *lift-pre-var* [*simp*]:
$\lceil var\ x \rceil_< = \$x$
⟨*proof*⟩

lemma *lift-post-var* [*simp*]:
$\lceil var\ x \rceil_> = \$x'$
⟨*proof*⟩

12.3 Substitution Laws

lemma *pre-var-subst* [*usubst*]:
$\sigma(\$x \mapsto_s \ll v \gg) \dagger \lceil P \rceil_< = \sigma \dagger \lceil P[\![\ll v \gg / \&x]\!] \rceil_<$
⟨*proof*⟩

12.4 Unrestriction laws

Crucially, the lifting operators allow us to demonstrate unrestriction properties. For example, we can show that no primed variable is restricted in an expression over only the first element of the state-space product type.

lemma *unrest-dash-var-pre* [*unrest*]:
fixes $x :: ('a \Longrightarrow '\alpha)$

shows $\$x' \sharp \lceil p \rceil_<$
⟨*proof*⟩

end

13 Predicate Calculus Laws

theory *utp-pred-laws*
imports *utp-pred*
begin

13.1 Propositional Logic

Showing that predicates form a Boolean Algebra (under the predicate operators as opposed to the lattice operators) gives us many useful laws.

interpretation *boolean-algebra diff-upred not-upred conj-upred* (≤) (<)
disj-upred false-upred true-upred
⟨*proof*⟩

lemma *taut-true* [*simp*]: *'true'*
⟨*proof*⟩

lemma *taut-false* [*simp*]: *'false'* = *False*
⟨*proof*⟩

lemma *taut-conj*: *'A* ∧ *B'* = (*'A'* ∧ *'B'*)
⟨*proof*⟩

lemma *taut-conj-elim* [*elim*!]:
⟦ *'A* ∧ *B'*; ⟦ *'A'*; *'B'* ⟧ ⟹ *P* ⟧ ⟹ *P*
⟨*proof*⟩

lemma *taut-refine-impl*: ⟦ *Q* ⊑ *P*; *'P'* ⟧ ⟹ *'Q'*
⟨*proof*⟩

lemma *taut-shEx-elim*:
⟦ *'*(∃ *x* · *P x*)*'*; ⋀ *x*. Σ ♯ *P x*; ⋀ *x*. *'P x'* ⟹ *Q* ⟧ ⟹ *Q*
⟨*proof*⟩

Linking refinement and HOL implication

lemma *refine-prop-intro*:
assumes Σ ♯ *P* Σ ♯ *Q* *'Q'* ⟹ *'P'*
shows *P* ⊑ *Q*
⟨*proof*⟩

lemma *taut-not*: Σ ♯ *P* ⟹ (¬ *'P'*) = *'¬ P'*
⟨*proof*⟩

lemma *taut-shAll-intro*:
∀ *x*. *'P x'* ⟹ *'*∀ *x* · *P x'*
⟨*proof*⟩

lemma *taut-shAll-intro-2*:
∀ *x y*. *'P x y'* ⟹ *'*∀ (*x*, *y*) · *P x y'*

⟨*proof*⟩

lemma *taut-impl-intro*:
⟦ Σ ♯ *P*; ‘*P*‘ ⟹ ‘*Q*‘ ⟧ ⟹ ‘*P* ⇒ *Q*‘
⟨*proof*⟩

lemma *upred-eval-taut*:
‘*P*⟦«*b*»/&**v**⟧‘ = ⟦*P*⟧$_e$*b*
⟨*proof*⟩

lemma *refBy-order*: *P* ⊑ *Q* = ‘*Q* ⇒ *P*‘
⟨*proof*⟩

lemma *conj-idem* [*simp*]: ((*P*::′α *upred*) ∧ *P*) = *P*
⟨*proof*⟩

lemma *disj-idem* [*simp*]: ((*P*::′α *upred*) ∨ *P*) = *P*
⟨*proof*⟩

lemma *conj-comm*: ((*P*::′α *upred*) ∧ *Q*) = (*Q* ∧ *P*)
⟨*proof*⟩

lemma *disj-comm*: ((*P*::′α *upred*) ∨ *Q*) = (*Q* ∨ *P*)
⟨*proof*⟩

lemma *conj-subst*: *P* = *R* ⟹ ((*P*::′α *upred*) ∧ *Q*) = (*R* ∧ *Q*)
⟨*proof*⟩

lemma *disj-subst*: *P* = *R* ⟹ ((*P*::′α *upred*) ∨ *Q*) = (*R* ∨ *Q*)
⟨*proof*⟩

lemma *conj-assoc*:(((*P*::′α *upred*) ∧ *Q*) ∧ *S*) = (*P* ∧ (*Q* ∧ *S*))
⟨*proof*⟩

lemma *disj-assoc*:(((*P*::′α *upred*) ∨ *Q*) ∨ *S*) = (*P* ∨ (*Q* ∨ *S*))
⟨*proof*⟩

lemma *conj-disj-abs*:((*P*::′α *upred*) ∧ (*P* ∨ *Q*)) = *P*
⟨*proof*⟩

lemma *disj-conj-abs*:((*P*::′α *upred*) ∨ (*P* ∧ *Q*)) = *P*
⟨*proof*⟩

lemma *conj-disj-distr*:((*P*::′α *upred*) ∧ (*Q* ∨ *R*)) = ((*P* ∧ *Q*) ∨ (*P* ∧ *R*))
⟨*proof*⟩

lemma *disj-conj-distr*:((*P*::′α *upred*) ∨ (*Q* ∧ *R*)) = ((*P* ∨ *Q*) ∧ (*P* ∨ *R*))
⟨*proof*⟩

lemma *true-disj-zero* [*simp*]:
(*P* ∨ *true*) = *true* (*true* ∨ *P*) = *true*
⟨*proof*⟩

lemma *true-conj-zero* [*simp*]:
(*P* ∧ *false*) = *false* (*false* ∧ *P*) = *false*

⟨*proof*⟩

lemma *false-sup* [*simp*]: $\mathit{false} \sqcap P = P$ $P \sqcap \mathit{false} = P$
⟨*proof*⟩

lemma *true-inf* [*simp*]: $\mathit{true} \sqcup P = P$ $P \sqcup \mathit{true} = P$
⟨*proof*⟩

lemma *imp-vacuous* [*simp*]: $(\mathit{false} \Rightarrow u) = \mathit{true}$
⟨*proof*⟩

lemma *imp-true* [*simp*]: $(p \Rightarrow \mathit{true}) = \mathit{true}$
⟨*proof*⟩

lemma *true-imp* [*simp*]: $(\mathit{true} \Rightarrow p) = p$
⟨*proof*⟩

lemma *impl-mp1* [*simp*]: $(P \wedge (P \Rightarrow Q)) = (P \wedge Q)$
⟨*proof*⟩

lemma *impl-mp2* [*simp*]: $((P \Rightarrow Q) \wedge P) = (Q \wedge P)$
⟨*proof*⟩

lemma *impl-adjoin*: $((P \Rightarrow Q) \wedge R) = ((P \wedge R \Rightarrow Q \wedge R) \wedge R)$
⟨*proof*⟩

lemma *impl-refine-intro*:
$[\![Q_1 \sqsubseteq P_1;\ P_2 \sqsubseteq (P_1 \wedge Q_2)]\!] \Longrightarrow (P_1 \Rightarrow P_2) \sqsubseteq (Q_1 \Rightarrow Q_2)$
⟨*proof*⟩

lemma *spec-refine*:
$Q \sqsubseteq (P \wedge R) \Longrightarrow (P \Rightarrow Q) \sqsubseteq R$
⟨*proof*⟩

lemma *impl-disjI*: $[\![\text{`}P \Rightarrow R\text{`};\ \text{`}Q \Rightarrow R\text{`}]\!] \Longrightarrow \text{`}(P \vee Q) \Rightarrow R\text{`}$
⟨*proof*⟩

lemma *conditional-iff*:
$(P \Rightarrow Q) = (P \Rightarrow R) \longleftrightarrow \text{`}P \Rightarrow (Q \Leftrightarrow R)\text{`}$
⟨*proof*⟩

lemma *p-and-not-p* [*simp*]: $(P \wedge \neg P) = \mathit{false}$
⟨*proof*⟩

lemma *p-or-not-p* [*simp*]: $(P \vee \neg P) = \mathit{true}$
⟨*proof*⟩

lemma *p-imp-p* [*simp*]: $(P \Rightarrow P) = \mathit{true}$
⟨*proof*⟩

lemma *p-iff-p* [*simp*]: $(P \Leftrightarrow P) = \mathit{true}$
⟨*proof*⟩

lemma *p-imp-false* [*simp*]: $(P \Rightarrow \mathit{false}) = (\neg P)$
⟨*proof*⟩

lemma *not-conj-deMorgans* [*simp*]: $(\neg\ ((P::'\alpha\ upred) \wedge\ Q)) = ((\neg\ P) \vee (\neg\ Q))$
⟨*proof*⟩

lemma *not-disj-deMorgans* [*simp*]: $(\neg\ ((P::'\alpha\ upred) \vee\ Q)) = ((\neg\ P) \wedge (\neg\ Q))$
⟨*proof*⟩

lemma *conj-disj-not-abs* [*simp*]: $((P::'\alpha\ upred) \wedge ((\neg P) \vee\ Q)) = (P \wedge\ Q)$
⟨*proof*⟩

lemma *subsumption1*:
$`P \Rightarrow Q` \Longrightarrow (P \vee\ Q) = Q$
⟨*proof*⟩

lemma *subsumption2*:
$`Q \Rightarrow P` \Longrightarrow (P \vee\ Q) = P$
⟨*proof*⟩

lemma *neg-conj-cancel1*: $(\neg\ P \wedge (P \vee\ Q)) = (\neg\ P \wedge\ Q :: '\alpha\ upred)$
⟨*proof*⟩

lemma *neg-conj-cancel2*: $(\neg\ Q \wedge (P \vee\ Q)) = (\neg\ Q \wedge\ P :: '\alpha\ upred)$
⟨*proof*⟩

lemma *double-negation* [*simp*]: $(\neg\ \neg\ (P::'\alpha\ upred)) = P$
⟨*proof*⟩

lemma *true-not-false* [*simp*]: $true \neq false\ false \neq true$
⟨*proof*⟩

lemma *closure-conj-distr*: $([P]_u \wedge\ [Q]_u) = [P \wedge\ Q]_u$
⟨*proof*⟩

lemma *closure-imp-distr*: $`[P \Rightarrow Q]_u \Rightarrow [P]_u \Rightarrow [Q]_u`$
⟨*proof*⟩

lemma *true-iff* [*simp*]: $(P \Leftrightarrow true) = P$
⟨*proof*⟩

lemma *taut-iff-eq*:
$`P \Leftrightarrow Q` \longleftrightarrow (P = Q)$
⟨*proof*⟩

lemma *impl-alt-def*: $(P \Rightarrow Q) = (\neg\ P \vee\ Q)$
⟨*proof*⟩

13.2 Lattice laws

lemma *uinf-or*:
fixes $P\ Q :: '\alpha\ upred$
shows $(P \sqcap\ Q) = (P \vee\ Q)$
⟨*proof*⟩

lemma *usup-and*:
fixes $P\ Q :: '\alpha\ upred$
shows $(P \sqcup\ Q) = (P \wedge\ Q)$

⟨*proof*⟩

lemma *UINF-alt-def*:
$(\bigsqcap\ i \mid A(i) \cdot P(i)) = (\bigsqcap\ i \cdot A(i) \wedge P(i))$
⟨*proof*⟩

lemma *USUP-true* [*simp*]: $(\bigsqcup\ P \mid F(P) \cdot true) = true$
⟨*proof*⟩

lemma *UINF-mem-UNIV* [*simp*]: $(\bigsqcap\ x{\in}UNIV \cdot P(x)) = (\bigsqcap\ x \cdot P(x))$
⟨*proof*⟩

lemma *USUP-mem-UNIV* [*simp*]: $(\bigsqcup\ x{\in}UNIV \cdot P(x)) = (\bigsqcup\ x \cdot P(x))$
⟨*proof*⟩

lemma *USUP-false* [*simp*]: $(\bigsqcup\ i \cdot false) = false$
⟨*proof*⟩

lemma *USUP-mem-false* [*simp*]: $I \neq \{\} \Longrightarrow (\bigsqcup\ i{\in}I \cdot false) = false$
⟨*proof*⟩

lemma *USUP-where-false* [*simp*]: $(\bigsqcup\ i \mid false \cdot P(i)) = true$
⟨*proof*⟩

lemma *UINF-true* [*simp*]: $(\bigsqcap\ i \cdot true) = true$
⟨*proof*⟩

lemma *UINF-ind-const* [*simp*]:
$(\bigsqcap\ i \cdot P) = P$
⟨*proof*⟩

lemma *UINF-mem-true* [*simp*]: $A \neq \{\} \Longrightarrow (\bigsqcap\ i{\in}A \cdot true) = true$
⟨*proof*⟩

lemma *UINF-false* [*simp*]: $(\bigsqcap\ i \mid P(i) \cdot false) = false$
⟨*proof*⟩

lemma *UINF-where-false* [*simp*]: $(\bigsqcap\ i \mid false \cdot P(i)) = false$
⟨*proof*⟩

lemma *UINF-cong-eq*:
$[\![\bigwedge x.\ P_1(x) = P_2(x);\ \bigwedge x.\ `P_1(x) \Rightarrow Q_1(x) =_u Q_2(x)`]\!] \Longrightarrow$
$(\bigsqcap\ x \mid P_1(x) \cdot Q_1(x)) = (\bigsqcap\ x \mid P_2(x) \cdot Q_2(x))$
⟨*proof*⟩

lemma *UINF-as-Sup*: $(\bigsqcap\ P \in \mathcal{P} \cdot P) = \bigsqcap\ \mathcal{P}$
⟨*proof*⟩

lemma *UINF-as-Sup-collect*: $(\bigsqcap P{\in}A \cdot f(P)) = (\bigsqcap P{\in}A.\ f(P))$
⟨*proof*⟩

lemma *UINF-as-Sup-collect'*: $(\bigsqcap P \cdot f(P)) = (\bigsqcap P.\ f(P))$
⟨*proof*⟩

lemma *UINF-as-Sup-image*: $(\bigsqcap\ P \mid «P» \in_u «A» \cdot f(P)) = \bigsqcap\ (f\ `\ A)$

⟨*proof*⟩

lemma *USUP-as-Inf*: $(\bigsqcup\ P \in \mathcal{P} \cdot P) = \bigsqcup\ \mathcal{P}$
⟨*proof*⟩

lemma *USUP-as-Inf-collect*: $(\bigsqcup P{\in}A \cdot f(P)) = (\bigsqcup P{\in}A.\ f(P))$
⟨*proof*⟩

lemma *USUP-as-Inf-collect'*: $(\bigsqcup P \cdot f(P)) = (\bigsqcup P.\ f(P))$
⟨*proof*⟩

lemma *USUP-as-Inf-image*: $(\bigsqcup\ P \in \mathcal{P} \cdot f(P)) = \bigsqcup\ (f\ `\ \mathcal{P})$
⟨*proof*⟩

lemma *USUP-image-eq* [*simp*]: $USUP\ (\lambda i.\ «i» \in_u «f\ `\ A»)\ g = (\bigsqcup\ i{\in}A \cdot g(f(i)))$
⟨*proof*⟩

lemma *UINF-image-eq* [*simp*]: $UINF\ (\lambda i.\ «i» \in_u «f\ `\ A»)\ g = (\bigsqcap\ i{\in}A \cdot g(f(i)))$
⟨*proof*⟩

lemma *subst-continuous* [*usubst*]: $\sigma \dagger (\bigsqcap\ A) = (\bigsqcap\ \{\sigma \dagger P \mid P.\ P \in A\})$
⟨*proof*⟩

lemma *not-UINF*: $(\neg\ (\bigsqcap\ i{\in}A{\cdot}\ P(i))) = (\bigsqcup\ i{\in}A{\cdot}\ \neg\ P(i))$
⟨*proof*⟩

lemma *not-USUP*: $(\neg\ (\bigsqcup\ i{\in}A{\cdot}\ P(i))) = (\bigsqcap\ i{\in}A{\cdot}\ \neg\ P(i))$
⟨*proof*⟩

lemma *not-UINF-ind*: $(\neg\ (\bigsqcap\ i \cdot P(i))) = (\bigsqcup\ i \cdot \neg\ P(i))$
⟨*proof*⟩

lemma *not-USUP-ind*: $(\neg\ (\bigsqcup\ i \cdot P(i))) = (\bigsqcap\ i \cdot \neg\ P(i))$
⟨*proof*⟩

lemma *UINF-empty* [*simp*]: $(\bigsqcap\ i \in \{\} \cdot P(i)) = false$
⟨*proof*⟩

lemma *UINF-insert* [*simp*]: $(\bigsqcap\ i{\in}insert\ x\ xs \cdot P(i)) = (P(x) \sqcap (\bigsqcap\ i{\in}xs \cdot P(i)))$
⟨*proof*⟩

lemma *UINF-atLeast-first*:
$P(n) \sqcap (\bigsqcap\ i \in \{Suc\ n..\} \cdot P(i)) = (\bigsqcap\ i \in \{n..\} \cdot P(i))$
⟨*proof*⟩

lemma *UINF-atLeast-Suc*:
$(\bigsqcap\ i \in \{Suc\ m..\} \cdot P(i)) = (\bigsqcap\ i \in \{m..\} \cdot P(Suc\ i))$
⟨*proof*⟩

lemma *USUP-empty* [*simp*]: $(\bigsqcup\ i \in \{\} \cdot P(i)) = true$
⟨*proof*⟩

lemma *USUP-insert* [*simp*]: $(\bigsqcup\ i{\in}insert\ x\ xs \cdot P(i)) = (P(x) \sqcup (\bigsqcup\ i{\in}xs \cdot P(i)))$
⟨*proof*⟩

lemma *USUP-atLeast-first*:
$(P(n) \wedge (\bigsqcup\ i \in \{Suc\ n..\} \cdot P(i))) = (\bigsqcup\ i \in \{n..\} \cdot P(i))$
⟨*proof*⟩

lemma *USUP-atLeast-Suc*:
$(\bigsqcup\ i \in \{Suc\ m..\} \cdot P(i)) = (\bigsqcup\ i \in \{m..\} \cdot P(Suc\ i))$
⟨*proof*⟩

lemma *conj-UINF-dist*:
$(P \wedge (\bigsqcap\ Q{\in}S \cdot F(Q))) = (\bigsqcap\ Q{\in}S \cdot P \wedge F(Q))$
⟨*proof*⟩

lemma *conj-UINF-ind-dist*:
$(P \wedge (\bigsqcap\ Q \cdot F(Q))) = (\bigsqcap\ Q \cdot P \wedge F(Q))$
⟨*proof*⟩

lemma *disj-UINF-dist*:
$S \neq \{\} \Longrightarrow (P \vee (\bigsqcap\ Q{\in}S \cdot F(Q))) = (\bigsqcap\ Q{\in}S \cdot P \vee F(Q))$
⟨*proof*⟩

lemma *UINF-conj-UINF* [*simp*]:
$((\bigsqcap\ i{\in}I \cdot P(i)) \vee (\bigsqcap\ i{\in}I \cdot Q(i))) = (\bigsqcap\ i{\in}I \cdot P(i) \vee Q(i))$
⟨*proof*⟩

lemma *conj-USUP-dist*:
$S \neq \{\} \Longrightarrow (P \wedge (\bigsqcup\ Q{\in}S \cdot F(Q))) = (\bigsqcup\ Q{\in}S \cdot P \wedge F(Q))$
⟨*proof*⟩

lemma *USUP-conj-USUP* [*simp*]: $((\bigsqcup\ P \in A \cdot F(P)) \wedge (\bigsqcup\ P \in A \cdot G(P))) = (\bigsqcup\ P \in A \cdot F(P) \wedge G(P))$
⟨*proof*⟩

lemma *UINF-all-cong* [*cong*]:
assumes $\bigwedge P.\ F(P) = G(P)$
shows $(\bigsqcap\ P \cdot F(P)) = (\bigsqcap\ P \cdot G(P))$
⟨*proof*⟩

lemma *UINF-cong*:
assumes $\bigwedge P.\ P \in A \Longrightarrow F(P) = G(P)$
shows $(\bigsqcap\ P{\in}A \cdot F(P)) = (\bigsqcap\ P{\in}A \cdot G(P))$
⟨*proof*⟩

lemma *USUP-all-cong*:
assumes $\bigwedge P.\ F(P) = G(P)$
shows $(\bigsqcup\ P \cdot F(P)) = (\bigsqcup\ P \cdot G(P))$
⟨*proof*⟩

lemma *USUP-cong*:
assumes $\bigwedge P.\ P \in A \Longrightarrow F(P) = G(P)$
shows $(\bigsqcup\ P{\in}A \cdot F(P)) = (\bigsqcup\ P{\in}A \cdot G(P))$
⟨*proof*⟩

lemma *UINF-subset-mono*: $A \subseteq B \Longrightarrow (\bigsqcap\ P{\in}B \cdot F(P)) \sqsubseteq (\bigsqcap\ P{\in}A \cdot F(P))$
⟨*proof*⟩

lemma *USUP-subset-mono*: $A \subseteq B \Longrightarrow (\bigsqcup\ P{\in}A \cdot F(P)) \sqsubseteq (\bigsqcup\ P{\in}B \cdot F(P))$
⟨*proof*⟩

lemma *UINF-impl*: $(\bigsqcap\ P{\in}A \cdot F(P) \Rightarrow G(P)) = ((\bigsqcup\ P{\in}A \cdot F(P)) \Rightarrow (\bigsqcap\ P{\in}A \cdot G(P)))$
⟨*proof*⟩

lemma *USUP-is-forall*: $(\bigsqcup\ x \cdot P(x)) = (\forall\ x \cdot P(x))$
⟨*proof*⟩

lemma *USUP-ind-is-forall*: $(\bigsqcup\ x{\in}A \cdot P(x)) = (\forall\ x{\in}«A» \cdot P(x))$
⟨*proof*⟩

lemma *UINF-is-exists*: $(\bigsqcap\ x \cdot P(x)) = (\exists\ x \cdot P(x))$
⟨*proof*⟩

lemma *UINF-all-nats* [*simp*]:
fixes $P :: nat \Rightarrow {'}\alpha\ upred$
shows $(\bigsqcap\ n \cdot \bigsqcap\ i{\in}\{0..n\} \cdot P(i)) = (\bigsqcap\ n \cdot P(n))$
⟨*proof*⟩

lemma *USUP-all-nats* [*simp*]:
fixes $P :: nat \Rightarrow {'}\alpha\ upred$
shows $(\bigsqcup\ n \cdot \bigsqcup\ i{\in}\{0..n\} \cdot P(i)) = (\bigsqcup\ n \cdot P(n))$
⟨*proof*⟩

lemma *UINF-upto-expand-first*:
$m < n \Longrightarrow (\bigsqcap\ i \in \{m..{<}n\} \cdot P(i)) = ((P(m) :: {'}\alpha\ upred) \vee (\bigsqcap\ i \in \{Suc\ m..{<}n\} \cdot P(i)))$
⟨*proof*⟩

lemma *UINF-upto-expand-last*:
$(\bigsqcap\ i \in \{0..{<}Suc(n)\} \cdot P(i)) = ((\bigsqcap\ i \in \{0..{<}n\} \cdot P(i)) \vee P(n))$
⟨*proof*⟩

lemma *UINF-Suc-shift*: $(\bigsqcap\ i \in \{Suc\ 0..{<}Suc\ n\} \cdot P(i)) = (\bigsqcap\ i \in \{0..{<}n\} \cdot P(Suc\ i))$
⟨*proof*⟩

lemma *USUP-upto-expand-first*:
$(\bigsqcup\ i \in \{0..{<}Suc(n)\} \cdot P(i)) = (P(0) \wedge (\bigsqcup\ i \in \{1..{<}Suc(n)\} \cdot P(i)))$
⟨*proof*⟩

lemma *USUP-Suc-shift*: $(\bigsqcup\ i \in \{Suc\ 0..{<}Suc\ n\} \cdot P(i)) = (\bigsqcup\ i \in \{0..{<}n\} \cdot P(Suc\ i))$
⟨*proof*⟩

lemma *UINF-list-conv*:
$(\bigsqcap\ i \in \{0..{<}length(xs)\} \cdot f\ (xs\ !\ i)) = foldr\ (\vee)\ (map\ f\ xs)\ false$
⟨*proof*⟩

lemma *USUP-list-conv*:
$(\bigsqcup\ i \in \{0..{<}length(xs)\} \cdot f\ (xs\ !\ i)) = foldr\ (\wedge)\ (map\ f\ xs)\ true$
⟨*proof*⟩

lemma *UINF-refines*:
$[\![\bigwedge i.\ i{\in}I \Longrightarrow P \sqsubseteq Q\ i\ ]\!] \Longrightarrow P \sqsubseteq (\bigsqcap\ i{\in}I \cdot Q\ i)$
⟨*proof*⟩

lemma *UINF-refines'*:
assumes $\bigwedge i.\ P \sqsubseteq Q(i)$
shows $P \sqsubseteq (\bigsqcap\ i \cdot Q(i))$
⟨*proof*⟩

lemma *UINF-pred-ueq* [*simp*]:
$(\bigsqcap\ x \mid \text{«}x\text{»} =_u\ v \cdot P(x)) = (P\ x)[\![x \rightarrow v]\!]$
⟨*proof*⟩

lemma *UINF-pred-lit-eq* [*simp*]:
$(\bigsqcap\ x \mid \text{«}x = v\text{»} \cdot P(x)) = (P\ v)$
⟨*proof*⟩

13.3 Equality laws

lemma *eq-upred-refl* [*simp*]: $(x =_u\ x) = true$
⟨*proof*⟩

lemma *eq-upred-sym*: $(x =_u\ y) = (y =_u\ x)$
⟨*proof*⟩

lemma *eq-cong-left*:
assumes *vwb-lens x* $\$x \sharp Q\ \$x' \sharp Q\ \$x \sharp R\ \$x' \sharp R$
shows $((\$x' =_u\ \$x \wedge Q) = (\$x' =_u\ \$x \wedge R)) \longleftrightarrow (Q = R)$
⟨*proof*⟩

lemma *conj-eq-in-var-subst*:
fixes $x :: ('a \Longrightarrow 'a)$
assumes *vwb-lens x*
shows $(P \wedge \$x =_u\ v) = (P[\![v/\$x]\!] \wedge \$x =_u\ v)$
⟨*proof*⟩

lemma *conj-eq-out-var-subst*:
fixes $x :: ('a \Longrightarrow 'a)$
assumes *vwb-lens x*
shows $(P \wedge \$x' =_u\ v) = (P[\![v/\$x']\!] \wedge \$x' =_u\ v)$
⟨*proof*⟩

lemma *conj-pos-var-subst*:
assumes *vwb-lens x*
shows $(\$x \wedge Q) = (\$x \wedge Q[\![true/\$x]\!])$
⟨*proof*⟩

lemma *conj-neg-var-subst*:
assumes *vwb-lens x*
shows $(\neg\ \$x \wedge Q) = (\neg\ \$x \wedge Q[\![false/\$x]\!])$
⟨*proof*⟩

lemma *upred-eq-true* [*simp*]: $(p =_u\ true) = p$
⟨*proof*⟩

lemma *upred-eq-false* [*simp*]: $(p =_u\ false) = (\neg\ p)$
⟨*proof*⟩

lemma *upred-true-eq* [*simp*]: $(true =_u\ p) = p$
⟨*proof*⟩

lemma *upred-false-eq* [*simp*]: $(false =_u p) = (\neg\ p)$
⟨*proof*⟩

lemma *conj-var-subst*:
assumes *vwb-lens x*
shows $(P \wedge var\ x =_u v) = (P[\![v/x]\!] \wedge var\ x =_u v)$
⟨*proof*⟩

13.4 HOL Variable Quantifiers

lemma *shEx-unbound* [*simp*]: $(\exists\ x \cdot P) = P$
⟨*proof*⟩

lemma *shEx-bool* [*simp*]: $shEx\ P = (P\ True \vee P\ False)$
⟨*proof*⟩

lemma *shEx-commute*: $(\exists\ x \cdot \exists\ y \cdot P\ x\ y) = (\exists\ y \cdot \exists\ x \cdot P\ x\ y)$
⟨*proof*⟩

lemma *shEx-cong*: $[\![\bigwedge x.\ P\ x = Q\ x\ ]\!] \Longrightarrow shEx\ P = shEx\ Q$
⟨*proof*⟩

lemma *shEx-insert*: $(\exists\ x \in insert_u\ y\ A \cdot P(x)) = (P(x)[\![x{\rightarrow}y]\!] \vee (\exists\ x \in A \cdot P(x)))$
⟨*proof*⟩

lemma *shEx-one-point*: $(\exists\ x \cdot «x» =_u v \wedge P(x)) = P(x)[\![x{\rightarrow}v]\!]$
⟨*proof*⟩

lemma *shAll-unbound* [*simp*]: $(\forall\ x \cdot P) = P$
⟨*proof*⟩

lemma *shAll-bool* [*simp*]: $shAll\ P = (P\ True \wedge P\ False)$
⟨*proof*⟩

lemma *shAll-cong*: $[\![\bigwedge x.\ P\ x = Q\ x\ ]\!] \Longrightarrow shAll\ P = shAll\ Q$
⟨*proof*⟩

Quantifier lifting

named-theorems *uquant-lift*

lemma *shEx-lift-conj-1* [*uquant-lift*]:
$((\exists\ x \cdot P(x)) \wedge Q) = (\exists\ x \cdot P(x) \wedge Q)$
⟨*proof*⟩

lemma *shEx-lift-conj-2* [*uquant-lift*]:
$(P \wedge (\exists\ x \cdot Q(x))) = (\exists\ x \cdot P \wedge Q(x))$
⟨*proof*⟩

13.5 Case Splitting

lemma *eq-split-subst*:
assumes *vwb-lens x*
shows $(P = Q) \longleftrightarrow (\forall\ v.\ P[\![«v»/x]\!] = Q[\![«v»/x]\!])$
⟨*proof*⟩

lemma *eq-split-substI*:
assumes *vwb-lens* x $\bigwedge$ v. $P[\![\ll v \gg / x]\!] = Q[\![\ll v \gg / x]\!]$
shows $P = Q$
⟨*proof*⟩

lemma *taut-split-subst*:
assumes *vwb-lens* x
shows ‘P‘ $\longleftrightarrow$ ($\forall$ v. ‘$P[\![\ll v \gg / x]\!]$‘)
⟨*proof*⟩

lemma *eq-split*:
assumes ‘$P \Rightarrow Q$‘ ‘$Q \Rightarrow P$‘
shows $P = Q$
⟨*proof*⟩

lemma *bool-eq-splitI*:
assumes *vwb-lens* x $P[\![true/x]\!] = Q[\![true/x]\!]$ $P[\![false/x]\!] = Q[\![false/x]\!]$
shows $P = Q$
⟨*proof*⟩

lemma *subst-bool-split*:
assumes *vwb-lens* x
shows ‘P‘ = ‘$(P[\![false/x]\!] \wedge P[\![true/x]\!])$‘
⟨*proof*⟩

lemma *subst-eq-replace*:
fixes $x :: ('a \Longrightarrow 'a)$
shows $(p[\![u/x]\!] \wedge u =_u v) = (p[\![v/x]\!] \wedge u =_u v)$
⟨*proof*⟩

13.6 UTP Quantifiers

lemma *one-point*:
assumes *mwb-lens* x $x \sharp v$
shows $(\exists\ x \cdot P \wedge var\ x =_u v) = P[\![v/x]\!]$
⟨*proof*⟩

lemma *exists-twice*: *mwb-lens* $x \Longrightarrow (\exists\ x \cdot \exists\ x \cdot P) = (\exists\ x \cdot P)$
⟨*proof*⟩

lemma *all-twice*: *mwb-lens* $x \Longrightarrow (\forall\ x \cdot \forall\ x \cdot P) = (\forall\ x \cdot P)$
⟨*proof*⟩

lemma *exists-sub*: $[\![$ *mwb-lens* y; $x \subseteq_L y$ $]\!] \Longrightarrow (\exists\ x \cdot \exists\ y \cdot P) = (\exists\ y \cdot P)$
⟨*proof*⟩

lemma *all-sub*: $[\![$ *mwb-lens* y; $x \subseteq_L y$ $]\!] \Longrightarrow (\forall\ x \cdot \forall\ y \cdot P) = (\forall\ y \cdot P)$
⟨*proof*⟩

lemma *ex-commute*:
assumes $x \bowtie y$
shows $(\exists\ x \cdot \exists\ y \cdot P) = (\exists\ y \cdot \exists\ x \cdot P)$
⟨*proof*⟩

lemma *all-commute*:
assumes $x \bowtie y$

shows $(\forall\ x \cdot \forall\ y \cdot P) = (\forall\ y \cdot \forall\ x \cdot P)$
⟨*proof*⟩

lemma *ex-equiv*:
assumes $x \approx_L y$
shows $(\exists\ x \cdot P) = (\exists\ y \cdot P)$
⟨*proof*⟩

lemma *all-equiv*:
assumes $x \approx_L y$
shows $(\forall\ x \cdot P) = (\forall\ y \cdot P)$
⟨*proof*⟩

lemma *ex-zero*:
$(\exists\ \emptyset \cdot P) = P$
⟨*proof*⟩

lemma *all-zero*:
$(\forall\ \emptyset \cdot P) = P$
⟨*proof*⟩

lemma *ex-plus*:
$(\exists\ y;x \cdot P) = (\exists\ x \cdot \exists\ y \cdot P)$
⟨*proof*⟩

lemma *all-plus*:
$(\forall\ y;x \cdot P) = (\forall\ x \cdot \forall\ y \cdot P)$
⟨*proof*⟩

lemma *closure-all*:
$[P]_u = (\forall\ \Sigma \cdot P)$
⟨*proof*⟩

lemma *unrest-as-exists*:
vwb-lens $x \Longrightarrow (x \sharp P) \longleftrightarrow ((\exists\ x \cdot P) = P)$
⟨*proof*⟩

lemma *ex-mono*: $P \sqsubseteq Q \Longrightarrow (\exists\ x \cdot P) \sqsubseteq (\exists\ x \cdot Q)$
⟨*proof*⟩

lemma *ex-weakens*: *wb-lens* $x \Longrightarrow (\exists\ x \cdot P) \sqsubseteq P$
⟨*proof*⟩

lemma *all-mono*: $P \sqsubseteq Q \Longrightarrow (\forall\ x \cdot P) \sqsubseteq (\forall\ x \cdot Q)$
⟨*proof*⟩

lemma *all-strengthens*: *wb-lens* $x \Longrightarrow P \sqsubseteq (\forall\ x \cdot P)$
⟨*proof*⟩

lemma *ex-unrest*: $x \sharp P \Longrightarrow (\exists\ x \cdot P) = P$
⟨*proof*⟩

lemma *all-unrest*: $x \sharp P \Longrightarrow (\forall\ x \cdot P) = P$
⟨*proof*⟩

lemma *not-ex-not*: $\neg\ (\exists\ x \cdot \neg\ P) = (\forall\ x \cdot P)$
⟨*proof*⟩

lemma *not-all-not*: $\neg\ (\forall\ x \cdot \neg\ P) = (\exists\ x \cdot P)$
⟨*proof*⟩

lemma *ex-conj-contr-left*: $x \sharp P \Longrightarrow (\exists\ x \cdot P \wedge Q) = (P \wedge (\exists\ x \cdot Q))$
⟨*proof*⟩

lemma *ex-conj-contr-right*: $x \sharp Q \Longrightarrow (\exists\ x \cdot P \wedge Q) = ((\exists\ x \cdot P) \wedge Q)$
⟨*proof*⟩

13.7 Variable Restriction

lemma *var-res-all*:
$P \upharpoonright_v \Sigma = P$
⟨*proof*⟩

lemma *var-res-twice*:
mwb-lens $x \Longrightarrow P \upharpoonright_v x \upharpoonright_v x = P \upharpoonright_v x$
⟨*proof*⟩

13.8 Conditional laws

lemma *cond-def*:
$(P \lhd b \rhd Q) = ((b \wedge P) \vee ((\neg\ b) \wedge Q))$
⟨*proof*⟩

lemma *cond-idem* [*simp*]:$(P \lhd b \rhd P) = P$ ⟨*proof*⟩

lemma *cond-true-false* [*simp*]: $true \lhd b \rhd false = b$ ⟨*proof*⟩

lemma *cond-symm*:$(P \lhd b \rhd Q) = (Q \lhd \neg\ b \rhd P)$ ⟨*proof*⟩

lemma *cond-assoc*: $((P \lhd b \rhd Q) \lhd c \rhd R) = (P \lhd b \wedge c \rhd (Q \lhd c \rhd R))$ ⟨*proof*⟩

lemma *cond-distr*: $(P \lhd b \rhd (Q \lhd c \rhd R)) = ((P \lhd b \rhd Q) \lhd c \rhd (P \lhd b \rhd R))$ ⟨*proof*⟩

lemma *cond-unit-T* [*simp*]:$(P \lhd true \rhd Q) = P$ ⟨*proof*⟩

lemma *cond-unit-F* [*simp*]:$(P \lhd false \rhd Q) = Q$ ⟨*proof*⟩

lemma *cond-conj-not*: $((P \lhd b \rhd Q) \wedge (\neg\ b)) = (Q \wedge (\neg\ b))$
⟨*proof*⟩

lemma *cond-and-T-integrate*:
$((P \wedge b) \vee (Q \lhd b \rhd R)) = ((P \vee Q) \lhd b \rhd R)$
⟨*proof*⟩

lemma *cond-L6*: $(P \lhd b \rhd (Q \lhd b \rhd R)) = (P \lhd b \rhd R)$ ⟨*proof*⟩

lemma *cond-L7*: $(P \lhd b \rhd (P \lhd c \rhd Q)) = (P \lhd b \vee c \rhd Q)$ ⟨*proof*⟩

lemma *cond-and-distr*: $((P \wedge Q) \lhd b \rhd (R \wedge S)) = ((P \lhd b \rhd R) \wedge (Q \lhd b \rhd S))$ ⟨*proof*⟩

lemma *cond-or-distr*: $((P \vee Q) \lhd b \rhd (R \vee S)) = ((P \lhd b \rhd R) \vee (Q \lhd b \rhd S))$ ⟨*proof*⟩

lemma *cond-imp-distr*:
$((P \Rightarrow Q) \lhd b \rhd (R \Rightarrow S)) = ((P \lhd b \rhd R) \Rightarrow (Q \lhd b \rhd S))$ ⟨*proof*⟩

lemma *cond-eq-distr*:
$((P \Leftrightarrow Q) \lhd b \rhd (R \Leftrightarrow S)) = ((P \lhd b \rhd R) \Leftrightarrow (Q \lhd b \rhd S))$ ⟨*proof*⟩

lemma *cond-conj-distr*:$(P \wedge (Q \lhd b \rhd S)) = ((P \wedge Q) \lhd b \rhd (P \wedge S))$ ⟨*proof*⟩

lemma *cond-disj-distr*:$(P \vee (Q \lhd b \rhd S)) = ((P \vee Q) \lhd b \rhd (P \vee S))$ ⟨*proof*⟩

lemma *cond-neg*: $\neg (P \lhd b \rhd Q) = ((\neg P) \lhd b \rhd (\neg Q))$ ⟨*proof*⟩

lemma *cond-conj*: $P \lhd b \wedge c \rhd Q = (P \lhd c \rhd Q) \lhd b \rhd Q$
⟨*proof*⟩

lemma *spec-cond-dist*: $(P \Rightarrow (Q \lhd b \rhd R)) = ((P \Rightarrow Q) \lhd b \rhd (P \Rightarrow R))$
⟨*proof*⟩

lemma *cond-USUP-dist*: $(\bigsqcup P{\in}S \cdot F(P)) \lhd b \rhd (\bigsqcup P{\in}S \cdot G(P)) = (\bigsqcup P{\in}S \cdot F(P) \lhd b \rhd G(P))$
⟨*proof*⟩

lemma *cond-UINF-dist*: $(\bigsqcap P{\in}S \cdot F(P)) \lhd b \rhd (\bigsqcap P{\in}S \cdot G(P)) = (\bigsqcap P{\in}S \cdot F(P) \lhd b \rhd G(P))$
⟨*proof*⟩

lemma *cond-var-subst-left*:
assumes *vwb-lens x*
shows $(P[\![true/x]\!] \lhd var\ x \rhd Q) = (P \lhd var\ x \rhd Q)$
⟨*proof*⟩

lemma *cond-var-subst-right*:
assumes *vwb-lens x*
shows $(P \lhd var\ x \rhd Q[\![false/x]\!]) = (P \lhd var\ x \rhd Q)$
⟨*proof*⟩

lemma *cond-var-split*:
vwb-lens x $\Longrightarrow (P[\![true/x]\!] \lhd var\ x \rhd P[\![false/x]\!]) = P$
⟨*proof*⟩

lemma *cond-assign-subst*:
vwb-lens x $\Longrightarrow (P \lhd utp\text{-}expr.var\ x =_u v \rhd Q) = (P[\![v/x]\!] \lhd utp\text{-}expr.var\ x =_u v \rhd Q)$
⟨*proof*⟩

lemma *conj-conds*:
$(P1 \lhd b \rhd Q1 \wedge P2 \lhd b \rhd Q2) = (P1 \wedge P2) \lhd b \rhd (Q1 \wedge Q2)$
⟨*proof*⟩

lemma *disj-conds*:
$(P1 \lhd b \rhd Q1 \vee P2 \lhd b \rhd Q2) = (P1 \vee P2) \lhd b \rhd (Q1 \vee Q2)$
⟨*proof*⟩

lemma *cond-mono*:
$[\![P_1 \sqsubseteq P_2;\ Q_1 \sqsubseteq Q_2]\!] \Longrightarrow (P_1 \lhd b \rhd Q_1) \sqsubseteq (P_2 \lhd b \rhd Q_2)$
⟨*proof*⟩

lemma *cond-monotonic*:
$[\![\textit{mono } P; \textit{mono } Q]\!] \Longrightarrow \textit{mono } (\lambda\ X.\ P\ X \lhd b \rhd Q\ X)$
⟨*proof*⟩

13.9 Additional Expression Laws

lemma *le-pred-refl* [*simp*]:
fixes $x :: ('a::\textit{preorder}, 'α)\ \textit{uexpr}$
shows $(x \leq_u x) = \textit{true}$
⟨*proof*⟩

lemma *uzero-le-laws* [*simp*]:
$(0 :: ('a::\{\textit{linordered-semidom}\}, 'α)\ \textit{uexpr}) \leq_u \textit{numeral } x = \textit{true}$
$(1 :: ('a::\{\textit{linordered-semidom}\}, 'α)\ \textit{uexpr}) \leq_u \textit{numeral } x = \textit{true}$
$(0 :: ('a::\{\textit{linordered-semidom}\}, 'α)\ \textit{uexpr}) \leq_u 1 = \textit{true}$
⟨*proof*⟩

lemma *unumeral-le-1* [*simp*]:
assumes $(\textit{numeral } i :: 'a::\{\textit{numeral},\textit{ord}\}) \leq \textit{numeral } j$
shows $(\textit{numeral } i :: ('a, 'α)\ \textit{uexpr}) \leq_u \textit{numeral } j = \textit{true}$
⟨*proof*⟩

lemma *unumeral-le-2* [*simp*]:
assumes $(\textit{numeral } i :: 'a::\{\textit{numeral},\textit{linorder}\}) > \textit{numeral } j$
shows $(\textit{numeral } i :: ('a, 'α)\ \textit{uexpr}) \leq_u \textit{numeral } j = \textit{false}$
⟨*proof*⟩

lemma *uset-laws* [*simp*]:
$x \in_u \{\}_u = \textit{false}$
$x \in_u \{m..n\}_u = (m \leq_u x \wedge x \leq_u n)$
⟨*proof*⟩

lemma *ulit-eq* [*simp*]: $x = y \Longrightarrow (\text{«}x\text{»} =_u \text{«}y\text{»}) = \textit{true}$
⟨*proof*⟩

lemma *ulit-neq* [*simp*]: $x \neq y \Longrightarrow (\text{«}x\text{»} =_u \text{«}y\text{»}) = \textit{false}$
⟨*proof*⟩

lemma *uset-mems* [*simp*]:
$x \in_u \{y\}_u = (x =_u y)$
$x \in_u A \cup_u B = (x \in_u A \vee x \in_u B)$
$x \in_u A \cap_u B = (x \in_u A \wedge x \in_u B)$
⟨*proof*⟩

13.10 Refinement By Observation

Function to obtain the set of observations of a predicate

definition *obs-upred* :: $'α\ \textit{upred} \Rightarrow 'α\ \textit{set}$ ($‹[\![\text{-}]\!]_o›$)
where [*upred-defs*]: $[\![P]\!]_o = \{b.\ [\![P]\!]_e b\}$

lemma *obs-upred-refine-iff*:
$P \sqsubseteq Q \longleftrightarrow [\![Q]\!]_o \subseteq [\![P]\!]_o$
⟨*proof*⟩

A refinement can be demonstrated by considering only the observations of the predicates which

are relevant, i.e. not unrestricted, for them. In other words, if the alphabet can be split into two disjoint segments, x and y, and neither predicate refers to y then only x need be considered when checking for observations.

lemma *refine-by-obs*:
assumes $x \bowtie y$ *bij-lens* $(x +_L y)$ $y \sharp P$ $y \sharp Q$ $\{v.\ `P[\![\langle\!\langle v \rangle\!\rangle / x]\!]`\} \subseteq \{v.\ `Q[\![\langle\!\langle v \rangle\!\rangle / x]\!]`\}$
shows $Q \sqsubseteq P$
⟨*proof*⟩

13.11 Cylindric Algebra

lemma *C1*: $(\exists\ x \cdot \mathit{false}) = \mathit{false}$
⟨*proof*⟩

lemma *C2*: *wb-lens* $x \Longrightarrow `P \Rightarrow (\exists\ x \cdot P)`$
⟨*proof*⟩

lemma *C3*: *mwb-lens* $x \Longrightarrow (\exists\ x \cdot (P \wedge (\exists\ x \cdot Q))) = ((\exists\ x \cdot P) \wedge (\exists\ x \cdot Q))$
⟨*proof*⟩

lemma *C4a*: $x \approx_L y \Longrightarrow (\exists\ x \cdot \exists\ y \cdot P) = (\exists\ y \cdot \exists\ x \cdot P)$
⟨*proof*⟩

lemma *C4b*: $x \bowtie y \Longrightarrow (\exists\ x \cdot \exists\ y \cdot P) = (\exists\ y \cdot \exists\ x \cdot P)$
⟨*proof*⟩

lemma *C5*:
fixes $x :: ('a \Longrightarrow 'a)$
shows $(\&x =_u \&x) = \mathit{true}$
⟨*proof*⟩

lemma *C6*:
assumes *wb-lens* x $x \bowtie y$ $x \bowtie z$
shows $(\&y =_u \&z) = (\exists\ x \cdot \&y =_u \&x \wedge \&x =_u \&z)$
⟨*proof*⟩

lemma *C7*:
assumes *weak-lens* x $x \bowtie y$
shows $((\exists\ x \cdot \&x =_u \&y \wedge P) \wedge (\exists\ x \cdot \&x =_u \&y \wedge \neg\ P)) = \mathit{false}$
⟨*proof*⟩

end

14 Healthiness Conditions

theory *utp-healthy*
imports *utp-pred-laws*
begin

14.1 Main Definitions

We collect closure laws for healthiness conditions in the following theorem attribute.

named-theorems *closure*

type-synonym $'\alpha$ *health* = $'\alpha$ *upred* ⇒ $'\alpha$ *upred*

A predicate P is healthy, under healthiness function H, if P is a fixed-point of H.

definition *Healthy* :: $'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *health* $\Rightarrow$ *bool* (**infix** ‹*is*› *30*)
where P *is* $H \equiv (H\ P = P)$

lemma *Healthy-def'*: P *is* $H \longleftrightarrow (H\ P = P)$
⟨*proof*⟩

lemma *Healthy-if*: P *is* $H \Longrightarrow (H\ P = P)$
⟨*proof*⟩

lemma *Healthy-intro*: $H(P) = P \Longrightarrow P$ *is* H
⟨*proof*⟩

declare *Healthy-def'* [*upred-defs*]

abbreviation *Healthy-carrier* :: $'\alpha$ *health* $\Rightarrow$ $'\alpha$ *upred set* (‹$[\![$-$]\!]_H$›)
where $[\![H]\!]_H \equiv \{P.\ P \text{ is } H\}$

lemma *Healthy-carrier-image*:
$A \subseteq [\![\mathcal{H}]\!]_H \Longrightarrow \mathcal{H}\ `\ A = A$
⟨*proof*⟩

lemma *Healthy-carrier-Collect*: $A \subseteq [\![H]\!]_H \Longrightarrow A = \{H(P) \mid P.\ P \in A\}$
⟨*proof*⟩

lemma *Healthy-func*:
$[\![\ F \in [\![\mathcal{H}_1]\!]_H \to [\![\mathcal{H}_2]\!]_H;\ P \text{ is } \mathcal{H}_1\ ]\!] \Longrightarrow \mathcal{H}_2(F(P)) = F(P)$
⟨*proof*⟩

lemma *Healthy-comp*:
$[\![\ P \text{ is } \mathcal{H}_1;\ P \text{ is } \mathcal{H}_2\ ]\!] \Longrightarrow P \text{ is } \mathcal{H}_1 \circ \mathcal{H}_2$
⟨*proof*⟩

lemma *Healthy-apply-closed*:
assumes $F \in [\![H]\!]_H \to [\![H]\!]_H$ P *is* H
shows $F(P)$ *is* H
⟨*proof*⟩

lemma *Healthy-set-image-member*:
$[\![\ P \in F\ `\ A;\ \bigwedge x.\ F\ x \text{ is } H\ ]\!] \Longrightarrow P \text{ is } H$
⟨*proof*⟩

lemma *Healthy-case-prod* [*closure*]:
$[\![\ \bigwedge x\ y.\ P\ x\ y \text{ is } H\ ]\!] \Longrightarrow$ *case-prod* $P\ v$ *is* H
⟨*proof*⟩

lemma *Healthy-SUPREMUM*:
$A \subseteq [\![H]\!]_H \Longrightarrow Sup\ (H\ `\ A) = \bigsqcap A$
⟨*proof*⟩

lemma *Healthy-INFIMUM*:
$A \subseteq [\![H]\!]_H \Longrightarrow Inf\ (H\ `\ A) = \bigsqcup A$
⟨*proof*⟩

lemma *Healthy-nu* [*closure*]:

assumes *mono F F* $\in [\![id]\!]_H \to [\![H]\!]_H$
shows ν *F is H*
⟨*proof*⟩

lemma *Healthy-mu* [*closure*]:
assumes *mono F F* $\in [\![id]\!]_H \to [\![H]\!]_H$
shows μ *F is H*
⟨*proof*⟩

lemma *Healthy-subset-member*: $[\![A \subseteq [\![H]\!]_H;\ P \in A]\!] \Longrightarrow H(P) = P$
⟨*proof*⟩

lemma *is-Healthy-subset-member*: $[\![A \subseteq [\![H]\!]_H;\ P \in A]\!] \Longrightarrow P$ *is H*
⟨*proof*⟩

14.2 Properties of Healthiness Conditions

definition *Idempotent* :: $'\alpha$ *health* $\Rightarrow$ *bool* **where**
Idempotent(*H*) $\longleftrightarrow$ ($\forall$ *P*. *H*(*H*(*P*)) = *H*(*P*))

abbreviation *Monotonic* :: $'\alpha$ *health* $\Rightarrow$ *bool* **where**
Monotonic(*H*) $\equiv$ *mono H*

definition *IMH* :: $'\alpha$ *health* $\Rightarrow$ *bool* **where**
IMH(*H*) $\longleftrightarrow$ *Idempotent*(*H*) $\wedge$ *Monotonic*(*H*)

definition *Antitone* :: $'\alpha$ *health* $\Rightarrow$ *bool* **where**
Antitone(*H*) $\longleftrightarrow$ ($\forall$ *P Q*. *Q* $\sqsubseteq$ *P* $\longrightarrow$ (*H*(*P*) $\sqsubseteq$ *H*(*Q*)))

definition *Conjunctive* :: $'\alpha$ *health* $\Rightarrow$ *bool* **where**
Conjunctive(*H*) $\longleftrightarrow$ ($\exists$ *Q*. $\forall$ *P*. *H*(*P*) = (*P* $\wedge$ *Q*))

definition *FunctionalConjunctive* :: $'\alpha$ *health* $\Rightarrow$ *bool* **where**
FunctionalConjunctive(*H*) $\longleftrightarrow$ ($\exists$ *F*. $\forall$ *P*. *H*(*P*) = (*P* $\wedge$ *F*(*P*)) $\wedge$ *Monotonic*(*F*))

definition *WeakConjunctive* :: $'\alpha$ *health* $\Rightarrow$ *bool* **where**
WeakConjunctive(*H*) $\longleftrightarrow$ ($\forall$ *P*. $\exists$ *Q*. *H*(*P*) = (*P* $\wedge$ *Q*))

definition *Disjunctuous* :: $'\alpha$ *health* $\Rightarrow$ *bool* **where**
[*upred-defs*]: *Disjunctuous H* = ($\forall$ *P Q*. *H*(*P* $\sqcap$ *Q*) = (*H*(*P*) $\sqcap$ *H*(*Q*)))

definition *Continuous* :: $'\alpha$ *health* $\Rightarrow$ *bool* **where**
[*upred-defs*]: *Continuous H* = ($\forall$ *A*. *A* $\neq$ {} $\longrightarrow$ *H* ($\bigsqcap$ *A*) = $\bigsqcap$ (*H* ‘ *A*))

lemma *Healthy-Idempotent* [*closure*]:
Idempotent H $\Longrightarrow$ *H*(*P*) *is H*
⟨*proof*⟩

lemma *Healthy-range*: *Idempotent H* $\Longrightarrow$ *range H* = $[\![H]\!]_H$
⟨*proof*⟩

lemma *Idempotent-id* [*simp*]: *Idempotent id*
⟨*proof*⟩

lemma *Idempotent-comp* [*intro*]:
$[\![$ *Idempotent f*; *Idempotent g*; *f* $\circ$ *g* = *g* $\circ$ *f* $]\!] \Longrightarrow$ *Idempotent* (*f* $\circ$ *g*)

⟨*proof*⟩

lemma *Idempotent-image*: *Idempotent* $f \Longrightarrow f \mathbin{`} f \mathbin{`} A = f \mathbin{`} A$
⟨*proof*⟩

lemma *Monotonic-id* [*simp*]: *Monotonic id*
⟨*proof*⟩

lemma *Monotonic-id'* [*closure*]:
mono $(\lambda\ X.\ X)$
⟨*proof*⟩

lemma *Monotonic-const* [*closure*]:
Monotonic $(\lambda\ x.\ c)$
⟨*proof*⟩

lemma *Monotonic-comp* [*intro*]:
$[\![$ *Monotonic* f; *Monotonic* g $]\!] \Longrightarrow$ *Monotonic* $(f \circ g)$
⟨*proof*⟩

lemma *Monotonic-inf* [*closure*]:
assumes *Monotonic* P *Monotonic* Q
shows *Monotonic* $(\lambda\ X.\ P(X) \sqcap Q(X))$
⟨*proof*⟩

lemma *Monotonic-cond* [*closure*]:
assumes *Monotonic* P *Monotonic* Q
shows *Monotonic* $(\lambda\ X.\ P(X) \mathbin{\triangleleft} b \mathbin{\triangleright} Q(X))$
⟨*proof*⟩

lemma *Conjuctive-Idempotent*:
Conjunctive$(H) \Longrightarrow$ *Idempotent*(H)
⟨*proof*⟩

lemma *Conjunctive-Monotonic*:
Conjunctive$(H) \Longrightarrow$ *Monotonic*(H)
⟨*proof*⟩

lemma *Conjunctive-conj*:
assumes *Conjunctive*(HC)
shows $HC(P \wedge Q) = (HC(P) \wedge Q)$
⟨*proof*⟩

lemma *Conjunctive-distr-conj*:
assumes *Conjunctive*(HC)
shows $HC(P \wedge Q) = (HC(P) \wedge HC(Q))$
⟨*proof*⟩

lemma *Conjunctive-distr-disj*:
assumes *Conjunctive*(HC)
shows $HC(P \vee Q) = (HC(P) \vee HC(Q))$
⟨*proof*⟩

lemma *Conjunctive-distr-cond*:
assumes *Conjunctive*(HC)

shows $HC(P \lhd b \rhd Q) = (HC(P) \lhd b \rhd HC(Q))$
⟨*proof*⟩

lemma *FunctionalConjunctive-Monotonic*:
$FunctionalConjunctive(H) \Longrightarrow Monotonic(H)$
⟨*proof*⟩

lemma *WeakConjunctive-Refinement*:
assumes $WeakConjunctive(HC)$
shows $P \sqsubseteq HC(P)$
⟨*proof*⟩

lemma *WeakCojunctive-Healthy-Refinement*:
assumes $WeakConjunctive(HC)$ **and** P *is* HC
shows $HC(P) \sqsubseteq P$
⟨*proof*⟩

lemma *WeakConjunctive-implies-WeakConjunctive*:
$Conjunctive(H) \Longrightarrow WeakConjunctive(H)$
⟨*proof*⟩

declare *Conjunctive-def* [*upred-defs*]
declare *mono-def* [*upred-defs*]

lemma *Disjunctuous-Monotonic*: *Disjunctuous* $H \Longrightarrow$ *Monotonic* H
⟨*proof*⟩

lemma *ContinuousD* [*dest*]: ⟦ *Continuous* H; $A \neq \{\}$ ⟧ $\Longrightarrow H\ (\bigsqcap\ A) = (\bigsqcap\ P{\in}A.\ H(P))$
⟨*proof*⟩

lemma *Continuous-Disjunctous*: *Continuous* $H \Longrightarrow$ *Disjunctuous* H
⟨*proof*⟩

lemma *Continuous-Monotonic* [*closure*]: *Continuous* $H \Longrightarrow$ *Monotonic* H
⟨*proof*⟩

lemma *Continuous-comp* [*intro*]:
⟦ *Continuous* f; *Continuous* g ⟧ $\Longrightarrow$ *Continuous* $(f \circ g)$
⟨*proof*⟩

lemma *Continuous-const* [*closure*]: *Continuous* $(\lambda\ X.\ P)$
⟨*proof*⟩

lemma *Continuous-cond* [*closure*]:
assumes *Continuous* F *Continuous* G
shows *Continuous* $(\lambda\ X.\ F(X) \lhd b \rhd G(X))$
⟨*proof*⟩

Closure laws derived from continuity

lemma *Sup-Continuous-closed* [*closure*]:
⟦ *Continuous* H; $\bigwedge i.\ i \in A \Longrightarrow P(i)$ *is* H; $A \neq \{\}$ ⟧ $\Longrightarrow (\bigsqcap\ i{\in}A.\ P(i))$ *is* H
⟨*proof*⟩

lemma *UINF-mem-Continuous-closed* [*closure*]:
⟦ *Continuous* H; $\bigwedge i.\ i \in A \Longrightarrow P(i)$ *is* H; $A \neq \{\}$ ⟧ $\Longrightarrow (\bigsqcap\ i{\in}A \cdot P(i))$ *is* H

⟨*proof*⟩

lemma *UINF-mem-Continuous-closed-pair* [*closure*]:
assumes *Continuous H* ⋀ *i j*. (*i*, *j*) ∈ *A* ⟹ *P i j is H A* ≠ {}
shows (⨅ (*i*,*j*)∈*A* · *P i j*) *is H*
⟨*proof*⟩

lemma *UINF-mem-Continuous-closed-triple* [*closure*]:
assumes *Continuous H* ⋀ *i j k*. (*i*, *j*, *k*) ∈ *A* ⟹ *P i j k is H A* ≠ {}
shows (⨅ (*i*,*j*,*k*)∈*A* · *P i j k*) *is H*
⟨*proof*⟩

lemma *UINF-mem-Continuous-closed-quad* [*closure*]:
assumes *Continuous H* ⋀ *i j k l*. (*i*, *j*, *k*, *l*) ∈ *A* ⟹ *P i j k l is H A* ≠ {}
shows (⨅ (*i*,*j*,*k*,*l*)∈*A* · *P i j k l*) *is H*
⟨*proof*⟩

lemma *UINF-mem-Continuous-closed-quint* [*closure*]:
assumes *Continuous H* ⋀ *i j k l m*. (*i*, *j*, *k*, *l*, *m*) ∈ *A* ⟹ *P i j k l m is H A* ≠ {}
shows (⨅ (*i*,*j*,*k*,*l*,*m*)∈*A* · *P i j k l m*) *is H*
⟨*proof*⟩

lemma *UINF-ind-closed* [*closure*]:
assumes *Continuous H* ⋀ *i*. *P i* = *true* ⋀ *i*. *Q i is H*
shows *UINF P Q is H*
⟨*proof*⟩

All continuous functions are also Scott-continuous

lemma *sup-continuous-Continuous* [*closure*]: *Continuous F* ⟹ *sup-continuous F*
⟨*proof*⟩

lemma *USUP-healthy*: *A* ⊆ ⟦*H*⟧$_H$ ⟹ (⨆ *P*∈*A* · *F*(*P*)) = (⨆ *P*∈*A* · *F*(*H*(*P*)))
⟨*proof*⟩

lemma *UINF-healthy*: *A* ⊆ ⟦*H*⟧$_H$ ⟹ (⨅ *P*∈*A* · *F*(*P*)) = (⨅ *P*∈*A* · *F*(*H*(*P*)))
⟨*proof*⟩

end

15 Alphabetised Relations

theory *utp-rel*
imports
utp-pred-laws
utp-healthy
utp-lift
utp-tactics
begin

An alphabetised relation is simply a predicate whose state-space is a product type. In this theory we construct the core operators of the relational calculus, and prove a libary of associated theorems, based on Chapters 2 and 5 of the UTP book [22].

15.1 Relational Alphabets

We set up convenient syntax to refer to the input and output parts of the alphabet, as is common in UTP. Since we are in a product space, these are simply the lenses fst_L and snd_L.

definition $in\alpha :: ('\alpha \Longrightarrow '\alpha \times '\beta)$ **where**
[*lens-defs*]: $in\alpha = fst_L$

definition $out\alpha :: ('\beta \Longrightarrow '\alpha \times '\beta)$ **where**
[*lens-defs*]: $out\alpha = snd_L$

lemma *in*α*-uvar* [*simp*]: *vwb-lens in*α
⟨*proof*⟩

lemma *out*α*-uvar* [*simp*]: *vwb-lens out*α
⟨*proof*⟩

lemma *var-in-alpha* [*simp*]: $x ;_L in\alpha = ivar\ x$
⟨*proof*⟩

lemma *var-out-alpha* [*simp*]: $x ;_L out\alpha = ovar\ x$
⟨*proof*⟩

lemma *drop-pre-inv* [*simp*]: $[\![out\alpha \sharp p]\!] \Longrightarrow \lceil \lfloor p \rfloor_< \rceil_< = p$
⟨*proof*⟩

lemma *usubst-lookup-ivar-unrest* [*usubst*]:
$in\alpha \sharp \sigma \Longrightarrow \langle\sigma\rangle_s\ (ivar\ x) = \$x$
⟨*proof*⟩

lemma *usubst-lookup-ovar-unrest* [*usubst*]:
$out\alpha \sharp \sigma \Longrightarrow \langle\sigma\rangle_s\ (ovar\ x) = \$x\acute{}$
⟨*proof*⟩

lemma *out-alpha-in-indep* [*simp*]:
$out\alpha \bowtie in\text{-}var\ x\ in\text{-}var\ x \bowtie out\alpha$
⟨*proof*⟩

lemma *in-alpha-out-indep* [*simp*]:
$in\alpha \bowtie out\text{-}var\ x\ out\text{-}var\ x \bowtie in\alpha$
⟨*proof*⟩

The following two functions lift a predicate substitution to a relational one.

abbreviation *usubst-rel-lift* $:: '\alpha\ usubst \Rightarrow ('\alpha \times '\beta)\ usubst\ (\langle\lceil\text{-}\rceil_s\rangle)$ **where**
$\lceil\sigma\rceil_s \equiv \sigma \oplus_s in\alpha$

abbreviation *usubst-rel-drop* $:: ('\alpha \times '\alpha)\ usubst \Rightarrow '\alpha\ usubst\ (\langle\lfloor\text{-}\rfloor_s\rangle)$ **where**
$\lfloor\sigma\rfloor_s \equiv \sigma \upharpoonright_s in\alpha$

The alphabet of a relation then consists wholly of the input and output portions.

lemma *alpha-in-out*:
$\Sigma \approx_L in\alpha +_L out\alpha$
⟨*proof*⟩

15.2 Relational Types and Operators

We create type synonyms for conditions (which are simply predicates) – i.e. relations without dashed variables –, alphabetised relations where the input and output alphabet can be different, and finally homogeneous relations.

type-synonym $'\alpha$ *cond* $= '\alpha$ *upred*
type-synonym $('\alpha, '\beta)$ *urel* $= ('\alpha \times '\beta)$ *upred*
type-synonym $'\alpha$ *hrel* $= ('\alpha \times '\alpha)$ *upred*
type-synonym $('a, '\alpha)$ *hexpr* $= ('a, '\alpha \times '\alpha)$ *uexpr*

translations
(*type*) $('\alpha, '\beta)$ *urel* <= (*type*) $('\alpha \times '\beta)$ *upred*

We set up some overloaded constants for sequential composition and the identity in case we want to overload their definitions later.

consts
useq :: $'a \Rightarrow 'b \Rightarrow 'c$ (**infixr** ‹;;› *61*)
uassigns :: $'a$ *usubst* $\Rightarrow 'b$ ($‹\langle\text{-}\rangle_a›$)
uskip :: $'a$ (‹*II*›)

We define a specialised version of the conditional where the condition can refer only to undashed variables, as is usually the case in programs, but not universally in UTP models. We implement this by lifting the condition predicate into the relational state-space with construction $\lceil b \rceil_<$.

definition *lift-rcond* ($‹\lceil\text{-}\rceil_{\leftarrow}›$) **where**
[*upred-defs*]: $\lceil b \rceil_{\leftarrow} = \lceil b \rceil_<$

abbreviation
rcond :: $('\alpha, '\beta)$ *urel* $\Rightarrow '\alpha$ *cond* $\Rightarrow ('\alpha, '\beta)$ *urel* $\Rightarrow ('\alpha, '\beta)$ *urel*
($‹(3\text{-} \lhd \text{-} \rhd_r/ \text{-})›$ [*52*,*0*,*53*] *52*)
where $(P \lhd b \rhd_r Q) \equiv (P \lhd \lceil b \rceil_{\leftarrow} \rhd Q)$

Sequential composition is heterogeneous, and simply requires that the output alphabet of the first matches then input alphabet of the second. We define it by lifting HOL's built-in relational composition operator ((O)). Since this returns a set, the definition states that the state binding b is an element of this set.

lift-definition *seqr*::$('\alpha, '\beta)$ *urel* $\Rightarrow ('\beta, '\gamma)$ *urel* $\Rightarrow ('\alpha \times '\gamma)$ *upred*
is $\lambda\ P\ Q\ b.\ b \in (\{p.\ P\ p\}\ O\ \{q.\ Q\ q\})$ ⟨*proof*⟩

adhoc-overloading
useq $\rightleftharpoons$ *seqr*

We also set up a homogeneous sequential composition operator, and versions of *true* and *false* that are explicitly typed by a homogeneous alphabet.

abbreviation *seqh* :: $'\alpha$ *hrel* $\Rightarrow '\alpha$ *hrel* $\Rightarrow '\alpha$ *hrel* (**infixr** ‹;;$_h$› *61*) **where**
seqh $P\ Q \equiv (P\ ;;\ Q)$

abbreviation *truer* :: $'\alpha$ *hrel* ($‹true_h›$) **where**
truer $\equiv$ *true*

abbreviation *falser* :: $'\alpha$ *hrel* ($‹false_h›$) **where**
falser $\equiv$ *false*

We define the relational converse operator as an alphabet extrusion on the bijective lens $swap_L$ that swaps the elements of the product state-space.

abbreviation *conv-r* :: $('a, 'a \times 'b)$ *uexpr* $\Rightarrow$ $('a, 'b \times 'a)$ *uexpr* ($\langle$-$^{-}\rangle$ [*999*] *999*)
where *conv-r* $e \equiv e \oplus_p swap_L$

Assignment is defined using substitutions, where latter defines what each variable should map to. This approach, which is originally due to Back [3], permits more general assignment expressions. The definition of the operator identifies the after state binding, b', with the substitution function applied to the before state binding b.

lift-definition *assigns-r* :: $'a$ *usubst* $\Rightarrow$ $'a$ *hrel*
is $\lambda\ \sigma\ (b, b').\ b' = \sigma(b)$ $\langle proof \rangle$

adhoc-overloading
uassigns $\rightleftharpoons$ *assigns-r*

Relational identity, or skip, is then simply an assignment with the identity substitution: it simply identifies all variables.

definition *skip-r* :: $'a$ *hrel* **where**
[*urel-defs*]: *skip-r* = *assigns-r id*

adhoc-overloading
uskip $\rightleftharpoons$ *skip-r*

Non-deterministic assignment, also known as "choose", assigns an arbitrarily chosen value to the given variable

definition *nd-assign* :: $('a \Longrightarrow 'a) \Rightarrow 'a$ *hrel* **where**
[*urel-defs*]: *nd-assign* $x = (\bigsqcap\ v \cdot$ *assigns-r* $[x \mapsto_s \ll v \gg])$

We set up iterated sequential composition which iterates an indexed predicate over the elements of a list.

definition *seqr-iter* :: $'a$ *list* $\Rightarrow$ $('a \Rightarrow 'b$ *hrel*$) \Rightarrow 'b$ *hrel* **where**
[*urel-defs*]: *seqr-iter xs* P = *foldr* $(\lambda\ i\ Q.\ P(i)\ ;;\ Q)$ *xs II*

A singleton assignment simply applies a singleton substitution function, and similarly for a double assignment.

abbreviation *assign-r* :: $('t \Longrightarrow 'a) \Rightarrow ('t, 'a)$ *uexpr* $\Rightarrow 'a$ *hrel*
where *assign-r* $x\ v \equiv \langle [x \mapsto_s v] \rangle_a$

abbreviation *assign-2-r* ::
$('t1 \Longrightarrow 'a) \Rightarrow ('t2 \Longrightarrow 'a) \Rightarrow ('t1, 'a)$ *uexpr* $\Rightarrow ('t2, 'a)$ *uexpr* $\Rightarrow 'a$ *hrel*
where *assign-2-r* $x\ y\ u\ v \equiv$ *assigns-r* $[x \mapsto_s u,\ y \mapsto_s v]$

We also define the alphabetised skip operator that identifies all input and output variables in the given alphabet lens. All other variables are unrestricted. We also set up syntax for it.

definition *skip-ra* :: $('\beta, 'a)$ *lens* $\Rightarrow 'a$ *hrel* **where**
[*urel-defs*]: *skip-ra* $v = (\$v' =_u \$v)$

Similarly, we define the alphabetised assignment operator.

definition *assigns-ra* :: $'a$ *usubst* $\Rightarrow ('\beta, 'a)$ *lens* $\Rightarrow 'a$ *hrel* ($\langle\langle$-$\rangle$_$\rangle$) **where**
$\langle\sigma\rangle_a = (\lceil\sigma\rceil_s \dagger$ *skip-ra a*$)$

Assumptions ($c^\top$) and assertions ($c_\bot$) are encoded as conditionals. An assumption behaves like skip if the condition is true, and otherwise behaves like *false* (miracle). An assertion is the same, but yields *true*, which is an abort. They are the same as tests, as in Kleene Algebra

with Tests [24, 1] (KAT), which embeds a Boolean algebra into a Kleene algebra to represent conditions.

definition *rassume* :: $'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *hrel* **where**
[*urel-defs*]: *rassume c* = *II* $\triangleleft$ *c* $\triangleright_r$ *false*

definition *rassert* :: $'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *hrel* **where**
[*urel-defs*]: *rassert c* = *II* $\triangleleft$ *c* $\triangleright_r$ *true*

We define two variants of while loops based on strongest and weakest fixed points. The former is *false* for an infinite loop, and the latter is *true*.

definition *while-top* :: $'\alpha$ *cond* $\Rightarrow$ $'\alpha$ *hrel* $\Rightarrow$ $'\alpha$ *hrel* **where**
[*urel-defs*]: *while-top b P* = (ν *X* $\cdot$ (*P* ;; *X*) $\triangleleft$ *b* $\triangleright_r$ *II*)

definition *while-bot* :: $'\alpha$ *cond* $\Rightarrow$ $'\alpha$ *hrel* $\Rightarrow$ $'\alpha$ *hrel* **where**
[*urel-defs*]: *while-bot b P* = (μ *X* $\cdot$ (*P* ;; *X*) $\triangleleft$ *b* $\triangleright_r$ *II*)

While loops with invariant decoration (cf. [1]) – partial correctness.

definition *while-inv* :: $'\alpha$ *cond* $\Rightarrow$ $'\alpha$ *cond* $\Rightarrow$ $'\alpha$ *hrel* $\Rightarrow$ $'\alpha$ *hrel* **where**
[*urel-defs*]: *while-inv b p S* = *while-top b S*

While loops with invariant decoration – total correctness.

definition *while-inv-bot* :: $'\alpha$ *cond* $\Rightarrow$ $'\alpha$ *cond* $\Rightarrow$ $'\alpha$ *hrel* $\Rightarrow$ $'\alpha$ *hrel* **where**
[*urel-defs*]: *while-inv-bot b p S* = *while-bot b S*

While loops with invariant and variant decorations – total correctness.

definition *while-vrt* ::
$'\alpha$ *cond* $\Rightarrow$ $'\alpha$ *cond* $\Rightarrow$ (*nat*, $'\alpha$) *uexpr* $\Rightarrow$ $'\alpha$ *hrel* $\Rightarrow$ $'\alpha$ *hrel* **where**
[*urel-defs*]: *while-vrt b p v S* = *while-bot b S*

syntax
-uassume :: *uexp* $\Rightarrow$ *logic* ($\langle[\text{-}]^\top\rangle$)
-uassume :: *uexp* $\Rightarrow$ *logic* ($\langle ?[\text{-}]\rangle$)
-uassert :: *uexp* $\Rightarrow$ *logic* ($\langle\{\text{-}\}_\perp\rangle$)
-uwhile :: *uexp* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\langle$*while*$^\top$ - *do* - *od*$\rangle$)
-uwhile-top :: *uexp* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\langle$*while* - *do* - *od*$\rangle$)
-uwhile-bot :: *uexp* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\langle$*while*$_\perp$ - *do* - *od*$\rangle$)
-uwhile-inv :: *uexp* $\Rightarrow$ *uexp* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\langle$*while* - *invr* - *do* - *od*$\rangle$)
-uwhile-inv-bot :: *uexp* $\Rightarrow$ *uexp* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\langle$*while*$_\perp$ - *invr* - *do* - *od*$\rangle$ *71*)
-uwhile-vrt :: *uexp* $\Rightarrow$ *uexp* $\Rightarrow$ *uexp* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\langle$*while* - *invr* - *vrt* - *do* - *od*$\rangle$)

syntax-consts
-uassume == *rassume* **and**
-uassert == *rassert* **and**
-uwhile -uwhile-top == *while-top* **and**
-uwhile-bot == *while-bot* **and**
-uwhile-inv == *while-inv* **and**
-uwhile-inv-bot == *while-inv-bot* **and**
-uwhile-vrt == *while-vrt*

translations
-uassume b == *CONST rassume b*
-uassert b == *CONST rassert b*
-uwhile b P == *CONST while-top b P*
-uwhile-top b P == *CONST while-top b P*

-uwhile-bot b P == CONST while-bot b P
-uwhile-inv b p S == CONST while-inv b p S
-uwhile-inv-bot b p S == CONST while-inv-bot b p S
-uwhile-vrt b p v S == CONST while-vrt b p v S

We implement a poor man's version of alphabet restriction that hides a variable within a relation.

definition *rel-var-res* :: *'α hrel ⇒ ('a ⟹ 'α) ⇒ 'α hrel* (**infix** ‹⌈$_\alpha$› *80*) **where**
[*urel-defs*]: *P ⌈$_\alpha$ x = (∃ $x • ∃ $x´ • P)*

Alphabet extension and restriction add additional variables by the given lens in both their primed and unprimed versions.

definition *rel-aext* :: *'β hrel ⇒ ('β ⟹ 'α) ⇒ 'α hrel*
where [*upred-defs*]: *rel-aext P a = P ⊕$_p$ (a ×$_L$ a)*

definition *rel-ares* :: *'α hrel ⇒ ('β ⟹ 'α) ⇒ 'β hrel*
where [*upred-defs*]: *rel-ares P a = (P ⌈$_p$ (a × a))*

We next describe frames and antiframes with the help of lenses. A frame states that P defines how variables in a changed, and all those outside of a remain the same. An antiframe describes the converse: all variables outside a are specified by P, and all those in remain the same. For more information please see [25].

definition *frame* :: *('a ⟹ 'α) ⇒ 'α hrel ⇒ 'α hrel* **where**
[*urel-defs*]: *frame a P = (P ∧ $v´ =$_u$ $v ⊕ $v´ on &a)*

definition *antiframe* :: *('a ⟹ 'α) ⇒ 'α hrel ⇒ 'α hrel* **where**
[*urel-defs*]: *antiframe a P = (P ∧ $v´ =$_u$ $v´ ⊕ $v on &a)*

Frame extension combines alphabet extension with the frame operator to both add additional variables and then frame those.

definition *rel-frext* :: *('β ⟹ 'α) ⇒ 'β hrel ⇒ 'α hrel* **where**
[*upred-defs*]: *rel-frext a P = frame a (rel-aext P a)*

The nameset operator can be used to hide a portion of the after-state that lies outside the lens a. It can be useful to partition a relation's variables in order to conjoin it with another relation.

definition *nameset* :: *('a ⟹ 'α) ⇒ 'α hrel ⇒ 'α hrel* **where**
[*urel-defs*]: *nameset a P = (P ⌈$_v$ {$v,$a´})*

15.3 Syntax Translations

syntax
— Alternative traditional conditional syntax
-utp-if :: *uexp ⇒ logic ⇒ logic ⇒ logic* (‹(*if*$_u$ (-)/ *then* (-)/ *else* (-))› [*0*, *0*, *71*] *71*)
— Iterated sequential composition
-seqr-iter :: *pttrn ⇒ 'a list ⇒ 'σ hrel ⇒ 'σ hrel* (‹(*3*;; - : - •/ -)› [*0*, *0*, *10*] *10*)
— Single and multiple assignement
-assignment :: *svids ⇒ uexprs ⇒ 'α hrel* (‹'(-') := '(-')›)
-assignment :: *svids ⇒ uexprs ⇒ 'α hrel* (**infixr** ‹:=› *62*)
— Non-deterministic assignment
-nd-assign :: *svids ⇒ logic* (‹- := *› [*62*] *62*)
— Substitution constructor
-mk-usubst :: *svids ⇒ uexprs ⇒ 'α usubst*
— Alphabetised skip

-skip-ra :: *salpha* ⇒ *logic* (‹II_-›)
— Frame
-frame :: *salpha* ⇒ *logic* ⇒ *logic* (‹-:[-]› [*99*,*0*] *100*)
— Antiframe
-antiframe :: *salpha* ⇒ *logic* ⇒ *logic* (‹-:[![-]!]› [*79*,*0*] *80*)
— Relational Alphabet Extension
-rel-aext :: *logic* ⇒ *salpha* ⇒ *logic* (**infixl** ‹$\oplus_r$› *90*)
— Relational Alphabet Restriction
-rel-ares :: *logic* ⇒ *salpha* ⇒ *logic* (**infixl** ‹$\restriction_r$› *90*)
— Frame Extension
-rel-frext :: *salpha* ⇒ *logic* ⇒ *logic* (‹-:[-]$^+$› [*99*,*0*] *100*)
— Nameset
-nameset :: *salpha* ⇒ *logic* ⇒ *logic* (‹*ns* - • -› [*0*,*999*] *999*)

translations
-utp-if b P Q => *P* ◃ *b* ▹$_r$ *Q*
;; *x* : *l* • *P* ⇌ (*CONST seqr-iter*) *l* (λ*x*. *P*)
-mk-usubst σ (*-svid-unit x*) *v* ⇌ σ(&*x* ↦$_s$ *v*)
-mk-usubst σ (*-svid-list x xs*) (*-uexprs v vs*) ⇌ (*-mk-usubst* (σ(&*x* ↦$_s$ *v*)) *xs vs*)
-assignment xs vs => *CONST uassigns* (*-mk-usubst* (*CONST id*) *xs vs*)
-assignment x v <= *CONST uassigns* (*CONST subst-upd* (*CONST id*) *x v*)
-assignment x v <= *-assignment* (*-spvar x*) *v*
-nd-assign x => *CONST nd-assign* (*-mk-svid-list x*)
-nd-assign x <= *CONST nd-assign x*
x,*y* := *u*,*v* <= *CONST uassigns* (*CONST subst-upd* (*CONST subst-upd* (*CONST id*) (*CONST svar x*) *u*) (*CONST svar y*) *v*)
-skip-ra v ⇌ *CONST skip-ra v*
-frame x P => *CONST frame x P*
-frame (*-salphaset* (*-salphamk x*)) *P* <= *CONST frame x P*
-antiframe x P => *CONST antiframe x P*
-antiframe (*-salphaset* (*-salphamk x*)) *P* <= *CONST antiframe x P*
-nameset x P == *CONST nameset x P*
-rel-aext P a == *CONST rel-aext P a*
-rel-ares P a == *CONST rel-ares P a*
-rel-frext a P == *CONST rel-frext a P*

The following code sets up pretty-printing for homogeneous relational expressions. We cannot do this via the "translations" command as we only want the rule to apply when the input and output alphabet types are the same. The code has to deconstruct a ('*a*, '*α*) *uexpr* type, determine that it is relational (product alphabet), and then checks if the types *alpha* and *beta* are the same. If they are, the type is printed as a *hexpr*. Otherwise, we have no match. We then set up a regular translation for the *hrel* type that uses this.

⟨*ML*⟩

translations
(*type*) '*α hrel* <= (*type*) (*bool*, '*α*) *hexpr*

15.4 Relation Properties

We describe some properties of relations, including functional and injective relations. We also provide operators for extracting the domain and range of a UTP relation.

definition *ufunctional* :: ('*a*, '*b*) *urel* ⇒ *bool*
where [*urel-defs*]: *ufunctional R* ⟷ *II* ⊑ R^- ;; *R*

definition *uinj* :: (*'a*, *'b*) *urel* ⇒ *bool*
where [*urel-defs*]: *uinj R* ⟷ *II* ⊑ *R* ;; *R*⁻

definition *Dom* :: *'α hrel* ⇒ *'α upred*
where [*upred-defs*]: *Dom P* = ⌊∃ $**v**´ • *P*⌋<

definition *Ran* :: *'α hrel* ⇒ *'α upred*
where [*upred-defs*]: *Ran P* = ⌊∃ $**v** • *P*⌋>

— Configuration for UTP tactics.

update-uexpr-rep-eq-thms — Reread *rep-eq* theorems.

15.5 Introduction laws

lemma *urel-refine-ext*:
⟦ ⋀ *s s'*. *P*⟦«*s*»,«*s'*»/$**v**,$**v**´⟧ ⊑ *Q*⟦«*s*»,«*s'*»/$**v**,$**v**´⟧ ⟧ ⟹ *P* ⊑ *Q*
⟨*proof*⟩

lemma *urel-eq-ext*:
⟦ ⋀ *s s'*. *P*⟦«*s*»,«*s'*»/$**v**,$**v**´⟧ = *Q*⟦«*s*»,«*s'*»/$**v**,$**v**´⟧ ⟧ ⟹ *P* = *Q*
⟨*proof*⟩

15.6 Unrestriction Laws

lemma *unrest-iuvar* [*unrest*]: *outα* ♯ $*x*
⟨*proof*⟩

lemma *unrest-ouvar* [*unrest*]: *inα* ♯ $*x*´
⟨*proof*⟩

lemma *unrest-semir-undash* [*unrest*]:
fixes *x* :: (*'a* ⟹ *'α*)
assumes $*x* ♯ *P*
shows $*x* ♯ *P* ;; *Q*
⟨*proof*⟩

lemma *unrest-semir-dash* [*unrest*]:
fixes *x* :: (*'a* ⟹ *'α*)
assumes $*x*´ ♯ *Q*
shows $*x*´ ♯ *P* ;; *Q*
⟨*proof*⟩

lemma *unrest-cond* [*unrest*]:
⟦ *x* ♯ *P*; *x* ♯ *b*; *x* ♯ *Q* ⟧ ⟹ *x* ♯ *P* ◁ *b* ▷ *Q*
⟨*proof*⟩

lemma *unrest-lift-rcond* [*unrest*]:
x ♯ ⌈*b*⌉< ⟹ *x* ♯ ⌈*b*⌉←
⟨*proof*⟩

lemma *unrest-inα-var* [*unrest*]:
⟦ *mwb-lens x*; *inα* ♯ (*P* :: (*'a*, (*'α* × *'β*)) *uexpr*) ⟧ ⟹ $*x* ♯ *P*
⟨*proof*⟩

lemma *unrest-outα-var* [*unrest*]:

$[\![\mathit{mwb\text{-}lens}\ x;\ \mathit{out}\alpha\ \sharp\ (P :: ('a, ('\alpha \times '\beta))\ \mathit{uexpr})]\!] \Longrightarrow \$x' \sharp P$
⟨*proof*⟩

lemma *unrest-pre-outα* [*unrest*]: $\mathit{out}\alpha \sharp \lceil b \rceil_<$
⟨*proof*⟩

lemma *unrest-post-inα* [*unrest*]: $\mathit{in}\alpha \sharp \lceil b \rceil_>$
⟨*proof*⟩

lemma *unrest-pre-in-var* [*unrest*]:
$x \sharp p1 \Longrightarrow \$x \sharp \lceil p1 \rceil_<$
⟨*proof*⟩

lemma *unrest-post-out-var* [*unrest*]:
$x \sharp p1 \Longrightarrow \$x' \sharp \lceil p1 \rceil_>$
⟨*proof*⟩

lemma *unrest-convr-outα* [*unrest*]:
$\mathit{in}\alpha \sharp p \Longrightarrow \mathit{out}\alpha \sharp p^-$
⟨*proof*⟩

lemma *unrest-convr-inα* [*unrest*]:
$\mathit{out}\alpha \sharp p \Longrightarrow \mathit{in}\alpha \sharp p^-$
⟨*proof*⟩

lemma *unrest-in-rel-var-res* [*unrest*]:
vwb-lens $x \Longrightarrow \$x \sharp (P \upharpoonright_\alpha x)$
⟨*proof*⟩

lemma *unrest-out-rel-var-res* [*unrest*]:
vwb-lens $x \Longrightarrow \$x' \sharp (P \upharpoonright_\alpha x)$
⟨*proof*⟩

lemma *unrest-out-alpha-usubst-rel-lift* [*unrest*]:
$\mathit{out}\alpha \sharp \lceil \sigma \rceil_s$
⟨*proof*⟩

lemma *unrest-in-rel-aext* [*unrest*]: $x \bowtie y \Longrightarrow \$y \sharp P \oplus_r x$
⟨*proof*⟩

lemma *unrest-out-rel-aext* [*unrest*]: $x \bowtie y \Longrightarrow \$y' \sharp P \oplus_r x$
⟨*proof*⟩

lemma *rel-aext-false* [*alpha*]:
$\mathit{false} \oplus_r a = \mathit{false}$
⟨*proof*⟩

lemma *rel-aext-seq* [*alpha*]:
weak-lens $a \Longrightarrow (P ;; Q) \oplus_r a = (P \oplus_r a ;; Q \oplus_r a)$
⟨*proof*⟩

lemma *rel-aext-cond* [*alpha*]:
$(P \lhd b \rhd_r Q) \oplus_r a = (P \oplus_r a \lhd b \oplus_p a \rhd_r Q \oplus_r a)$
⟨*proof*⟩

15.7 Substitution laws

lemma *subst-seq-left* [*usubst*]:
$out\alpha \sharp \sigma \Longrightarrow \sigma \dagger (P ;; Q) = (\sigma \dagger P) ;; Q$
⟨*proof*⟩

lemma *subst-seq-right* [*usubst*]:
$in\alpha \sharp \sigma \Longrightarrow \sigma \dagger (P ;; Q) = P ;; (\sigma \dagger Q)$
⟨*proof*⟩

The following laws support substitution in heterogeneous relations for polymorphically typed literal expressions. These cannot be supported more generically due to limitations in HOL's type system. The laws are presented in a slightly strange way so as to be as general as possible.

lemma *bool-seqr-laws* [*usubst*]:
fixes $x :: (bool \Longrightarrow 'a)$
shows
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x \mapsto_s true) \dagger (P ;; Q) = \sigma \dagger (P[\![true/\$x]\!] ;; Q)$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x \mapsto_s false) \dagger (P ;; Q) = \sigma \dagger (P[\![false/\$x]\!] ;; Q)$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x' \mapsto_s true) \dagger (P ;; Q) = \sigma \dagger (P ;; Q[\![true/\$x']\!])$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x' \mapsto_s false) \dagger (P ;; Q) = \sigma \dagger (P ;; Q[\![false/\$x']\!])$
⟨*proof*⟩

lemma *zero-one-seqr-laws* [*usubst*]:
fixes $x :: (- \Longrightarrow 'a)$
shows
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x \mapsto_s 0) \dagger (P ;; Q) = \sigma \dagger (P[\![0/\$x]\!] ;; Q)$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x \mapsto_s 1) \dagger (P ;; Q) = \sigma \dagger (P[\![1/\$x]\!] ;; Q)$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x' \mapsto_s 0) \dagger (P ;; Q) = \sigma \dagger (P ;; Q[\![0/\$x']\!])$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x' \mapsto_s 1) \dagger (P ;; Q) = \sigma \dagger (P ;; Q[\![1/\$x']\!])$
⟨*proof*⟩

lemma *numeral-seqr-laws* [*usubst*]:
fixes $x :: (- \Longrightarrow 'a)$
shows
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x \mapsto_s numeral\ n) \dagger (P ;; Q) = \sigma \dagger (P[\![numeral\ n/\$x]\!] ;; Q)$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x' \mapsto_s numeral\ n) \dagger (P ;; Q) = \sigma \dagger (P ;; Q[\![numeral\ n/\$x']\!])$
⟨*proof*⟩

lemma *usubst-condr* [*usubst*]:
$\sigma \dagger (P \lhd b \rhd Q) = (\sigma \dagger P \lhd \sigma \dagger b \rhd \sigma \dagger Q)$
⟨*proof*⟩

lemma *subst-skip-r* [*usubst*]:
$out\alpha \sharp \sigma \Longrightarrow \sigma \dagger II = \langle \lfloor \sigma \rfloor_s \rangle_a$
⟨*proof*⟩

lemma *subst-pre-skip* [*usubst*]: $\lceil \sigma \rceil_s \dagger II = \langle \sigma \rangle_a$
⟨*proof*⟩

lemma *subst-rel-lift-seq* [*usubst*]:
$\lceil \sigma \rceil_s \dagger (P ;; Q) = (\lceil \sigma \rceil_s \dagger P) ;; Q$
⟨*proof*⟩

lemma *subst-rel-lift-comp* [*usubst*]:
$\lceil \sigma \rceil_s \circ \lceil \varrho \rceil_s = \lceil \sigma \circ \varrho \rceil_s$
⟨*proof*⟩

lemma *usubst-upd-in-comp* [*usubst*]:
$\sigma(\&in\alpha{:}x \mapsto_s v) = \sigma(\$x \mapsto_s v)$
⟨*proof*⟩

lemma *usubst-upd-out-comp* [*usubst*]:
$\sigma(\&out\alpha{:}x \mapsto_s v) = \sigma(\$x\acute{} \mapsto_s v)$
⟨*proof*⟩

lemma *subst-lift-upd* [*alpha*]:
fixes $x :: ('a \Longrightarrow 'a)$
shows $\lceil \sigma(x \mapsto_s v) \rceil_s = \lceil \sigma \rceil_s(\$x \mapsto_s \lceil v \rceil_<)$
⟨*proof*⟩

lemma *subst-drop-upd* [*alpha*]:
fixes $x :: ('a \Longrightarrow 'a)$
shows $\lfloor \sigma(\$x \mapsto_s v) \rfloor_s = \lfloor \sigma \rfloor_s(x \mapsto_s \lfloor v \rfloor_<)$
⟨*proof*⟩

lemma *subst-lift-pre* [*usubst*]: $\lceil \sigma \rceil_s \dagger \lceil b \rceil_< = \lceil \sigma \dagger b \rceil_<$
⟨*proof*⟩

lemma *unrest-usubst-lift-in* [*unrest*]:
$x \sharp P \Longrightarrow \$x \sharp \lceil P \rceil_s$
⟨*proof*⟩

lemma *unrest-usubst-lift-out* [*unrest*]:
fixes $x :: ('a \Longrightarrow 'a)$
shows $\$x\acute{} \sharp \lceil P \rceil_s$
⟨*proof*⟩

lemma *subst-lift-cond* [*usubst*]: $\lceil \sigma \rceil_s \dagger \lceil s \rceil_\leftarrow = \lceil \sigma \dagger s \rceil_\leftarrow$
⟨*proof*⟩

lemma *msubst-seq* [*usubst*]: $(P(x) ;; Q(x))[\![x \rightarrow « v »]\!] = ((P(x))[\![x \rightarrow « v »]\!] ;; (Q(x))[\![x \rightarrow « v »]\!])$
⟨*proof*⟩

15.8 Alphabet laws

lemma *aext-cond* [*alpha*]:
$(P \lhd b \rhd Q) \oplus_p a = ((P \oplus_p a) \lhd (b \oplus_p a) \rhd (Q \oplus_p a))$
⟨*proof*⟩

lemma *aext-seq* [*alpha*]:
wb-lens $a \Longrightarrow ((P ;; Q) \oplus_p (a \times_L a)) = ((P \oplus_p (a \times_L a)) ;; (Q \oplus_p (a \times_L a)))$
⟨*proof*⟩

lemma *rcond-lift-true* [*simp*]:
$\lceil true \rceil_\leftarrow = true$
⟨*proof*⟩

lemma *rcond-lift-false* [*simp*]:
$\lceil false \rceil_\leftarrow = false$
⟨*proof*⟩

lemma *rel-ares-aext* [*alpha*]:

vwb-lens $a \Longrightarrow (P \oplus_r a) \upharpoonright_r a = P$
⟨*proof*⟩

lemma *rel-aext-ares* [*alpha*]:
$\{\$a, \$a\acute{}\} \natural P \Longrightarrow P \upharpoonright_r a \oplus_r a = P$
⟨*proof*⟩

lemma *rel-aext-uses* [*unrest*]:
vwb-lens $a \Longrightarrow \{\$a, \$a\acute{}\} \natural (P \oplus_r a)$
⟨*proof*⟩

15.9 Relational unrestriction

Relational unrestriction states that a variable is both unchanged by a relation, and is not "read" by the relation.

definition *RID* :: $('a \Longrightarrow 'c) \Rightarrow 'c$ *hrel* $\Rightarrow 'c$ *hrel*
where *RID* $x\ P = ((\exists\ \$x \cdot \exists\ \$x\acute{} \cdot P) \wedge \$x\acute{} =_u \$x)$

declare *RID-def* [*urel-defs*]

lemma *RID1*: *vwb-lens* $x \Longrightarrow (\forall\ v.\ x := \ll v \gg\ ;;\ P = P\ ;;\ x := \ll v \gg) \Longrightarrow RID(x)(P) = P$
⟨*proof*⟩

lemma *RID2*: *vwb-lens* $x \Longrightarrow x := \ll v \gg\ ;;\ RID(x)(P) = RID(x)(P)\ ;;\ x := \ll v \gg$
⟨*proof*⟩

lemma *RID-assign-commute*:
vwb-lens $x \Longrightarrow P = RID(x)(P) \longleftrightarrow (\forall\ v.\ x := \ll v \gg\ ;;\ P = P\ ;;\ x := \ll v \gg)$
⟨*proof*⟩

lemma *RID-idem*:
mwb-lens $x \Longrightarrow RID(x)(RID(x)(P)) = RID(x)(P)$
⟨*proof*⟩

lemma *RID-mono*:
$P \sqsubseteq Q \Longrightarrow RID(x)(P) \sqsubseteq RID(x)(Q)$
⟨*proof*⟩

lemma *RID-pr-var* [*simp*]:
RID (*pr-var* x) = *RID* x
⟨*proof*⟩

lemma *RID-skip-r*:
vwb-lens $x \Longrightarrow RID(x)(II) = II$
⟨*proof*⟩

lemma *skip-r-RID* [*closure*]: *vwb-lens* $x \Longrightarrow II$ *is* $RID(x)$
⟨*proof*⟩

lemma *RID-disj*:
$RID(x)(P \vee Q) = (RID(x)(P) \vee RID(x)(Q))$
⟨*proof*⟩

lemma *disj-RID* [*closure*]: $[\![\ P$ *is* $RID(x);\ Q$ *is* $RID(x)\ ]\!] \Longrightarrow (P \vee Q)$ *is* $RID(x)$
⟨*proof*⟩

lemma *RID-conj*:
vwb-lens x $\Longrightarrow$ *RID*(*x*)(*RID*(*x*)(*P*) $\wedge$ *RID*(*x*)(*Q*)) = (*RID*(*x*)(*P*) $\wedge$ *RID*(*x*)(*Q*))
⟨*proof*⟩

lemma *conj-RID* [*closure*]: ⟦ *vwb-lens x*; *P is RID*(*x*); *Q is RID*(*x*) ⟧ $\Longrightarrow$ (*P* $\wedge$ *Q*) *is RID*(*x*)
⟨*proof*⟩

lemma *RID-assigns-r-diff*:
⟦ *vwb-lens x*; *x* ♯ σ ⟧ $\Longrightarrow$ *RID*(*x*)(⟨σ⟩$_a$) = ⟨σ⟩$_a$
⟨*proof*⟩

lemma *assigns-r-RID* [*closure*]: ⟦ *vwb-lens x*; *x* ♯ σ ⟧ $\Longrightarrow$ ⟨σ⟩$_a$ *is RID*(*x*)
⟨*proof*⟩

lemma *RID-assign-r-same*:
vwb-lens x $\Longrightarrow$ *RID*(*x*)(*x* := *v*) = *II*
⟨*proof*⟩

lemma *RID-seq-left*:
assumes *vwb-lens x*
shows *RID*(*x*)(*RID*(*x*)(*P*) ;; *Q*) = (*RID*(*x*)(*P*) ;; *RID*(*x*)(*Q*))
⟨*proof*⟩

lemma *RID-seq-right*:
assumes *vwb-lens x*
shows *RID*(*x*)(*P* ;; *RID*(*x*)(*Q*)) = (*RID*(*x*)(*P*) ;; *RID*(*x*)(*Q*))
⟨*proof*⟩

lemma *seqr-RID-closed* [*closure*]: ⟦ *vwb-lens x*; *P is RID*(*x*); *Q is RID*(*x*) ⟧ $\Longrightarrow$ *P* ;; *Q is RID*(*x*)
⟨*proof*⟩

definition *unrest-relation* :: ('*a* $\Longrightarrow$ 'α) $\Rightarrow$ 'α *hrel* $\Rightarrow$ *bool* (**infix** ‹♯♯› *20*)
where (*x* ♯♯ *P*) $\longleftrightarrow$ (*P is RID*(*x*))

declare *unrest-relation-def* [*urel-defs*]

lemma *runrest-assign-commute*:
⟦ *vwb-lens x*; *x* ♯♯ *P* ⟧ $\Longrightarrow$ *x* := «*v*» ;; *P* = *P* ;; *x* := «*v*»
⟨*proof*⟩

lemma *runrest-ident-var*:
assumes *x* ♯♯ *P*
shows ($*x* $\wedge$ *P*) = (*P* $\wedge$ $*x´*)
⟨*proof*⟩

lemma *skip-r-runrest* [*unrest*]:
vwb-lens x $\Longrightarrow$ *x* ♯♯ *II*
⟨*proof*⟩

lemma *assigns-r-runrest*:
⟦ *vwb-lens x*; *x* ♯ σ ⟧ $\Longrightarrow$ *x* ♯♯ ⟨σ⟩$_a$
⟨*proof*⟩

lemma *seq-r-runrest* [*unrest*]:

assumes *vwb-lens x* *x* ♯♯ *P* *x* ♯♯ *Q*
shows *x* ♯♯ (*P* ;; *Q*)
⟨*proof*⟩

lemma *false-runrest* [*unrest*]: *x* ♯♯ *false*
⟨*proof*⟩

lemma *and-runrest* [*unrest*]: ⟦ *vwb-lens x*; *x* ♯♯ *P*; *x* ♯♯ *Q* ⟧ ⟹ *x* ♯♯ (*P* ∧ *Q*)
⟨*proof*⟩

lemma *or-runrest* [*unrest*]: ⟦ *x* ♯♯ *P*; *x* ♯♯ *Q* ⟧ ⟹ *x* ♯♯ (*P* ∨ *Q*)
⟨*proof*⟩

end

16 Fixed-points and Recursion

theory *utp-recursion*
imports
utp-pred-laws
utp-rel
begin

16.1 Fixed-point Laws

lemma *mu-id*: (*μ X* · *X*) = *true*
⟨*proof*⟩

lemma *mu-const*: (*μ X* · *P*) = *P*
⟨*proof*⟩

lemma *nu-id*: (*ν X* · *X*) = *false*
⟨*proof*⟩

lemma *nu-const*: (*ν X* · *P*) = *P*
⟨*proof*⟩

lemma *mu-refine-intro*:
assumes (*C* ⇒ *S*) ⊑ *F*(*C* ⇒ *S*) (*C* ∧ *μ F*) = (*C* ∧ *ν F*)
shows (*C* ⇒ *S*) ⊑ *μ F*
⟨*proof*⟩

16.2 Obtaining Unique Fixed-points

Obtaining termination proofs via approximation chains. Theorems and proofs adapted from Chapter 2, page 63 of the UTP book [22].

type-synonym *'a chain* = *nat* ⇒ *'a upred*

definition *chain* :: *'a chain* ⇒ *bool* **where**
chain Y = ((*Y 0* = *false*) ∧ (∀ *i*. *Y* (*Suc i*) ⊑ *Y i*))

lemma *chain0* [*simp*]: *chain Y* ⟹ *Y 0* = *false*
⟨*proof*⟩

lemma *chainI*:
assumes $Y\ 0 = false \bigwedge i.\ Y\ (Suc\ i) \sqsubseteq Y\ i$
shows *chain Y*
⟨*proof*⟩

lemma *chainE*:
assumes $chain\ Y \bigwedge i.\ [\![\ Y\ 0 = false;\ Y\ (Suc\ i) \sqsubseteq Y\ i\ ]\!] \Longrightarrow P$
shows *P*
⟨*proof*⟩

lemma *L274*:
assumes $\forall\ n.\ ((E\ n \wedge_p X) = (E\ n \wedge Y))$
shows $(\bigsqcap\ (range\ E) \wedge X) = (\bigsqcap\ (range\ E) \wedge Y)$
⟨*proof*⟩

Constructive chains

definition *constr* ::
$('a\ upred \Rightarrow 'a\ upred) \Rightarrow 'a\ chain \Rightarrow bool$ **where**
$constr\ F\ E \longleftrightarrow chain\ E \wedge (\forall\ X\ n.\ ((F(X) \wedge E(n+1)) = (F(X \wedge E(n)) \wedge E\ (n+1))))$

lemma *constrI*:
assumes $chain\ E \bigwedge X\ n.\ ((F(X) \wedge E(n+1)) = (F(X \wedge E(n)) \wedge E\ (n+1)))$
shows *constr F E*
⟨*proof*⟩

This lemma gives a way of showing that there is a unique fixed-point when the predicate function can be built using a constructive function F over an approximation chain E

lemma *chain-pred-terminates*:
assumes *constr F E mono F*
shows $(\bigsqcap\ (range\ E) \wedge \mu\ F) = (\bigsqcap\ (range\ E) \wedge \nu\ F)$
⟨*proof*⟩

theorem *constr-fp-uniq*:
assumes *constr F E mono F* $\bigsqcap\ (range\ E) = C$
shows $(C \wedge \mu\ F) = (C \wedge \nu\ F)$
⟨*proof*⟩

16.3 Noetherian Induction Instantiation

Contribution from Yakoub Nemouchi.The following generalization was used by Tobias Nipkow and Peter Lammich in *Refine_Monadic*

lemma *wf-fixp-uinduct-pure-ueq-gen*:
assumes *fixp-unfold*: *fp B = B (fp B)*
and *WF*: *wf R*
and *induct-step*:
$\bigwedge f\ st.\ [\![\bigwedge st'.\ (st',st) \in R \Longrightarrow (((Pre \wedge \lceil e \rceil_< =_u «st'») \Rightarrow Post) \sqsubseteq f)]\!]$
$\Longrightarrow fp\ B = f \Longrightarrow ((Pre \wedge \lceil e \rceil_< =_u «st») \Rightarrow Post) \sqsubseteq (B\ f)$
shows $((Pre \Rightarrow Post) \sqsubseteq fp\ B)$
⟨*proof*⟩

The next lemma shows that using substitution also work. However it is not that generic nor practical for proof automation ...

lemma *refine-usubst-to-ueq*:
$vwb\text{-}lens\ E \Longrightarrow (Pre \Rightarrow Post)[\![«st'»/\$E]\!] \sqsubseteq f[\![«st'»/\$E]\!] = (((Pre \wedge \$E =_u «st'») \Rightarrow Post) \sqsubseteq f)$

⟨*proof*⟩

By instantiation of ⟦*?fp ?B = ?B (?fp ?B)*; *wf ?R*; ⋀*f st.* ⟦⋀*st′*. (*st′*, *st*) ∈ *?R* ⟹ (*?Pre* ∧ ⌈*?e*⌉$_<$ =$_u$ «*st′*» ⇒ *?Post*) ⊑ *f*; *?fp ?B = f*⟧ ⟹ (*?Pre* ∧ ⌈*?e*⌉$_<$ =$_u$ «*st*» ⇒ *?Post*) ⊑ *?B f*⟧ ⟹ (*?Pre* ⇒ *?Post*) ⊑ *?fp ?B* with *μ* and lifting of the well-founded relation we have ...

lemma *mu-rec-total-pure-rule*:
assumes *WF*: *wf R*
and *M*: *mono B*
and *induct-step*:
⋀ *f st.* ⟦(*Pre* ∧ (⌈*e*⌉$_<$,«*st*»)$_u$ ∈$_u$ «*R*» ⇒ *Post*) ⊑ *f*⟧
⟹ *μ B* = *f* ⟹(*Pre* ∧ ⌈*e*⌉$_<$ =$_u$ «*st*» ⇒ *Post*) ⊑ (*B f*)
shows (*Pre* ⇒ *Post*) ⊑ *μ B*
⟨*proof*⟩

lemma *nu-rec-total-pure-rule*:
assumes *WF*: *wf R*
and *M*: *mono B*
and *induct-step*:
⋀ *f st.* ⟦(*Pre* ∧ (⌈*e*⌉$_<$,«*st*»)$_u$ ∈$_u$ «*R*» ⇒ *Post*) ⊑ *f*⟧
⟹ *ν B* = *f* ⟹(*Pre* ∧ ⌈*e*⌉$_<$ =$_u$ «*st*» ⇒ *Post*) ⊑ (*B f*)
shows (*Pre* ⇒ *Post*) ⊑ *ν B*
⟨*proof*⟩

Since *B* (*Pre* ∧ (⌈*E*⌉$_<$, «*st*»)$_u$ ∈$_u$ «*R*» ⇒ *Post*) ⊑ *B* (*μ B*) and *mono B*, thus, ⟦*wf ?R*; *Monotonic ?B*; ⋀*f st.* ⟦(*?Pre* ∧ (⌈*?e*⌉$_<$, «*st*»)$_u$ ∈$_u$ «*?R*» ⇒ *?Post*) ⊑ *f*; *μ ?B* = *f*⟧ ⟹ (*?Pre* ∧ ⌈*?e*⌉$_<$ =$_u$ «*st*» ⇒ *?Post*) ⊑ *?B f*⟧ ⟹ (*?Pre* ⇒ *?Post*) ⊑ *μ ?B* can be expressed as follows

lemma *mu-rec-total-utp-rule*:
assumes *WF*: *wf R*
and *M*: *mono B*
and *induct-step*:
⋀*st.* (*Pre* ∧ ⌈*e*⌉$_<$ =$_u$ «*st*» ⇒ *Post*) ⊑ (*B* ((*Pre* ∧ (⌈*e*⌉$_<$,«*st*»)$_u$ ∈$_u$ «*R*» ⇒ *Post*)))
shows (*Pre* ⇒ *Post*) ⊑ *μ B*
⟨*proof*⟩

lemma *nu-rec-total-utp-rule*:
assumes *WF*: *wf R*
and *M*: *mono B*
and *induct-step*:
⋀*st.* (*Pre* ∧ ⌈*e*⌉$_<$ =$_u$ «*st*» ⇒ *Post*) ⊑ (*B* ((*Pre* ∧ (⌈*e*⌉$_<$,«*st*»)$_u$ ∈$_u$ «*R*» ⇒ *Post*)))
shows (*Pre* ⇒ *Post*) ⊑ *ν B*
⟨*proof*⟩

end

17 Sequent Calculus

theory *utp-sequent*
imports *utp-pred-laws*
begin

definition *sequent* :: *′α upred* ⇒ *′α upred* ⇒ *bool* (**infixr** ‹⊩› *15*) **where**
[*upred-defs*]: *sequent P Q* = (*Q* ⊑ *P*)

abbreviation *sequent-triv* (‹⊩ -› [*15*] *15*) **where** ⊩ *P* ≡ (*true* ⊩ *P*)

translations
$\Vdash\ P <= \mathit{true} \Vdash P$

lemma *sTrue*: $P \Vdash \mathit{true}$
⟨*proof*⟩

lemma *sAx*: $P \Vdash P$
⟨*proof*⟩

lemma *sNotI*: $\Gamma \wedge P \Vdash \mathit{false} \Longrightarrow \Gamma \Vdash \neg P$
⟨*proof*⟩

lemma *sConjI*: $[\![\Gamma \Vdash P; \Gamma \Vdash Q]\!] \Longrightarrow \Gamma \Vdash P \wedge Q$
⟨*proof*⟩

lemma *sImplI*: $[\![(\Gamma \wedge P) \Vdash Q]\!] \Longrightarrow \Gamma \Vdash (P \Rightarrow Q)$
⟨*proof*⟩

end

18 Relational Calculus Laws

theory *utp-rel-laws*
imports
utp-rel
utp-recursion
begin

18.1 Conditional Laws

lemma *comp-cond-left-distr*:
$((P \lhd b \rhd_r Q) ;; R) = ((P ;; R) \lhd b \rhd_r (Q ;; R))$
⟨*proof*⟩

lemma *cond-seq-left-distr*:
$out\alpha \sharp b \Longrightarrow ((P \lhd b \rhd Q) ;; R) = ((P ;; R) \lhd b \rhd (Q ;; R))$
⟨*proof*⟩

lemma *cond-seq-right-distr*:
$in\alpha \sharp b \Longrightarrow (P ;; (Q \lhd b \rhd R)) = ((P ;; Q) \lhd b \rhd (P ;; R))$
⟨*proof*⟩

Alternative expression of conditional using assumptions and choice

lemma *rcond-rassume-expand*: $P \lhd b \rhd_r Q = ([b]^\top ;; P) \sqcap ([(\neg b)]^\top ;; Q)$
⟨*proof*⟩

18.2 Precondition and Postcondition Laws

theorem *precond-equiv*:
$P = (P ;; \mathit{true}) \longleftrightarrow (out\alpha \sharp P)$
⟨*proof*⟩

theorem *postcond-equiv*:
$P = (\mathit{true} ;; P) \longleftrightarrow (in\alpha \sharp P)$

⟨*proof*⟩

lemma *precond-right-unit*: $out\alpha \sharp p \Longrightarrow (p ;; true) = p$
⟨*proof*⟩

lemma *postcond-left-unit*: $in\alpha \sharp p \Longrightarrow (true ;; p) = p$
⟨*proof*⟩

theorem *precond-left-zero*:
assumes $out\alpha \sharp p\ p \neq false$
shows $(true ;; p) = true$
⟨*proof*⟩

theorem *feasibile-iff-true-right-zero*:
$P ;; true = true \longleftrightarrow \exists\ out\alpha \cdot P$‘
⟨*proof*⟩

18.3 Sequential Composition Laws

lemma *seqr-assoc*: $(P ;; Q) ;; R = P ;; (Q ;; R)$
⟨*proof*⟩

lemma *seqr-left-unit* [*simp*]:
$II ;; P = P$
⟨*proof*⟩

lemma *seqr-right-unit* [*simp*]:
$P ;; II = P$
⟨*proof*⟩

lemma *seqr-left-zero* [*simp*]:
$false ;; P = false$
⟨*proof*⟩

lemma *seqr-right-zero* [*simp*]:
$P ;; false = false$
⟨*proof*⟩

lemma *impl-seqr-mono*: $[\![\text{‘}P \Rightarrow Q\text{‘}; \text{‘}R \Rightarrow S\text{‘}]\!] \Longrightarrow \text{‘}(P ;; R) \Rightarrow (Q ;; S)\text{‘}$
⟨*proof*⟩

lemma *seqr-mono*:
$[\![P_1 \sqsubseteq P_2; Q_1 \sqsubseteq Q_2]\!] \Longrightarrow (P_1 ;; Q_1) \sqsubseteq (P_2 ;; Q_2)$
⟨*proof*⟩

lemma *seqr-monotonic*:
$[\![mono\ P; mono\ Q]\!] \Longrightarrow mono\ (\lambda\ X.\ P\ X ;; Q\ X)$
⟨*proof*⟩

lemma *Monotonic-seqr-tail* [*closure*]:
assumes *Monotonic F*
shows *Monotonic* $(\lambda\ X.\ P ;; F(X))$
⟨*proof*⟩

lemma *seqr-exists-left*:
$((\exists\ \$x \cdot P) ;; Q) = (\exists\ \$x \cdot (P ;; Q))$

⟨*proof*⟩

lemma *seqr-exists-right*:
$(P \;;; (\exists\ \$x' \cdot Q)) = (\exists\ \$x' \cdot (P \;;; Q))$
⟨*proof*⟩

lemma *seqr-or-distl*:
$((P \lor Q) \;;; R) = ((P \;;; R) \lor (Q \;;; R))$
⟨*proof*⟩

lemma *seqr-or-distr*:
$(P \;;; (Q \lor R)) = ((P \;;; Q) \lor (P \;;; R))$
⟨*proof*⟩

lemma *seqr-inf-distl*:
$((P \sqcap Q) \;;; R) = ((P \;;; R) \sqcap (Q \;;; R))$
⟨*proof*⟩

lemma *seqr-inf-distr*:
$(P \;;; (Q \sqcap R)) = ((P \;;; Q) \sqcap (P \;;; R))$
⟨*proof*⟩

lemma *seqr-and-distr-ufunc*:
ufunctional $P \Longrightarrow (P \;;; (Q \land R)) = ((P \;;; Q) \land (P \;;; R))$
⟨*proof*⟩

lemma *seqr-and-distl-uinj*:
uinj $R \Longrightarrow ((P \land Q) \;;; R) = ((P \;;; R) \land (Q \;;; R))$
⟨*proof*⟩

lemma *seqr-unfold*:
$(P \;;; Q) = (\exists\ v \cdot P[\![\ll v \gg / \$\mathbf{v}']\!] \land Q[\![\ll v \gg / \$\mathbf{v}]\!])$
⟨*proof*⟩

lemma *seqr-middle*:
assumes *vwb-lens* x
shows $(P \;;; Q) = (\exists\ v \cdot P[\![\ll v \gg / \$x']\!] \;;; Q[\![\ll v \gg / \$x]\!])$
⟨*proof*⟩

lemma *seqr-left-one-point*:
assumes *vwb-lens* x
shows $((P \land \$x' =_u \ll v \gg) \;;; Q) = (P[\![\ll v \gg / \$x']\!] \;;; Q[\![\ll v \gg / \$x]\!])$
⟨*proof*⟩

lemma *seqr-right-one-point*:
assumes *vwb-lens* x
shows $(P \;;; (\$x =_u \ll v \gg \land Q)) = (P[\![\ll v \gg / \$x']\!] \;;; Q[\![\ll v \gg / \$x]\!])$
⟨*proof*⟩

lemma *seqr-left-one-point-true*:
assumes *vwb-lens* x
shows $((P \land \$x') \;;; Q) = (P[\![true/\$x']\!] \;;; Q[\![true/\$x]\!])$
⟨*proof*⟩

lemma *seqr-left-one-point-false*:

assumes *vwb-lens x*
shows $((P \wedge \neg \$x\acute{}) \mathbin{;;} Q) = (P[\![false/\$x\acute{}]\!] \mathbin{;;} Q[\![false/\$x]\!])$
⟨*proof*⟩

lemma *seqr-right-one-point-true*:
assumes *vwb-lens x*
shows $(P \mathbin{;;} (\$x \wedge Q)) = (P[\![true/\$x\acute{}]\!] \mathbin{;;} Q[\![true/\$x]\!])$
⟨*proof*⟩

lemma *seqr-right-one-point-false*:
assumes *vwb-lens x*
shows $(P \mathbin{;;} (\neg \$x \wedge Q)) = (P[\![false/\$x\acute{}]\!] \mathbin{;;} Q[\![false/\$x]\!])$
⟨*proof*⟩

lemma *seqr-insert-ident-left*:
assumes *vwb-lens x* $\$x\acute{} \sharp P$ $\$x \sharp Q$
shows $((\$x\acute{} =_u \$x \wedge P) \mathbin{;;} Q) = (P \mathbin{;;} Q)$
⟨*proof*⟩

lemma *seqr-insert-ident-right*:
assumes *vwb-lens x* $\$x\acute{} \sharp P$ $\$x \sharp Q$
shows $(P \mathbin{;;} (\$x\acute{} =_u \$x \wedge Q)) = (P \mathbin{;;} Q)$
⟨*proof*⟩

lemma *seq-var-ident-lift*:
assumes *vwb-lens x* $\$x\acute{} \sharp P$ $\$x \sharp Q$
shows $((\$x\acute{} =_u \$x \wedge P) \mathbin{;;} (\$x\acute{} =_u \$x \wedge Q)) = (\$x\acute{} =_u \$x \wedge (P \mathbin{;;} Q))$
⟨*proof*⟩

lemma *seqr-bool-split*:
assumes *vwb-lens x*
shows $P \mathbin{;;} Q = (P[\![true/\$x\acute{}]\!] \mathbin{;;} Q[\![true/\$x]\!] \vee P[\![false/\$x\acute{}]\!] \mathbin{;;} Q[\![false/\$x]\!])$
⟨*proof*⟩

lemma *cond-inter-var-split*:
assumes *vwb-lens x*
shows $(P \lhd \$x\acute{} \rhd Q) \mathbin{;;} R = (P[\![true/\$x\acute{}]\!] \mathbin{;;} R[\![true/\$x]\!] \vee Q[\![false/\$x\acute{}]\!] \mathbin{;;} R[\![false/\$x]\!])$
⟨*proof*⟩

theorem *seqr-pre-transfer*: $in\alpha \sharp q \Longrightarrow ((P \wedge q) \mathbin{;;} R) = (P \mathbin{;;} (q^- \wedge R))$
⟨*proof*⟩

theorem *seqr-pre-transfer'*:
$((P \wedge \lceil q \rceil_>) \mathbin{;;} R) = (P \mathbin{;;} (\lceil q \rceil_< \wedge R))$
⟨*proof*⟩

theorem *seqr-post-out*: $in\alpha \sharp r \Longrightarrow (P \mathbin{;;} (Q \wedge r)) = ((P \mathbin{;;} Q) \wedge r)$
⟨*proof*⟩

lemma *seqr-post-var-out*:
fixes $x :: (bool \Longrightarrow 'a)$
shows $(P \mathbin{;;} (Q \wedge \$x\acute{})) = ((P \mathbin{;;} Q) \wedge \$x\acute{})$
⟨*proof*⟩

theorem *seqr-post-transfer*: $out\alpha \sharp q \Longrightarrow (P \mathbin{;;} (q \wedge R)) = ((P \wedge q^-) \mathbin{;;} R)$

⟨*proof*⟩

lemma *seqr-pre-out*: $out\alpha \sharp p \Longrightarrow ((p \wedge Q) ;; R) = (p \wedge (Q ;; R))$
⟨*proof*⟩

lemma *seqr-pre-var-out*:
fixes $x :: (bool \Longrightarrow {'\alpha})$
shows $((\$x \wedge P) ;; Q) = (\$x \wedge (P ;; Q))$
⟨*proof*⟩

lemma *seqr-true-lemma*:
$(P = (\neg ((\neg P) ;; true))) = (P = (P ;; true))$
⟨*proof*⟩

lemma *seqr-to-conj*: $[\![out\alpha \sharp P; in\alpha \sharp Q]\!] \Longrightarrow (P ;; Q) = (P \wedge Q)$
⟨*proof*⟩

lemma *shEx-lift-seq-1* [*uquant-lift*]:
$((\exists\ x \cdot P\ x) ;; Q) = (\exists\ x \cdot (P\ x ;; Q))$
⟨*proof*⟩

lemma *shEx-mem-lift-seq-1* [*uquant-lift*]:
assumes $out\alpha \sharp A$
shows $((\exists\ x \in A \cdot P\ x) ;; Q) = (\exists\ x \in A \cdot (P\ x ;; Q))$
⟨*proof*⟩

lemma *shEx-lift-seq-2* [*uquant-lift*]:
$(P ;; (\exists\ x \cdot Q\ x)) = (\exists\ x \cdot (P ;; Q\ x))$
⟨*proof*⟩

lemma *shEx-mem-lift-seq-2* [*uquant-lift*]:
assumes $in\alpha \sharp A$
shows $(P ;; (\exists\ x \in A \cdot Q\ x)) = (\exists\ x \in A \cdot (P ;; Q\ x))$
⟨*proof*⟩

18.4 Iterated Sequential Composition Laws

lemma *iter-seqr-nil* [*simp*]: $(;; i : [] \cdot P(i)) = II$
⟨*proof*⟩

lemma *iter-seqr-cons* [*simp*]: $(;; i : (x \# xs) \cdot P(i)) = P(x) ;; (;; i : xs \cdot P(i))$
⟨*proof*⟩

18.5 Quantale Laws

lemma *seq-Sup-distl*: $P ;; (\bigsqcap A) = (\bigsqcap Q{\in}A.\ P ;; Q)$
⟨*proof*⟩

lemma *seq-Sup-distr*: $(\bigsqcap A) ;; Q = (\bigsqcap P{\in}A.\ P ;; Q)$
⟨*proof*⟩

lemma *seq-UINF-distl*: $P ;; (\bigsqcap Q{\in}A \cdot F(Q)) = (\bigsqcap Q{\in}A \cdot P ;; F(Q))$
⟨*proof*⟩

lemma *seq-UINF-distl'*: $P ;; (\bigsqcap Q \cdot F(Q)) = (\bigsqcap Q \cdot P ;; F(Q))$
⟨*proof*⟩

lemma *seq-UINF-distr*: $(\bigsqcap\ P{\in}A \cdot F(P)) ;; Q = (\bigsqcap\ P{\in}A \cdot F(P) ;; Q)$
⟨*proof*⟩

lemma *seq-UINF-distr'*: $(\bigsqcap\ P \cdot F(P)) ;; Q = (\bigsqcap\ P \cdot F(P) ;; Q)$
⟨*proof*⟩

lemma *seq-SUP-distl*: $P ;; (\bigsqcap i{\in}A.\ Q(i)) = (\bigsqcap i{\in}A.\ P ;; Q(i))$
⟨*proof*⟩

lemma *seq-SUP-distr*: $(\bigsqcap i{\in}A.\ P(i)) ;; Q = (\bigsqcap i{\in}A.\ P(i) ;; Q)$
⟨*proof*⟩

18.6 Skip Laws

lemma *cond-skip*: $out\alpha \sharp b \Longrightarrow (b \wedge II) = (II \wedge b^-)$
⟨*proof*⟩

lemma *pre-skip-post*: $(\lceil b \rceil_< \wedge II) = (II \wedge \lceil b \rceil_>)$
⟨*proof*⟩

lemma *skip-var*:
fixes $x :: (bool \Longrightarrow {'\alpha})$
shows $(\$x \wedge II) = (II \wedge \$x')$
⟨*proof*⟩

lemma *skip-r-unfold*:
vwb-lens $x \Longrightarrow II = (\$x' =_u \$x \wedge II{\restriction}_\alpha x)$
⟨*proof*⟩

lemma *skip-r-alpha-eq*:
$II = (\$\mathbf{v}' =_u \$\mathbf{v})$
⟨*proof*⟩

lemma *skip-ra-unfold*:
$II_{x;y} = (\$x' =_u \$x \wedge II_y)$
⟨*proof*⟩

lemma *skip-res-as-ra*:
$[\![$ *vwb-lens* y; $x +_L y \approx_L 1_L$; $x \bowtie y\ ]\!] \Longrightarrow II{\restriction}_\alpha x = II_y$
⟨*proof*⟩

18.7 Assignment Laws

lemma *assigns-subst* [*usubst*]:
$\lceil \sigma \rceil_s \dagger \langle \varrho \rangle_a = \langle \varrho \circ \sigma \rangle_a$
⟨*proof*⟩

lemma *assigns-r-comp*: $(\langle \sigma \rangle_a ;; P) = (\lceil \sigma \rceil_s \dagger P)$
⟨*proof*⟩

lemma *assigns-r-feasible*:
$(\langle \sigma \rangle_a ;; true) = true$
⟨*proof*⟩

lemma *assign-subst* [*usubst*]:

⟦ *mwb-lens x*; *mwb-lens y* ⟧ ⟹ [$x ↦$_s_ ⌈*u*⌉$_<$] † (*y* := *v*) = (*x*, *y*) := (*u*, [*x* ↦$_s$ *u*] † *v*)
⟨*proof*⟩

lemma *assign-vacuous-skip*:
assumes *vwb-lens x*
shows (*x* := &*x*) = *II*
⟨*proof*⟩

The following law shows the case for the above law when x is only mainly-well behaved. We require that the state is one of those in which x is well defined using and assumption.

lemma *assign-vacuous-assume*:
assumes *mwb-lens x*
shows [(&**v** ∈$_u$ «$\mathcal{S}_x$»)]$^\top$;; (*x* := &*x*) = [(&**v** ∈$_u$ «$\mathcal{S}_x$»)]$^\top$
⟨*proof*⟩

lemma *assign-simultaneous*:
assumes *vwb-lens y x* ⋈ *y*
shows (*x*,*y*) := (*e*, &*y*) = (*x* := *e*)
⟨*proof*⟩

lemma *assigns-idem*: *mwb-lens x* ⟹ (*x*,*x*) := (*u*,*v*) = (*x* := *v*)
⟨*proof*⟩

lemma *assigns-comp*: (⟨*f*⟩$_a$;; ⟨*g*⟩$_a$) = ⟨*g* ∘ *f*⟩$_a$
⟨*proof*⟩

lemma *assigns-cond*: (⟨*f*⟩$_a$ ◁ *b* ▷$_r$ ⟨*g*⟩$_a$) = ⟨*f* ◁ *b* ▷$_s$ *g*⟩$_a$
⟨*proof*⟩

lemma *assigns-r-conv*:
bij f ⟹ ⟨*f*⟩$_a{}^-$ = ⟨*inv f*⟩$_a$
⟨*proof*⟩

lemma *assign-pred-transfer*:
fixes *x* :: ('*a* ⟹ '*α*)
assumes $*x* ♯ *b* *out*α ♯ *b*
shows (*b* ∧ *x* := *v*) = (*x* := *v* ∧ *b*$^-$)
⟨*proof*⟩

lemma *assign-r-comp*: *x* := *u* ;; *P* = *P*⟦⌈*u*⌉$_<$/$*x*⟧
⟨*proof*⟩

lemma *assign-test*: *mwb-lens x* ⟹ (*x* := «*u*» ;; *x* := «*v*») = (*x* := «*v*»)
⟨*proof*⟩

lemma *assign-twice*: ⟦ *mwb-lens x*; *x* ♯ *f* ⟧ ⟹ (*x* := *e* ;; *x* := *f*) = (*x* := *f*)
⟨*proof*⟩

lemma *assign-commute*:
assumes *x* ⋈ *y x* ♯ *f y* ♯ *e*
shows (*x* := *e* ;; *y* := *f*) = (*y* := *f* ;; *x* := *e*)
⟨*proof*⟩

lemma *assign-cond*:
fixes *x* :: ('*a* ⟹ '*α*)

assumes $out\alpha \sharp b$
shows $(x := e \;;; (P \lhd b \rhd Q)) = ((x := e \;;; P) \lhd (b[\![\lceil e \rceil_< / \$x]\!]) \rhd (x := e \;;; Q))$
⟨*proof*⟩

lemma *assign-rcond*:
fixes $x :: ('a \Longrightarrow 'a)$
shows $(x := e \;;; (P \lhd b \rhd_r Q)) = ((x := e \;;; P) \lhd (b[\![e/x]\!]) \rhd_r (x := e \;;; Q))$
⟨*proof*⟩

lemma *assign-r-alt-def*:
fixes $x :: ('a \Longrightarrow 'a)$
shows $x := v = II[\![\lceil v \rceil_< / \$x]\!]$
⟨*proof*⟩

lemma *assigns-r-ufunc*: *ufunctional* $\langle f \rangle_a$
⟨*proof*⟩

lemma *assigns-r-uinj*: *inj f* $\Longrightarrow$ *uinj* $\langle f \rangle_a$
⟨*proof*⟩

lemma *assigns-r-swap-uinj*:
$[\![$ *vwb-lens x*; *vwb-lens y*; $x \bowtie y \,]\!] \Longrightarrow$ *uinj* $((x,y) := (\&y,\&x))$
⟨*proof*⟩

lemma *assign-unfold*:
vwb-lens x $\Longrightarrow (x := v) = (\$x' =_u \lceil v \rceil_< \wedge II \restriction_\alpha x)$
⟨*proof*⟩

18.8 Non-deterministic Assignment Laws

lemma *nd-assign-comp*:
$x \bowtie y \Longrightarrow x := * \;;; y := * = x,y := *$
⟨*proof*⟩

lemma *nd-assign-assign*:
$[\![$ *vwb-lens x*; $x \sharp e \,]\!] \Longrightarrow x := * \;;; x := e = x := e$
⟨*proof*⟩

18.9 Converse Laws

lemma *convr-invol* [*simp*]: $p^{--} = p$
⟨*proof*⟩

lemma *lit-convr* [*simp*]: $\ll v \gg^- = \ll v \gg$
⟨*proof*⟩

lemma *uivar-convr* [*simp*]:
fixes $x :: ('a \Longrightarrow 'a)$
shows $(\$x)^- = \$x'$
⟨*proof*⟩

lemma *uovar-convr* [*simp*]:
fixes $x :: ('a \Longrightarrow 'a)$
shows $(\$x')^- = \$x$
⟨*proof*⟩

lemma *uop-convr* [*simp*]: $(uop\ f\ u)^- = uop\ f\ (u^-)$
⟨*proof*⟩

lemma *bop-convr* [*simp*]: $(bop\ f\ u\ v)^- = bop\ f\ (u^-)\ (v^-)$
⟨*proof*⟩

lemma *eq-convr* [*simp*]: $(p =_u q)^- = (p^- =_u q^-)$
⟨*proof*⟩

lemma *not-convr* [*simp*]: $(\neg\ p)^- = (\neg\ p^-)$
⟨*proof*⟩

lemma *disj-convr* [*simp*]: $(p \vee q)^- = (q^- \vee p^-)$
⟨*proof*⟩

lemma *conj-convr* [*simp*]: $(p \wedge q)^- = (q^- \wedge p^-)$
⟨*proof*⟩

lemma *seqr-convr* [*simp*]: $(p\ ;;\ q)^- = (q^-\ ;;\ p^-)$
⟨*proof*⟩

lemma *pre-convr* [*simp*]: $\lceil p \rceil_<{}^- = \lceil p \rceil_>$
⟨*proof*⟩

lemma *post-convr* [*simp*]: $\lceil p \rceil_>{}^- = \lceil p \rceil_<$
⟨*proof*⟩

18.10 Assertion and Assumption Laws

declare *sublens-def* [*lens-defs del*]

lemma *assume-false*: $[false]^\top = false$
⟨*proof*⟩

lemma *assume-true*: $[true]^\top = II$
⟨*proof*⟩

lemma *assume-seq*: $[b]^\top\ ;;\ [c]^\top = [(b \wedge c)]^\top$
⟨*proof*⟩

lemma *assert-false*: $\{false\}_\perp = true$
⟨*proof*⟩

lemma *assert-true*: $\{true\}_\perp = II$
⟨*proof*⟩

lemma *assert-seq*: $\{b\}_\perp\ ;;\ \{c\}_\perp = \{(b \wedge c)\}_\perp$
⟨*proof*⟩

18.11 Frame and Antiframe Laws

named-theorems *frame*

lemma *frame-all* [*frame*]: $\Sigma{:}[P] = P$
⟨*proof*⟩

lemma *frame-none* [*frame*]:
$\emptyset$:[P] = ($P \wedge II$)
⟨*proof*⟩

lemma *frame-commute*:
assumes \$$y$ ♯ P \$$y'$ ♯ P \$$x$ ♯ Q \$$x'$ ♯ Q $x \bowtie y$
shows x:[P] ;; y:[Q] = y:[Q] ;; x:[P]
⟨*proof*⟩

lemma *frame-contract-RID*:
assumes *vwb-lens x P is RID*(x) $x \bowtie y$
shows (x;y):[P] = y:[P]
⟨*proof*⟩

lemma *frame-miracle* [*simp*]:
x:[*false*] = *false*
⟨*proof*⟩

lemma *frame-skip* [*simp*]:
vwb-lens x ⟹ x:[II] = II
⟨*proof*⟩

lemma *frame-assign-in* [*frame*]:
⟦ *vwb-lens a*; $x \subseteq_L a$ ⟧ ⟹ a:[$x := v$] = $x := v$
⟨*proof*⟩

lemma *frame-conj-true* [*frame*]:
⟦ {\$$x$,\$$x'$} ♮ P; *vwb-lens x* ⟧ ⟹ ($P \wedge x$:[*true*]) = x:[P]
⟨*proof*⟩

lemma *frame-is-assign* [*frame*]:
vwb-lens x ⟹ x:[\$$x'$ =$_u$ ⌈v⌉$_<$] = $x := v$
⟨*proof*⟩

lemma *frame-seq* [*frame*]:
⟦ *vwb-lens x*; {\$$x$,\$$x'$} ♮ P; {\$$x$,\$$x'$} ♮ Q ⟧ ⟹ x:[P ;; Q] = x:[P] ;; x:[Q]
⟨*proof*⟩

lemma *frame-to-antiframe* [*frame*]:
⟦ $x \bowtie y$; $x +_L y = 1_L$ ⟧ ⟹ x:[P] = y:[⟦P⟧]
⟨*proof*⟩

lemma *rel-frext-miracle* [*frame*]:
a:[*false*]$^+$ = *false*
⟨*proof*⟩

lemma *rel-frext-skip* [*frame*]:
vwb-lens a ⟹ a:[II]$^+$ = II
⟨*proof*⟩

lemma *rel-frext-seq* [*frame*]:
vwb-lens a ⟹ a:[P ;; Q]$^+$ = (a:[P]$^+$;; a:[Q]$^+$)
⟨*proof*⟩

lemma *rel-frext-assigns* [*frame*]:

vwb-lens $a \Longrightarrow a{:}[\langle\sigma\rangle_a]^+ = \langle\sigma \oplus_s a\rangle_a$
⟨*proof*⟩

lemma *rel-frext-rcond* [*frame*]:
$a{:}[P \lhd b \rhd_r Q]^+ = (a{:}[P]^+ \lhd b \oplus_p a \rhd_r a{:}[Q]^+)$
⟨*proof*⟩

lemma *rel-frext-commute*:
$x \bowtie y \Longrightarrow x{:}[P]^+ ;; y{:}[Q]^+ = y{:}[Q]^+ ;; x{:}[P]^+$
⟨*proof*⟩

lemma *antiframe-disj* [*frame*]: $(x{:}[\![P]\!] \vee x{:}[\![Q]\!]) = x{:}[\![P \vee Q]\!]$
⟨*proof*⟩

lemma *antiframe-seq* [*frame*]:
$[\![$ *vwb-lens* x; $\$x' \sharp P$; $\$x \sharp Q$ $]\!] \Longrightarrow (x{:}[\![P]\!] ;; x{:}[\![Q]\!]) = x{:}[\![P ;; Q]\!]$
⟨*proof*⟩

lemma *nameset-skip*: *vwb-lens* $x \Longrightarrow (ns\ x \cdot II) = II_x$
⟨*proof*⟩

lemma *nameset-skip-ra*: *vwb-lens* $x \Longrightarrow (ns\ x \cdot II_x) = II_x$
⟨*proof*⟩

declare *sublens-def* [*lens-defs*]

18.12 While Loop Laws

theorem *while-unfold*:
while b *do* P *od* $= ((P ;;$ *while* b *do* P *od*$) \lhd b \rhd_r II)$
⟨*proof*⟩

theorem *while-false*: *while false do* P *od* $= II$
⟨*proof*⟩

theorem *while-true*: *while true do* P *od* $=$ *false*
⟨*proof*⟩

theorem *while-bot-unfold*:
$while_\bot$ b *do* P *od* $= ((P ;;$ $while_\bot$ b *do* P *od*$) \lhd b \rhd_r II)$
⟨*proof*⟩

theorem *while-bot-false*: $while_\bot$ *false do* P *od* $= II$
⟨*proof*⟩

theorem *while-bot-true*: $while_\bot$ *true do* P *od* $= (\mu\ X \cdot P ;; X)$
⟨*proof*⟩

An infinite loop with a feasible body corresponds to a program error (non-termination).

theorem *while-infinite*: $P ;; true_h = true \Longrightarrow while_\bot$ *true do* P *od* $=$ *true*
⟨*proof*⟩

18.13 Algebraic Properties

interpretation *upred-semiring*: *semiring-1*
where *times* $=$ *seqr* **and** *one* $=$ *skip-r* **and** *zero* $= false_h$ **and** *plus* $=$ *Lattices.sup*

⟨*proof*⟩

declare *upred-semiring.power-Suc* [*simp del*]

We introduce the power syntax derived from semirings

abbreviation *upower* :: 'α *hrel* ⇒ *nat* ⇒ 'α *hrel* (**infixr** ‹$\frown$› *80*) **where**
upower P n ≡ *upred-semiring.power P n*

translations
P $\frown$ *i* <= *CONST power.power II op* ;; *P i*
P $\frown$ *i* <= (*CONST power.power II op* ;; *P*) *i*

Set up transfer tactic for powers

lemma *upower-rep-eq*:
⟦*P* $\frown$ *i*⟧$_e$ = (λ *b*. *b* ∈ ({*p*. ⟦*P*⟧$_e$ *p*} $\frown\frown$ *i*))
⟨*proof*⟩

lemma *upower-rep-eq-alt*:
⟦*power.power* ⟨*id*⟩$_a$ (;;) *P i*⟧$_e$ = (λ*b*. *b* ∈ ({*p*. ⟦*P*⟧$_e$ *p*} $\frown\frown$ *i*))
⟨*proof*⟩

update-uexpr-rep-eq-thms

lemma *Sup-power-expand*:
fixes *P* :: *nat* ⇒ '*a*::*complete-lattice*
shows *P*(*0*) ⊓ (⨅ *i*. *P*(*i*+*1*)) = (⨅ *i*. *P*(*i*))
⟨*proof*⟩

lemma *Sup-upto-Suc*: (⨅ *i*∈{*0*..*Suc n*}. *P* $\frown$ *i*) = (⨅ *i*∈{*0*..*n*}. *P* $\frown$ *i*) ⊓ *P* $\frown$ *Suc n*
⟨*proof*⟩

The following two proofs are adapted from the AFP entry Kleene Algebra. See also [2, 1].

lemma *upower-inductl*: *Q* ⊑ ((*P* ;; *Q*) ⊓ *R*) ⟹ *Q* ⊑ *P* $\frown$ *n* ;; *R*
⟨*proof*⟩

lemma *upower-inductr*:
assumes *Q* ⊑ *R* ⊓ (*Q* ;; *P*)
shows *Q* ⊑ *R* ;; (*P* $\frown$ *n*)
⟨*proof*⟩

lemma *SUP-atLeastAtMost-first*:
fixes *P* :: *nat* ⇒ '*a*::*complete-lattice*
assumes *m* ≤ *n*
shows (⨅ *i*∈{*m*..*n*}. *P*(*i*)) = *P*(*m*) ⊓ (⨅ *i*∈{*Suc m*..*n*}. *P*(*i*))
⟨*proof*⟩

lemma *upower-seqr-iter*: *P* $\frown$ *n* = (;; *Q* : *replicate n P* • *Q*)
⟨*proof*⟩

lemma *assigns-power*: ⟨*f*⟩$_a$ $\frown$ *n* = ⟨*f* $\frown\frown$ *n*⟩$_a$
⟨*proof*⟩

18.14 Kleene Star

definition *ustar* :: 'α *hrel* ⇒ 'α *hrel* (‹-*› [*999*] *999*) **where**

$P^\star = (\prod i{\in}\{0..\} \cdot P \hat{} i)$

lemma *ustar-rep-eq*:
$[\![P^\star]\!]_e = (\lambda b.\ b \in (\{p.\ [\![P]\!]_e\ p\}^*))$
⟨*proof*⟩

update-uexpr-rep-eq-thms

18.15 Kleene Plus

purge-notation *trancl* (‹(‹*notation*=‹*postfix* +››-$^+$)› [*1000*] *999*)

definition *uplus* :: $'\alpha$ *hrel* ⇒ $'\alpha$ *hrel* (‹-$^+$› [*999*] *999*) **where**
[*upred-defs*]: $P^+ = P ;; P^\star$

lemma *uplus-power-def*: $P^+ = (\prod\ i \cdot P \hat{}\ (Suc\ i))$
⟨*proof*⟩

18.16 Omega

definition *uomega* :: $'\alpha$ *hrel* ⇒ $'\alpha$ *hrel* (‹-$^\omega$› [*999*] *999*) **where**
$P^\omega = (\mu\ X \cdot P ;; X)$

18.17 Relation Algebra Laws

theorem *RA1*: $(P ;; (Q ;; R)) = ((P ;; Q) ;; R)$
⟨*proof*⟩

theorem *RA2*: $(P ;; II) = P\ (II ;; P) = P$
⟨*proof*⟩

theorem *RA3*: $P^{--} = P$
⟨*proof*⟩

theorem *RA4*: $(P ;; Q)^- = (Q^- ;; P^-)$
⟨*proof*⟩

theorem *RA5*: $(P \vee Q)^- = (P^- \vee Q^-)$
⟨*proof*⟩

theorem *RA6*: $((P \vee Q) ;; R) = (P;;R \vee Q;;R)$
⟨*proof*⟩

theorem *RA7*: $((P^- ;; (\neg(P ;; Q))) \vee (\neg Q)) = (\neg Q)$
⟨*proof*⟩

18.18 Kleene Algebra Laws

lemma *ustar-alt-def*: $P^\star = (\prod\ i \cdot P \hat{}\ i)$
⟨*proof*⟩

theorem *ustar-sub-unfoldl*: $P^\star \sqsubseteq II \sqcap (P;;P^\star)$
⟨*proof*⟩

theorem *ustar-inductl*:
assumes $Q \sqsubseteq R$ $Q \sqsubseteq P ;; Q$

shows $Q \sqsubseteq P^\star \;;;\; R$
⟨*proof*⟩

theorem *ustar-inductr*:
assumes $Q \sqsubseteq R$ $Q \sqsubseteq Q \;;;\; P$
shows $Q \sqsubseteq R \;;;\; P^\star$
⟨*proof*⟩

lemma *ustar-refines-nu*: $(\nu\ X \cdot (P \;;;\; X) \sqcap II) \sqsubseteq P^\star$
⟨*proof*⟩

lemma *ustar-as-nu*: $P^\star = (\nu\ X \cdot (P \;;;\; X) \sqcap II)$
⟨*proof*⟩

lemma *ustar-unfoldl*: $P^\star = II \sqcap (P \;;;\; P^\star)$
⟨*proof*⟩

While loop can be expressed using Kleene star

lemma *while-star-form*:
while b *do* P *od* $= (P \lhd b \rhd_r II)^\star \;;;\; [(\neg b)]^\top$
⟨*proof*⟩

18.19 Omega Algebra Laws

lemma *uomega-induct*:
$P \;;;\; P^\omega \sqsubseteq P^\omega$
⟨*proof*⟩

18.20 Refinement Laws

lemma *skip-r-refine*:
$(p \Rightarrow p) \sqsubseteq II$
⟨*proof*⟩

lemma *conj-refine-left*:
$(Q \Rightarrow P) \sqsubseteq R \Longrightarrow P \sqsubseteq (Q \wedge R)$
⟨*proof*⟩

lemma *pre-weak-rel*:
assumes ‘$Pre \Rightarrow I$‘
and $(I \Rightarrow Post) \sqsubseteq P$
shows $(Pre \Rightarrow Post) \sqsubseteq P$
⟨*proof*⟩

lemma *cond-refine-rel*:
assumes $S \sqsubseteq (\lceil b \rceil_< \wedge P)$ $S \sqsubseteq (\lceil \neg b \rceil_< \wedge Q)$
shows $S \sqsubseteq P \lhd b \rhd_r Q$
⟨*proof*⟩

lemma *seq-refine-pred*:
assumes $(\lceil b \rceil_< \Rightarrow \lceil s \rceil_>) \sqsubseteq P$ **and** $(\lceil s \rceil_< \Rightarrow \lceil c \rceil_>) \sqsubseteq Q$
shows $(\lceil b \rceil_< \Rightarrow \lceil c \rceil_>) \sqsubseteq (P \;;;\; Q)$
⟨*proof*⟩

lemma *seq-refine-unrest*:
assumes $out\alpha \sharp b$ $in\alpha \sharp c$

assumes $(b \Rightarrow \lceil s \rceil_>) \sqsubseteq P$ **and** $(\lceil s \rceil_< \Rightarrow c) \sqsubseteq Q$
shows $(b \Rightarrow c) \sqsubseteq (P \;;;\; Q)$
⟨*proof*⟩

18.21 Domain and Range Laws

lemma *Dom-conv-Ran*:
$Dom(P^-) = Ran(P)$
⟨*proof*⟩

lemma *Ran-conv-Dom*:
$Ran(P^-) = Dom(P)$
⟨*proof*⟩

lemma *Dom-skip*:
$Dom(II) = true$
⟨*proof*⟩

lemma *Dom-assigns*:
$Dom(\langle\sigma\rangle_a) = true$
⟨*proof*⟩

lemma *Dom-miracle*:
$Dom(false) = false$
⟨*proof*⟩

lemma *Dom-assume*:
$Dom([b]^\top) = b$
⟨*proof*⟩

lemma *Dom-seq*:
$Dom(P \;;;\; Q) = Dom(P \;;;\; [Dom(Q)]^\top)$
⟨*proof*⟩

lemma *Dom-disj*:
$Dom(P \vee Q) = (Dom(P) \vee Dom(Q))$
⟨*proof*⟩

lemma *Dom-inf*:
$Dom(P \sqcap Q) = (Dom(P) \vee Dom(Q))$
⟨*proof*⟩

lemma *assume-Dom*:
$[Dom(P)]^\top \;;;\; P = P$
⟨*proof*⟩

end

19 UTP Theories

theory *utp-theory*
imports *utp-rel-laws*
begin

Here, we mechanise a representation of UTP theories using locales [4]. We also link them to

the HOL-Algebra library [5], which allows us to import properties from complete lattices and Galois connections.

19.1 Complete lattice of predicates

definition *upred-lattice* :: ('α *upred*) *gorder* (‹$\mathcal{P}$›) **where**
upred-lattice = (| *carrier* = *UNIV*, *eq* = (=), *le* = ($\sqsubseteq$) |)

$\mathcal{P}$ is the complete lattice of alphabetised predicates. All other theories will be defined relative to it.

interpretation *upred-lattice*: *complete-lattice* $\mathcal{P}$
⟨*proof*⟩

lemma *upred-weak-complete-lattice* [*simp*]: *weak-complete-lattice* $\mathcal{P}$
⟨*proof*⟩

lemma *upred-lattice-eq* [*simp*]:
($.=_{\mathcal{P}}$) = (=)
⟨*proof*⟩

lemma *upred-lattice-le* [*simp*]:
le $\mathcal{P}$ *P Q* = (*P* $\sqsubseteq$ *Q*)
⟨*proof*⟩

lemma *upred-lattice-carrier* [*simp*]:
carrier $\mathcal{P}$ = *UNIV*
⟨*proof*⟩

lemma *Healthy-fixed-points* [*simp*]: *fps* $\mathcal{P}$ *H* = $[\![H]\!]_H$
⟨*proof*⟩

lemma *upred-lattice-Idempotent* [*simp*]: $Idem_{\mathcal{P}}$ *H* = *Idempotent H*
⟨*proof*⟩

lemma *upred-lattice-Monotonic* [*simp*]: $Mono_{\mathcal{P}}$ *H* = *Monotonic H*
⟨*proof*⟩

19.2 UTP theories hierarchy

definition *utp-order* :: ('α × 'α) *health* ⇒ 'α *hrel gorder* **where**
utp-order H = (| *carrier* = {*P*. *P is H*}, *eq* = (=), *le* = ($\sqsubseteq$) |)

Constant *utp-order* obtains the order structure associated with a UTP theory. Its carrier is the set of healthy predicates, equality is HOL equality, and the order is refinement.

lemma *utp-order-carrier* [*simp*]:
carrier (*utp-order H*) = $[\![H]\!]_H$
⟨*proof*⟩

lemma *utp-order-eq* [*simp*]:
eq (*utp-order T*) = (=)
⟨*proof*⟩

lemma *utp-order-le* [*simp*]:
le (*utp-order T*) = ($\sqsubseteq$)
⟨*proof*⟩

lemma *utp-partial-order*: *partial-order* (*utp-order* T)
⟨*proof*⟩

lemma *utp-weak-partial-order*: *weak-partial-order* (*utp-order* T)
⟨*proof*⟩

lemma *mono-Monotone-utp-order*:
mono $f \Longrightarrow$ *Monotone* (*utp-order* T) f
⟨*proof*⟩

lemma *isotone-utp-orderI*: *Monotonic* $H \Longrightarrow$ *isotone* (*utp-order* X) (*utp-order* Y) H
⟨*proof*⟩

lemma *Mono-utp-orderI*:
$[\![\bigwedge P\ Q.\ [\![P \sqsubseteq Q;\ P\ is\ H;\ Q\ is\ H]\!] \Longrightarrow F(P) \sqsubseteq F(Q)]\!] \Longrightarrow Mono_{utp\text{-}order\ H}\ F$
⟨*proof*⟩

The UTP order can equivalently be characterised as the fixed point lattice, *fpl*.

lemma *utp-order-fpl*: *utp-order* H = *fpl* $\mathcal{P}$ H
⟨*proof*⟩

19.3 UTP theory hierarchy

We next define a hierarchy of locales that characterise different classes of UTP theory. Minimally we require that a UTP theory's healthiness condition is idempotent.

locale *utp-theory* =
fixes *hcond* :: $'\alpha$ *hrel* ⇒ $'\alpha$ *hrel* (‹$\mathcal{H}$›)
assumes *HCond-Idem*: $\mathcal{H}(\mathcal{H}(P)) = \mathcal{H}(P)$
begin

abbreviation *thy-order* :: $'\alpha$ *hrel gorder* **where**
thy-order ≡ *utp-order* $\mathcal{H}$

lemma *HCond-Idempotent* [*closure*,*intro*]: *Idempotent* $\mathcal{H}$
⟨*proof*⟩

sublocale *utp-po*: *partial-order utp-order* $\mathcal{H}$
⟨*proof*⟩

We need to remove some transitivity rules to stop them being applied in calculations

declare *utp-po.trans* [*trans del*]

end

locale *utp-theory-lattice* = *utp-theory* +
assumes *uthy-lattice*: *complete-lattice* (*utp-order* $\mathcal{H}$)
begin

sublocale *complete-lattice utp-order* $\mathcal{H}$
⟨*proof*⟩

declare *top-closed* [*simp del*]
declare *bottom-closed* [*simp del*]

The healthiness conditions of a UTP theory lattice form a complete lattice, and allows us to make use of complete lattice results from HOL-Algebra [5], such as the Knaster-Tarski theorem. We can also retrieve lattice operators as below.

abbreviation *utp-top* (‹⊤›)
where *utp-top* ≡ *top* (*utp-order* $\mathcal{H}$)

abbreviation *utp-bottom* (‹⊥›)
where *utp-bottom* ≡ *bottom* (*utp-order* $\mathcal{H}$)

abbreviation *utp-join* (**infixl** ‹⊔› *65*) **where**
utp-join ≡ *join* (*utp-order* $\mathcal{H}$)

abbreviation *utp-meet* (**infixl** ‹⊓› *70*) **where**
utp-meet ≡ *meet* (*utp-order* $\mathcal{H}$)

abbreviation *utp-sup* (‹⨆ -› [*90*] *90*) **where**
utp-sup ≡ *Lattice.sup* (*utp-order* $\mathcal{H}$)

abbreviation *utp-inf* (‹⨅ -› [*90*] *90*) **where**
utp-inf ≡ *Lattice.inf* (*utp-order* $\mathcal{H}$)

abbreviation *utp-gfp* (‹$\boldsymbol{\nu}$›) **where**
utp-gfp ≡ *GREATEST-FP* (*utp-order* $\mathcal{H}$)

abbreviation *utp-lfp* (‹$\boldsymbol{\mu}$›) **where**
utp-lfp ≡ *LEAST-FP* (*utp-order* $\mathcal{H}$)

end

syntax
-tmu :: *logic* ⇒ *pttrn* ⇒ *logic* ⇒ *logic* (‹$\boldsymbol{\mu}_1$ - · -› [*0*, *10*] *10*)
-tnu :: *logic* ⇒ *pttrn* ⇒ *logic* ⇒ *logic* (‹$\boldsymbol{\nu}_1$ - · -› [*0*, *10*] *10*)

syntax-consts
-tmu == *LEAST-FP* **and**
-tnu == *GREATEST-FP*

notation *gfp* (‹μ›)
notation *lfp* (‹ν›)

translations
$\boldsymbol{\mu}_H$ *X* · *P* == *CONST LEAST-FP* (*CONST utp-order H*) (λ *X*. *P*)
$\boldsymbol{\nu}_H$ *X* · *P* == *CONST GREATEST-FP* (*CONST utp-order H*) (λ *X*. *P*)

lemma *upred-lattice-inf*:
Lattice.inf $\mathcal{P}$ *A* = ⨅ *A*
⟨*proof*⟩

We can then derive a number of properties about these operators, as below.

context *utp-theory-lattice*
begin

lemma *LFP-healthy-comp*: $\boldsymbol{\mu}$ *F* = $\boldsymbol{\mu}$ (*F* ∘ $\mathcal{H}$)
⟨*proof*⟩

lemma *GFP-healthy-comp*: $\boldsymbol{\nu}\ F = \boldsymbol{\nu}\ (F \circ \mathcal{H})$
⟨*proof*⟩

lemma *top-healthy* [*closure*]: $\top$ *is* $\mathcal{H}$
⟨*proof*⟩

lemma *bottom-healthy* [*closure*]: $\bot$ *is* $\mathcal{H}$
⟨*proof*⟩

lemma *utp-top*: P *is* $\mathcal{H} \Longrightarrow P \sqsubseteq \top$
⟨*proof*⟩

lemma *utp-bottom*: P *is* $\mathcal{H} \Longrightarrow \bot \sqsubseteq P$
⟨*proof*⟩

end

lemma *upred-top*: $\top_{\mathcal{P}} = \mathit{false}$
⟨*proof*⟩

lemma *upred-bottom*: $\bot_{\mathcal{P}} = \mathit{true}$
⟨*proof*⟩

One way of obtaining a complete lattice is showing that the healthiness conditions are monotone, which the below locale characterises.

locale *utp-theory-mono* = *utp-theory* +
assumes *HCond-Mono* [*closure*,*intro*]: *Monotonic* $\mathcal{H}$

sublocale *utp-theory-mono* ⊆ *utp-theory-lattice*
⟨*proof*⟩

In a monotone theory, the top and bottom can always be obtained by applying the healthiness condition to the predicate top and bottom, respectively.

context *utp-theory-mono*
begin

lemma *healthy-top*: $\top = \mathcal{H}(\mathit{false})$
⟨*proof*⟩

lemma *healthy-bottom*: $\bot = \mathcal{H}(\mathit{true})$
⟨*proof*⟩

lemma *healthy-inf*:
assumes $A \subseteq [\![\mathcal{H}]\!]_H$
shows $\bigsqcap\ A = \mathcal{H}\ (\bigsqcap\ A)$
⟨*proof*⟩

end

locale *utp-theory-continuous* = *utp-theory* +
assumes *HCond-Cont* [*closure*,*intro*]: *Continuous* $\mathcal{H}$

sublocale *utp-theory-continuous* ⊆ *utp-theory-mono*
⟨*proof*⟩

context *utp-theory-continuous*
begin

lemma *healthy-inf-cont*:
assumes $A \subseteq [\![\mathcal{H}]\!]_H$ $A \neq \{\}$
shows $\bigsqcap A = \bigsqcap A$
⟨*proof*⟩

lemma *healthy-inf-def*:
assumes $A \subseteq [\![\mathcal{H}]\!]_H$
shows $\bigsqcap A = (\mathit{if}\ (A = \{\})\ \mathit{then}\ \top\ \mathit{else}\ (\bigsqcap A))$
⟨*proof*⟩

lemma *healthy-meet-cont*:
assumes P *is* $\mathcal{H}$ Q *is* $\mathcal{H}$
shows $P \sqcap Q = P \sqcap Q$
⟨*proof*⟩

lemma *meet-is-healthy* [*closure*]:
assumes P *is* $\mathcal{H}$ Q *is* $\mathcal{H}$
shows $P \sqcap Q$ *is* $\mathcal{H}$
⟨*proof*⟩

lemma *meet-bottom* [*simp*]:
assumes P *is* $\mathcal{H}$
shows $P \sqcap \bot = \bot$
⟨*proof*⟩

lemma *meet-top* [*simp*]:
assumes P *is* $\mathcal{H}$
shows $P \sqcap \top = P$
⟨*proof*⟩

The UTP theory lfp operator can be rewritten to the alphabetised predicate lfp when in a continuous context.

theorem *utp-lfp-def*:
assumes *Monotonic* F $F \in [\![\mathcal{H}]\!]_H \to [\![\mathcal{H}]\!]_H$
shows $\boldsymbol{\mu}\ F = (\mu\ X \cdot F(\mathcal{H}(X)))$
⟨*proof*⟩

lemma *UINF-ind-Healthy* [*closure*]:
assumes $\bigwedge i.\ P(i)$ *is* $\mathcal{H}$
shows $(\bigsqcap i \cdot P(i))$ *is* $\mathcal{H}$
⟨*proof*⟩

end

In another direction, we can also characterise UTP theories that are relational. Minimally this requires that the healthiness condition is closed under sequential composition.

locale *utp-theory-rel* =
utp-theory +
assumes *Healthy-Sequence* [*closure*]: $[\![\ P$ *is* $\mathcal{H}$; Q *is* $\mathcal{H}\ ]\!] \Longrightarrow (P\ ;;\ Q)$ *is* $\mathcal{H}$
begin

lemma *upower-Suc-Healthy* [*closure*]:

assumes P *is* $\mathcal{H}$
shows $P \hat{} \mathit{Suc}\ n$ *is* $\mathcal{H}$
⟨*proof*⟩

end

locale *utp-theory-cont-rel* = *utp-theory-rel* + *utp-theory-continuous*
begin

lemma *seq-cont-Sup-distl*:
assumes P *is* $\mathcal{H}$ $A \subseteq [\![\mathcal{H}]\!]_H$ $A \neq \{\}$
shows $P ;; (\bigsqcap A) = \bigsqcap \{P ;; Q \mid Q.\ Q \in A\}$
⟨*proof*⟩

lemma *seq-cont-Sup-distr*:
assumes Q *is* $\mathcal{H}$ $A \subseteq [\![\mathcal{H}]\!]_H$ $A \neq \{\}$
shows $(\bigsqcap A) ;; Q = \bigsqcap \{P ;; Q \mid P.\ P \in A\}$
⟨*proof*⟩

lemma *uplus-healthy* [*closure*]:
assumes P *is* $\mathcal{H}$
shows P^+ *is* $\mathcal{H}$
⟨*proof*⟩

end

There also exist UTP theories with units. Not all theories have both a left and a right unit (e.g. H1-H2 designs) and so we split up the locale into two cases.

locale *utp-theory-units* =
utp-theory-rel +
fixes *utp-unit* (‹$\mathcal{II}$›)
assumes *Healthy-Unit* [*closure*]: $\mathcal{II}$ *is* $\mathcal{H}$
begin

We can characterise the theory Kleene star by lifting the relational one.

definition *utp-star* (‹-⋆›) [*999*] *999*) **where**
[*upred-defs*]: *utp-star* $P = (P^\star ;; \mathcal{II})$

We can then characterise tests as refinements of units.

definition *utp-test* :: *'a hrel* ⇒ *bool* **where**
[*upred-defs*]: *utp-test* $b = (\mathcal{II} \sqsubseteq b)$

end

locale *utp-theory-left-unital* =
utp-theory-units +
assumes *Unit-Left*: P *is* $\mathcal{H} \Longrightarrow (\mathcal{II} ;; P) = P$

locale *utp-theory-right-unital* =
utp-theory-units +
assumes *Unit-Right*: P *is* $\mathcal{H} \Longrightarrow (P ;; \mathcal{II}) = P$

locale *utp-theory-unital* =
utp-theory-left-unital + *utp-theory-right-unital*
begin

lemma *Unit-self* [*simp*]:
II ;; II = II
⟨*proof*⟩

lemma *utest-intro*:
$\mathit{II} \sqsubseteq P \Longrightarrow$ *utp-test* P
⟨*proof*⟩

lemma *utest-Unit* [*closure*]:
utp-test II
⟨*proof*⟩

end

locale *utp-theory-mono-unital* = *utp-theory-unital* + *utp-theory-mono*
begin

lemma *utest-Top* [*closure*]: *utp-test* ⊤
⟨*proof*⟩

end

locale *utp-theory-cont-unital* = *utp-theory-cont-rel* + *utp-theory-unital*

sublocale *utp-theory-cont-unital* ⊆ *utp-theory-mono-unital*
⟨*proof*⟩

locale *utp-theory-unital-zerol* =
utp-theory-unital +
utp-theory-lattice +
assumes *Top-Left-Zero*: P *is* $\mathcal{H} \Longrightarrow$ ⊤ ;; P = ⊤

locale *utp-theory-cont-unital-zerol* =
utp-theory-cont-unital + *utp-theory-unital-zerol*
begin

lemma *Top-test-Right-Zero*:
assumes b *is* $\mathcal{H}$ *utp-test* b
shows b ;; ⊤ = ⊤
⟨*proof*⟩

end

19.4 Theory of relations

interpretation *rel-theory*: *utp-theory-mono-unital id skip-r*
rewrites *rel-theory.utp-top* = *false*
and *rel-theory.utp-bottom* = *true*
and *carrier* (*utp-order id*) = *UNIV*
and (P *is id*) = *True*
⟨*proof*⟩

thm *rel-theory.GFP-unfold*

19.5 Theory links

We can also describe links between theories, such a Galois connections and retractions, using the following notation.

definition *mk-conn* (‹- ⇐⟨-,-⟩⇒ -› [*90*,*0*,*0*,*91*] *91*) **where**
H1 ⇐⟨$\mathcal{H}_1$,$\mathcal{H}_2$⟩⇒ *H2* ≡ (| *orderA* = *utp-order H1*, *orderB* = *utp-order H2*, *lower* = $\mathcal{H}_2$, *upper* = $\mathcal{H}_1$ |)

lemma *mk-conn-orderA* [*simp*]: $\mathcal{X}_{H1}$ ⇐⟨$\mathcal{H}_1$,$\mathcal{H}_2$⟩⇒ *H2* = *utp-order H1*
⟨*proof*⟩

lemma *mk-conn-orderB* [*simp*]: $\mathcal{Y}_{H1}$ ⇐⟨$\mathcal{H}_1$,$\mathcal{H}_2$⟩⇒ *H2* = *utp-order H2*
⟨*proof*⟩

lemma *mk-conn-lower* [*simp*]: π_{*H1} ⇐⟨$\mathcal{H}_1$,$\mathcal{H}_2$⟩⇒ *H2* = $\mathcal{H}_1$
⟨*proof*⟩

lemma *mk-conn-upper* [*simp*]: $\pi^*{}_{H1}$ ⇐⟨$\mathcal{H}_1$,$\mathcal{H}_2$⟩⇒ *H2* = $\mathcal{H}_2$
⟨*proof*⟩

lemma *galois-comp*: (*H*$_2$ ⇐⟨$\mathcal{H}_3$,$\mathcal{H}_4$⟩⇒ *H*$_3$) ∘$_g$ (*H*$_1$ ⇐⟨$\mathcal{H}_1$,$\mathcal{H}_2$⟩⇒ *H*$_2$) = *H*$_1$ ⇐⟨$\mathcal{H}_1$∘$\mathcal{H}_3$,$\mathcal{H}_4$∘$\mathcal{H}_2$⟩⇒ *H*$_3$
⟨*proof*⟩

Example Galois connection / retract: Existential quantification

lemma *Idempotent-ex*: *mwb-lens x* ⟹ *Idempotent* (*ex x*)
⟨*proof*⟩

lemma *Monotonic-ex*: *mwb-lens x* ⟹ *Monotonic* (*ex x*)
⟨*proof*⟩

lemma *ex-closed-unrest*:
vwb-lens x ⟹ ⟦*ex x*⟧$_H$ = {*P*. *x* ♯ *P*}
⟨*proof*⟩

Any theory can be composed with an existential quantification to produce a Galois connection

theorem *ex-retract*:
assumes *vwb-lens x Idempotent H ex x* ∘ *H* = *H* ∘ *ex x*
shows *retract* ((*ex x* ∘ *H*) ⇐⟨*ex x*, *H*⟩⇒ *H*)
⟨*proof*⟩

corollary *ex-retract-id*:
assumes *vwb-lens x*
shows *retract* (*ex x* ⇐⟨*ex x*, *id*⟩⇒ *id*)
⟨*proof*⟩
end

20 Relational Hoare calculus

theory *utp-hoare*
imports
utp-rel-laws
utp-theory
begin

20.1 Hoare Triple Definitions and Tactics

definition *hoare-r* :: $'\alpha$ *cond* $\Rightarrow$ $'\alpha$ *hrel* $\Rightarrow$ $'\alpha$ *cond* $\Rightarrow$ *bool* ($\langle\{\!|$-$|\!\}$/ -/ $\{\!|$-$|\!\}_u\rangle$) **where**
$\{\!|p|\!\}\, Q \,\{\!|r|\!\}_u = ((\lceil p \rceil_< \Rightarrow \lceil r \rceil_>) \sqsubseteq Q)$

declare *hoare-r-def* [*upred-defs*]

named-theorems *hoare* **and** *hoare-safe*

method *hoare-split* **uses** *hr* =
((*simp add*: *assigns-comp*)*?*, — Combine Assignments where possible
(*auto*
intro: *hoare intro*!: *hoare-safe hr*
simp add: *conj-comm conj-assoc usubst unrest*))[*1*] — Apply Hoare logic laws

method *hoare-auto* **uses** *hr* = (*hoare-split hr*: *hr*; (*rel-simp*)*?*, *auto?*)

20.2 Basic Laws

lemma *hoare-meaning*:
$\{\!|P|\!\}\, S \,\{\!|Q|\!\}_u = (\forall\ s\ s'.\ [\![P]\!]_e\ s \wedge [\![S]\!]_e\ (s,\ s') \longrightarrow [\![Q]\!]_e\ s')$
⟨*proof*⟩

lemma *hoare-assume*: $\{\!|P|\!\}\, S \,\{\!|Q|\!\}_u \Longrightarrow\ ?[P]\ ;;\ S = ?[P]\ ;;\ S\ ;;\ ?[Q]$
⟨*proof*⟩

lemma *hoare-r-conj* [*hoare-safe*]: $[\![\ \{\!|p|\!\}\, Q \,\{\!|r|\!\}_u;\ \{\!|p|\!\}\, Q \,\{\!|s|\!\}_u\ ]\!] \Longrightarrow \{\!|p|\!\}\, Q \,\{\!|r \wedge s|\!\}_u$
⟨*proof*⟩

lemma *hoare-r-weaken-pre* [*hoare*]:
$\{\!|p|\!\}\, Q \,\{\!|r|\!\}_u \Longrightarrow \{\!|p \wedge q|\!\}\, Q \,\{\!|r|\!\}_u$
$\{\!|q|\!\}\, Q \,\{\!|r|\!\}_u \Longrightarrow \{\!|p \wedge q|\!\}\, Q \,\{\!|r|\!\}_u$
⟨*proof*⟩

lemma *pre-str-hoare-r*:
assumes ‘$p_1 \Rightarrow p_2$‘ **and** $\{\!|p_2|\!\}\, C \,\{\!|q|\!\}_u$
shows $\{\!|p_1|\!\}\, C \,\{\!|q|\!\}_u$
⟨*proof*⟩

lemma *post-weak-hoare-r*:
assumes $\{\!|p|\!\}\, C \,\{\!|q_2|\!\}_u$ **and** ‘$q_2 \Rightarrow q_1$‘
shows $\{\!|p|\!\}\, C \,\{\!|q_1|\!\}_u$
⟨*proof*⟩

lemma *hoare-r-conseq*: $[\![$ ‘$p_1 \Rightarrow p_2$‘; $\{\!|p_2|\!\}\, S \,\{\!|q_2|\!\}_u$; ‘$q_2 \Rightarrow q_1$‘ $]\!] \Longrightarrow \{\!|p_1|\!\}\, S \,\{\!|q_1|\!\}_u$
⟨*proof*⟩

20.3 Assignment Laws

lemma *assigns-hoare-r* [*hoare-safe*]: ‘$p \Rightarrow \sigma \dagger q$‘ $\Longrightarrow \{\!|p|\!\}\, \langle\sigma\rangle_a \,\{\!|q|\!\}_u$
⟨*proof*⟩

lemma *assigns-backward-hoare-r*:
$\{\!|\sigma \dagger p|\!\}\, \langle\sigma\rangle_a \,\{\!|p|\!\}_u$
⟨*proof*⟩

lemma *assign-floyd-hoare-r*:
assumes *vwb-lens x*
shows $\{\!|p|\!\}$ *assign-r x e* $\{\!|\exists\, v \cdot p[\![\ll v \gg / x]\!] \wedge \&x =_u e[\![\ll v \gg / x]\!]|\!\}_u$
⟨*proof*⟩

lemma *assigns-init-hoare* [*hoare-safe*]:
$[\![$ *vwb-lens x*; $x \sharp p$; $x \sharp v$; $\{\!|\&x =_u v \wedge p|\!\} S \{\!|q|\!\}_u \,]\!] \Longrightarrow \{\!|p|\!\} x := v \;;;\; S \{\!|q|\!\}_u$
⟨*proof*⟩

lemma *skip-hoare-r* [*hoare-safe*]: $\{\!|p|\!\} II \{\!|p|\!\}_u$
⟨*proof*⟩

lemma *skip-hoare-impl-r* [*hoare-safe*]: $`p \Rightarrow q` \Longrightarrow \{\!|p|\!\} II \{\!|q|\!\}_u$
⟨*proof*⟩

20.4 Sequence Laws

lemma *seq-hoare-r*: $[\![\ \{\!|p|\!\} Q_1 \{\!|s|\!\}_u \ ;\ \{\!|s|\!\} Q_2 \{\!|r|\!\}_u \,]\!] \Longrightarrow \{\!|p|\!\} Q_1 \;;;\; Q_2 \{\!|r|\!\}_u$
⟨*proof*⟩

lemma *seq-hoare-invariant* [*hoare-safe*]: $[\![\ \{\!|p|\!\} Q_1 \{\!|p|\!\}_u \ ;\ \{\!|p|\!\} Q_2 \{\!|p|\!\}_u \,]\!] \Longrightarrow \{\!|p|\!\} Q_1 \;;;\; Q_2 \{\!|p|\!\}_u$
⟨*proof*⟩

lemma *seq-hoare-stronger-pre-1* [*hoare-safe*]:
$[\![\ \{\!|p \wedge q|\!\} Q_1 \{\!|p \wedge q|\!\}_u \ ;\ \{\!|p \wedge q|\!\} Q_2 \{\!|q|\!\}_u \,]\!] \Longrightarrow \{\!|p \wedge q|\!\} Q_1 \;;;\; Q_2 \{\!|q|\!\}_u$
⟨*proof*⟩

lemma *seq-hoare-stronger-pre-2* [*hoare-safe*]:
$[\![\ \{\!|p \wedge q|\!\} Q_1 \{\!|p \wedge q|\!\}_u \ ;\ \{\!|p \wedge q|\!\} Q_2 \{\!|p|\!\}_u \,]\!] \Longrightarrow \{\!|p \wedge q|\!\} Q_1 \;;;\; Q_2 \{\!|p|\!\}_u$
⟨*proof*⟩

lemma *seq-hoare-inv-r-2* [*hoare*]: $[\![\ \{\!|p|\!\} Q_1 \{\!|q|\!\}_u \ ;\ \{\!|q|\!\} Q_2 \{\!|q|\!\}_u \,]\!] \Longrightarrow \{\!|p|\!\} Q_1 \;;;\; Q_2 \{\!|q|\!\}_u$
⟨*proof*⟩

lemma *seq-hoare-inv-r-3* [*hoare*]: $[\![\ \{\!|p|\!\} Q_1 \{\!|p|\!\}_u \ ;\ \{\!|p|\!\} Q_2 \{\!|q|\!\}_u \,]\!] \Longrightarrow \{\!|p|\!\} Q_1 \;;;\; Q_2 \{\!|q|\!\}_u$
⟨*proof*⟩

20.5 Conditional Laws

lemma *cond-hoare-r* [*hoare-safe*]: $[\![\ \{\!|b \wedge p|\!\} S \{\!|q|\!\}_u \ ;\ \{\!|\neg b \wedge p|\!\} T \{\!|q|\!\}_u \,]\!] \Longrightarrow \{\!|p|\!\} S \lhd b \rhd_r T \{\!|q|\!\}_u$
⟨*proof*⟩

lemma *cond-hoare-r-wp*:
assumes $\{\!|p'|\!\} S \{\!|q|\!\}_u$ **and** $\{\!|p''|\!\} T \{\!|q|\!\}_u$
shows $\{\!|(b \wedge p') \vee (\neg b \wedge p'')|\!\} S \lhd b \rhd_r T \{\!|q|\!\}_u$
⟨*proof*⟩

lemma *cond-hoare-r-sp*:
assumes $\langle \{\!|b \wedge p|\!\} S \{\!|q|\!\}_u \rangle$ **and** $\langle \{\!|\neg b \wedge p|\!\} T \{\!|s|\!\}_u \rangle$
shows $\langle \{\!|p|\!\} S \lhd b \rhd_r T \{\!|q \vee s|\!\}_u \rangle$
⟨*proof*⟩

20.6 Recursion Laws

lemma *nu-hoare-r-partial*:
assumes *induct-step*:

$\bigwedge st\ P.\ \{\!|p|\!\}P\{\!|q|\!\}_u \Longrightarrow \{\!|p|\!\}F\ P\{\!|q|\!\}_u$
shows $\{\!|p|\!\}\nu\ F\ \{\!|q|\!\}_u$
⟨*proof*⟩

lemma *mu-hoare-r*:
assumes *WF*: *wf R*
assumes *M*:*mono F*
assumes *induct-step*:
$\bigwedge st\ P.\ \{\!|p \wedge (e,\langle\!\langle st\rangle\!\rangle)_u \in_u \langle\!\langle R\rangle\!\rangle|\!\}P\{\!|q|\!\}_u \Longrightarrow \{\!|p \wedge e =_u \langle\!\langle st\rangle\!\rangle|\!\}F\ P\{\!|q|\!\}_u$
shows $\{\!|p|\!\}\mu\ F\ \{\!|q|\!\}_u$
⟨*proof*⟩

lemma *mu-hoare-r′*:
assumes *WF*: *wf R*
assumes *M*:*mono F*
assumes *induct-step*:
$\bigwedge st\ P.\ \{\!|p \wedge (e,\langle\!\langle st\rangle\!\rangle)_u \in_u \langle\!\langle R\rangle\!\rangle|\!\}\ P\ \{\!|q|\!\}_u \Longrightarrow \{\!|p \wedge e =_u \langle\!\langle st\rangle\!\rangle|\!\}\ F\ P\ \{\!|q|\!\}_u$
assumes *I0*: *‘p′ ⇒ p‘*
shows $\{\!|p'|\!\}\ \mu\ F\ \{\!|q|\!\}_u$
⟨*proof*⟩

20.7 Iteration Rules

lemma *iter-hoare-r*: $\{\!|P|\!\}S\{\!|P|\!\}_u \Longrightarrow \{\!|P|\!\}S^\star\{\!|P|\!\}_u$
⟨*proof*⟩

lemma *while-hoare-r* [*hoare-safe*]:
assumes $\{\!|p \wedge b|\!\}S\{\!|p|\!\}_u$
shows $\{\!|p|\!\}$*while b do S od*$\{\!|\neg b \wedge p|\!\}_u$
⟨*proof*⟩

lemma *while-invr-hoare-r* [*hoare-safe*]:
assumes $\{\!|p \wedge b|\!\}S\{\!|p|\!\}_u$ *‘pre ⇒ p‘ ‘(¬b ∧ p) ⇒ post‘*
shows $\{\!|$*pre*$|\!\}$*while b invr p do S od*$\{\!|$*post*$|\!\}_u$
⟨*proof*⟩

lemma *while-r-minimal-partial*:
assumes *seq-step*: *‘p ⇒ invar‘*
assumes *induct-step*: $\{\!|$*invar*$\wedge\ b|\!\}\ C\ \{\!|$*invar*$|\!\}_u$
shows $\{\!|p|\!\}$*while b do C od*$\{\!|\neg b \wedge$ *invar*$|\!\}_u$
⟨*proof*⟩

lemma *approx-chain*:
$(\bigsqcap n{::}nat.\ \lceil p \wedge v <_u \langle\!\langle n\rangle\!\rangle\rceil_<) = \lceil p\rceil_<$
⟨*proof*⟩

Total correctness law for Hoare logic, based on constructive chains. This is limited to variants that have naturals numbers as their range.

lemma *while-term-hoare-r*:
assumes $\bigwedge z{::}nat.\ \{\!|p \wedge b \wedge v =_u \langle\!\langle z\rangle\!\rangle|\!\}S\{\!|p \wedge v <_u \langle\!\langle z\rangle\!\rangle|\!\}_u$
shows $\{\!|p|\!\}$*while*$_\perp$ *b do S od*$\{\!|\neg b \wedge p|\!\}_u$
⟨*proof*⟩

lemma *while-vrt-hoare-r* [*hoare-safe*]:
assumes $\bigwedge z{::}nat.\ \{\!|p \wedge b \wedge v =_u \langle\!\langle z\rangle\!\rangle|\!\}S\{\!|p \wedge v <_u \langle\!\langle z\rangle\!\rangle|\!\}_u$ *‘pre ⇒ p‘ ‘(¬b ∧ p) ⇒ post‘*

shows $\{\!|pre|\!\}\ while\ b\ invr\ p\ vrt\ v\ do\ S\ od\{\!|post|\!\}_u$
⟨*proof*⟩

General total correctness law based on well-founded induction

lemma *while-wf-hoare-r*:
assumes *WF*: *wf R*
assumes *I0*: ‘*pre* ⇒ *p*‘
assumes *induct-step*:$\bigwedge$ *st*. $\{\!|b \wedge p \wedge e =_u «st»|\!\}\ Q\{\!|p \wedge (e, «st»)_u \in_u «R»|\!\}_u$
assumes *PHI*:‘(¬*b* ∧ *p*) ⇒ *post*‘
shows $\{\!|pre|\!\}\ while_\perp\ b\ invr\ p\ do\ Q\ od\{\!|post|\!\}_u$
⟨*proof*⟩

20.8 Frame Rules

Frame rule: If starting S in a state satisfying *pestablishesq* in the final state, then we can insert an invariant predicate r when S is framed by a, provided that r does not refer to variables in the frame, and q does not refer to variables outside the frame.

lemma *frame-hoare-r*:
assumes *vwb-lens a* $a \sharp r$ $a \natural q$ $\{\!|p|\!\}P\{\!|q|\!\}_u$
shows $\{\!|p \wedge r|\!\}a{:}[P]\{\!|q \wedge r|\!\}_u$
⟨*proof*⟩

lemma *frame-strong-hoare-r* [*hoare-safe*]:
assumes *vwb-lens a* $a \sharp r$ $a \natural q$ $\{\!|p \wedge r|\!\}S\{\!|q|\!\}_u$
shows $\{\!|p \wedge r|\!\}a{:}[S]\{\!|q \wedge r|\!\}_u$
⟨*proof*⟩

lemma *frame-hoare-r′* [*hoare-safe*]:
assumes *vwb-lens a* $a \sharp r$ $a \natural q$ $\{\!|r \wedge p|\!\}S\{\!|q|\!\}_u$
shows $\{\!|r \wedge p|\!\}a{:}[S]\{\!|r \wedge q|\!\}_u$
⟨*proof*⟩

lemma *antiframe-hoare-r*:
assumes *vwb-lens a* $a \natural r$ $a \sharp q$ $\{\!|p|\!\}P\{\!|q|\!\}_u$
shows $\{\!|p \wedge r|\!\}\ a{:}[\![P]\!]\ \{\!|q \wedge r|\!\}_u$
⟨*proof*⟩

lemma *antiframe-strong-hoare-r*:
assumes *vwb-lens a* $a \natural r$ $a \sharp q$ $\{\!|p \wedge r|\!\}P\{\!|q|\!\}_u$
shows $\{\!|p \wedge r|\!\}\ a{:}[\![P]\!]\ \{\!|q \wedge r|\!\}_u$
⟨*proof*⟩

end

21 Weakest (Liberal) Precondition Calculus

theory *utp-wp*
imports *utp-hoare*
begin

A very quick implementation of wlp – more laws still needed!

named-theorems *wp*

method *wp-tac* = (*simp add*: *wp*)

consts
uwp :: $'a \Rightarrow 'b \Rightarrow 'c$

syntax
-uwp :: *logic* $\Rightarrow$ *uexp* $\Rightarrow$ *logic* (**infix** ‹*wp*› *60*)

syntax-consts
-uwp == *uwp*

translations
-uwp P b == *CONST uwp P b*

definition *wp-upred* :: $('\alpha, '\beta)$ *urel* $\Rightarrow$ $'\beta$ *cond* $\Rightarrow$ $'\alpha$ *cond* **where**
wp-upred Q r = $\lfloor \neg\ (Q\ ;;\ (\neg\ \lceil r \rceil_<)) :: ('\alpha, '\beta)\ urel \rfloor_<$

adhoc-overloading
uwp $\rightleftharpoons$ *wp-upred*

declare *wp-upred-def* [*urel-defs*]

lemma *wp-true* [*wp*]: *p wp true* = *true*
⟨*proof*⟩

theorem *wp-assigns-r* [*wp*]:
$\langle\sigma\rangle_a$ *wp r* = σ † *r*
⟨*proof*⟩

theorem *wp-skip-r* [*wp*]:
II wp r = *r*
⟨*proof*⟩

theorem *wp-abort* [*wp*]:
r ≠ *true* ⟹ *true wp r* = *false*
⟨*proof*⟩

theorem *wp-conj* [*wp*]:
P wp (*q* ∧ *r*) = (*P wp q* ∧ *P wp r*)
⟨*proof*⟩

theorem *wp-seq-r* [*wp*]: (*P* ;; *Q*) *wp r* = *P wp* (*Q wp r*)
⟨*proof*⟩

theorem *wp-choice* [*wp*]: (*P* ⊓ *Q*) *wp R* = (*P wp R* ∧ *Q wp R*)
⟨*proof*⟩

theorem *wp-cond* [*wp*]: (*P* ◁ *b* ▷$_r$ *Q*) *wp r* = ((*b* ⇒ *P wp r*) ∧ ((¬ *b*) ⇒ *Q wp r*))
⟨*proof*⟩

lemma *wp-USUP-pre* [*wp*]: *P wp* ($\bigsqcup i{\in}\{0..n\} \cdot Q(i)$) = ($\bigsqcup i{\in}\{0..n\} \cdot$ *P wp* $Q(i)$)
⟨*proof*⟩

theorem *wp-hoare-link*:
$\{\!|p|\!\}\ Q\ \{\!|r|\!\}_u \longleftrightarrow$ (*Q wp r* ⊑ *p*)
⟨*proof*⟩

If two programs have the same weakest precondition for any postcondition then the programs are the same.

theorem *wp-eq-intro*: $[\![\bigwedge r.\ P\ wp\ r = Q\ wp\ r\ ]\!] \Longrightarrow P = Q$
⟨*proof*⟩
end

22 Dynamic Logic

theory *utp-dynlog*
imports *utp-sequent utp-wp*
begin

22.1 Definitions

named-theorems *dynlog-simp* **and** *dynlog-intro*

definition *dBox* :: *'s hrel* ⇒ *'s upred* ⇒ *'s upred* (‹[-]› [*0*,*999*] *999*)
where [*upred-defs*]: *dBox A* Φ = *A wp* Φ

definition *dDia* :: *'s hrel* ⇒ *'s upred* ⇒ *'s upred* (‹<->› [*0*,*999*] *999*)
where [*upred-defs*]: *dDia A* Φ = (¬ [*A*] (¬ Φ))

22.2 Box Laws

lemma *dBox-false* [*dynlog-simp*]: [*false*]Φ = *true*
⟨*proof*⟩

lemma *dBox-skip* [*dynlog-simp*]: [*II*]Φ = Φ
⟨*proof*⟩

lemma *dBox-assigns* [*dynlog-simp*]: $[\langle\sigma\rangle_a]\Phi = (\sigma \dagger \Phi)$
⟨*proof*⟩

lemma *dBox-choice* [*dynlog-simp*]: [*P* ⊓ *Q*]Φ = ([*P*]Φ ∧ [*Q*]Φ)
⟨*proof*⟩

lemma *dBox-seq*: [*P* ;; *Q*]Φ = [*P*][*Q*]Φ
⟨*proof*⟩

lemma *dBox-star-unfold*: $[P^\star]\Phi = (\Phi \wedge [P][P^\star]\Phi)$
⟨*proof*⟩

lemma *dBox-star-induct*: ‘(Φ ∧ $[P^\star]$(Φ ⇒ [*P*]Φ)) ⇒ $[P^\star]\Phi$‘
⟨*proof*⟩

lemma *dBox-test*: [*?*[*p*]]Φ = (*p* ⇒ Φ)
⟨*proof*⟩

22.3 Diamond Laws

lemma *dDia-false* [*dynlog-simp*]: <*false*>Φ = *false*
⟨*proof*⟩

lemma *dDia-skip* [*dynlog-simp*]: <*II*>Φ = Φ
⟨*proof*⟩

lemma *dDia-assigns* [*dynlog-simp*]: $<\langle\sigma\rangle_a>\Phi = (\sigma \dagger \Phi)$
⟨*proof*⟩

lemma *dDia-choice*: $<P \sqcap Q>\Phi = (<P>\Phi \vee <Q>\Phi)$
⟨*proof*⟩

lemma *dDia-seq*: $<P ;; Q>\Phi = <P><Q>\Phi$
⟨*proof*⟩

lemma *dDia-test*: $<?[p]>\Phi = (p \wedge \Phi)$
⟨*proof*⟩

22.4 Sequent Laws

lemma *sBoxSeq* [*dynlog-simp*]: $\Gamma \Vdash [P ;; Q]\Phi \equiv \Gamma \Vdash [P][Q]\Phi$
⟨*proof*⟩

lemma *sBoxTest* [*dynlog-intro*]: $\Gamma \Vdash (b \Rightarrow \Psi) \Longrightarrow \Gamma \Vdash [?[b]]\Psi$
⟨*proof*⟩

lemma *sBoxAssignFwd* [*dynlog-simp*]: $[\![\textit{vwb-lens } x;\ x \sharp v;\ x \sharp \Gamma]\!] \Longrightarrow (\Gamma \Vdash [x := v]\Phi) = ((\&x =_u v \wedge \Gamma) \Vdash \Phi)$
⟨*proof*⟩

lemma *sBoxIndStar*: $\Vdash [\Phi \Rightarrow [P]\Phi]_u \Longrightarrow \Phi \Vdash [P^\star]\Phi$
⟨*proof*⟩

lemma *hoare-as-dynlog*: $\{\!|p|\!\}\, Q\, \{\!|r|\!\}_u = (p \Vdash [Q]r)$
⟨*proof*⟩

end

23 State Variable Declaration Parser

theory *utp-state-parser*
imports *utp-rel*
begin

This theory sets up a parser for state blocks, as an alternative way of providing lenses to a predicate. A program with local variables can be represented by a predicate indexed by a tuple of lenses, where each lens represents a variable. These lenses must then be supplied with respect to a suitable state space. Instead of creating a type to represent this alphabet, we can create a product type for the state space, with an entry for each variable. Then each variable becomes a composition of the fst_L and snd_L lenses to index the correct position in the variable vector.

We first creation a vacuous definition that will mark when an indexed predicate denotes a state block.

definition *state-block* :: $('v \Rightarrow 'p) \Rightarrow 'v \Rightarrow 'p$ **where**
[*upred-defs*]: *state-block f x* = *f x*

We declare a number of syntax translations to produce lens and product types, to obtain a type for the overall state space, to construct a tuple that denotes the lens vector parameter, to construct the vector itself, and finally to construct the state declaration.

syntax

-lensT :: *type* ⇒ *type* ⇒ *type* (‹*LENSTYPE'*(-, -')›)
-pairT :: *type* ⇒ *type* ⇒ *type* (‹*PAIRTYPE'*(-, -')›)
-state-type :: *pttrn* ⇒ *type*
-state-tuple :: *type* ⇒ *pttrn* ⇒ *logic*
-state-lenses :: *pttrn* ⇒ *logic*
-state-decl :: *pttrn* ⇒ *logic* ⇒ *logic* (‹*LOCAL* - • ->› [*0*, *10*] *10*)

syntax-types
-lensT ⇌ *lens* **and**
-pairT ⇌ *prod*

translations
(*type*) *PAIRTYPE*('*a*, '*b*) => (*type*) '*a* × '*b*
(*type*) *LENSTYPE*('*a*, '*b*) => (*type*) '*a* ⟹ '*b*

-state-type (*-constrain x t*) => *t*
-state-type (*CONST Pair* (*-constrain x t*) *vs*) => *-pairT t* (*-state-type vs*)

-state-tuple st (*-constrain x t*) => *-constrain x* (*-lensT t st*)
-state-tuple st (*CONST Pair* (*-constrain x t*) *vs*) =>
CONST Product-Type.Pair (*-constrain x* (*-lensT t st*)) (*-state-tuple st vs*)

-state-decl vs P =>
CONST state-block (*-abs* (*-state-tuple* (*-state-type vs*) *vs*) *P*) (*-state-lenses vs*)
-state-decl vs P <= *CONST state-block* (*-abs vs P*) *k*

⟨*ML*⟩

23.1 Examples

term *LOCAL* (*x*::*int*, *y*::*real*, *z*::*int*) • *x* := (&*x* + &*z*)

lemma *LOCAL p* • *II* = *II*
⟨*proof*⟩

end

24 Relational Operational Semantics

theory *utp-rel-opsem*
imports
utp-rel-laws
utp-hoare
begin

This theory uses the laws of relational calculus to create a basic operational semantics. It is based on Chapter 10 of the UTP book [22].

fun *trel* :: '*α usubst* × '*α hrel* ⇒ '*α usubst* × '*α hrel* ⇒ *bool* (**infix** ‹→$_u$› *85*) **where**
(*σ*, *P*) →$_u$ (*ϱ*, *Q*) ⟷ (⟨*σ*⟩$_a$;; *P*) ⊑ (⟨*ϱ*⟩$_a$;; *Q*)

lemma *trans-trel*:
⟦ (*σ*, *P*) →$_u$ (*ϱ*, *Q*); (*ϱ*, *Q*) →$_u$ (*φ*, *R*) ⟧ ⟹ (*σ*, *P*) →$_u$ (*φ*, *R*)
⟨*proof*⟩

lemma *skip-trel*: $(\sigma,\ II) \rightarrow_u (\sigma,\ II)$
⟨*proof*⟩

lemma *assigns-trel*: $(\sigma,\ \langle\varrho\rangle_a) \rightarrow_u (\varrho \circ \sigma,\ II)$
⟨*proof*⟩

lemma *assign-trel*:
$(\sigma,\ x := v) \rightarrow_u (\sigma(\&x \mapsto_s \sigma\ \dagger\ v),\ II)$
⟨*proof*⟩

lemma *seq-trel*:
assumes $(\sigma,\ P) \rightarrow_u (\varrho,\ Q)$
shows $(\sigma,\ P\ ;;\ R) \rightarrow_u (\varrho,\ Q\ ;;\ R)$
⟨*proof*⟩

lemma *seq-skip-trel*:
$(\sigma,\ II\ ;;\ P) \rightarrow_u (\sigma,\ P)$
⟨*proof*⟩

lemma *nondet-left-trel*:
$(\sigma,\ P \sqcap Q) \rightarrow_u (\sigma,\ P)$
⟨*proof*⟩

lemma *nondet-right-trel*:
$(\sigma,\ P \sqcap Q) \rightarrow_u (\sigma,\ Q)$
⟨*proof*⟩

lemma *rcond-true-trel*:
assumes $\sigma\ \dagger\ b = true$
shows $(\sigma,\ P \lhd b \rhd_r Q) \rightarrow_u (\sigma,\ P)$
⟨*proof*⟩

lemma *rcond-false-trel*:
assumes $\sigma\ \dagger\ b = false$
shows $(\sigma,\ P \lhd b \rhd_r Q) \rightarrow_u (\sigma,\ Q)$
⟨*proof*⟩

lemma *while-true-trel*:
assumes $\sigma\ \dagger\ b = true$
shows $(\sigma,\ while\ b\ do\ P\ od) \rightarrow_u (\sigma,\ P\ ;;\ while\ b\ do\ P\ od)$
⟨*proof*⟩

lemma *while-false-trel*:
assumes $\sigma\ \dagger\ b = false$
shows $(\sigma,\ while\ b\ do\ P\ od) \rightarrow_u (\sigma,\ II)$
⟨*proof*⟩

Theorem linking Hoare calculus and operational semantics. If we start Q in a state σ_0 satisfying p, and Q reaches final state σ_1 then r holds in this final state.

theorem *hoare-opsem-link*:
$\{\!|p|\!\}\, Q \,\{\!|r|\!\}_u = (\forall\ \sigma_0\ \sigma_1.\ `\sigma_0\ \dagger\ p` \wedge (\sigma_0,\ Q) \rightarrow_u (\sigma_1,\ II) \longrightarrow `\sigma_1\ \dagger\ r`)$
⟨*proof*⟩

declare *trel.simps* [*simp del*]

end

25 Symbolic Evaluation of Relational Programs

theory *utp-sym-eval*
imports *utp-rel-opsem*
begin

The following operator applies a variable context Γ as an assignment, and composes it with a relation P for the purposes of evaluation.

definition *utp-sym-eval* :: *'s usubst* ⇒ *'s hrel* ⇒ *'s hrel* (**infixr** ‹⊨› *55*) **where**
[*upred-defs*]: *utp-sym-eval* Γ P = ($\langle\Gamma\rangle_a$;; P)

named-theorems *symeval*

lemma *seq-symeval* [*symeval*]: $\Gamma \models P ;; Q = (\Gamma \models P) ;; Q$
⟨*proof*⟩

lemma *assigns-symeval* [*symeval*]: $\Gamma \models \langle\sigma\rangle_a = (\sigma \circ \Gamma) \models II$
⟨*proof*⟩

lemma *term-symeval* [*symeval*]: $(\Gamma \models II) ;; P = \Gamma \models P$
⟨*proof*⟩

lemma *if-true-symeval* [*symeval*]: $[\![\Gamma \dagger b = true]\!] \Longrightarrow \Gamma \models (P \lhd b \rhd_r Q) = \Gamma \models P$
⟨*proof*⟩

lemma *if-false-symeval* [*symeval*]: $[\![\Gamma \dagger b = false]\!] \Longrightarrow \Gamma \models (P \lhd b \rhd_r Q) = \Gamma \models Q$
⟨*proof*⟩

lemma *while-true-symeval* [*symeval*]: $[\![\Gamma \dagger b = true]\!] \Longrightarrow \Gamma \models$ *while b do P od* = $\Gamma \models$ (*P* ;; *while b do P od*)
⟨*proof*⟩

lemma *while-false-symeval* [*symeval*]: $[\![\Gamma \dagger b = false]\!] \Longrightarrow \Gamma \models$ *while b do P od* = $\Gamma \models II$
⟨*proof*⟩

lemma *while-inv-true-symeval* [*symeval*]: $[\![\Gamma \dagger b = true]\!] \Longrightarrow \Gamma \models$ *while b invr S do P od* = $\Gamma \models$ (*P* ;; *while b do P od*)
⟨*proof*⟩

lemma *while-inv-false-symeval* [*symeval*]: $[\![\Gamma \dagger b = false]\!] \Longrightarrow \Gamma \models$ *while b invr S do P od* = $\Gamma \models II$
⟨*proof*⟩

method *sym-eval* = (*simp add*: *symeval usubst lit-simps*[*THEN sym*]), (*simp del*: *One-nat-def add*: *One-nat-def*[*THEN sym*])*?*

syntax
-terminated :: *logic* ⇒ *logic* (‹*terminated*: -› [*999*] *999*)

translations
terminated: Γ == $\Gamma \models II$

end

26 Strong Postcondition Calculus

theory *utp-sp*
imports *utp-wp*
begin

named-theorems *sp*

method *sp-tac* = (*simp add*: *sp*)

consts
  usp :: $'a \Rightarrow 'b \Rightarrow 'c$ (**infix** ‹*sp*› *60*)

definition *sp-upred* :: $'\alpha\ cond \Rightarrow ('\alpha, '\beta)\ urel \Rightarrow '\beta\ cond$ **where**
sp-upred p Q = $\lfloor(\lceil p \rceil_> ;; Q) :: ('\alpha, '\beta)\ urel\rfloor_>$

adhoc-overloading
  usp $\rightleftharpoons$ *sp-upred*

declare *sp-upred-def* [*upred-defs*]

lemma *sp-false* [*sp*]: *p sp false* = *false*
  ⟨*proof*⟩

lemma *sp-true* [*sp*]: *q* $\neq$ *false* $\Longrightarrow$ *q sp true* = *true*
  ⟨*proof*⟩

lemma *sp-assigns-r* [*sp*]:
  vwb-lens x $\Longrightarrow$ (*p sp x* := *e*) = ($\exists\ v \cdot p[\![«v»/x]\!] \wedge \&x =_u e[\![«v»/x]\!]$)
  ⟨*proof*⟩

lemma *sp-it-is-post-condition*:
  $\{\!|p|\!\}\ C\ \{\!|p\ sp\ C|\!\}_u$
  ⟨*proof*⟩

lemma *sp-it-is-the-strongest-post*:
  ‘*p sp C* ⇒ *Q*‘ $\Longrightarrow \{\!|p|\!\}\ C\ \{\!|Q|\!\}_u$
  ⟨*proof*⟩

lemma *sp-so*:
  ‘*p sp C* ⇒ *Q*‘ = $\{\!|p|\!\}\ C\ \{\!|Q|\!\}_u$
  ⟨*proof*⟩

theorem *sp-hoare-link*:
  $\{\!|p|\!\}\ Q\ \{\!|r|\!\}_u \longleftrightarrow (r \sqsubseteq p\ sp\ Q)$
  ⟨*proof*⟩

lemma *sp-while-r* [*sp*]:
  assumes ‹‘*pre* ⇒ *I*‘› **and** ‹$\{\!|I \wedge b|\!\}\ C\ \{\!|I'|\!\}_u$› **and** ‹‘*I*′ ⇒ *I*‘›
  shows (*pre sp invar I while*$_\perp$ *b do C od*) = ($\neg b \wedge I$)
  ⟨*proof*⟩

theorem *sp-eq-intro*: $[\![\bigwedge r.\ r\ sp\ P = r\ sp\ Q]\!] \Longrightarrow P = Q$
  ⟨*proof*⟩

lemma *wp-sp-sym*:

‘prog wp (true sp prog)‘
⟨proof⟩

lemma *it-is-pre-condition*:⦇C wp Q⦈ C⦇Q⦈$_u$
⟨proof⟩

lemma *it-is-the-weakest-pre*:‘P ⇒ C wp Q‘ = ⦇P⦈ C⦇Q⦈$_u$
⟨proof⟩

lemma *s-pre*:‘P ⇒ C wp Q‘=⦇P⦈ C⦇Q⦈$_u$
⟨proof⟩

end

27 Concurrent Programming

theory *utp-concurrency*
imports
utp-hoare
utp-rel
utp-tactics
utp-theory
begin

In this theory we describe the UTP scheme for concurrency, *parallel-by-merge*, which provides a general parallel operator parametrised by a “merge predicate” that explains how to merge the after states of the composed predicates. It can thus be applied to many languages and concurrency schemes, with this theory providing a number of generic laws. The operator is explained in more detail in Chapter 7 of the UTP book [22].

27.1 Variable Renamings

In parallel-by-merge constructions, a merge predicate defines the behaviour following execution of of parallel processes, $P \parallel Q$, as a relation that merges the output of P and Q. In order to achieve this we need to separate the variable values output from P and Q, and in addition the variable values before execution. The following three constructs do these separations. The initial state-space before execution is $'\alpha$, the final state-space after the first parallel process is $'\beta_0$, and the final state-space for the second is $'\beta_1$. These three functions lift variables on these three state-spaces, respectively.

alphabet ($'\alpha$, $'\beta_0$, $'\beta_1$) *mrg* =
mrg-prior :: $'\alpha$
mrg-left :: $'\beta_0$
mrg-right :: $'\beta_1$

definition *pre-uvar* :: ($'a \Longrightarrow '\alpha$) ⇒ ($'a \Longrightarrow$ ($'\alpha$, $'\beta_0$, $'\beta_1$) *mrg*) **where**
[*upred-defs*]: *pre-uvar x* = *x* ;$_L$ *mrg-prior*

definition *left-uvar* :: ($'a \Longrightarrow '\beta_0$) ⇒ ($'a \Longrightarrow$ ($'\alpha$, $'\beta_0$, $'\beta_1$) *mrg*) **where**
[*upred-defs*]: *left-uvar x* = *x* ;$_L$ *mrg-left*

definition *right-uvar* :: ($'a \Longrightarrow '\beta_1$) ⇒ ($'a \Longrightarrow$ ($'\alpha$, $'\beta_0$, $'\beta_1$) *mrg*) **where**
[*upred-defs*]: *right-uvar x* = *x* ;$_L$ *mrg-right*

We set up syntax for the three variable classes using a subscript $<$, 0-x, and 1-x, respectively.

syntax
-svarpre :: *svid* ⇒ *svid* (‹-$_<$› [*995*] *995*)
-svarleft :: *svid* ⇒ *svid* (‹*0*−-› [*995*] *995*)
-svarright :: *svid* ⇒ *svid* (‹*1*−-› [*995*] *995*)

syntax-consts
-svarpre ⇌ *pre-uvar* **and**
-svarleft ⇌ *left-uvar* **and**
-svarright ⇌ *right-uvar*

translations
-svarpre x == *CONST pre-uvar x*
-svarleft x == *CONST left-uvar x*
-svarright x == *CONST right-uvar x*
-svarpre Σ <= *CONST pre-uvar* 1_L
-svarleft Σ <= *CONST left-uvar* 1_L
-svarright Σ <= *CONST right-uvar* 1_L

We proved behavedness closure properties about the lenses.

lemma *left-uvar* [*simp*]: *vwb-lens x* ⟹ *vwb-lens* (*left-uvar x*)
⟨*proof*⟩

lemma *right-uvar* [*simp*]: *vwb-lens x* ⟹ *vwb-lens* (*right-uvar x*)
⟨*proof*⟩

lemma *pre-uvar* [*simp*]: *vwb-lens x* ⟹ *vwb-lens* (*pre-uvar x*)
⟨*proof*⟩

lemma *left-uvar-mwb* [*simp*]: *mwb-lens x* ⟹ *mwb-lens* (*left-uvar x*)
⟨*proof*⟩

lemma *right-uvar-mwb* [*simp*]: *mwb-lens x* ⟹ *mwb-lens* (*right-uvar x*)
⟨*proof*⟩

lemma *pre-uvar-mwb* [*simp*]: *mwb-lens x* ⟹ *mwb-lens* (*pre-uvar x*)
⟨*proof*⟩

We prove various independence laws about the variable classes.

lemma *left-uvar-indep-right-uvar* [*simp*]:
left-uvar x ⋈ *right-uvar y*
⟨*proof*⟩

lemma *left-uvar-indep-pre-uvar* [*simp*]:
left-uvar x ⋈ *pre-uvar y*
⟨*proof*⟩

lemma *left-uvar-indep-left-uvar* [*simp*]:
x ⋈ *y* ⟹ *left-uvar x* ⋈ *left-uvar y*
⟨*proof*⟩

lemma *right-uvar-indep-left-uvar* [*simp*]:
right-uvar x ⋈ *left-uvar y*
⟨*proof*⟩

lemma *right-uvar-indep-pre-uvar* [*simp*]:
right-uvar x ⋈ *pre-uvar y*
⟨*proof*⟩

lemma *right-uvar-indep-right-uvar* [*simp*]:
x ⋈ *y* ⟹ *right-uvar x* ⋈ *right-uvar y*
⟨*proof*⟩

lemma *pre-uvar-indep-left-uvar* [*simp*]:
pre-uvar x ⋈ *left-uvar y*
⟨*proof*⟩

lemma *pre-uvar-indep-right-uvar* [*simp*]:
pre-uvar x ⋈ *right-uvar y*
⟨*proof*⟩

lemma *pre-uvar-indep-pre-uvar* [*simp*]:
x ⋈ *y* ⟹ *pre-uvar x* ⋈ *pre-uvar y*
⟨*proof*⟩

27.2 Merge Predicates

A merge predicate is a relation whose input has three parts: the prior variables, the output variables of the left predicate, and the output of the right predicate.

type-synonym *'α merge* = (('*α*, '*α*, '*α*) *mrg*, '*α*) *urel*

skip is the merge predicate which ignores the output of both parallel predicates

definition *skip*$_m$ :: *'α merge* **where**
[*upred-defs*]: *skip*$_m$ = ($**v**´ =_u $**v**$_<$)

swap is a predicate that the swaps the left and right indices; it is used to specify commutativity of the parallel operator

definition *swap*$_m$ :: (('*α*, '*β*, '*β*) *mrg*) *hrel* **where**
[*upred-defs*]: *swap*$_m$ = (*0*−**v**,*1*−**v**) := (&*1*−**v**,&*0*−**v**)

A symmetric merge is one for which swapping the order of the merged concurrent predicates has no effect. We represent this by the following healthiness condition that states that *swap*$_m$ is a left-unit.

abbreviation *SymMerge* :: *'α merge* ⇒ *'α merge* **where**
SymMerge(*M*) ≡ (*swap*$_m$;; *M*)

27.3 Separating Simulations

U0 and U1 are relations modify the variables of the input state-space such that they become indexed with 0 and 1, respectively.

definition *U0* :: ('*β*$_0$, ('*α*, '*β*$_0$, '*β*$_1$) *mrg*) *urel* **where**
[*upred-defs*]: *U0* = ($*0*−**v**´ =_u $**v**)

definition *U1* :: ('*β*$_1$, ('*α*, '*β*$_0$, '*β*$_1$) *mrg*) *urel* **where**
[*upred-defs*]: *U1* = ($*1*−**v**´ =_u $**v**)

lemma *U0-swap*: (*U0* ;; *swap*$_m$) = *U1*
⟨*proof*⟩

lemma *U1-swap*: $(U1 \;;;\; swap_m) = U0$
⟨*proof*⟩

As shown below, separating simulations can also be expressed using the following two alphabet extrusions

definition $U0\alpha$ **where** [*upred-defs*]: $U0\alpha = (1_L \times_L mrg\text{-}left)$

definition $U1\alpha$ **where** [*upred-defs*]: $U1\alpha = (1_L \times_L mrg\text{-}right)$

We then create the following intuitive syntax for separating simulations.

abbreviation *U0-alpha-lift* ($\langle\lceil\text{-}\rceil_0\rangle$) **where** $\lceil P\rceil_0 \equiv P \oplus_p U0\alpha$

abbreviation *U1-alpha-lift* ($\langle\lceil\text{-}\rceil_1\rangle$) **where** $\lceil P\rceil_1 \equiv P \oplus_p U1\alpha$

$\lceil P\rceil_0$ is predicate P where all variables are indexed by 0, and $\lceil P\rceil_1$ is where all variables are indexed by 1. We can thus equivalently express separating simulations using alphabet extrusion.

lemma *U0-as-alpha*: $(P \;;;\; U0) = \lceil P\rceil_0$
⟨*proof*⟩

lemma *U1-as-alpha*: $(P \;;;\; U1) = \lceil P\rceil_1$
⟨*proof*⟩

lemma $U0\alpha$-*vwb-lens* [*simp*]: *vwb-lens* $U0\alpha$
⟨*proof*⟩

lemma $U1\alpha$-*vwb-lens* [*simp*]: *vwb-lens* $U1\alpha$
⟨*proof*⟩

lemma $U0\alpha$-*indep-right-uvar* [*simp*]: *vwb-lens* $x \Longrightarrow U0\alpha \bowtie$ *out-var* (*right-uvar* x)
⟨*proof*⟩

lemma $U1\alpha$-*indep-left-uvar* [*simp*]: *vwb-lens* $x \Longrightarrow U1\alpha \bowtie$ *out-var* (*left-uvar* x)
⟨*proof*⟩

lemma *U0-alpha-lift-bool-subst* [*usubst*]:
$\sigma(\$0\text{-}x' \mapsto_s true) \dagger \lceil P\rceil_0 = \sigma \dagger \lceil P[\![true/\$x']\!]\rceil_0$
$\sigma(\$0\text{-}x' \mapsto_s false) \dagger \lceil P\rceil_0 = \sigma \dagger \lceil P[\![false/\$x']\!]\rceil_0$
⟨*proof*⟩

lemma *U1-alpha-lift-bool-subst* [*usubst*]:
$\sigma(\$1\text{-}x' \mapsto_s true) \dagger \lceil P\rceil_1 = \sigma \dagger \lceil P[\![true/\$x']\!]\rceil_1$
$\sigma(\$1\text{-}x' \mapsto_s false) \dagger \lceil P\rceil_1 = \sigma \dagger \lceil P[\![false/\$x']\!]\rceil_1$
⟨*proof*⟩

lemma *U0-alpha-out-var* [*alpha*]: $\lceil \$x'\rceil_0 = \$0\text{-}x'$
⟨*proof*⟩

lemma *U1-alpha-out-var* [*alpha*]: $\lceil \$x'\rceil_1 = \$1\text{-}x'$
⟨*proof*⟩

lemma *U0-skip* [*alpha*]: $\lceil II\rceil_0 = (\$0\text{-}\mathbf{v}' =_u \$\mathbf{v})$
⟨*proof*⟩

lemma *U1-skip* [*alpha*]: $\lceil II\rceil_1 = (\$1\text{-}\mathbf{v}' =_u \$\mathbf{v})$

⟨*proof*⟩

lemma *U0-seqr* [*alpha*]: $\lceil P \;;;\; Q \rceil_0 = P \;;;\; \lceil Q \rceil_0$
⟨*proof*⟩

lemma *U1-seqr* [*alpha*]: $\lceil P \;;;\; Q \rceil_1 = P \;;;\; \lceil Q \rceil_1$
⟨*proof*⟩

lemma *U0α-comp-in-var* [*alpha*]: $(\textit{in-var}\ x) \;;_L\ U0\alpha = \textit{in-var}\ x$
⟨*proof*⟩

lemma *U0α-comp-out-var* [*alpha*]: $(\textit{out-var}\ x) \;;_L\ U0\alpha = \textit{out-var}\ (\textit{left-uvar}\ x)$
⟨*proof*⟩

lemma *U1α-comp-in-var* [*alpha*]: $(\textit{in-var}\ x) \;;_L\ U1\alpha = \textit{in-var}\ x$
⟨*proof*⟩

lemma *U1α-comp-out-var* [*alpha*]: $(\textit{out-var}\ x) \;;_L\ U1\alpha = \textit{out-var}\ (\textit{right-uvar}\ x)$
⟨*proof*⟩

27.4 Associative Merges

Associativity of a merge means that if we construct a three way merge from a two way merge and then rotate the three inputs of the merge to the left, then we get exactly the same three way merge back.

We first construct the operator that constructs the three way merge by effectively wiring up the two way merge in an appropriate way.

definition *ThreeWayMerge* :: $'\alpha\ \textit{merge} \Rightarrow ((('\alpha, '\alpha)\ (('\alpha, '\alpha, '\alpha)\ \textit{mrg}, '\alpha)\ \textit{mrg}, '\alpha)\ \textit{urel}\ (\langle \mathbf{M}3\,'(\text{-}')\rangle)$ **where**
[*upred-defs*]: $\textit{ThreeWayMerge}\ M = ((\$0\text{−}\mathbf{v}' =_u \$0\text{−}\mathbf{v} \wedge \$1\text{−}\mathbf{v}' =_u \$1\text{−}0\text{−}\mathbf{v} \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v}_<) \;;;\; M \;;;\; U0 \wedge \$1\text{−}\mathbf{v}' =_u \$1\text{−}1\text{−}\mathbf{v} \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v}_<) \;;;\; M$

The next definition rotates the inputs to a three way merge to the left one place.

abbreviation $\textit{rotate}_m$ **where** $\textit{rotate}_m \equiv (0\text{−}\mathbf{v}, 1\text{−}0\text{−}\mathbf{v}, 1\text{−}1\text{−}\mathbf{v}) := (\&1\text{−}0\text{−}\mathbf{v}, \&1\text{−}1\text{−}\mathbf{v}, \&0\text{−}\mathbf{v})$

Finally, a merge is associative if rotating the inputs does not effect the output.

definition *AssocMerge* :: $'\alpha\ \textit{merge} \Rightarrow \textit{bool}$ **where**
[*upred-defs*]: $\textit{AssocMerge}\ M = (\textit{rotate}_m \;;;\; \mathbf{M}3(M) = \mathbf{M}3(M))$

27.5 Parallel Operators

We implement the following useful abbreviation for separating of two parallel processes and copying of the before variables, all to act as input to the merge predicate.

abbreviation *par-sep* (**infixr** $\langle \|_s \rangle$ *85*) **where**
$P \parallel_s Q \equiv (P \;;;\; U0) \wedge (Q \;;;\; U1) \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v}$

The following implementation of parallel by merge is less general than the book version, in that it does not properly partition the alphabet into two disjoint segments. We could actually achieve this specifying lenses into the larger alphabet, but this would complicate the definition of programs. May reconsider later.

definition
$\textit{par-by-merge} :: ('\alpha, '\beta)\ \textit{urel} \Rightarrow (('\alpha, '\beta, '\gamma)\ \textit{mrg}, '\delta)\ \textit{urel} \Rightarrow ('\alpha, '\gamma)\ \textit{urel} \Rightarrow ('\alpha, '\delta)\ \textit{urel}$
$(\langle \text{-} \|_\text{-} \text{-}\rangle\ [85,0,86]\ 85)$

where [*upred-defs*]: $P \parallel_M Q = (P \parallel_s Q ;; M)$

lemma *par-by-merge-alt-def*: $P \parallel_M Q = (\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v}) ;; M$
⟨*proof*⟩

lemma *shEx-pbm-left*: $((\exists\ x \cdot P\ x) \parallel_M Q) = (\exists\ x \cdot (P\ x \parallel_M Q))$
⟨*proof*⟩

lemma *shEx-pbm-right*: $(P \parallel_M (\exists\ x \cdot Q\ x)) = (\exists\ x \cdot (P \parallel_M Q\ x))$
⟨*proof*⟩

27.6 Unrestriction Laws

lemma *unrest-in-par-by-merge* [*unrest*]:
$[\![\$x \sharp P; \$x_< \sharp M; \$x \sharp Q]\!] \Longrightarrow \$x \sharp P \parallel_M Q$
⟨*proof*⟩

lemma *unrest-out-par-by-merge* [*unrest*]:
$[\![\$x' \sharp M]\!] \Longrightarrow \$x' \sharp P \parallel_M Q$
⟨*proof*⟩

27.7 Substitution laws

Substitution is a little tricky because when we push the expression through the composition operator the alphabet of the expression must also change. Consequently for now we only support literal substitution, though this could be generalised with suitable alphabet coercsions. We need quite a number of variants to support this which are below.

lemma *U0-seq-subst*: $(P ;; U0)[\![\langle\!\langle v \rangle\!\rangle/\$0{-}x']\!] = (P[\![\langle\!\langle v \rangle\!\rangle/\$x']\!] ;; U0)$
⟨*proof*⟩

lemma *U1-seq-subst*: $(P ;; U1)[\![\langle\!\langle v \rangle\!\rangle/\$1{-}x']\!] = (P[\![\langle\!\langle v \rangle\!\rangle/\$x']\!] ;; U1)$
⟨*proof*⟩

lemma *lit-pbm-subst* [*usubst*]:
fixes $x :: (- \Longrightarrow 'a)$
shows
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x \mapsto_s \langle\!\langle v \rangle\!\rangle) \dagger (P \parallel_M Q) = \sigma \dagger ((P[\![\langle\!\langle v \rangle\!\rangle/\$x]\!]) \parallel_{M[\![\langle\!\langle v \rangle\!\rangle/\$x_<]\!]} (Q[\![\langle\!\langle v \rangle\!\rangle/\$x]\!]))$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x' \mapsto_s \langle\!\langle v \rangle\!\rangle) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M[\![\langle\!\langle v \rangle\!\rangle/\$x']\!]} Q)$
⟨*proof*⟩

lemma *bool-pbm-subst* [*usubst*]:
fixes $x :: (- \Longrightarrow 'a)$
shows
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x \mapsto_s \mathit{false}) \dagger (P \parallel_M Q) = \sigma \dagger ((P[\![\mathit{false}/\$x]\!]) \parallel_{M[\![\mathit{false}/\$x_<]\!]} (Q[\![\mathit{false}/\$x]\!]))$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x \mapsto_s \mathit{true}) \dagger (P \parallel_M Q) = \sigma \dagger ((P[\![\mathit{true}/\$x]\!]) \parallel_{M[\![\mathit{true}/\$x_<]\!]} (Q[\![\mathit{true}/\$x]\!]))$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x' \mapsto_s \mathit{false}) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M[\![\mathit{false}/\$x']\!]} Q)$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x' \mapsto_s \mathit{true}) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M[\![\mathit{true}/\$x']\!]} Q)$
⟨*proof*⟩

lemma *zero-one-pbm-subst* [*usubst*]:
fixes $x :: (- \Longrightarrow 'a)$
shows
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x \mapsto_s 0) \dagger (P \parallel_M Q) = \sigma \dagger ((P[\![0/\$x]\!]) \parallel_{M[\![0/\$x_<]\!]} (Q[\![0/\$x]\!]))$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x \mapsto_s 1) \dagger (P \parallel_M Q) = \sigma \dagger ((P[\![1/\$x]\!]) \parallel_{M[\![1/\$x_<]\!]} (Q[\![1/\$x]\!]))$

$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x' \mapsto_s 0) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M[\![0/\$x']\!]} Q)$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x' \mapsto_s 1) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M[\![1/\$x']\!]} Q)$
⟨*proof*⟩

lemma *numeral-pbm-subst* [*usubst*]:
fixes $x :: (- \Longrightarrow 'a)$
shows
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x \mapsto_s \mathit{numeral}\ n) \dagger (P \parallel_M Q) = \sigma \dagger ((P[\![\mathit{numeral}\ n/\$x]\!]) \parallel_{M[\![\mathit{numeral}\ n/\$x_<]\!]} (Q[\![\mathit{numeral}\ n/\$x]\!]))$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x' \mapsto_s \mathit{numeral}\ n) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M[\![\mathit{numeral}\ n/\$x']\!]} Q)$
⟨*proof*⟩

27.8 Parallel-by-merge laws

lemma *par-by-merge-false* [*simp*]:
$P \parallel_{false} Q = false$
⟨*proof*⟩

lemma *par-by-merge-left-false* [*simp*]:
$false \parallel_M Q = false$
⟨*proof*⟩

lemma *par-by-merge-right-false* [*simp*]:
$P \parallel_M false = false$
⟨*proof*⟩

lemma *par-by-merge-seq-add*: $(P \parallel_M Q) ;; R = (P \parallel_{M\ ;;\ R} Q)$
⟨*proof*⟩

A skip parallel-by-merge yields a skip whenever the parallel predicates are both feasible.

lemma *par-by-merge-skip*:
assumes $P ;; true = true\ Q ;; true = true$
shows $P \parallel_{skip_m} Q = II$
⟨*proof*⟩

lemma *skip-merge-swap*: $swap_m ;; skip_m = skip_m$
⟨*proof*⟩

lemma *par-sep-swap*: $P \parallel_s Q ;; swap_m = Q \parallel_s P$
⟨*proof*⟩

Parallel-by-merge commutes when the merge predicate is unchanged by swap

lemma *par-by-merge-commute-swap*:
shows $P \parallel_M Q = Q \parallel_{swap_m\ ;;\ M} P$
⟨*proof*⟩

theorem *par-by-merge-commute*:
assumes *M is SymMerge*
shows $P \parallel_M Q = Q \parallel_M P$
⟨*proof*⟩

lemma *par-by-merge-mono-1*:
assumes $P_1 \sqsubseteq P_2$
shows $P_1 \parallel_M Q \sqsubseteq P_2 \parallel_M Q$

⟨*proof*⟩

lemma *par-by-merge-mono-2*:
assumes $Q_1 \sqsubseteq Q_2$
shows $(P \parallel_M Q_1) \sqsubseteq (P \parallel_M Q_2)$
⟨*proof*⟩

lemma *par-by-merge-mono*:
assumes $P_1 \sqsubseteq P_2$ $Q_1 \sqsubseteq Q_2$
shows $P_1 \parallel_M Q_1 \sqsubseteq P_2 \parallel_M Q_2$
⟨*proof*⟩

theorem *par-by-merge-assoc*:
assumes *M is SymMerge AssocMerge M*
shows $(P \parallel_M Q) \parallel_M R = P \parallel_M (Q \parallel_M R)$
⟨*proof*⟩

theorem *par-by-merge-choice-left*:
$(P \sqcap Q) \parallel_M R = (P \parallel_M R) \sqcap (Q \parallel_M R)$
⟨*proof*⟩

theorem *par-by-merge-choice-right*:
$P \parallel_M (Q \sqcap R) = (P \parallel_M Q) \sqcap (P \parallel_M R)$
⟨*proof*⟩

theorem *par-by-merge-or-left*:
$(P \vee Q) \parallel_M R = (P \parallel_M R \vee Q \parallel_M R)$
⟨*proof*⟩

theorem *par-by-merge-or-right*:
$P \parallel_M (Q \vee R) = (P \parallel_M Q \vee P \parallel_M R)$
⟨*proof*⟩

theorem *par-by-merge-USUP-mem-left*:
$(\bigsqcap i{\in}I \cdot P(i)) \parallel_M Q = (\bigsqcap i{\in}I \cdot P(i) \parallel_M Q)$
⟨*proof*⟩

theorem *par-by-merge-USUP-ind-left*:
$(\bigsqcap i \cdot P(i)) \parallel_M Q = (\bigsqcap i \cdot P(i) \parallel_M Q)$
⟨*proof*⟩

theorem *par-by-merge-USUP-mem-right*:
$P \parallel_M (\bigsqcap i{\in}I \cdot Q(i)) = (\bigsqcap i{\in}I \cdot P \parallel_M Q(i))$
⟨*proof*⟩

theorem *par-by-merge-USUP-ind-right*:
$P \parallel_M (\bigsqcap i \cdot Q(i)) = (\bigsqcap i \cdot P \parallel_M Q(i))$
⟨*proof*⟩

27.9 Example: Simple State-Space Division

The following merge predicate divides the state space using a pair of independent lenses.

definition *StateMerge* :: $('a \Longrightarrow 'a) \Rightarrow ('b \Longrightarrow 'a) \Rightarrow 'a$ *merge* $(\langle M[\text{-}|\text{-}]_\sigma \rangle)$ **where**
[*upred-defs*]: $M[a|b]_\sigma = (\$\mathbf{v}' =_u (\$\mathbf{v}_< \oplus \$0{-}\mathbf{v}$ *on* $\&a) \oplus \$1{-}\mathbf{v}$ *on* $\&b)$

lemma *swap-StateMerge*: $a \bowtie b \Longrightarrow (swap_m \mathbin{;;} M[a|b]_\sigma) = M[b|a]_\sigma$
⟨*proof*⟩

abbreviation *StateParallel* :: $'\alpha$ *hrel* ⇒ ($'a \Longrightarrow '\alpha$) ⇒ ($'b \Longrightarrow '\alpha$) ⇒ $'\alpha$ *hrel* ⇒ $'\alpha$ *hrel* (‹- |-|-|$_\sigma$ -› [85,0,0,86] 86)
where $P \ |a|b|_\sigma \ Q \equiv P \parallel_{M[a|b]_\sigma} Q$

lemma *StateParallel-commute*: $a \bowtie b \Longrightarrow P \ |a|b|_\sigma \ Q = Q \ |b|a|_\sigma \ P$
⟨*proof*⟩

lemma *StateParallel-form*:
$P \ |a|b|_\sigma \ Q = (\exists \ (st_0, \ st_1) \cdot P[\![«st_0»/\$\mathbf{v}´]\!] \wedge Q[\![«st_1»/\$\mathbf{v}´]\!] \wedge \$\mathbf{v}´ =_u (\$\mathbf{v} \oplus «st_0» \ on \ \&a) \oplus «st_1» \ on \ \&b)$
⟨*proof*⟩

lemma *StateParallel-form′*:
assumes *vwb-lens a vwb-lens b* $a \bowtie b$
shows $P \ |a|b|_\sigma \ Q = \{\&a,\&b\}:[(P \upharpoonright_v \{\$\mathbf{v},\$a´\}) \wedge (Q \upharpoonright_v \{\$\mathbf{v},\$b´\})]$
⟨*proof*⟩

We can frame all the variables that the parallel operator refers to

lemma *StateParallel-frame*:
assumes *vwb-lens a vwb-lens b* $a \bowtie b$
shows $\{\&a,\&b\}:[P \ |a|b|_\sigma \ Q] = P \ |a|b|_\sigma \ Q$
⟨*proof*⟩

Parallel Hoare logic rule. This employs something similar to separating conjunction in the postcondition, but we explicitly require that the two conjuncts only refer to variables on the left and right of the parallel composition explicitly.

theorem *StateParallel-hoare* [*hoare*]:
assumes $\{\!|c|\!\}P\{\!|d_1|\!\}_u \ \{\!|c|\!\}Q\{\!|d_2|\!\}_u \ a \bowtie b \ a \natural d_1 \ b \natural d_2$
shows $\{\!|c|\!\}P \ |a|b|_\sigma \ Q\{\!|d_1 \wedge d_2|\!\}_u$
⟨*proof*⟩

Specialised version of the above law where an invariant expression referring to variables outside the frame is preserved.

theorem *StateParallel-frame-hoare* [*hoare*]:
assumes *vwb-lens a vwb-lens b* $a \bowtie b \ a \natural d_1 \ b \natural d_2 \ a \sharp c_1 \ b \sharp c_1 \ \{\!|c_1 \wedge c_2|\!\}P\{\!|d_1|\!\}_u \ \{\!|c_1 \wedge c_2|\!\}Q\{\!|d_2|\!\}_u$
shows $\{\!|c_1 \wedge c_2|\!\}P \ |a|b|_\sigma \ Q\{\!|c_1 \wedge d_1 \wedge d_2|\!\}_u$
⟨*proof*⟩

end

28 Meta-theory for the Standard Core

theory *utp*
imports
utp-var
utp-expr
utp-expr-insts
utp-expr-funcs
utp-unrest
utp-usedby
utp-subst

utp-meta-subst
utp-alphabet
utp-lift
utp-pred
utp-pred-laws
utp-recursion
utp-dynlog
utp-rel
utp-rel-laws
utp-sequent
utp-state-parser
utp-sym-eval
utp-tactics
utp-hoare
utp-wp
utp-sp
utp-theory
utp-concurrency
utp-rel-opsem
begin end

29 Overloaded Expression Constructs

theory *utp-expr-ovld*
imports *utp*
begin

29.1 Overloadable Constants

For convenience, we often want to utilise the same expression syntax for multiple constructs. This can be achieved using ad-hoc overloading. We create a number of polymorphic constants and then overload their definitions using appropriate implementations. In order for this to work, each collection must have its own unique type. Thus we do not use the HOL map type directly, but rather our own partial function type, for example.

consts
— Empty elements, for example empty set, nil list, 0...
uempty :: $'f$
— Function application, map application, list application...
uapply :: $'f \Rightarrow 'k \Rightarrow 'v$
— Function update, map update, list update...
uupd :: $'f \Rightarrow 'k \Rightarrow 'v \Rightarrow 'f$
— Domain of maps, lists...
udom :: $'f \Rightarrow 'a\ set$
— Range of maps, lists...
uran :: $'f \Rightarrow 'b\ set$
— Domain restriction
udomres :: $'a\ set \Rightarrow 'f \Rightarrow 'f$
— Range restriction
uranres :: $'f \Rightarrow 'b\ set \Rightarrow 'f$
— Collection cardinality
ucard :: $'f \Rightarrow nat$
— Collection summation
usums :: $'f \Rightarrow 'a$
— Construct a collection from a list of entries

uentries :: *'k set* ⇒ (*'k* ⇒ *'v*) ⇒ *'f*

We need a function corresponding to function application in order to overload.

definition *fun-apply* :: (*'a* ⇒ *'b*) ⇒ (*'a* ⇒ *'b*)
where *fun-apply f x* = *f x*

declare *fun-apply-def* [*simp*]

definition *ffun-entries* :: *'k set* ⇒ (*'k* ⇒ *'v*) ⇒ (*'k*, *'v*) *ffun* **where**
ffun-entries d f = *graph-ffun* {(*k*, *f k*) | *k*. *k* ∈ *d*}

We then set up the overloading for a number of useful constructs for various collections.

adhoc-overloading
uempty ⇌ *0* **and**
uapply ⇌ *fun-apply* **and** *uapply* ⇌ *nth* **and** *uapply* ⇌ *pfun-app* **and**
uapply ⇌ *ffun-app* **and**
uupd ⇌ *pfun-upd* **and** *uupd* ⇌ *ffun-upd* **and** *uupd* ⇌ *list-augment* **and**
udom ⇌ *Domain* **and** *udom* ⇌ *pdom* **and** *udom* ⇌ *fdom* **and** *udom* ⇌ *seq-dom* **and**
udom ⇌ *Range* **and** *uran* ⇌ *pran* **and** *uran* ⇌ *fran* **and** *uran* ⇌ *set* **and**
udomres ⇌ *pdom-res* **and** *udomres* ⇌ *fdom-res* **and**
uranres ⇌ *pran-res* **and** *udomres* ⇌ *fran-res* **and**
ucard ⇌ *card* **and** *ucard* ⇌ *pcard* **and** *ucard* ⇌ *length* **and**
usums ⇌ *list-sum* **and** *usums* ⇌ *Sum* **and** *usums* ⇌ *pfun-sum* **and**
uentries ⇌ *pfun-entries* **and** *uentries* ⇌ *ffun-entries*

29.2 Syntax Translations

syntax
-uundef :: *logic* (‹$\perp_u$›)
-umap-empty :: *logic* (‹$[]_u$›)
-uapply :: (*'a* ⇒ *'b*, *'α*) *uexpr* ⇒ *utuple-args* ⇒ (*'b*, *'α*) *uexpr* (‹-'(-')$_a$› [*999*,*0*] *999*)
-umaplet :: [*logic*, *logic*] => *umaplet* (‹- /↦/ -›)
:: *umaplet* => *umaplets* (‹-›)
-UMaplets :: [*umaplet*, *umaplets*] => *umaplets* (‹-,/ -›)
-UMapUpd :: [*logic*, *umaplets*] => *logic* (‹-/'(-')$_u$› [*900*,*0*] *900*)
-UMap :: *umaplets* => *logic* (‹(*1*[-]$_u$)›)
-ucard :: *logic* ⇒ *logic* (‹#$_u$'(-')›)
-udom :: *logic* ⇒ *logic* (‹*dom*$_u$'(-')›)
-uran :: *logic* ⇒ *logic* (‹*ran*$_u$'(-')›)
-usum :: *logic* ⇒ *logic* (‹*sum*$_u$'(-')›)
-udom-res :: *logic* ⇒ *logic* ⇒ *logic* (**infixl** ‹◁$_u$› *85*)
-uran-res :: *logic* ⇒ *logic* ⇒ *logic* (**infixl** ‹▷$_u$› *85*)
-uentries :: *logic* ⇒ *logic* ⇒ *logic* (‹*entr*$_u$'(-,-')›)

translations
— Pretty printing for adhoc-overloaded constructs
f(*x*)$_a$ <= *CONST uapply f x*
dom$_u$(*f*) <= *CONST udom f*
ran$_u$(*f*) <= *CONST uran f*
A ◁$_u$ *f* <= *CONST udomres A f*
f ▷$_u$ *A* <= *CONST uranres f A*
#$_u$(*f*) <= *CONST ucard f*
f(*k* ↦ *v*)$_u$ <= *CONST uupd f k v*
0 <= *CONST uempty* — We have to do this so we don't see uempty. Is there a better way of printing?

— Overloaded construct translations
$f(x,y,z,u)_a$ == *CONST bop CONST uapply f* $(x,y,z,u)_u$
$f(x,y,z)_a$ == *CONST bop CONST uapply f* $(x,y,z)_u$
$f(x,y)_a$ == *CONST bop CONST uapply f* $(x,y)_u$
$f(x)_a$ == *CONST bop CONST uapply f x*
$\#_u(xs)$ == *CONST uop CONST ucard xs*
$sum_u(A)$ == *CONST uop CONST usums A*
$dom_u(f)$ == *CONST uop CONST udom f*
$ran_u(f)$ == *CONST uop CONST uran f*
$[]_u$ => «*CONST uempty*»
$\bot_u$ == «*CONST undefined*»
$A \lhd_u f$ == *CONST bop* (*CONST udomres*) *A f*
$f \rhd_u A$ == *CONST bop* (*CONST uranres*) *f A*
$entr_u(d,f)$ == *CONST bop CONST uentries d* «*f*»
-UMapUpd m (*-UMaplets xy ms*) == *-UMapUpd* (*-UMapUpd m xy*) *ms*
-UMapUpd m (*-umaplet x y*) == *CONST trop CONST uupd m x y*
-UMap ms == *-UMapUpd* $[]_u$ *ms*
-UMap (*-UMaplets ms1 ms2*) <= *-UMapUpd* (*-UMap ms1*) *ms2*
-UMaplets ms1 (*-UMaplets ms2 ms3*) <= *-UMaplets* (*-UMaplets ms1 ms2*) *ms3*

29.3 Simplifications

lemma *ufun-apply-lit* [*simp*]:
«*f*»(«*x*»)$_a$ = «*f*(*x*)»
⟨*proof*⟩

lemma *lit-plus-appl* [*lit-norm*]: «(+)»(*x*)$_a$(*y*)$_a$ = *x* + *y* ⟨*proof*⟩
lemma *lit-minus-appl* [*lit-norm*]: «(−)»(*x*)$_a$(*y*)$_a$ = *x* − *y* ⟨*proof*⟩
lemma *lit-mult-appl* [*lit-norm*]: «*times*»(*x*)$_a$(*y*)$_a$ = *x* ∗ *y* ⟨*proof*⟩
lemma *lit-divide-apply* [*lit-norm*]: «(/)»(*x*)$_a$(*y*)$_a$ = *x* / *y* ⟨*proof*⟩

lemma *pfun-entries-apply* [*simp*]:
(*entr*$_u$(*d*,*f*) :: (('*k*, '*v*) *pfun*, 'α) *uexpr*)(*i*)$_a$ = ((«*f*»(*i*)$_a$) ◁ *i* ∈$_u$ *d* ▷ $\bot_u$)
⟨*proof*⟩

lemma *udom-uupdate-pfun* [*simp*]:
fixes *m* :: (('*k*, '*v*) *pfun*, 'α) *uexpr*
shows *dom*$_u$(*m*(*k* ↦ *v*)$_u$) = {*k*}$_u$ ∪$_u$ *dom*$_u$(*m*)
⟨*proof*⟩

lemma *uapply-uupdate-pfun* [*simp*]:
fixes *m* :: (('*k*, '*v*) *pfun*, 'α) *uexpr*
shows (*m*(*k* ↦ *v*)$_u$)(*i*)$_a$ = *v* ◁ *i* =$_u$ *k* ▷ *m*(*i*)$_a$
⟨*proof*⟩

29.4 Indexed Assignment

syntax
— Indexed assignment
-assignment-upd :: *svid* ⇒ *uexp* ⇒ *uexp* ⇒ *logic* (‹(-[-] :=/ -)› [*63*, *0*, *0*] *62*)

translations
— Indexed assignment uses the overloaded collection update function *uupd*.
-assignment-upd x k v => *x* := &*x*(*k* ↦ *v*)$_u$

end

30 Meta-theory for the Standard Core with Overloaded Constructs

theory *utp-full*
imports *utp utp-expr-ovld*
begin end

31 UTP Easy Expression Parser

theory *utp-easy-parser*
imports *utp-full*
begin

31.1 Replacing the Expression Grammar

The following theory provides an easy to use expression parser that is primarily targetted towards expressing programs. Unlike the built-in UTP expression syntax, this uses a closed grammar separate to the HOL *logic* nonterminal, that gives more freedom in what can be expressed. In particular, identifiers are interpreted as UTP variables rather than HOL variables and functions do not require subscripts and other strange decorations.

The first step is to remove the from the UTP parse the following grammar rule that uses arbitrary HOL logic to represent expressions. Instead, we will populate the *uexp* grammar manually.

purge-syntax
-uexp-l :: *logic* ⇒ *uexp* (‹-› [*64*] *64*)

31.2 Expression Operators

syntax
-ue-quote :: *uexp* ⇒ *logic* (‹'(-')$_e$›)
-ue-tuple :: *uexprs* ⇒ *uexp* (‹'(-')›)
-ue-lit :: *logic* ⇒ *uexp* (‹«-»›)
-ue-var :: *svid* ⇒ *uexp* (‹-›)
-ue-eq :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infix** ‹=› *150*)
-ue-uop :: *id* ⇒ *uexp* ⇒ *uexp* (‹-'(-')› [*999*,*0*] *999*)
-ue-bop :: *id* ⇒ *uexp* ⇒ *uexp* ⇒ *uexp* (‹-'(-, -')› [*999*,*0*,*0*] *999*)
-ue-trop :: *id* ⇒ *uexp* ⇒ *uexp* ⇒ *uexp* ⇒ *uexp* (‹-'(-, -, -')› [*999*,*0*,*0*,*0*] *999*)
-ue-apply :: *uexp* ⇒ *uexp* ⇒ *uexp* (‹-[-]› [*999*] *999*)

translations
-ue-quote e => *e*
-ue-tuple (*-uexprs x* (*-uexprs y z*)) => *-ue-tuple* (*-uexprs x* (*-ue-tuple* (*-uexprs y z*)))
-ue-tuple (*-uexprs x y*) => *CONST bop CONST Pair x y*
-ue-tuple x => *x*
-ue-lit x => *CONST lit x*
-ue-var x => *CONST utp-expr.var* (*CONST pr-var x*)
-ue-eq x y => *x* =$_u$ *y*
-ue-uop f x => *CONST uop f x*
-ue-bop f x y => *CONST bop f x y*
-ue-trop f x y => *CONST trop f x y*
-ue-apply f x => *f*(*x*)$_a$

31.3 Predicate Operators

syntax

-ue-true :: *uexp* (‹*true*›)
-ue-false :: *uexp* (‹*false*›)
-ue-not :: *uexp* ⇒ *uexp* (‹¬ -› [*40*] *40*)
-ue-conj :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infixr** ‹∧› *135*)
-ue-disj :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infixr** ‹∨› *130*)
-ue-impl :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infixr** ‹⇒› *125*)
-ue-iff :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infixr** ‹⇒› *125*)
-ue-mem :: *uexp* ⇒ *uexp* ⇒ *uexp* (‹(-/ ∈ -)› [*151*, *151*] *150*)
-ue-nmem :: *uexp* ⇒ *uexp* ⇒ *uexp* (‹(-/ ∉ -)› [*151*, *151*] *150*)

translations

-ue-true => *CONST true-upred*
-ue-false => *CONST false-upred*
-ue-not p => *CONST not-upred p*
-ue-conj p q => $p \wedge_p q$
-ue-disj p q => $p \vee_p q$
-ue-impl p q => $p \Rightarrow q$
-ue-iff p q => $p \Leftrightarrow q$
-ue-mem x A => $x \in_u A$
-ue-nmem x A => $x \notin_u A$

31.4 Arithmetic Operators

syntax

-ue-num :: *num-const* ⇒ *uexp* (‹-›)
-ue-size :: *uexp* ⇒ *uexp* (‹#-› [*999*] *999*)
-ue-eq :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infix** ‹=› *150*)
-ue-le :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infix** ‹≤› *150*)
-ue-lt :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infix** ‹<› *150*)
-ue-ge :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infix** ‹≥› *150*)
-ue-gt :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infix** ‹>› *150*)
-ue-zero :: *uexp* (‹*0*›)
-ue-one :: *uexp* (‹*1*›)
-ue-plus :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infixl** ‹+› *165*)
-ue-uminus :: *uexp* ⇒ *uexp* (‹− -› [*181*] *180*)
-ue-minus :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infixl** ‹−› *165*)
-ue-times :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infixl** ‹∗› *170*)
-ue-div :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infixl** ‹*div*› *170*)

translations

-ue-num x => *-Numeral x*
-ue-size e => $\#_u(e)$
-ue-le x y => $x \leq_u y$
-ue-lt x y => $x <_u y$
-ue-ge x y => $x \geq_u y$
-ue-gt x y => $x >_u y$
-ue-zero => *0*
-ue-one => *1*
-ue-plus x y => $x + y$
-ue-uminus x => $- x$
-ue-minus x y => $x - y$
-ue-times x y => $x * y$
-ue-div x y => *CONST divide x y*

31.5 Sets

syntax
-ue-empset :: *uexp* (‹{}›)
-ue-setprod :: *uexp* ⇒ *uexp* ⇒ *uexp* (**infixr** ‹×› *80*)
-ue-atLeastAtMost :: *uexp* ⇒ *uexp* ⇒ *uexp* (‹(*1*{-..-})›)
-ue-atLeastLessThan :: *uexp* ⇒ *uexp* ⇒ *uexp* (‹(*1*{-..<-})›)

translations
-ue-empset => $\{\}_u$
-ue-setprod e f => *CONST bop* (*CONST Product-Type.Times*) *e f*
-ue-atLeastAtMost m n => $\{m..n\}_u$
-ue-atLeastLessThan m n => $\{m..<n\}_u$

31.6 Imperative Program Syntax

syntax
-ue-if-then :: *uexp* ⇒ *logic* ⇒ *logic* ⇒ *logic* (‹*if* - *then* - *else* - *fi*›)
-ue-hoare :: *uexp* ⇒ *logic* ⇒ *uexp* ⇒ *logic* (‹{{-}} / - / {{-}}›)
-ue-wp :: *logic* ⇒ *uexp* ⇒ *uexp* (**infix** ‹*wp*› *60*)

translations
-ue-if-then b P Q => $P \lhd b \rhd_r Q$
-ue-hoare b P c => $\{\!|b|\!\}P\{\!|c|\!\}_u$
-ue-wp P b => *P wp b*

end

32 Example: Summing a List

theory *sum-list*
imports *../utp-easy-parser*
begin

This theory exemplifies the use of the Isabelle/UTP Hoare logic verification component. We first create a state space with the variables the program needs.

alphabet *st-sum-list* =
i :: *nat*
xs :: *int list*
ans :: *int*

Next, we define the program as by a homogeneous relation over the state-space type.

abbreviation *Sum-List* :: *st-sum-list hrel* **where**
Sum-List ≡
i := *0* ;;
ans := *0* ;;
while (*i* < #*xs*) *invr* (*ans* = *list-sum*(*take*(*i*, *xs*)))
do
ans := *ans* + *xs*[*i*] ;;
i := *i* + *1*
od

Next, we symbolically evaluate some examples.

lemma *TRY*([&*xs* ↦$_s$ «[*4*,*3*,*7*,*1*,*12*,*8*]»] ⊨ *Sum-List*)
⟨*proof*⟩

Finally, we verify the program.

theorem *Sum-List-sums*:
{{$xs = «XS»$}} *Sum-List* {{$ans = list\text{-}sum(xs)$}}
⟨*proof*⟩

end

33 Simple UTP real-time theory

theory *utp-simple-time* **imports** *../utp* **begin**

In this section we give a small example UTP theory, and show how Isabelle/UTP can be used to automate production of programming laws.

33.1 Observation Space and Signature

We first declare the observation space for our theory of timed relations. It consists of two variables, to denote time and the program state, respectively.

alphabet *'s st-time* =
clock :: *nat* *st* :: *'s*

A timed relation is a homogeneous relation over the declared observation space.

type-synonym *'s time-rel* = *'s st-time hrel*

We introduce the following operator for adding an n-unit delay to a timed relation.

definition *Wait* :: *nat* ⇒ *'s time-rel* **where**
[*upred-defs*]: $Wait(n) = (\$clock' =_u \$clock + «n» \wedge \$st' =_u \$st)$

33.2 UTP Theory

We define a single healthiness condition which ensures that the clock monotonically advances, and so forbids reverse time travel.

definition *HT* :: *'s time-rel* ⇒ *'s time-rel* **where**
[*upred-defs*]: $HT(P) = (P \wedge \$clock \leq_u \$clock')$

This healthiness condition is idempotent, monotonic, and also continuous, meaning it distributes through arbitary non-empty infima.

theorem *HT-idem*: $HT(HT(P)) = HT(P)$ ⟨*proof*⟩

theorem *HT-mono*: $P \sqsubseteq Q \Longrightarrow HT(P) \sqsubseteq HT(Q)$ ⟨*proof*⟩

theorem *HT-continuous*: *Continuous HT* ⟨*proof*⟩

We now create the UTP theory object for timed relations. This is done using a local interpretation *utp-theory-continuous HT*. This raises the proof obligations that *HT* is both idempotent and continuous, which we have proved already. The result of this command is a collection of theorems that can be derived from these facts. Notably, we obtain a complete lattice of timed relations via the Knaster-Tarski theorem. We also apply some locale rewrites so that the theorems that are exports have a more intuitive form.

interpretation *time-theory*: *utp-theory-continuous HT*
rewrites $P \in carrier\ time\text{-}theory.thy\text{-}order \longleftrightarrow P\ is\ HT$

and *carrier time-theory.thy-order* $\to$ *carrier time-theory.thy-order* $\equiv [\![HT]\!]_H \to [\![HT]\!]_H$
and *le time-theory.thy-order* $= (\sqsubseteq)$
and *eq time-theory.thy-order* $= (=)$
⟨*proof*⟩

The object *time-theory* is a new namespace that contains both definitions and theorems. Since the theory forms a complete lattice, we obtain a top element, bottom element, and a least fixed-point constructor. We give all of these some intuitive syntax.

notation *time-theory.utp-top* $(\langle \top_t \rangle)$
notation *time-theory.utp-bottom* $(\langle \bot_t \rangle)$
notation *time-theory.utp-lfp* $(\langle \mu_t \rangle)$

Below is a selection of theorems that have been exported by the locale interpretation.

thm *time-theory.bottom-healthy*
thm *time-theory.top-higher*
thm *time-theory.meet-bottom*
thm *time-theory.LFP-unfold*

33.3 Closure Laws

HT applied to *Wait* has no affect, since the latter always advances time.

lemma *HT-Wait*: $HT(Wait(n)) = Wait(n)$ ⟨*proof*⟩

lemma *HT-Wait-closed* [*closure*]: *Wait*(*n*) *is HT*
⟨*proof*⟩

Relational identity, *II*, is likewise HT-healthy.

lemma *HT-skip-closed* [*closure*]: *II is HT*
⟨*proof*⟩

HT is closed under sequential composition, which can be shown by transitivity of $(\leq)$.

lemma *HT-seqr-closed* [*closure*]:
$[\![P \text{ is } HT;\ Q \text{ is } HT]\!] \Longrightarrow P ;; Q \text{ is } HT$
⟨*proof*⟩

Assignment is also healthy, provided that the clock variable is not assigned.

lemma *HT-assign-closed* [*closure*]: $[\![vwb\text{-}lens\ x;\ clock \bowtie x]\!] \Longrightarrow x := v \text{ is } HT$
⟨*proof*⟩

An alternative characterisation of the above is that x is within the state space lens.

lemma *HT-assign-closed′* [*closure*]: $[\![vwb\text{-}lens\ x;\ x \subseteq_L st]\!] \Longrightarrow x := v \text{ is } HT$
⟨*proof*⟩

33.4 Algebraic Laws

Finally, we prove some useful algebraic laws.

theorem *Wait-skip*: $Wait(0) = II$ ⟨*proof*⟩

theorem *Wait-Wait*: $Wait(m) ;; Wait(n) = Wait\ (m + n)$ ⟨*proof*⟩

theorem *Wait-cond*: $Wait(m) ;; (P \lhd b \rhd_r Q) = (Wait\ m ;; P) \lhd b[\![\&clock+\!\!\ll\!m\!\gg/\&clock]\!] \rhd_r (Wait\ m ;; Q)$

⟨*proof*⟩

end

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