The Twelvefold Way

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December 14, 2021

Abstract

This entry provides all cardinality theorems of the Twelvefold Way. The Twelvefold Way [1, 5, 6] systematically classifies twelve related combinatorial problems concerning two finite sets, which include counting permutations, combinations, multisets, set partitions and number partitions. This development builds upon the existing formal developments [2, 3, 4] with cardinality theorems for those structures. It provides twelve bijections from the various structures to different equivalence classes on finite functions, and hence, proves cardinality formulae for these equivalence classes on finite functions.

Contents

1 Preliminaries

theory *Preliminaries*
imports
Main
HOL−Library.Multiset
HOL−Library.FuncSet
HOL−Combinatorics.Permutations
HOL−ex.Birthday-Paradox
Card-Partitions.Card-Partitions
Bell-Numbers-Spivey.Bell-Numbers
Card-Multisets.Card-Multisets
Card-Number-Partitions.Card-Number-Partitions
begin

1.1 Additions to Finite Set Theory

lemma *subset-with-given-card-exists*:
assumes $n \leq card\ A$
shows $\exists B \subseteq A.\ card\ B = n$
⟨*proof*⟩

1.2 Additions to Equiv Relation Theory

lemmas *univ-commute′* = *univ-commute*[*unfolded Equiv-Relations.proj-def*]

lemma *univ-predicate-impl-forall*:
assumes $equiv\ A\ R$
assumes $P\ respects\ R$
assumes $X \in A\ //\ R$
assumes $univ\ P\ X$
shows $\forall x \in X.\ P\ x$
⟨*proof*⟩

lemma *univ-preserves-predicate*:
assumes $equiv\ A\ r$
assumes $P\ respects\ r$
shows $\{x \in A.\ P\ x\}\ //\ r = \{X \in A\ //\ r.\ univ\ P\ X\}$
⟨*proof*⟩

lemma *Union-quotient-restricted*:
  assumes *equiv A r*
  assumes *P respects r*
  shows $\bigcup(\{x \in A.\ P\ x\}\ //\ r) = \{x \in A.\ P\ x\}$
⟨*proof*⟩

lemma *finite-equiv-implies-finite-carrier*:
  assumes *equiv A R*
  assumes *finite* $(A\ //\ R)$
  assumes $\forall X \in A\ //\ R.$ *finite* X
  shows *finite A*
⟨*proof*⟩

lemma *finite-quotient-iff*:
  assumes *equiv A R*
  shows *finite* $A \longleftrightarrow$ (*finite* $(A\ //\ R) \wedge (\forall X \in A\ //\ R.$ *finite* X))
⟨*proof*⟩

1.2.1 Counting Sets by Splitting into Equivalence Classes

lemma *card-equiv-class-restricted*:
  assumes *finite* $\{x \in A.\ P\ x\}$
  assumes *equiv A R*
  assumes *P respects R*
  shows *card* $\{x \in A.\ P\ x\}$ = *sum card* $(\{x \in A.\ P\ x\}\ //\ R)$
⟨*proof*⟩

lemma *card-equiv-class-restricted-same-size*:
  assumes *equiv A R*
  assumes *P respects R*
  assumes $\bigwedge F.\ F \in \{x \in A.\ P\ x\}\ //\ R \Longrightarrow$ *card* $F = k$
  shows *card* $\{x \in A.\ P\ x\} = k\ *$ *card* $(\{x \in A.\ P\ x\}\ //\ R)$
⟨*proof*⟩

lemma *card-equiv-class*:
  assumes *finite A*
  assumes *equiv A R*
  shows *card A* = *sum card* $(A\ //\ R)$
⟨*proof*⟩

lemma *card-equiv-class-same-size*:
  assumes *equiv A R*
  assumes $\bigwedge F.\ F \in A\ //\ R \Longrightarrow$ *card* $F = k$
  shows *card* $A = k\ *$ *card* $(A\ //\ R)$
⟨*proof*⟩

1.3 Additions to FuncSet Theory

lemma *finite-same-card-bij-on-ext-funcset*:
  assumes *finite A finite B card A = card B*

shows $\exists f.\ f \in A \to_E B \wedge$ *bij-betw f A B*
⟨*proof*⟩

lemma *card-extensional-funcset*:
assumes *finite A*
shows *card* $(A \to_E B) =$ *card* B ^ *card* A
⟨*proof*⟩

lemma *bij-betw-implies-inj-on-and-card-eq*:
assumes *finite B*
assumes $f \in A \to_E B$
shows *bij-betw f A B* $\longleftrightarrow$ *inj-on f A* $\wedge$ *card A* $=$ *card B*
⟨*proof*⟩

lemma *bij-betw-implies-surj-on-and-card-eq*:
assumes *finite A*
assumes $f \in A \to_E B$
shows *bij-betw f A B* $\longleftrightarrow f$ ‘ $A = B \wedge$ *card A* $=$ *card B*
⟨*proof*⟩

1.4 Additions to Permutations Theory

lemma
assumes $f \in A \to_E B$ f ‘ $A = B$
assumes *p permutes B* $(\forall x.\ f'\ x = p\ (f\ x))$
shows $(\lambda b.\ \{x{\in}A.\ f\ x = b\})$ ‘ $B = (\lambda b.\ \{x{\in}A.\ f'\ x = b\})$ ‘ B
⟨*proof*⟩

1.5 Additions to List Theory

The theorem *card-lists-length-eq* contains the superfluous assumption *finite A*. Here, we derive that fact without that unnecessary assumption.

lemma *lists-length-eq-Suc-eq-image-Cons*:
$\{xs.\ set\ xs \subseteq A \wedge length\ xs = Suc\ n\} = (\lambda(x, xs).\ x\#xs)$ ‘ $(A \times \{xs.\ set\ xs \subseteq A \wedge length\ xs = n\})$
(**is** $?A = ?B$)
⟨*proof*⟩

lemma *lists-length-eq-Suc-eq-empty-iff*:
$\{xs.\ set\ xs \subseteq A \wedge length\ xs = Suc\ n\} = \{\} \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *lists-length-eq-eq-empty-iff*:
$\{xs.\ set\ xs \subseteq A \wedge length\ xs = n\} = \{\} \longleftrightarrow (A = \{\} \wedge n > 0)$
⟨*proof*⟩

lemma *finite-lists-length-eq-iff*:
finite $\{xs.\ set\ xs \subseteq A \wedge length\ xs = n\} \longleftrightarrow$ (*finite* $A \vee n = 0$)
⟨*proof*⟩

lemma *card-lists-length-eq*:
shows $card\ \{xs.\ set\ xs \subseteq B \wedge length\ xs = n\} = card\ B\ \widehat{}\ n$
⟨*proof*⟩

1.6 Additions to Disjoint Set Theory

lemma *bij-betw-congI*:
assumes $bij\text{-}betw\ f\ A\ A'$
assumes $\forall\, a \in A.\ f\ a = g\ a$
shows $bij\text{-}betw\ g\ A\ A'$
⟨*proof*⟩

lemma *disjoint-family-onI*[*intro*]:
assumes $\bigwedge m\ n.\ m \in S \Longrightarrow n \in S \Longrightarrow m \neq n \Longrightarrow A\ m \cap A\ n = \{\}$
shows $disjoint\text{-}family\text{-}on\ A\ S$
⟨*proof*⟩

The following lemma is not needed for this development, but is useful and could be moved to Disjoint Set theory or Equiv Relation theory if translated from set partitions to equivalence relations.

lemma *infinite-partition-on*:
assumes $infinite\ A$
shows $infinite\ \{P.\ partition\text{-}on\ A\ P\}$
⟨*proof*⟩

lemma *finitely-many-partition-on-iff*:
$finite\ \{P.\ partition\text{-}on\ A\ P\} \longleftrightarrow finite\ A$
⟨*proof*⟩

1.7 Additions to Multiset Theory

lemma *mset-set-subseteq-mset-set*:
assumes $finite\ B\ A \subseteq B$
shows $mset\text{-}set\ A \subseteq\#\ mset\text{-}set\ B$
⟨*proof*⟩

lemma *mset-set-set-mset*:
assumes $M \subseteq\#\ mset\text{-}set\ A$
shows $mset\text{-}set\ (set\text{-}mset\ M) = M$
⟨*proof*⟩

lemma *mset-set-set-mset'*:
assumes $\forall\, x.\ count\ M\ x \leq 1$
shows $mset\text{-}set\ (set\text{-}mset\ M) = M$
⟨*proof*⟩

lemma *card-set-mset*:
assumes $M \subseteq\#\ mset\text{-}set\ A$

shows *card* (*set-mset* M) = *size* M
⟨*proof*⟩

lemma *card-set-mset′*:
assumes $\forall x.$ *count* M $x \leq 1$
shows *card* (*set-mset* M) = *size* M
⟨*proof*⟩

lemma *count-mset-set-leq*:
assumes *finite* A
shows *count* (*mset-set* A) $x \leq 1$
⟨*proof*⟩

lemma *count-mset-set-leq′*:
assumes *finite* A
shows *count* (*mset-set* A) $x \leq$ *Suc 0*
⟨*proof*⟩

lemma *msubset-mset-set-iff*:
assumes *finite* A
shows *set-mset* $M \subseteq A \wedge (\forall x.$ *count* M $x \leq 1) \longleftrightarrow (M \subseteq\#$ *mset-set* $A)$
⟨*proof*⟩

lemma *image-mset-fun-upd*:
assumes $x \notin\# M$
shows *image-mset* $(f(x := y))$ M = *image-mset* f M
⟨*proof*⟩

1.8 Additions to Number Partitions Theory

lemma *Partition-diag*:
shows *Partition* n n = *1*
⟨*proof*⟩

1.9 Cardinality Theorems with Iverson Function

definition *iverson* :: *bool* ⇒ *nat*
where
iverson b = (*if* b *then 1 else 0*)

lemma *card-partition-on-size1-eq-iverson*:
assumes *finite* A
shows *card* $\{P.$ *partition-on* A $P \wedge$ *card* $P \leq k \wedge (\forall X \in P.$ *card* $X = 1)\}$ = *iverson* (*card* $A \leq k$)
⟨*proof*⟩

lemma *card-number-partitions-with-only-parts-1*:
card $\{N.$ $(\forall n.$ $n \in\# N \longrightarrow n = 1) \wedge$ *number-partition* n $N \wedge$ *size* $N \leq x\}$ = *iverson* $(n \leq x)$
⟨*proof*⟩

end

2 Main Observations on Operations and Permutations

theory *Twelvefold-Way-Core*
imports *Preliminaries*
begin

2.1 Range Multiset

2.1.1 Existence of a Suitable Finite Function

lemma *obtain-function*:
  assumes *finite A*
  assumes *size M* = *card A*
  shows $\exists f.$ *image-mset f* (*mset-set A*) = *M*
⟨*proof*⟩

lemma *obtain-function-on-ext-funcset*:
  assumes *finite A*
  assumes *size M* = *card A*
  shows $\exists f \in A \rightarrow_E$ *set-mset M*. *image-mset f* (*mset-set A*) = *M*
⟨*proof*⟩

2.1.2 Existence of Permutation

lemma *image-mset-eq-implies-bij-betw*:
  fixes f :: $'a1 \Rightarrow 'b$ **and** f' :: $'a2 \Rightarrow 'b$
  assumes *finite A finite A'*
  assumes *mset-eq*: *image-mset f* (*mset-set A*) = *image-mset f'* (*mset-set A'*)
  obtains *bij* **where** *bij-betw bij A A'* **and** $\forall x \in A.\ f\ x = f'\ (bij\ x)$
⟨*proof*⟩

lemma *image-mset-eq-implies-permutes*:
  fixes f :: $'a \Rightarrow 'b$
  assumes *finite A*
  assumes *mset-eq*: *image-mset f* (*mset-set A*) = *image-mset f'* (*mset-set A*)
  obtains *p* **where** *p permutes A* **and** $\forall x \in A.\ f\ x = f'\ (p\ x)$
⟨*proof*⟩

2.2 Domain Partition

2.2.1 Existence of a Suitable Finite Function

lemma *obtain-function-with-partition*:
  assumes *finite A finite B*
  assumes *partition-on A P*

assumes $\mathit{card}\ P \leq \mathit{card}\ B$
shows $\exists f \in A \rightarrow_E B.\ (\lambda b.\ \{x \in A.\ f\ x = b\})\ `\ B - \{\{\}\} = P$
$\langle \mathit{proof} \rangle$

2.2.2 Equality under Permutation Application

lemma *permutes-implies-inv-image-on-eq*:
assumes $p\ \mathit{permutes}\ B$
shows $(\lambda b.\ \{x \in A.\ p\ (f\ x) = b\})\ `\ B = (\lambda b.\ \{x \in A.\ f\ x = b\})\ `\ B$
$\langle \mathit{proof} \rangle$

2.2.3 Existence of Permutation

lemma *the-elem*:
assumes $f \in A \rightarrow_E B\ f' \in A \rightarrow_E B$
assumes *partitions-eq*: $(\lambda b.\ \{x \in A.\ f\ x = b\})\ `\ B - \{\{\}\} = (\lambda b.\ \{x \in A.\ f'\ x = b\})\ `\ B - \{\{\}\}$
assumes $x \in A$
shows $\mathit{the\text{-}elem}\ (f\ `\ \{xa \in A.\ f'\ xa = f'\ x\}) = f\ x$
$\langle \mathit{proof} \rangle$

lemma *the-elem-eq*:
assumes $f \in A \rightarrow_E B$
assumes $b \in f\ `\ A$
shows $\mathit{the\text{-}elem}\ (f\ `\ \{x' \in A.\ f\ x' = b\}) = b$
$\langle \mathit{proof} \rangle$

lemma *partitions-eq-implies*:
assumes $f \in A \rightarrow_E B\ f' \in A \rightarrow_E B$
assumes *partitions-eq*: $(\lambda b.\ \{x \in A.\ f\ x = b\})\ `\ B - \{\{\}\} = (\lambda b.\ \{x \in A.\ f'\ x = b\})\ `\ B - \{\{\}\}$
assumes $x \in A\ x' \in A$
assumes $f\ x = f\ x'$
shows $f'\ x = f'\ x'$
$\langle \mathit{proof} \rangle$

lemma *card-domain-partitions*:
assumes $f \in A \rightarrow_E B$
assumes *finite* B
shows $\mathit{card}\ ((\lambda b.\ \{x \in A.\ f\ x = b\})\ `\ B - \{\{\}\}) = \mathit{card}\ (f\ `\ A)$
$\langle \mathit{proof} \rangle$

lemma *partitions-eq-implies-permutes*:
assumes $f \in A \rightarrow_E B\ f' \in A \rightarrow_E B$
assumes *finite* B
assumes *partitions-eq*: $(\lambda b.\ \{x \in A.\ f\ x = b\})\ `\ B - \{\{\}\} = (\lambda b.\ \{x \in A.\ f'\ x = b\})\ `\ B - \{\{\}\}$
shows $\exists p.\ p\ \mathit{permutes}\ B \wedge (\forall x{\in}A.\ f\ x = p\ (f'\ x))$
$\langle \mathit{proof} \rangle$

2.3 Number Partition of Range

2.3.1 Existence of a Suitable Finite Function

lemma *obtain-partition*:
  assumes *finite A*
  assumes *number-partition* (*card A*) *N*
  shows $\exists P$. *partition-on A P* $\wedge$ *image-mset card* (*mset-set P*) = *N*
⟨*proof*⟩

lemma *obtain-extensional-function-from-number-partition*:
  assumes *finite A finite B*
  assumes *number-partition* (*card A*) *N*
  assumes *size N* $\leq$ *card B*
  shows $\exists f \in A \rightarrow_E B$. *image-mset* ($\lambda X$. *card X*) (*mset-set* (((λb. $\{x \in A.\ f\ x = b\}$)) ‘ $B - \{\{\}\}$)) = *N*
⟨*proof*⟩

2.3.2 Equality under Permutation Application

lemma *permutes-implies-multiset-of-partition-cards-eq*:
  assumes p_A *permutes A* p_B *permutes B*
  shows *image-mset card* (*mset-set* ((λb. $\{x \in A.\ p_B\ (f'\ (p_A\ x)) = b\}$) ‘ $B - \{\{\}\}$)) = *image-mset card* (*mset-set* ((λb. $\{x \in A.\ f'\ x = b\}$) ‘ $B - \{\{\}\}$))
⟨*proof*⟩

2.3.3 Existence of Permutation

lemma *partition-implies-permutes*:
  assumes *finite A*
  assumes *partition-on A P partition-on A P'*
  assumes *image-mset card* (*mset-set P'*) = *image-mset card* (*mset-set P*)
  obtains *p* **where** *p permutes A* $P' = (\lambda X.\ p$ ‘ $X)$ ‘ P
⟨*proof*⟩

lemma *permutes-domain-partition-eq*:
  assumes $f \in A \rightarrow B$
  assumes p_A *permutes A*
  assumes $b \in B$
  shows p_A ‘ $\{x \in A.\ f\ x = b\} = \{x \in A.\ f\ (inv\ p_A\ x) = b\}$
⟨*proof*⟩

lemma *image-domain-partition-eq*:
  assumes $f \in A \rightarrow_E B$
  assumes p_A *permutes A*
  shows (λX. p_A ‘ X) ‘ ((λb. $\{x \in A.\ f\ x = b\}$) ‘ B) = (λb. $\{x \in A.\ f\ (inv\ p_A\ x) = b\}$) ‘ B
⟨*proof*⟩

lemma *multiset-of-partition-cards-eq-implies-permutes*:

assumes *finite* A *finite* B $f \in A \to_E B$ $f' \in A \to_E B$
assumes *eq*: *image-mset card* (*mset-set* ((λb. $\{x \in A.\ f\ x = b\}$) ‘ $B - \{\{\}\}$)) = *image-mset card* (*mset-set* ((λb. $\{x \in A.\ f'\ x = b\}$) ‘ $B - \{\{\}\}$))
obtains p_A p_B **where** p_A *permutes* A p_B *permutes* B $\forall x{\in}A.\ f\ x = p_B\ (f'\ (p_A\ x))$
⟨*proof*⟩

2.4 Bijections on Same Domain and Range

2.4.1 Existence of Domain Permutation

lemma *obtain-domain-permutation-for-two-bijections*:
assumes *bij-betw* f A B *bij-betw* f' A B
obtains p **where** p *permutes* A **and** $\forall a{\in}A.\ f\ a = f'\ (p\ a)$
⟨*proof*⟩

2.4.2 Existence of Range Permutation

lemma *obtain-range-permutation-for-two-bijections*:
assumes *bij-betw* f A B *bij-betw* f' A B
obtains p **where** p *permutes* B **and** $\forall a{\in}A.\ f\ a = p\ (f'\ a)$
⟨*proof*⟩

end

3 Definition of Equivalence Classes

theory *Equiv-Relations-on-Functions*
imports
Preliminaries
Twelvefold-Way-Core
begin

3.1 Permutation on the Domain

definition *domain-permutation*
where
domain-permutation A B = $\{(f, f') \in (A \to_E B) \times (A \to_E B).\ \exists p.\ p$ *permutes* $A \wedge (\forall x \in A.\ f\ x = f'\ (p\ x))\}$

lemma *equiv-domain-permutation*:
equiv ($A \to_E B$) (*domain-permutation* A B)
⟨*proof*⟩

3.1.1 Respecting Functions

lemma *inj-on-respects-domain-permutation*:
(λf. *inj-on* f A) *respects* *domain-permutation* A B
⟨*proof*⟩

lemma *image-respects-domain-permutation*:
  $(\lambda f.\ f\ \text{`}\ A)$ *respects* (*domain-permutation* $A\ B$)
⟨*proof*⟩

lemma *surjective-respects-domain-permutation*:
  $(\lambda f.\ f\ \text{`}\ A = B)$ *respects domain-permutation* $A\ B$
⟨*proof*⟩

lemma *bij-betw-respects-domain-permutation*:
  $(\lambda f.$ *bij-betw* $f\ A\ B)$ *respects domain-permutation* $A\ B$
⟨*proof*⟩

lemma *image-mset-respects-domain-permutation*:
  shows $(\lambda f.$ *image-mset* f (*mset-set* A)) *respects* (*domain-permutation* $A\ B$)
⟨*proof*⟩

3.2 Permutation on the Range

definition *range-permutation*
where
  range-permutation $A\ B = \{(f, f') \in (A \to_E B) \times (A \to_E B).\ \exists p.\ p$ *permutes* B $\wedge (\forall x \in A.\ f\ x = p\ (f'\ x))\}$

lemma *equiv-range-permutation*:
  equiv $(A \to_E B)$ (*range-permutation* $A\ B$)
⟨*proof*⟩

3.2.1 Respecting Functions

lemma *inj-on-respects-range-permutation*:
  $(\lambda f.$ *inj-on* $f\ A)$ *respects range-permutation* $A\ B$
⟨*proof*⟩

lemma *surj-on-respects-range-permutation*:
  $(\lambda f.\ f\ \text{`}\ A = B)$ *respects range-permutation* $A\ B$
⟨*proof*⟩

lemma *bij-betw-respects-range-permutation*:
  $(\lambda f.$ *bij-betw* $f\ A\ B)$ *respects range-permutation* $A\ B$
⟨*proof*⟩

lemma *domain-partitions-respects-range-permutation*:
  $(\lambda f.\ (\lambda b.\ \{x \in A.\ f\ x = b\})\ \text{`}\ B - \{\{\}\})$ *respects range-permutation* $A\ B$
⟨*proof*⟩

3.3 Permutation on the Domain and the Range

definition *domain-and-range-permutation*
where
  domain-and-range-permutation $A\ B = \{(f, f') \in (A \to_E B) \times (A \to_E B).$

$\exists\, p_A\ p_B.\ p_A\ \mathit{permutes}\ A \wedge p_B\ \mathit{permutes}\ B \wedge (\forall\, x \in A.\ f\, x = p_B\ (f'\ (p_A\ x)))\}$

lemma *equiv-domain-and-range-permutation*:
$\mathit{equiv}\ (A \rightarrow_E B)\ (\mathit{domain\text{-}and\text{-}range\text{-}permutation}\ A\ B)$
⟨*proof*⟩

3.3.1 Respecting Functions

lemma *inj-on-respects-domain-and-range-permutation*:
$(\lambda f.\ \mathit{inj\text{-}on}\ f\ A)\ \mathit{respects}\ \mathit{domain\text{-}and\text{-}range\text{-}permutation}\ A\ B$
⟨*proof*⟩

lemma *surjective-respects-domain-and-range-permutation*:
$(\lambda f.\ f\ `\ A = B)\ \mathit{respects}\ \mathit{domain\text{-}and\text{-}range\text{-}permutation}\ A\ B$
⟨*proof*⟩

lemma *bij-betw-respects-domain-and-range-permutation*:
$(\lambda f.\ \mathit{bij\text{-}betw}\ f\ A\ B)\ \mathit{respects}\ \mathit{domain\text{-}and\text{-}range\text{-}permutation}\ A\ B$
⟨*proof*⟩

lemma *count-image-mset'*:
$\mathit{count}\ (\mathit{image\text{-}mset}\ f\ A)\ x = \mathit{sum}\ (\mathit{count}\ A)\ \{x' \in \mathit{set\text{-}mset}\ A.\ f\, x' = x\}$
⟨*proof*⟩

lemma *multiset-of-partition-cards-respects-domain-and-range-permutation*:
assumes *finite B*
shows $(\lambda f.\ \mathit{image\text{-}mset}\ (\lambda X.\ \mathit{card}\ X)\ (\mathit{mset\text{-}set}\ (((\lambda b.\ \{x \in A.\ f\, x = b\}))\ `\ B - \{\{\}\})))\ \mathit{respects}\ \mathit{domain\text{-}and\text{-}range\text{-}permutation}\ A\ B$
⟨*proof*⟩

end

4 Functions from A to B

theory *Twelvefold-Way-Entry1*
imports *Preliminaries*
begin

Note that the cardinality theorems of both structures, lists and finite functions, are already available. Hence, this development creates the bijection between those two structures and transfers the one cardinality theorem to the other structures and vice versa, although not strictly needed as both cardinality theorems were already available.

4.1 Definition of Bijections

definition *sequence-of* :: $'a\ \mathit{set} \Rightarrow (\mathit{nat} \Rightarrow {}'a) \Rightarrow ({}'a \Rightarrow {}'b) \Rightarrow {}'b\ \mathit{list}$
where

$\textit{sequence-of } A \textit{ enum } f = \textit{map } (\lambda n.\ f \ (\textit{enum } n)) \ [0..<\textit{card } A]$

definition $\textit{function-of} :: 'a \textit{ set} \Rightarrow (\textit{nat} \Rightarrow 'a) \Rightarrow 'b \textit{ list} \Rightarrow ('a \Rightarrow 'b)$
where
$\textit{function-of } A \textit{ enum } xs = (\lambda a.\ \textit{if } a \in A \textit{ then } xs \ ! \ \textit{inv-into } \{0..<\textit{length } xs\} \textit{ enum } a \textit{ else undefined})$

4.2 Properties for Bijections

lemma *nth-sequence-of*:
assumes $i < \textit{card } A$
shows $(\textit{sequence-of } A \textit{ enum } f) \ ! \ i = f \ (\textit{enum } i)$
⟨*proof*⟩

lemma *nth-sequence-of-inv-into*:
assumes $\textit{bij-betw enum } \{0..<\textit{card } A\} \ A$
assumes $a \in A$
shows $(\textit{sequence-of } A \textit{ enum } f) \ ! \ (\textit{inv-into } \{0..<\textit{card } A\} \textit{ enum } a) = f \ a$
⟨*proof*⟩

lemma *set-sequence-of*:
assumes $\textit{bij-betw enum } \{0..<\textit{card } A\} \ A$
assumes $f \in A \rightarrow_E B$
shows $\textit{set } (\textit{sequence-of } A \textit{ enum } f) \subseteq B$
⟨*proof*⟩

lemma *length-sequence-of*:
assumes $\textit{bij-betw enum } \{0..<\textit{card } A\} \ A$
assumes $f \in A \rightarrow_E B$
shows $\textit{length } (\textit{sequence-of } A \textit{ enum } f) = \textit{card } A$
⟨*proof*⟩

lemma *function-of-enum*:
assumes $\textit{bij-betw enum } \{0..<\textit{card } A\} \ A$
assumes $\textit{length } xs = \textit{card } A$
assumes $i < \textit{card } A$
shows $\textit{function-of } A \textit{ enum } xs \ (\textit{enum } i) = xs \ ! \ i$
⟨*proof*⟩

lemma *function-of-in-extensional-funcset*:
assumes $\textit{bij-betw enum } \{0..<\textit{card } A\} \ A$
assumes $\textit{set } xs \subseteq B \ \textit{length } xs = \textit{card } A$
shows $\textit{function-of } A \textit{ enum } xs \in A \rightarrow_E B$
⟨*proof*⟩

lemma *sequence-of-function-of*:
assumes $\textit{bij-betw enum } \{0..<\textit{card } A\} \ A$
assumes $\textit{set } xs \subseteq B \ \textit{length } xs = \textit{card } A$
shows $\textit{sequence-of } A \textit{ enum } (\textit{function-of } A \textit{ enum } xs) = xs$

$\langle proof \rangle$

lemma *function-of-sequence-of*:
  assumes *bij-betw enum* $\{0..<card\ A\}$ *A*
  assumes $f \in A \rightarrow_E B$
  shows *function-of A enum* (*sequence-of A enum f*) = *f*
$\langle proof \rangle$

4.3 Bijections

lemma *bij-betw-sequence-of*:
  assumes *bij-betw enum* $\{0..<card\ A\}$ *A*
  shows *bij-betw* (*sequence-of A enum*) $(A \rightarrow_E B)$ $\{xs.\ set\ xs \subseteq B \wedge length\ xs = card\ A\}$
$\langle proof \rangle$

lemma *bij-betw-function-of*:
  assumes *bij-betw enum* $\{0..<card\ A\}$ *A*
  shows *bij-betw* (*function-of A enum*) $\{xs.\ set\ xs \subseteq B \wedge length\ xs = card\ A\}$ $(A \rightarrow_E B)$
$\langle proof \rangle$

4.4 Cardinality

lemma
  assumes *finite A*
  shows $card\ (A \rightarrow_E B) = card\ B\ \widehat{}\ card\ A$
$\langle proof \rangle$

lemma *card-sequences*:
  assumes *finite A*
  shows $card\ \{xs.\ set\ xs \subseteq B \wedge length\ xs = card\ A\} = card\ B\ \widehat{}\ card\ A$
$\langle proof \rangle$

lemma
  shows $card\ \{xs.\ set\ xs \subseteq A \wedge length\ xs = n\} = card\ A\ \widehat{}\ n$
$\langle proof \rangle$

end

5 Injections from A to B

theory *Twelvefold-Way-Entry2*
imports *Twelvefold-Way-Entry1*
begin

Note that the cardinality theorems of both structures, distinct lists and finite injective functions, are already available. Hence, this development creates the bijection between those two structures and transfers the one cardinality theorem to the other structures and vice versa, although not strictly needed as both cardinality theorems were already available.

5.1 Properties for Bijections

lemma *inj-on-implies-distinct*:
  assumes *bij-betw enum* $\{0..<$*card A*$\}$ *A*
  assumes $f \in A \rightarrow_E B$
  assumes *inj-on f A*
  shows *distinct* (*sequence-of A enum f*)
$\langle$*proof*$\rangle$

lemma *distinct-implies-inj-on*:
  assumes *bij-betw enum* $\{0..<$*card A*$\}$ *A*
  assumes *length xs* = *card A*
  assumes *distinct xs*
  shows *inj-on* (*function-of A enum xs*) *A*
$\langle$*proof*$\rangle$

lemma *image-sequence-of-inj*:
  assumes *bij-betw enum* $\{0..<$*card A*$\}$ *A*
  shows *sequence-of A enum* ‘ $\{f \in A \rightarrow_E B.$ *inj-on f A*$\} \subseteq \{$*xs. set xs* $\subseteq B \wedge$ *length xs* = *card A* $\wedge$ *distinct xs*$\}$
$\langle$*proof*$\rangle$

lemma *image-function-of-distinct*:
  assumes *bij-betw enum* $\{0..<$*card A*$\}$ *A*
  shows *function-of A enum* ‘ $\{$*xs. set xs* $\subseteq B \wedge$ *length xs* = *card A* $\wedge$ *distinct xs*$\}$ $\subseteq \{f \in A \rightarrow_E B.$ *inj-on f A*$\}$
$\langle$*proof*$\rangle$

5.2 Bijections

lemma *bij-betw-sequence-of*:
  assumes *bij-betw enum* $\{0..<$*card A*$\}$ *A*
  shows *bij-betw* (*sequence-of A enum*) $\{f.\ f \in A \rightarrow_E B \wedge$ *inj-on f A*$\}$ $\{$*xs. set xs* $\subseteq B \wedge$ *length xs* = *card A* $\wedge$ *distinct xs*$\}$
$\langle$*proof*$\rangle$

lemma *bij-betw-function-of*:
  assumes *bij-betw enum* $\{0..<$*card A*$\}$ *A*
  shows *bij-betw* (*function-of A enum*) $\{$*xs. set xs* $\subseteq B \wedge$ *length xs* = *card A* $\wedge$ *distinct xs*$\}$ $\{f \in A \rightarrow_E B.$ *inj-on f A*$\}$
$\langle$*proof*$\rangle$

5.3 Cardinality

lemma
  assumes *finite A finite B card A* $\leq$ *card B*

shows *card* $\{f \in A \rightarrow_E B.\ \textit{inj-on}\ f\ A\} = \prod\{\textit{card}\ B - \textit{card}\ A + 1..\textit{card}\ B\}$
⟨*proof*⟩

lemma *card-sequences*:
assumes *finite A finite B card* $A \leq$ *card B*
shows *card* $\{xs.\ \textit{set}\ xs \subseteq B \wedge \textit{length}\ xs = \textit{card}\ A \wedge \textit{distinct}\ xs\} = \textit{fact}\ (\textit{card}\ B)$ $\textit{div fact}\ (\textit{card}\ B - \textit{card}\ A)$
⟨*proof*⟩

end

6 Functions from A to B, up to a Permutation of A

theory *Twelvefold-Way-Entry4*
imports *Equiv-Relations-on-Functions*
begin

6.1 Definition of Bijections

definition *msubset-of* :: $'a\ \textit{set} \Rightarrow ('a \Rightarrow 'b)\ \textit{set} \Rightarrow 'b\ \textit{multiset}$
where
msubset-of $A\ F = \textit{univ}\ (\lambda f.\ \textit{image-mset}\ f\ (\textit{mset-set}\ A))\ F$

definition *functions-of* :: $'a\ \textit{set} \Rightarrow 'b\ \textit{multiset} \Rightarrow ('a \Rightarrow 'b)\ \textit{set}$
where
functions-of $A\ B = \{f \in A \rightarrow_E \textit{set-mset}\ B.\ \textit{image-mset}\ f\ (\textit{mset-set}\ A) = B\}$

6.2 Properties for Bijections

lemma *msubset-of*:
assumes $F \in (A \rightarrow_E B)\ //\ \textit{domain-permutation}\ A\ B$
shows *size* $(\textit{msubset-of}\ A\ F) = \textit{card}\ A$
and *set-mset* $(\textit{msubset-of}\ A\ F) \subseteq B$
⟨*proof*⟩

lemma *functions-of*:
assumes *finite A*
assumes *set-mset* $M \subseteq B$
assumes *size* $M = \textit{card}\ A$
shows *functions-of* $A\ M \in (A \rightarrow_E B)\ //\ \textit{domain-permutation}\ A\ B$
⟨*proof*⟩

lemma *functions-of-msubset-of*:
assumes *finite A*
assumes $F \in (A \rightarrow_E B)\ //\ \textit{domain-permutation}\ A\ B$
shows *functions-of* $A\ (\textit{msubset-of}\ A\ F) = F$
⟨*proof*⟩

lemma *msubset-of-functions-of*:
assumes *set-mset* $M \subseteq B$ *size* $M =$ *card* A *finite* A
shows *msubset-of* A (*functions-of* A M) $= M$
⟨*proof*⟩

6.3 Bijections

lemma *bij-betw-msubset-of*:
assumes *finite* A
shows *bij-betw* (*msubset-of* A) (($A \rightarrow_E B$) // *domain-permutation* A B) $\{M.$ *set-mset* $M \subseteq B \wedge$ *size* $M =$ *card* $A\}$
⟨*proof*⟩

6.4 Cardinality

lemma
assumes *finite* A *finite* B
shows *card* (($A \rightarrow_E B$) // *domain-permutation* A B) $=$ *card* B $+$ *card* A $-$ *1* *choose card* A
⟨*proof*⟩

end

7 Injections from A to B up to a Permutation of A

theory *Twelvefold-Way-Entry5*
imports
Equiv-Relations-on-Functions
begin

7.1 Definition of Bijections

definition *subset-of* :: $'a$ *set* $\Rightarrow$ ($'a \Rightarrow 'b$) *set* $\Rightarrow$ $'b$ *set*
where
subset-of A F $=$ *univ* ($\lambda f.\ f$ ‘ A) F

definition *functions-of* :: $'a$ *set* $\Rightarrow$ $'b$ *set* $\Rightarrow$ ($'a \Rightarrow 'b$) *set*
where
functions-of A $B = \{f \in A \rightarrow_E B.\ f$ ‘ $A = B\}$

7.2 Properties for Bijections

lemma *functions-of-eq*:
assumes *finite* A
assumes $f \in \{f \in A \rightarrow_E B.$ *inj-on* f $A\}$
shows *functions-of* A (f ‘ A) $=$ *domain-permutation* A B ‘‘ $\{f\}$
⟨*proof*⟩

lemma *subset-of*:
  assumes $F \in \{f \in A \to_E B.\ \textit{inj-on}\ f\ A\}\ //\ \textit{domain-permutation}\ A\ B$
  shows $\textit{subset-of}\ A\ F \subseteq B$ **and** $\textit{card}\ (\textit{subset-of}\ A\ F) = \textit{card}\ A$
⟨*proof*⟩

lemma *functions-of*:
  assumes $\textit{finite}\ A\ \textit{finite}\ B\ X \subseteq B\ \textit{card}\ X = \textit{card}\ A$
  shows $\textit{functions-of}\ A\ X \in \{f \in A \to_E B.\ \textit{inj-on}\ f\ A\}\ //\ \textit{domain-permutation}\ A\ B$
⟨*proof*⟩

lemma *subset-of-functions-of*:
  assumes $\textit{finite}\ A\ \textit{finite}\ X\ \textit{card}\ A = \textit{card}\ X$
  shows $\textit{subset-of}\ A\ (\textit{functions-of}\ A\ X) = X$
⟨*proof*⟩

lemma *functions-of-subset-of*:
  assumes $\textit{finite}\ A$
  assumes $F \in \{f \in A \to_E B.\ \textit{inj-on}\ f\ A\}\ //\ \textit{domain-permutation}\ A\ B$
  shows $\textit{functions-of}\ A\ (\textit{subset-of}\ A\ F) = F$
⟨*proof*⟩

7.3 Bijections

lemma *bij-betw-subset-of*:
  assumes $\textit{finite}\ A\ \textit{finite}\ B$
  shows $\textit{bij-betw}\ (\textit{subset-of}\ A)\ (\{f \in A \to_E B.\ \textit{inj-on}\ f\ A\}\ //\ \textit{domain-permutation}\ A\ B)\ \{X.\ X \subseteq B \wedge \textit{card}\ X = \textit{card}\ A\}$
⟨*proof*⟩

lemma *bij-betw-functions-of*:
  assumes $\textit{finite}\ A\ \textit{finite}\ B$
  shows $\textit{bij-betw}\ (\textit{functions-of}\ A)\ \{X.\ X \subseteq B \wedge \textit{card}\ X = \textit{card}\ A\}\ (\{f \in A \to_E B.\ \textit{inj-on}\ f\ A\}\ //\ \textit{domain-permutation}\ A\ B)$
⟨*proof*⟩

lemma *bij-betw-mset-set*:
  shows $\textit{bij-betw}\ \textit{mset-set}\ \{A.\ \textit{finite}\ A\}\ \{M.\ \forall x.\ \textit{count}\ M\ x \leq 1\}$
⟨*proof*⟩

lemma *bij-betw-mset-set-card*:
  assumes $\textit{finite}\ A$
  shows $\textit{bij-betw}\ \textit{mset-set}\ \{X.\ X \subseteq A \wedge \textit{card}\ X = k\}\ \{M.\ M \subseteq\#\ \textit{mset-set}\ A \wedge \textit{size}\ M = k\}$
⟨*proof*⟩

lemma *bij-betw-mset-set-card′*:
  assumes $\textit{finite}\ A$
  shows $\textit{bij-betw}\ \textit{mset-set}\ \{X.\ X \subseteq A \wedge \textit{card}\ X = k\}\ \{M.\ \textit{set-mset}\ M \subseteq A \wedge \textit{size}$

$M = k \wedge (\forall x.\ count\ M\ x \leq 1)\}$
⟨*proof*⟩

7.4 Cardinality

lemma *card-injective-functions-domain-permutation*:
  assumes *finite A finite B*
  shows $card\ (\{f \in A \rightarrow_E B.\ inj\text{-}on\ f\ A\}\ //\ domain\text{-}permutation\ A\ B) = card\ B$ $choose\ card\ A$
⟨*proof*⟩

lemma *card-multiset-only-sets*:
  assumes *finite A*
  shows $card\ \{M.\ M \subseteq\#\ mset\text{-}set\ A \wedge size\ M = k\} = card\ A\ choose\ k$
⟨*proof*⟩

lemma *card-multiset-only-sets'*:
  assumes *finite A*
  shows $card\ \{M.\ set\text{-}mset\ M \subseteq A \wedge size\ M = k \wedge (\forall x.\ count\ M\ x \leq 1)\} = card$ $A\ choose\ k$
⟨*proof*⟩

end

8 Surjections from A to B up to a Permutation on A

theory *Twelvefold-Way-Entry6*
imports *Twelvefold-Way-Entry4*
begin

8.1 Properties for Bijections

lemma *set-mset-eq-implies-surj-on*:
  assumes *finite A*
  assumes $size\ M = card\ A\ set\text{-}mset\ M = B$
  assumes $f \in functions\text{-}of\ A\ M$
  shows $f\ `\ A = B$
⟨*proof*⟩

lemma *surj-on-implies-set-mset-eq*:
  assumes *finite A*
  assumes $F \in (A \rightarrow_E B)\ //\ domain\text{-}permutation\ A\ B$
  assumes $univ\ (\lambda f.\ f\ `\ A = B)\ F$
  shows $set\text{-}mset\ (msubset\text{-}of\ A\ F) = B$
⟨*proof*⟩

lemma *functions-of-is-surj-on*:
  assumes *finite A*

assumes *size* M = *card* A *set-mset* M = B
shows *univ* ($\lambda f.\ f$ ‘ $A = B$) (*functions-of* A M)
⟨*proof*⟩

8.2 Bijections

lemma *bij-betw-msubset-of*:
assumes *finite* A
shows *bij-betw* (*msubset-of* A) ($\{f \in A \rightarrow_E B.\ f$ ‘ $A = B\}$ // *domain-permutation* A B)
$\{M.$ *set-mset* $M = B \wedge$ *size* M = *card* $A\}$
(**is** *bij-betw* - *?FSet ?MSet*)
⟨*proof*⟩

8.3 Cardinality

lemma *card-surjective-functions-domain-permutation*:
assumes *finite* A *finite* B
assumes *card* $B \leq$ *card* A
shows *card* ($\{f \in A \rightarrow_E B.\ f$ ‘ $A = B\}$ // *domain-permutation* A B) = (*card* A $-$ *1*) *choose* (*card* A $-$ *card* B)
⟨*proof*⟩

end

9 Functions from A to B up to a Permutation on B

theory *Twelvefold-Way-Entry7*
imports *Equiv-Relations-on-Functions*
begin

9.1 Definition of Bijections

definition *partitions-of* :: $'a$ *set* $\Rightarrow$ $'b$ *set* $\Rightarrow$ ($'a \Rightarrow$ $'b$) *set* $\Rightarrow$ $'a$ *set set*
where
partitions-of A B F = *univ* ($\lambda f.\ (\lambda b.\ \{x \in A.\ f\,x = b\})$ ‘ $B - \{\{\}\}$) F

definition *functions-of* :: $'a$ *set set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $'b$ *set* $\Rightarrow$ ($'a \Rightarrow$ $'b$) *set*
where
functions-of P A B = $\{f \in A \rightarrow_E B.\ (\lambda b.\ \{x \in A.\ f\,x = b\})$ ‘ $B - \{\{\}\} = P\}$

9.2 Properties for Bijections

lemma *partitions-of*:
assumes *finite* B
assumes $F \in (A \rightarrow_E B)$ // *range-permutation* A B
shows *card* (*partitions-of* A B F) $\leq$ *card* B
and *partition-on* A (*partitions-of* A B F)

⟨*proof*⟩

lemma *functions-of*:
  assumes *finite A finite B*
  assumes *partition-on A P*
  assumes *card P* $\leq$ *card B*
  shows *functions-of P A B* $\in$ ($A \rightarrow_E B$) // *range-permutation A B*
⟨*proof*⟩

lemma *functions-of-partitions-of*:
  assumes *finite B*
  assumes $F \in (A \rightarrow_E B)$ // *range-permutation A B*
  shows *functions-of* (*partitions-of A B F*) *A B* = *F*
⟨*proof*⟩

lemma *partitions-of-functions-of*:
  assumes *finite A finite B*
  assumes *partition-on A P*
  assumes *card P* $\leq$ *card B*
  shows *partitions-of A B* (*functions-of P A B*) = *P*
⟨*proof*⟩

9.3 Bijections

lemma *bij-betw-partitions-of*:
  assumes *finite A finite B*
  shows *bij-betw* (*partitions-of A B*) (($A \rightarrow_E B$) // *range-permutation A B*) {*P*. *partition-on A P* $\wedge$ *card P* $\leq$ *card B*}
⟨*proof*⟩

9.4 Cardinality

lemma
  assumes *finite A finite B*
  shows *card* (($A \rightarrow_E B$) // *range-permutation A B*) = ($\sum$*j*$\leq$*card B*. *Stirling* (*card A*) *j*)
⟨*proof*⟩

end

10 Injections from A to B up to a Permutation on B

theory *Twelvefold-Way-Entry8*
imports *Twelvefold-Way-Entry7*
begin

10.1 Properties for Bijections

lemma *inj-on-implies-partitions-of*:
 assumes $F \in (A \rightarrow_E B)$ // *range-permutation* $A\ B$
 assumes *univ* $(\lambda f.$ *inj-on* $f\ A)\ F$
 shows $\forall X \in$ *partitions-of* $A\ B\ F.$ *card* $X = 1$
⟨*proof*⟩

lemma *unique-part-eq-singleton*:
 assumes *partition-on* $A\ P$
 assumes $\forall X{\in}P.$ *card* $X = 1$
 assumes $x \in A$
 shows $(THE\ X.\ x \in X \wedge X \in P) = \{x\}$
⟨*proof*⟩

lemma *functions-of-is-inj-on*:
 assumes *finite* A *finite* B *partition-on* $A\ P$ *card* $P \leq$ *card* B
 assumes $\forall X{\in}P.$ *card* $X = 1$
 shows *univ* $(\lambda f.$ *inj-on* $f\ A)$ (*functions-of* $P\ A\ B$)
⟨*proof*⟩

10.2 Bijections

lemma *bij-betw-partitions-of*:
 assumes *finite* A *finite* B
 shows *bij-betw* (*partitions-of* $A\ B$) $(\{f \in A \rightarrow_E B.$ *inj-on* $f\ A\}$ // *range-permutation* $A\ B)$ $\{P.$ *partition-on* $A\ P \wedge$ *card* $P \leq$ *card* $B \wedge (\forall X{\in}P.$ *card* $X = 1)\}$
⟨*proof*⟩

10.3 Cardinality

lemma *card-injective-functions-range-permutation*:
 assumes *finite* A *finite* B
 shows *card* $(\{f \in A \rightarrow_E B.$ *inj-on* $f\ A\}$ // *range-permutation* $A\ B) =$ *iverson* (*card* $A \leq$ *card* B)
⟨*proof*⟩

end

11 Surjections from A to B up to a Permutation on B

theory *Twelvefold-Way-Entry9*
imports *Twelvefold-Way-Entry7*
begin

11.1 Properties for Bijections

lemma *surjective-on-implies-card-eq*:

assumes f ‘ $A = B$
shows *card* $((\lambda b.\ \{x \in A.\ f\ x = b\})$ ‘ $B - \{\{\}\}) =$ *card* B
⟨*proof*⟩

lemma *card-eq-implies-surjective-on*:
assumes *finite* B $f \in A \rightarrow_E B$
assumes *card-eq*: *card* $((\lambda b.\ \{x \in A.\ f\ x = b\})$ ‘ $B - \{\{\}\}) =$ *card* B
shows f ‘ $A = B$
⟨*proof*⟩

lemma *card-partitions-of*:
assumes $F \in (A \rightarrow_E B)$ // *range-permutation* A B
assumes *univ* $(\lambda f.\ f$ ‘ $A = B)$ F
shows *card* (*partitions-of* A B F) $=$ *card* B
⟨*proof*⟩

lemma *functions-of-is-surj-on*:
assumes *finite* A *finite* B
assumes *partition-on* A P *card* $P =$ *card* B
shows *univ* $(\lambda f.\ f$ ‘ $A = B)$ (*functions-of* P A B)
⟨*proof*⟩

11.2 Bijections

lemma *bij-betw-partitions-of*:
assumes *finite* A *finite* B
shows *bij-betw* (*partitions-of* A B) $(\{f \in A \rightarrow_E B.\ f$ ‘ $A = B\}$ // *range-permutation* A $B)$ $\{P.$ *partition-on* A $P \wedge$ *card* $P =$ *card* $B\}$
⟨*proof*⟩

11.3 Cardinality

lemma *card-surjective-functions-range-permutation*:
assumes *finite* A *finite* B
shows *card* $(\{f \in A \rightarrow_E B.\ f$ ‘ $A = B\}$ // *range-permutation* A $B) =$ *Stirling* (*card* A) (*card* B)
⟨*proof*⟩

end

12 Surjections from A to B

theory *Twelvefold-Way-Entry3*
imports
Twelvefold-Way-Entry9
begin

lemma *card-of-equiv-class*:
assumes *finite* B

assumes $F \in \{f \in A \rightarrow_E B.\ f\ `\ A = B\}\ //\ \mathit{range\text{-}permutation}\ A\ B$
shows $\mathit{card}\ F = \mathit{fact}\ (\mathit{card}\ B)$
⟨*proof*⟩

lemma *card-extensional-funcset-surj-on*:
assumes *finite A finite B*
shows $\mathit{card}\ \{f \in A \rightarrow_E B.\ f\ `\ A = B\} = \mathit{fact}\ (\mathit{card}\ B) * \mathit{Stirling}\ (\mathit{card}\ A)\ (\mathit{card}\ B)$ (**is** *card ?F = -*)
⟨*proof*⟩

end

13 Functions from A to B up to a Permutation on A and B

theory *Twelvefold-Way-Entry10*
imports *Equiv-Relations-on-Functions*
begin

13.1 Definition of Bijections

definition $\mathit{number\text{-}partition\text{-}of} :: 'a\ \mathit{set} \Rightarrow 'b\ \mathit{set} \Rightarrow ('a \Rightarrow 'b)\ \mathit{set} \Rightarrow \mathit{nat\ multiset}$
where
$\mathit{number\text{-}partition\text{-}of}\ A\ B\ F = \mathit{univ}\ (\lambda f.\ \mathit{image\text{-}mset}\ (\lambda X.\ \mathit{card}\ X)\ (\mathit{mset\text{-}set}\ ((\lambda b.\ \{x \in A.\ f\ x = b\})\ `\ B - \{\{\}\})))\ F$

definition $\mathit{functions\text{-}of} :: 'a\ \mathit{set} \Rightarrow 'b\ \mathit{set} \Rightarrow \mathit{nat\ multiset} \Rightarrow ('a \Rightarrow 'b)\ \mathit{set}$
where
$\mathit{functions\text{-}of}\ A\ B\ N = \{f \in A \rightarrow_E B.\ \mathit{image\text{-}mset}\ (\lambda X.\ \mathit{card}\ X)\ (\mathit{mset\text{-}set}\ ((\lambda b.\ \{x \in A.\ f\ x = b\})\ `\ B - \{\{\}\})) = N\}$

13.2 Properties for Bijections

lemma *card-setsum-partition*:
assumes *finite A finite B* $f \in A \rightarrow_E B$
shows $\mathit{sum}\ \mathit{card}\ ((\lambda b.\ \{x \in A.\ f\ x = b\})\ `\ B - \{\{\}\}) = \mathit{card}\ A$
⟨*proof*⟩

lemma *number-partition-of*:
assumes *finite A finite B*
assumes $F \in (A \rightarrow_E B)\ //\ \mathit{domain\text{-}and\text{-}range\text{-}permutation}\ A\ B$
shows *number-partition* (*card A*) (*number-partition-of A B F*)
and $\mathit{size}\ (\mathit{number\text{-}partition\text{-}of}\ A\ B\ F) \leq \mathit{card}\ B$
⟨*proof*⟩

lemma *functions-of*:
assumes *finite A finite B*
assumes *number-partition* (*card A*) *N*
assumes $\mathit{size}\ N \leq \mathit{card}\ B$

shows *functions-of A B N* $\in$ $(A \rightarrow_E B)$ // *domain-and-range-permutation A B*
⟨*proof*⟩

lemma *functions-of-number-partition-of*:
assumes *finite A finite B*
assumes $F \in (A \rightarrow_E B)$ // *domain-and-range-permutation A B*
shows *functions-of A B* (*number-partition-of A B F*) = *F*
⟨*proof*⟩

lemma *number-partition-of-functions-of*:
assumes *finite A finite B*
assumes *number-partition* (*card A*) *N size N* $\leq$ *card B*
shows *number-partition-of A B* (*functions-of A B N*) = *N*
⟨*proof*⟩

13.3 Bijections

lemma *bij-betw-number-partition-of*:
assumes *finite A finite B*
shows *bij-betw* (*number-partition-of A B*) $((A \rightarrow_E B)$ // *domain-and-range-permutation A B*) {*N. number-partition* (*card A*) *N* $\wedge$ *size N* $\leq$ *card B*}
⟨*proof*⟩

13.4 Cardinality

lemma *card-domain-and-range-permutation*:
assumes *finite A finite B*
shows *card* $((A \rightarrow_E B)$ // *domain-and-range-permutation A B*) = *Partition* (*card A* + *card B*) (*card B*)
⟨*proof*⟩

end

14 Injections from A to B up to a permutation on A and B

theory *Twelvefold-Way-Entry11*
imports *Twelvefold-Way-Entry10*
begin

14.1 Properties for Bijections

lemma *all-one-implies-inj-on*:
assumes *finite A finite B*
assumes $\forall n.\ n \in\# N \longrightarrow n = 1$ *number-partition* (*card A*) *N size N* $\leq$ *card B*
assumes $f \in$ *functions-of A B N*
shows *inj-on f A*
⟨*proof*⟩

lemma *inj-on-implies-all-one*:
assumes *finite A finite B*
assumes $F \in (A \rightarrow_E B)$ // *domain-and-range-permutation A B*
assumes *univ* ($\lambda f.$ *inj-on f A*) *F*
shows $\forall\, n.$ *n*∈# *number-partition-of A B F* $\longrightarrow n = 1$
⟨*proof*⟩

lemma *functions-of-is-inj-on*:
assumes *finite A finite B*
assumes $\forall\, n.$ *n*∈# *N* $\longrightarrow n = 1$ *number-partition* (*card A*) *N size N* $\leq$ *card B*
shows *univ* ($\lambda f.$ *inj-on f A*) (*functions-of A B N*)
⟨*proof*⟩

14.2 Bijections

lemma *bij-betw-number-partition-of*:
assumes *finite A finite B*
shows *bij-betw* (*number-partition-of A B*) ($\{f \in A \rightarrow_E B.$ *inj-on f A*$\}$ // *domain-and-range-permutation A B*) $\{N.$ ($\forall\, n.$ *n*∈# *N* $\longrightarrow n = 1$) $\wedge$ *number-partition* (*card A*) *N* $\wedge$ *size N* $\leq$ *card B*$\}$
⟨*proof*⟩

lemma *bij-betw-functions-of*:
assumes *finite A finite B*
shows *bij-betw* (*functions-of A B*) $\{N.$ ($\forall\, n.$ *n*∈# *N* $\longrightarrow n = 1$) $\wedge$ *number-partition* (*card A*) *N* $\wedge$ *size N* $\leq$ *card B*$\}$ ($\{f \in A \rightarrow_E B.$ *inj-on f A*$\}$ // *domain-and-range-permutation A B*)
⟨*proof*⟩

14.3 Cardinality

lemma *card-injective-functions-domain-and-range-permutation*:
assumes *finite A finite B*
shows *card* ($\{f \in A \rightarrow_E B.$ *inj-on f A*$\}$ // *domain-and-range-permutation A B*) = *iverson* (*card A* $\leq$ *card B*)
⟨*proof*⟩

end

15 Surjections from A to B up to a Permutation on A and B

theory *Twelvefold-Way-Entry12*
imports *Twelvefold-Way-Entry9 Twelvefold-Way-Entry10*
begin

15.1 Properties for Bijections

lemma *size-eq-card-implies-surj-on*:

assumes *finite A finite B*
assumes *size N = card B*
assumes $f \in$ *functions-of A B N*
shows f ' $A = B$
⟨*proof*⟩

lemma *surj-on-implies-size-eq-card*:
assumes *finite A finite B*
assumes $F \in (A \rightarrow_E B)$ // *domain-and-range-permutation A B*
assumes *univ* (λf. f ' $A = B$) F
shows *size* (*number-partition-of A B F*) = *card B*
⟨*proof*⟩

lemma *functions-of-is-surj-on*:
assumes *finite A finite B*
assumes *number-partition* (*card A*) *N size N = card B*
shows *univ* (λf. f ' $A = B$) (*functions-of A B N*)
⟨*proof*⟩

15.2 Bijections

lemma *bij-betw-number-partition-of*:
assumes *finite A finite B*
shows *bij-betw* (*number-partition-of A B*) ($\{f \in A \rightarrow_E B.\ f$ ' $A = B\}$ // *domain-and-range-permutation A B*) $\{N$. *number-partition* (*card A*) $N \wedge$ *size* $N =$ *card B*$\}$
⟨*proof*⟩

lemma *bij-betw-functions-of*:
assumes *finite A finite B*
shows *bij-betw* (*functions-of A B*) $\{N$. *number-partition* (*card A*) $N \wedge$ *size* $N =$ *card B*$\}$ ($\{f \in A \rightarrow_E B.\ f$ ' $A = B\}$ // *domain-and-range-permutation A B*)
⟨*proof*⟩

15.3 Cardinality

lemma *card-surjective-functions-domain-and-range-permutation*:
assumes *finite A finite B*
shows *card* ($\{f \in A \rightarrow_E B.\ f$ ' $A = B\}$ // *domain-and-range-permutation A B*) = *Partition* (*card A*) (*card B*)
⟨*proof*⟩

end

16 Cardinality of Bijections

theory *Card-Bijections*
imports
Twelvefold-Way-Entry2

Twelvefold-Way-Entry3
Twelvefold-Way-Entry5
Twelvefold-Way-Entry6
Twelvefold-Way-Entry8
Twelvefold-Way-Entry9
Twelvefold-Way-Entry11
Twelvefold-Way-Entry12
begin

16.1 Bijections from A to B

lemma *bij-betw-set-is-empty*:
assumes *finite A finite B*
assumes $card\ A \neq card\ B$
shows $\{f \in A \rightarrow_E B.\ bij\text{-}betw\ f\ A\ B\} = \{\}$
⟨*proof*⟩

lemma *card-bijections-eq-zero*:
assumes *finite A finite B*
assumes $card\ A \neq card\ B$
shows $card\ \{f \in A \rightarrow_E B.\ bij\text{-}betw\ f\ A\ B\} = 0$
⟨*proof*⟩

Two alternative proofs for the cardinality of bijections up to a permutation on A.

lemma
assumes *finite A finite B*
assumes $card\ A = card\ B$
shows $card\ \{f \in A \rightarrow_E B.\ bij\text{-}betw\ f\ A\ B\} = fact\ (card\ B)$
⟨*proof*⟩

lemma *card-bijections*:
assumes *finite A finite B*
assumes $card\ A = card\ B$
shows $card\ \{f \in A \rightarrow_E B.\ bij\text{-}betw\ f\ A\ B\} = fact\ (card\ B)$
⟨*proof*⟩

16.2 Bijections from A to B up to a Permutation on A

lemma *bij-betw-quotient-domain-permutation-eq-empty*:
assumes $card\ A \neq card\ B$
shows $\{f \in A \rightarrow_E B.\ bij\text{-}betw\ f\ A\ B\}\ //\ domain\text{-}permutation\ A\ B = \{\}$
⟨*proof*⟩

lemma *card-bijections-domain-permutation-eq-0*:
assumes $card\ A \neq card\ B$
shows $card\ (\{f \in A \rightarrow_E B.\ bij\text{-}betw\ f\ A\ B\}\ //\ domain\text{-}permutation\ A\ B) = 0$
⟨*proof*⟩

Two alternative proofs for the cardinality of bijections up to a permutation on A.

lemma
assumes *finite A finite B*
assumes *card A = card B*
shows *card* ($\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *domain-permutation A B*) = *1*
⟨*proof*⟩

lemma *card-bijections-domain-permutation-eq-1*:
assumes *finite A finite B*
assumes *card A = card B*
shows *card* ($\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *domain-permutation A B*) = *1*
⟨*proof*⟩

lemma *card-bijections-domain-permutation*:
assumes *finite A finite B*
shows *card* ($\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *domain-permutation A B*) = *iverson* (*card A = card B*)
⟨*proof*⟩

16.3 Bijections from A to B up to a Permutation on B

lemma *bij-betw-quotient-range-permutation-eq-empty*:
assumes *card A* $\neq$ *card B*
shows $\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *range-permutation A B* = $\{\}$
⟨*proof*⟩

lemma *card-bijections-range-permutation-eq-0*:
assumes *card A* $\neq$ *card B*
shows *card* ($\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *range-permutation A B*) = *0*
⟨*proof*⟩

Two alternative proofs for the cardinality of bijections up to a permutation on B.

lemma
assumes *finite A finite B*
assumes *card A = card B*
shows *card* ($\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *range-permutation A B*) = *1*
⟨*proof*⟩

lemma *card-bijections-range-permutation-eq-1*:
assumes *finite A finite B*
assumes *card A = card B*
shows *card* ($\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *range-permutation A B*) = *1*
⟨*proof*⟩

lemma *card-bijections-range-permutation*:
assumes *finite A finite B*

shows *card* ({*f* ∈ *A* $\to_E$ *B*. *bij-betw f A B*} // *range-permutation A B*) = *iverson* (*card A* = *card B*)
⟨*proof*⟩

16.4 Bijections from A to B up to a Permutation on A and B

lemma *bij-betw-quotient-domain-and-range-permutation-eq-empty*:
assumes *card A* ≠ *card B*
shows {*f* ∈ *A* $\to_E$ *B*. *bij-betw f A B*} // *domain-and-range-permutation A B* = {}
⟨*proof*⟩

lemma *card-bijections-domain-and-range-permutation-eq-0*:
assumes *card A* ≠ *card B*
shows *card* ({*f* ∈ *A* $\to_E$ *B*. *bij-betw f A B*} // *domain-and-range-permutation A B*) = *0*
⟨*proof*⟩

Two alternative proofs for the cardinality of bijections up to a permutation on A and B.

lemma
assumes *finite A finite B*
assumes *card A* = *card B*
shows *card* ({*f* ∈ *A* $\to_E$ *B*. *bij-betw f A B*} // *domain-and-range-permutation A B*) = *1*
⟨*proof*⟩

lemma *card-bijections-domain-and-range-permutation-eq-1*:
assumes *finite A finite B*
assumes *card A* = *card B*
shows *card* ({*f* ∈ *A* $\to_E$ *B*. *bij-betw f A B*} // *domain-and-range-permutation A B*) = *1*
⟨*proof*⟩

lemma *card-bijections-domain-and-range-permutation*:
assumes *finite A finite B*
shows *card* ({*f* ∈ *A* $\to_E$ *B*. *bij-betw f A B*} // *domain-and-range-permutation A B*) = *iverson* (*card A* = *card B*)
⟨*proof*⟩

end

17 Direct Proofs for Cardinality of Bijections

theory *Card-Bijections-Direct*
imports
Equiv-Relations-on-Functions

Twelvefold-Way-Core
begin

17.1 Bijections from A to B up to a Permutation on A

17.1.1 Equivalence Class

lemma *bijections-in-domain-permutation*:
assumes *finite A finite B*
assumes *card A = card B*
shows $\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\} \in \{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *domain-permutation A B*
⟨*proof*⟩

lemma *bij-betw-quotient-domain-permutation-eq*:
assumes *finite A finite B*
assumes *card A = card B*
shows $\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *domain-permutation A B* = $\{\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}\}$
⟨*proof*⟩

17.1.2 Cardinality

lemma
assumes *finite A finite B*
assumes *card A = card B*
shows *card* $(\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *domain-permutation A B*$) = 1$
⟨*proof*⟩

17.2 Bijections from A to B up to a Permutation on B

17.2.1 Equivalence Class

lemma *bijections-in-range-permutation*:
assumes *finite A finite B*
assumes *card A = card B*
shows $\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\} \in \{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *range-permutation A B*
⟨*proof*⟩

lemma *bij-betw-quotient-range-permutation-eq*:
assumes *finite A finite B*
assumes *card A = card B*
shows $\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}$ // *range-permutation A B* = $\{\{f \in A \rightarrow_E B.$ *bij-betw f A B*$\}\}$
⟨*proof*⟩

17.2.2 Cardinality

lemma *card-bijections-range-permutation-eq-1*:
assumes *finite A finite B*

assumes *card* A = *card* B
shows *card* ($\{f \in A \to_E B.$ *bij-betw* f A $B\}$ // *range-permutation* A B) = *1*
⟨*proof*⟩

17.3 Bijections from A to B up to a Permutation on A and B

17.3.1 Equivalence Class

lemma *bijections-in-domain-and-range-permutation*:
assumes *finite* A *finite* B
assumes *card* A = *card* B
shows $\{f \in A \to_E B.$ *bij-betw* f A $B\} \in \{f \in A \to_E B.$ *bij-betw* f A $B\}$ // *domain-and-range-permutation* A B
⟨*proof*⟩

lemma *bij-betw-quotient-domain-and-range-permutation-eq*:
assumes *finite* A *finite* B
assumes *card* A = *card* B
shows $\{f \in A \to_E B.$ *bij-betw* f A $B\}$ // *domain-and-range-permutation* A B = $\{\{f \in A \to_E B.$ *bij-betw* f A $B\}\}$
⟨*proof*⟩

17.3.2 Cardinality

lemma *card-bijections-domain-and-range-permutation-eq-1*:
assumes *finite* A *finite* B
assumes *card* A = *card* B
shows *card* ($\{f \in A \to_E B.$ *bij-betw* f A $B\}$ // *domain-and-range-permutation* A B) = *1*
⟨*proof*⟩

end

18 The Twelvefold Way

theory *Twelvefold-Way*
imports
Preliminaries
Twelvefold-Way-Core
Equiv-Relations-on-Functions
Twelvefold-Way-Entry1
Twelvefold-Way-Entry2
Twelvefold-Way-Entry4
Twelvefold-Way-Entry5
Twelvefold-Way-Entry6
Twelvefold-Way-Entry7
Twelvefold-Way-Entry8
Twelvefold-Way-Entry9

Twelvefold-Way-Entry3
Twelvefold-Way-Entry10
Twelvefold-Way-Entry11
Twelvefold-Way-Entry12
Card-Bijections
Card-Bijections-Direct
begin

end

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