

The Twelfold Way

Lukas Bulwahn

April 20, 2020

Abstract

This entry provides all cardinality theorems of the Twelfold Way. The Twelfold Way [1, 5, 6] systematically classifies twelve related combinatorial problems concerning two finite sets, which include counting permutations, combinations, multisets, set partitions and number partitions. This development builds upon the existing formal developments [2, 3, 4] with cardinality theorems for those structures. It provides twelve bijections from the various structures to different equivalence classes on finite functions, and hence, proves cardinality formulae for these equivalence classes on finite functions.

Contents

1 Preliminaries	4
1.1 Additions to Finite Set Theory	4
1.2 Additions to Equiv Relation Theory	4
1.2.1 Counting Sets by Splitting into Equivalence Classes	5
1.3 Additions to FuncSet Theory	5
1.4 Additions to Permutations Theory	6
1.5 Additions to List Theory	6
1.6 Additions to Disjoint Set Theory	7
1.7 Additions to Multiset Theory	7
1.8 Additions to Number Partitions Theory	8
1.9 Cardinality Theorems with Iverson Function	8
2 Main Observations on Operations and Permutations	9
2.1 Range Multiset	9
2.1.1 Existence of a Suitable Finite Function	9
2.1.2 Existence of Permutation	9
2.2 Domain Partition	9
2.2.1 Existence of a Suitable Finite Function	9
2.2.2 Equality under Permutation Application	10
2.2.3 Existence of Permutation	10
2.3 Number Partition of Range	11

2.3.1	Existence of a Suitable Finite Function	11
2.3.2	Equality under Permutation Application	11
2.3.3	Existence of Permutation	11
2.4	Bijections on Same Domain and Range	12
2.4.1	Existence of Domain Permutation	12
2.4.2	Existence of Range Permutation	12
3	Definition of Equivalence Classes	12
3.1	Permutation on the Domain	12
3.1.1	Respecting Functions	12
3.2	Permutation on the Range	13
3.2.1	Respecting Functions	13
3.3	Permutation on the Domain and the Range	13
3.3.1	Respecting Functions	14
4	Functions from A to B	14
4.1	Definition of Bijections	14
4.2	Properties for Bijections	15
4.3	Bijections	16
4.4	Cardinality	16
5	Injections from A to B	16
5.1	Properties for Bijections	17
5.2	Bijections	17
5.3	Cardinality	17
6	Functions from A to B, up to a Permutation of A	18
6.1	Definition of Bijections	18
6.2	Properties for Bijections	18
6.3	Bijections	19
6.4	Cardinality	19
7	Injections from A to B up to a Permutation of A	19
7.1	Definition of Bijections	19
7.2	Properties for Bijections	19
7.3	Bijections	20
7.4	Cardinality	21
8	Surjections from A to B up to a Permutation on A	21
8.1	Properties for Bijections	21
8.2	Bijections	22
8.3	Cardinality	22

9	Functions from A to B up to a Permutation on B	22
9.1	Definition of Bijections	22
9.2	Properties for Bijections	22
9.3	Bijections	23
9.4	Cardinality	23
10	Injections from A to B up to a Permutation on B	23
10.1	Properties for Bijections	24
10.2	Bijections	24
10.3	Cardinality	24
11	Surjections from A to B up to a Permutation on B	24
11.1	Properties for Bijections	24
11.2	Bijections	25
11.3	Cardinality	25
12	Surjections from A to B	25
13	Functions from A to B up to a Permutation on A and B	26
13.1	Definition of Bijections	26
13.2	Properties for Bijections	26
13.3	Bijections	27
13.4	Cardinality	27
14	Injections from A to B up to a permutation on A and B	27
14.1	Properties for Bijections	27
14.2	Bijections	28
14.3	Cardinality	28
15	Surjections from A to B up to a Permutation on A and B	28
15.1	Properties for Bijections	28
15.2	Bijections	29
15.3	Cardinality	29
16	Cardinality of Bijections	29
16.1	Bijections from A to B	30
16.2	Bijections from A to B up to a Permutation on A	30
16.3	Bijections from A to B up to a Permutation on B	31
16.4	Bijections from A to B up to a Permutation on A and B	32
17	Direct Proofs for Cardinality of Bijections	32
17.1	Bijections from A to B up to a Permutation on A	33
17.1.1	Equivalence Class	33
17.1.2	Cardinality	33
17.2	Bijections from A to B up to a Permutation on B	33

17.2.1	Equivalence Class	33
17.2.2	Cardinality	33
17.3	Bijections from A to B up to a Permutation on A and B . . .	34
17.3.1	Equivalence Class	34
17.3.2	Cardinality	34
18	The Twelffold Way	34

1 Preliminaries

```

theory Preliminaries
imports
  Main
  HOL-Library.Multiset
  HOL-Library.FuncSet
  HOL-Library.Permutations
  HOL-ex.Birthday-Paradox
  Card-Partitions.Card-Partitions
  Bell-Numbers-Spivey.Bell-Numbers
  Card-Multisets.Card-Multisets
  Card-Number-Partitions.Card-Number-Partitions
begin

```

1.1 Additions to Finite Set Theory

```

lemma subset-with-given-card-exists:
  assumes  $n \leq \text{card } A$ 
  shows  $\exists B \subseteq A. \text{card } B = n$ 
  <proof>

```

1.2 Additions to Equiv Relation Theory

```

lemmas univ-commute' = univ-commute[unfolded Equiv-Relations.proj-def]

```

```

lemma univ-predicate-impl-forall:
  assumes equiv A R
  assumes P respects R
  assumes  $X \in A // R$ 
  assumes univ P X
  shows  $\forall x \in X. P x$ 
  <proof>

```

```

lemma univ-preserves-predicate:
  assumes equiv A r
  assumes P respects r
  shows  $\{x \in A. P x\} // r = \{X \in A // r. \text{univ } P X\}$ 
  <proof>

```

lemma *Union-quotient-restricted:*
assumes *equiv A r*
assumes *P respects r*
shows $\bigcup (\{x \in A. P x\} // r) = \{x \in A. P x\}$
<proof>

lemma *finite-equiv-implies-finite-carrier:*
assumes *equiv A R*
assumes *finite (A // R)*
assumes $\forall X \in A // R. \text{finite } X$
shows *finite A*
<proof>

lemma *finite-quotient-iff:*
assumes *equiv A R*
shows $\text{finite } A \longleftrightarrow (\text{finite } (A // R) \wedge (\forall X \in A // R. \text{finite } X))$
<proof>

1.2.1 Counting Sets by Splitting into Equivalence Classes

lemma *card-equiv-class-restricted:*
assumes *finite {x ∈ A. P x}*
assumes *equiv A R*
assumes *P respects R*
shows $\text{card } \{x \in A. P x\} = \text{sum card } (\{x \in A. P x\} // R)$
<proof>

lemma *card-equiv-class-restricted-same-size:*
assumes *equiv A R*
assumes *P respects R*
assumes $\bigwedge F. F \in \{x \in A. P x\} // R \implies \text{card } F = k$
shows $\text{card } \{x \in A. P x\} = k * \text{card } (\{x \in A. P x\} // R)$
<proof>

lemma *card-equiv-class:*
assumes *finite A*
assumes *equiv A R*
shows $\text{card } A = \text{sum card } (A // R)$
<proof>

lemma *card-equiv-class-same-size:*
assumes *equiv A R*
assumes $\bigwedge F. F \in A // R \implies \text{card } F = k$
shows $\text{card } A = k * \text{card } (A // R)$
<proof>

1.3 Additions to FuncSet Theory

lemma *finite-same-card-bij-on-ext-funcset:*
assumes *finite A finite B card A = card B*

shows $\exists f. f \in A \rightarrow_E B \wedge \text{bij-betw } f A B$
 ⟨proof⟩

lemma *card-extensional-funcset*:

assumes *finite A*
shows $\text{card } (A \rightarrow_E B) = \text{card } B \wedge \text{card } A$
 ⟨proof⟩

lemma *bij-betw-implies-inj-on-and-card-eq*:

assumes *finite B*
assumes $f \in A \rightarrow_E B$
shows $\text{bij-betw } f A B \longleftrightarrow \text{inj-on } f A \wedge \text{card } A = \text{card } B$
 ⟨proof⟩

lemma *bij-betw-implies-surj-on-and-card-eq*:

assumes *finite A*
assumes $f \in A \rightarrow_E B$
shows $\text{bij-betw } f A B \longleftrightarrow f ' A = B \wedge \text{card } A = \text{card } B$
 ⟨proof⟩

1.4 Additions to Permutations Theory

lemma

assumes $f \in A \rightarrow_E B$ $f ' A = B$
assumes p *permutes B* $(\forall x. f' x = p (f x))$
shows $(\lambda b. \{x \in A. f x = b\}) ' B = (\lambda b. \{x \in A. f' x = b\}) ' B$
 ⟨proof⟩

1.5 Additions to List Theory

The theorem *card-lists-length-eq* contains the superfluous assumption *finite A*. Here, we derive that fact without that unnecessary assumption.

lemma *lists-length-eq-Suc-eq-image-Cons*:

$\{xs. \text{set } xs \subseteq A \wedge \text{length } xs = \text{Suc } n\} = (\lambda(x, xs). x \# xs) ' (A \times \{xs. \text{set } xs \subseteq A \wedge \text{length } xs = n\})$
 (is ?A = ?B)
 ⟨proof⟩

lemma *lists-length-eq-Suc-eq-empty-iff*:

$\{xs. \text{set } xs \subseteq A \wedge \text{length } xs = \text{Suc } n\} = \{\} \longleftrightarrow A = \{\}$
 ⟨proof⟩

lemma *lists-length-eq-eq-empty-iff*:

$\{xs. \text{set } xs \subseteq A \wedge \text{length } xs = n\} = \{\} \longleftrightarrow (A = \{\} \wedge n > 0)$
 ⟨proof⟩

lemma *finite-lists-length-eq-iff*:

$\text{finite } \{xs. \text{set } xs \subseteq A \wedge \text{length } xs = n\} \longleftrightarrow (\text{finite } A \vee n = 0)$
 ⟨proof⟩

lemma *card-lists-length-eq*:
shows $\text{card } \{xs. \text{set } xs \subseteq B \wedge \text{length } xs = n\} = \text{card } B \wedge n$
<proof>

1.6 Additions to Disjoint Set Theory

lemma *bij-betw-congI*:
assumes *bij-betw* f A A'
assumes $\forall a \in A. f\ a = g\ a$
shows *bij-betw* g A A'
<proof>

lemma *disjoint-family-onI*[*intro*]:
assumes $\bigwedge m\ n. m \in S \implies n \in S \implies m \neq n \implies A\ m \cap A\ n = \{\}$
shows *disjoint-family-on* A S
<proof>

The following lemma is not needed for this development, but is useful and could be moved to Disjoint Set theory or Equiv Relation theory if translated from set partitions to equivalence relations.

lemma *infinite-partition-on*:
assumes *infinite* A
shows *infinite* $\{P. \text{partition-on } A\ P\}$
<proof>

lemma *finitely-many-partition-on-iff*:
finite $\{P. \text{partition-on } A\ P\} \longleftrightarrow \text{finite } A$
<proof>

1.7 Additions to Multiset Theory

lemma *mset-set-subseteq-mset-set*:
assumes *finite* B $A \subseteq B$
shows *mset-set* $A \subseteq\#$ *mset-set* B
<proof>

lemma *mset-set-set-mset*:
assumes $M \subseteq\#$ *mset-set* A
shows *mset-set* (*set-mset* M) = M
<proof>

lemma *mset-set-set-mset'*:
assumes $\forall x. \text{count } M\ x \leq 1$
shows *mset-set* (*set-mset* M) = M
<proof>

lemma *card-set-mset*:
assumes $M \subseteq\#$ *mset-set* A

shows $\text{card } (\text{set-mset } M) = \text{size } M$
<proof>

lemma *card-set-mset'*:
assumes $\forall x. \text{count } M x \leq 1$
shows $\text{card } (\text{set-mset } M) = \text{size } M$
<proof>

lemma *count-mset-set-leq*:
assumes *finite A*
shows $\text{count } (\text{mset-set } A) x \leq 1$
<proof>

lemma *count-mset-set-leq'*:
assumes *finite A*
shows $\text{count } (\text{mset-set } A) x \leq \text{Suc } 0$
<proof>

lemma *msubset-mset-set-iff*:
assumes *finite A*
shows $\text{set-mset } M \subseteq A \wedge (\forall x. \text{count } M x \leq 1) \longleftrightarrow (M \subseteq\# \text{mset-set } A)$
<proof>

lemma *image-mset-fun-upd*:
assumes $x \notin\# M$
shows $\text{image-mset } (f(x := y)) M = \text{image-mset } f M$
<proof>

1.8 Additions to Number Partitions Theory

lemma *Partition-diag*:
shows $\text{Partition } n \ n = 1$
<proof>

1.9 Cardinality Theorems with Iverson Function

definition *iverson* :: $\text{bool} \Rightarrow \text{nat}$
where
 $\text{iverson } b = (\text{if } b \text{ then } 1 \text{ else } 0)$

lemma *card-partition-on-size1-eq-iverson*:
assumes *finite A*
shows $\text{card } \{P. \text{partition-on } A P \wedge \text{card } P \leq k \wedge (\forall X \in P. \text{card } X = 1)\} = \text{iverson } (\text{card } A \leq k)$
<proof>

lemma *card-number-partitions-with-only-parts-1*:
 $\text{card } \{N. (\forall n. n \in\# N \longrightarrow n = 1) \wedge \text{number-partition } n N \wedge \text{size } N \leq x\} = \text{iverson } (n \leq x)$
<proof>

end

2 Main Observations on Operations and Permutations

theory *Twelvefold-Way-Core*
imports *Preliminaries*
begin

2.1 Range Multiset

2.1.1 Existence of a Suitable Finite Function

lemma *obtain-function*:
 assumes *finite A*
 assumes *size M = card A*
 shows $\exists f. \text{image-mset } f \text{ (mset-set } A) = M$
(*proof*)

lemma *obtain-function-on-ext-funcset*:
 assumes *finite A*
 assumes *size M = card A*
 shows $\exists f \in A \rightarrow_E \text{set-mset } M. \text{image-mset } f \text{ (mset-set } A) = M$
(*proof*)

2.1.2 Existence of Permutation

lemma *image-mset-eq-implies-bij-betw*:
 fixes $f :: 'a1 \Rightarrow 'b$ and $f' :: 'a2 \Rightarrow 'b$
 assumes *finite A finite A'*
 assumes *mset-eq: image-mset f (mset-set A) = image-mset f' (mset-set A')*
 obtains *bij where bij-betw bij A A' and $\forall x \in A. f x = f' (bij x)$*
(*proof*)

lemma *image-mset-eq-implies-permutes*:
 fixes $f :: 'a \Rightarrow 'b$
 assumes *finite A*
 assumes *mset-eq: image-mset f (mset-set A) = image-mset f' (mset-set A)*
 obtains *p where p permutes A and $\forall x \in A. f x = f' (p x)$*
(*proof*)

2.2 Domain Partition

2.2.1 Existence of a Suitable Finite Function

lemma *obtain-function-with-partition*:
 assumes *finite A finite B*
 assumes *partition-on A P*

assumes $\text{card } P \leq \text{card } B$
shows $\exists f \in A \rightarrow_E B. (\lambda b. \{x \in A. f x = b\}) ' B - \{\{\}\} = P$
 <proof>

2.2.2 Equality under Permutation Application

lemma *permutes-implies-inv-image-on-eq*:
assumes p permutes B
shows $(\lambda b. \{x \in A. p (f x) = b\}) ' B = (\lambda b. \{x \in A. f x = b\}) ' B$
 <proof>

2.2.3 Existence of Permutation

lemma *the-elem*:
assumes $f \in A \rightarrow_E B$ $f' \in A \rightarrow_E B$
assumes *partitions-eq*: $(\lambda b. \{x \in A. f x = b\}) ' B - \{\{\}\} = (\lambda b. \{x \in A. f' x = b\}) ' B - \{\{\}\}$
assumes $x \in A$
shows *the-elem* $(f ' \{x a \in A. f' x a = f' x\}) = f x$
 <proof>

lemma *the-elem-eq*:
assumes $f \in A \rightarrow_E B$
assumes $b \in f ' A$
shows *the-elem* $(f ' \{x' \in A. f x' = b\}) = b$
 <proof>

lemma *partitions-eq-implies*:
assumes $f \in A \rightarrow_E B$ $f' \in A \rightarrow_E B$
assumes *partitions-eq*: $(\lambda b. \{x \in A. f x = b\}) ' B - \{\{\}\} = (\lambda b. \{x \in A. f' x = b\}) ' B - \{\{\}\}$
assumes $x \in A$ $x' \in A$
assumes $f x = f x'$
shows $f' x = f' x'$
 <proof>

lemma *card-domain-partitions*:
assumes $f \in A \rightarrow_E B$
assumes *finite* B
shows $\text{card} ((\lambda b. \{x \in A. f x = b\}) ' B - \{\{\}\}) = \text{card} (f ' A)$
 <proof>

lemma *partitions-eq-implies-permutes*:
assumes $f \in A \rightarrow_E B$ $f' \in A \rightarrow_E B$
assumes *finite* B
assumes *partitions-eq*: $(\lambda b. \{x \in A. f x = b\}) ' B - \{\{\}\} = (\lambda b. \{x \in A. f' x = b\}) ' B - \{\{\}\}$
shows $\exists p. p$ permutes $B \wedge (\forall x \in A. f x = p (f' x))$
 <proof>

2.3 Number Partition of Range

2.3.1 Existence of a Suitable Finite Function

lemma *obtain-partition:*

assumes *finite A*

assumes *number-partition (card A) N*

shows $\exists P. \text{partition-on } A \ P \wedge \text{image-mset card (mset-set } P) = N$

<proof>

lemma *obtain-extensional-function-from-number-partition:*

assumes *finite A finite B*

assumes *number-partition (card A) N*

assumes *size N ≤ card B*

shows $\exists f \in A \rightarrow_E B. \text{image-mset } (\lambda X. \text{card } X) (\text{mset-set } (((\lambda b. \{x \in A. f \ x = b\}) \text{ ' } B - \{\{\}\})) = N$

<proof>

2.3.2 Equality under Permutation Application

lemma *permutes-implies-multiset-of-partition-cards-eq:*

assumes *p_A permutes A p_B permutes B*

shows $\text{image-mset card (mset-set } ((\lambda b. \{x \in A. p_B (f' (p_A \ x)) = b\}) \text{ ' } B - \{\{\}\})) = \text{image-mset card (mset-set } ((\lambda b. \{x \in A. f' \ x = b\}) \text{ ' } B - \{\{\}\}))$

<proof>

2.3.3 Existence of Permutation

lemma *partition-implies-permutes:*

assumes *finite A*

assumes *partition-on A P partition-on A P'*

assumes *image-mset card (mset-set P') = image-mset card (mset-set P)*

obtains *p where p permutes A P' = (λX. p ' X) ' P*

<proof>

lemma *permutes-domain-partition-eq:*

assumes *f ∈ A → B*

assumes *p_A permutes A*

assumes *b ∈ B*

shows $p_A \text{ ' } \{x \in A. f \ x = b\} = \{x \in A. f (inv \ p_A \ x) = b\}$

<proof>

lemma *image-domain-partition-eq:*

assumes *f ∈ A →_E B*

assumes *p_A permutes A*

shows $(\lambda X. p_A \text{ ' } X) \text{ ' } ((\lambda b. \{x \in A. f \ x = b\}) \text{ ' } B) = (\lambda b. \{x \in A. f (inv \ p_A \ x) = b\}) \text{ ' } B$

<proof>

lemma *multiset-of-partition-cards-eq-implies-permutes:*

assumes *finite A finite B* $f \in A \rightarrow_E B$ $f' \in A \rightarrow_E B$
assumes *eq: image-mset card* $(\text{mset-set } ((\lambda b. \{x \in A. f x = b\}) ' B - \{\{\}\})) =$
 $\text{image-mset card } (\text{mset-set } ((\lambda b. \{x \in A. f' x = b\}) ' B - \{\{\}\}))$
obtains $p_A p_B$ **where** p_A *permutes A* p_B *permutes B* $\forall x \in A. f x = p_B (f' (p_A x))$
 $\langle \text{proof} \rangle$

2.4 Bijections on Same Domain and Range

2.4.1 Existence of Domain Permutation

lemma *obtain-domain-permutation-for-two-bijections:*
assumes *bij-betw f A B* *bij-betw f' A B*
obtains p **where** p *permutes A* **and** $\forall a \in A. f a = f' (p a)$
 $\langle \text{proof} \rangle$

2.4.2 Existence of Range Permutation

lemma *obtain-range-permutation-for-two-bijections:*
assumes *bij-betw f A B* *bij-betw f' A B*
obtains p **where** p *permutes B* **and** $\forall a \in A. f a = p (f' a)$
 $\langle \text{proof} \rangle$

end

3 Definition of Equivalence Classes

theory *Equiv-Relations-on-Functions*

imports

Preliminaries

Twelvefold-Way-Core

begin

3.1 Permutation on the Domain

definition *domain-permutation*

where

$\text{domain-permutation } A B = \{(f, f') \in (A \rightarrow_E B) \times (A \rightarrow_E B). \exists p. p \text{ permutes } A \wedge (\forall x \in A. f x = f' (p x))\}$

lemma *equiv-domain-permutation:*

$\text{equiv } (A \rightarrow_E B) (\text{domain-permutation } A B)$
 $\langle \text{proof} \rangle$

3.1.1 Respecting Functions

lemma *inj-on-respects-domain-permutation:*

$(\lambda f. \text{inj-on } f A) \text{ respects domain-permutation } A B$
 $\langle \text{proof} \rangle$

lemma *image-respects-domain-permutation:*
 $(\lambda f. f \text{ ' } A)$ respects (domain-permutation $A B$)
 ⟨proof⟩

lemma *surjective-respects-domain-permutation:*
 $(\lambda f. f \text{ ' } A = B)$ respects domain-permutation $A B$
 ⟨proof⟩

lemma *bij-betw-respects-domain-permutation:*
 $(\lambda f. \text{bij-betw } f A B)$ respects domain-permutation $A B$
 ⟨proof⟩

lemma *image-mset-respects-domain-permutation:*
 shows $(\lambda f. \text{image-mset } f (\text{mset-set } A))$ respects (domain-permutation $A B$)
 ⟨proof⟩

3.2 Permutation on the Range

definition *range-permutation*

where

$\text{range-permutation } A B = \{(f, f') \in (A \rightarrow_E B) \times (A \rightarrow_E B). \exists p. p \text{ permutes } B$
 $\wedge (\forall x \in A. f x = p (f' x))\}$

lemma *equiv-range-permutation:*
 $\text{equiv } (A \rightarrow_E B)$ (range-permutation $A B$)
 ⟨proof⟩

3.2.1 Respecting Functions

lemma *inj-on-respects-range-permutation:*
 $(\lambda f. \text{inj-on } f A)$ respects range-permutation $A B$
 ⟨proof⟩

lemma *surj-on-respects-range-permutation:*
 $(\lambda f. f \text{ ' } A = B)$ respects range-permutation $A B$
 ⟨proof⟩

lemma *bij-betw-respects-range-permutation:*
 $(\lambda f. \text{bij-betw } f A B)$ respects range-permutation $A B$
 ⟨proof⟩

lemma *domain-partitions-respects-range-permutation:*
 $(\lambda f. (\lambda b. \{x \in A. f x = b\}) \text{ ' } B - \{\{\}\})$ respects range-permutation $A B$
 ⟨proof⟩

3.3 Permutation on the Domain and the Range

definition *domain-and-range-permutation*

where

$\text{domain-and-range-permutation } A B = \{(f, f') \in (A \rightarrow_E B) \times (A \rightarrow_E B).$

$\exists p_A p_B. p_A \text{ permutes } A \wedge p_B \text{ permutes } B \wedge (\forall x \in A. f x = p_B (f' (p_A x)))\}$

lemma *equiv-domain-and-range-permutation:*

equiv ($A \rightarrow_E B$) (*domain-and-range-permutation* $A B$)
 ⟨*proof*⟩

3.3.1 Respecting Functions

lemma *inj-on-respects-domain-and-range-permutation:*

($\lambda f. \text{inj-on } f A$) *respects domain-and-range-permutation* $A B$
 ⟨*proof*⟩

lemma *surjective-respects-domain-and-range-permutation:*

($\lambda f. f ' A = B$) *respects domain-and-range-permutation* $A B$
 ⟨*proof*⟩

lemma *bij-betw-respects-domain-and-range-permutation:*

($\lambda f. \text{bij-betw } f A B$) *respects domain-and-range-permutation* $A B$
 ⟨*proof*⟩

lemma *count-image-mset':*

count (*image-mset* $f A$) $x = \text{sum}$ (*count* A) $\{x' \in \text{set-mset } A. f x' = x\}$
 ⟨*proof*⟩

lemma *multiset-of-partition-cards-respects-domain-and-range-permutation:*

assumes *finite* B
shows ($\lambda f. \text{image-mset } (\lambda X. \text{card } X) (\text{mset-set } (((\lambda b. \{x \in A. f x = b\})) ' B - \{\{\}\}))$) *respects domain-and-range-permutation* $A B$
 ⟨*proof*⟩

end

4 Functions from A to B

theory *Twelvefold-Way-Entry1*

imports *Preliminaries*

begin

Note that the cardinality theorems of both structures, lists and finite functions, are already available. Hence, this development creates the bijection between those two structures and transfers the one cardinality theorem to the other structures and vice versa, although not strictly needed as both cardinality theorems were already available.

4.1 Definition of Bijections

definition *sequence-of* :: $'a \text{ set} \Rightarrow (\text{nat} \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b \text{ list}$
where

$sequence-of\ A\ enum\ f = map\ (\lambda n. f\ (enum\ n))\ [0..<card\ A]$

definition $function-of :: 'a\ set \Rightarrow (nat \Rightarrow 'a) \Rightarrow 'b\ list \Rightarrow ('a \Rightarrow 'b)$

where

$function-of\ A\ enum\ xs = (\lambda a. if\ a \in A\ then\ xs\ !\ inv-into\ \{0..<length\ xs\}\ enum\ a\ else\ undefined)$

4.2 Properties for Bijections

lemma $nth-sequence-of$:

assumes $i < card\ A$

shows $(sequence-of\ A\ enum\ f)\ !\ i = f\ (enum\ i)$

$\langle proof \rangle$

lemma $nth-sequence-of-inv-into$:

assumes $bij-betw\ enum\ \{0..<card\ A\}\ A$

assumes $a \in A$

shows $(sequence-of\ A\ enum\ f)\ !\ (inv-into\ \{0..<card\ A\}\ enum\ a) = f\ a$

$\langle proof \rangle$

lemma $set-sequence-of$:

assumes $bij-betw\ enum\ \{0..<card\ A\}\ A$

assumes $f \in A \rightarrow_E B$

shows $set\ (sequence-of\ A\ enum\ f) \subseteq B$

$\langle proof \rangle$

lemma $length-sequence-of$:

assumes $bij-betw\ enum\ \{0..<card\ A\}\ A$

assumes $f \in A \rightarrow_E B$

shows $length\ (sequence-of\ A\ enum\ f) = card\ A$

$\langle proof \rangle$

lemma $function-of-enum$:

assumes $bij-betw\ enum\ \{0..<card\ A\}\ A$

assumes $length\ xs = card\ A$

assumes $i < card\ A$

shows $function-of\ A\ enum\ xs\ (enum\ i) = xs\ !\ i$

$\langle proof \rangle$

lemma $function-of-in-extensional-funcset$:

assumes $bij-betw\ enum\ \{0..<card\ A\}\ A$

assumes $set\ xs \subseteq B\ length\ xs = card\ A$

shows $function-of\ A\ enum\ xs \in A \rightarrow_E B$

$\langle proof \rangle$

lemma $sequence-of-function-of$:

assumes $bij-betw\ enum\ \{0..<card\ A\}\ A$

assumes $set\ xs \subseteq B\ length\ xs = card\ A$

shows $sequence-of\ A\ enum\ (function-of\ A\ enum\ xs) = xs$

<proof>

lemma *function-of-sequence-of*:

assumes *bij-betw enum* $\{0..<card\ A\}$ *A*

assumes $f \in A \rightarrow_E B$

shows *function-of A enum* (*sequence-of A enum f*) = *f*

<proof>

4.3 Bijections

lemma *bij-betw-sequence-of*:

assumes *bij-betw enum* $\{0..<card\ A\}$ *A*

shows *bij-betw* (*sequence-of A enum*) ($A \rightarrow_E B$) $\{xs. set\ xs \subseteq B \wedge length\ xs = card\ A\}$

<proof>

lemma *bij-betw-function-of*:

assumes *bij-betw enum* $\{0..<card\ A\}$ *A*

shows *bij-betw* (*function-of A enum*) $\{xs. set\ xs \subseteq B \wedge length\ xs = card\ A\}$ ($A \rightarrow_E B$)

<proof>

4.4 Cardinality

lemma

assumes *finite A*

shows $card\ (A \rightarrow_E B) = card\ B \wedge card\ A$

<proof>

lemma *card-sequences*:

assumes *finite A*

shows $card\ \{xs. set\ xs \subseteq B \wedge length\ xs = card\ A\} = card\ B \wedge card\ A$

<proof>

lemma

shows $card\ \{xs. set\ xs \subseteq A \wedge length\ xs = n\} = card\ A \wedge n$

<proof>

end

5 Injections from A to B

theory *Twelfefold-Way-Entry2*

imports *Twelfefold-Way-Entry1*

begin

Note that the cardinality theorems of both structures, distinct lists and finite injective functions, are already available. Hence, this development creates the bijection between those two structures and transfers the one cardinality

theorem to the other structures and vice versa, although not strictly needed as both cardinality theorems were already available.

5.1 Properties for Bijections

lemma *inj-on-implies-distinct*:

assumes *bij-betw enum* $\{0..<card\ A\}$ *A*

assumes $f \in A \rightarrow_E B$

assumes *inj-on* $f\ A$

shows *distinct* (*sequence-of* *A* *enum* f)

<proof>

lemma *distinct-implies-inj-on*:

assumes *bij-betw enum* $\{0..<card\ A\}$ *A*

assumes *length* $xs = card\ A$

assumes *distinct* xs

shows *inj-on* (*function-of* *A* *enum* xs) *A*

<proof>

lemma *image-sequence-of-inj*:

assumes *bij-betw enum* $\{0..<card\ A\}$ *A*

shows *sequence-of* *A* *enum* $\{f \in A \rightarrow_E B. inj-on\ f\ A\} \subseteq \{xs. set\ xs \subseteq B \wedge length\ xs = card\ A \wedge distinct\ xs\}$

<proof>

lemma *image-function-of-distinct*:

assumes *bij-betw enum* $\{0..<card\ A\}$ *A*

shows *function-of* *A* *enum* $\{xs. set\ xs \subseteq B \wedge length\ xs = card\ A \wedge distinct\ xs\} \subseteq \{f \in A \rightarrow_E B. inj-on\ f\ A\}$

<proof>

5.2 Bijections

lemma *bij-betw-sequence-of*:

assumes *bij-betw enum* $\{0..<card\ A\}$ *A*

shows *bij-betw* (*sequence-of* *A* *enum*) $\{f. f \in A \rightarrow_E B \wedge inj-on\ f\ A\} \{xs. set\ xs \subseteq B \wedge length\ xs = card\ A \wedge distinct\ xs\}$

<proof>

lemma *bij-betw-function-of*:

assumes *bij-betw enum* $\{0..<card\ A\}$ *A*

shows *bij-betw* (*function-of* *A* *enum*) $\{xs. set\ xs \subseteq B \wedge length\ xs = card\ A \wedge distinct\ xs\} \{f \in A \rightarrow_E B. inj-on\ f\ A\}$

<proof>

5.3 Cardinality

lemma

assumes *finite* *A* *finite* *B* $card\ A \leq card\ B$

shows $\text{card } \{f \in A \rightarrow_E B. \text{inj-on } f A\} = \prod \{\text{card } B - \text{card } A + 1.. \text{card } B\}$
 <proof>

lemma *card-sequences*:

assumes *finite A finite B card A ≤ card B*

shows $\text{card } \{xs. \text{set } xs \subseteq B \wedge \text{length } xs = \text{card } A \wedge \text{distinct } xs\} = \text{fact } (\text{card } B)$
div fact (card B - card A)

<proof>

end

6 Functions from A to B, up to a Permutation of A

theory *Twelvefold-Way-Entry4*

imports *Equiv-Relations-on-Functions*

begin

6.1 Definition of Bijections

definition *msubset-of* :: *'a set ⇒ ('a ⇒ 'b) set ⇒ 'b multiset*

where

msubset-of A F = univ (λf. image-mset f (mset-set A)) F

definition *functions-of* :: *'a set ⇒ 'b multiset ⇒ ('a ⇒ 'b) set*

where

functions-of A B = {f ∈ A →_E set-mset B. image-mset f (mset-set A) = B}

6.2 Properties for Bijections

lemma *msubset-of*:

assumes *F ∈ (A →_E B) // domain-permutation A B*

shows *size (msubset-of A F) = card A*

and *set-mset (msubset-of A F) ⊆ B*

<proof>

lemma *functions-of*:

assumes *finite A*

assumes *set-mset M ⊆ B*

assumes *size M = card A*

shows *functions-of A M ∈ (A →_E B) // domain-permutation A B*

<proof>

lemma *functions-of-msubset-of*:

assumes *finite A*

assumes *F ∈ (A →_E B) // domain-permutation A B*

shows *functions-of A (msubset-of A F) = F*

<proof>

lemma *msubset-of-functions-of*:
assumes *set-mset* $M \subseteq B$ *size* $M = \text{card } A$ *finite* A
shows *msubset-of* A (*functions-of* A M) = M
<proof>

6.3 Bijections

lemma *bij-betw-msubset-of*:
assumes *finite* A
shows *bij-betw* (*msubset-of* A) (($A \rightarrow_E B$) // *domain-permutation* A B) { M .
set-mset $M \subseteq B \wedge \text{size } M = \text{card } A$ }
<proof>

6.4 Cardinality

lemma
assumes *finite* A *finite* B
shows *card* (($A \rightarrow_E B$) // *domain-permutation* A B) = $\text{card } B + \text{card } A - 1$
choose *card* A
<proof>

end

7 Injections from A to B up to a Permutation of A

theory *Twelffold-Way-Entry5*
imports
Equiv-Relations-on-Functions
begin

7.1 Definition of Bijections

definition *subset-of* :: ' a set \Rightarrow (' $a \Rightarrow$ ' b) set \Rightarrow ' b set
where
subset-of A $F = \text{univ } (\lambda f. f \text{ ' } A) F$

definition *functions-of* :: ' a set \Rightarrow ' b set \Rightarrow (' $a \Rightarrow$ ' b) set
where
functions-of A $B = \{f \in A \rightarrow_E B. f \text{ ' } A = B\}$

7.2 Properties for Bijections

lemma *functions-of-eq*:
assumes *finite* A
assumes $f \in \{f \in A \rightarrow_E B. \text{inj-on } f A\}$
shows *functions-of* A ($f \text{ ' } A$) = *domain-permutation* A B “ { f }
<proof>

lemma *subset-of*:

assumes $F \in \{f \in A \rightarrow_E B. \text{inj-on } f A\}$ // domain-permutation $A B$

shows $\text{subset-of } A F \subseteq B$ **and** $\text{card } (\text{subset-of } A F) = \text{card } A$

<proof>

lemma *functions-of*:

assumes *finite* A *finite* B $X \subseteq B$ $\text{card } X = \text{card } A$

shows $\text{functions-of } A X \in \{f \in A \rightarrow_E B. \text{inj-on } f A\}$ // domain-permutation $A B$

<proof>

lemma *subset-of-functions-of*:

assumes *finite* A *finite* X $\text{card } A = \text{card } X$

shows $\text{subset-of } A (\text{functions-of } A X) = X$

<proof>

lemma *functions-of-subset-of*:

assumes *finite* A

assumes $F \in \{f \in A \rightarrow_E B. \text{inj-on } f A\}$ // domain-permutation $A B$

shows $\text{functions-of } A (\text{subset-of } A F) = F$

<proof>

7.3 Bijections

lemma *bij-betw-subset-of*:

assumes *finite* A *finite* B

shows $\text{bij-betw } (\text{subset-of } A) (\{f \in A \rightarrow_E B. \text{inj-on } f A\} // \text{domain-permutation } A B) \{X. X \subseteq B \wedge \text{card } X = \text{card } A\}$

<proof>

lemma *bij-betw-functions-of*:

assumes *finite* A *finite* B

shows $\text{bij-betw } (\text{functions-of } A) \{X. X \subseteq B \wedge \text{card } X = \text{card } A\} (\{f \in A \rightarrow_E B. \text{inj-on } f A\} // \text{domain-permutation } A B)$

<proof>

lemma *bij-betw-mset-set*:

shows $\text{bij-betw mset-set } \{A. \text{finite } A\} \{M. \forall x. \text{count } M x \leq 1\}$

<proof>

lemma *bij-betw-mset-set-card*:

assumes *finite* A

shows $\text{bij-betw mset-set } \{X. X \subseteq A \wedge \text{card } X = k\} \{M. M \subseteq\# \text{mset-set } A \wedge \text{size } M = k\}$

<proof>

lemma *bij-betw-mset-set-card'*:

assumes *finite* A

shows $\text{bij-betw mset-set } \{X. X \subseteq A \wedge \text{card } X = k\} \{M. \text{set-mset } M \subseteq A \wedge \text{size}$

$M = k \wedge (\forall x. \text{count } M \ x \leq 1)\}$
 ⟨proof⟩

7.4 Cardinality

lemma *card-injective-functions-domain-permutation:*

assumes *finite A finite B*
shows $\text{card } \{f \in A \rightarrow_E B. \text{inj-on } f \ A\} // \text{domain-permutation } A \ B) = \text{card } B$
choose card A
 ⟨proof⟩

lemma *card-multiset-only-sets:*

assumes *finite A*
shows $\text{card } \{M. M \subseteq\# \text{mset-set } A \wedge \text{size } M = k\} = \text{card } A \ \text{choose } k$
 ⟨proof⟩

lemma *card-multiset-only-sets':*

assumes *finite A*
shows $\text{card } \{M. \text{set-mset } M \subseteq A \wedge \text{size } M = k \wedge (\forall x. \text{count } M \ x \leq 1)\} = \text{card } A \ \text{choose } k$
 ⟨proof⟩

end

8 Surjections from A to B up to a Permutation on A

theory *Twelfefold-Way-Entry6*
imports *Twelfefold-Way-Entry4*
begin

8.1 Properties for Bijections

lemma *set-mset-eq-implies-surj-on:*

assumes *finite A*
assumes $\text{size } M = \text{card } A \ \text{set-mset } M = B$
assumes $f \in \text{functions-of } A \ M$
shows $f \ ' \ A = B$
 ⟨proof⟩

lemma *surj-on-implies-set-mset-eq:*

assumes *finite A*
assumes $F \in (A \rightarrow_E B) // \text{domain-permutation } A \ B$
assumes $\text{univ } (\lambda f. f \ ' \ A = B) \ F$
shows $\text{set-mset } (\text{msubset-of } A \ F) = B$
 ⟨proof⟩

lemma *functions-of-is-surj-on:*

assumes *finite A*

assumes $size\ M = card\ A\ set\ mset\ M = B$
shows $univ\ (\lambda f. f\ 'A = B)$ (*functions-of A M*)
 <proof>

8.2 Bijections

lemma *bij-betw-msubset-of*:
assumes $finite\ A$
shows $bij\ betw\ (msubset\ of\ A)\ (\{f \in A \rightarrow_E B. f\ 'A = B\} //\ domain\ permutation\ A\ B)$
 $\{M. set\ mset\ M = B \wedge size\ M = card\ A\}$
(is *bij-betw - ?FSet ?MSet*)
 <proof>

8.3 Cardinality

lemma *card-surjective-functions-domain-permutation*:
assumes $finite\ A\ finite\ B$
assumes $card\ B \leq card\ A$
shows $card\ (\{f \in A \rightarrow_E B. f\ 'A = B\} //\ domain\ permutation\ A\ B) = (card\ A - 1)$ *choose (card A - card B)*
 <proof>

end

9 Functions from A to B up to a Permutation on B

theory *Twelffold-Way-Entry7*
imports *Equiv-Relations-on-Functions*
begin

9.1 Definition of Bijections

definition *partitions-of* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \Rightarrow 'b)\ set \Rightarrow 'a\ set\ set$
where

$partitions\ of\ A\ B\ F = univ\ (\lambda f. (\lambda b. \{x \in A. f\ x = b\})\ 'B - \{\{\}\})\ F$

definition *functions-of* :: $'a\ set\ set \Rightarrow 'a\ set \Rightarrow 'b\ set \Rightarrow ('a \Rightarrow 'b)\ set$
where

$functions\ of\ P\ A\ B = \{f \in A \rightarrow_E B. (\lambda b. \{x \in A. f\ x = b\})\ 'B - \{\{\}\} = P\}$

9.2 Properties for Bijections

lemma *partitions-of*:
assumes $finite\ B$
assumes $F \in (A \rightarrow_E B) //\ range\ permutation\ A\ B$
shows $card\ (partitions\ of\ A\ B\ F) \leq card\ B$
and *partition-on A (partitions-of A B F)*

<proof>

lemma *functions-of:*

assumes *finite A finite B*

assumes *partition-on A P*

assumes *card P ≤ card B*

shows *functions-of P A B ∈ (A →_E B) // range-permutation A B*

<proof>

lemma *functions-of-partitions-of:*

assumes *finite B*

assumes *F ∈ (A →_E B) // range-permutation A B*

shows *functions-of (partitions-of A B F) A B = F*

<proof>

lemma *partitions-of-functions-of:*

assumes *finite A finite B*

assumes *partition-on A P*

assumes *card P ≤ card B*

shows *partitions-of A B (functions-of P A B) = P*

<proof>

9.3 Bijections

lemma *bij-betw-partitions-of:*

assumes *finite A finite B*

shows *bij-betw (partitions-of A B) ((A →_E B) // range-permutation A B) {P.
partition-on A P ∧ card P ≤ card B}*

<proof>

9.4 Cardinality

lemma

assumes *finite A finite B*

shows *card ((A →_E B) // range-permutation A B) = (∑<sub>j≤card B. Stirling
(card A) j)</sub>*

<proof>

end

10 Injections from A to B up to a Permutation on B

theory *Twelfold-Way-Entry8*

imports *Twelfold-Way-Entry7*

begin

10.1 Properties for Bijections

lemma *inj-on-implies-partitions-of*:

assumes $F \in (A \rightarrow_E B)$ // *range-permutation* $A B$

assumes *univ* $(\lambda f. \text{inj-on } f A) F$

shows $\forall X \in \text{partitions-of } A B F. \text{card } X = 1$

<proof>

lemma *unique-part-eq-singleton*:

assumes *partition-on* $A P$

assumes $\forall X \in P. \text{card } X = 1$

assumes $x \in A$

shows $(\text{THE } X. x \in X \wedge X \in P) = \{x\}$

<proof>

lemma *functions-of-is-inj-on*:

assumes *finite* A *finite* B *partition-on* $A P$ $\text{card } P \leq \text{card } B$

assumes $\forall X \in P. \text{card } X = 1$

shows *univ* $(\lambda f. \text{inj-on } f A)$ $(\text{functions-of } P A B)$

<proof>

10.2 Bijections

lemma *bij-betw-partitions-of*:

assumes *finite* A *finite* B

shows *bij-betw* $(\text{partitions-of } A B)$ $(\{f \in A \rightarrow_E B. \text{inj-on } f A\} // \text{range-permutation } A B)$ $\{P. \text{partition-on } A P \wedge \text{card } P \leq \text{card } B \wedge (\forall X \in P. \text{card } X = 1)\}$

<proof>

10.3 Cardinality

lemma *card-injective-functions-range-permutation*:

assumes *finite* A *finite* B

shows $\text{card } (\{f \in A \rightarrow_E B. \text{inj-on } f A\} // \text{range-permutation } A B) = \text{iverson } (\text{card } A \leq \text{card } B)$

<proof>

end

11 Surjections from A to B up to a Permutation on B

theory *Twelfold-Way-Entry9*

imports *Twelfold-Way-Entry7*

begin

11.1 Properties for Bijections

lemma *surjective-on-implies-card-eq*:

assumes $f \text{ ' } A = B$
shows $\text{card } ((\lambda b. \{x \in A. f x = b\}) \text{ ' } B - \{\{\}\}) = \text{card } B$
 <proof>

lemma *card-eq-implies-surjective-on*:
assumes *finite B f ∈ A →_E B*
assumes *card-eq: card ((λb. {x ∈ A. f x = b}) ' B - {{}}) = card B*
shows $f \text{ ' } A = B$
 <proof>

lemma *card-partitions-of*:
assumes $F \in (A \rightarrow_E B) // \text{range-permutation } A B$
assumes *univ (λf. f ' A = B) F*
shows $\text{card } (\text{partitions-of } A B F) = \text{card } B$
 <proof>

lemma *functions-of-is-surj-on*:
assumes *finite A finite B*
assumes *partition-on A P card P = card B*
shows *univ (λf. f ' A = B) (functions-of P A B)*
 <proof>

11.2 Bijections

lemma *bij-betw-partitions-of*:
assumes *finite A finite B*
shows *bij-betw (partitions-of A B) ({f ∈ A →_E B. f ' A = B} // range-permutation A B) {P. partition-on A P ∧ card P = card B}*
 <proof>

11.3 Cardinality

lemma *card-surjective-functions-range-permutation*:
assumes *finite A finite B*
shows $\text{card } (\{f \in A \rightarrow_E B. f \text{ ' } A = B\} // \text{range-permutation } A B) = \text{Stirling } (\text{card } A) (\text{card } B)$
 <proof>

end

12 Surjections from A to B

theory *Twelvefold-Way-Entry3*
imports *Twelvefold-Way-Entry9*
begin

lemma *card-of-equiv-class*:
assumes *finite B*

assumes $F \in \{f \in A \rightarrow_E B. f \text{ ' } A = B\}$ // *range-permutation A B*
shows $\text{card } F = \text{fact } (\text{card } B)$
 <proof>

lemma *card-extensional-funcset-surj-on*:
assumes *finite A finite B*
shows $\text{card } \{f \in A \rightarrow_E B. f \text{ ' } A = B\} = \text{fact } (\text{card } B) * \text{Stirling } (\text{card } A) (\text{card } B)$ (*is card ?F = -*)
 <proof>

end

13 Functions from A to B up to a Permutation on A and B

theory *Twelvefold-Way-Entry10*
imports *Equiv-Relations-on-Functions*
begin

13.1 Definition of Bijections

definition *number-partition-of* :: $'a \text{ set} \Rightarrow 'b \text{ set} \Rightarrow ('a \Rightarrow 'b) \text{ set} \Rightarrow \text{nat multiset}$
where
 $\text{number-partition-of } A \ B \ F = \text{univ } (\lambda f. \text{image-mset } (\lambda X. \text{card } X) (\text{mset-set } ((\lambda b. \{x \in A. f \ x = b\}) \text{ ' } B - \{\{\}\}))) \ F$

definition *functions-of* :: $'a \text{ set} \Rightarrow 'b \text{ set} \Rightarrow \text{nat multiset} \Rightarrow ('a \Rightarrow 'b) \text{ set}$
where
 $\text{functions-of } A \ B \ N = \{f \in A \rightarrow_E B. \text{image-mset } (\lambda X. \text{card } X) (\text{mset-set } ((\lambda b. \{x \in A. f \ x = b\}) \text{ ' } B - \{\{\}\})) = N\}$

13.2 Properties for Bijections

lemma *card-setsum-partition*:
assumes *finite A finite B* $f \in A \rightarrow_E B$
shows $\text{sum card } ((\lambda b. \{x \in A. f \ x = b\}) \text{ ' } B - \{\{\}\}) = \text{card } A$
 <proof>

lemma *number-partition-of*:
assumes *finite A finite B*
assumes $F \in (A \rightarrow_E B)$ // *domain-and-range-permutation A B*
shows *number-partition* $(\text{card } A)$ (*number-partition-of A B F*)
and *size* $(\text{number-partition-of } A \ B \ F) \leq \text{card } B$
 <proof>

lemma *functions-of*:
assumes *finite A finite B*
assumes *number-partition* $(\text{card } A) \ N$
assumes *size* $N \leq \text{card } B$

shows *functions-of A B N* $\in (A \rightarrow_E B)$ // *domain-and-range-permutation A B*
<proof>

lemma *functions-of-number-partition-of*:
assumes *finite A finite B*
assumes $F \in (A \rightarrow_E B)$ // *domain-and-range-permutation A B*
shows *functions-of A B* (*number-partition-of A B F*) = F
<proof>

lemma *number-partition-of-functions-of*:
assumes *finite A finite B*
assumes *number-partition* (*card A*) N *size* $N \leq \text{card } B$
shows *number-partition-of A B* (*functions-of A B N*) = N
<proof>

13.3 Bijections

lemma *bij-betw-number-partition-of*:
assumes *finite A finite B*
shows *bij-betw* (*number-partition-of A B*) ($(A \rightarrow_E B)$ // *domain-and-range-permutation A B*) { N . *number-partition* (*card A*) $N \wedge \text{size } N \leq \text{card } B$ }
<proof>

13.4 Cardinality

lemma *card-domain-and-range-permutation*:
assumes *finite A finite B*
shows *card* ($(A \rightarrow_E B)$ // *domain-and-range-permutation A B*) = *Partition* (*card A* + *card B*) (*card B*)
<proof>

end

14 Injections from A to B up to a permutation on A and B

theory *Twelfold-Way-Entry11*
imports *Twelfold-Way-Entry10*
begin

14.1 Properties for Bijections

lemma *all-one-implies-inj-on*:
assumes *finite A finite B*
assumes $\forall n. n \in \# N \longrightarrow n = 1$ *number-partition* (*card A*) N *size* $N \leq \text{card } B$
assumes $f \in \text{functions-of } A B N$
shows *inj-on* $f A$
<proof>

lemma *inj-on-implies-all-one*:
assumes *finite A finite B*
assumes $F \in (A \rightarrow_E B)$ // *domain-and-range-permutation A B*
assumes *univ* $(\lambda f. \text{inj-on } f \ A)$ F
shows $\forall n. n \in \# \text{ number-partition-of } A \ B \ F \longrightarrow n = 1$
<proof>

lemma *functions-of-is-inj-on*:
assumes *finite A finite B*
assumes $\forall n. n \in \# \ N \longrightarrow n = 1$ *number-partition* $(\text{card } A) \ N$ *size* $N \leq \text{card } B$
shows *univ* $(\lambda f. \text{inj-on } f \ A)$ $(\text{functions-of } A \ B \ N)$
<proof>

14.2 Bijections

lemma *bij-betw-number-partition-of*:
assumes *finite A finite B*
shows *bij-betw* $(\text{number-partition-of } A \ B)$ $(\{f \in A \rightarrow_E B. \text{inj-on } f \ A\} //$
domain-and-range-permutation A B) $\{N. (\forall n. n \in \# \ N \longrightarrow n = 1) \wedge \text{number-partition}$
 $(\text{card } A) \ N \wedge \text{size } N \leq \text{card } B\}$
<proof>

lemma *bij-betw-functions-of*:
assumes *finite A finite B*
shows *bij-betw* $(\text{functions-of } A \ B)$ $\{N. (\forall n. n \in \# \ N \longrightarrow n = 1) \wedge \text{number-partition}$
 $(\text{card } A) \ N \wedge \text{size } N \leq \text{card } B\}$ $(\{f \in A \rightarrow_E B. \text{inj-on } f \ A\} //$
domain-and-range-permutation A B)
<proof>

14.3 Cardinality

lemma *card-injective-functions-domain-and-range-permutation*:
assumes *finite A finite B*
shows *card* $(\{f \in A \rightarrow_E B. \text{inj-on } f \ A\} //$
domain-and-range-permutation A B)
 $= \text{iverson } (\text{card } A \leq \text{card } B)$
<proof>

end

15 Surjections from A to B up to a Permutation on A and B

theory *Twelffold-Way-Entry12*
imports *Twelffold-Way-Entry9 Twelffold-Way-Entry10*
begin

15.1 Properties for Bijections

lemma *size-eq-card-implies-surj-on*:

```

assumes finite A finite B
assumes size N = card B
assumes f ∈ functions-of A B N
shows f ‘ A = B
⟨proof⟩

```

```

lemma surj-on-implies-size-eq-card:
assumes finite A finite B
assumes F ∈ (A →E B) // domain-and-range-permutation A B
assumes univ (λf. f ‘ A = B) F
shows size (number-partition-of A B F) = card B
⟨proof⟩

```

```

lemma functions-of-is-surj-on:
assumes finite A finite B
assumes number-partition (card A) N size N = card B
shows univ (λf. f ‘ A = B) (functions-of A B N)
⟨proof⟩

```

15.2 Bijections

```

lemma bij-betw-number-partition-of:
assumes finite A finite B
shows bij-betw (number-partition-of A B) ({f ∈ A →E B. f ‘ A = B} // domain-and-range-permutation A B) {N. number-partition (card A) N ∧ size N = card B}
⟨proof⟩

```

```

lemma bij-betw-functions-of:
assumes finite A finite B
shows bij-betw (functions-of A B) {N. number-partition (card A) N ∧ size N = card B} ({f ∈ A →E B. f ‘ A = B} // domain-and-range-permutation A B)
⟨proof⟩

```

15.3 Cardinality

```

lemma card-surjective-functions-domain-and-range-permutation:
assumes finite A finite B
shows card ({f ∈ A →E B. f ‘ A = B} // domain-and-range-permutation A B) = Partition (card A) (card B)
⟨proof⟩

```

end

16 Cardinality of Bijections

```

theory Card-Bijections
imports
  Twelvefold-Way-Entry2

```

Twelfold-Way-Entry3
Twelfold-Way-Entry5
Twelfold-Way-Entry6
Twelfold-Way-Entry8
Twelfold-Way-Entry9
Twelfold-Way-Entry11
Twelfold-Way-Entry12

begin

16.1 Bijections from A to B

lemma *bij-betw-set-is-empty:*
assumes *finite A finite B*
assumes *card A \neq card B*
shows $\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} = \{\}$
<proof>

lemma *card-bijections-eq-zero:*
assumes *finite A finite B*
assumes *card A \neq card B*
shows $\text{card } \{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} = 0$
<proof>

Two alternative proofs for the cardinality of bijections up to a permutation on A.

lemma
assumes *finite A finite B*
assumes *card A = card B*
shows $\text{card } \{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} = \text{fact } (\text{card } B)$
<proof>

lemma *card-bijections:*
assumes *finite A finite B*
assumes *card A = card B*
shows $\text{card } \{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} = \text{fact } (\text{card } B)$
<proof>

16.2 Bijections from A to B up to a Permutation on A

lemma *bij-betw-quotient-domain-permutation-eq-empty:*
assumes *card A \neq card B*
shows $\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} // \text{domain-permutation } A \ B = \{\}$
<proof>

lemma *card-bijections-domain-permutation-eq-0:*
assumes *card A \neq card B*
shows $\text{card } (\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} // \text{domain-permutation } A \ B) = 0$
<proof>

Two alternative proofs for the cardinality of bijections up to a permutation on A.

lemma

assumes *finite A finite B*

assumes *card A = card B*

shows *card ({f ∈ A →_E B. bij-betw f A B} // domain-permutation A B) = 1*
<proof>

lemma *card-bijections-domain-permutation-eq-1:*

assumes *finite A finite B*

assumes *card A = card B*

shows *card ({f ∈ A →_E B. bij-betw f A B} // domain-permutation A B) = 1*
<proof>

lemma *card-bijections-domain-permutation:*

assumes *finite A finite B*

shows *card ({f ∈ A →_E B. bij-betw f A B} // domain-permutation A B) =*
iverson (card A = card B)
<proof>

16.3 Bijections from A to B up to a Permutation on B

lemma *bij-betw-quotient-range-permutation-eq-empty:*

assumes *card A ≠ card B*

shows *{f ∈ A →_E B. bij-betw f A B} // range-permutation A B = {}*
<proof>

lemma *card-bijections-range-permutation-eq-0:*

assumes *card A ≠ card B*

shows *card ({f ∈ A →_E B. bij-betw f A B} // range-permutation A B) = 0*
<proof>

Two alternative proofs for the cardinality of bijections up to a permutation on B.

lemma

assumes *finite A finite B*

assumes *card A = card B*

shows *card ({f ∈ A →_E B. bij-betw f A B} // range-permutation A B) = 1*
<proof>

lemma *card-bijections-range-permutation-eq-1:*

assumes *finite A finite B*

assumes *card A = card B*

shows *card ({f ∈ A →_E B. bij-betw f A B} // range-permutation A B) = 1*
<proof>

lemma *card-bijections-range-permutation:*

assumes *finite A finite B*

shows $\text{card } (\{f \in A \rightarrow_E B. \text{bij-betw } f A B\} // \text{range-permutation } A B) = \text{iverson } (\text{card } A = \text{card } B)$
 <proof>

16.4 Bijections from A to B up to a Permutation on A and B

lemma *bij-betw-quotient-domain-and-range-permutation-eq-empty*:

assumes $\text{card } A \neq \text{card } B$
shows $\{f \in A \rightarrow_E B. \text{bij-betw } f A B\} // \text{domain-and-range-permutation } A B = \{\}$
 <proof>

lemma *card-bijections-domain-and-range-permutation-eq-0*:

assumes $\text{card } A \neq \text{card } B$
shows $\text{card } (\{f \in A \rightarrow_E B. \text{bij-betw } f A B\} // \text{domain-and-range-permutation } A B) = 0$
 <proof>

Two alternative proofs for the cardinality of bijections up to a permutation on A and B.

lemma

assumes $\text{finite } A \text{ finite } B$
assumes $\text{card } A = \text{card } B$
shows $\text{card } (\{f \in A \rightarrow_E B. \text{bij-betw } f A B\} // \text{domain-and-range-permutation } A B) = 1$
 <proof>

lemma *card-bijections-domain-and-range-permutation-eq-1*:

assumes $\text{finite } A \text{ finite } B$
assumes $\text{card } A = \text{card } B$
shows $\text{card } (\{f \in A \rightarrow_E B. \text{bij-betw } f A B\} // \text{domain-and-range-permutation } A B) = 1$
 <proof>

lemma *card-bijections-domain-and-range-permutation*:

assumes $\text{finite } A \text{ finite } B$
shows $\text{card } (\{f \in A \rightarrow_E B. \text{bij-betw } f A B\} // \text{domain-and-range-permutation } A B) = \text{iverson } (\text{card } A = \text{card } B)$
 <proof>

end

17 Direct Proofs for Cardinality of Bijections

theory *Card-Bijections-Direct*

imports

Equiv-Relations-on-Functions

Twelfefold-Way-Core
begin

17.1 Bijections from A to B up to a Permutation on A

17.1.1 Equivalence Class

lemma *bijections-in-domain-permutation:*

assumes *finite A finite B*
assumes *card A = card B*
shows $\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} \in \{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} //$
domain-permutation A B
<proof>

lemma *bij-betw-quotient-domain-permutation-eq:*

assumes *finite A finite B*
assumes *card A = card B*
shows $\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} //$ *domain-permutation A B* = $\{\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\}\}$
<proof>

17.1.2 Cardinality

lemma

assumes *finite A finite B*
assumes *card A = card B*
shows *card* ($\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} //$ *domain-permutation A B*) = 1
<proof>

17.2 Bijections from A to B up to a Permutation on B

17.2.1 Equivalence Class

lemma *bijections-in-range-permutation:*

assumes *finite A finite B*
assumes *card A = card B*
shows $\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} \in \{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} //$
range-permutation A B
<proof>

lemma *bij-betw-quotient-range-permutation-eq:*

assumes *finite A finite B*
assumes *card A = card B*
shows $\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} //$ *range-permutation A B* = $\{\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\}\}$
<proof>

17.2.2 Cardinality

lemma *card-bijections-range-permutation-eq-1:*

assumes *finite A finite B*

assumes $\text{card } A = \text{card } B$
shows $\text{card } (\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} // \text{range-permutation } A \ B) = 1$
 <proof>

17.3 Bijections from A to B up to a Permutation on A and B

17.3.1 Equivalence Class

lemma *bijections-in-domain-and-range-permutation:*

assumes $\text{finite } A \ \text{finite } B$
assumes $\text{card } A = \text{card } B$
shows $\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} \in \{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} // \text{domain-and-range-permutation } A \ B$
 <proof>

lemma *bij-betw-quotient-domain-and-range-permutation-eq:*

assumes $\text{finite } A \ \text{finite } B$
assumes $\text{card } A = \text{card } B$
shows $\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} // \text{domain-and-range-permutation } A \ B = \{\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\}\}$
 <proof>

17.3.2 Cardinality

lemma *card-bijections-domain-and-range-permutation-eq-1:*

assumes $\text{finite } A \ \text{finite } B$
assumes $\text{card } A = \text{card } B$
shows $\text{card } (\{f \in A \rightarrow_E B. \text{bij-betw } f \ A \ B\} // \text{domain-and-range-permutation } A \ B) = 1$
 <proof>

end

18 The Twelfold Way

theory *Twelfold-Way*

imports

Preliminaries
Twelfold-Way-Core
Equiv-Relations-on-Functions
Twelfold-Way-Entry1
Twelfold-Way-Entry2
Twelfold-Way-Entry4
Twelfold-Way-Entry5
Twelfold-Way-Entry6
Twelfold-Way-Entry7
Twelfold-Way-Entry8
Twelfold-Way-Entry9

Twelfold-Way-Entry3
Twelfold-Way-Entry10
Twelfold-Way-Entry11
Twelfold-Way-Entry12
Card-Bijections
Card-Bijections-Direct
begin

end

References

- [1] K. P. Bogart. *Combinatorics Through Guided Discovery*. 2004.
- [2] L. Bulwahn. Cardinality of set partitions. *Archive of Formal Proofs*, Dec. 2015. http://isa-afp.org/entries/Card_Partitions.shtml, Formal proof development.
- [3] L. Bulwahn. Cardinality of multisets. *Archive of Formal Proofs*, June 2016. http://isa-afp.org/entries/Card_Multisets.shtml, Formal proof development.
- [4] L. Bulwahn. Cardinality of number partitions. *Archive of Formal Proofs*, Jan. 2016. http://isa-afp.org/entries/Card_Number_Partitions.shtml, Formal proof development.
- [5] R. P. Stanley. *Enumerative Combinatorics. Volume 1*. Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge, New York, second edition, 2012.
- [6] Wikipedia. Twelfold way — wikipedia, the free encyclopedia, 2016. [Online; accessed 4-October-2016].