# Basic Geometric Properties of Triangles 

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#### Abstract

In this work, we define angles between vectors and between three points. Building on this, we prove basic geometric properties of triangles, such as the Isosceles Triangle Theorem, the Law of Sines and the Law of Cosines, that the sum of the angles of a triangle is $\pi$, and the congruence theorems for triangles.

The definitions and proofs were developed following those by John Harrison in HOL Light. However, due to Isabelle's type class system, all definitions and theorems in the Isabelle formalisation hold for all real inner product spaces.


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## 1 Definition of angles

```
theory Angles
imports
    HOL-Analysis.Multivariate-Analysis
begin
lemma collinear-translate-iff: collinear \(\left(((+) a)^{\prime} A\right) \longleftrightarrow\) collinear \(A\)
    \(\langle p r o o f\rangle\)
```

definition vangle where
vangle $u v=($ if $u=0 \vee v=0$ then pi／2 else $\arccos (u \cdot v /$（norm $u *$ norm $v)$ ））
definition angle where
angle a $b c=$ vangle $(a-b)(c-b)$
lemma angle－altdef：angle a $b c=\arccos ((a-b) \cdot(c-b) /($ dist $a b *$ dist $c$ b）） $\langle p r o o f\rangle$
lemma vangle－0－left［simp］：vangle $0 v=p i / 2$ and vangle－0－right［simp］：vangle $u 0=p i / 2$〈proof〉
lemma vangle－refl $[$ simp $]: u \neq 0 \Longrightarrow$ vangle $u u=0$ $\langle p r o o f\rangle$
lemma angle－refl［simp］：angle a a b＝pi／2 angle abb＝pi／2 $\langle p r o o f\rangle$
lemma angle－refl－mid［simp］：$a \neq b \Longrightarrow$ angle a $b a=0$ $\langle p r o o f\rangle$
lemma cos－vangle：cos $($ vangle $u v)=u \cdot v /($ norm $u *$ norm $v)$ $\langle p r o o f\rangle$
lemma cos－angle： $\cos ($ angle $a b c)=(a-b) \cdot(c-b) /($ dist $a b *$ dist $c b)$ $\langle p r o o f\rangle$
lemma inner－conv－angle：$(a-b) \cdot(c-b)=$ dist $a b *$ dist $c b * \cos$（angle a $b$ c） $\langle p r o o f\rangle$
lemma vangle－commute：vangle $u v=$ vangle $v u$ $\langle p r o o f\rangle$
lemma angle－commute：angle a $b c=$ angle $c b a$ $\langle$ proof $\rangle$
lemma vangle－nonneg：vangle $u v \geq 0$ and vangle－le－pi：vangle $u v \leq p i$〈proof〉
lemmas vangle－bounds $=$ vangle－nonneg vangle－le－pi
lemma angle－nonneg：angle a $b c \geq 0$ and angle－le－pi：angle a $b c \leq p i$ $\langle p r o o f\rangle$

```
lemmas angle-bounds \(=\) angle-nonneg angle-le-pi
lemma sin-vangle-nonneg: \(\sin (\) vangle \(u v) \geq 0\)
    \(\langle p r o o f\rangle\)
lemma sin-angle-nonneg: sin (angle abc) \(\geq 0\)
    〈proof〉
lemma vangle-eq-0D:
    assumes vangle \(u v=0\)
    shows norm \(u *_{R} v=\operatorname{norm} v *_{R} u\)
\(\langle p r o o f\rangle\)
lemma vangle-eq-piD:
    assumes vangle \(u v=p i\)
    shows norm \(u *_{R} v+\operatorname{norm} v *_{R} u=0\)
\(\langle p r o o f\rangle\)
lemma dist-triangle-eq:
    fixes \(a b c::\) ' \(a\) :: real-inner
    shows \((\) dist \(a c=\) dist \(a b+\operatorname{dist} b c) \longleftrightarrow \operatorname{dist} a b *_{R}(c-b)+\operatorname{dist} b c *_{R}(a\)
\(-b)=0\)
    \(\langle p r o o f\rangle\)
lemma angle-eq-pi-imp-dist-additive:
    assumes angle a bc=pi
    shows dist \(a c=\) dist \(a b+\) dist \(b c\)
    〈proof〉
lemma orthogonal-iff-vangle: orthogonal \(u v \longleftrightarrow\) vangle \(u v=p i / 2\)
    \(\langle p r o o f\rangle\)
lemma cos-minus1-imp-pi:
    assumes \(\cos x=-1 x \geq 0 x<3 * p i\)
    shows \(\quad x=p i\)
\(\langle p r o o f\rangle\)
lemma vangle－eqI：
    assumes \(u \neq 0 v \neq 0 w \neq 0 x \neq 0\)
    assumes \((u \cdot v) *\) norm \(w *\) norm \(x=(w \cdot x) *\) norm \(u *\) norm \(v\)
    shows vangle \(u v=\) vangle \(w x\)
    〈proof〉
lemma angle-eqI:
    assumes \(a \neq b a \neq c d \neq e d \neq f\)
    assumes \(((b-a) \cdot(c-a)) *\) dist \(d e *\) dist \(d f=((e-d) \cdot(f-d)) *\) dist \(a b *\)
```

```
dist a c
    shows angle b a c= angle e d f
    <proof\rangle
```

lemma cos-vangle-eqD: $\cos ($ vangle $u v)=\cos ($ vangle $w x) \Longrightarrow$ vangle $u v=$
vangle $w x$
〈proof〉
lemma cos-angle-eqD: cos (angle abc)= cos (angle def) angle abc=
angle $d e f$
$\langle p r o o f\rangle$
lemma sin-vangle-zero-iff: sin (vangle $u v)=0 \longleftrightarrow$ vangle $u v \in\{0, p i\}$
$\langle p r o o f\rangle$
lemma sin-angle-zero-iff: sin (angle a bc)=0 $\longleftrightarrow$ angle a $b c \in\{0, p i\}$
$\langle p r o o f\rangle$
lemma vangle-collinear: vangle $u v \in\{0, p i\} \Longrightarrow$ collinear $\{0, u, v\}$
$\langle p r o o f\rangle$
lemma angle-collinear: angle $a b c \in\{0, p i\} \Longrightarrow$ collinear $\{a, b, c\}$
$\langle p r o o f\rangle$
lemma not-collinear-vangle: $\neg$ collinear $\{0, u, v\} \Longrightarrow$ vangle $u v \in\{0<. .<p i\}$
$\langle p r o o f\rangle$
lemma not-collinear-angle: $\neg$ collinear $\{a, b, c\} \Longrightarrow$ angle a $b c \in\{0<. .<p i\}$
$\langle p r o o f\rangle$

### 1.1 Contributions from Lukas Bulwahn

```
lemma vangle-scales:
    assumes 0<c
    shows vangle (c*R
<proof\rangle
lemma vangle-inverse:
    vangle (- v
<proof>
lemma orthogonal-iff-angle:
    shows orthogonal (A-B) (C-B)\longleftrightarrow angle A B C=pi/2
\langleproof\rangle
lemma angle-inverse:
    assumes between (A,C)B
    assumes }A\not=BB\not=
    shows angle A B D=pi- angle C B D
```

```
\langleproof\rangle
```

lemma strictly－between－implies－angle－eq－pi：
assumes between $(A, C) B$
assumes $A \neq B B \neq C$
shows angle $A B C=p i$
$\langle p r o o f\rangle$
end

## 2 Basic Properties of Triangles

## theory Triangle imports <br> Angles <br> begin

We prove a number of basic geometric properties of triangles．All theorems hold in any real inner product space．

## 2．1 Thales＇theorem

## theorem thales：

fixes $A B C$ ：：＇$a$ ：：real－inner
assumes dist $B($ midpoint $A C)=\operatorname{dist} A C / 2$
shows orthogonal $(A-B)(C-B)$
$\langle p r o o f\rangle$

## 2．2 Sine and cosine laws

The proof of the Law of Cosines follows trivially from the definition of the angle，the definition of the norm in vector spaces with an inner product and the bilinearity of the inner product．

```
lemma cosine-law-vector:
    norm \((u-v)^{\wedge} 2=\operatorname{norm} u^{\wedge} 2+\operatorname{norm} v \wedge 2-2 *\) norm \(u *\) norm \(v * \cos\)
(vangle \(u v\) )
    〈proof〉
lemma cosine-law-triangle:
    dist \(b c\) へ2 \(=\) dist \(a b へ 2+\operatorname{dist} a c \wedge 2-2 *\) dist \(a b *\) dist \(a c * \cos\) (angle
\(b a c\) )
    \(\langle p r o o f\rangle\)
```

According to our definition，angles are always between 0 and $\pi$ and therefore， the sign of an angle is always non－negative．We can therefore look at $\sin (\alpha)^{2}$ ， which we can express in terms of $\cos (\alpha)$ using the identity $\sin (\alpha)^{2}+\cos (\alpha)^{2}=$ 1．The remaining proof is then a trivial consequence of the definitions．

## lemma sine-law-triangle:

$$
\sin (\text { angle } a b c) * \text { dist } b c=\sin (\text { angle } b a c) * \text { dist } a c(\text { is } ? A=? B)
$$

$\langle p r o o f\rangle$
The following forms of the Law of Sines/Cosines are more convenient for eliminating sines/cosines from a goal completely.

```
lemma cosine-law-triangle':
    \(2 *\) dist \(a b *\) dist \(a c * \cos (\) angle \(b a c)=(\) dist \(a b へ 2+\) dist \(a c \wedge 2-d i s t b\)
\(c^{\wedge}\) 2)
    \(\langle p r o o f\rangle\)
```

lemma cosine-law-triangle ${ }^{\prime \prime}$ :
$\cos ($ angle $b a c)=\left(\right.$ dist $\left.a b{ }^{\wedge} 2+\operatorname{dist} a c \wedge 2-d i s t b c \wedge 2\right) /(2 *$ dist $a b *$
dist a c)
$\langle p r o o f\rangle$
lemma sine-law-triangle':
$b \neq c \Longrightarrow \sin ($ angle $a b c)=\sin ($ angle $b a c) *$ dist $a c /$ dist $b c$
$\langle p r o o f\rangle$
lemma sine-law-triangle":
$b \neq c \Longrightarrow \sin ($ angle $c b a)=\sin ($ angle $b a c) *$ dist $a c / d i s t b c$ $\langle$ proof $\rangle$

### 2.3 Sum of angles

## context

begin

```
private lemma gather-squares: a* (a*b)=a^2 * (b :: real)
    \langleproof\rangle lemma eval-power: x^ numeral n =x*x` pred-numeral n
    <proof\rangle
```

The proof that the sum of the angles in a triangle is $\pi$ is somewhat more involved. Following the HOL Light proof by John Harrison, we first prove that $\cos (\alpha+\beta+\gamma)=-1$ and $\alpha+\beta+\gamma \in[0 ; 3 \pi)$, which then implies the theorem.
The main work is proving $\cos (\alpha+\beta+\gamma)$. This is done using the addition theorems for the sine and cosine, then using the Laws of Sines to eliminate all sin terms save $\sin (\gamma)^{2}$, which only appears squared in the remaining goal. We then use $\sin (\gamma)^{2}=1-\cos (\gamma)^{2}$ to eliminate this term and apply the law of cosines to eliminate this term as well.

The remaining goal is a non-linear equation containing only the length of the sides of the triangle. It can be shown by simple algebraic rewriting.

```
lemma angle-sum-triangle:
    assumes }a\not=b\veeb\not=c\vee\\mp@code{l}
    shows angle c a b angle a b c + angle b c a = pi
```

```
\langleproof\rangle
```

end

### 2.4 Congruence Theorems

If two triangles agree on two angles at a non-degenerate side, the third angle must also be equal.

```
lemma similar-triangle-aa:
    assumes b1 =c1 b2 f=c2
    assumes angle a1 b1 c1 = angle a2 b2 c2
    assumes angle b1 c1 a1 = angle b2 c2 a2
    shows angle b1 a1 c1 = angle b2 a2 c2
<proof\rangle
```

A triangle is defined by its three angles and the lengths of three sides up to congruence. Two triangles are congruent if they have their angles are the same and their sides have the same length.

```
locale congruent-triangle =
    fixes a1 b1 c1 ::' 'a :: real-inner and a2 b2 c2 :: 'b :: real-inner
    assumes sides': dist a1 b1 = dist a2 b2 dist a1 c1 = dist a2 c2 dist b1 c1 =
dist b2 c2
            and angles': angle b1 a1 c1 = angle b2 a2 c2 angle a1 b1 c1 = angle a2 b2 c2
                angle a1 c1 b1 = angle a2 c2 b2
begin
lemma sides:
    dist a1 b1 = dist a2 b2 dist a1 c1 = dist a2 c2 dist b1 c1 = dist b2 c2
    dist b1 a1 = dist a2 b2 dist c1 a1 = dist a2 c2 dist c1 b1 = dist b2 c2
    dist a1 b1 = dist b2 a2 dist a1 c1 = dist c2 a2 dist b1 c1 = dist c2 b2
    dist b1 a1 = dist b2 a2 dist c1 a1 = dist c2 a2 dist c1 b1 = dist c2 b2
    \langleproof\rangle
lemma angles:
    angle b1 a1 c1 = angle b2 a2 c2 angle a1 b1 c1 = angle a2 b2 c2 angle a1 c1 b1
= angle a2 c2 b2
    angle c1 a1 b1 = angle b2 a2 c2 angle c1 b1 a1 = angle a2 b2 c2 angle b1 c1 a1
= angle a2 c2 b2
    angle b1 a1 c1 = angle c2 a2 b2 angle a1 b1 c1 = angle c2 b2 a2 angle a1 c1 b1
= angle b2 c2 a2
    angle c1 a1 b1 = angle c2 a2 b2 angle c1 b1 a1 = angle c2 b2 a2 angle b1 c1 a1
= angle b2 c2 a2
    <proof>
```

end
lemmas congruent-triangle $D=$ congruent-triangle.sides congruent-triangle.angles
Given two triangles that agree on a subset of its side lengths and angles
that are sufficient to define a triangle uniquely up to congruence, one can conclude that they must also agree on all remaining quantities, i.e. that they are congruent.
The following four congruence theorems state what constitutes such a uniquelydefining subset of quantities. Each theorem states in its name which quantities are required and in which order (clockwise or counter-clockwise): an "s" stands for a side, an "a" stands for an angle.

The lemma "congruent-triangleI-sas, for example, requires that two adjacent sides and the angle inbetween are the same in both triangles.

```
lemma congruent-triangleI-sss:
    fixes a1 b1 c1 :: 'a :: real-inner and a2 b2 c2 :: 'b :: real-inner
    assumes dist a1 b1 = dist a2 b2
    assumes dist b1 c1 = dist b2 c2
    assumes dist a1 c1 = dist a2 c2
    shows congruent-triangle a1 b1 c1 a2 b2 c2
<proof\rangle
lemmas congruent-triangle-sss = congruent-triangleD[OF congruent-triangleI-sss]
lemma congruent-triangleI-sas:
    assumes dist a1 b1 = dist a2 b2
    assumes dist b1 c1 = dist b2 c2
    assumes angle a1 b1 c1 = angle a2 b2 c2
    shows congruent-triangle a1 b1 c1 a2 b2 c2
<proof\rangle
lemmas congruent-triangle-sas = congruent-triangleD[OF congruent-triangleI-sas]
lemma congruent-triangleI-aas:
    assumes angle a1 b1 c1 = angle a2 b2 c2
    assumes angle b1 c1 a1 = angle b2 c2 a2
    assumes dist a1 b1 = dist a2 b2
    assumes \negcollinear {a1,b1,c1}
    shows congruent-triangle a1 b1 c1 a2 b2 c2
<proof\rangle
lemmas congruent-triangle-aas = congruent-triangleD[OF congruent-triangleI-aas]
lemma congruent-triangleI-asa:
    assumes angle a1 b1 c1 = angle a2 b2 c2
    assumes dist a1 b1 = dist a2 b2
    assumes angle b1 a1 c1 = angle b2 a2 c2
    assumes \negcollinear {a1, b1, c1}
    shows congruent-triangle a1 b1 c1 a2 b2 c2
<proof\rangle
lemmas congruent-triangle-asa = congruent-triangleD[OF congruent-triangleI-asa]
```


### 2.5 Isosceles Triangle Theorem

We now prove the Isosceles Triangle Theorem: in a triangle where two sides have the same length, the two angles that are adjacent to only one of the two sides must be equal.
lemma isosceles-triangle:
assumes dist a $c=$ dist $b c$
shows angle $b a c=$ angle $a b c$
〈proof〉
For the non-degenerate case (i.e. the three points are not collinear), We also prove the converse.

```
lemma isosceles-triangle-converse:
    assumes angle a b c= angle b a c\negcollinear {a,b,c}
    shows dist a c= dist bc
    <proof\rangle
```


### 2.6 Contributions by Lukas Bulwahn

lemma Pythagoras:
fixes $A B C$ :: ' $a$ :: real-inner
assumes orthogonal $(A-C)(B-C)$
shows $(\text { dist } B C)^{\wedge} 2+(\text { dist } C A)^{\wedge} 2=(\operatorname{dist} A B)^{\wedge} 2$
$\langle p r o o f\rangle$
lemma isosceles-triangle-orthogonal-on-midpoint:
fixes $A B C$ :: 'a :: euclidean-space
assumes dist $C A=\operatorname{dist} C B$
shows orthogonal $(C-$ midpoint $A B)(A-$ midpoint $A B)$
$\langle p r o o f\rangle$
end

