Basic Geometric Properties of Triangles

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Abstract

In this work, we define angles between vectors and between three points. Building on this, we prove basic geometric properties of triangles, such as the Isosceles Triangle Theorem, the Law of Sines and the Law of Cosines, that the sum of the angles of a triangle is $\pi$, and the congruence theorems for triangles.

The definitions and proofs were developed following those by John Harrison in HOL Light. However, due to Isabelle’s type class system, all definitions and theorems in the Isabelle formalisation hold for all real inner product spaces.

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1 Definition of angles

theory Angles
imports
  HOL–Analysis.Analysis
begin

lemma collinear-translate-iff: collinear (((+ a) · A) ←→ collinear A
  ⟨proof⟩
**definition vangle where**
\[
vangle u v = (if u = 0 \lor v = 0 then \pi / 2 else \arccos (u \cdot v / (\|u\| \times \|v\|)))
\]

**definition angle where**
\[
angle a b c = vangle (a - b) (c - b)
\]

**lemma angle-altdef:** \(\angle a b c = \arccos (((a - b) \cdot (c - b)) / (\text{dist } a b \times \text{dist } c b))\)

\[
\langle \text{proof} \rangle
\]

**lemma vangle-0-left [simp]:** \(vangle 0 v = \pi / 2\)

**and** \(\text{vangle-0-right [simp]: vangle u 0 = \pi / 2}\)

\[
\langle \text{proof} \rangle
\]

**lemma vangle-refl [simp]:** \(u \neq 0 \implies vangle u u = 0\)

\[
\langle \text{proof} \rangle
\]

**lemma vangle-refl-mid [simp]:** \(a \neq b \implies \angle a b a = 0\)

\[
\langle \text{proof} \rangle
\]

**lemma cos-vangle:** \(\cos (vangle u v) = u \cdot v / (\|u\| \times \|v\|)\)

\[
\langle \text{proof} \rangle
\]

**lemma cos-angle:** \(\cos (\angle a b c) = ((a - b) \cdot (c - b)) / (\text{dist } a b \times \text{dist } c b)\)

\[
\langle \text{proof} \rangle
\]

**lemma inner-conv-angle:** \((a - b) \cdot (c - b) = \text{dist } a b \times \text{dist } c b \times \cos (\angle a b c)\)

\[
\langle \text{proof} \rangle
\]

**lemma vangle-commute:** \(vangle u v = vangle v u\)

\[
\langle \text{proof} \rangle
\]

**lemma angle-commute:** \(\angle a b c = \angle c b a\)

\[
\langle \text{proof} \rangle
\]

**lemma vangle-nonneg:** \(vangle u v \geq 0\) and \(\text{vangle-le-pi: vangle u v \leq \pi}\)

\[
\langle \text{proof} \rangle
\]

**lemmas vangle-bounds = vangle-nonneg vangle-le-pi**

**lemma angle-nonneg:** \(\angle a b c \geq 0\) and \(\text{angle-le-pi: angle a b c \leq \pi}\)

\[
\langle \text{proof} \rangle
\]
lemmas angle-bounds = angle-nonneg angle-le-pi

lemma sin-vangle-nonneg: sin (vangle u v) ≥ 0
⟨proof⟩

lemma sin-angle-nonneg: sin (angle a b c) ≥ 0
⟨proof⟩

lemma vangle-eq-0D:
  assumes vangle u v = 0
  shows norm u *R v = norm v *R u
⟨proof⟩

lemma vangle-eq-piD:
  assumes vangle u v = pi
  shows norm u *R v + norm v *R u = 0
⟨proof⟩

lemma dist-triangle-eq:
  fixes a b c :: 'a :: real-inner
  shows (dist a c = dist a b + dist b c) ←→ dist a b *R (c - b) + dist b c *R (a - b) = 0
⟨proof⟩

lemma angle-eq-pi-imp-dist-additive:
  assumes angle a b c = pi
  shows dist a c = dist a b + dist b c
⟨proof⟩

lemma orthogonal-iff-vangle: orthogonal u v ←→ vangle u v = pi / 2
⟨proof⟩

lemma cos-minus1-imp-pi:
  assumes cos x = −1 x ≥ 0 x < 3 * pi
  shows x = pi
⟨proof⟩

lemma vangle-eql:
  assumes u ≠ 0 v ≠ 0 w ≠ 0 x ≠ 0
  assumes (u · v) * norm w * norm x = (w · x) * norm u * norm v
  shows vangle u v = vangle w x
⟨proof⟩

lemma angle-eql:
  assumes a ≠ b a ≠ c d ≠ e d ≠ f
  assumes ((b−a) · (c−a)) * dist d e * dist d f = ((e−d) · (f−d)) * dist a b *
\[
dist a c \quad \text{shows} \quad \angle a b c = \angle e d f
\]

\textbf{lemma \ cos-vangle-eqD:} \ \cos (\angle u v) = \cos (\angle w x) \implies \angle u v = \angle w x

\textbf{lemma \ cos-angle-eqD:} \ \cos (\angle a b c) = \cos (\angle d e f) \implies \angle a b c = \angle d e f

\textbf{lemma \ sin-vangle-zero-iff:} \ \sin (\angle u v) = 0 \iff \angle u v \in \{0, \pi\}

\textbf{lemma \ sin-angle-zero-iff:} \ \sin (\angle a b c) = 0 \iff \angle a b c \in \{0, \pi\}

\textbf{lemma \ vangle-collinear:} \ \angle u v \in \{0, \pi\} \implies \text{collinear} \{0, u, v\}

\textbf{lemma \ angle-collinear:} \ \angle a b c \in \{0, \pi\} \implies \text{collinear} \{a, b, c\}

\textbf{lemma \ not-collinear-vangle:} \ \neg \text{collinear} \{0, u, v\} \implies \angle u v \in \{0 <..< \pi\}

\textbf{lemma \ not-collinear-angle:} \ \neg \text{collinear} \{a, b, c\} \implies \angle a b c \in \{0 <..< \pi\}

\textbf{1.1 Contributions from Lukas Bulwahn}

\textbf{lemma \ vangle-scales:}
\begin{itemize}
  \item \textbf{assumes} \ 0 < c
  \item \textbf{shows} \ \angle (c \ast v_1) v_2 = \angle v_1 v_2
\end{itemize}

\textbf{lemma \ vangle-inverse:}
\begin{itemize}
  \item \ \angle (- v_1) v_2 = \pi - \angle v_1 v_2
\end{itemize}

\textbf{lemma \ orthogonal-iff-angle:}
\begin{itemize}
  \item \textbf{shows} \ \text{orthogonal} \ (A - B) (C - B) \iff \angle A B C = \pi / 2
\end{itemize}

\textbf{lemma \ angle-inverse:}
\begin{itemize}
  \item \textbf{assumes} \ \text{between} \ (A, C) B
  \item \textbf{assumes} \ A \neq B \ B \neq C
  \item \textbf{shows} \ \angle A B D = \pi - \angle C B D
\end{itemize}
lemma strictly-between-implies-angle-eq-pi:
  assumes between (A, C) B
  assumes A ≠ B B ≠ C
  shows angle A B C = pi
⟨proof⟩
end

2 Basic Properties of Triangles

We prove a number of basic geometric properties of triangles. All theorems
hold in any real inner product space.

2.1 Thales’ theorem

theorem thales:
  fixes A B C :: 'a :: real-inner
  assumes dist B (midpoint A C) = dist A C / 2
  shows   orthogonal (A − B) (C − B)
⟨proof⟩

2.2 Sine and cosine laws

The proof of the Law of Cosines follows trivially from the definition of the
angle, the definition of the norm in vector spaces with an inner product and
the bilinearity of the inner product.

lemma cosine-law-vector:
  norm (u − v) ^ 2 = norm u ^ 2 + norm v ^ 2 − 2 * norm u * norm v * cos
                     (vangle u v)
  ⟨proof⟩

lemma cosine-law-triangle:
  dist b c ^ 2 = dist a b ^ 2 + dist a c ^ 2 − 2 * dist a b * dist a c * cos (angle
                     b a c)
  ⟨proof⟩

According to our definition, angles are always between 0 and π and therefore,
the sign of an angle is always non-negative. We can therefore look at sin(α)^2,
which we can express in terms of $\cos(\alpha)$ using the identity $\sin(\alpha)^2 + \cos(\alpha)^2 = 1$. The remaining proof is then a trivial consequence of the definitions.

**lemma sine-law-triangle:**

\[
\sin (\text{angle } a \ b \ c) \ast \text{dist } b \ c = \sin (\text{angle } b \ a \ c) \ast \text{dist } a \ c \ (\text{is } ?A = ?B)
\]

⟨proof⟩

The following forms of the Law of Sines/Cosines are more convenient for eliminating sines/cosines from a goal completely.

**lemma cosine-law-triangle':**

\[
2 \ast \text{dist } a \ b \ast \text{dist } a \ c \ast \cos (\text{angle } b \ a \ c) = (\text{dist } a \ b \ast 2 + \text{dist } a \ c \ast 2 - \text{dist } b \ c \ast 2)
\]

⟨proof⟩

**lemma cosine-law-triangle'':**

\[
\cos (\text{angle } b \ a \ c) = (\text{dist } a \ b \ast 2 + \text{dist } a \ c \ast 2 - \text{dist } b \ c \ast 2) / (2 \ast \text{dist } a \ b \ast \text{dist } a \ c)
\]

⟨proof⟩

**lemma sine-law-triangle':**

\[
b \neq c \implies \sin (\text{angle } a \ b \ c) = \sin (\text{angle } b \ a \ c) \ast \text{dist } a \ c / \text{dist } b \ c
\]

⟨proof⟩

**lemma sine-law-triangle'':**

\[
b \neq c \implies \sin (\text{angle } c \ b \ a) = \sin (\text{angle } b \ a \ c) \ast \text{dist } a \ c / \text{dist } b \ c
\]

⟨proof⟩

### 2.3 Sum of angles

**context**

**begin**

**private lemma gather-squares:**

\[
a \ast (a \ast b) = a \ast 2 \ast (b :: \text{real})
\]

⟨proof⟩

**lemma eval-power:**

\[
x \ast \text{numeral } n = x \ast x \ast \text{pred-numeral } n
\]

⟨proof⟩

The proof that the sum of the angles in a triangle is $\pi$ is somewhat more involved. Following the HOL Light proof by John Harrison, we first prove that $\cos(\alpha + \beta + \gamma) = -1$ and $\alpha + \beta + \gamma \in [0; 3\pi)$, which then implies the theorem.

The main work is proving $\cos(\alpha + \beta + \gamma)$. This is done using the addition theorems for the sine and cosine, then using the Laws of Sines to eliminate all sin terms save $\sin(\gamma)^2$, which only appears squared in the remaining goal. We then use $\sin(\gamma)^2 = 1 - \cos(\gamma)^2$ to eliminate this term and apply the law of cosines to eliminate this term as well.

The remaining goal is a non-linear equation containing only the length of the sides of the triangle. It can be shown by simple algebraic rewriting.

**lemma angle-sum-triangle:**
\[ a \neq b \lor b \neq c \lor a \neq c \]

shows \( \angle c + \angle a b + \angle b c + \angle a = \pi \)

\[ \langle \text{proof} \rangle \]

end

2.4 Congruence Theorems

If two triangles agree on two angles at a non-degenerate side, the third angle must also be equal.

\[ \text{lemma similar-triangle-aa:} \]

\[ \text{assumes } b_1 \neq c_1 \land b_2 \neq c_2 \]

\[ \text{assumes } \angle a_1 b_1 c_1 = \angle a_2 b_2 c_2 \]

\[ \text{assumes } \angle b_1 c_1 a_1 = \angle b_2 c_2 a_2 \]

\[ \text{shows } \angle b_1 a_1 c_1 = \angle b_2 a_2 c_2 \]

\[ \langle \text{proof} \rangle \]

A triangle is defined by its three angles and the lengths of three sides up to congruence. Two triangles are congruent if they have their angles are the same and their sides have the same length.

\[ \text{locale congruent-triangle =} \]

\[ \text{fixes } a_1 b_1 c_1 :: \text{real-inner} \land a_2 b_2 c_2 :: \text{real-inner} \]

\[ \text{assumes sides': } \text{dist } a_1 b_1 = \text{dist } a_2 b_2 \land \text{dist } a_1 c_1 = \text{dist } a_2 c_2 \land \text{dist } b_1 c_1 = \text{dist } b_2 c_2 \]

\[ \text{and angles': } \angle b_1 a_1 c_1 = \angle b_2 a_2 c_2 \land \angle a_1 b_1 c_1 = \angle a_2 b_2 c_2 \]

\[ \angle a_1 c_1 b_1 = \angle a_2 c_2 b_2 \]

\[ \text{begin} \]

\[ \text{lemma sides:} \]

\[ \text{dist } a_1 b_1 = \text{dist } a_2 b_2 \land \text{dist } a_1 c_1 = \text{dist } a_2 c_2 \land \text{dist } b_1 c_1 = \text{dist } b_2 c_2 \]

\[ \text{dist } b_1 a_1 = \text{dist } a_2 b_2 \land \text{dist } c_1 a_1 = \text{dist } a_2 c_2 \land \text{dist } c_1 b_1 = \text{dist } b_2 c_2 \]

\[ \text{dist } b_1 c_1 = \text{dist } b_2 c_2 \land \text{dist } b_1 a_1 = \text{dist } b_2 a_2 \land \text{dist } c_1 a_1 = \text{dist } c_2 a_2 \land \text{dist } c_1 b_1 = \text{dist } c_2 b_2 \]

\[ \langle \text{proof} \rangle \]

\[ \text{lemma angles:} \]

\[ \angle b_1 a_1 c_1 = \angle b_2 a_2 c_2 \land \angle a_1 b_1 c_1 = \angle a_2 b_2 c_2 \land \angle a_1 c_1 b_1 = \angle a_2 c_2 b_2 \]

\[ \angle c_1 a_1 b_1 = \angle c_2 a_2 b_2 \land \angle c_1 b_1 a_1 = \angle c_2 b_2 a_2 \land \angle b_1 c_1 a_1 = \angle b_2 c_2 a_2 \]

\[ \langle \text{proof} \rangle \]

end
lemmas congruent-triangleD = congruent-triangle.sides congruent-triangle.angles

Given two triangles that agree on a subset of its side lengths and angles that are sufficient to define a triangle uniquely up to congruence, one can conclude that they must also agree on all remaining quantities, i.e. that they are congruent.

The following four congruence theorems state what constitutes such a uniquely-defining subset of quantities. Each theorem states in its name which quantities are required and in which order (clockwise or counter-clockwise): an “s” stands for a side, an “a” stands for an angle.

The lemma “congruent-triangleI-sas, for example, requires that two adjacent sides and the angle inbetween are the same in both triangles.

lemma congruent-triangleI-sss:
fixtures a1 b1 c1 :: 'a :: real-inner and a2 b2 c2 :: 'b :: real-inner
assumes dist a1 b1 = dist a2 b2
assumes dist b1 c1 = dist b2 c2
assumes dist a1 c1 = dist a2 c2
shows congruent-triangle a1 b1 c1 a2 b2 c2
⟨proof⟩

lemmas congruent-triangle-sss = congruent-triangleD[OF congruent-triangleI-sss]

lemma congruent-triangleI-sas:
assumes dist a1 b1 = dist a2 b2
assumes dist b1 c1 = dist b2 c2
assumes angle a1 b1 c1 = angle a2 b2 c2
shows congruent-triangle a1 b1 c1 a2 b2 c2
⟨proof⟩

lemmas congruent-triangle-sas = congruent-triangleD[OF congruent-triangleI-sas]

lemma congruent-triangleI-aas:
assumes angle a1 b1 c1 = angle a2 b2 c2
assumes angle b1 c1 a1 = angle b2 c2 a2
assumes dist a1 b1 = dist a2 b2
assumes ¬collinear {a1, b1, c1}
shows congruent-triangle a1 b1 c1 a2 b2 c2
⟨proof⟩

lemmas congruent-triangle-aas = congruent-triangleD[OF congruent-triangleI-aas]

lemma congruent-triangleI-asa:
assumes angle a1 b1 c1 = angle a2 b2 c2
assumes dist a1 b1 = dist a2 b2
assumes angle b1 a1 c1 = angle b2 a2 c2
assumes ¬collinear {a1, b1, c1}
shows congruent-triangle a1 b1 c1 a2 b2 c2
⟨proof⟩

lemmas congruent-triangle-asa = congruent-triangleD[OF congruent-triangleI-asa]
lemmas congruent-triangle-asa = congruent-triangleD[OF congruent-triangleI-asa]

2.5 Isosceles Triangle Theorem

We now prove the Isosceles Triangle Theorem: in a triangle where two sides have the same length, the two angles that are adjacent to only one of the two sides must be equal.

lemma isosceles-triangle:
  assumes dist a c = dist b c
  shows angle b a c = angle a b c
  ⟨proof⟩

For the non-degenerate case (i.e. the three points are not collinear), We also prove the converse.

lemma isosceles-triangle-converse:
  assumes angle a b c = angle b a c ¬collinear {a,b,c}
  shows dist a c = dist b c
  ⟨proof⟩

2.6 Contributions by Lukas Bulwahn

lemma Pythagoras:
  fixes A B C :: 'a :: real-inner
  assumes orthogonal (A − C) (B − C)
  shows (dist B C) ^ 2 + (dist C A) ^ 2 = (dist A B) ^ 2
  ⟨proof⟩

lemma isosceles-triangle-orthogonal-on-midpoint:
  fixes A B C :: 'a :: euclidean-space
  assumes dist C A = dist C B
  shows orthogonal (C − midpoint A B) (A − midpoint A B)
  ⟨proof⟩

end