

Basic Geometric Properties of Triangles

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Abstract

In this work, we define angles between vectors and between three points. Building on this, we prove basic geometric properties of triangles, such as the Isosceles Triangle Theorem, the Law of Sines and the Law of Cosines, that the sum of the angles of a triangle is π , and the congruence theorems for triangles.

The definitions and proofs were developed following those by John Harrison in HOL Light. However, due to Isabelle's type class system, all definitions and theorems in the Isabelle formalisation hold for all real inner product spaces.

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1 Definition of angles

theory *Angles*

imports

~/src/HOL/Analysis/Analysis

begin

lemma *collinear-translate-iff*: $collinear ((op + a) ' A) \longleftrightarrow collinear A$
by (*auto simp: collinear-def*)

definition *vangle* **where**

vangle $u\ v = (if\ u = 0 \vee v = 0\ then\ pi / 2\ else\ arccos\ (u \cdot v / (norm\ u * norm\ v)))$

definition *angle where*

angle a b c = vangle (a - b) (c - b)

lemma *angle-altdef: angle a b c = arccos ((a - b) · (c - b) / (dist a b * dist c b))*

by (*simp add: angle-def vangle-def dist-norm*)

lemma *vangle-0-left [simp]: vangle 0 v = pi / 2*

and *vangle-0-right [simp]: vangle u 0 = pi / 2*

by (*simp-all add: vangle-def*)

lemma *vangle-refl [simp]: u ≠ 0 ⇒ vangle u u = 0*

by (*simp add: vangle-def dot-square-norm power2-eq-square*)

lemma *angle-refl [simp]: angle a a b = pi / 2 angle a b b = pi / 2*

by (*simp-all add: angle-def*)

lemma *angle-refl-mid [simp]: a ≠ b ⇒ angle a b a = 0*

by (*simp add: angle-def*)

lemma *cos-vangle: cos (vangle u v) = u · v / (norm u * norm v)*

unfolding *vangle-def using Cauchy-Schwarz-ineq2[of u v] by (auto simp: field-simps)*

lemma *cos-angle: cos (angle a b c) = (a - b) · (c - b) / (dist a b * dist c b)*

by (*simp add: angle-def cos-vangle dist-norm*)

lemma *inner-conv-angle: (a - b) · (c - b) = dist a b * dist c b * cos (angle a b c)*

by (*simp add: cos-angle*)

lemma *vangle-commute: vangle u v = vangle v u*

by (*simp add: vangle-def inner-commute mult.commute*)

lemma *angle-commute: angle a b c = angle c b a*

by (*simp add: angle-def vangle-commute*)

lemma *vangle-nonneg: vangle u v ≥ 0 and vangle-le-pi: vangle u v ≤ pi*

using *Cauchy-Schwarz-ineq2[of u v]*

by (*auto simp: vangle-def field-simps intro!: arccos-lbound arccos-ubound*)

lemmas *vangle-bounds = vangle-nonneg vangle-le-pi*

lemma *angle-nonneg: angle a b c ≥ 0 and angle-le-pi: angle a b c ≤ pi*

using *vangle-bounds unfolding angle-def by blast+*

lemmas *angle-bounds = angle-nonneg angle-le-pi*

lemma *sin-vangle-nonneg*: $\sin (\text{vangle } u \ v) \geq 0$
using *vangle-bounds* **by** (*rule sin-ge-zero*)

lemma *sin-angle-nonneg*: $\sin (\text{angle } a \ b \ c) \geq 0$
using *angle-bounds* **by** (*rule sin-ge-zero*)

lemma *vangle-eq-0D*:
assumes $\text{vangle } u \ v = 0$
shows $\text{norm } u \ *_R \ v = \text{norm } v \ *_R \ u$
proof –
from *assms* **have** $u \cdot v = \text{norm } u \ * \ \text{norm } v$
using *arccos-eq-iff*[*of* $(u \cdot v) / (\text{norm } u \ * \ \text{norm } v) \ 1$] *Cauchy-Schwarz-ineq2*[*of*
 $u \ v$]
by (*fastforce simp: vangle-def split: if-split-asm*)
thus *?thesis* **by** (*subst (asm) norm-cauchy-schwarz-eq*) *simp-all*
qed

lemma *vangle-eq-piD*:
assumes $\text{vangle } u \ v = \pi$
shows $\text{norm } u \ *_R \ v + \text{norm } v \ *_R \ u = 0$
proof –
from *assms* **have** $(-u) \cdot v = \text{norm } (-u) \ * \ \text{norm } v$
using *arccos-eq-iff*[*of* $(u \cdot v) / (\text{norm } u \ * \ \text{norm } v) \ -1$] *Cauchy-Schwarz-ineq2*[*of*
 $u \ v$]
by (*simp add: field-simps vangle-def split: if-split-asm*)
thus *?thesis* **by** (*subst (asm) norm-cauchy-schwarz-eq*) *simp-all*
qed

lemma *dist-triangle-eq*:
fixes $a \ b \ c :: 'a :: \text{real-inner}$
shows $(\text{dist } a \ c = \text{dist } a \ b + \text{dist } b \ c) \longleftrightarrow \text{dist } a \ b \ *_R \ (c - b) + \text{dist } b \ c \ *_R \ (a - b) = 0$
using *norm-triangle-eq*[*of* $b - a \ c - b$]
by (*simp add: dist-norm norm-minus-commute algebra-simps*)

lemma *angle-eq-pi-imp-dist-additive*:
assumes $\text{angle } a \ b \ c = \pi$
shows $\text{dist } a \ c = \text{dist } a \ b + \text{dist } b \ c$
using *vangle-eq-piD*[*OF* *assms*[*unfolded angle-def*]]
by (*subst dist-triangle-eq*) (*simp add: dist-norm norm-minus-commute*)

lemma *orthogonal-iff-vangle*: $\text{orthogonal } u \ v \longleftrightarrow \text{vangle } u \ v = \pi / 2$
using *arccos-eq-iff*[*of* $u \cdot v / (\text{norm } u \ * \ \text{norm } v) \ 0$] *Cauchy-Schwarz-ineq2*[*of* $u \ v$]
by (*auto simp: vangle-def orthogonal-def*)

lemma *cos-minus1-imp-pi*:

assumes $\cos x = -1$ $x \geq 0$ $x < 3 * \pi$
shows $x = \pi$
proof –
have $\cos (x - \pi) = 1$ **by** (*simp add: assms*)
then obtain $n :: \text{int}$ **where** n : $\text{of-int } n = (x / \pi - 1) / 2$
by (*subst (asm) cos-one-2pi-int*) (*auto simp: field-simps*)
also from *assms* **have** $\dots \in \{-1 < .. < 1\}$ **by** (*auto simp: field-simps*)
finally have $n = 0$ **by** *simp*
with n **show** *?thesis* **by** *simp*
qed

lemma *vangle-eqI*:
assumes $u \neq 0$ $v \neq 0$ $w \neq 0$ $x \neq 0$
assumes $(u \cdot v) * \text{norm } w * \text{norm } x = (w \cdot x) * \text{norm } u * \text{norm } v$
shows $\text{vangle } u \ v = \text{vangle } w \ x$
using *assms* *Cauchy-Schwarz-ineq2*[*of u v*] *Cauchy-Schwarz-ineq2*[*of w x*]
unfolding *vangle-def* **by** (*auto simp: arccos-eq-iff field-simps*)

lemma *angle-eqI*:
assumes $a \neq b$ $a \neq c$ $d \neq e$ $d \neq f$
assumes $((b-a) \cdot (c-a)) * \text{dist } d \ e * \text{dist } d \ f = ((e-d) \cdot (f-d)) * \text{dist } a \ b * \text{dist } a \ c$
shows $\text{angle } b \ a \ c = \text{angle } e \ d \ f$
using *assms* **unfolding** *angle-def*
by (*intro vangle-eqI*) (*simp-all add: dist-norm norm-minus-commute*)

lemma *cos-vangle-eqD*: $\cos (\text{vangle } u \ v) = \cos (\text{vangle } w \ x) \implies \text{vangle } u \ v = \text{vangle } w \ x$
by (*rule cos-inj-pi*) (*simp-all add: vangle-bounds*)

lemma *cos-angle-eqD*: $\cos (\text{angle } a \ b \ c) = \cos (\text{angle } d \ e \ f) \implies \text{angle } a \ b \ c = \text{angle } d \ e \ f$
unfolding *angle-def* **by** (*rule cos-vangle-eqD*)

lemma *sin-vangle-zero-iff*: $\sin (\text{vangle } u \ v) = 0 \iff \text{vangle } u \ v \in \{0, \pi\}$
proof
assume $\sin (\text{vangle } u \ v) = 0$
then obtain $n :: \text{int}$ **where** n : $\text{of-int } n = \text{vangle } u \ v / \pi$
by (*subst (asm) sin-zero-iff-int2*) *auto*
also have $\dots \in \{0..1\}$ **using** *vangle-bounds* **by** (*auto simp: field-simps*)
finally have $n \in \{0, 1\}$ **by** *auto*
thus $\text{vangle } u \ v \in \{0, \pi\}$ **using** n **by** (*auto simp: field-simps*)
qed *auto*

lemma *sin-angle-zero-iff*: $\sin (\text{angle } a \ b \ c) = 0 \iff \text{angle } a \ b \ c \in \{0, \pi\}$
unfolding *angle-def* **by** (*simp only: sin-vangle-zero-iff*)

lemma *vangle-collinear*: $\text{vangle } u \ v \in \{0, \pi\} \implies \text{collinear } \{0, u, v\}$

```

apply (subst norm-cauchy-schwarz-equal [symmetric])
apply (subst norm-cauchy-schwarz-abs-eq)
apply (auto dest!: vangle-eq-0D vangle-eq-piD simp: eq-neg-iff-add-eq-0)
done

```

```

lemma angle-collinear: angle a b c ∈ {0, pi} ⇒ collinear {a, b, c}
apply (unfold angle-def, drule vangle-collinear)
apply (subst collinear-translate-iff[symmetric, of - -b])
apply (auto simp: insert-commute)
done

```

```

lemma not-collinear-vangle: ¬collinear {0,u,v} ⇒ vangle u v ∈ {0<..using vangle-bounds[of u v] vangle-collinear[of u v]
  by (cases vangle u v = 0 ∨ vangle u v = pi) auto

```

```

lemma not-collinear-angle: ¬collinear {a,b,c} ⇒ angle a b c ∈ {0<..using angle-bounds[of a b c] angle-collinear[of a b c]
  by (cases angle a b c = 0 ∨ angle a b c = pi) auto

```

end

2 Basic Properties of Triangles

```

theory Triangle
imports
  Complex-Main
  ~~/src/HOL/Analysis/Topology-Euclidean-Space
  Angles
begin

```

We prove a number of basic geometric properties of triangles. All theorems hold in any real inner product space.

2.1 Sine and cosine laws

The proof of the Law of Cosines follows trivially from the definition of the angle, the definition of the norm in vector spaces with an inner product and the bilinearity of the inner product.

```

lemma cosine-law-vector:
  norm (u - v) ^ 2 = norm u ^ 2 + norm v ^ 2 - 2 * norm u * norm v * cos
  (vangle u v)
  by (simp add: power2-norm-eq-inner cos-vangle algebra-simps inner-commute)

```

```

lemma cosine-law-triangle:
  dist b c ^ 2 = dist a b ^ 2 + dist a c ^ 2 - 2 * dist a b * dist a c * cos (angle
  b a c)
  using cosine-law-vector[of b - a c - a]
  by (simp add: dist-norm angle-def vangle-commute norm-minus-commute)

```

According to our definition, angles are always between 0 and π and therefore, the sign of an angle is always non-negative. We can therefore look at $\sin(\alpha)^2$, which we can express in terms of $\cos(\alpha)$ using the identity $\sin(\alpha)^2 + \cos(\alpha)^2 = 1$. The remaining proof is then a trivial consequence of the definitions.

lemma *sine-law-triangle*:

$\sin (\text{angle } a \ b \ c) * \text{dist } b \ c = \sin (\text{angle } b \ a \ c) * \text{dist } a \ c$ (**is** $?A = ?B$)

proof (*cases a = b*)

assume *neq*: $a \neq b$

show *?thesis*

proof (*rule power2-eq-imp-eq*)

from *neq* **have** $(\sin (\text{angle } a \ b \ c) * \text{dist } b \ c) ^ 2 * \text{dist } a \ b ^ 2 =$
 $\text{dist } a \ b ^ 2 * \text{dist } b \ c ^ 2 - ((a - b) \cdot (c - b)) ^ 2$

by (*simp add: sin-squared-eq cos-angle dist-commute field-simps*)

also **have** $\dots = \text{dist } a \ b ^ 2 * \text{dist } a \ c ^ 2 - ((b - a) \cdot (c - a)) ^ 2$

by (*simp only: dist-norm power2-norm-eq-inner*)

(*simp add: power2-eq-square algebra-simps inner-commute*)

also **from** *neq* **have** $\dots = (\sin (\text{angle } b \ a \ c) * \text{dist } a \ c) ^ 2 * \text{dist } a \ b ^ 2$

by (*simp add: sin-squared-eq cos-angle dist-commute field-simps*)

finally **show** $?A ^ 2 = ?B ^ 2$ **using** *neq* **by** (*subst (asm) mult-cancel-right*)

simp-all

qed (*auto intro!: mult-nonneg-nonneg sin-angle-nonneg*)

qed *simp-all*

The following forms of the Law of Sines/Cosines are more convenient for eliminating sines/cosines from a goal completely.

lemma *cosine-law-triangle'*:

$2 * \text{dist } a \ b * \text{dist } a \ c * \cos (\text{angle } b \ a \ c) = (\text{dist } a \ b ^ 2 + \text{dist } a \ c ^ 2 - \text{dist } b \ c ^ 2)$

using *cosine-law-triangle[of b c a]* **by** *simp*

lemma *cosine-law-triangle''*:

$\cos (\text{angle } b \ a \ c) = (\text{dist } a \ b ^ 2 + \text{dist } a \ c ^ 2 - \text{dist } b \ c ^ 2) / (2 * \text{dist } a \ b * \text{dist } a \ c)$

using *cosine-law-triangle[of b c a]* **by** *simp*

lemma *sine-law-triangle'*:

$b \neq c \implies \sin (\text{angle } a \ b \ c) = \sin (\text{angle } b \ a \ c) * \text{dist } a \ c / \text{dist } b \ c$

using *sine-law-triangle[of a b c]* **by** (*simp add: divide-simps*)

lemma *sine-law-triangle''*:

$b \neq c \implies \sin (\text{angle } c \ b \ a) = \sin (\text{angle } b \ a \ c) * \text{dist } a \ c / \text{dist } b \ c$

using *sine-law-triangle[of a b c]* **by** (*simp add: divide-simps angle-commute*)

2.2 Sum of angles

context

begin

private lemma *gather-squares*: $a * (a * b) = a^2 * (b :: \text{real})$
by (*simp-all add: power2-eq-square*)

private lemma *eval-power*: $x \hat{=} \text{numeral } n = x * x \hat{=} \text{pred-numeral } n$
by (*subst numeral-eq-Suc, subst power-Suc*) *simp*

The proof that the sum of the angles in a triangle is π is somewhat more involved. Following the HOL Light proof by John Harrison, we first prove that $\cos(\alpha + \beta + \gamma) = -1$ and $\alpha + \beta + \gamma \in [0; 3\pi)$, which then implies the theorem.

The main work is proving $\cos(\alpha + \beta + \gamma)$. This is done using the addition theorems for the sine and cosine, then using the Laws of Sines to eliminate all sin terms save $\sin(\gamma)^2$, which only appears squared in the remaining goal. We then use $\sin(\gamma)^2 = 1 - \cos(\gamma)^2$ to eliminate this term and apply the law of cosines to eliminate this term as well.

The remaining goal is a non-linear equation containing only the length of the sides of the triangle. It can be shown by simple algebraic rewriting.

lemma *angle-sum-triangle*:
assumes $a \neq b \vee b \neq c \vee a \neq c$
shows $\text{angle } c \ a \ b + \text{angle } a \ b \ c + \text{angle } b \ c \ a = \text{pi}$
proof (*rule cos-minus1-imp-pi*)
show $\cos(\text{angle } c \ a \ b + \text{angle } a \ b \ c + \text{angle } b \ c \ a) = -1$
proof (*cases a ≠ b*)
case *True*
thus $\cos(\text{angle } c \ a \ b + \text{angle } a \ b \ c + \text{angle } b \ c \ a) = -1$
apply (*simp add: cos-add sin-add cosine-law-triangle'' field-simps*
sine-law-triangle''[of a b c] sine-law-triangle''[of b a c]
angle-commute dist-commute gather-squares sin-squared-eq)
apply (*simp add: eval-power algebra-simps dist-commute*)
done
qed (*insert assms, auto*)

show $\text{angle } c \ a \ b + \text{angle } a \ b \ c + \text{angle } b \ c \ a < 3 * \text{pi}$
proof (*rule ccontr*)
assume $\neg(\text{angle } c \ a \ b + \text{angle } a \ b \ c + \text{angle } b \ c \ a < 3 * \text{pi})$
with *angle-le-pi[of c a b] angle-le-pi[of a b c] angle-le-pi[of b c a]*
have $A: \text{angle } c \ a \ b = \text{pi} \ \text{angle } a \ b \ c = \text{pi}$ **by** *simp-all*
thus *False* **using** *angle-eq-pi-imp-dist-additive[of c a b]*
angle-eq-pi-imp-dist-additive[of a b c] **by** (*simp add: dist-commute*)
qed
qed (*auto intro!: add-nonneg-nonneg angle-nonneg*)

end

2.3 Congruence Theorems

If two triangles agree on two angles at a non-degenerate side, the third angle must also be equal.

lemma *similar-triangle-aa*:

assumes $b1 \neq c1$ $b2 \neq c2$

assumes $\text{angle } a1 \ b1 \ c1 = \text{angle } a2 \ b2 \ c2$

assumes $\text{angle } b1 \ c1 \ a1 = \text{angle } b2 \ c2 \ a2$

shows $\text{angle } b1 \ a1 \ c1 = \text{angle } b2 \ a2 \ c2$

proof –

from *assms angle-sum-triangle*[of $a1 \ b1 \ c1$] *angle-sum-triangle*[of $a2 \ b2 \ c2$, *sym-metric*]

show *?thesis* **by** (*auto simp: algebra-simps angle-commute*)

qed

A triangle is defined by its three angles and the lengths of three sides up to congruence. Two triangles are congruent if they have their angles are the same and their sides have the same length.

locale *congruent-triangle* =

fixes $a1 \ b1 \ c1 :: 'a :: \text{real-inner}$ **and** $a2 \ b2 \ c2 :: 'b :: \text{real-inner}$

assumes *sides'*: $\text{dist } a1 \ b1 = \text{dist } a2 \ b2$ $\text{dist } a1 \ c1 = \text{dist } a2 \ c2$ $\text{dist } b1 \ c1 = \text{dist } b2 \ c2$

and *angles'*: $\text{angle } b1 \ a1 \ c1 = \text{angle } b2 \ a2 \ c2$ $\text{angle } a1 \ b1 \ c1 = \text{angle } a2 \ b2 \ c2$
 $\text{angle } a1 \ c1 \ b1 = \text{angle } a2 \ c2 \ b2$

begin

lemma *sides*:

$\text{dist } a1 \ b1 = \text{dist } a2 \ b2$ $\text{dist } a1 \ c1 = \text{dist } a2 \ c2$ $\text{dist } b1 \ c1 = \text{dist } b2 \ c2$

$\text{dist } b1 \ a1 = \text{dist } a2 \ b2$ $\text{dist } c1 \ a1 = \text{dist } a2 \ c2$ $\text{dist } c1 \ b1 = \text{dist } b2 \ c2$

$\text{dist } a1 \ b1 = \text{dist } b2 \ a2$ $\text{dist } a1 \ c1 = \text{dist } c2 \ a2$ $\text{dist } b1 \ c1 = \text{dist } c2 \ b2$

$\text{dist } b1 \ a1 = \text{dist } b2 \ a2$ $\text{dist } c1 \ a1 = \text{dist } c2 \ a2$ $\text{dist } c1 \ b1 = \text{dist } c2 \ b2$

using *sides'* **by** (*simp-all add: dist-commute*)

lemma *angles*:

$\text{angle } b1 \ a1 \ c1 = \text{angle } b2 \ a2 \ c2$ $\text{angle } a1 \ b1 \ c1 = \text{angle } a2 \ b2 \ c2$ $\text{angle } a1 \ c1 \ b1 = \text{angle } a2 \ c2 \ b2$

$\text{angle } c1 \ a1 \ b1 = \text{angle } b2 \ a2 \ c2$ $\text{angle } c1 \ b1 \ a1 = \text{angle } a2 \ b2 \ c2$ $\text{angle } b1 \ c1 \ a1 = \text{angle } a2 \ c2 \ b2$

$\text{angle } b1 \ a1 \ c1 = \text{angle } c2 \ a2 \ b2$ $\text{angle } a1 \ b1 \ c1 = \text{angle } c2 \ b2 \ a2$ $\text{angle } a1 \ c1 \ b1 = \text{angle } b2 \ c2 \ a2$

$\text{angle } c1 \ a1 \ b1 = \text{angle } c2 \ a2 \ b2$ $\text{angle } c1 \ b1 \ a1 = \text{angle } c2 \ b2 \ a2$ $\text{angle } b1 \ c1 \ a1 = \text{angle } b2 \ c2 \ a2$

using *angles'* **by** (*simp-all add: angle-commute*)

end

lemmas *congruent-triangleD* = *congruent-triangle.sides congruent-triangle.angles*

Given two triangles that agree on a subset of its side lengths and angles

that are sufficient to define a triangle uniquely up to congruence, one can conclude that they must also agree on all remaining quantities, i.e. that they are congruent.

The following four congruence theorems state what constitutes such a uniquely-defining subset of quantities. Each theorem states in its name which quantities are required and in which order (clockwise or counter-clockwise): an “s” stands for a side, an “a” stands for an angle.

The lemma “congruent-triangleI-sas, for example, requires that two adjacent sides and the angle inbetween are the same in both triangles.

lemma *congruent-triangleI-sss*:

fixes $a1\ b1\ c1 :: 'a :: \text{real-inner}$ **and** $a2\ b2\ c2 :: 'b :: \text{real-inner}$

assumes $\text{dist } a1\ b1 = \text{dist } a2\ b2$

assumes $\text{dist } b1\ c1 = \text{dist } b2\ c2$

assumes $\text{dist } a1\ c1 = \text{dist } a2\ c2$

shows *congruent-triangle* $a1\ b1\ c1\ a2\ b2\ c2$

proof –

have $A: \text{angle } a1\ b1\ c1 = \text{angle } a2\ b2\ c2$

if $\text{dist } a1\ b1 = \text{dist } a2\ b2\ \text{dist } b1\ c1 = \text{dist } b2\ c2\ \text{dist } a1\ c1 = \text{dist } a2\ c2$

for $a1\ b1\ c1 :: 'a$ **and** $a2\ b2\ c2 :: 'b$

proof –

from *that cosine-law-triangle* "[of $a1\ b1\ c1$] *cosine-law-triangle*" [of $a2\ b2\ c2$]

show *?thesis* **by** (*intro cos-angle-eqD*) (*simp add: dist-commute*)

qed

from *assms* **show** *?thesis* **by** *unfold-locales (auto intro!: A simp: dist-commute)*

qed

lemmas *congruent-triangle-sss = congruent-triangleD* [*OF congruent-triangleI-sss*]

lemma *congruent-triangleI-sas*:

assumes $\text{dist } a1\ b1 = \text{dist } a2\ b2$

assumes $\text{dist } b1\ c1 = \text{dist } b2\ c2$

assumes $\text{angle } a1\ b1\ c1 = \text{angle } a2\ b2\ c2$

shows *congruent-triangle* $a1\ b1\ c1\ a2\ b2\ c2$

proof (*rule congruent-triangleI-sss*)

show $\text{dist } a1\ c1 = \text{dist } a2\ c2$

proof (*rule power2-eq-imp-eq*)

from *cosine-law-triangle* [of $a1\ c1\ b1$] *cosine-law-triangle* [of $a2\ c2\ b2$] *assms*

show $(\text{dist } a1\ c1)^2 = (\text{dist } a2\ c2)^2$ **by** (*simp add: dist-commute*)

qed *simp-all*

qed *fact+*

lemmas *congruent-triangle-sas = congruent-triangleD* [*OF congruent-triangleI-sas*]

lemma *congruent-triangleI-aas*:

assumes $\text{angle } a1\ b1\ c1 = \text{angle } a2\ b2\ c2$

assumes $\text{angle } b1\ c1\ a1 = \text{angle } b2\ c2\ a2$

assumes $\text{dist } a1\ b1 = \text{dist } a2\ b2$

```

assumes  $\neg$ collinear {a1,b1,c1}
shows congruent-triangle a1 b1 c1 a2 b2 c2
proof (rule congruent-triangleI-sas)
from  $\langle \neg$ collinear {a1,b1,c1}  $\rangle$  have neq: a1  $\neq$  b1 by auto
with assms(3) have neq': a2  $\neq$  b2 by auto
have A: angle c1 a1 b1 = angle c2 a2 b2 using neq neq' assms
using angle-sum-triangle[of a1 b1 c1] angle-sum-triangle[of a2 b2 c2]
by simp
from assms have B: angle b1 a1 c1  $\in$  {0<.. $\pi$ }
by (intro not-collinear-angle) (simp-all add: insert-commute)
from sine-law-triangle[of c1 a1 b1] sine-law-triangle[of c2 a2 b2] assms A B
show dist b1 c1 = dist b2 c2
by (auto simp: angle-commute dist-commute sin-angle-zero-iff)
qed fact+

```

lemmas congruent-triangle-aas = congruent-triangleD[OF congruent-triangleI-aas]

lemma congruent-triangleI-asa:

```

assumes angle a1 b1 c1 = angle a2 b2 c2
assumes dist a1 b1 = dist a2 b2
assumes angle b1 a1 c1 = angle b2 a2 c2
assumes  $\neg$ collinear {a1, b1, c1}
shows congruent-triangle a1 b1 c1 a2 b2 c2
proof (rule congruent-triangleI-aas)
from assms have neq: a1  $\neq$  b1 a2  $\neq$  b2 by auto
show angle b1 c1 a1 = angle b2 c2 a2
by (rule similar-triangle-aa) (insert assms neq, simp-all add: angle-commute)
qed fact+

```

lemmas congruent-triangle-asa = congruent-triangleD[OF congruent-triangleI-asa]

2.4 Isosceles Triangle Theorem

We now prove the Isosceles Triangle Theorem: in a triangle where two sides have the same length, the two angles that are adjacent to only one of the two sides must be equal.

lemma isosceles-triangle:

```

assumes dist a c = dist b c
shows angle b a c = angle a b c
by (rule congruent-triangle-sss) (insert assms, simp-all add: dist-commute)

```

For the non-degenerate case (i.e. the three points are not collinear), We also prove the converse.

lemma isosceles-triangle-converse:

```

assumes angle a b c = angle b a c  $\neg$ collinear {a,b,c}
shows dist a c = dist b c
by (rule congruent-triangle-asa[OF assms(1) - - assms(2)])
    (simp-all add: dist-commute angle-commute assms)

```

end