Transitive Union-Closed Families

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March 17, 2025

Abstract

We formalise a proof by Aaronson, Ellis and Leader showing that the Union-Closed Conjecture holds for the union-closed family generated by the cyclic translates of any fixed set.

Contents

1 Transitive Union-Closed Families

A family of sets is union-closed if the union of any two sets from the family is in the family. The Union-Closed Conjecture is an open problem in combinatorics posed by Frankl in 1979. It states that for every finite, union-closed family of sets (other than the family containing only the empty set) there exists an element that belongs to at least half of the sets in the family. We formalise a proof by Aaronson, Ellis and Leader showing that the Union-Closed Conjecture holds for the union-closed family generated by the cyclic translates of any fixed set [1].

theory *Transitive-Union-Closed-Families*
  imports *Pluennecke-Ruzsa-Inequality.Pluennecke-Ruzsa-Inequality*

begin

no-notation *equivalence.Partition* (**infixl** *'/ 75*)

definition *union-closed*:: *'a set set ⇒ bool*
  where *union-closed* $\mathcal{F} \equiv (\forall A \in \mathcal{F}.\ \forall\ B \in \mathcal{F}.\ A \cup B \in \mathcal{F})$

abbreviation *set-difference* :: $['a\ set,'a\ set] \Rightarrow 'a\ set$ (**infixl** $\setminus$ *65*)
  where $A \setminus B \equiv A-B$

locale *Family* = *additive-abelian-group* +
  fixes R
  assumes *finG*: *finite G*
  assumes *RG*: $R \subseteq G$
  assumes *R-nonempty*: $R \neq \{\}$

begin

definition *union-closed-conjecture-property*:: $'a\ set\ set \Rightarrow bool$
  where *union-closed-conjecture-property* $\mathcal{F}$
    $\equiv \exists \mathcal{X}\subseteq\mathcal{F}.\ \exists x\in G.\ x \in \bigcap \mathcal{X} \wedge card\ \mathcal{X} \geq card\ \mathcal{F}\ /\ 2$

definition $Neighbd \equiv \lambda A.\ sumset\ A\ R$

definition $Interior \equiv \lambda A.\ \{x\in G.\ sumset\ \{x\}\ R \subseteq A\}$

definition $\mathcal{F} \equiv Neighbd$ ‘ $Pow\ G$

We show that the family $\mathcal{F}$ as defined above and appears in the statement of the theorem [1] is actually a finite, nonempty union-closed family indeed.

lemma *card$\mathcal{F}$-gt0* [*simp*]: $card\ \mathcal{F} > 0$ **and** *finite$\mathcal{F}$*: *finite* $\mathcal{F}$
  ⟨*proof*⟩

lemma *union-closed* $\mathcal{F}$
⟨*proof*⟩

lemma *cardG-gt0*: $card\ G > 0$
  ⟨*proof*⟩

lemma *$\mathcal{F}$-subset*: $\mathcal{F} \subseteq Pow\ G$
  ⟨*proof*⟩

1.1 Proof of the main theorem

lemma *card-Interior-le*:
  assumes $S \subseteq G$
  shows $card\ (Interior\ S) \leq card\ S$
⟨*proof*⟩

lemma *Interior-subset-G* [*iff*]: $Interior\ S \subseteq G$
  ⟨*proof*⟩

lemma *Neighbd-subset-G* [*iff*]: $Neighbd\ S \subseteq G$
  ⟨*proof*⟩

lemma *average-ge*:

shows $(\sum S{\in}\mathcal{F}.(card\ S))\ /\ card\ \mathcal{F} \geq card\ G\ /\ 2$
$\langle proof \rangle$

We have thus shown that the average size of a set in the family $\mathcal{F}$ is at least $|G|/2$, proving the first part of Theorem 2 in the paper [1]. Using this, we will now show the main statement, i.e. that the Union-Closed Conjecture holds for the family $\mathcal{F}$.

theorem *Aaronson-Ellis-Leader-union-closed-conjecture*:
shows *union-closed-conjecture-property* $\mathcal{F}$
$\langle proof \rangle$

end

end

References

[1] J. Aaronson, D. Ellis, and I. Leader. A note on transitive union-closed families. 28(2), 2021. doihttps://doi.org/10.37236/9956.