# Transitive Union-Closed Families

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**Abstract**

We formalise a proof by Aaronson, Ellis and Leader showing that the Union-Closed Conjecture holds for the union-closed family generated by the cyclic translates of any fixed set.

## Contents



# 1 Transitive Union-Closed Families

A family of sets is union-closed if the union of any two sets from the family is in the family. The Union-Closed Conjecture is an open problem in combinatorics posed by Frankl in 1979. It states that for every finite, union-closed family of sets (other than the family containing only the empty set) there exists an element that belongs to at least half of the sets in the family. We formalise a proof by Aaronson, Ellis and Leader showing that the Union-Closed Conjecture holds for the union-closed family generated by the cyclic translates of any fixed set [1].

**theory** *Transitive-Union-Closed-Families*
**imports** *Pluennecke-Ruzsa-Inequality.Pluennecke-Ruzsa-Inequality*

**begin**

**no-notation** *equivalence.Partition* (**infixl** *'/ 75*)

**definition** *union-closed*:: *'a set set* $\Rightarrow$ *bool*
**where** *union-closed* $\mathcal{F} \equiv (\forall A{\in}\mathcal{F}.\ \forall\ B{\in}\mathcal{F}.\ A \cup B \in \mathcal{F})$

**abbreviation** *set-difference* :: $['a\ set,'a\ set] \Rightarrow\ 'a\ set$ (**infixl** \ *65*)
  **where** $A \setminus B \equiv A{-}B$

**locale** *Family* = *additive-abelian-group* +
  **fixes** $R$
  **assumes** *finG*: *finite* $G$
  **assumes** *RG*: $R \subseteq G$
  **assumes** *R-nonempty*: $R \neq \{\}$

**begin**

**definition** *union-closed-conjecture-property*:: $'a\ set\ set \Rightarrow bool$
  **where** *union-closed-conjecture-property* $\mathcal{F}$
    $\equiv \exists \mathcal{X}{\subseteq}\mathcal{F}.\ \exists x{\in}G.\ x \in \bigcap \mathcal{X} \wedge card\ \mathcal{X} \geq card\ \mathcal{F}\ /\ 2$

**definition** *Neighbd* $\equiv \lambda A.\ sumset\ A\ R$

**definition** *Interior* $\equiv \lambda A.\ \{x{\in}G.\ sumset\ \{x\}\ R \subseteq A\}$

**definition** $\mathcal{F} \equiv$ *Neighbd* ` *Pow G*

We show that the family $\mathcal{F}$ as defined above and appears in the statement of the theorem [1] is actually a finite, nonempty union-closed family indeed.

**lemma** *card*$\mathcal{F}$*-gt0* [*simp*]: *card* $\mathcal{F} > 0$ **and** *finite*$\mathcal{F}$: *finite* $\mathcal{F}$
  **using** $\mathcal{F}$*-def finG* **by** *fastforce*+

**lemma** *union-closed* $\mathcal{F}$
**proof**−
  **have** $*$:$\forall\ A \subseteq G.\ \forall\ B \subseteq G.\ (sumset\ A\ R) \cup (sumset\ B\ R) = sumset\ (A \cup B)\ R$
    **by** (*simp add*: *sumset-subset-Un1*)
  **show** *?thesis* **using** $*$
    **by** (*auto simp*: *union-closed-def* $\mathcal{F}$*-def Neighbd-def*)
**qed**

**lemma** *cardG-gt0*: *card* $G > 0$
  **using** *RG R-nonempty card-0-eq finG* **by** *blast*

**lemma** $\mathcal{F}$*-subset*: $\mathcal{F} \subseteq$ *Pow G*
  **by** (*simp add*: *Neighbd-def PowI* $\mathcal{F}$*-def image-subset-iff sumset-subset-carrier*)

## 1.1 Proof of the main theorem

**lemma** *card-Interior-le*:
  **assumes** $S \subseteq G$
  **shows** *card* (*Interior S*) $\leq$ *card S*
**proof** −
  **obtain** $r$ **where** $r \in R$
    **using** *R-nonempty* **by** *blast*
  **show** *?thesis*

**proof** (*intro card-inj-on-le*)
**let** *?f* = (λ*x*. *x* ⊕ *r*)
**show** *inj-on ?f* (*Interior S*) *?f* ' *Interior S* ⊆ *S*
**using** *RG* ‹*r* ∈ *R*› **by** (*auto simp*: *Interior-def inj-on-def*)
**show** *finite S*
**using** *assms finG finite-subset* **by** *blast*
**qed**
**qed**

**lemma** *Interior-subset-G* [*iff*]: *Interior S* ⊆ *G*
**using** *Interior-def* **by** *auto*

**lemma** *Neighbd-subset-G* [*iff*]: *Neighbd S* ⊆ *G*
**by** (*simp add*: *Neighbd-def sumset-subset-carrier*)

**lemma** *average-ge*:
**shows** (∑ *S*∈𝓕.(*card S*)) / *card* 𝓕 ≥ *card G* / *2*
**proof**−
**define** *f* **where** *f* ≡ λ*S*. *minusset* (*G* \ *Interior S*)

The following corresponds to (1) in the paper.

**have** *1*: *card S* + *card* (*f S*) ≥ *card G* **if** *S* ⊆ *G* **for** *S*
**proof**−
**have** *card* (*f S*) = *card G* − *card* (*Interior S*)
**unfolding** *f-def*
**by** (*metis Diff-subset Interior-subset-G card-Diff-subset card-minusset′ finG finite-subset*)
**with** *that* **show** *?thesis* **using** *card-Interior-le*
**by** (*metis* (*no-types*, *lifting*) *add.commute diff-le-mono2 le-diff-conv*)
**qed**

The following corresponds to (2) in the paper.

**have** *2*: *f S* = *sumset* (*minusset* (*G* \ *S*)) *R* **if** *S* ⊆ *G* **for** *S*
**proof**−
**have** ∗: *x* ∈ *f S* ⟷ *x* ∈ *sumset* (*minusset* (*G* \ *S*)) *R* **if** *x* ∈ *G* **for** *x*
**proof** −
**have** *x* ∈ *f S* ⟷ *inverse x* ∉ *Interior S*
**using** *that minusset.simps* **by** (*fastforce simp*: *f-def*)+
**also have** . . . ⟷ (*sumset* {*inverse x*} *R*) ∩ (*G**S*) ≠ {}
**using** *sumset-subset-carrier that* **by** (*auto simp*: *Interior-def*)
**also have** . . . ⟷ *x* ∈ *sumset* (*minusset* (*G**S*)) *R*
**proof**
**assume** *L*: *sumset* {*inverse x*} *R* ∩ (*G* \ *S*) ≠ {}
**then obtain** *r* **where** *r*: *inverse x* ⊕ *r* ∉ *S* **and** *r* ∈ *R*
**using** ‹*S* ⊆ *G*› ‹*x* ∈ *G*› **by** (*auto simp*: *sumset-eq minusset-eq*)
**then have** *inverse* (*inverse x* ⊕ *r*) ∈ *minusset* (*G* \ *S*)
**using** *RG that* **by** *auto*
**moreover have** *x* = *inverse* (*inverse x* ⊕ *r*) ⊕ *r*
**using** *RG* ‹*r* ∈ *R*› *that commutative inverse-composition-commute invertible-right-inverse2*

**by** *auto*
**ultimately show** $x \in$ *sumset* (*minusset* $(G \setminus S)$) $R$
**by** (*metis RG* ‹$r \in R$› *minusset-subset-carrier subset-eq sumset.simps*)
**next**
**assume** $R$: $x \in$ *sumset* (*minusset* $(G \setminus S)$) $R$
**then obtain** $g$ $r$ **where** $*$: $g{\in}G$ $g \notin S$ $r{\in}R$ $x =$ *inverse* $g \oplus r$
**by** (*metis Diff-iff minusset.simps sumset.cases*)
**show** *sumset* $\{$*inverse* $x\}$ $R \cap (G \setminus S) \neq \{\}$
**proof**
**assume** *sumset* $\{$*inverse* $x\}$ $R \cap (G \setminus S) = \{\}$
**then have** $g \notin$ *sumset* $\{$*inverse* $x\}$ $R$
**using** ‹$g \notin S$› *sumset-subset-carrier that* **by** *fastforce*
**then have** $g \neq$ *local.inverse* (*local.inverse* $g \oplus r$) $\oplus$ $r$
**using** $*$ *RG that* **by** (*auto simp*: *sumset-eq*)
**with** $*$ *RG that* **show** *False*
**by** (*metis commutative invertible invertible-left-inverse2 invertible-right-inverse2 subset-eq*)
**qed**
**qed**
**finally show** *?thesis* **.**
**qed**
**show** *?thesis*
**proof**
**show** $f$ $S \subseteq$ *sumset* (*minusset* $(G \setminus S)$) $R$
**using** $*$ *f-def minusset-subset-carrier* **by** *blast*
**next**
**show** *sumset* (*minusset* $(G \setminus S)$) $R \subseteq f$ $S$
**by** (*meson* $*$ *subset-iff sumset-subset-carrier*)
**qed**
**qed**
**then have** $f$ ‘ *Pow* $G \subseteq \mathcal{F}$
**by** (*auto simp*: *Neighbd-def* $\mathcal{F}$*-def minusset-subset-carrier*)

The following corresponds to (3) in the paper.

**have** *3*: *Neighbd* (*Interior* (*sumset* $A$ $R$)) $=$ *sumset* $A$ $R$
**if** $A \subseteq G$ **for** $A$
**using** *that* **by** (*force simp*: *sumset-eq Neighbd-def Interior-def*)

"Putting everything together":

**moreover**
**have** *sumset* $X$ $R =$ *sumset* $Y$ $R$
**if** $X \subseteq G$ $Y \subseteq G$
*minusset* $(G \setminus$ *Interior* (*sumset* $X$ $R$)) $=$ *minusset* $(G \setminus$ *Interior* (*sumset* $Y$ $R$))
**for** $X$ $Y$
**using** *that 3*
**by** (*metis Diff-Diff-Int Int-absorb2 Interior-subset-G inf-commute minus-minusset*)
**ultimately have** *inj-on* $f$ $\mathcal{F}$
**by** (*auto simp*: *inj-on-def* $\mathcal{F}$*-def f-def Neighbd-def*)

**moreover have** $f$ ‘ $\mathcal{F} \subseteq \mathcal{F}$
  **using** *2* $\mathcal{F}$*-def* ‹$f$ ‘ *Pow* $G \subseteq \mathcal{F}$› **by** *force*
**moreover have** $\mathcal{F} \subseteq f$ ‘ $\mathcal{F}$
  **by** (*metis* ‹*inj-on* $f$ $\mathcal{F}$› ‹$f$ ‘ $\mathcal{F} \subseteq \mathcal{F}$› *endo-inj-surj finite*$\mathcal{F}$)
**ultimately have** *bij-betw* $f$ $\mathcal{F}$ $\mathcal{F}$
  **by** (*simp add*: *bij-betw-def*)
**then have** *sum-card-eq*: $(\sum S{\in}\mathcal{F}.\ card\ (f\ S)) = (\sum S{\in}\mathcal{F}.\ card\ S)$
  **by** (*simp add*: *sum.reindex-bij-betw*)

**have** $card\ G\ /\ 2 = (1\ /\ (2 * card\ \mathcal{F})) * (\sum S{\in}\mathcal{F}.\ card\ G)$
  **by** *simp*
**also have** $\ldots \leq (1\ /\ (2 * card\ \mathcal{F})) * (\sum S{\in}\mathcal{F}.\ card\ S + card\ (f\ S))$
  **by** (*intro sum-mono mult-left-mono of-nat-mono 1*) (*auto simp*: $\mathcal{F}$*-def*)
**also have** $\ldots = (1\ /\ card\ \mathcal{F}) * (\sum S{\in}\mathcal{F}.\ card\ S)$
  **by** (*simp add*: *sum-card-eq sum.distrib*)
**finally show** *?thesis*
  **by** *argo*
**qed**

We have thus shown that the average size of a set in the family $\mathcal{F}$ is at least $|G|/2$, proving the first part of Theorem 2 in the paper [1]. Using this, we will now show the main statement, i.e. that the Union-Closed Conjecture holds for the family $\mathcal{F}$.

**theorem** *Aaronson-Ellis-Leader-union-closed-conjecture*:
  **shows** *union-closed-conjecture-property* $\mathcal{F}$
**proof** −
— First, quite a big calculation not mentioned in the article: counting all the elements in two different ways.
**have** ∗: $(\sum S{\in}\mathcal{F}.(card\ S)) = (\sum x{\in}G.\ card\ \{S{\in}\mathcal{F}.\ x{\in}S\})$
  **using** *finite*$\mathcal{F}$ $\mathcal{F}$*-subset*
**proof** *induction*
  **case** *empty*
  **then show** *?case*
    **by** *simp*
**next**
  **case** (*insert* $S$ $\mathcal{G}$)
  **then have** $A$: $\{T.\ (T = S \lor T{\in}\mathcal{G}) \land x{\in}T\}$
      $= \{T{\in}\mathcal{G}.\ x{\in}T\} \cup (if\ x{\in}S\ then\ \{S\}\ else\ \{\})$
    **for** $x$
    **by** *auto*
  **have** $B$: $card\ \{T.\ (T = S \lor T{\in}\mathcal{G}) \land x{\in}T\}$
      $= card\ \{T{\in}\mathcal{G}.\ x{\in}T\} + (if\ x{\in}S\ then\ 1\ else\ 0)$
    **for** $x$
    **by** (*simp add*: *A card-insert-if insert*)
  **have** $S = (\bigcup x{\in}G.\ if\ x \in S\ then\ \{x\}\ else\ \{\})$
    **using** *insert.prems* **by** *auto*
  **then have** $card\ S = card\ (\bigcup x{\in}G.\ if\ x \in S\ then\ \{x\}\ else\ \{\})$
    **by** *simp*
  **also have** $\ldots = (\sum i{\in}G.\ card\ (if\ i \in S\ then\ \{i\}\ else\ \{\}))$

**by** (*intro card-UN-disjoint*) (*auto simp*: *finG*)
**also have** ... = ($\sum x{\in}G$. *if* $x \in S$ *then 1 else 0*)
**by** (*force intro*: *sum.cong*)
**finally have** *C*: *card* $S$ = ($\sum x{\in}G$. *if* $x \in S$ *then 1 else 0*) **.**
**show** *?case*
**using** *insert* **by** (*auto simp*: *sum.distrib B C*)
**qed**

**have** $1/2 \le$ (*sum card* $\mathcal{F}$) / (*card* $\mathcal{F}$ $*$ *card* $G$)
**using** *mult-right-mono* [*OF average-ge*, *of 1* / *card* $G$]
**using** *cardG-gt0* **by** (*simp add*: *divide-simps split*: *if-splits*)
**also have** ... = ($\sum x{\in}G$. ((*card* $\{S{\in}\mathcal{F}.\ x{\in}S\}$) / (*card* $\mathcal{F}$))) / *card* $G$
**by** (*simp add*: $*$ *sum-divide-distrib*)
**finally have** $**$: $1/2 \le$ ($\sum x{\in}G$. *card* $\{S{\in}\mathcal{F}.\ x{\in}S\}$ / *card* $\mathcal{F}$) / *card* $G$ **.**
— There is a typo in the paper (bottom of page): instead of $x \in S$ it says $x \in \mathcal{F}$.
**show** *?thesis*
**proof** (*rule ccontr*) — Contradict the inequality proved above
**assume** $\neg$ *union-closed-conjecture-property* $\mathcal{F}$
**then have** *A*: $\bigwedge \mathcal{X}\ x.\ [\![\mathcal{X}{\subseteq}\mathcal{F};\ x{\in}G;\ x \in \bigcap \mathcal{X}]\!] \Longrightarrow$ *card* $\mathcal{X}$ < *card* $\mathcal{F}$ / *2*
**by** (*fastforce simp*: *union-closed-conjecture-property-def*)
**have** ($\sum x{\in}G$. *real* (*card* $\{S{\in}\mathcal{F}.\ x{\in}S\}$)) < ($\sum x{\in}G$. *card* $\mathcal{F}$ / *2*)
**proof** (*intro sum-strict-mono*)
**fix** $x :: 'a$
**assume** $x \in G$
**then have** *card* $\{S{\in}\mathcal{F}.\ x{\in}S\}$ < *card* $\mathcal{F}$ / *2*
**by** (*intro A*) *auto*
**then show** *real* (*card* $\{S{\in}\mathcal{F}.\ x{\in}S\}$) < *real* (*card* $\mathcal{F}$) / *2*
**by** *blast*
**qed** (*use unit-closed finG* **in** *auto*)
**also have** ... = *card* $\mathcal{F}$ $*$ (*card* $G$ / *2*)
**by** *simp*
**finally have** *B*: ($\sum x{\in}G$. *real* (*card* $\{S{\in}\mathcal{F}.\ x{\in}S\}$)) < *card* $\mathcal{F}$ $*$ (*card* $G$ / *2*) **.**
**have** ($\sum x{\in}G$. *card* $\{S{\in}\mathcal{F}.\ x{\in}S\}$ / *card* $\mathcal{F}$) / *card* $G$ < *1/2*
**using** *divide-strict-right-mono* [*OF B*, *of card* $\mathcal{F}$ $*$ *card* $G$]
**using** *cardG-gt0*
**by** (*simp add*: *divide-simps sum-divide-distrib*)
**with** $**$ **show** *False*
**by** *argo*
**qed**
**qed**

**end**

**end**

## References

[1] J. Aaronson, D. Ellis, and I. Leader. A note on transitive union-closed families. 28(2), 2021. doihttps://doi.org/10.37236/9956.