

# Executable Transitive Closures of Finite Relations\*

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## Abstract

We provide a generic work-list algorithm to compute the transitive closure of finite relations where only successors of newly detected states are generated. This algorithm is then instantiated for lists over arbitrary carriers and red black trees [1] (which are faster but require a linear order on the carrier), respectively.

Our formalization was performed as part of the *IsaFoR/CeTA* project<sup>1</sup> [2], where reflexive transitive closures of large tree automata have to be computed.

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<sup>1</sup><http://cl-informatik.uibk.ac.at/software/ceta>

# 1 A Generic Work-List Algorithm

```
theory Transitive-Closure-Impl
imports Main
begin
```

Let  $R$  be some finite relation. We start to present a standard work-list algorithm to compute all elements that are reachable from some initial set by at most  $n$   $R$ -steps. Then, we obtain algorithms for the (reflexive) transitive closure from a given starting set by exploiting the fact that for finite relations we have to iterate at most  $\text{card } R$  times. The presented algorithms are generic in the sense that the underlying data structure can freely be chosen, you just have to provide certain operations like union, membership, etc.

## 1.1 Bounded Reachability

We provide an algorithm *relpow-impl* that computes all states that are reachable from an initial set of states *new* by at most  $n$  steps. The algorithm also stores a set of states that have already been visited *have*, and then show, do not have to be expanded a second time. The algorithm is parametric in the underlying data structure, it just requires operations for union and membership as well as a function to compute the successors of a list.

```
fun
  relpow-impl ::
    ('a list  $\Rightarrow$  'a list)  $\Rightarrow$ 
    ('a list  $\Rightarrow$  'b  $\Rightarrow$  'b)  $\Rightarrow$  ('a  $\Rightarrow$  'b  $\Rightarrow$  bool)  $\Rightarrow$  'a list  $\Rightarrow$  'b  $\Rightarrow$  nat  $\Rightarrow$  'b
where
  relpow-impl succ un memb new have 0 = un new have |
  relpow-impl succ un memb new have (Suc m) =
    (if new = [] then have
     else
      let
        maybe = succ new;
        have' = un new have;
        new' = filter ( $\lambda n. \neg \text{memb } n \text{ have}'$ ) maybe
      in relpow-impl succ un memb new' have' m)
```

We need to know that the provided operations behave correctly.

```
locale set-access =
fixes un :: 'a list  $\Rightarrow$  'b  $\Rightarrow$  'b
and set-of :: 'b  $\Rightarrow$  'a set
and memb :: 'a  $\Rightarrow$  'b  $\Rightarrow$  bool
and empty :: 'b
assumes un: set-of (un as bs) = set as  $\cup$  set-of bs
and memb: memb a bs  $\longleftrightarrow$  (a  $\in$  set-of bs)
and empty: set-of empty = {}
```

**locale** *set-access-succ* = *set-access un*  
**for** *un* :: 'a list  $\Rightarrow$  'b  $\Rightarrow$  'b +  
**fixes** *succ* :: 'a list  $\Rightarrow$  'a list  
**and** *rel* :: ('a  $\times$  'a) set  
**assumes** *succ*: set (*succ as*) = {b.  $\exists$  a  $\in$  set *as*. (a, b)  $\in$  *rel*}  
**begin**

**abbreviation** *relpow-i*  $\equiv$  *relpow-impl succ un memb*

What follows is the main technical result of the *relpow-impl* algorithm: what it computes for arbitrary values of *new* and *have*.

**lemma** *relpow-impl-main*:

*set-of (relpow-i new have n)* =  
 {b | a b m. a  $\in$  set *new*  $\wedge$  m  $\leq$  n  $\wedge$  (a, b)  $\in$  (*rel*  $\cap$  {(a, b). b  $\notin$  set-of *have*})  
 $\widetilde{m}$ }  $\cup$   
 set-of *have*  
 (is ?l *new have n* = ?r *new have n*)  
 <proof>

From the previous lemma we can directly derive that *relpow-impl* works correctly if *have* is initially set to *empty*

**lemma** *relpow-impl*:

*set-of (relpow-i new empty n)* = {b | a b m. a  $\in$  set *new*  $\wedge$  m  $\leq$  n  $\wedge$  (a, b)  $\in$  *rel*  
 $\widetilde{m}$ }  
 <proof>

**end**

## 1.2 Reflexive Transitive Closure and Transitive closure

Using *relpow-impl* it is now easy to obtain algorithms for the reflexive transitive closure and the transitive closure by restricting the number of steps to the size of the finite relation. Note that *relpow-impl* will abort the computation as soon as no new states are detected. Hence, there is no penalty in using this large bound.

**definition**

*rtrancl-impl* ::  
 (('a  $\times$  'a) list  $\Rightarrow$  'a list  $\Rightarrow$  'a list)  $\Rightarrow$   
 ('a list  $\Rightarrow$  'b  $\Rightarrow$  'b)  $\Rightarrow$  ('a  $\Rightarrow$  'b  $\Rightarrow$  bool)  $\Rightarrow$  'b  $\Rightarrow$  ('a  $\times$  'a) list  $\Rightarrow$  'a list  $\Rightarrow$  'b

**where**

*rtrancl-impl gen-succ un memb emp rel* =  
 (let  
   *succ* = *gen-succ rel*;  
   *n* = *length rel*  
 in ( $\lambda$  *as*. *relpow-impl succ un memb as emp n*))

**definition**

```

trancl-impl ::
  (('a × 'a) list ⇒ 'a list ⇒ 'a list) ⇒
  ('a list ⇒ 'b ⇒ 'b) ⇒ ('a ⇒ 'b ⇒ bool) ⇒ 'b ⇒ ('a × 'a) list ⇒ 'a list ⇒ 'b
where
  trancl-impl gen-succ un memb emp rel =
  (let
    succ = gen-succ rel;
    n = length rel
  in (λ as. relpow-impl succ un memb (succ as) emp n))

```

The soundness of both *rtrancl-impl* and *trancl-impl* follows from the soundness of *relpow-impl* and the fact that for finite relations, we can limit the number of steps to explore all elements in the reflexive transitive closure.

**lemma** *rtrancl-finite-relpow*:

```

(a, b) ∈ (set rel)* ↔ (∃ n ≤ length rel. (a, b) ∈ set rel ~n) (is ?l = ?r)
⟨proof⟩

```

**locale** *set-access-gen* = *set-access un*

```

for un :: 'a list ⇒ 'b ⇒ 'b +
fixes gen-succ :: ('a × 'a) list ⇒ 'a list ⇒ 'a list
assumes gen-succ: set (gen-succ rel as) = {b. ∃ a ∈ set as. (a, b) ∈ set rel}
begin

```

**abbreviation** *rtrancl-i* ≡ *rtrancl-impl gen-succ un memb empty*

**abbreviation** *trancl-i* ≡ *trancl-impl gen-succ un memb empty*

**lemma** *rtrancl-impl*:

```

set-of (rtrancl-i rel as) = {b. (∃ a ∈ set as. (a, b) ∈ (set rel)*)}
⟨proof⟩

```

**lemma** *trancl-impl*:

```

set-of (trancl-i rel as) = {b. (∃ a ∈ set as. (a, b) ∈ (set rel)+)}
⟨proof⟩

```

**end**

**end**

## 2 Closure Computation using Lists

**theory** *Transitive-Closure-List-Impl*

**imports** *Transitive-Closure-Impl*

**begin**

We provide two algorithms for the computation of the reflexive transitive closure which internally work on lists. The first one (*rtrancl-list-impl*) computes the closure on demand for a given set of initial states. The second one (*memo-list-rtrancl*) precomputes the closure for each individual state, stores the result, and then only does a look-up.

For the transitive closure there are the corresponding algorithms *trancl-list-impl* and *memo-list-trancl*.

## 2.1 Computing Closures from Sets On-The-Fly

The algorithms are based on the generic algorithms *rtrancl-impl* and *trancl-impl* instantiated by list operations. Here, after computing the successors in a straightforward way, we use *remdups* to not have duplicates in the results. Moreover, also in the union operation we filter to those elements that have not yet been seen. The use of *filter* in the union operation is preferred over *remdups* since by construction the latter set will not contain duplicates.

**definition** *rtrancl-list-impl* :: ('a × 'a) list ⇒ 'a list ⇒ 'a list

**where**

```
rtrancl-list-impl = rtrancl-impl
  (λ r as. remdups (map snd (filter (λ (a, b). a ∈ set as) r)))
  (λ xs ys. (filter (λ x. x ∉ set ys) xs) @ ys)
  (λ x xs. x ∈ set xs)
  []
```

**definition** *trancl-list-impl* :: ('a × 'a) list ⇒ 'a list ⇒ 'a list

**where**

```
trancl-list-impl = trancl-impl
  (λ r as. remdups (map snd (filter (λ (a, b). a ∈ set as) r)))
  (λ xs ys. (filter (λ x. x ∉ set ys) xs) @ ys)
  (λ x xs. x ∈ set xs)
  []
```

**lemma** *rtrancl-list-impl*:

```
set (rtrancl-list-impl r as) = {b. ∃ a ∈ set as. (a, b) ∈ (set r)*}
⟨proof⟩
```

**lemma** *trancl-list-impl*:

```
set (trancl-list-impl r as) = {b. ∃ a ∈ set as. (a, b) ∈ (set r)+}
⟨proof⟩
```

## 2.2 Precomputing Closures for Single States

Storing all relevant entries is done by mapping all left-hand sides of the relation to their closure. To avoid redundant entries, *remdups* is used.

**definition** *memo-list-rtrancl* :: ('a × 'a) list ⇒ ('a ⇒ 'a list)

**where**

```
memo-list-rtrancl r =
  (let
    tr = rtrancl-list-impl r;
    rm = map (λa. (a, tr [a])) ((remdups ∘ map fst) r)
  in
  (λa. case map-of rm a of
```

```

    None ⇒ [a]
  | Some as ⇒ as))

```

**lemma** *memo-list-rtrancl*:

```

  set (memo-list-rtrancl r a) = {b. (a, b) ∈ (set r)*} (is ?l = ?r)
⟨proof⟩

```

**definition** *memo-list-trancl* :: ('a × 'a) list ⇒ ('a ⇒ 'a list)

**where**

```

memo-list-trancl r =
  (let
    tr = trancl-list-impl r;
    rm = map (λa. (a, tr [a])) ((remdups ∘ map fst) r)
  in
  (λa. case map-of rm a of
    None ⇒ []
  | Some as ⇒ as))

```

**lemma** *memo-list-trancl*:

```

  set (memo-list-trancl r a) = {b. (a, b) ∈ (set r)+} (is ?l = ?r)
⟨proof⟩

```

**end**

### 3 Accessing Values via Keys

**theory** *RBT-Map-Set-Extension*

**imports**

*Collections.RBTMapImpl*

*Collections.RBTSetImpl*

*Matrix.Utility*

**begin**

We provide two extensions of the red black tree implementation.

The first extension provides two convenience methods on sets which are represented by red black trees: a check on subsets and the big union operator.

The second extension is to provide two operations *elem-list-to-rm* and *rm-set-lookup* which can be used to index a set of values via keys. More precisely, given a list of values of type 'v and a key function of type 'v ⇒ 'k, *elem-list-to-rm* will generate a map of type 'k ⇒ 'v set. Then with *rs-set-lookup* we can efficiently access all values which match a given key.

#### 3.1 Subset and Union

For the subset operation  $r \subseteq s$  we provide two implementations. The first one (*rs-subset*) traverses over  $r$  and then performs membership tests  $\in s$ . Its complexity is  $\mathcal{O}(|r| \cdot \log(|s|))$ . The second one (*rs-subset-list*) generates

sorted lists for both  $r$  and  $s$  and then linearly checks the subset condition. Its complexity is  $\mathcal{O}(|r| + |s|)$ .

As union operator we use the standard fold function. Note that the order of the union is important so that new sets are added to the big union.

**definition**  $rs\text{-subset} :: ('a :: \text{linorder}) rs \Rightarrow 'a rs \Rightarrow 'a \text{ option}$

**where**

```

rs-subset as bs = rs.iteratei
  as
  (\ maybe. case maybe of None  $\Rightarrow$  True | Some -  $\Rightarrow$  False)
  (\ a -. if rs.memb a bs then None else Some a)
  None

```

**lemma**  $rs\text{-subset}$  [simp]:

```
rs-subset as bs = None  $\longleftrightarrow$  rs. $\alpha$  as  $\subseteq$  rs. $\alpha$  bs
```

$\langle$ proof $\rangle$

**definition**  $rs\text{-subset-list} :: ('a :: \text{linorder}) rs \Rightarrow 'a rs \Rightarrow 'a \text{ option}$

**where**

```
rs-subset-list as bs = sorted-list-subset (rs.to-sorted-list as) (rs.to-sorted-list bs)
```

**lemma**  $rs\text{-subset-list}$  [simp]:

```
rs-subset-list as bs = None  $\longleftrightarrow$  rs. $\alpha$  as  $\subseteq$  rs. $\alpha$  bs
```

$\langle$ proof $\rangle$

**definition**  $rs\text{-Union} :: ('q :: \text{linorder}) rs \text{ list} \Rightarrow 'q rs$

**where**

```
rs-Union = foldl rs.union (rs.empty ())
```

**lemma**  $rs\text{-Union}$  [simp]:

```
rs. $\alpha$  (rs-Union qs) =  $\bigcup$  (rs. $\alpha$  ' set qs)
```

$\langle$ proof $\rangle$

### 3.2 Grouping Values via Keys

The functions to produce the index ( $elem\text{-list-to-rm}$ ) and the lookup function ( $rm\text{-set-lookup}$ ) are straight-forward, however it requires some tedious reasoning that they perform as they should.

**fun**  $elem\text{-list-to-rm} :: ('d \Rightarrow 'k :: \text{linorder}) \Rightarrow 'd \text{ list} \Rightarrow ('k, 'd \text{ list}) \text{ rm}$

**where**

```

elem-list-to-rm key [] = rm.empty () |
elem-list-to-rm key (d # ds) =
  (let
    rm = elem-list-to-rm key ds;
    k = key d
  in
  (case rm. $\alpha$  rm k of
    None  $\Rightarrow$  rm.update-dj k [d] rm

```

| *Some data*  $\Rightarrow$  *rm.update k (d # data) rm*)

**definition** *rm-set-lookup* *rm* = ( $\lambda$  *a*. (*case rm.alpha rm a of None*  $\Rightarrow$  [] | *Some rules*  $\Rightarrow$  *rules*))

**lemma** *rm-to-list-empty* [*simp*]:  
*rm.to-list (rm.empty ())* = []  
 <*proof*>

**locale** *rm-set* =  
**fixes** *rm* :: ('*k* :: *linorder*, '*d list*) *rm*  
**and** *key* :: '*d*  $\Rightarrow$  '*k*  
**and** *data* :: '*d set*  
**assumes** *rm-set-lookup*:  $\bigwedge$  *k*. *set (rm-set-lookup rm k)* = {*d*  $\in$  *data*. *key d = k*}  
**begin**

**lemma** *data-lookup*:  
*data* =  $\bigcup$  {*set (rm-set-lookup rm k) | k. True*} (**is** - = ?*R*)  
 <*proof*>

**lemma** *finite-data*:  
*finite data*  
 <*proof*>

**end**

**interpretation** *elem-list-to-rm*: *rm-set elem-list-to-rm key ds key set ds*  
 <*proof*>

**end**

## 4 Closure Computation via Red Black Trees

**theory** *Transitive-Closure-RBT-Impl*

**imports**

*Transitive-Closure-Impl*

*RBT-Map-Set-Extension*

**begin**

We provide two algorithms to compute the reflexive transitive closure which internally work on red black trees. Therefore, the carrier has to be linear ordered. The first one (*rtrancl-rbt-impl*) computes the closure on demand for a given set of initial states. The second one (*memo-rbt-rtrancl*) precomputes the closure for each individual state, stores the results, and then only does a look-up.

For the transitive closure there are the corresponding algorithms *trancl-rbt-impl* and *memo-rbt-trancl*



## 4.1 Computing Closures from Sets On-The-Fly

The algorithms are based on the generic algorithms *rtrancl-impl* and *trancl-impl* using red black trees. To compute the successors efficiently, all successors of a state are collected and stored in a red black tree map by using *elem-list-to-rm*. Then, to lift the successor relation for single states to lists of states, all results are united using *rs-Union*. The rest is standard.

**interpretation** *set-access*  $\lambda$  *as* *bs*. *rs.union* *bs* (*rs.from-list* *as*) *rs.alpha* *rs.memb* *rs.empty* ()  
 ⟨*proof*⟩

**abbreviation** *rm-succ* :: ('a :: linorder × 'a) list ⇒ 'a list ⇒ 'a list

**where**

*rm-succ* ≡ (λ *r*. let *rm* = *elem-list-to-rm* *fst* *r* in  
 (λ *as*. *rs.to-list* (*rs-Union* (*map* (λ *a*. *rs.from-list* (*map* *snd* (*rm-set-lookup* *rm* *a*)))) *as*))))

**definition** *rtrancl-rbt-impl* :: ('a :: linorder × 'a) list ⇒ 'a list ⇒ 'a rs

**where**

*rtrancl-rbt-impl* = *rtrancl-impl* *rm-succ*  
 (λ *as* *bs*. *rs.union* *bs* (*rs.from-list* *as*)) *rs.memb* (*rs.empty* ())

**definition** *trancl-rbt-impl* :: ('a :: linorder × 'a) list ⇒ 'a list ⇒ 'a rs

**where**

*trancl-rbt-impl* = *trancl-impl* *rm-succ*  
 (λ *as* *bs*. *rs.union* *bs* (*rs.from-list* *as*)) *rs.memb* (*rs.empty* ())

**lemma** *rtrancl-rbt-impl*:

*rs.alpha* (*rtrancl-rbt-impl* *r* *as*) = {*b*. ∃ *a* ∈ *set as*. (*a*,*b*) ∈ (*set r*)\*}  
 ⟨*proof*⟩

**lemma** *trancl-rbt-impl*:

*rs.alpha* (*trancl-rbt-impl* *r* *as*) = {*b*. ∃ *a* ∈ *set as*. (*a*,*b*) ∈ (*set r*)+}  
 ⟨*proof*⟩

## 4.2 Precomputing Closures for Single States

Storing all relevant entries is done by mapping all left-hand sides of the relation to their closure. Since we assume a linear order on the carrier, for the lookup we can use maps that are implemented as red black trees.

**definition** *memo-rbt-rtrancl* :: ('a :: linorder × 'a) list ⇒ ('a ⇒ 'a rs)

**where**

*memo-rbt-rtrancl* *r* =  
 (let  
   *tr* = *rtrancl-rbt-impl* *r*;  
   *rm* = *rm.to-map* (*map* (λ *a*. (*a*, *tr* [*a*])) ((*rs.to-list* ∘ *rs.from-list* ∘ *map* *fst*)  
*r*))  
 in

```
(λa. case rm.lookup a rm of
  None ⇒ rs.from-list [a]
  | Some as ⇒ as))
```

**lemma** *memo-rbt-rtrancl*:

```
rs.α (memo-rbt-rtrancl r a) = {b. (a, b) ∈ (set r)*} (is ?l = ?r)
⟨proof⟩
```

**definition** *memo-rbt-trancl* :: ('a :: linorder × 'a) list ⇒ ('a ⇒ 'a rs)

**where**

```
memo-rbt-trancl r =
  (let
    tr = trancl-rbt-impl r;
    rm = rm.to-map (map (λ a. (a, tr [a])) ((rs.to-list ∘ rs.from-list ∘ map fst)
r))
  in (λ a.
    (case rm.lookup a rm of
      None ⇒ rs.empty ()
      | Some as ⇒ as)))
```

**lemma** *memo-rbt-trancl*:

```
rs.α (memo-rbt-trancl r a) = {b. (a, b) ∈ (set r)+} (is ?l = ?r)
⟨proof⟩
```

**end**

## 5 Computing Images of Finite Transitive Closures

**theory** *Finite-Transitive-Closure-Simprocs*

**imports** *Transitive-Closure-List-Impl*

**begin**

**lemma** *rtrancl-Image-eq*:

```
assumes r = set r' and x = set x'
shows r* “ x = set (rtrancl-list-impl r' x')
⟨proof⟩
```

**lemma** *trancl-Image-eq*:

```
assumes r = set r' and x = set x'
shows r+ “ x = set (trancl-list-impl r' x')
⟨proof⟩
```

### 5.1 A Simproc for Computing the Images of Finite Transitive Closures

⟨ML⟩

## 5.2 Example

The images of (reflexive) transitive closures are computed by evaluation.

**lemma**

$\{(1::nat, 2), (2, 3), (3, 4), (4, 5)\}^*$  “  $\{1\} = \{1, 2, 3, 4, 5\}$   
 $\{(1::nat, 2), (2, 3), (3, 4), (4, 5)\}^+$  “  $\{1\} = \{2, 3, 4, 5\}$   
*<proof>*

Evaluation does not allow for free variables and thus fails in their presence.

**lemma**

$\{(x, y)\}^*$  “  $\{x\} = \{x, y\}$   
*<proof>*

**end**

## References

- [1] P. Lammich and A. Lochbihler. The Isabelle collections framework. In *Proc. ITP'10*, volume 6172 of *LNCS*, pages 339–354, 2010.
- [2] R. Thiemann and C. Sternagel. Certification of termination proofs using CeTA. In *Proc. TPHOLs'09*, volume 5674 of *LNCS*, pages 452–468, 2009.