

Transition Systems and Automata

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Abstract

This entry provides a very abstract theory of transition systems that can be instantiated to express various types of automata. A transition system is typically instantiated by providing a set of initial states, a predicate for enabled transitions, and a transition execution function. From this, it defines the concepts of finite and infinite paths as well as the set of reachable states, among other things. Many useful theorems, from basic path manipulation rules to coinduction and run construction rules, are proven in this abstract transition system context. The library comes with instantiations for DFAs, NFAs, and Büchi automata.

Contents

1	Basics	4
1.1	Miscellaneous	4
2	Finite and Infinite Sequences	5
2.1	List Basics	5
2.2	Stream Basics	6
2.3	The scan Function	10
2.4	Transposing Streams	11
2.5	Distinct Streams	12
2.6	Sorted Streams	13
3	Linear Temporal Logic on Streams	14
3.1	Basics	14
3.2	Infinite Occurrence	15
4	Zippering Sequences	16
4.1	Zippering Lists	16
4.2	Zippering Streams	17
5	Maps	18

6	Basics	18
6.1	Expanding set functions to sets of functions	19
6.2	Expanding set maps into sets of maps	19
7	Transition Systems	22
7.1	Universal Transition Systems	22
7.2	Transition Systems	22
7.3	Transition Systems with Initial States	24
8	Additional Theorems for Transition Systems	25
9	Constructing Paths and Runs in Transition Systems	26
10	Deterministic Automata	27
11	Deterministic Finite Automata	38
12	Nondeterministic Automata	39
13	Nondeterministic Finite Automata	52
14	Deterministic Büchi Automata	53
15	Deterministic Generalized Büchi Automata	53
16	Deterministic Büchi Automata Combinations	54
17	Deterministic Büchi Transition Automata	56
18	Deterministic Generalized Büchi Transition Automata	57
19	Deterministic Büchi Transition Automata Combinations	58
20	Deterministic Co-Büchi Automata	59
21	Deterministic Co-Generalized Co-Büchi Automata	60
22	Deterministic Co-Büchi Automata Combinations	60
23	Deterministic Rabin Automata	63
24	Deterministic Rabin Automata Combinations	63
25	Relations and Refinement	64
25.1	Predicate to Set Conversion Setup	64
25.2	Relation Composition	65
25.3	Relation Basics	66

25.4 Parametricity	66
25.5 Lists	67
25.6 Streams	67
25.7 Functional Relations	69
26 Refinement for Transition Systems	70
27 Relations on Deterministic Rabin Automata	70
28 Implementation	72
28.1 Syntax	72
28.2 Monadic Refinement	72
28.3 Implementations for Sets Represented by Lists	73
28.4 Autoref Setup	74
29 Implementation of Deterministic Rabin Automata	78
30 Exploration of Deterministic Rabin Automata	79
31 Explicit Deterministic Rabin Automata	80
32 Explore and Enumerate Nodes of Deterministic Rabin Automata	83
32.1 Syntax	84
33 Image on Explicit Automata	84
34 Exploration and Translation	84
35 Nondeterministic Büchi Automata	87
36 Nondeterministic Generalized Büchi Automata	88
37 Nondeterministic Büchi Automata Combinations	89
38 Connecting Nondeterministic Büchi Automata to CAVA Automata Structures	91
38.1 Regular Graphs	91
38.2 Indexed Generalized Büchi Graphs	91
39 Relations on Nondeterministic Büchi Automata	92
40 Implementation of Nondeterministic Büchi Automata	93

41 Algorithms on Nondeterministic Büchi Automata	95
41.1 Miscellaneous Amendments	95
41.2 Operations	95
41.3 Implementations	95
42 Explicit Nondeterministic Büchi Automata	97
43 Explore and Enumerate Nodes of Nondeterministic Büchi Automata	100
43.1 Syntax	100
44 Image on Explicit Automata	101
45 Exploration and Translation	101
46 Connecting Nondeterministic Generalized Büchi Automata to CAVA Automata Structures	103
46.1 Regular Graphs	103
46.2 Indexed Generalized Büchi Graphs	104
47 Relations on Nondeterministic Generalized Büchi Automata	105
48 Implementation of Nondeterministic Generalized Büchi Automata	106
49 Algorithms on Nondeterministic Generalized Büchi Automata	108
49.1 Operations	108
49.2 Implementations	108
50 Nondeterministic Büchi Transition Automata	111
51 Nondeterministic Generalized Büchi Transition Automata	112
52 Nondeterministic Büchi Transition Automata Combinations	113

1 Basics

```
theory Basic
imports Main
begin
```

1.1 Miscellaneous

abbreviation (*input*) *const* $x \equiv \lambda -. x$

lemmas [*simp*] = *map-prod.id map-prod.comp[symmetric]*

lemma *prod-UNIV[iff]*: $A \times B = UNIV \longleftrightarrow A = UNIV \wedge B = UNIV$ *<proof>*

```

lemma prod-singleton:
  fst ‘  $A = \{x\} \implies A = \text{fst} \text{ ‘ } A \times \text{snd} \text{ ‘ } A$ 
  snd ‘  $A = \{y\} \implies A = \text{fst} \text{ ‘ } A \times \text{snd} \text{ ‘ } A$ 
  ⟨proof⟩

lemma infinite-subset[trans]:  $\text{infinite } A \implies A \subseteq B \implies \text{infinite } B$  ⟨proof⟩
lemma finite-subset[trans]:  $A \subseteq B \implies \text{finite } B \implies \text{finite } A$  ⟨proof⟩

declare infinite-coinduct[case-names infinite, coinduct pred: infinite]
lemma infinite-psubset-coinduct[case-names infinite, consumes 1]:
  assumes  $R A$ 
  assumes  $\bigwedge A. R A \implies \exists B \subset A. R B$ 
  shows  $\text{infinite } A$ 
  ⟨proof⟩

thm inj-on-subset subset-inj-on

lemma inj-inj-on[dest]:  $\text{inj } f \implies \text{inj-on } f S$  ⟨proof⟩

end

```

2 Finite and Infinite Sequences

```

theory Sequence
imports
  Basic
  HOL-Library.Stream
  HOL-Library.Monad-Syntax
begin

```

2.1 List Basics

```

declare upt-Suc[simp del]
declare last.simps[simp del]
declare butlast.simps[simp del]
declare Cons-nth-drop-Suc[simp]
declare list.pred-True[simp]

lemma list-pred-cases:
  assumes  $\text{list-all } P xs$ 
  obtains  $(\text{nil}) \text{ } xs = [] \mid (\text{cons}) \text{ } y \text{ } ys$  where  $xs = y \# ys \text{ } P \text{ } y \text{ } \text{list-all } P \text{ } ys$ 
  ⟨proof⟩

lemma lists-iff-set:  $w \in \text{lists } A \iff \text{set } w \subseteq A$  ⟨proof⟩

lemma fold-const:  $\text{fold } \text{const } xs \text{ } a = \text{last } (a \# xs)$ 
  ⟨proof⟩

```

lemma *take-Suc*: $take (Suc\ n)\ xs = (if\ xs = []\ then\ []\ else\ hd\ xs\ \# \ take\ n\ (tl\ xs))$
 ⟨proof⟩

lemma *bind-map[simp]*: $map\ f\ xs \ggg\ g = xs \ggg\ g \circ f$ ⟨proof⟩

lemma *ball-bind[iff]*: $Ball\ (set\ (xs \ggg\ f))\ P \longleftrightarrow (\forall\ x \in set\ xs.\ \forall\ y \in set\ (f\ x).\ P\ y)$
 ⟨proof⟩

lemma *bex-bind[iff]*: $Bex\ (set\ (xs \ggg\ f))\ P \longleftrightarrow (\exists\ x \in set\ xs.\ \exists\ y \in set\ (f\ x).\ P\ y)$
 ⟨proof⟩

lemma *list-choice*: $list\ all\ (\lambda\ x.\ \exists\ y.\ P\ x\ y)\ xs \longleftrightarrow (\exists\ ys.\ list\ all2\ P\ xs\ ys)$
 ⟨proof⟩

lemma *listset-member*: $ys \in listset\ XS \longleftrightarrow list\ all2\ (\in)\ ys\ XS$
 ⟨proof⟩

lemma *listset-empty[iff]*: $listset\ XS = \{\}\ \longleftrightarrow \neg\ list\ all\ (\lambda\ A.\ A \neq \{\})\ XS$
 ⟨proof⟩

lemma *listset-finite[iff]*:
 assumes $list\ all\ (\lambda\ A.\ A \neq \{\})\ XS$
 shows $finite\ (listset\ XS) \longleftrightarrow list\ all\ finite\ XS$
 ⟨proof⟩

lemma *listset-finite'[intro]*:
 assumes $list\ all\ finite\ XS$
 shows $finite\ (listset\ XS)$
 ⟨proof⟩

lemma *listset-card[simp]*: $card\ (listset\ XS) = prod\ list\ (map\ card\ XS)$
 ⟨proof⟩

2.2 Stream Basics

declare *stream.map-id[simp]*
declare *stream.set-map[simp]*
declare *stream.set-sel(1)[intro!, simp]*
declare *stream.pred-True[simp]*
declare *stream.pred-map[iff]*
declare *stream.rel-map[iff]*
declare *shift-simps[simp del]*
declare *stake-sdrop[simp]*
declare *stake-siterate[simp del]*
declare *sdrop-snth[simp]*

lemma *stream-pred-cases*:
 assumes $pred\ stream\ P\ xs$
 obtains $(scons)\ y\ ys$ **where** $xs = y\ \#\#\ ys\ P\ y\ pred\ stream\ P\ ys$
 ⟨proof⟩

lemma *stream-rel-coinduct[case-names stream-rel, coinduct pred: stream-all2]*:

assumes $R\ u\ v$
assumes $\bigwedge a\ u\ b\ v. R\ (a\ \#\#\ u)\ (b\ \#\#\ v) \implies P\ a\ b \wedge R\ u\ v$
shows $stream\text{-}all2\ P\ u\ v$
 $\langle proof \rangle$

lemma $stream\text{-}rel\text{-}coinduct\text{-}shift[case\text{-}names\ stream\text{-}rel, consumes\ 1]:$
assumes $R\ u\ v$
assumes $\bigwedge u\ v. R\ u\ v \implies$
 $\exists u_1\ u_2\ v_1\ v_2. u = u_1\ @-\ u_2 \wedge v = v_1\ @-\ v_2 \wedge u_1 \neq [] \wedge v_1 \neq [] \wedge list\text{-}all2$
 $P\ u_1\ v_1 \wedge R\ u_2\ v_2$
shows $stream\text{-}all2\ P\ u\ v$
 $\langle proof \rangle$

lemma $stream\text{-}pred\text{-}coinduct[case\text{-}names\ stream\text{-}pred, coinduct\ pred: pred\text{-}stream]:$
assumes $R\ w$
assumes $\bigwedge a\ w. R\ (a\ \#\#\ w) \implies P\ a \wedge R\ w$
shows $pred\text{-}stream\ P\ w$
 $\langle proof \rangle$

lemma $stream\text{-}pred\text{-}coinduct\text{-}shift[case\text{-}names\ stream\text{-}pred, consumes\ 1]:$
assumes $R\ w$
assumes $\bigwedge w. R\ w \implies \exists u\ v. w = u\ @-\ v \wedge u \neq [] \wedge list\text{-}all\ P\ u \wedge R\ v$
shows $pred\text{-}stream\ P\ w$
 $\langle proof \rangle$

lemma $stream\text{-}pred\text{-}flat\text{-}coinduct[case\text{-}names\ stream\text{-}pred, consumes\ 1]:$
assumes $R\ ws$
assumes $\bigwedge w\ ws. R\ (w\ \#\#\ ws) \implies w \neq [] \wedge list\text{-}all\ P\ w \wedge R\ ws$
shows $pred\text{-}stream\ P\ (flat\ ws)$
 $\langle proof \rangle$

lemmas $stream\text{-}eq\text{-}coinduct[case\text{-}names\ stream\text{-}eq, coinduct\ pred: HOL.eq] =$
 $stream\text{-}rel\text{-}coinduct[\mathbf{where}\ ?P = HOL.eq, unfolded\ stream.rel\text{-}eq]$
lemmas $stream\text{-}eq\text{-}coinduct\text{-}shift[case\text{-}names\ stream\text{-}eq, consumes\ 1] =$
 $stream\text{-}rel\text{-}coinduct\text{-}shift[\mathbf{where}\ ?P = HOL.eq, unfolded\ stream.rel\text{-}eq\ list.rel\text{-}eq]$

lemma $stream\text{-}pred\text{-}shift[iff]: pred\text{-}stream\ P\ (u\ @-\ v) \longleftrightarrow list\text{-}all\ P\ u \wedge pred\text{-}stream$
 $P\ v$
 $\langle proof \rangle$

lemma $stream\text{-}rel\text{-}shift[iff]:$
assumes $length\ u_1 = length\ v_1$
shows $stream\text{-}all2\ P\ (u_1\ @-\ u_2)\ (v_1\ @-\ v_2) \longleftrightarrow list\text{-}all2\ P\ u_1\ v_1 \wedge stream\text{-}all2$
 $P\ u_2\ v_2$
 $\langle proof \rangle$

lemma $sset\text{-}subset\text{-}stream\text{-}pred: sset\ w \subseteq A \longleftrightarrow pred\text{-}stream\ (\lambda a. a \in A)\ w$
 $\langle proof \rangle$

lemma $eq\text{-}scons: w = a\ \#\#\ v \longleftrightarrow a = shd\ w \wedge v = stl\ w\ \langle proof \rangle$
lemma $scons\text{-}eq: a\ \#\#\ v = w \longleftrightarrow shd\ w = a \wedge stl\ w = v\ \langle proof \rangle$
lemma $eq\text{-}shift: w = u\ @-\ v \longleftrightarrow stake\ (length\ u)\ w = u \wedge sdrop\ (length\ u)\ w$
 $= v$

<proof>

lemma *shift-eq*: $u @- v = w \longleftrightarrow u = \text{stake } (\text{length } u) \ w \wedge v = \text{sdrop } (\text{length } u) \ w$

<proof>

lemma *scons-eq-shift*: $a \#\# w = u @- v \longleftrightarrow (\square = u \wedge a \#\# w = v) \vee (\exists u'. a \# u' = u \wedge w = u' @- v)$

<proof>

lemma *shift-eq-scons*: $u @- v = a \#\# w \longleftrightarrow (u = \square \wedge v = a \#\# w) \vee (\exists u'. u = a \# u' \wedge u' @- v = w)$

<proof>

lemma *stream-all2-sset1*:

assumes *stream-all2* $P \ x \ ys$

shows $\forall x \in \text{sset } xs. \exists y \in \text{sset } ys. P \ x \ y$

<proof>

lemma *stream-all2-sset2*:

assumes *stream-all2* $P \ xs \ ys$

shows $\forall y \in \text{sset } ys. \exists x \in \text{sset } xs. P \ x \ y$

<proof>

lemma *smap-eq-scons*[*iff*]: $\text{smap } f \ xs = y \#\# \ ys \longleftrightarrow f \ (\text{shd } xs) = y \wedge \text{smap } f \ (\text{stl } xs) = ys$

<proof>

lemma *scons-eq-smap*[*iff*]: $y \#\# \ ys = \text{smap } f \ xs \longleftrightarrow y = f \ (\text{shd } xs) \wedge ys = \text{smap } f \ (\text{stl } xs)$

<proof>

lemma *smap-eq-shift*[*iff*]:

$\text{smap } f \ w = u @- v \longleftrightarrow (\exists w_1 \ w_2. w = w_1 @- w_2 \wedge \text{map } f \ w_1 = u \wedge \text{smap } f \ w_2 = v)$

<proof>

lemma *shift-eq-smap*[*iff*]:

$u @- v = \text{smap } f \ w \longleftrightarrow (\exists w_1 \ w_2. w = w_1 @- w_2 \wedge u = \text{map } f \ w_1 \wedge v = \text{smap } f \ w_2)$

<proof>

lemma *szip-eq-scons*[*iff*]: $\text{szip } xs \ ys = z \#\# \ zs \longleftrightarrow (\text{shd } xs, \text{shd } ys) = z \wedge \text{szip } (\text{stl } xs) \ (\text{stl } ys) = zs$

<proof>

lemma *scons-eq-szip*[*iff*]: $z \#\# \ zs = \text{szip } xs \ ys \longleftrightarrow z = (\text{shd } xs, \text{shd } ys) \wedge zs = \text{szip } (\text{stl } xs) \ (\text{stl } ys)$

<proof>

lemma *siterate-eq-scons*[*iff*]: $\text{siterate } f \ s = a \#\# \ w \longleftrightarrow s = a \wedge \text{siterate } f \ (f \ s) = w$

<proof>

lemma *scons-eq-siterate*[*iff*]: $a \#\# \ w = \text{siterate } f \ s \longleftrightarrow a = s \wedge w = \text{siterate } f \ (f \ s)$

<proof>

lemma *snth-0*: $(a \#\# w) !! 0 = a$ \langle proof \rangle

lemma *eqI-snth*:

assumes $\bigwedge i. u !! i = v !! i$

shows $u = v$

\langle proof \rangle

lemma *stream-pred-snth*: $\text{pred-stream } P \ w \longleftrightarrow (\forall i. P (w !! i))$

\langle proof \rangle

lemma *stream-rel-snth*: $\text{stream-all2 } P \ u \ v \longleftrightarrow (\forall i. P (u !! i) (v !! i))$

\langle proof \rangle

lemma *stream-rel-pred-szip*: $\text{stream-all2 } P \ u \ v \longleftrightarrow \text{pred-stream } (\text{case-prod } P)$
 $(\text{szip } u \ v)$

\langle proof \rangle

lemma *sconst-eq*[*iff*]: $\text{sconst } x = \text{sconst } y \longleftrightarrow x = y$ \langle proof \rangle

lemma *stream-pred--sconst*[*iff*]: $\text{pred-stream } P (\text{sconst } x) \longleftrightarrow P \ x$

\langle proof \rangle

lemma *stream-rel-sconst*[*iff*]: $\text{stream-all2 } P (\text{sconst } x) (\text{sconst } y) \longleftrightarrow P \ x \ y$

\langle proof \rangle

lemma *set-sset-stake*[*intro!*, *simp*]: $\text{set } (\text{stake } n \ xs) \subseteq \text{sset } xs$

\langle proof \rangle

lemma *sset-sdrop*[*intro!*, *simp*]: $\text{sset } (\text{sdrop } n \ xs) \subseteq \text{sset } xs$

\langle proof \rangle

lemma *set-stake-snth*: $x \in \text{set } (\text{stake } n \ xs) \longleftrightarrow (\exists i < n. xs !! i = x)$

\langle proof \rangle

lemma *szip-transfer*[*transfer-rule*]:

includes *lifting-syntax*

shows $(\text{stream-all2 } A \implies \text{stream-all2 } B \implies \text{stream-all2 } (\text{rel-prod } A \ B))$

szip szip

\langle proof \rangle

lemma *siterate-transfer*[*transfer-rule*]:

includes *lifting-syntax*

shows $((A \implies A) \implies A \implies \text{stream-all2 } A) \text{ siterate siterate}$

\langle proof \rangle

lemma *split-stream-first*:

assumes $A \cap \text{sset } xs \neq \{\}$

obtains $ys \ a \ zs$

where $xs = ys @- a \#\# zs \ A \cap \text{set } ys = \{\} \ a \in A$

\langle proof \rangle

lemma *split-stream-first'*:

assumes $x \in \text{sset } xs$

obtains $ys \ zs$

where $xs = ys @ - x \# \# zs \ x \notin \text{set } ys$
 ⟨proof⟩

lemma *streams-UNIV*[*iff*]: $\text{streams } A = \text{UNIV} \longleftrightarrow A = \text{UNIV}$
 ⟨proof⟩

lemma *streams-int*[*simp*]: $\text{streams } (A \cap B) = \text{streams } A \cap \text{streams } B$ ⟨proof⟩

lemma *streams-Int*[*simp*]: $\text{streams } (\bigcap S) = \bigcap (\text{streams } ` S)$ ⟨proof⟩

lemma *pred-list-listsp*[*pred-set-conv*]: $\text{list-all} = \text{listsp}$
 ⟨proof⟩

lemma *pred-stream-streamsp*[*pred-set-conv*]: $\text{pred-stream} = \text{streamsp}$
 ⟨proof⟩

2.3 The scan Function

primrec (*transfer*) *scan* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'a \text{ list} \Rightarrow 'b \Rightarrow 'b \text{ list}$ **where**
 $\text{scan } f \ [] \ a = [] \mid \text{scan } f \ (x \# xs) \ a = f \ x \ a \ \# \ \text{scan } f \ xs \ (f \ x \ a)$

lemma *scan-append*[*simp*]: $\text{scan } f \ (xs @ ys) \ a = \text{scan } f \ xs \ a @ \text{scan } f \ ys \ (fold \ f \ xs \ a)$
 ⟨proof⟩

lemma *scan-eq-nil*[*iff*]: $\text{scan } f \ xs \ a = [] \longleftrightarrow xs = []$ ⟨proof⟩

lemma *scan-eq-cons*[*iff*]:
 $\text{scan } f \ xs \ a = b \ \# \ w \longleftrightarrow (\exists \ y \ ys. xs = y \# ys \wedge f \ y \ a = b \wedge \text{scan } f \ ys \ (f \ y \ a) = w)$
 ⟨proof⟩

lemma *scan-eq-append*[*iff*]:
 $\text{scan } f \ xs \ a = u @ v \longleftrightarrow (\exists \ ys \ zs. xs = ys @ zs \wedge \text{scan } f \ ys \ a = u \wedge \text{scan } f \ zs \ (fold \ f \ ys \ a) = v)$
 ⟨proof⟩

lemma *scan-length*[*simp*]: $\text{length } (\text{scan } f \ xs \ a) = \text{length } xs$
 ⟨proof⟩

lemma *scan-last*: $\text{last } (a \# \text{scan } f \ xs \ a) = \text{fold } f \ xs \ a$
 ⟨proof⟩

lemma *scan-butlast*[*simp*]: $\text{scan } f \ (\text{butlast } xs) \ a = \text{butlast } (\text{scan } f \ xs \ a)$
 ⟨proof⟩

lemma *scan-const*[*simp*]: $\text{scan } \text{const } xs \ a = xs$
 ⟨proof⟩

lemma *scan-nth*[*simp*]:
assumes $i < \text{length } (\text{scan } f \ xs \ a)$
shows $\text{scan } f \ xs \ a ! i = \text{fold } f \ (\text{take } (\text{Suc } i) \ xs) \ a$
 ⟨proof⟩

lemma *scan-map*[*simp*]: $\text{scan } f \ (\text{map } g \ xs) \ a = \text{scan } (f \circ g) \ xs \ a$
 ⟨proof⟩

lemma *scan-take*[*simp*]: $\text{take } k \ (\text{scan } f \ xs \ a) = \text{scan } f \ (\text{take } k \ xs) \ a$

<proof>

lemma *scan-drop[simp]*: $\text{drop } k (\text{scan } f \text{ } xs \ a) = \text{scan } f (\text{drop } k \ xs) (\text{fold } f (\text{take } k \ xs) \ a)$

<proof>

primcorec (*transfer*) *sscan* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'a \ \text{stream} \Rightarrow 'b \Rightarrow 'b \ \text{stream}$
where

$\text{sscan } f \ xs \ a = f (\text{shd } xs) \ a \ \#\#\ \text{sscan } f (\text{stl } xs) (f (\text{shd } xs) \ a)$

lemma *sscan-scons[simp]*: $\text{sscan } f (x \ \#\#\ xs) \ a = f \ x \ a \ \#\#\ \text{sscan } f \ xs (f \ x \ a)$

<proof>

lemma *sscan-shift[simp]*: $\text{sscan } f (xs \ @- \ ys) \ a = \text{scan } f \ xs \ a \ @- \ \text{sscan } f \ ys (\text{fold } f \ xs \ a)$

<proof>

lemma *sscan-eq-scons[iff]*:

$\text{sscan } f \ xs \ a = b \ \#\#\ w \longleftrightarrow f (\text{shd } xs) \ a = b \wedge \text{sscan } f (\text{stl } xs) (f (\text{shd } xs) \ a) = w$

<proof>

lemma *scons-eq-sscan[iff]*:

$b \ \#\#\ w = \text{sscan } f \ xs \ a \longleftrightarrow b = f (\text{shd } xs) \ a \wedge w = \text{sscan } f (\text{stl } xs) (f (\text{shd } xs) \ a)$

<proof>

lemma *sscan-const[simp]*: $\text{sscan } \text{const } xs \ a = xs$

<proof>

lemma *sscan-snth[simp]*: $\text{sscan } f \ xs \ a \ !! \ i = \text{fold } f (\text{stake } (Suc \ i) \ xs) \ a$

<proof>

lemma *sscan-scons-snth[simp]*: $(a \ \#\#\ \text{sscan } f \ xs \ a) \ !! \ i = \text{fold } f (\text{stake } i \ xs) \ a$

<proof>

lemma *sscan-smap[simp]*: $\text{sscan } f (\text{smap } g \ xs) \ a = \text{sscan } (f \circ g) \ xs \ a$

<proof>

lemma *sscan-stake[simp]*: $\text{stake } k (\text{sscan } f \ xs \ a) = \text{scan } f (\text{stake } k \ xs) \ a$

<proof>

lemma *sscan-sdrop[simp]*: $\text{sdrop } k (\text{sscan } f \ xs \ a) = \text{sscan } f (\text{sdrop } k \ xs) (\text{fold } f (\text{stake } k \ xs) \ a)$

<proof>

2.4 Transposing Streams

primcorec (*transfer*) *stranspose* :: $'a \ \text{stream} \ \text{list} \Rightarrow 'a \ \text{list} \ \text{stream}$ **where**

$\text{stranspose } ws = \text{map } \text{shd } ws \ \#\#\ \text{stranspose } (\text{map } \text{stl } ws)$

lemma *stranspose-eq-scons[iff]*: $\text{stranspose } ws = a \ \#\#\ w \longleftrightarrow \text{map } \text{shd } ws = a \wedge \text{stranspose } (\text{map } \text{stl } ws) = w$

<proof>

lemma *scons-eq-stranspose[iff]*: $a \ \#\#\ w = \text{stranspose } ws \longleftrightarrow a = \text{map } \text{shd } ws \wedge w = \text{stranspose } (\text{map } \text{stl } ws)$

<proof>

lemma *stranspose-nil*[simp]: $\text{stranspose } [] = \text{sconst } []$ *<proof>*
lemma *stranspose-cons*[simp]: $\text{stranspose } (w \# ws) = \text{smap2 } \text{Cons } w (\text{stranspose } ws)$
<proof>

lemma *snth-stranspose*[simp]: $\text{stranspose } ws !! k = \text{map } (\lambda w. w !! k) ws$ *<proof>*

lemma *stranspose-nth*[simp]:

assumes $k < \text{length } ws$

shows $\text{smap } (\lambda xs. xs ! k) (\text{stranspose } ws) = ws ! k$

<proof>

2.5 Distinct Streams

coinductive *sdistinct* :: 'a stream \Rightarrow bool **where**

$\text{scons}[\text{intro!}]: x \notin \text{sset } xs \Longrightarrow \text{sdistinct } xs \Longrightarrow \text{sdistinct } (x \#\# xs)$

lemma *sdistinct-scons-elim*[elim!]:

assumes $\text{sdistinct } (x \#\# xs)$

obtains $x \notin \text{sset } xs \text{ sdistinct } xs$

<proof>

lemma *sdistinct-coinduct*[case-names *sdistinct*, *coinduct pred: sdistinct*]:

assumes $P xs$

assumes $\bigwedge x xs. P (x \#\# xs) \Longrightarrow x \notin \text{sset } xs \wedge P xs$

shows $\text{sdistinct } xs$

<proof>

lemma *sdistinct-shift*[intro!]:

assumes $\text{distinct } xs \text{ sdistinct } ys \text{ set } xs \cap \text{sset } ys = \{\}$

shows $\text{sdistinct } (xs @- ys)$

<proof>

lemma *sdistinct-shift-elim*[elim!]:

assumes $\text{sdistinct } (xs @- ys)$

obtains $\text{distinct } xs \text{ sdistinct } ys \text{ set } xs \cap \text{sset } ys = \{\}$

<proof>

lemma *sdistinct-infinite-sset*:

assumes $\text{sdistinct } w$

shows $\text{infinite } (\text{sset } w)$

<proof>

lemma *not-sdistinct-decomp*:

assumes $\neg \text{sdistinct } w$

obtains $u v a w'$

where $w = u @- a \#\# v @- a \#\# w'$

<proof>

2.6 Sorted Streams

coinductive (in order) *sascending* :: 'a stream \Rightarrow bool **where**
 $a \leq b \implies \text{sascending } (b \## w) \implies \text{sascending } (a \## b \## w)$

coinductive (in order) *sdescending* :: 'a stream \Rightarrow bool **where**
 $a \geq b \implies \text{sdescending } (b \## w) \implies \text{sdescending } (a \## b \## w)$

lemma *sdescending-coinduct*[case-names *sdescending*, coinduct pred: *sdescending*]:

assumes $P w$
assumes $\bigwedge a b w. P (a \## b \## w) \implies a \geq b \wedge P (b \## w)$
shows *sdescending* w
 $\langle \text{proof} \rangle$

lemma *sdescending-scons*:

assumes *sdescending* $(a \## w)$
shows *sdescending* w
 $\langle \text{proof} \rangle$

lemma *sdescending-sappend*:

assumes *sdescending* $(u @- v)$
obtains *sdescending* v
 $\langle \text{proof} \rangle$

lemma *sdescending-sdrop*:

assumes *sdescending* w
shows *sdescending* $(\text{sdrop } k w)$
 $\langle \text{proof} \rangle$

lemma *sdescending-sset-scons*:

assumes *sdescending* $(a \## w)$
assumes $b \in \text{sset } w$
shows $a \geq b$
 $\langle \text{proof} \rangle$

lemma *sdescending-sset-sappend*:

assumes *sdescending* $(u @- v)$
assumes $a \in \text{set } u \ b \in \text{sset } v$
shows $a \geq b$
 $\langle \text{proof} \rangle$

lemma *sdescending-snth-antimono*:

assumes *sdescending* w
shows *antimono* $(\text{snth } w)$
 $\langle \text{proof} \rangle$

lemma *sdescending-stuck*:

fixes $w :: 'a :: \text{wellorder stream}$
assumes *sdescending* w
obtains $u a$
where $w = u @- \text{sconst } a$
 $\langle \text{proof} \rangle$

end

3 Linear Temporal Logic on Streams

theory *Sequence-LTL*

imports

Sequence

HOL-Library.Linear-Temporal-Logic-on-Streams

begin

3.1 Basics

Avoid destroying the constant *holds* prematurely.

lemmas [simp del] = holds.simps holds-eq1 holds-eq2 not-holds-eq

lemma *ev-smap*[iff]: $ev\ P\ (smap\ f\ w) \longleftrightarrow ev\ (P\ \circ\ smap\ f)\ w$ *<proof>*

lemma *alw-smap*[iff]: $alw\ P\ (smap\ f\ w) \longleftrightarrow alw\ (P\ \circ\ smap\ f)\ w$ *<proof>*

lemma *holds-smap*[iff]: $holds\ P\ (smap\ f\ w) \longleftrightarrow holds\ (P\ \circ\ f)\ w$ *<proof>*

lemmas [iff] = *ev-sconst alw-sconst hld-smap'*

lemmas [iff] = *alw-ev-stl*

lemma *alw-ev-sdrop*[iff]: $alw\ (ev\ P)\ (sdrop\ n\ w) \longleftrightarrow alw\ (ev\ P)\ w$ *<proof>*

lemma *alw-ev-scons*[iff]: $alw\ (ev\ P)\ (a\ \#\#\ w) \longleftrightarrow alw\ (ev\ P)\ w$ *<proof>*

lemma *alw-ev-shift*[iff]: $alw\ (ev\ P)\ (u\ @-\ v) \longleftrightarrow alw\ (ev\ P)\ v$ *<proof>*

lemmas [simp del, iff] = *ev-alw-stl*

lemma *ev-alw-sdrop*[iff]: $ev\ (alw\ P)\ (sdrop\ n\ w) \longleftrightarrow ev\ (alw\ P)\ w$ *<proof>*

lemma *ev-alw-scons*[iff]: $ev\ (alw\ P)\ (a\ \#\#\ w) \longleftrightarrow ev\ (alw\ P)\ w$ *<proof>*

lemma *ev-alw-shift*[iff]: $ev\ (alw\ P)\ (u\ @-\ v) \longleftrightarrow ev\ (alw\ P)\ v$ *<proof>*

lemma *holds-sconst*[iff]: $holds\ P\ (sconst\ a) \longleftrightarrow P\ a$ *<proof>*

lemma *HLD-sconst*[iff]: $HLD\ A\ (sconst\ a) \longleftrightarrow a \in A$ *<proof>*

lemma *ev-alt-def*: $ev\ \varphi\ w \longleftrightarrow (\exists\ u\ v.\ w = u\ @-\ v \wedge \varphi\ v)$ *<proof>*

lemma *ev-stl-alt-def*: $ev\ \varphi\ (stl\ w) \longleftrightarrow (\exists\ u\ v.\ w = u\ @-\ v \wedge u \neq [] \wedge \varphi\ v)$ *<proof>*

lemma *ev-HLD-sset*: $ev\ (HLD\ A)\ w \longleftrightarrow sset\ w \cap A \neq \{\}$ *<proof>*

lemma *alw-ev-coinduct*[case-names *alw-ev*, consumes 1]:

assumes *R w*

assumes $\bigwedge w.\ R\ w \implies ev\ \varphi\ w \wedge ev\ R\ (stl\ w)$

shows $alw (ev \varphi) w$
 $\langle proof \rangle$

3.2 Infinite Occurrence

abbreviation $infs P w \equiv alw (ev (holds P)) w$

abbreviation $fins P w \equiv \neg infs P w$

lemma $infs\text{-suffix}$: $infs P w \longleftrightarrow (\forall u v. w = u @- v \longrightarrow Bex (sset v) P)$
 $\langle proof \rangle$

lemma $infs\text{-snth}$: $infs P w \longleftrightarrow (\forall n. \exists k \geq n. P (w !! k))$
 $\langle proof \rangle$

lemma $infs\text{-infm}$: $infs P w \longleftrightarrow (\exists_{\infty} i. P (w !! i))$
 $\langle proof \rangle$

lemma $infs\text{-coinduct}$ [$case\text{-names } infs, coinduct\ pred: infs$]:

assumes $R w$

assumes $\bigwedge w. R w \implies Bex (sset w) P \wedge ev R (stl w)$

shows $infs P w$

$\langle proof \rangle$

lemma $infs\text{-coinduct-shift}$ [$case\text{-names } infs, consumes 1$]:

assumes $R w$

assumes $\bigwedge w. R w \implies \exists u v. w = u @- v \wedge Bex (set u) P \wedge R v$

shows $infs P w$

$\langle proof \rangle$

lemma $infs\text{-flat-coinduct}$ [$case\text{-names } infs\text{-flat}, consumes 1$]:

assumes $R w$

assumes $\bigwedge u v. R (u \#\# v) \implies Bex (set u) P \wedge R v$

shows $infs P (flat w)$

$\langle proof \rangle$

lemma $infs\text{-sscan-coinduct}$ [$case\text{-names } infs\text{-sscan}, consumes 1$]:

assumes $R w a$

assumes $\bigwedge w a. R w a \implies P a \wedge (\exists u v. w = u @- v \wedge u \neq [] \wedge R v (fold f u a))$

shows $infs P (a \#\# sscan f w a)$

$\langle proof \rangle$

lemma $infs\text{-mono}$: $(\bigwedge a. a \in sset w \implies P a \implies Q a) \implies infs P w \implies infs Q w$

$\langle proof \rangle$

lemma $infs\text{-mono-strong}$: $stream\text{-all2} (\lambda a b. P a \longrightarrow Q b) u v \implies infs P u \implies infs Q v$

$\langle proof \rangle$

lemma $infs\text{-all}$: $Ball (sset w) P \implies infs P w$ $\langle proof \rangle$

lemma $infs\text{-any}$: $infs P w \implies Bex (sset w) P$ $\langle proof \rangle$

lemma $infs\text{-bot}$ [iff]: $infs bot w \longleftrightarrow False$ $\langle proof \rangle$

lemma $infs\text{-top}$ [iff]: $infs top w \longleftrightarrow True$ $\langle proof \rangle$

lemma *infs-disj*[*iff*]: $\text{infs } (\lambda a. P a \vee Q a) w \longleftrightarrow \text{infs } P w \vee \text{infs } Q w$
 <proof>
lemma *infs-bex*[*iff*]:
assumes *finite S*
shows $\text{infs } (\lambda a. \exists x \in S. P x a) w \longleftrightarrow (\exists x \in S. \text{infs } (P x) w)$
 <proof>
lemma *infs-bex-le-nat*[*iff*]: $\text{infs } (\lambda a. \exists k < n :: \text{nat}. P k a) w \longleftrightarrow (\exists k < n. \text{infs } (P k) w)$
 <proof>

lemma *infs-cycle*[*iff*]:
assumes $w \neq []$
shows $\text{infs } P (\text{cycle } w) \longleftrightarrow \text{Bex } (\text{set } w) P$
 <proof>

end

4 Zipping Sequences

theory *Sequence-Zip*
imports *Sequence-LTL*
begin

4.1 Zipping Lists

notation *zip* (**infixr** || 51)

lemmas [*simp*] = *zip-map-fst-snd*

lemma *split-zip*[*no-atp*]: $(\bigwedge x. \text{PROP } P x) \equiv (\bigwedge y z. \text{length } y = \text{length } z \implies \text{PROP } P (y \parallel z))$
 <proof>

lemma *split-zip-all*[*no-atp*]: $(\forall x. P x) \longleftrightarrow (\forall y z. \text{length } y = \text{length } z \longrightarrow P (y \parallel z))$
 <proof>

lemma *split-zip-ex*[*no-atp*]: $(\exists x. P x) \longleftrightarrow (\exists y z. \text{length } y = \text{length } z \wedge P (y \parallel z))$
 <proof>

lemma *zip-eq*[*iff*]:
assumes $\text{length } u = \text{length } v \text{ length } r = \text{length } s$
shows $u \parallel v = r \parallel s \longleftrightarrow u = r \wedge v = s$
 <proof>

lemma *list-rel-pred-zip*: $\text{list-all2 } P xs ys \longleftrightarrow \text{length } xs = \text{length } ys \wedge \text{list-all } (\text{case-prod } P) (xs \parallel ys)$
 <proof>

lemma *list-choice-zip*: $\text{list-all } (\lambda x. \exists y. P x y) xs \longleftrightarrow$

$(\exists ys. \text{length } ys = \text{length } xs \wedge \text{list-all } (\text{case-prod } P) (xs \parallel ys))$
 $\langle \text{proof} \rangle$

lemma *list-choice-pair*: $\text{list-all } (\lambda xy. \text{case-prod } (\lambda x y. \exists z. P x y z) xy) (xs \parallel ys) \longleftrightarrow$
 $(\exists zs. \text{length } zs = \min (\text{length } xs) (\text{length } ys) \wedge \text{list-all } (\lambda (x, y, z). P x y z) (xs \parallel ys \parallel zs))$
 $\langle \text{proof} \rangle$

lemma *list-rel-zip*[*iff*]:

assumes $\text{length } u = \text{length } v \text{ length } r = \text{length } s$
shows $\text{list-all2 } (\text{rel-prod } A B) (u \parallel v) (r \parallel s) \longleftrightarrow \text{list-all2 } A u r \wedge \text{list-all2 } B v s$
 $\langle \text{proof} \rangle$

lemma *zip-last*[*simp*]:

assumes $xs \parallel ys \neq [] \text{ length } xs = \text{length } ys$
shows $\text{last } (xs \parallel ys) = (\text{last } xs, \text{last } ys)$
 $\langle \text{proof} \rangle$

4.2 Zipping Streams

notation *szip* (**infixr** $|||$ 51)

lemmas [*simp*] = *szip-unfold*

lemma *smap-szip-same*: $\text{smap } f (xs ||| xs) = \text{smap } (\lambda x. f (x, x)) xs \langle \text{proof} \rangle$

lemma *szip-smap*[*simp*]: $\text{smap } \text{fst } zs ||| \text{smap } \text{snd } zs = zs \langle \text{proof} \rangle$

lemma *szip-smap-fst*[*simp*]: $\text{smap } \text{fst } (xs ||| ys) = xs \langle \text{proof} \rangle$

lemma *szip-smap-snd*[*simp*]: $\text{smap } \text{snd } (xs ||| ys) = ys \langle \text{proof} \rangle$

lemma *szip-smap-both*: $\text{smap } f xs ||| \text{smap } g ys = \text{smap } (\text{map-prod } f g) (xs ||| ys)$
 $\langle \text{proof} \rangle$

lemma *szip-smap-left*: $\text{smap } f xs ||| ys = \text{smap } (\text{apfst } f) (xs ||| ys) \langle \text{proof} \rangle$

lemma *szip-smap-right*: $xs ||| \text{smap } f ys = \text{smap } (\text{apsnd } f) (xs ||| ys) \langle \text{proof} \rangle$

lemmas *szip-smap-fold* = *szip-smap-both szip-smap-left szip-smap-right*

lemma *szip-sconst-smap-fst*: $\text{sconst } a ||| xs = \text{smap } (\text{Pair } a) xs$
 $\langle \text{proof} \rangle$

lemma *szip-sconst-smap-snd*: $xs ||| \text{sconst } a = \text{smap } (\text{prod.swap} \circ \text{Pair } a) xs$
 $\langle \text{proof} \rangle$

lemma *split-szip*[*no-atp*]: $(\bigwedge x. \text{PROP } P x) \equiv (\bigwedge y z. \text{PROP } P (y ||| z))$
 $\langle \text{proof} \rangle$

lemma *split-szip-all*[*no-atp*]: $(\forall x. P x) \longleftrightarrow (\forall y z. P (y ||| z)) \langle \text{proof} \rangle$

lemma *split-szip-ex*[*no-atp*]: $(\exists x. P x) \longleftrightarrow (\exists y z. P (y ||| z)) \langle \text{proof} \rangle$

lemma *szip-eq*[*iff*]: $u ||| v = r ||| s \longleftrightarrow u = r \wedge v = s$
 $\langle \text{proof} \rangle$

lemma *stream-rel-szip*[*iff*]:
 $stream\text{-}all2\ (rel\text{-}prod\ A\ B)\ (u\ |||\ v)\ (r\ |||\ s) \longleftrightarrow stream\text{-}all2\ A\ u\ r \wedge stream\text{-}all2\ B\ v\ s$
 $\langle proof \rangle$

lemma *szip-shift*[*simp*]:
assumes $length\ u = length\ s$
shows $u\ @-\ v\ |||\ s\ @-\ t = (u\ ||\ s)\ @-\ (v\ |||\ t)$
 $\langle proof \rangle$

lemma *szip-sset-fst*[*simp*]: $fst\ 'sset\ (u\ |||\ v) = sset\ u\ \langle proof \rangle$
lemma *szip-sset-snd*[*simp*]: $snd\ 'sset\ (u\ |||\ v) = sset\ v\ \langle proof \rangle$
lemma *szip-sset-elim*[*elim*]:
assumes $(a, b) \in sset\ (u\ |||\ v)$
obtains $a \in sset\ u\ b \in sset\ v$
 $\langle proof \rangle$

lemma *szip-sset*[*simp*]: $sset\ (u\ |||\ v) \subseteq sset\ u \times sset\ v\ \langle proof \rangle$

lemma *sset-szip-finite*[*iff*]: $finite\ (sset\ (u\ |||\ v)) \longleftrightarrow finite\ (sset\ u) \wedge finite\ (sset\ v)$
 $\langle proof \rangle$

lemma *infs-szip-fst*[*iff*]: $infs\ (P \circ fst)\ (u\ |||\ v) \longleftrightarrow infs\ P\ u$
 $\langle proof \rangle$

lemma *infs-szip-snd*[*iff*]: $infs\ (P \circ snd)\ (u\ |||\ v) \longleftrightarrow infs\ P\ v$
 $\langle proof \rangle$

end

5 Maps

theory *Maps*
imports *Sequence-Zip*
begin

6 Basics

lemma *fun-upd-None*[*simp*]:
assumes $p \notin dom\ f$
shows $f\ (p := None) = f$
 $\langle proof \rangle$

lemma *finite-set-of-finite-maps'*:
assumes $finite\ A\ finite\ B$
shows $finite\ \{m.\ dom\ m \subseteq A \wedge ran\ m \subseteq B\}$
 $\langle proof \rangle$

lemma *fold-map-of*:
assumes *distinct xs*
shows $\text{fold } (\lambda x (k, m). (\text{Suc } k, m (x \mapsto k))) \text{ } xs (k, m) =$
 $(k + \text{length } xs, m ++ \text{map-of } (xs \parallel [k ..< k + \text{length } xs]))$
 $\langle \text{proof} \rangle$

6.1 Expanding set functions to sets of functions

definition *expand* :: $('a \Rightarrow 'b \text{ set}) \Rightarrow ('a \Rightarrow 'b) \text{ set}$ **where**
 $\text{expand } f = \{g. \forall x. g \ x \in f \ x\}$

lemma *expand-update[simp]*:
assumes $f \ x \neq \{\}$
shows $\text{expand } (f (x := S)) = (\bigcup y \in S. (\lambda g. g (x := y))) \text{ } \text{'expand } f$
 $\langle \text{proof} \rangle$

6.2 Expanding set maps into sets of maps

definition *expand-map* :: $('a \rightarrow 'b \text{ set}) \Rightarrow ('a \rightarrow 'b) \text{ set}$ **where**
 $\text{expand-map } f \equiv \text{expand } (\text{case-option } \{\text{None}\} (\text{image } \text{Some}) \circ f)$

lemma *expand-map-alt-def*: $\text{expand-map } f =$
 $\{g. \text{dom } g = \text{dom } f \wedge (\forall x \ S \ y. f \ x = \text{Some } S \longrightarrow g \ x = \text{Some } y \longrightarrow y \in S)\}$
 $\langle \text{proof} \rangle$

lemma *expand-map-dom*:
assumes $g \in \text{expand-map } f$
shows $\text{dom } g = \text{dom } f$
 $\langle \text{proof} \rangle$

lemma *expand-map-empty[simp]*: $\text{expand-map } \text{Map.empty} = \{\text{Map.empty}\}$ $\langle \text{proof} \rangle$

lemma *expand-map-update[simp]*:
 $\text{expand-map } (f (x \mapsto S)) = (\bigcup y \in S. (\lambda g. g (x \mapsto y))) \text{ } \text{'expand-map } (f (x :=$
 $\text{None}))$
 $\langle \text{proof} \rangle$

end

theory *Acceptance*

imports *Sequence-LTL*

begin

type-synonym $'a \text{ pred} = 'a \Rightarrow \text{bool}$
type-synonym $'a \text{ rabin} = 'a \text{ pred} \times 'a \text{ pred}$
type-synonym $'a \text{ gen} = 'a \text{ list}$

definition *rabin* :: $'a \text{ rabin} \Rightarrow 'a \text{ stream pred}$ **where**
 $\text{rabin} \equiv \lambda (I, F) w. \text{infs } I \ w \wedge \text{fins } F \ w$

lemma *rabin[intro]*:

assumes $IF = (I, F)$ *infs* I *w fins* F w
shows *rabin* IF w
 \langle *proof* \rangle

lemma *rabin-elim*[*elim*]:

assumes *rabin* IF w
obtains I F
where $IF = (I, F)$ *infs* I *w fins* F w
 \langle *proof* \rangle

definition *gen* :: $('a \Rightarrow 'b \text{ pred}) \Rightarrow ('a \text{ gen} \Rightarrow 'b \text{ pred})$ **where**
gen P cs $w \equiv \forall c \in \text{set } cs. P c w$

lemma *gen*[*intro*]:

assumes $\bigwedge c. c \in \text{set } cs \Longrightarrow P c w$
shows *gen* P cs w
 \langle *proof* \rangle

lemma *gen-elim*[*elim*]:

assumes *gen* P cs w
obtains $\bigwedge c. c \in \text{set } cs \Longrightarrow P c w$
 \langle *proof* \rangle

definition *cogen* :: $('a \Rightarrow 'b \text{ pred}) \Rightarrow ('a \text{ cogen} \Rightarrow 'b \text{ pred})$ **where**
cogen P cs $w \equiv \exists c \in \text{set } cs. P c w$

lemma *cogen*[*intro*]:

assumes $c \in \text{set } cs$ $P c w$
shows *cogen* P cs w
 \langle *proof* \rangle

lemma *cogen-elim*[*elim*]:

assumes *cogen* P cs w
obtains c
where $c \in \text{set } cs$ $P c w$
 \langle *proof* \rangle

lemma *cogen-alt-def*: *cogen* P cs $w \longleftrightarrow \neg \text{gen } (\lambda c w. \text{Not } (P c w))$ cs w \langle *proof* \rangle

end

theory *Degeneralization*

imports

Acceptance

Sequence-Zip

begin

type-synonym $'a \text{ degen} = 'a \times \text{nat}$

definition *degen* :: $'a \text{ pred} \text{ gen} \Rightarrow 'a \text{ degen} \text{ pred}$ **where**

degen $cs \equiv \lambda (a, k). k \geq \text{length } cs \vee (cs ! k) a$

lemma *degen-simps*[*iff*]: *degen* cs $(a, k) \longleftrightarrow k \geq \text{length } cs \vee (cs ! k) a$ \langle *proof* \rangle

definition *count* :: 'a pred gen \Rightarrow 'a \Rightarrow nat \Rightarrow nat **where**

count cs a k \equiv
 if $k < \text{length } cs$
 then if $(cs ! k) a$ then $\text{Suc } k \bmod \text{length } cs$ else k
 else if $cs = []$ then k else 0

lemma *count-empty[simp]*: *count [] a k = k* $\langle \text{proof} \rangle$

lemma *count-nonempty[simp]*: $cs \neq [] \Longrightarrow \text{count } cs a k < \text{length } cs$ $\langle \text{proof} \rangle$

lemma *count-constant-1*:

assumes $k < \text{length } cs$
assumes $\bigwedge a. a \in \text{set } w \Longrightarrow \neg (cs ! k) a$
shows $\text{fold } (\text{count } cs) w k = k$
 $\langle \text{proof} \rangle$

lemma *count-constant-2*:

assumes $k < \text{length } cs$
assumes $\bigwedge a. a \in \text{set } (w \parallel k \# \text{scan } (\text{count } cs) w k) \Longrightarrow \neg \text{degen } cs a$
shows $\text{fold } (\text{count } cs) w k = k$
 $\langle \text{proof} \rangle$

lemma *count-step*:

assumes $k < \text{length } cs$
assumes $(cs ! k) a$
shows $\text{count } cs a k = \text{Suc } k \bmod \text{length } cs$
 $\langle \text{proof} \rangle$

lemma *degen-skip-condition*:

assumes $k < \text{length } cs$
assumes $\text{infs } (\text{degen } cs) (w \parallel\parallel k \#\#\text{sscan } (\text{count } cs) w k)$
obtains $u a v$
where $w = u @- a \#\#\text{fold } (\text{count } cs) u k = k (cs ! k) a$
 $\langle \text{proof} \rangle$

lemma *degen-skip-arbitrary*:

assumes $k < \text{length } cs$ $l < \text{length } cs$
assumes $\text{infs } (\text{degen } cs) (w \parallel\parallel k \#\#\text{sscan } (\text{count } cs) w k)$
obtains $u v$
where $w = u @- v \text{fold } (\text{count } cs) u k = l$
 $\langle \text{proof} \rangle$

lemma *degen-skip-arbitrary-condition*:

assumes $l < \text{length } cs$
assumes $\text{infs } (\text{degen } cs) (w \parallel\parallel k \#\#\text{sscan } (\text{count } cs) w k)$
obtains $u a v$
where $w = u @- a \#\#\text{fold } (\text{count } cs) u k = l (cs ! l) a$
 $\langle \text{proof} \rangle$

lemma *gen-degen-step*:

assumes $\text{gen infs } cs w$
obtains $u a v$
where $w = u @- a \#\#\text{degen } cs (a, \text{fold } (\text{count } cs) u k)$
 $\langle \text{proof} \rangle$

lemma *degen-infs[iff]*: *infs (degen cs) (w ||| k ## sscan (count cs) w k) \longleftrightarrow gen infs cs w*
<proof>

end

7 Transition Systems

theory *Transition-System*
imports *../Basic/Sequence*
begin

7.1 Universal Transition Systems

locale *transition-system-universal* =
fixes *execute* :: *'transition \Rightarrow 'state \Rightarrow 'state*
begin

abbreviation *target* \equiv *fold execute*
abbreviation *states* \equiv *scan execute*
abbreviation *trace* \equiv *sscan execute*

lemma *target-alt-def*: *target r p = last (p # states r p) <proof>*

end

7.2 Transition Systems

locale *transition-system* =
transition-system-universal execute
for *execute* :: *'transition \Rightarrow 'state \Rightarrow 'state*
+
fixes *enabled* :: *'transition \Rightarrow 'state \Rightarrow bool*
begin

abbreviation *successors* *p* \equiv *{execute a p | a. enabled a p}*

inductive *path* :: *'transition list \Rightarrow 'state \Rightarrow bool* **where**
nil[intro!]: *path [] p |*
cons[intro!]: *enabled a p \implies path r (execute a p) \implies path (a # r) p*

inductive-cases *path-cons-elim[elim!]*: *path (a # r) p*

lemma *path-append[intro!]*:
assumes *path r p path s (target r p)*
shows *path (r @ s) p*
<proof>

lemma *path-append-elim[elim!]*:
assumes *path (r @ s) p*

obtains $path\ r\ p\ path\ s\ (target\ r\ p)$
 $\langle proof \rangle$

coinductive $run :: 'transition\ stream \Rightarrow 'state \Rightarrow bool$ **where**
 $scons[intro!]:\ enabled\ a\ p \Longrightarrow run\ r\ (execute\ a\ p) \Longrightarrow run\ (a\ \#\#\ r)\ p$

inductive-cases $run-scons-elim[elim!]:\ run\ (a\ \#\#\ r)\ p$

lemma $run-shift[intro!]:$
assumes $path\ r\ p\ run\ s\ (target\ r\ p)$
shows $run\ (r\ @-\ s)\ p$
 $\langle proof \rangle$

lemma $run-shift-elim[elim!]:$
assumes $run\ (r\ @-\ s)\ p$
obtains $path\ r\ p\ run\ s\ (target\ r\ p)$
 $\langle proof \rangle$

lemma $run-coinduct[case-names\ run,\ coinduct\ pred:\ run]:$
assumes $R\ r\ p$
assumes $\bigwedge a\ r\ p.\ R\ (a\ \#\#\ r)\ p \Longrightarrow enabled\ a\ p \wedge R\ r\ (execute\ a\ p)$
shows $run\ r\ p$
 $\langle proof \rangle$

lemma $run-coinduct-shift[case-names\ run,\ consumes\ 1]:$
assumes $R\ r\ p$
assumes $\bigwedge r\ p.\ R\ r\ p \Longrightarrow \exists\ s\ t.\ r = s\ @-\ t \wedge s \neq [] \wedge path\ s\ p \wedge R\ t\ (target\ s\ p)$
shows $run\ r\ p$
 $\langle proof \rangle$

lemma $run-flat-coinduct[case-names\ run,\ consumes\ 1]:$
assumes $R\ rs\ p$
assumes $\bigwedge r\ rs\ p.\ R\ (r\ \#\#\ rs)\ p \Longrightarrow r \neq [] \wedge path\ r\ p \wedge R\ rs\ (target\ r\ p)$
shows $run\ (flat\ rs)\ p$
 $\langle proof \rangle$

inductive-set $reachable :: 'state \Rightarrow 'state\ set$ **for** p **where**
 $reflexive[intro!]:\ p \in reachable\ p \mid$
 $execute[intro!]:\ q \in reachable\ p \Longrightarrow enabled\ a\ q \Longrightarrow execute\ a\ q \in reachable\ p$

inductive-cases $reachable-elim[elim!]:\ q \in reachable\ p$

lemma $reachable-execute'[intro]:$
assumes $enabled\ a\ p\ q \in reachable\ (execute\ a\ p)$
shows $q \in reachable\ p$
 $\langle proof \rangle$

lemma $reachable-elim'[elim]:$
assumes $q \in reachable\ p$
obtains $q = p \mid a$ **where** $enabled\ a\ p\ q \in reachable\ (execute\ a\ p)$
 $\langle proof \rangle$

lemma *reachable-target*[*intro*]:
assumes $q \in \text{reachable } p \text{ path } r \ q$
shows $\text{target } r \ q \in \text{reachable } p$
 $\langle \text{proof} \rangle$

lemma *reachable-target-elim*[*elim*]:
assumes $q \in \text{reachable } p$
obtains r
where $\text{path } r \ p \ q = \text{target } r \ p$
 $\langle \text{proof} \rangle$

lemma *reachable-alt-def*: $\text{reachable } p = \{ \text{target } r \ p \mid r. \text{path } r \ p \}$ $\langle \text{proof} \rangle$

lemma *reachable-trans*[*trans*]: $q \in \text{reachable } p \implies s \in \text{reachable } q \implies s \in \text{reachable } p$ $\langle \text{proof} \rangle$

lemma *reachable-successors*[*intro!*]: $\text{successors } p \subseteq \text{reachable } p$ $\langle \text{proof} \rangle$

lemma *reachable-step*: $\text{reachable } p = \text{insert } p \ (\bigcup (\text{reachable } \text{' successors } p))$
 $\langle \text{proof} \rangle$

end

7.3 Transition Systems with Initial States

locale *transition-system-initial* =
transition-system *execute* *enabled*
for $\text{execute} :: \text{'transition} \Rightarrow \text{'state} \Rightarrow \text{'state}$
and $\text{enabled} :: \text{'transition} \Rightarrow \text{'state} \Rightarrow \text{bool}$
 $+$
fixes $\text{initial} :: \text{'state} \Rightarrow \text{bool}$
begin

inductive-set $\text{nodes} :: \text{'state} \text{ set}$ **where**
initial[*intro*]: $\text{initial } p \implies p \in \text{nodes} \mid$
execute[*intro!*]: $p \in \text{nodes} \implies \text{enabled } a \ p \implies \text{execute } a \ p \in \text{nodes}$

lemma *nodes-target*[*intro*]:
assumes $p \in \text{nodes} \text{ path } r \ p$
shows $\text{target } r \ p \in \text{nodes}$
 $\langle \text{proof} \rangle$

lemma *nodes-target-elim*[*elim*]:
assumes $q \in \text{nodes}$
obtains $r \ p$
where $\text{initial } p \text{ path } r \ p \ q = \text{target } r \ p$
 $\langle \text{proof} \rangle$

lemma *nodes-alt-def*: $\text{nodes} = \bigcup (\text{reachable } \text{' Collect } \text{initial})$ $\langle \text{proof} \rangle$

lemma *nodes-trans*[*trans*]: $p \in \text{nodes} \implies q \in \text{reachable } p \implies q \in \text{nodes}$ $\langle \text{proof} \rangle$

end

end

8 Additional Theorems for Transition Systems

theory *Transition-System-Extra*

imports

../Basic/Sequence-LTL

Transition-System

begin

context *transition-system*

begin

definition *enablers* $p \equiv \{a. \text{enabled } a \ p\}$

definition *paths* $p \equiv \{r. \text{path } r \ p\}$

definition *runs* $p \equiv \{r. \text{run } r \ p\}$

lemma *stake-run*:

assumes $\bigwedge k. \text{path } (\text{stake } k \ r) \ p$

shows $\text{run } r \ p$

<proof>

lemma *snth-run*:

assumes $\bigwedge k. \text{enabled } (r \ !! \ k) \ (\text{target } (\text{stake } k \ r) \ p)$

shows $\text{run } r \ p$

<proof>

lemma *run-stake*:

assumes $\text{run } r \ p$

shows $\text{path } (\text{stake } k \ r) \ p$

<proof>

lemma *run-sdrop*:

assumes $\text{run } r \ p$

shows $\text{run } (\text{sdrop } k \ r) \ (\text{target } (\text{stake } k \ r) \ p)$

<proof>

lemma *run-snth*:

assumes $\text{run } r \ p$

shows $\text{enabled } (r \ !! \ k) \ (\text{target } (\text{stake } k \ r) \ p)$

<proof>

lemma *run-alt-def-snth*: $\text{run } r \ p \longleftrightarrow (\forall k. \text{enabled } (r \ !! \ k) \ (\text{target } (\text{stake } k \ r) \ p))$

<proof>

lemma *reachable-states*:

assumes $q \in \text{reachable } p \ \text{path } r \ q$

shows $\text{set } (\text{states } r \ q) \subseteq \text{reachable } p$

```

    <proof>
lemma reachable-trace:
  assumes  $q \in \text{reachable } p \text{ run } r \ q$ 
  shows  $\text{sset } (\text{trace } r \ q) \subseteq \text{reachable } p$ 
  <proof>

end

context transition-system-initial
begin

  lemma nodes-states:
    assumes  $p \in \text{nodes } \text{path } r \ p$ 
    shows  $\text{set } (\text{states } r \ p) \subseteq \text{nodes}$ 
    <proof>
  lemma nodes-trace:
    assumes  $p \in \text{nodes } \text{run } r \ p$ 
    shows  $\text{sset } (\text{trace } r \ p) \subseteq \text{nodes}$ 
    <proof>

end

end

```

9 Constructing Paths and Runs in Transition Systems

```

theory Transition-System-Construction
imports
  ../Basic/Sequence-LTL
  Transition-System
begin

  context transition-system
  begin

    lemma invariant-run:
      assumes  $P \ p \ \wedge \ p. \ P \ p \implies \exists \ a. \ \text{enabled } a \ p \ \wedge \ P \ (\text{execute } a \ p) \ \wedge \ Q \ p \ a$ 
      obtains  $r$ 
      where  $\text{run } r \ p \ \text{pred-stream } P \ (p \ \#\# \ \text{trace } r \ p) \ \text{stream-all2 } Q \ (p \ \#\# \ \text{trace } r$ 
     $p) \ r$ 
      <proof>
    lemma recurring-condition:
      assumes  $P \ p \ \wedge \ p. \ P \ p \implies \exists \ r. \ r \neq [] \ \wedge \ \text{path } r \ p \ \wedge \ P \ (\text{target } r \ p)$ 
      obtains  $r$ 
      where  $\text{run } r \ p \ \text{infs } P \ (p \ \#\# \ \text{trace } r \ p)$ 
      <proof>
  end

```

lemma *invariant-run-index*:
assumes $P\ n\ p \wedge n\ p. P\ n\ p \implies \exists a. \text{enabled } a\ p \wedge P\ (\text{Suc } n)$ (*execute a p*)
 $\wedge Q\ n\ p\ a$
obtains r
where
 $\text{run } r\ p$
 $\wedge i. P\ (n + i)\ (\text{target } (\text{stake } i\ r)\ p)$
 $\wedge i. Q\ (n + i)\ (\text{target } (\text{stake } i\ r)\ p)\ (r\ !!\ i)$
 $\langle \text{proof} \rangle$

lemma *koenig*:
assumes *infinite* (*reachable p*)
assumes $\bigwedge q. q \in \text{reachable } p \implies \text{finite } (\text{successors } q)$
obtains r
where $\text{run } r\ p$
 $\langle \text{proof} \rangle$

end

end

10 Deterministic Automata

theory *Deterministic*

imports

$\dots/ \text{Transition-Systems}/ \text{Transition-System}$
 $\dots/ \text{Transition-Systems}/ \text{Transition-System-Extra}$
 $\dots/ \text{Transition-Systems}/ \text{Transition-System-Construction}$
 $\dots/ \text{Basic}/ \text{Degeneralization}$

begin

locale *automaton* =
fixes $\text{automaton} :: 'label\ set \Rightarrow 'state \Rightarrow ('label \Rightarrow 'state \Rightarrow 'state) \Rightarrow 'condition$
 $\Rightarrow 'automaton$
fixes $\text{alphabet } \text{initial } \text{transition } \text{condition}$
assumes $\text{automaton}[\text{simp}]: \text{automaton } (\text{alphabet } A)\ (\text{initial } A)\ (\text{transition } A)$
 $(\text{condition } A) = A$
assumes $\text{alphabet}[\text{simp}]: \text{alphabet } (\text{automaton } a\ i\ t\ c) = a$
assumes $\text{initial}[\text{simp}]: \text{initial } (\text{automaton } a\ i\ t\ c) = i$
assumes $\text{transition}[\text{simp}]: \text{transition } (\text{automaton } a\ i\ t\ c) = t$
assumes $\text{condition}[\text{simp}]: \text{condition } (\text{automaton } a\ i\ t\ c) = c$
begin

sublocale *transition-system-initial*

$\text{transition } A\ \lambda\ a\ p. a \in \text{alphabet } A\ \lambda\ p. p = \text{initial } A$

for A

defines $\text{path}' = \text{path}$ **and** $\text{run}' = \text{run}$ **and** $\text{reachable}' = \text{reachable}$ **and** nodes'
 $= \text{nodes}$

$\langle \text{proof} \rangle$

lemma *path-alt-def*: $\text{path } A \ w \ p \longleftrightarrow w \in \text{lists } (\text{alphabet } A)$
 $\langle \text{proof} \rangle$

lemma *run-alt-def*: $\text{run } A \ w \ p \longleftrightarrow w \in \text{streams } (\text{alphabet } A)$
 $\langle \text{proof} \rangle$

end

locale *automaton-path* =
automaton automaton alphabet initial transition condition
for *automaton* :: 'label set \Rightarrow 'state \Rightarrow ('label \Rightarrow 'state \Rightarrow 'state) \Rightarrow 'condition
 \Rightarrow 'automaton
and *alphabet initial transition condition*
+
fixes *test* :: 'condition \Rightarrow 'label list \Rightarrow 'state list \Rightarrow 'state \Rightarrow bool
begin

definition *language* :: 'automaton \Rightarrow 'label list set **where**
language $A \equiv \{w. \text{path } A \ w \ (\text{initial } A) \wedge \text{test } (\text{condition } A) \ w \ (\text{states } A \ w \ (\text{initial } A)) \ (\text{initial } A)\}$

lemma *language[intro]*:
assumes *path* $A \ w \ (\text{initial } A) \ \text{test } (\text{condition } A) \ w \ (\text{states } A \ w \ (\text{initial } A))$
 $(\text{initial } A)$
shows $w \in \text{language } A$
 $\langle \text{proof} \rangle$

lemma *language-elim[elim]*:
assumes $w \in \text{language } A$
obtains *path* $A \ w \ (\text{initial } A) \ \text{test } (\text{condition } A) \ w \ (\text{states } A \ w \ (\text{initial } A))$
 $(\text{initial } A)$
 $\langle \text{proof} \rangle$

lemma *language-alphabet*: $\text{language } A \subseteq \text{lists } (\text{alphabet } A) \ \langle \text{proof} \rangle$

end

locale *automaton-run* =
automaton automaton alphabet initial transition condition
for *automaton* :: 'label set \Rightarrow 'state \Rightarrow ('label \Rightarrow 'state \Rightarrow 'state) \Rightarrow 'condition
 \Rightarrow 'automaton
and *alphabet initial transition condition*
+
fixes *test* :: 'condition \Rightarrow 'label stream \Rightarrow 'state stream \Rightarrow 'state \Rightarrow bool
begin

definition *language* :: 'automaton \Rightarrow 'label stream set **where**
language $A \equiv \{w. \text{run } A \ w \ (\text{initial } A) \wedge \text{test } (\text{condition } A) \ w \ (\text{trace } A \ w \ (\text{initial } A)) \ (\text{initial } A)\}$

lemma *language*[*intro*]:
 assumes *run A w (initial A) test (condition A) w (trace A w (initial A))*
 (*initial A*)
 shows *w ∈ language A*
 ⟨*proof*⟩

lemma *language-elim*[*elim*]:
 assumes *w ∈ language A*
 obtains *run A w (initial A) test (condition A) w (trace A w (initial A))*
 (*initial A*)
 ⟨*proof*⟩

lemma *language-alphabet*: *language A ⊆ streams (alphabet A)* ⟨*proof*⟩

end

locale *automaton-degeneralization* =
 a: *automaton automaton₁ alphabet₁ initial₁ transition₁ condition₁* +
 b: *automaton automaton₂ alphabet₂ initial₂ transition₂ condition₂*
 for *automaton₁ :: 'label set ⇒ 'state ⇒ ('label ⇒ 'state ⇒ 'state) ⇒ 'item pred*
gen ⇒ 'automaton₁
 and *alphabet₁ initial₁ transition₁ condition₁*
 and *automaton₂ :: 'label set ⇒ 'state degen ⇒ ('label ⇒ 'state degen ⇒ 'state*
degen) ⇒ 'item-degen pred ⇒ 'automaton₂
 and *alphabet₂ initial₂ transition₂ condition₂*
 +
 fixes *item :: 'state × 'label × 'state ⇒ 'item*
 fixes *translate :: 'item-degen ⇒ 'item degen*
begin

definition *degeneralize* :: *'automaton₁ ⇒ 'automaton₂ where*
degeneralize A ≡ automaton₂
 (*alphabet₁ A*)
 (*initial₁ A, 0*)
 ($\lambda a (p, k). (transition_1 A a p, count (condition_1 A) (item (p, a, transition_1 A a p)) k)$)
 (*degen (condition₁ A) ◦ translate*)

lemma *degeneralize-simps*[*simp*]:
alphabet₂ (degeneralize A) = alphabet₁ A
initial₂ (degeneralize A) = (initial₁ A, 0)
transition₂ (degeneralize A) a (p, k) =
 (*transition₁ A a p, count (condition₁ A) (item (p, a, transition₁ A a p)) k*)
condition₂ (degeneralize A) = degen (condition₁ A) ◦ translate
 ⟨*proof*⟩

lemma *degeneralize-target*[*simp*]: *b.target (degeneralize A) w (p, k) =*
 (*a.target A w p, fold (count (condition₁ A) ◦ item) (p # a.states A w p) || w*
 || *a.states A w p*) *k*)

$\langle \text{proof} \rangle$
lemma *degeneralize-states[simp]*: $b.\text{states} (\text{degeneralize } A) w (p, k) =$
 $a.\text{states } A w p \parallel \text{scan} (\text{count} (\text{condition}_1 A) \circ \text{item}) (p \# a.\text{states } A w p \parallel w$
 $\parallel a.\text{states } A w p) k$
 $\langle \text{proof} \rangle$
lemma *degeneralize-trace[simp]*: $b.\text{trace} (\text{degeneralize } A) w (p, k) =$
 $a.\text{trace } A w p \parallel \text{sscan} (\text{count} (\text{condition}_1 A) \circ \text{item}) (p \#\# a.\text{trace } A w p \parallel \parallel$
 $w \parallel a.\text{trace } A w p) k$
 $\langle \text{proof} \rangle$

lemma *degeneralize-path[iff]*: $b.\text{path} (\text{degeneralize } A) w (p, k) \longleftrightarrow a.\text{path } A w$
 p
 $\langle \text{proof} \rangle$
lemma *degeneralize-run[iff]*: $b.\text{run} (\text{degeneralize } A) w (p, k) \longleftrightarrow a.\text{run } A w p$
 $\langle \text{proof} \rangle$

lemma *degeneralize-reachable-fst[simp]*: $\text{fst} \text{ ` } b.\text{reachable} (\text{degeneralize } A) (p, k)$
 $= a.\text{reachable } A p$
 $\langle \text{proof} \rangle$
lemma *degeneralize-reachable-snd-empty[simp]*:
assumes $\text{condition}_1 A = []$
shows $\text{snd} \text{ ` } b.\text{reachable} (\text{degeneralize } A) (p, k) = \{k\}$
 $\langle \text{proof} \rangle$
lemma *degeneralize-reachable-empty[simp]*:
assumes $\text{condition}_1 A = []$
shows $b.\text{reachable} (\text{degeneralize } A) (p, k) = a.\text{reachable } A p \times \{k\}$
 $\langle \text{proof} \rangle$
lemma *degeneralize-reachable-snd*:
 $\text{snd} \text{ ` } b.\text{reachable} (\text{degeneralize } A) (p, k) \subseteq \text{insert } k \{0 \dots < \text{length} (\text{condition}_1$
 $A)\}$
 $\langle \text{proof} \rangle$
lemma *degeneralize-reachable*:
 $b.\text{reachable} (\text{degeneralize } A) (p, k) \subseteq a.\text{reachable } A p \times \text{insert } k \{0 \dots < \text{length}$
 $(\text{condition}_1 A)\}$
 $\langle \text{proof} \rangle$

lemma *degeneralize-nodes-fst[simp]*: $\text{fst} \text{ ` } b.\text{nodes} (\text{degeneralize } A) = a.\text{nodes } A$
 $\langle \text{proof} \rangle$
lemma *degeneralize-nodes-snd-empty*:
assumes $\text{condition}_1 A = []$
shows $\text{snd} \text{ ` } b.\text{nodes} (\text{degeneralize } A) = \{0\}$
 $\langle \text{proof} \rangle$
lemma *degeneralize-nodes-empty*:
assumes $\text{condition}_1 A = []$
shows $b.\text{nodes} (\text{degeneralize } A) = a.\text{nodes } A \times \{0\}$
 $\langle \text{proof} \rangle$
lemma *degeneralize-nodes-snd*:
 $\text{snd} \text{ ` } b.\text{nodes} (\text{degeneralize } A) \subseteq \text{insert } 0 \{0 \dots < \text{length} (\text{condition}_1 A)\}$
 $\langle \text{proof} \rangle$

lemma *degeneralize-nodes*:
 $b.\text{nodes } (\text{degeneralize } A) \subseteq a.\text{nodes } A \times \text{insert } 0 \{0 \dots < \text{length } (\text{condition}_1 A)\}$
 ⟨proof⟩

lemma *degeneralize-nodes-finite*[iff]: $\text{finite } (b.\text{nodes } (\text{degeneralize } A)) \longleftrightarrow \text{finite } (a.\text{nodes } A)$
 ⟨proof⟩

lemma *degeneralize-nodes-card*: $\text{card } (b.\text{nodes } (\text{degeneralize } A)) \leq \max 1 (\text{length } (\text{condition}_1 A)) * \text{card } (a.\text{nodes } A)$
 ⟨proof⟩

end

locale *automaton-degeneralization-run* =

automaton-degeneralization

*automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁

*automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂

item translate +

a: *automaton-run* *automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁ *test*₁ +

b: *automaton-run* *automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂ *test*₂

for *automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁ *test*₁

and *automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂ *test*₂

and *item translate*

+

assumes *test*[iff]: *test*₂ (*degen cs* ◦ *translate*) *w*

(*r* ||| *sscan* (*count cs* ◦ *item*) (*p* ## *r* ||| *w* ||| *r*) *k*) (*p*, *k*) \longleftrightarrow *test*₁ *cs w r p*

begin

lemma *degeneralize-language*[simp]: $b.\text{language } (\text{degeneralize } A) = a.\text{language } A$ ⟨proof⟩

end

locale *automaton-product* =

a: *automaton* *automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁ +

b: *automaton* *automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂ +

c: *automaton* *automaton*₃ *alphabet*₃ *initial*₃ *transition*₃ *condition*₃

for *automaton*₁ :: 'label set \Rightarrow 'state₁ \Rightarrow ('label \Rightarrow 'state₁ \Rightarrow 'state₁) \Rightarrow 'condition₁ \Rightarrow 'automaton₁

and *alphabet*₁ *initial*₁ *transition*₁ *condition*₁

and *automaton*₂ :: 'label set \Rightarrow 'state₂ \Rightarrow ('label \Rightarrow 'state₂ \Rightarrow 'state₂) \Rightarrow 'condition₂ \Rightarrow 'automaton₂

and *alphabet*₂ *initial*₂ *transition*₂ *condition*₂

and *automaton*₃ :: 'label set \Rightarrow 'state₁ \times 'state₂ \Rightarrow ('label \Rightarrow 'state₁ \times 'state₂) \Rightarrow 'state₁ \times 'state₂ \Rightarrow 'condition₃ \Rightarrow 'automaton₃

and *alphabet*₃ *initial*₃ *transition*₃ *condition*₃

+

fixes *condition* :: 'condition₁ \Rightarrow 'condition₂ \Rightarrow 'condition₃

begin

definition *product* :: 'automaton₁ ⇒ 'automaton₂ ⇒ 'automaton₃ **where**
 product *A B* ≡ *automaton₃*
 (*alphabet₁* *A* ∩ *alphabet₂* *B*)
 (*initial₁* *A*, *initial₂* *B*)
 (λ *a* (*p*, *q*). (*transition₁* *A* *a* *p*, *transition₂* *B* *a* *q*))
 (*condition* (*condition₁* *A*) (*condition₂* *B*))

lemma *product-simps[simp]*:
 alphabet₃ (*product* *A B*) = *alphabet₁* *A* ∩ *alphabet₂* *B*
 initial₃ (*product* *A B*) = (*initial₁* *A*, *initial₂* *B*)
 transition₃ (*product* *A B*) *a* (*p*, *q*) = (*transition₁* *A* *a* *p*, *transition₂* *B* *a* *q*)
 condition₃ (*product* *A B*) = *condition* (*condition₁* *A*) (*condition₂* *B*)
 ⟨*proof*⟩

lemma *product-target[simp]*: *c.target* (*product* *A B*) *w* (*p*, *q*) = (*a.target* *A* *w* *p*, *b.target* *B* *w* *q*)
 ⟨*proof*⟩

lemma *product-states[simp]*: *c.states* (*product* *A B*) *w* (*p*, *q*) = *a.states* *A* *w* *p* || *b.states* *B* *w* *q*
 ⟨*proof*⟩

lemma *product-trace[simp]*: *c.trace* (*product* *A B*) *w* (*p*, *q*) = *a.trace* *A* *w* *p* ||| *b.trace* *B* *w* *q*
 ⟨*proof*⟩

lemma *product-path[iff]*: *c.path* (*product* *A B*) *w* (*p*, *q*) ⟷ *a.path* *A* *w* *p* ∧ *b.path* *B* *w* *q*
 ⟨*proof*⟩

lemma *product-run[iff]*: *c.run* (*product* *A B*) *w* (*p*, *q*) ⟷ *a.run* *A* *w* *p* ∧ *b.run* *B* *w* *q*
 ⟨*proof*⟩

lemma *product-reachable[simp]*: *c.reachable* (*product* *A B*) (*p*, *q*) ⊆ *a.reachable* *A* *p* × *b.reachable* *B* *q*
 ⟨*proof*⟩

lemma *product-nodes[simp]*: *c.nodes* (*product* *A B*) ⊆ *a.nodes* *A* × *b.nodes* *B*
 ⟨*proof*⟩

lemma *product-reachable-fst[simp]*:
 assumes *alphabet₁* *A* ⊆ *alphabet₂* *B*
 shows *fst* ' *c.reachable* (*product* *A B*) (*p*, *q*) = *a.reachable* *A* *p*
 ⟨*proof*⟩

lemma *product-reachable-snd[simp]*:
 assumes *alphabet₁* *A* ⊇ *alphabet₂* *B*
 shows *snd* ' *c.reachable* (*product* *A B*) (*p*, *q*) = *b.reachable* *B* *q*
 ⟨*proof*⟩

lemma *product-nodes-fst[simp]*:
 assumes *alphabet₁* *A* ⊆ *alphabet₂* *B*
 shows *fst* ' *c.nodes* (*product* *A B*) = *a.nodes* *A*


```

    <proof>
lemma product-nodes-snd[simp]:
  assumes alphabet1 A  $\supseteq$  alphabet2 B
  shows snd ' c.nodes (product A B) = b.nodes B
  <proof>

lemma product-nodes-finite[intro]:
  assumes finite (a.nodes A) finite (b.nodes B)
  shows finite (c.nodes (product A B))
  <proof>
lemma product-nodes-finite-strong[iff]:
  assumes alphabet1 A = alphabet2 B
  shows finite (c.nodes (product A B))  $\longleftrightarrow$  finite (a.nodes A)  $\wedge$  finite (b.nodes
B)
  <proof>
lemma product-nodes-card[intro]:
  assumes finite (a.nodes A) finite (b.nodes B)
  shows card (c.nodes (product A B))  $\leq$  card (a.nodes A) * card (b.nodes B)
  <proof>
lemma product-nodes-card-strong[intro]:
  assumes alphabet1 A = alphabet2 B
  shows card (c.nodes (product A B))  $\leq$  card (a.nodes A) * card (b.nodes B)
  <proof>

end

locale automaton-intersection-path =
  automaton-product
    automaton1 alphabet1 initial1 transition1 condition1
    automaton2 alphabet2 initial2 transition2 condition2
    automaton3 alphabet3 initial3 transition3 condition3
    condition +
  a: automaton-path automaton1 alphabet1 initial1 transition1 condition1 test1 +
  b: automaton-path automaton2 alphabet2 initial2 transition2 condition2 test2 +
  c: automaton-path automaton3 alphabet3 initial3 transition3 condition3 test3
  for automaton1 alphabet1 initial1 transition1 condition1 test1
  and automaton2 alphabet2 initial2 transition2 condition2 test2
  and automaton3 alphabet3 initial3 transition3 condition3 test3
  and condition
  +
  assumes test[iff]: length r = length s  $\implies$ 
    test3 (condition c1 c2) w (r || s) (p, q)  $\longleftrightarrow$  test1 c1 w r p  $\wedge$  test2 c2 w s q
begin

  lemma product-language[simp]: c.language (product A B) = a.language A  $\cap$ 
b.language B <proof>

end

```

```

locale automaton-union-path =
  automaton-product
    automaton1 alphabet1 initial1 transition1 condition1
    automaton2 alphabet2 initial2 transition2 condition2
    automaton3 alphabet3 initial3 transition3 condition3
    condition +
  a: automaton-path automaton1 alphabet1 initial1 transition1 condition1 test1 +
  b: automaton-path automaton2 alphabet2 initial2 transition2 condition2 test2 +
  c: automaton-path automaton3 alphabet3 initial3 transition3 condition3 test3
  for automaton1 alphabet1 initial1 transition1 condition1 test1
  and automaton2 alphabet2 initial2 transition2 condition2 test2
  and automaton3 alphabet3 initial3 transition3 condition3 test3
  and condition
  +
  assumes test[iff]: length r = length s  $\implies$ 
    test3 (condition c1 c2) w (r || s) (p, q)  $\longleftrightarrow$  test1 c1 w r p  $\vee$  test2 c2 w s q
begin

  lemma product-language[simp]:
    assumes alphabet1 A = alphabet2 B
    shows c.language (product A B) = a.language A  $\cup$  b.language B
    <proof>

end

locale automaton-intersection-run =
  automaton-product
    automaton1 alphabet1 initial1 transition1 condition1
    automaton2 alphabet2 initial2 transition2 condition2
    automaton3 alphabet3 initial3 transition3 condition3
    condition +
  a: automaton-run automaton1 alphabet1 initial1 transition1 condition1 test1 +
  b: automaton-run automaton2 alphabet2 initial2 transition2 condition2 test2 +
  c: automaton-run automaton3 alphabet3 initial3 transition3 condition3 test3
  for automaton1 alphabet1 initial1 transition1 condition1 test1
  and automaton2 alphabet2 initial2 transition2 condition2 test2
  and automaton3 alphabet3 initial3 transition3 condition3 test3
  and condition
  +
  assumes test[iff]: test3 (condition c1 c2) w (r ||| s) (p, q)  $\longleftrightarrow$  test1 c1 w r p
   $\wedge$  test2 c2 w s q
begin

  lemma product-language[simp]: c.language (product A B) = a.language A  $\cap$ 
  b.language B <proof>

end

locale automaton-union-run =

```

```

automaton-product
  automaton1 alphabet1 initial1 transition1 condition1
  automaton2 alphabet2 initial2 transition2 condition2
  automaton3 alphabet3 initial3 transition3 condition3
  condition +
  a: automaton-run automaton1 alphabet1 initial1 transition1 condition1 test1 +
  b: automaton-run automaton2 alphabet2 initial2 transition2 condition2 test2 +
  c: automaton-run automaton3 alphabet3 initial3 transition3 condition3 test3
  for automaton1 alphabet1 initial1 transition1 condition1 test1
  and automaton2 alphabet2 initial2 transition2 condition2 test2
  and automaton3 alphabet3 initial3 transition3 condition3 test3
  and condition
  +
  assumes test[iff]: test3 (condition c1 c2) w (r ||| s) (p, q)  $\longleftrightarrow$  test1 c1 w r p
 $\vee$  test2 c2 w s q
  begin

    lemma product-language[simp]:
      assumes alphabet1 A = alphabet2 B
      shows c.language (product A B) = a.language A  $\cup$  b.language B
      <proof>

    end

  locale automaton-product-list =
    a: automaton automaton1 alphabet1 initial1 transition1 condition1 +
    b: automaton automaton2 alphabet2 initial2 transition2 condition2
    for automaton1 :: 'label set  $\Rightarrow$  'state  $\Rightarrow$  ('label  $\Rightarrow$  'state  $\Rightarrow$  'state)  $\Rightarrow$  'condition1
     $\Rightarrow$  'automaton1
    and alphabet1 initial1 transition1 condition1
    and automaton2 :: 'label set  $\Rightarrow$  'state list  $\Rightarrow$  ('label  $\Rightarrow$  'state list  $\Rightarrow$  'state list)
     $\Rightarrow$  'condition2  $\Rightarrow$  'automaton2
    and alphabet2 initial2 transition2 condition2
    +
    fixes condition :: 'condition1 list  $\Rightarrow$  'condition2
  begin

    definition product :: 'automaton1 list  $\Rightarrow$  'automaton2 where
      product AA  $\equiv$  automaton2
        ( $\bigcap$  (alphabet1 ' set AA))
        (map initial1 AA)
        ( $\lambda$  a ps. map2 ( $\lambda$  A p. transition1 A a p) AA ps)
        (condition (map condition1 AA))

    lemma product-simps[simp]:
      alphabet2 (product AA) =  $\bigcap$  (alphabet1 ' set AA)
      initial2 (product AA) = map initial1 AA
      transition2 (product AA) a ps = map2 ( $\lambda$  A p. transition1 A a p) AA ps

```

$condition_2$ (product AA) = condition (map $condition_1$ AA)
 ⟨proof⟩

lemma *product-trace-smap*:

assumes $length\ ps = length\ AA\ k < length\ AA$

shows $smap\ (\lambda\ ps.\ ps\ !\ k)\ (b.trace\ (product\ AA)\ w\ ps) = a.trace\ (AA\ !\ k)\ w$
 ($ps\ !\ k$)
 ⟨proof⟩

lemma *product-nodes*: $b.nodes\ (product\ AA) \subseteq listset\ (map\ a.nodes\ AA)$

⟨proof⟩

lemma *product-nodes-finite*[intro]:

assumes $list-all\ (finite\ \circ\ a.nodes)\ AA$

shows $finite\ (b.nodes\ (product\ AA))$

⟨proof⟩

lemma *product-nodes-card*:

assumes $list-all\ (finite\ \circ\ a.nodes)\ AA$

shows $card\ (b.nodes\ (product\ AA)) \leq prod-list\ (map\ (card\ \circ\ a.nodes)\ AA)$

⟨proof⟩

end

locale *automaton-intersection-list-run* =

automaton-product-list

$automaton_1\ alphabet_1\ initial_1\ transition_1\ condition_1$

$automaton_2\ alphabet_2\ initial_2\ transition_2\ condition_2$

$condition +$

a : *automaton-run* $automaton_1\ alphabet_1\ initial_1\ transition_1\ condition_1\ test_1 +$

b : *automaton-run* $automaton_2\ alphabet_2\ initial_2\ transition_2\ condition_2\ test_2$

for $automaton_1\ alphabet_1\ initial_1\ transition_1\ condition_1\ test_1$

and $automaton_2\ alphabet_2\ initial_2\ transition_2\ condition_2\ test_2$

and $condition$

$+$

assumes $test[iff]$: $test_2\ (condition\ cs)\ w\ rs\ ps \longleftrightarrow$

$(\forall\ k < length\ cs.\ test_1\ (cs\ !\ k)\ w\ (smap\ (\lambda\ ps.\ ps\ !\ k)\ rs)\ (ps\ !\ k))$

begin

lemma *product-language*[simp]: $b.language\ (product\ AA) = \bigcap\ (a.language\ 'set\ AA)$

⟨proof⟩

end

locale *automaton-union-list-run* =

automaton-product-list

$automaton_1\ alphabet_1\ initial_1\ transition_1\ condition_1$

$automaton_2\ alphabet_2\ initial_2\ transition_2\ condition_2$

```

    condition +
a: automaton-run automaton1 alphabet1 initial1 transition1 condition1 test1 +
b: automaton-run automaton2 alphabet2 initial2 transition2 condition2 test2
for automaton1 alphabet1 initial1 transition1 condition1 test1
and automaton2 alphabet2 initial2 transition2 condition2 test2
and condition
+
assumes test[iff]: test2 (condition cs) w rs ps  $\longleftrightarrow$ 
  ( $\exists k < \text{length } cs. \text{test}_1 (cs ! k) w (\text{smap } (\lambda ps. ps ! k) rs) (ps ! k)$ )
begin

  lemma product-language[simp]:
    assumes  $\bigcap (alphabet_1 \text{ 'set } AA) = \bigcup (alphabet_1 \text{ 'set } AA)$ 
    shows b.language (product AA) =  $\bigcup (a.language \text{ 'set } AA)$ 
    <proof>

end

locale automaton-complement =
  a: automaton automaton1 alphabet1 initial1 transition1 condition1 +
  b: automaton automaton2 alphabet2 initial2 transition2 condition2
for automaton1 :: 'label set  $\Rightarrow$  'state  $\Rightarrow$  ('label  $\Rightarrow$  'state  $\Rightarrow$  'state)  $\Rightarrow$  'condition1
 $\Rightarrow$  'automaton1
and alphabet1 initial1 transition1 condition1
and automaton2 :: 'label set  $\Rightarrow$  'state  $\Rightarrow$  ('label  $\Rightarrow$  'state  $\Rightarrow$  'state)  $\Rightarrow$  'condition2
 $\Rightarrow$  'automaton2
and alphabet2 initial2 transition2 condition2
+
fixes condition :: 'condition1  $\Rightarrow$  'condition2
begin

  definition complement :: 'automaton1  $\Rightarrow$  'automaton2 where
    complement A  $\equiv$  automaton2 (alphabet1 A) (initial1 A) (transition1 A)
    (condition (condition1 A))

  lemma combine-simps[simp]:
    alphabet2 (complement A) = alphabet1 A
    initial2 (complement A) = initial1 A
    transition2 (complement A) = transition1 A
    condition2 (complement A) = condition (condition1 A)
    <proof>

end

locale automaton-complement-path =
  automaton-complement
  automaton1 alphabet1 initial1 transition1 condition1
  automaton2 alphabet2 initial2 transition2 condition2
  condition +

```

```

a: automaton-path automaton1 alphabet1 initial1 transition1 condition1 test1 +
b: automaton-path automaton2 alphabet2 initial2 transition2 condition2 test2
for automaton1 alphabet1 initial1 transition1 condition1 test1
and automaton2 alphabet2 initial2 transition2 condition2 test2
and condition
+
assumes test[iff]: test2 (condition c) w r p  $\longleftrightarrow$   $\neg$  test1 c w r p
begin

lemma complement-language[simp]: b.language (complement A) = lists (alphabet1
A) - a.language A
  <proof>

end

locale automaton-complement-run =
  automaton-complement
  automaton1 alphabet1 initial1 transition1 condition1
  automaton2 alphabet2 initial2 transition2 condition2
  condition +
  a: automaton-run automaton1 alphabet1 initial1 transition1 condition1 test1 +
  b: automaton-run automaton2 alphabet2 initial2 transition2 condition2 test2
  for automaton1 alphabet1 initial1 transition1 condition1 test1
  and automaton2 alphabet2 initial2 transition2 condition2 test2
  and condition
  +
  assumes test[iff]: test2 (condition c) w r p  $\longleftrightarrow$   $\neg$  test1 c w r p
begin

  lemma complement-language[simp]: b.language (complement A) = streams
  (alphabet1 A) - a.language A
    <proof>

end

end

```

11 Deterministic Finite Automata

```

theory DFA
imports ../Deterministic
begin

datatype ('label, 'state) dfa = dfa
  (alphabet: 'label set)
  (initial: 'state)
  (transition: 'label  $\Rightarrow$  'state  $\Rightarrow$  'state)
  (accepting: 'state pred)

```

global-interpretation *dfa: automaton dfa alphabet initial transition accepting*
defines *path = dfa.path and run = dfa.run and reachable = dfa.reachable and*
nodes = dfa.nodes
 <proof>

global-interpretation *dfa: automaton-path dfa alphabet initial transition accept-*
ing $\lambda P w r p. P$ (last (p # r))
defines *language = dfa.language*
 <proof>

abbreviation *target where target \equiv dfa.target*
abbreviation *states where states \equiv dfa.states*
abbreviation *trace where trace \equiv dfa.trace*
abbreviation *successors where successors \equiv dfa.successors TYPE('label)*

global-interpretation *intersection: automaton-intersection-path*
dfa alphabet initial transition accepting $\lambda P w r p. P$ (last (p # r))
dfa alphabet initial transition accepting $\lambda P w r p. P$ (last (p # r))
dfa alphabet initial transition accepting $\lambda P w r p. P$ (last (p # r))
 $\lambda c_1 c_2 (p, q). c_1 p \wedge c_2 q$
defines *intersect = intersection.product*
 <proof>

global-interpretation *union: automaton-union-path*
dfa alphabet initial transition accepting $\lambda P w r p. P$ (last (p # r))
dfa alphabet initial transition accepting $\lambda P w r p. P$ (last (p # r))
dfa alphabet initial transition accepting $\lambda P w r p. P$ (last (p # r))
 $\lambda c_1 c_2 (p, q). c_1 p \vee c_2 q$
defines *union = union.product*
 <proof>

global-interpretation *complement: automaton-complement-path*
dfa alphabet initial transition accepting $\lambda P w r p. P$ (last (p # r))
dfa alphabet initial transition accepting $\lambda P w r p. P$ (last (p # r))
 $\lambda c p. \neg c p$
defines *complement = complement.complement*
 <proof>

end

12 Nondeterministic Automata

theory *Nondeterministic*

imports

../Transition-Systems/Transition-System
 ../Transition-Systems/Transition-System-Extra
 ../Transition-Systems/Transition-System-Construction
 ../Basic/Degeneralization

begin

```

locale automaton =
  fixes automaton :: 'label set  $\Rightarrow$  'state set  $\Rightarrow$  ('label  $\Rightarrow$  'state  $\Rightarrow$  'state set)  $\Rightarrow$ 
'condition  $\Rightarrow$  'automaton
  fixes alphabet initial transition condition
  assumes automaton[simp]: automaton (alphabet A) (initial A) (transition A)
(condition A) = A
  assumes alphabet[simp]: alphabet (automaton a i t c) = a
  assumes initial[simp]: initial (automaton a i t c) = i
  assumes transition[simp]: transition (automaton a i t c) = t
  assumes condition[simp]: condition (automaton a i t c) = c
begin

  sublocale transition-system-initial
     $\lambda$  a p. snd a  $\lambda$  a p. fst a  $\in$  alphabet A  $\wedge$  snd a  $\in$  transition A (fst a) p  $\lambda$  p. p
 $\in$  initial A
    for A
    defines path' = path and run' = run and reachable' = reachable and nodes'
= nodes
     $\langle$ proof $\rangle$ 

  lemma states-alt-def: states r p = map snd r  $\langle$ proof $\rangle$ 
  lemma trace-alt-def: trace r p = smap snd r  $\langle$ proof $\rangle$ 

  lemma successors-alt-def: successors A p = ( $\bigcup$  a  $\in$  alphabet A. transition A a
p)  $\langle$ proof $\rangle$ 

  lemma reachable-transition[intro]:
    assumes a  $\in$  alphabet A q  $\in$  reachable A p r  $\in$  transition A a q
    shows r  $\in$  reachable A p
     $\langle$ proof $\rangle$ 
  lemma nodes-transition[intro]:
    assumes a  $\in$  alphabet A p  $\in$  nodes A q  $\in$  transition A a p
    shows q  $\in$  nodes A
     $\langle$ proof $\rangle$ 

  lemma path-alphabet:
    assumes length r = length w path A (w || r) p
    shows w  $\in$  lists (alphabet A)
     $\langle$ proof $\rangle$ 
  lemma run-alphabet:
    assumes run A (w ||| r) p
    shows w  $\in$  streams (alphabet A)
     $\langle$ proof $\rangle$ 

  definition restrict :: 'automaton  $\Rightarrow$  'automaton where
    restrict A  $\equiv$  automaton
      (alphabet A)
      (initial A)
      ( $\lambda$  a p. if a  $\in$  alphabet A then transition A a p else {})

```


(*condition A*)

lemma *restrict-simps*[*simp*]:

alphabet (*restrict A*) = *alphabet A*

initial (*restrict A*) = *initial A*

transition (*restrict A*) *a p* = (if *a* ∈ *alphabet A* then *transition A a p* else {})

condition (*restrict A*) = *condition A*

⟨*proof*⟩

lemma *restrict-path*[*simp*]: *path* (*restrict A*) = *path A*

⟨*proof*⟩

lemma *restrict-run*[*simp*]: *run* (*restrict A*) = *run A*

⟨*proof*⟩

end

locale *automaton-path* =

automaton automaton alphabet initial transition condition

for *automaton* :: '*label set* ⇒ '*state set* ⇒ ('*label* ⇒ '*state* ⇒ '*state set*) ⇒ '*condition* ⇒ '*automaton*

and *alphabet initial transition condition*

+

fixes *test* :: '*condition* ⇒ '*label list* ⇒ '*state list* ⇒ '*state* ⇒ *bool*

begin

definition *language* :: '*automaton* ⇒ '*label list set* **where**

language A ≡ {*w* | *w r p. length r = length w* ∧ *p* ∈ *initial A* ∧ *path A (w || r) p* ∧ *test (condition A) w r p*}

lemma *language*[*intro*]:

assumes *length r = length w p* ∈ *initial A path A (w || r) p test (condition A) w r p*

shows *w* ∈ *language A*

⟨*proof*⟩

lemma *language-elim*[*elim*]:

assumes *w* ∈ *language A*

obtains *r p*

where *length r = length w p* ∈ *initial A path A (w || r) p test (condition A) w r p*

⟨*proof*⟩

lemma *language-alphabet*: *language A* ⊆ *lists (alphabet A)* ⟨*proof*⟩

lemma *restrict-language*[*simp*]: *language (restrict A)* = *language A* ⟨*proof*⟩

end

locale *automaton-run* =

automaton automaton alphabet initial transition condition

for *automaton* :: 'label set \Rightarrow 'state set \Rightarrow ('label \Rightarrow 'state \Rightarrow 'state set) \Rightarrow
'condition \Rightarrow 'automaton
and *alphabet initial transition condition*
+
fixes *test* :: 'condition \Rightarrow 'label stream \Rightarrow 'state stream \Rightarrow 'state \Rightarrow bool
begin

definition *language* :: 'automaton \Rightarrow 'label stream set **where**
language $A \equiv \{w \mid w \ r \ p. p \in \text{initial } A \wedge \text{run } A \ (w \ ||| \ r) \ p \wedge \text{test} \ (\text{condition} \ A) \ w \ r \ p\}$

lemma *language[intro]*:
assumes $p \in \text{initial } A \ \text{run } A \ (w \ ||| \ r) \ p \ \text{test} \ (\text{condition } A) \ w \ r \ p$
shows $w \in \text{language } A$
 $\langle \text{proof} \rangle$

lemma *language-elim[elim]*:
assumes $w \in \text{language } A$
obtains $r \ p$
where $p \in \text{initial } A \ \text{run } A \ (w \ ||| \ r) \ p \ \text{test} \ (\text{condition } A) \ w \ r \ p$
 $\langle \text{proof} \rangle$

lemma *language-alphabet*: $\text{language } A \subseteq \text{streams} \ (\text{alphabet } A) \ \langle \text{proof} \rangle$

lemma *restrict-language[simp]*: $\text{language} \ (\text{restrict } A) = \text{language } A \ \langle \text{proof} \rangle$

end

locale *automaton-degeneralization* =
a: *automaton* *automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁ +
b: *automaton* *automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂
for *automaton*₁ :: 'label set \Rightarrow 'state set \Rightarrow ('label \Rightarrow 'state \Rightarrow 'state set) \Rightarrow
'item pred gen \Rightarrow 'automaton₁
and *alphabet*₁ *initial*₁ *transition*₁ *condition*₁
and *automaton*₂ :: 'label set \Rightarrow 'state degen set \Rightarrow ('label \Rightarrow 'state degen \Rightarrow
'state degen set) \Rightarrow 'item-degen pred \Rightarrow 'automaton₂
and *alphabet*₂ *initial*₂ *transition*₂ *condition*₂
+
fixes *item* :: 'state \times 'label \times 'state \Rightarrow 'item
fixes *translate* :: 'item-degen \Rightarrow 'item degen
begin

definition *degeneralize* :: 'automaton₁ \Rightarrow 'automaton₂ **where**
degeneralize $A \equiv \text{automaton}_2$
(*alphabet*₁ A)
(*initial*₁ $A \times \{0\}$)
 $(\lambda a \ (p, k). \{(q, \text{count} \ (\text{condition}_1 \ A) \ (\text{item} \ (p, a, q)) \ k) \mid q. q \in \text{transition}_1 \ A \ a \ p\})$
(*degen* (*condition*₁ A) \circ *translate*)

lemma *degeneralize-simps*[simp]:

*alphabet*₂ (*degeneralize* *A*) = *alphabet*₁ *A*

*initial*₂ (*degeneralize* *A*) = *initial*₁ *A* × {0}

*transition*₂ (*degeneralize* *A*) *a* (*p*, *k*) =

{(*q*, *count* (*condition*₁ *A*) (*item* (*p*, *a*, *q*)) *k*) | *q*. *q* ∈ *transition*₁ *A* *a* *p*}

*condition*₂ (*degeneralize* *A*) = *degen* (*condition*₁ *A*) ∘ *translate*

⟨*proof*⟩

lemma *run-degeneralize*:

assumes *a.run* *A* (*w* ||| *r*) *p*

shows *b.run* (*degeneralize* *A*) (*w* ||| *r* ||| *sscan* (*count* (*condition*₁ *A*) ∘ *item*)

(*p* ## *r* ||| *w* ||| *r*) *k*) (*p*, *k*)

⟨*proof*⟩

lemma *degeneralize-run*:

assumes *b.run* (*degeneralize* *A*) (*w* ||| *rs*) *pk*

obtains *r* *s* *p* *k*

where *rs* = *r* ||| *s* *pk* = (*p*, *k*) *a.run* *A* (*w* ||| *r*) *p* *s* = *sscan* (*count* (*condition*₁ *A*) ∘ *item*) (*p* ## *r* ||| *w* ||| *r*) *k*

⟨*proof*⟩

lemma *degeneralize-nodes*:

b.nodes (*degeneralize* *A*) ⊆ *a.nodes* *A* × *insert* 0 {0 ..< *length* (*condition*₁ *A*)}

⟨*proof*⟩

lemma *nodes-degeneralize*: *a.nodes* *A* ⊆ *fst* ‘ *b.nodes* (*degeneralize* *A*)

⟨*proof*⟩

lemma *degeneralize-nodes-finite*[iff]: *finite* (*b.nodes* (*degeneralize* *A*)) ↔ *finite* (*a.nodes* *A*)

⟨*proof*⟩

end

locale *automaton-degeneralization-run* =

automaton-degeneralization

*automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁

*automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂

item *translate* +

a: *automaton-run* *automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁ *test*₁ +

b: *automaton-run* *automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂ *test*₂

for *automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁ *test*₁

and *automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂ *test*₂

and *item* *translate*

+

assumes *test*[iff]: *test*₂ (*degen* *cs* ∘ *translate*) *w*

(*r* ||| *sscan* (*count* *cs* ∘ *item*) (*p* ## *r* ||| *w* ||| *r*) *k*) (*p*, *k*) ↔ *test*₁ *cs* *w* *r* *p*

begin

lemma *degeneralize-language*[simp]: *b.language* (*degeneralize* *A*) = *a.language*

A

⟨proof⟩

end

locale *automaton-product* =

a: *automaton* *automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁ +

b: *automaton* *automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂ +

c: *automaton* *automaton*₃ *alphabet*₃ *initial*₃ *transition*₃ *condition*₃

for *automaton*₁ :: 'label set ⇒ 'state₁ set ⇒ ('label ⇒ 'state₁ ⇒ 'state₁ set) ⇒
'condition₁ ⇒ 'automaton₁

and *alphabet*₁ *initial*₁ *transition*₁ *condition*₁

and *automaton*₂ :: 'label set ⇒ 'state₂ set ⇒ ('label ⇒ 'state₂ ⇒ 'state₂ set)
⇒ 'condition₂ ⇒ 'automaton₂

and *alphabet*₂ *initial*₂ *transition*₂ *condition*₂

and *automaton*₃ :: 'label set ⇒ ('state₁ × 'state₂) set ⇒ ('label ⇒ 'state₁ ×
'state₂ ⇒ ('state₁ × 'state₂) set) ⇒ 'condition₃ ⇒ 'automaton₃

and *alphabet*₃ *initial*₃ *transition*₃ *condition*₃

+

fixes *condition* :: 'condition₁ ⇒ 'condition₂ ⇒ 'condition₃

begin

definition *product* :: 'automaton₁ ⇒ 'automaton₂ ⇒ 'automaton₃ **where**

product *A* *B* ≡ *automaton*₃

(*alphabet*₁ *A* ∩ *alphabet*₂ *B*)

(*initial*₁ *A* × *initial*₂ *B*)

(λ *a* (*p*, *q*). *transition*₁ *A* *a* *p* × *transition*₂ *B* *a* *q*)

(*condition* (*condition*₁ *A*) (*condition*₂ *B*))

lemma *product-simps*[*simp*]:

*alphabet*₃ (*product* *A* *B*) = *alphabet*₁ *A* ∩ *alphabet*₂ *B*

*initial*₃ (*product* *A* *B*) = *initial*₁ *A* × *initial*₂ *B*

*transition*₃ (*product* *A* *B*) *a* (*p*, *q*) = *transition*₁ *A* *a* *p* × *transition*₂ *B* *a* *q*

*condition*₃ (*product* *A* *B*) = *condition* (*condition*₁ *A*) (*condition*₂ *B*)

⟨proof⟩

lemma *product-target*[*simp*]:

assumes *length* *w* = *length* *r* *length* *r* = *length* *s*

shows *c.target* (*w* || *r* || *s*) (*p*, *q*) = (*a.target* (*w* || *r*) *p*, *b.target* (*w* || *s*) *q*)

⟨proof⟩

lemma *product-path*[*iff*]:

assumes *length* *w* = *length* *r* *length* *r* = *length* *s*

shows *c.path* (*product* *A* *B*) (*w* || *r* || *s*) (*p*, *q*) ↔

a.path *A* (*w* || *r*) *p* ∧ *b.path* *B* (*w* || *s*) *q*

⟨proof⟩

lemma *product-run*[*iff*]: *c.run* (*product* *A* *B*) (*w* ||| *r* ||| *s*) (*p*, *q*) ↔

a.run *A* (*w* ||| *r*) *p* ∧ *b.run* *B* (*w* ||| *s*) *q*

⟨proof⟩

lemma *product-nodes*: $c.nodes (product A B) \subseteq a.nodes A \times b.nodes B$
<proof>

lemma *product-nodes-finite*[intro]:
assumes *finite* ($a.nodes A$) *finite* ($b.nodes B$)
shows *finite* ($c.nodes (product A B)$)
<proof>

end

locale *automaton-intersection-path* =
automaton-product
*automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁
*automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂
*automaton*₃ *alphabet*₃ *initial*₃ *transition*₃ *condition*₃
condition +
a: *automaton-path* *automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁ *test*₁ +
b: *automaton-path* *automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂ *test*₂ +
c: *automaton-path* *automaton*₃ *alphabet*₃ *initial*₃ *transition*₃ *condition*₃ *test*₃
for *automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁ *test*₁
and *automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂ *test*₂
and *automaton*₃ *alphabet*₃ *initial*₃ *transition*₃ *condition*₃ *test*₃
and *condition*
+
assumes *test*[*iff*]: $length\ r = length\ w \implies length\ s = length\ w \implies$
 $test_3 (condition\ c_1\ c_2)\ w\ (r\ ||\ s)\ (p,\ q) \longleftrightarrow test_1\ c_1\ w\ r\ p \wedge test_2\ c_2\ w\ s\ q$
begin

lemma *product-language*[*simp*]: $c.language (product A B) = a.language A \cap b.language B$
<proof>

end

locale *automaton-intersection-run* =
automaton-product
*automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁
*automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂
*automaton*₃ *alphabet*₃ *initial*₃ *transition*₃ *condition*₃
condition +
a: *automaton-run* *automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁ *test*₁ +
b: *automaton-run* *automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂ *test*₂ +
c: *automaton-run* *automaton*₃ *alphabet*₃ *initial*₃ *transition*₃ *condition*₃ *test*₃
for *automaton*₁ *alphabet*₁ *initial*₁ *transition*₁ *condition*₁ *test*₁
and *automaton*₂ *alphabet*₂ *initial*₂ *transition*₂ *condition*₂ *test*₂
and *automaton*₃ *alphabet*₃ *initial*₃ *transition*₃ *condition*₃ *test*₃
and *condition*
+

assumes *test*[*iff*]: $test_3$ (condition c_1 c_2) w ($r \parallel s$) (p , q) \longleftrightarrow $test_1$ c_1 w r p
 \wedge $test_2$ c_2 w s q

begin

lemma *product-language*[*simp*]: c .*language* (product A B) = a .*language* $A \cap$
 b .*language* B
 ⟨*proof*⟩

end

locale *automaton-sum* =

a : *automaton* $automaton_1$ $alphabet_1$ $initial_1$ $transition_1$ $condition_1$ +

b : *automaton* $automaton_2$ $alphabet_2$ $initial_2$ $transition_2$ $condition_2$ +

c : *automaton* $automaton_3$ $alphabet_3$ $initial_3$ $transition_3$ $condition_3$

for $automaton_1$:: 'label set \Rightarrow 'state₁ set \Rightarrow ('label \Rightarrow 'state₁ \Rightarrow 'state₁ set) \Rightarrow
 'condition₁ \Rightarrow 'automaton₁

and $alphabet_1$ $initial_1$ $transition_1$ $condition_1$

and $automaton_2$:: 'label set \Rightarrow 'state₂ set \Rightarrow ('label \Rightarrow 'state₂ \Rightarrow 'state₂ set)
 \Rightarrow 'condition₂ \Rightarrow 'automaton₂

and $alphabet_2$ $initial_2$ $transition_2$ $condition_2$

and $automaton_3$:: 'label set \Rightarrow ('state₁ + 'state₂) set \Rightarrow ('label \Rightarrow 'state₁ +
 'state₂ \Rightarrow ('state₁ + 'state₂) set) \Rightarrow 'condition₃ \Rightarrow 'automaton₃

and $alphabet_3$ $initial_3$ $transition_3$ $condition_3$

+

fixes $condition$:: 'condition₁ \Rightarrow 'condition₂ \Rightarrow 'condition₃

begin

definition sum :: 'automaton₁ \Rightarrow 'automaton₂ \Rightarrow 'automaton₃ **where**

sum A $B \equiv automaton_3$

($alphabet_1$ $A \cup alphabet_2$ B)

($initial_1$ A $\langle + \rangle$ $initial_2$ B)

($\lambda a. \lambda Inl p \Rightarrow Inl$ ' $transition_1$ A a $p \mid Inr q \Rightarrow Inr$ ' $transition_2$ B a q)

($condition$ ($condition_1$ A) ($condition_2$ B))

lemma *sum-simps*[*simp*]:

$alphabet_3$ (sum A B) = $alphabet_1$ $A \cup alphabet_2$ B

$initial_3$ (sum A B) = $initial_1$ A $\langle + \rangle$ $initial_2$ B

$transition_3$ (sum A B) a (Inl p) = Inl ' $transition_1$ A a p

$transition_3$ (sum A B) a (Inr q) = Inr ' $transition_2$ B a q

$condition_3$ (sum A B) = $condition$ ($condition_1$ A) ($condition_2$ B)

⟨*proof*⟩

lemma *path-sum-a*:

assumes $length$ r = $length$ w a .*path* A ($w \parallel r$) p

shows c .*path* (sum A B) ($w \parallel map$ Inl r) (Inl p)

⟨*proof*⟩

lemma *path-sum-b*:

assumes $length$ s = $length$ w b .*path* B ($w \parallel s$) q

shows c .*path* (sum A B) ($w \parallel map$ Inr s) (Inr q)

⟨proof⟩
lemma *sum-path*:
 assumes $\text{alphabet}_1 A = \text{alphabet}_2 B$
 assumes $\text{length } rs = \text{length } w \text{ c.path } (\text{sum } A B) (w \parallel rs) pq$
 obtains
 (a) $r p$ **where** $rs = \text{map } \text{Inl } r pq = \text{Inl } p \text{ a.path } A (w \parallel r) p \mid$
 (b) $s q$ **where** $rs = \text{map } \text{Inr } s pq = \text{Inr } q \text{ b.path } B (w \parallel s) q$
 ⟨proof⟩

lemma *run-sum-a*:
 assumes $a.\text{run } A (w \parallel\parallel r) p$
 shows $c.\text{run } (\text{sum } A B) (w \parallel\parallel \text{smap } \text{Inl } r) (\text{Inl } p)$
 ⟨proof⟩

lemma *run-sum-b*:
 assumes $b.\text{run } B (w \parallel\parallel s) q$
 shows $c.\text{run } (\text{sum } A B) (w \parallel\parallel \text{smap } \text{Inr } s) (\text{Inr } q)$
 ⟨proof⟩

lemma *sum-run*:
 assumes $\text{alphabet}_1 A = \text{alphabet}_2 B$
 assumes $c.\text{run } (\text{sum } A B) (w \parallel\parallel rs) pq$
 obtains
 (a) $r p$ **where** $rs = \text{smap } \text{Inl } r pq = \text{Inl } p \text{ a.run } A (w \parallel\parallel r) p \mid$
 (b) $s q$ **where** $rs = \text{smap } \text{Inr } s pq = \text{Inr } q \text{ b.run } B (w \parallel\parallel s) q$
 ⟨proof⟩

lemma *sum-nodes*:
 assumes $\text{alphabet}_1 A = \text{alphabet}_2 B$
 shows $c.\text{nodes } (\text{sum } A B) \subseteq a.\text{nodes } A <+> b.\text{nodes } B$
 ⟨proof⟩

lemma *sum-nodes-finite[intro]*:
 assumes $\text{alphabet}_1 A = \text{alphabet}_2 B$
 assumes $\text{finite } (a.\text{nodes } A) \text{ finite } (b.\text{nodes } B)$
 shows $\text{finite } (c.\text{nodes } (\text{sum } A B))$
 ⟨proof⟩

end

locale *automaton-union-path* =
 automaton-sum
 automaton₁ alphabet₁ initial₁ transition₁ condition₁
 automaton₂ alphabet₂ initial₂ transition₂ condition₂
 automaton₃ alphabet₃ initial₃ transition₃ condition₃
 condition +
 a: automaton-path automaton₁ alphabet₁ initial₁ transition₁ condition₁ test₁ +
 b: automaton-path automaton₂ alphabet₂ initial₂ transition₂ condition₂ test₂ +
 c: automaton-path automaton₃ alphabet₃ initial₃ transition₃ condition₃ test₃
for automaton₁ alphabet₁ initial₁ transition₁ condition₁ test₁
and automaton₂ alphabet₂ initial₂ transition₂ condition₂ test₂

```

and automaton3 alphabet3 initial3 transition3 condition3 test3
and condition
+
assumes test1[iff]: length r = length w  $\implies$  test3 (condition c1 c2) w (map Inl
r) (Inl p)  $\longleftrightarrow$  test1 c1 w r p
assumes test2[iff]: length s = length w  $\implies$  test3 (condition c1 c2) w (map Inr
s) (Inr q)  $\longleftrightarrow$  test2 c2 w s q
begin

lemma sum-language[simp]:
assumes alphabet1 A = alphabet2 B
shows c.language (sum A B) = a.language A  $\cup$  b.language B
<proof>

end

```

```

locale automaton-union-run =
  automaton-sum
  automaton1 alphabet1 initial1 transition1 condition1
  automaton2 alphabet2 initial2 transition2 condition2
  automaton3 alphabet3 initial3 transition3 condition3
  condition +
  a: automaton-run automaton1 alphabet1 initial1 transition1 condition1 test1 +
  b: automaton-run automaton2 alphabet2 initial2 transition2 condition2 test2 +
  c: automaton-run automaton3 alphabet3 initial3 transition3 condition3 test3
for automaton1 alphabet1 initial1 transition1 condition1 test1
and automaton2 alphabet2 initial2 transition2 condition2 test2
and automaton3 alphabet3 initial3 transition3 condition3 test3
and condition
+
assumes test1[iff]: test3 (condition c1 c2) w (smap Inl r) (Inl p)  $\longleftrightarrow$  test1 c1
w r p
assumes test2[iff]: test3 (condition c1 c2) w (smap Inr s) (Inr q)  $\longleftrightarrow$  test2 c2
w s q
begin

lemma sum-language[simp]:
assumes alphabet1 A = alphabet2 B
shows c.language (sum A B) = a.language A  $\cup$  b.language B
<proof>

end

```

```

locale automaton-product-list =
  a: automaton automaton1 alphabet1 initial1 transition1 condition1 +
  b: automaton automaton2 alphabet2 initial2 transition2 condition2
for automaton1 :: 'label set  $\Rightarrow$  'state set  $\Rightarrow$  ('label  $\Rightarrow$  'state  $\Rightarrow$  'state set)  $\Rightarrow$ 
'condition1  $\Rightarrow$  'automaton1
and alphabet1 initial1 transition1 condition1

```


and *automaton₂* :: 'label set ⇒ 'state list set ⇒ ('label ⇒ 'state list ⇒ 'state list set) ⇒ 'condition₂ ⇒ 'automaton₂
and *alphabet₂ initial₂ transition₂ condition₂*
 +
fixes *condition* :: 'condition₁ list ⇒ 'condition₂
begin

definition *product* :: 'automaton₁ list ⇒ 'automaton₂ **where**

product AA ≡ *automaton₂*
 (∩ (alphabet₁ ' set AA))
 (listset (map initial₁ AA))
 (λ a ps. listset (map2 (λ A p. transition₁ A a p) AA ps))
 (condition (map condition₁ AA))

lemma *product-simps*[simp]:

alphabet₂ (product AA) = ∩ (alphabet₁ ' set AA)
initial₂ (product AA) = listset (map initial₁ AA)
transition₂ (product AA) a ps = listset (map2 (λ A p. transition₁ A a p) AA ps)
condition₂ (product AA) = condition (map condition₁ AA)
 ⟨proof⟩

lemma *product-run-length*:

assumes *length ps = length AA*
assumes *b.run (product AA) (w ||| r) ps*
assumes *qs ∈ sset r*
shows *length qs = length AA*

⟨proof⟩

lemma *product-run-stranspose*:

assumes *length ps = length AA*
assumes *b.run (product AA) (w ||| r) ps*
obtains *rs where r = stranspose rs length rs = length AA*

⟨proof⟩

lemma *run-product*:

assumes *length rs = length AA length ps = length AA*
assumes $\bigwedge k. k < \text{length } AA \implies a.\text{run } (AA ! k) (w ||| rs ! k) (ps ! k)$
shows *b.run (product AA) (w ||| stranspose rs) ps*

⟨proof⟩

lemma *product-run*:

assumes *length rs = length AA length ps = length AA*
assumes *b.run (product AA) (w ||| stranspose rs) ps*
shows $k < \text{length } AA \implies a.\text{run } (AA ! k) (w ||| rs ! k) (ps ! k)$

⟨proof⟩

lemma *product-nodes*: *b.nodes (product AA) ⊆ listset (map a.nodes AA)*

⟨proof⟩

lemma *product-nodes-finite*[intro]:

```

assumes list-all (finite ◦ a.nodes) AA
shows finite (b.nodes (product AA))
  ⟨proof⟩
lemma product-nodes-card:
assumes list-all (finite ◦ a.nodes) AA
shows card (b.nodes (product AA)) ≤ prod-list (map (card ◦ a.nodes) AA)
  ⟨proof⟩

```

end

```

locale automaton-intersection-list-run =
  automaton-product-list
  automaton1 alphabet1 initial1 transition1 condition1
  automaton2 alphabet2 initial2 transition2 condition2
  condition +
  a: automaton-run automaton1 alphabet1 initial1 transition1 condition1 test1 +
  b: automaton-run automaton2 alphabet2 initial2 transition2 condition2 test2
for automaton1 alphabet1 initial1 transition1 condition1 test1
and automaton2 alphabet2 initial2 transition2 condition2 test2
and condition
  +
assumes test[iff]: length rs = length cs ⇒ length ps = length cs ⇒
  test2 (condition cs) w (stranspose rs) ps ⇔ list-all (λ (c, r, p). test1 c w r
  p) (cs || rs || ps)
begin

```

```

lemma product-language[simp]: b.language (product AA) = ⋂ (a.language ‘ set
  AA)
  ⟨proof⟩

```

end

```

locale automaton-sum-list =
  a: automaton automaton1 alphabet1 initial1 transition1 condition1 +
  b: automaton automaton2 alphabet2 initial2 transition2 condition2
for automaton1 :: ‘label set ⇒ ‘state set ⇒ (‘label ⇒ ‘state ⇒ ‘state set) ⇒
  ‘condition1 ⇒ ‘automaton1
and alphabet1 initial1 transition1 condition1
and automaton2 :: ‘label set ⇒ (nat × ‘state) set ⇒ (‘label ⇒ nat × ‘state ⇒
  (nat × ‘state) set) ⇒ ‘condition2 ⇒ ‘automaton2
and alphabet2 initial2 transition2 condition2
  +
fixes condition :: ‘condition1 list ⇒ ‘condition2
begin

```

```

definition sum :: ‘automaton1 list ⇒ ‘automaton2 where
  sum AA ≡ automaton2
  (⋃ (alphabet1 ‘ set AA))
  (⋃ k < length AA. {k} × initial1 (AA ! k))

```

$(\lambda a (k, p). \{k\} \times \text{transition}_1 (AA ! k) a p)$
 $(\text{condition} (\text{map condition}_1 AA))$

lemma *sum-simps[simp]*:

$\text{alphabet}_2 (\text{sum } AA) = \bigcup (\text{alphabet}_1 \text{ ' set } AA)$
 $\text{initial}_2 (\text{sum } AA) = (\bigcup k < \text{length } AA. \{k\} \times \text{initial}_1 (AA ! k))$
 $\text{transition}_2 (\text{sum } AA) a (k, p) = \{k\} \times \text{transition}_1 (AA ! k) a p$
 $\text{condition}_2 (\text{sum } AA) = \text{condition} (\text{map condition}_1 AA)$
 $\langle \text{proof} \rangle$

lemma *run-sum*:

assumes $\bigcap (\text{alphabet}_1 \text{ ' set } AA) = \bigcup (\text{alphabet}_1 \text{ ' set } AA)$
assumes $A \in \text{set } AA$
assumes $a.\text{run } A (w \parallel s) p$
obtains k **where** $k < \text{length } AA$ $A = AA ! k$ $b.\text{run} (\text{sum } AA) (w \parallel \text{sconst } k$
 $\parallel s) (k, p)$
 $\langle \text{proof} \rangle$

lemma *sum-run*:

assumes $\bigcap (\text{alphabet}_1 \text{ ' set } AA) = \bigcup (\text{alphabet}_1 \text{ ' set } AA)$
assumes $k < \text{length } AA$
assumes $b.\text{run} (\text{sum } AA) (w \parallel r) (k, p)$
obtains s **where** $r = \text{sconst } k \parallel s$ $a.\text{run} (AA ! k) (w \parallel s) p$
 $\langle \text{proof} \rangle$

lemma *sum-nodes*:

assumes $\bigcap (\text{alphabet}_1 \text{ ' set } AA) = \bigcup (\text{alphabet}_1 \text{ ' set } AA)$
shows $b.\text{nodes} (\text{sum } AA) \subseteq (\bigcup k < \text{length } AA. \{k\} \times a.\text{nodes} (AA ! k))$
 $\langle \text{proof} \rangle$

lemma *sum-nodes-finite[intro]*:

assumes $\bigcap (\text{alphabet}_1 \text{ ' set } AA) = \bigcup (\text{alphabet}_1 \text{ ' set } AA)$
assumes $\text{list-all} (\text{finite} \circ a.\text{nodes}) AA$
shows $\text{finite} (b.\text{nodes} (\text{sum } AA))$
 $\langle \text{proof} \rangle$

end

locale *automaton-union-list-run* =

automaton-sum-list

$\text{automaton}_1 \text{ alphabet}_1 \text{ initial}_1 \text{ transition}_1 \text{ condition}_1$

$\text{automaton}_2 \text{ alphabet}_2 \text{ initial}_2 \text{ transition}_2 \text{ condition}_2$

$\text{condition} +$

$a: \text{automaton-run } \text{automaton}_1 \text{ alphabet}_1 \text{ initial}_1 \text{ transition}_1 \text{ condition}_1 \text{ test}_1 +$

$b: \text{automaton-run } \text{automaton}_2 \text{ alphabet}_2 \text{ initial}_2 \text{ transition}_2 \text{ condition}_2 \text{ test}_2$

for $\text{automaton}_1 \text{ alphabet}_1 \text{ initial}_1 \text{ transition}_1 \text{ condition}_1 \text{ test}_1$

and $\text{automaton}_2 \text{ alphabet}_2 \text{ initial}_2 \text{ transition}_2 \text{ condition}_2 \text{ test}_2$

and condition

$+$

assumes $\text{test}[iff]: k < \text{length } cs \implies \text{test}_2 (\text{condition } cs) w (\text{sconst } k \parallel r) (k,$

```

p)  $\longleftrightarrow$  test1 (cs ! k) w r p
begin

  lemma sum-language[simp]:
    assumes  $\bigcap$  (alphabet1 ' set AA) =  $\bigcup$  (alphabet1 ' set AA)
    shows b.language (sum AA) =  $\bigcup$  (a.language ' set AA)
    <proof>

end

end

```

13 Nondeterministic Finite Automata

```

theory NFA
imports ../Nondeterministic
begin

datatype ('label, 'state) nfa = nfa
  (alphabet: 'label set)
  (initial: 'state set)
  (transition: 'label  $\Rightarrow$  'state  $\Rightarrow$  'state set)
  (accepting: 'state pred)

global-interpretation nfa: automaton nfa alphabet initial transition accepting
  defines path = nfa.path and run = nfa.run and reachable = nfa.reachable and
nodes = nfa.nodes
  <proof>

global-interpretation nfa: automaton-path nfa alphabet initial transition ac-
cepting  $\lambda$  P w r p. P (last (p # r))
  defines language = nfa.language
  <proof>

abbreviation target where target  $\equiv$  nfa.target
abbreviation states where states  $\equiv$  nfa.states
abbreviation trace where trace  $\equiv$  nfa.trace
abbreviation successors where successors  $\equiv$  nfa.successors TYPE('label)

global-interpretation nfa: automaton-intersection-path
  nfa alphabet initial transition accepting  $\lambda$  P w r p. P (last (p # r))
  nfa alphabet initial transition accepting  $\lambda$  P w r p. P (last (p # r))
  nfa alphabet initial transition accepting  $\lambda$  P w r p. P (last (p # r))
   $\lambda$  c1 c2 (p, q). c1 p  $\wedge$  c2 q
  defines intersect = nfa.product
  <proof>

global-interpretation nfa: automaton-union-path
  nfa alphabet initial transition accepting  $\lambda$  P w r p. P (last (p # r))

```

```

nfa alphabet initial transition accepting  $\lambda P w r p. P$  (last (p # r))
nfa alphabet initial transition accepting  $\lambda P w r p. P$  (last (p # r))
case-sum
defines union = nfa.sum
<proof>

```

end

14 Deterministic Büchi Automata

```

theory DBA
imports ../Deterministic
begin

```

```

datatype ('label, 'state) dba = dba
  (alphabet: 'label set)
  (initial: 'state)
  (transition: 'label  $\Rightarrow$  'state  $\Rightarrow$  'state)
  (accepting: 'state pred)

```

```

global-interpretation dba: automaton dba alphabet initial transition accepting
defines path = dba.path and run = dba.run and reachable = dba.reachable
and nodes = dba.nodes
<proof>

```

```

global-interpretation dba: automaton-run dba alphabet initial transition accept-
ing  $\lambda P w r p. \text{infs } P (p \#\# r)$ 
defines language = dba.language
<proof>

```

```

abbreviation target where target  $\equiv$  dba.target
abbreviation states where states  $\equiv$  dba.states
abbreviation trace where trace  $\equiv$  dba.trace

```

```

abbreviation successors where successors  $\equiv$  dba.successors TYPE('label)

```

end

15 Deterministic Generalized Büchi Automata

```

theory DGBA
imports ../Deterministic
begin

```

```

datatype ('label, 'state) dgba = dgba
  (alphabet: 'label set)
  (initial: 'state)
  (transition: 'label  $\Rightarrow$  'state  $\Rightarrow$  'state)

```

(accepting: 'state pred gen)

global-interpretation *dgba: automaton dgba alphabet initial transition accepting*
defines *path = dgba.path and run = dgba.run and reachable = dgba.reachable*
and *nodes = dgba.nodes*

<proof>

global-interpretation *dgba: automaton-run dgba alphabet initial transition ac-*
cepting $\lambda P w r p$. gen infs $P (p \#\# r)$

defines *language = dgba.language*

<proof>

abbreviation *target where target \equiv dgba.target*

abbreviation *states where states \equiv dgba.states*

abbreviation *trace where trace \equiv dgba.trace*

abbreviation *successors where successors \equiv dgba.successors TYPE('label)*

end

16 Deterministic Büchi Automata Combinations

theory *DBA-Combine*

imports *DBA DGBA*

begin

global-interpretation *degeneralization: automaton-degeneralization-run*

dgba dgba.alphabet dgba.initial dgba.transition dgba.accepting $\lambda P w r p$. gen infs
 $P (p \#\# r)$

dba dba.alphabet dba.initial dba.transition dba.accepting $\lambda P w r p$. infs $P (p$
 $\#\# r)$

fst id

defines *degeneralize = degeneralization.degeneralize*

<proof>

lemmas *degeneralize-language[simp] = degeneralization.degeneralize-language[folded*
DBA.language-def]

lemmas *degeneralize-nodes-finite[iff] = degeneralization.degeneralize-nodes-finite[folded*
DBA.nodes-def]

lemmas *degeneralize-nodes-card = degeneralization.degeneralize-nodes-card[folded*
DBA.nodes-def]

global-interpretation *intersection: automaton-intersection-run*

dba.dba dba.alphabet dba.initial dba.transition dba.accepting $\lambda P w r p$. infs P
 $(p \#\# r)$

dba.dba dba.alphabet dba.initial dba.transition dba.accepting $\lambda P w r p$. infs P
 $(p \#\# r)$

dgba.dgba dgba.alphabet dgba.initial dgba.transition dgba.accepting $\lambda P w r p$.
gen infs $P (p \#\# r)$

$\lambda c_1 c_2. [c_1 \circ fst, c_2 \circ snd]$

defines *intersect' = intersection.product*

<proof>

lemmas *intersect'-language*[simp] = *intersection.product-language*[folded *DGBA.language-def*]

lemmas *intersect'-nodes-finite* = *intersection.product-nodes-finite*[folded *DGBA.nodes-def*]

lemmas *intersect'-nodes-card* = *intersection.product-nodes-card*[folded *DGBA.nodes-def*]

global-interpretation *union: automaton-union-run*

dba.dba dba.alphabet dba.initial dba.transition dba.accepting $\lambda P w r p. \text{infs } P$

(*p ## r*)

dba.dba dba.alphabet dba.initial dba.transition dba.accepting $\lambda P w r p. \text{infs } P$

(*p ## r*)

dba.dba dba.alphabet dba.initial dba.transition dba.accepting $\lambda P w r p. \text{infs } P$

(*p ## r*)

$\lambda c_1 c_2 pq. (c_1 \circ \text{fst}) pq \vee (c_2 \circ \text{snd}) pq$

defines *union* = *union.product*

<proof>

lemmas *union-language* = *union.product-language*

lemmas *union-nodes-finite* = *union.product-nodes-finite*

lemmas *union-nodes-card* = *union.product-nodes-card*

global-interpretation *intersection-list: automaton-intersection-list-run*

dba.dba dba.alphabet dba.initial dba.transition dba.accepting $\lambda P w r p. \text{infs } P$

(*p ## r*)

dgba.dgba dgba.alphabet dgba.initial dgba.transition dgba.accepting $\lambda P w r p.$

gen infs P (p ## r)

$\lambda cs. \text{map } (\lambda k pp. (cs ! k) (pp ! k)) [0 .. < \text{length } cs]$

defines *intersect-list'* = *intersection-list.product*

<proof>

lemmas *intersect-list'-language*[simp] = *intersection-list.product-language*[folded *DGBA.language-def*]

lemmas *intersect-list'-nodes-finite* = *intersection-list.product-nodes-finite*[folded *DGBA.nodes-def*]

lemmas *intersect-list'-nodes-card* = *intersection-list.product-nodes-card*[folded *DGBA.nodes-def*]

global-interpretation *union-list: automaton-union-list-run*

dba.dba dba.alphabet dba.initial dba.transition dba.accepting $\lambda P w r p. \text{infs } P$

(*p ## r*)

dba.dba dba.alphabet dba.initial dba.transition dba.accepting $\lambda P w r p. \text{infs } P$

(*p ## r*)

$\lambda cs pp. \exists k < \text{length } cs. (cs ! k) (pp ! k)$

defines *union-list* = *union-list.product*

<proof>

lemmas *union-list-language* = *union-list.product-language*

lemmas *union-list-nodes-finite* = *union-list.product-nodes-finite*

lemmas *union-list-nodes-card* = *union-list.product-nodes-card*

abbreviation *intersect where* $intersect\ A\ B \equiv degeneralize\ (intersect'\ A\ B)$

lemma *intersect-language[simp]*: $DBA.language\ (intersect\ A\ B) = DBA.language\ A \cap DBA.language\ B$

<proof>

lemma *intersect-nodes-finite*:

assumes *finite* $(DBA.nodes\ A)$ *finite* $(DBA.nodes\ B)$

shows *finite* $(DBA.nodes\ (intersect\ A\ B))$

<proof>

lemma *intersect-nodes-card*:

assumes *finite* $(DBA.nodes\ A)$ *finite* $(DBA.nodes\ B)$

shows $card\ (DBA.nodes\ (intersect\ A\ B)) \leq 2 * card\ (DBA.nodes\ A) * card\ (DBA.nodes\ B)$

<proof>

abbreviation *intersect-list where* $intersect-list\ AA \equiv degeneralize\ (intersect-list'\ AA)$

lemma *intersect-list-language[simp]*: $DBA.language\ (intersect-list\ AA) = \bigcap\ (DBA.language\ 'set\ AA)$

<proof>

lemma *intersect-list-nodes-finite*:

assumes *list-all* $(finite \circ DBA.nodes)\ AA$

shows *finite* $(DBA.nodes\ (intersect-list\ AA))$

<proof>

lemma *intersect-list-nodes-card*:

assumes *list-all* $(finite \circ DBA.nodes)\ AA$

shows $card\ (DBA.nodes\ (intersect-list\ AA)) \leq max\ 1\ (length\ AA) * prod-list\ (map\ (card \circ DBA.nodes)\ AA)$

<proof>

end

17 Deterministic Büchi Transition Automata

theory *DBTA*

imports *../Deterministic*

begin

datatype $(label, state)\ dbta = dbta$

(alphabet: 'label set)

(initial: 'state)

(transition: 'label \Rightarrow 'state \Rightarrow 'state)

(accepting: ('state \times 'label \times 'state) pred)

global-interpretation *dbta: automaton dbta alphabet initial transition accepting*


```

defines path = dbta.path and run = dbta.run and reachable = dbta.reachable
and nodes = dbta.nodes
  <proof>
global-interpretation dbta: automaton-run dbta alphabet initial transition ac-
cepting
   $\lambda P w r p. \text{infs } P (p \#\# r \|\| w \|\| r)$ 
defines language = dbta.language
  <proof>

abbreviation target where target  $\equiv$  dbta.target
abbreviation states where states  $\equiv$  dbta.states
abbreviation trace where trace  $\equiv$  dbta.trace
abbreviation successors where successors  $\equiv$  dbta.successors TYPE('label)

end

```

18 Deterministic Generalized Büchi Transition Automata

```

theory DGBTA
imports ../Deterministic
begin

datatype ('label, 'state) dgba = dgba
  (alphabet: 'label set)
  (initial: 'state)
  (transition: 'label  $\Rightarrow$  'state  $\Rightarrow$  'state)
  (accepting: ('state  $\times$  'label  $\times$  'state) pred gen)

global-interpretation dgba: automaton dgba alphabet initial transition accept-
ing
defines path = dgba.path and run = dgba.run and reachable = dgba.reachable
and nodes = dgba.nodes
  <proof>
global-interpretation dgba: automaton-run dgba alphabet initial transition
accepting
   $\lambda P w r p. \text{gen infs } P (p \#\# r \|\| w \|\| r)$ 
defines language = dgba.language
  <proof>

abbreviation target where target  $\equiv$  dgba.target
abbreviation states where states  $\equiv$  dgba.states
abbreviation trace where trace  $\equiv$  dgba.trace
abbreviation successors where successors  $\equiv$  dgba.successors TYPE('label)

end

```

19 Deterministic Büchi Transition Automata Combinations

theory *DBTA-Combine*
imports *DBTA DGBTA*
begin

global-interpretation *degeneralization: automaton-degeneralization-run*
dgba dgba.alphabet dgba.initial dgba.transition dgba.accepting $\lambda P w r p. \text{gen}$
infs $P (p \#\# r \|\| w \|\| r)$
dbta dbta.alphabet dbta.initial dbta.transition dbta.accepting $\lambda P w r p. \text{infs } P$
 $(p \#\# r \|\| w \|\| r)$
id $\lambda ((p, k), a, (q, l)). ((p, a, q), k)$
defines *degeneralize* = *degeneralization.degeneralize*
 $\langle \text{proof} \rangle$

lemmas *degeneralize-language[simp]* = *degeneralization.degeneralize-language[folded DBTA.language-def]*
lemmas *degeneralize-nodes-finite[iff]* = *degeneralization.degeneralize-nodes-finite[folded DBTA.nodes-def]*
lemmas *degeneralize-nodes-card* = *degeneralization.degeneralize-nodes-card[folded DBTA.nodes-def]*

global-interpretation *intersection: automaton-intersection-run*
dbta.dbta dbta.alphabet dbta.initial dbta.transition dbta.accepting $\lambda P w r p. \text{infs}$
 $P (p \#\# r \|\| w \|\| r)$
dbta.dbta dbta.alphabet dbta.initial dbta.transition dbta.accepting $\lambda P w r p. \text{infs}$
 $P (p \#\# r \|\| w \|\| r)$
dgba.dgba dgba.alphabet dgba.initial dgba.transition dgba.accepting $\lambda P w r$
 $p. \text{gen infs } P (p \#\# r \|\| w \|\| r)$
 $\lambda c_1 c_2. [c_1 \circ (\lambda ((p_1, p_2), a, (q_1, q_2)). (p_1, a, q_1)), c_2 \circ (\lambda ((p_1, p_2), a, (q_1, q_2)). (p_2, a, q_2))]$
defines *intersect'* = *intersection.product*
 $\langle \text{proof} \rangle$

lemmas *intersect'-language[simp]* = *intersection.product-language[folded DGBTA.language-def]*
lemmas *intersect'-nodes-finite* = *intersection.product-nodes-finite[folded DGBTA.nodes-def]*
lemmas *intersect'-nodes-card* = *intersection.product-nodes-card[folded DGBTA.nodes-def]*

global-interpretation *union: automaton-union-run*
dbta.dbta dbta.alphabet dbta.initial dbta.transition dbta.accepting $\lambda P w r p. \text{infs}$
 $P (p \#\# r \|\| w \|\| r)$
dbta.dbta dbta.alphabet dbta.initial dbta.transition dbta.accepting $\lambda P w r p. \text{infs}$
 $P (p \#\# r \|\| w \|\| r)$
dbta.dbta dbta.alphabet dbta.initial dbta.transition dbta.accepting $\lambda P w r p. \text{infs}$
 $P (p \#\# r \|\| w \|\| r)$
 $\lambda c_1 c_2 pq. (c_1 \circ (\lambda ((p_1, p_2), a, (q_1, q_2)). (p_1, a, q_1))) pq \vee (c_2 \circ (\lambda ((p_1, p_2),$

```

a, (q1, q2)). (p2, a, q2))) pq
defines union = union.product
⟨proof⟩

lemmas union-language = union.product-language
lemmas union-nodes-finite = union.product-nodes-finite
lemmas union-nodes-card = union.product-nodes-card

abbreviation intersect where intersect A B ≡ degeneralize (intersect' A B)

lemma intersect-language[simp]: DBTA.language (intersect A B) = DBTA.language
A ∩ DBTA.language B
⟨proof⟩
lemma intersect-nodes-finite:
assumes finite (DBTA.nodes A) finite (DBTA.nodes B)
shows finite (DBTA.nodes (intersect A B))
⟨proof⟩
lemma intersect-nodes-card:
assumes finite (DBTA.nodes A) finite (DBTA.nodes B)
shows card (DBTA.nodes (intersect A B)) ≤ 2 * card (DBTA.nodes A) * card
(DBTA.nodes B)
⟨proof⟩

end

```

20 Deterministic Co-Büchi Automata

```

theory DCA
imports ../Deterministic
begin

datatype ('label, 'state) dca = dca
  (alphabet: 'label set)
  (initial: 'state)
  (transition: 'label ⇒ 'state ⇒ 'state)
  (rejecting: 'state ⇒ bool)

global-interpretation dca: automaton dca alphabet initial transition rejecting
defines path = dca.path and run = dca.run and reachable = dca.reachable
and nodes = dca.nodes
⟨proof⟩
global-interpretation dca: automaton-run dca alphabet initial transition reject-
ing λ P w r p. fins P (p ## r)
defines language = dca.language
⟨proof⟩

abbreviation target where target ≡ dca.target
abbreviation states where states ≡ dca.states
abbreviation trace where trace ≡ dca.trace

```

abbreviation *successors* **where** *successors* \equiv *dca.successors* *TYPE*('label)

end

21 Deterministic Co-Generalized Co-Büchi Automata

theory *DGCA*

imports *../Deterministic*

begin

datatype ('label, 'state) *dgca* = *dgca*
(*alphabet*: 'label set)
(*initial*: 'state)
(*transition*: 'label \Rightarrow 'state \Rightarrow 'state)
(*rejecting*: 'state pred gen)

global-interpretation *dgca*: *automaton dgca alphabet initial transition rejecting*
defines *path* = *dgca.path* **and** *run* = *dgca.run* **and** *reachable* = *dgca.reachable*
and *nodes* = *dgca.nodes*

<proof>

global-interpretation *dgca*: *automaton-run dgca alphabet initial transition re-*
jecting $\lambda P w r p.$ *cogen fins* *P* (*p* ## *r*)

defines *language* = *dgca.language*

<proof>

abbreviation *target* **where** *target* \equiv *dgca.target*

abbreviation *states* **where** *states* \equiv *dgca.states*

abbreviation *trace* **where** *trace* \equiv *dgca.trace*

abbreviation *successors* **where** *successors* \equiv *dgca.successors* *TYPE*('label)

end

22 Deterministic Co-Büchi Automata Combinations

theory *DCA-Combine*

imports *DCA DGCA*

begin

global-interpretation *degeneralization: automaton-degeneralization-run*
dgca dgca.alphabet dgca.initial dgca.transition dgca.rejecting $\lambda P w r p.$ *cogen*
fins *P* (*p* ## *r*)

dca dca.alphabet dca.initial dca.transition dca.rejecting $\lambda P w r p.$ *fins* *P* (*p* ##
r)

fst id

defines *degeneralize* = *degeneralization.degeneralize*

<proof>

lemmas *degeneralize-language*[simp] = *degeneralization.degeneralize-language*[folded *DCA.language-def*]
lemmas *degeneralize-nodes-finite*[iff] = *degeneralization.degeneralize-nodes-finite*[folded *DCA.nodes-def*]
lemmas *degeneralize-nodes-card* = *degeneralization.degeneralize-nodes-card*[folded *DCA.nodes-def*]

global-interpretation *intersection: automaton-intersection-run*
dca.dca dca.alphabet dca.initial dca.transition dca.rejecting $\lambda P w r p. \text{fins } P (p \# \# r)$
dca.dca dca.alphabet dca.initial dca.transition dca.rejecting $\lambda P w r p. \text{fins } P (p \# \# r)$
dca.dca dca.alphabet dca.initial dca.transition dca.rejecting $\lambda P w r p. \text{fins } P (p \# \# r)$
 $\lambda c_1 c_2 pq. (c_1 \circ \text{fst}) pq \vee (c_2 \circ \text{snd}) pq$
defines *intersect* = *intersection.product*
 $\langle \text{proof} \rangle$

lemmas *intersect-language* = *intersection.product-language*
lemmas *intersect-nodes-finite* = *intersection.product-nodes-finite*
lemmas *intersect-nodes-card* = *intersection.product-nodes-card*

global-interpretation *union: automaton-union-run*
dca.dca dca.alphabet dca.initial dca.transition dca.rejecting $\lambda P w r p. \text{fins } P (p \# \# r)$
dca.dca dca.alphabet dca.initial dca.transition dca.rejecting $\lambda P w r p. \text{fins } P (p \# \# r)$
dgca.dgca dgca.alphabet dgca.initial dgca.transition dgca.rejecting $\lambda P w r p. \text{cogen fins } P (p \# \# r)$
 $\lambda c_1 c_2. [c_1 \circ \text{fst}, c_2 \circ \text{snd}]$
defines *union'* = *union.product*
 $\langle \text{proof} \rangle$

lemmas *union'-language*[simp] = *union.product-language*[folded *DGCA.language-def*]
lemmas *union'-nodes-finite* = *union.product-nodes-finite*[folded *DGCA.nodes-def*]
lemmas *union'-nodes-card* = *union.product-nodes-card*[folded *DGCA.nodes-def*]

global-interpretation *intersection-list: automaton-intersection-list-run*
dca.dca dca.alphabet dca.initial dca.transition dca.rejecting $\lambda P w r p. \text{fins } P (p \# \# r)$
dca.dca dca.alphabet dca.initial dca.transition dca.rejecting $\lambda P w r p. \text{fins } P (p \# \# r)$
 $\lambda cs pp. \exists k < \text{length } cs. (cs ! k) (pp ! k)$
defines *intersect-list* = *intersection-list.product*
 $\langle \text{proof} \rangle$

lemmas *intersect-list-language* = *intersection-list.product-language*
lemmas *intersect-list-nodes-finite* = *intersection-list.product-nodes-finite*
lemmas *intersect-list-nodes-card* = *intersection-list.product-nodes-card*

global-interpretation *union-list: automaton-union-list-run*
dca.dca dca.alphabet dca.initial dca.transition dca.rejecting $\lambda P w r p. \text{fins } P (p \text{ \#\# } r)$
dgca.dgca dgca.alphabet dgca.initial dgca.transition dgca.rejecting $\lambda P w r p. \text{cogen fins } P (p \text{ \#\# } r)$
 $\lambda cs. \text{map } (\lambda k pp. (cs ! k) (pp ! k)) [0 ..< \text{length } cs]$
defines *union-list'* = *union-list.product*
 $\langle \text{proof} \rangle$

lemmas *union-list'-language[simp]* = *union-list.product-language[folded DGCA.language-def]*
lemmas *union-list'-nodes-finite* = *union-list.product-nodes-finite[folded DGCA.nodes-def]*
lemmas *union-list'-nodes-card* = *union-list.product-nodes-card[folded DGCA.nodes-def]*

abbreviation *union where union A B* $\equiv \text{degeneralize } (\text{union}' A B)$

lemma *union-language[simp]*:

assumes *dca.alphabet A = dca.alphabet B*

shows *DCA.language (union A B) = DCA.language A \cup DCA.language B*

$\langle \text{proof} \rangle$

lemma *union-nodes-finite*:

assumes *finite (DCA.nodes A) finite (DCA.nodes B)*

shows *finite (DCA.nodes (union A B))*

$\langle \text{proof} \rangle$

lemma *union-nodes-card*:

assumes *finite (DCA.nodes A) finite (DCA.nodes B)*

shows *card (DCA.nodes (union A B)) $\leq 2 * \text{card } (DCA.nodes A) * \text{card } (DCA.nodes B)$*

$\langle \text{proof} \rangle$

abbreviation *union-list where union-list AA* $\equiv \text{degeneralize } (\text{union-list}' AA)$

lemma *union-list-language[simp]*:

assumes $\bigcap (dca.alphabet \text{ ' set } AA) = \bigcup (dca.alphabet \text{ ' set } AA)$

shows *DCA.language (union-list AA) = $\bigcup (DCA.language \text{ ' set } AA)$*

$\langle \text{proof} \rangle$

lemma *union-list-nodes-finite*:

assumes *list-all (finite \circ DCA.nodes) AA*

shows *finite (DCA.nodes (union-list AA))*

$\langle \text{proof} \rangle$

lemma *union-list-nodes-card*:

assumes *list-all (finite \circ DCA.nodes) AA*

shows *card (DCA.nodes (union-list AA)) $\leq \text{max } 1 (\text{length } AA) * \text{prod-list } (\text{map } (\text{card } \circ \text{DCA.nodes}) AA)$*

$\langle \text{proof} \rangle$

end

23 Deterministic Rabin Automata

```

theory DRA
imports ../Deterministic
begin

  datatype ('label, 'state) dra = dra
    (alphabet: 'label set)
    (initial: 'state)
    (transition: 'label  $\Rightarrow$  'state  $\Rightarrow$  'state)
    (condition: 'state rabin gen)

  global-interpretation dra: automaton dra alphabet initial transition condition
    defines path = dra.path and run = dra.run and reachable = dra.reachable
and nodes = dra.nodes
    <proof>
  global-interpretation dra: automaton-run dra alphabet initial transition condi-
    tion  $\lambda P w r p$ . cogen rabin P (p ## r)
    defines language = dra.language
    <proof>

  abbreviation target where target  $\equiv$  dra.target
  abbreviation states where states  $\equiv$  dra.states
  abbreviation trace where trace  $\equiv$  dra.trace
  abbreviation successors where successors  $\equiv$  dra.successors TYPE('label)

end

```

24 Deterministic Rabin Automata Combinations

```

theory DRA-Combine
imports DRA ../DBA/DBA ../DCA/DCA
begin

  global-interpretation intersection-bc: automaton-intersection-run
    dba.dba dba.alphabet dba.initial dba.transition dba.accepting  $\lambda P w r p$ . infs P
    (p ## r)
    dca.dca dca.alphabet dca.initial dca.transition dca.rejecting  $\lambda P w r p$ . fins P (p
    ## r)
    dra.dra dra.alphabet dra.initial dra.transition dra.condition  $\lambda P w r p$ . cogen
    rabin P (p ## r)
     $\lambda c_1 c_2$ . [(c1  $\circ$  fst, c2  $\circ$  snd)]
    defines intersect-bc = intersection-bc.product
    <proof>

  lemmas intersect-bc-language[simp] = intersection-bc.product-language[folded DCA.language-def
    DRA.language-def]
  lemmas intersect-bc-nodes-finite = intersection-bc.product-nodes-finite[folded DCA.nodes-def]

```

DRA.nodes-def]
lemmas *intersect-bc-nodes-card = intersection-bc.product-nodes-card*[*folded DCA.nodes-def*
DRA.nodes-def]

global-interpretation *union-list: automaton-union-list-run*
dra.dra dra.alphabet dra.initial dra.transition dra.condition $\lambda P w r p$. *cogen*
rabin P (p ## r)
dra.dra dra.alphabet dra.initial dra.transition dra.condition $\lambda P w r p$. *cogen*
rabin P (p ## r)
 λcs . *do* { $k \leftarrow [0 ..< \text{length } cs]$; $(f, g) \leftarrow cs ! k$; $[(\lambda pp. f (pp ! k), \lambda pp. g (pp$
 $! k))]$ }
defines *union-list = union-list.product*
 $\langle \text{proof} \rangle$

lemmas *union-list-language = union-list.product-language*
lemmas *union-list-nodes-finite = union-list.product-nodes-finite*
lemmas *union-list-nodes-card = union-list.product-nodes-card*

end

25 Relations and Refinement

theory *Refine*

imports

Automatic-Refinement.Automatic-Refinement

Refine-Monadic.Refine-Foreach

Sequence-LTL

Maps

begin

25.1 Predicate to Set Conversion Setup

lemma *right-unique-pred-set-conv*[*pred-set-conv*]: *right-unique = single-valuedp*
 $\langle \text{proof} \rangle$

lemma *bi-unique-pred-set-conv*[*pred-set-conv*]: *bi-unique* $(\lambda x y. (x, y) \in R) \longleftrightarrow$
bijective R
 $\langle \text{proof} \rangle$

useful for unfolding equality constants in theorems about predicates

lemma *pred-Id*: *HOL.eq* = $(\lambda x y. (x, y) \in \text{Id})$ $\langle \text{proof} \rangle$

lemma *pred-bool-Id*: *HOL.eq* = $(\lambda x y. (x, y) \in (\text{Id} :: \text{bool rel}))$ $\langle \text{proof} \rangle$

lemma *pred-nat-Id*: *HOL.eq* = $(\lambda x y. (x, y) \in (\text{Id} :: \text{nat rel}))$ $\langle \text{proof} \rangle$

lemma *pred-set-Id*: *HOL.eq* = $(\lambda x y. (x, y) \in (\text{Id} :: 'a \text{ set rel}))$ $\langle \text{proof} \rangle$

lemma *pred-list-Id*: *HOL.eq* = $(\lambda x y. (x, y) \in (\text{Id} :: 'a \text{ list rel}))$ $\langle \text{proof} \rangle$

lemma *pred-stream-Id*: *HOL.eq* = $(\lambda x y. (x, y) \in (\text{Id} :: 'a \text{ stream rel}))$ $\langle \text{proof} \rangle$

lemma *eq-onp-Id-on-eq*[*pred-set-conv*]: $eq\text{-onp } (\lambda a. a \in A) = (\lambda x y. (x, y) \in Id\text{-on } A)$

<proof>

lemma *rel-fun-fun-rel-eq*[*pred-set-conv*]:

$rel\text{-fun } (\lambda x y. (x, y) \in A) (\lambda x y. (x, y) \in B) = (\lambda f g. (f, g) \in A \rightarrow B)$

<proof>

lemma *rel-prod-prod-rel-eq*[*pred-set-conv*]:

$rel\text{-prod } (\lambda x y. (x, y) \in A) (\lambda x y. (x, y) \in B) = (\lambda f g. (f, g) \in A \times_r B)$

<proof>

lemma *rel-sum-sum-rel-eq*[*pred-set-conv*]:

$rel\text{-sum } (\lambda x y. (x, y) \in A) (\lambda x y. (x, y) \in B) = (\lambda f g. (f, g) \in \langle A, B \rangle \text{ sum-rel})$

<proof>

lemma *rel-set-set-rel-eq*[*pred-set-conv*]:

$rel\text{-set } (\lambda x y. (x, y) \in A) = (\lambda f g. (f, g) \in \langle A \rangle \text{ set-rel})$

<proof>

lemma *rel-option-option-rel-eq*[*pred-set-conv*]:

$rel\text{-option } (\lambda x y. (x, y) \in A) = (\lambda f g. (f, g) \in \langle A \rangle \text{ option-rel})$

<proof>

thm *image-transfer image-transfer*[*to-set*]

thm *fun-upd-transfer fun-upd-transfer*[*to-set*]

25.2 Relation Composition

lemma *relcomp-trans-1*[*trans*]:

assumes $(f, g) \in A_1$

assumes $(g, h) \in A_2$

shows $(f, h) \in A_1 \circ A_2$

<proof>

lemma *relcomp-trans-2*[*trans*]:

assumes $(f, g) \in A_1 \rightarrow B_1$

assumes $(g, h) \in A_2 \rightarrow B_2$

shows $(f, h) \in A_1 \circ A_2 \rightarrow B_1 \circ B_2$

<proof>

lemma *relcomp-trans-3*[*trans*]:

assumes $(f, g) \in A_1 \rightarrow B_1 \rightarrow C_1$

assumes $(g, h) \in A_2 \rightarrow B_2 \rightarrow C_2$

shows $(f, h) \in A_1 \circ A_2 \rightarrow B_1 \circ B_2 \rightarrow C_1 \circ C_2$

<proof>

lemma *relcomp-trans-4*[*trans*]:

assumes $(f, g) \in A_1 \rightarrow B_1 \rightarrow C_1 \rightarrow D_1$

assumes $(g, h) \in A_2 \rightarrow B_2 \rightarrow C_2 \rightarrow D_2$

shows $(f, h) \in A_1 \circ A_2 \rightarrow B_1 \circ B_2 \rightarrow C_1 \circ C_2 \rightarrow D_1 \circ D_2$

<proof>

lemma *relcomp-trans-5*[*trans*]:

assumes $(f, g) \in A_1 \rightarrow B_1 \rightarrow C_1 \rightarrow D_1 \rightarrow E_1$

assumes $(g, h) \in A_2 \rightarrow B_2 \rightarrow C_2 \rightarrow D_2 \rightarrow E_2$

shows $(f, h) \in A_1 \circ A_2 \rightarrow B_1 \circ B_2 \rightarrow C_1 \circ C_2 \rightarrow D_1 \circ D_2 \rightarrow E_1 \circ E_2$

<proof>

25.3 Relation Basics

lemma *inv-fun-rel-eq[simp]*: $(A \rightarrow B)^{-1} = A^{-1} \rightarrow B^{-1}$

<proof>

lemma *inv-option-rel-eq[simp]*: $(\langle K \rangle \text{option-rel})^{-1} = \langle K^{-1} \rangle \text{option-rel}$

<proof>

lemma *inv-prod-rel-eq[simp]*: $(P \times_r Q)^{-1} = P^{-1} \times_r Q^{-1}$

<proof>

lemma *inv-sum-rel-eq[simp]*: $(\langle P, Q \rangle \text{sum-rel})^{-1} = \langle P^{-1}, Q^{-1} \rangle \text{sum-rel}$

<proof>

lemma *set-rel-converse[simp]*: $(\langle A \rangle \text{set-rel})^{-1} = \langle A^{-1} \rangle \text{set-rel}$ *<proof>*

lemma *build-rel-domain[simp]*: $\text{Domain } (br \ \alpha \ I) = \text{Collect } I$ *<proof>*

lemma *build-rel-range[simp]*: $\text{Range } (br \ \alpha \ I) = \alpha \ ` \ \text{Collect } I$ *<proof>*

lemma *build-rel-image[simp]*: $br \ \alpha \ I \ \text{`` } A = \alpha \ ` \ (A \cap \text{Collect } I)$ *<proof>*

lemma *prod-rel-domain[simp]*: $\text{Domain } (A \times_r B) = \text{Domain } A \times \text{Domain } B$
<proof>

lemma *prod-rel-range[simp]*: $\text{Range } (A \times_r B) = \text{Range } A \times \text{Range } B$ *<proof>*

lemma *member-Id-on[iff]*: $(x, y) \in \text{Id-on } A \iff x = y \wedge y \in A$ *<proof>*

lemma *bijjective-Id-on[intro!, simp]*: *bijjective* $(\text{Id-on } A)$ *<proof>*

lemma *relcomp-Id-on[simp]*: $\text{Id-on } A \ O \ \text{Id-on } B = \text{Id-on } (A \cap B)$ *<proof>*

lemma *prod-rel-Id-on[simp]*: $\text{Id-on } A \times_r \text{Id-on } B = \text{Id-on } (A \times B)$ *<proof>*

lemma *set-rel-Id-on[simp]*: $\langle \text{Id-on } S \rangle \text{set-rel} = \text{Id-on } (\text{Pow } S)$ *<proof>*

25.4 Parametricity

lemmas *basic-param[param]* =

option.rel-transfer[unfolded pred-bool-Id, to-set]

All-transfer[unfolded pred-bool-Id, to-set]

Ex-transfer[unfolded pred-bool-Id, to-set]

Union-transfer[to-set]

image-transfer[to-set]

Image-parametric[to-set]

lemma *Sigma-param[param]*: $(\text{Sigma}, \text{Sigma}) \in \langle A \rangle \text{set-rel} \rightarrow (A \rightarrow \langle B \rangle \text{set-rel})$
 $\rightarrow \langle A \times_r B \rangle \text{set-rel}$
<proof>

lemma *set-filter-param[param]*:

$(\text{Set.filter}, \text{Set.filter}) \in (A \rightarrow \text{bool-rel}) \rightarrow \langle A \rangle \text{set-rel} \rightarrow \langle A \rangle \text{set-rel}$

<proof>

lemma *is-singleton-param[param]*:

assumes *bijjective* A

shows $(\text{is-singleton}, \text{is-singleton}) \in \langle A \rangle \text{set-rel} \rightarrow \text{bool-rel}$

<proof>

lemma *the-elem-param*[*param*]:
assumes *is-singleton S is-singleton T*
assumes $(S, T) \in \langle A \rangle$ *set-rel*
shows $(\text{the-elem } S, \text{the-elem } T) \in A$
 $\langle \text{proof} \rangle$

25.5 Lists

lemma *list-all2-list-rel-conv*[*pred-set-conv*]:
 $\text{list-all2 } (\lambda x y. (x, y) \in R) = (\lambda x y. (x, y) \in \langle R \rangle)$ *list-rel*
 $\langle \text{proof} \rangle$

lemmas *list-rel-single-valued*[*iff*] = *list-rel-sv-iff*

lemmas *list-rel-simps*[*simp*] =
 list.rel-eq-onp [*to-set*]
 $\text{list.rel-conversep}$ [*to-set, symmetric*]
 list.rel-compp [*to-set*]

lemmas *list-rel-param*[*param*] =
 list.set-transfer [*to-set*]
 $\text{list.pred-transfer}$ [*unfolded pred-bool-Id, to-set, folded pred-list-listsp*]
 list.rel-transfer [*unfolded pred-bool-Id, to-set*]

lemmas *null-param*[*param*] = *null-transfer*[*unfolded pred-bool-Id, to-set*]

thm *param-set list.set-transfer*[*to-set*]

lemmas *scan-param*[*param*] = *scan.transfer*[*to-set*]
lemma *bind-param*[*param*]: $(\text{List.bind}, \text{List.bind}) \in \langle A \rangle$ *list-rel* $\rightarrow (A \rightarrow \langle B \rangle)$
list-rel $\rightarrow \langle B \rangle$ *list-rel*
 $\langle \text{proof} \rangle$

lemma *set-id-param*[*param*]: $(\text{set}, \text{id}) \in \langle A \rangle$ *list-set-rel* $\rightarrow \langle A \rangle$ *set-rel*
 $\langle \text{proof} \rangle$

25.6 Streams

definition *stream-rel* :: $('a \times 'b)$ *set* $\Rightarrow ('a$ *stream* $\times 'b$ *stream)* *set* **where**
 $[\text{to-relAPP}]$: $\text{stream-rel } R \equiv \{(x, y). \text{stream-all2 } (\lambda x y. (x, y) \in R) x y\}$

lemma *stream-all2-stream-rel-conv*[*pred-set-conv*]:
 $\text{stream-all2 } (\lambda x y. (x, y) \in R) = (\lambda x y. (x, y) \in \langle R \rangle)$ *stream-rel*
 $\langle \text{proof} \rangle$

lemmas *stream-rel-coinduct'*[*case-names stream-rel, coinduct set: stream-rel*] =
 $\text{stream-rel-coinduct}$ [*to-set*]

lemmas *stream-rel-intros* = *stream.rel-intros*[*to-set*]

lemmas *stream-rel-cases* = *stream.rel-cases*[*to-set*]
lemmas *stream-rel-inject*[*iff*] = *stream.rel-inject*[*to-set*]

lemma *stream-rel-single-valued*[*iff*]: *single-valued* ($\langle A \rangle$ *stream-rel*) \longleftrightarrow *single-valued* *A*
 \langle *proof* \rangle

lemmas *stream-rel-simps*[*simp*] =
stream.rel-eq[*unfolded pred-Id, THEN IdD, to-set*]
stream.rel-eq-onp[*to-set*]
stream.rel-conversep[*to-set*]
stream.rel-compp[*to-set*]

lemmas *stream-rel-param*[*param*] =
stream.ctr-transfer[*to-set*]
stream.sel-transfer[*to-set*]
stream.pred-transfer[*unfolded pred-bool-Id, to-set, folded pred-stream-streamsp*]
stream.rel-transfer[*unfolded pred-bool-Id, to-set*]
stream.map-transfer[*to-set*]
stream.set-transfer[*to-set*]
stream.case-transfer[*to-set*]
stream.corec-transfer[*unfolded pred-bool-Id, to-set*]

lemma *stream-Rangep-rel*: *Rangep* (*stream-all2* *R*) = *pred-stream* (*Rangep* *R*)
 \langle *proof* \rangle

lemmas *stream-rel-domain*[*simp*] = *stream.Domainp-rel*[*to-set*]
lemmas *stream-rel-range*[*simp*] = *stream-Rangep-rel*[*to-set*]

lemma *stream-param*[*param*]:
assumes (*HOL.eq, HOL.eq*) $\in R \rightarrow R \rightarrow \text{bool-rel}$
shows (*HOL.eq, HOL.eq*) $\in \langle R \rangle$ *stream-rel* $\rightarrow \langle R \rangle$ *stream-rel* $\rightarrow \text{bool-rel}$
 \langle *proof* \rangle

lemmas *szip-param*[*param*] = *szip-transfer*[*to-set*]
lemmas *siterate-param*[*param*] = *siterate-transfer*[*to-set*]
lemmas *sscan-param*[*param*] = *sscan.transfer*[*to-set*]

lemma *streams-param*[*param*]: (*streams, streams*) $\in \langle A \rangle$ *set-rel* $\rightarrow \langle \langle A \rangle$ *stream-rel* \rangle
set-rel
 \langle *proof* \rangle

lemma *holds-param*[*param*]: (*holds, holds*) $\in (A \rightarrow \text{bool-rel}) \rightarrow \langle \langle A \rangle$ *stream-rel* \rangle
 $\rightarrow \text{bool-rel}$
 \langle *proof* \rangle

lemma *HLD-param*[*param*]:
assumes *single-valued A single-valued* (A^{-1})

shows $(HLD, HLD) \in \langle A \rangle \text{ set-rel} \rightarrow \langle A \rangle \text{ stream-rel} \rightarrow \text{bool-rel}$
<proof>

lemma $ev\text{-param}[param]: (ev, ev) \in (\langle A \rangle \text{ stream-rel} \rightarrow \text{bool-rel}) \rightarrow (\langle A \rangle \text{ stream-rel} \rightarrow \text{bool-rel})$
<proof>

lemma $alw\text{-param}[param]: (alw, alw) \in (\langle A \rangle \text{ stream-rel} \rightarrow \text{bool-rel}) \rightarrow (\langle A \rangle \text{ stream-rel} \rightarrow \text{bool-rel})$
<proof>

25.7 Functional Relations

lemma $br\text{-set-rel}: \langle br\ f\ P \rangle \text{ set-rel} = br\ (image\ f)\ (\lambda\ A.\ Ball\ A\ P)$
<proof>

lemma $br\text{-list-rel}: \langle br\ f\ P \rangle \text{ list-rel} = br\ (map\ f)\ (list\text{-all}\ P)$
<proof>

lemma $br\text{-list-set-rel}: \langle br\ f\ P \rangle \text{ list-set-rel} = br\ (set\ \circ\ map\ f)\ (\lambda\ s.\ list\text{-all}\ P\ s\ \wedge\ distinct\ (map\ f\ s))$
<proof>

lemma $br\text{-fun-rel1}: Id \rightarrow br\ f\ P = br\ (comp\ f)\ (All\ \circ\ comp\ P)$
<proof>

term $set\ \circ\ map\ f\ \circ\ map\ g\ \circ\ map\ h$

term $set\ \circ\ sort$

end

theory *Acceptance-Refine*

imports *Acceptance Refine*

begin

abbreviation $(input)\ pred\text{-rel}\ A \equiv A \rightarrow \text{bool-rel}$

abbreviation $(input)\ rabin\text{-rel}\ A \equiv pred\text{-rel}\ A \times_r pred\text{-rel}\ A$

lemma $rabin\text{-param}[param]: (rabin, rabin) \in rabin\text{-rel}\ A \rightarrow pred\text{-rel}\ (\langle A \rangle \text{ stream-rel})$
<proof>

lemma $gen\text{-param}[param]: (gen, gen) \in (A \rightarrow pred\text{-rel}\ B) \rightarrow (\langle A \rangle \text{ list-rel} \rightarrow pred\text{-rel}\ B)$

$\langle proof \rangle$
lemma *cogen-param*[*param*]: $(cogen, cogen) \in (A \rightarrow pred\text{-}rel\ B) \rightarrow (\langle A \rangle list\text{-}rel \rightarrow pred\text{-}rel\ B)$
 $\langle proof \rangle$

end

26 Refinement for Transition Systems

theory *Transition-System-Refine*

imports

Transition-System

Transition-System-Extra

../Basic/Refine

begin

lemma *path-param*[*param*]: $(transition\text{-}system.path, transition\text{-}system.path) \in (T \rightarrow S \rightarrow S) \rightarrow (T \rightarrow S \rightarrow bool\text{-}rel) \rightarrow \langle T \rangle list\text{-}rel \rightarrow S \rightarrow bool\text{-}rel$
 $\langle proof \rangle$

lemma *run-param*[*param*]: $(transition\text{-}system.run, transition\text{-}system.run) \in (T \rightarrow S \rightarrow S) \rightarrow (T \rightarrow S \rightarrow bool\text{-}rel) \rightarrow \langle T \rangle stream\text{-}rel \rightarrow S \rightarrow bool\text{-}rel$
 $\langle proof \rangle$

lemma *paths-param*[*param*]:
assumes [*param*]: $(exa, exb) \in T \rightarrow S \rightarrow S$
assumes $(transition\text{-}system.enableds\ ena, transition\text{-}system.enableds\ enb) \in S \rightarrow \langle T \rangle set\text{-}rel$
shows $(transition\text{-}system.paths\ exa\ ena, transition\text{-}system.paths\ exb\ enb) \in S \rightarrow \langle \langle T \rangle list\text{-}rel \rangle set\text{-}rel$
 $\langle proof \rangle$

lemma *runs-param*[*param*]:
assumes $(exa, exb) \in T \rightarrow S \rightarrow S$
assumes $(transition\text{-}system.enableds\ ena, transition\text{-}system.enableds\ enb) \in S \rightarrow \langle T \rangle set\text{-}rel$
shows $(transition\text{-}system.runs\ exa\ ena, transition\text{-}system.runs\ exb\ enb) \in S \rightarrow \langle \langle T \rangle stream\text{-}rel \rangle set\text{-}rel$
 $\langle proof \rangle$

end

27 Relations on Deterministic Rabin Automata

theory *DRA-Refine*

imports

DRA

../Basic/Acceptance-Refine

../Transition-Systems/Transition-System-Refine

begin

definition $\text{dra-rel} :: ('label_1 \times 'label_2) \text{ set} \Rightarrow ('state_1 \times 'state_2) \text{ set} \Rightarrow$
 $(('label_1, 'state_1) \text{ dra} \times ('label_2, 'state_2) \text{ dra}) \text{ set}$ **where**
 $[\text{to-relAPP}]: \text{dra-rel } L \ S \equiv \{(A_1, A_2).$
 $(\text{alphabet } A_1, \text{alphabet } A_2) \in \langle L \rangle \text{ set-rel} \wedge$
 $(\text{initial } A_1, \text{initial } A_2) \in S \wedge$
 $(\text{transition } A_1, \text{transition } A_2) \in L \rightarrow S \rightarrow S \wedge$
 $(\text{condition } A_1, \text{condition } A_2) \in \langle \text{rabin-rel } S \rangle \text{ list-rel}\}$

lemma $\text{dra-param}[\text{param}]:$
 $(\text{dra}, \text{dra}) \in \langle L \rangle \text{ set-rel} \rightarrow S \rightarrow (L \rightarrow S \rightarrow S) \rightarrow \langle \text{rabin-rel } S \rangle \text{ list-rel} \rightarrow$
 $\langle L, S \rangle \text{ dra-rel}$
 $(\text{alphabet}, \text{alphabet}) \in \langle L, S \rangle \text{ dra-rel} \rightarrow \langle L \rangle \text{ set-rel}$
 $(\text{initial}, \text{initial}) \in \langle L, S \rangle \text{ dra-rel} \rightarrow S$
 $(\text{transition}, \text{transition}) \in \langle L, S \rangle \text{ dra-rel} \rightarrow L \rightarrow S \rightarrow S$
 $(\text{condition}, \text{condition}) \in \langle L, S \rangle \text{ dra-rel} \rightarrow \langle \text{rabin-rel } S \rangle \text{ list-rel}$
 $\langle \text{proof} \rangle$

lemma $\text{dra-rel-id}[\text{simp}]: \langle \text{Id}, \text{Id} \rangle \text{ dra-rel} = \text{Id} \langle \text{proof} \rangle$

lemma $\text{dra-rel-comp}[\text{trans}]:$

assumes $[\text{param}]: (A, B) \in \langle L_1, S_1 \rangle \text{ dra-rel} (B, C) \in \langle L_2, S_2 \rangle \text{ dra-rel}$
shows $(A, C) \in \langle L_1 \ O \ L_2, S_1 \ O \ S_2 \rangle \text{ dra-rel}$

$\langle \text{proof} \rangle$

lemma $\text{dra-rel-converse}[\text{simp}]: (\langle L, S \rangle \text{ dra-rel})^{-1} = \langle L^{-1}, S^{-1} \rangle \text{ dra-rel}$

$\langle \text{proof} \rangle$

lemma $\text{dra-rel-eq}: (A, A) \in \langle \text{Id-on } (\text{alphabet } A), \text{Id-on } (\text{nodes } A) \rangle \text{ dra-rel}$
 $\langle \text{proof} \rangle$

lemma $\text{enableds-param}[\text{param}]: (\text{dra.enableds}, \text{dra.enableds}) \in \langle L, S \rangle \text{ dra-rel} \rightarrow$
 $S \rightarrow \langle L \rangle \text{ set-rel}$
 $\langle \text{proof} \rangle$

lemma $\text{paths-param}[\text{param}]: (\text{dra.paths}, \text{dra.paths}) \in \langle L, S \rangle \text{ dra-rel} \rightarrow S \rightarrow \langle \langle L \rangle$
 $\text{list-rel} \rangle \text{ set-rel}$

$\langle \text{proof} \rangle$

lemma $\text{runs-param}[\text{param}]: (\text{dra.runs}, \text{dra.runs}) \in \langle L, S \rangle \text{ dra-rel} \rightarrow S \rightarrow \langle \langle L \rangle$
 $\text{stream-rel} \rangle \text{ set-rel}$

$\langle \text{proof} \rangle$

lemma $\text{reachable-param}[\text{param}]: (\text{reachable}, \text{reachable}) \in \langle L, S \rangle \text{ dra-rel} \rightarrow S \rightarrow$
 $\langle S \rangle \text{ set-rel}$

$\langle \text{proof} \rangle$

lemma $\text{nodes-param}[\text{param}]: (\text{nodes}, \text{nodes}) \in \langle L, S \rangle \text{ dra-rel} \rightarrow \langle S \rangle \text{ set-rel}$

$\langle \text{proof} \rangle$

lemma $\text{language-param}[\text{param}]: (\text{language}, \text{language}) \in \langle L, S \rangle \text{ dra-rel} \rightarrow \langle \langle L \rangle$
 $\text{stream-rel} \rangle \text{ set-rel}$

$\langle \text{proof} \rangle$

end

28 Implementation

```
theory Implement
imports
  HOL-Library.Monad-Syntax
  Collections.Refine-Dflt
  Refine
begin
```

28.1 Syntax

```
no-syntax -do-let :: [pttrn, 'a]  $\Rightarrow$  do-bind (( $\lambda$ let - =/ -) [1000, 13] 13)
syntax -do-let :: [pttrn, 'a]  $\Rightarrow$  do-bind (( $\lambda$ let - =/ -) 13)
```

28.2 Monadic Refinement

```
lemmas [refine] = plain-nres-relI
```

```
lemma vcg0:
  assumes  $(f, g) \in \langle Id \rangle$  nres-rel
  shows  $g \leq h \Longrightarrow f \leq h$ 
   $\langle$ proof $\rangle$ 
```

```
lemma vcg1:
  assumes  $(f, g) \in Id \rightarrow \langle Id \rangle$  nres-rel
  shows  $g\ x \leq h\ x \Longrightarrow f\ x \leq h\ x$ 
   $\langle$ proof $\rangle$ 
```

```
lemma vcg2:
  assumes  $(f, g) \in Id \rightarrow Id \rightarrow \langle Id \rangle$  nres-rel
  shows  $g\ x\ y \leq h\ x\ y \Longrightarrow f\ x\ y \leq h\ x\ y$ 
   $\langle$ proof $\rangle$ 
```

```
lemma RETURN-nres-relD:
  assumes  $(RETURN\ x, RETURN\ y) \in \langle A \rangle$  nres-rel
  shows  $(x, y) \in A$ 
   $\langle$ proof $\rangle$ 
```

```
lemma FOREACH-rule-insert:
  assumes finite S
  assumes  $I\ \{\} s$ 
  assumes  $\bigwedge s. I\ S\ s \Longrightarrow P\ s$ 
  assumes  $\bigwedge T\ x\ s. T \subseteq S \Longrightarrow I\ T\ s \Longrightarrow x \in S \Longrightarrow x \notin T \Longrightarrow f\ x\ s \leq SPEC$ 
  ( $I$  (insert  $x\ T$ ))
  shows FOREACH  $S\ f\ s \leq SPEC\ P$ 
   $\langle$ proof $\rangle$ 
```

```
lemma FOREACH-rule-map:
  assumes finite (dom  $g$ )
  assumes  $I\ Map.empty\ s$ 
```


assumes $\bigwedge s. I g s \implies P s$
assumes $\bigwedge h k v s. h \subseteq_m g \implies I h s \implies g k = \text{Some } v \implies k \notin \text{dom } h \implies$
 $f (k, v) s \leq \text{SPEC } (I (h (k \mapsto v)))$
shows $\text{FOREACH } (\text{map-to-set } g) f s \leq \text{SPEC } P$
 $\langle \text{proof} \rangle$
lemma *FOREACH-rule-insert-eq*:
assumes *finite* S
assumes $X \{\} = s$
assumes $X S = t$
assumes $\bigwedge T x. T \subseteq S \implies x \in S \implies x \notin T \implies f x (X T) \leq \text{SPEC } (\text{HOL.eq } (X (\text{insert } x T)))$
shows $\text{FOREACH } S f s \leq \text{SPEC } (\text{HOL.eq } t)$
 $\langle \text{proof} \rangle$
lemma *FOREACH-rule-map-eq*:
assumes *finite* $(\text{dom } g)$
assumes $X \text{Map.empty} = s$
assumes $X g = t$
assumes $\bigwedge h k v. h \subseteq_m g \implies g k = \text{Some } v \implies k \notin \text{dom } h \implies$
 $f (k, v) (X h) \leq \text{SPEC } (\text{HOL.eq } (X (h (k \mapsto v))))$
shows $\text{FOREACH } (\text{map-to-set } g) f s \leq \text{SPEC } (\text{HOL.eq } t)$
 $\langle \text{proof} \rangle$

lemma *FOREACH-rule-map-map*: $(\text{FOREACH } (\text{map-to-set } m) (\lambda (k, v). F k (f k v)),$
 $\text{FOREACH } (\text{map-to-set } (\lambda k. \text{map-option } (f k) (m k))) (\lambda (k, v). F k v)) \in \text{Id}$
 $\rightarrow \langle \text{Id} \rangle \text{nres-rel}$
 $\langle \text{proof} \rangle$

28.3 Implementations for Sets Represented by Lists

lemma *list-set-rel-Id-on[simp]*: $\langle \text{Id-on } A \rangle \text{list-set-rel} = \langle \text{Id} \rangle \text{list-set-rel} \cap \text{UNIV}$
 $\times \text{Pow } A$
 $\langle \text{proof} \rangle$

lemma *list-set-card[param]*: $(\text{length}, \text{card}) \in \langle A \rangle \text{list-set-rel} \rightarrow \text{nat-rel}$
 $\langle \text{proof} \rangle$

lemma *list-set-insert[param]*:
assumes $y \notin Y$
assumes $(x, y) \in A (xs, Y) \in \langle A \rangle \text{list-set-rel}$
shows $(x \# xs, \text{insert } y Y) \in \langle A \rangle \text{list-set-rel}$
 $\langle \text{proof} \rangle$

lemma *list-set-union[param]*:
assumes $X \cap Y = \{\}$
assumes $(xs, X) \in \langle A \rangle \text{list-set-rel } (ys, Y) \in \langle A \rangle \text{list-set-rel}$
shows $(xs @ ys, X \cup Y) \in \langle A \rangle \text{list-set-rel}$
 $\langle \text{proof} \rangle$

lemma *list-set-Union[param]*:
assumes $\bigwedge X Y. X \in S \implies Y \in S \implies X \neq Y \implies X \cap Y = \{\}$
assumes $(xs, S) \in \langle \langle A \rangle \text{list-set-rel} \rangle \text{list-set-rel}$

shows ($\text{concat } xs, \text{Union } S$) $\in \langle A \rangle \text{ list-set-rel}$
 $\langle \text{proof} \rangle$
lemma $\text{list-set-image}[param]$:
assumes $\text{inj-on } g \ S$
assumes $(f, g) \in A \rightarrow B$ $(xs, S) \in \langle A \rangle \text{ list-set-rel}$
shows $(\text{map } f \ xs, g \ ' \ S) \in \langle B \rangle \text{ list-set-rel}$
 $\langle \text{proof} \rangle$
lemma $\text{list-set-bind}[param]$:
assumes $\bigwedge x \ y. x \in S \implies y \in S \implies x \neq y \implies g \ x \cap g \ y = \{\}$
assumes $(xs, S) \in \langle A \rangle \text{ list-set-rel}$ $(f, g) \in A \rightarrow \langle B \rangle \text{ list-set-rel}$
shows $(xs \gg= f, S \gg= g) \in \langle B \rangle \text{ list-set-rel}$
 $\langle \text{proof} \rangle$

28.4 Autoref Setup

lemma $\text{dflt-ahm-rel-finite-nat}$: $\text{finite-map-rel} (\langle \text{nat-rel}, V \rangle \text{dflt-ahm-rel}) \langle \text{proof} \rangle$

context
begin

interpretation $\text{autoref-syn} \langle \text{proof} \rangle$

lemma $[\text{autoref-op-pat}]$: $(\text{Some } \circ f) \mid ' X \equiv OP (\lambda f \ X. (\text{Some } \circ f) \mid ' X) f \ X$
 $\langle \text{proof} \rangle$

lemma $[\text{autoref-op-pat}]$: $\bigcup (m \ ' \ S) \equiv OP (\lambda S \ m. \bigcup (m \ ' \ S)) S \ m \langle \text{proof} \rangle$

definition gen-UNION **where**

$\text{gen-UNION } \text{tol } \text{emp } \text{un } X \ f \equiv \text{fold} (\text{un } \circ f) (\text{tol } X) \ \text{emp}$

lemma $\text{gen-UNION}[\text{autoref-rules-raw}]$:

assumes PRIO-TAG-GEN-ALGO

assumes to-list : $\text{SIDE-GEN-ALGO} (\text{is-set-to-list } A \ Rs1 \ \text{tol})$

assumes empty : $\text{GEN-OP } \text{emp} \ \{\} (\langle B \rangle \ Rs3)$

assumes union : $\text{GEN-OP } \text{un } \text{union} (\langle B \rangle \ Rs2 \rightarrow \langle B \rangle \ Rs3 \rightarrow \langle B \rangle \ Rs3)$

shows $(\text{gen-UNION } \text{tol } \text{emp } \text{un}, \lambda A \ f. \bigcup (f \ ' \ A)) \in \langle A \rangle \ Rs1 \rightarrow (A \rightarrow \langle B \rangle \ Rs2) \rightarrow \langle B \rangle \ Rs3$
 $\langle \text{proof} \rangle$

definition gen-Image **where**

$\text{gen-Image } \text{tol1 } \text{mem2 } \text{emp3 } \text{ins3 } X \ Y \equiv \text{fold}$

$(\lambda (a, b). \text{if } \text{mem2 } a \ Y \ \text{then } \text{ins3 } b \ \text{else } \text{id}) (\text{tol1 } X) \ \text{emp3}$

lemma $\text{gen-Image}[\text{autoref-rules}]$:

assumes PRIO-TAG-GEN-ALGO

assumes to-list : $\text{SIDE-GEN-ALGO} (\text{is-set-to-list} (A \times_r B) \ Rs1 \ \text{tol1})$

assumes member : $\text{GEN-OP } \text{mem2} (\in) (A \rightarrow \langle A \rangle \ Rs2 \rightarrow \text{bool-rel})$

assumes empty : $\text{GEN-OP } \text{emp3} \ \{\} (\langle B \rangle \ Rs3)$

assumes insert : $\text{GEN-OP } \text{ins3} \ \text{Set.insert} (B \rightarrow \langle B \rangle \ Rs3 \rightarrow \langle B \rangle \ Rs3)$

shows $(\text{gen-Image } \text{tol1 } \text{mem2 } \text{emp3 } \text{ins3}, \text{Image}) \in \langle A \times_r B \rangle \ Rs1 \rightarrow \langle A \rangle \ Rs2 \rightarrow \langle B \rangle \ Rs3$

$\langle \text{proof} \rangle$

lemma *list-set-union-autoref*[*autoref-rules*]:

assumes *PRIO-TAG-OPTIMIZATION*

assumes *SIDE-PRECOND-OPT* ($a' \cap b' = \{\}$)

assumes $(a, a') \in \langle R \rangle \text{ list-set-rel}$

assumes $(b, b') \in \langle R \rangle \text{ list-set-rel}$

shows $(a @ b,$

$(OP \text{ union} :: \langle R \rangle \text{ list-set-rel} \rightarrow \langle R \rangle \text{ list-set-rel} \rightarrow \langle R \rangle \text{ list-set-rel}) \$ a' \$ b')$

\in

$\langle R \rangle \text{ list-set-rel}$

$\langle \text{proof} \rangle$

lemma *list-set-image-autoref*[*autoref-rules*]:

assumes *PRIO-TAG-OPTIMIZATION*

assumes *INJ: SIDE-PRECOND-OPT* (*inj-on* f s)

assumes $\bigwedge xi x. (xi, x) \in Ra \implies x \in s \implies (fi\ xi, f\ \$\ x) \in Rb$

assumes *LP: (l,s) ∈ ⟨Ra⟩ list-set-rel*

shows (*map* fi $l,$

$(OP \text{ image} :: (Ra \rightarrow Rb) \rightarrow \langle Ra \rangle \text{ list-set-rel} \rightarrow \langle Rb \rangle \text{ list-set-rel}) \$ f \$ s) \in$

$\langle Rb \rangle \text{ list-set-rel}$

$\langle \text{proof} \rangle$

lemma *list-set-UNION-autoref*[*autoref-rules*]:

assumes *PRIO-TAG-OPTIMIZATION*

assumes *SIDE-PRECOND-OPT* ($\forall x \in S. \forall y \in S. x \neq y \longrightarrow g\ x \cap g\ y =$

$\{\}$)

assumes $(xs, S) \in \langle A \rangle \text{ list-set-rel}$ $(f, g) \in A \rightarrow \langle B \rangle \text{ list-set-rel}$

shows $(xs \ggg f,$

$(OP (\lambda A f. \bigcup (f ' A)) :: \langle A \rangle \text{ list-set-rel} \rightarrow (A \rightarrow \langle B \rangle \text{ list-set-rel}) \rightarrow \langle B \rangle$

$\text{list-set-rel}) \$ S \$ g) \in$

$\langle B \rangle \text{ list-set-rel}$

$\langle \text{proof} \rangle$

definition *gen-equals where*

gen-equals ball lu eq f g \equiv

$\text{ball } f (\lambda (k, v). \text{rel-option eq } (lu\ k\ g) (\text{Some } v)) \wedge$

$\text{ball } g (\lambda (k, v). \text{rel-option eq } (lu\ k\ f) (\text{Some } v))$

lemma *gen-equals*[*autoref-rules*]:

assumes *PRIO-TAG-GEN-ALGO*

assumes *BALL: GEN-OP ball op-map-ball* ($\langle Rk, Rv \rangle Rm \rightarrow (Rk \times_r Rv \rightarrow \text{bool-rel}) \rightarrow \text{bool-rel}$)

assumes *LU: GEN-OP lu op-map-lookup* ($Rk \rightarrow \langle Rk, Rv \rangle Rm \rightarrow \langle Rv \rangle \text{option-rel}$)

assumes *EQ: GEN-OP eq HOL.eq* ($Rv \rightarrow Rv \rightarrow \text{bool-rel}$)

shows (*gen-equals ball lu eq, HOL.eq*) $\in \langle Rk, Rv \rangle Rm \rightarrow \langle Rk, Rv \rangle Rm \rightarrow \text{bool-rel}$

$\langle \text{proof} \rangle$

definition *op-set-enumerate* :: 'a set \Rightarrow ('a \rightarrow nat) nres **where**
op-set-enumerate S \equiv SPEC (λ f. dom f = S \wedge inj-on f S)

lemma [*autoref-itype*]: *op-set-enumerate* ::_i $\langle A \rangle_i$ i-set \rightarrow_i $\langle \langle A, i\text{-nat} \rangle_i$ i-map \rangle_i
i-nres \langle proof \rangle

lemma [*autoref-hom*]: CONSTRAINT *op-set-enumerate* ($\langle A \rangle$ Rs \rightarrow $\langle \langle A, \text{nat-rel} \rangle$
Rm \rangle nres-rel) \langle proof \rangle

definition *gen-enumerate* **where**

gen-enumerate tol upd emp S \equiv snd (fold (λ x (k, m). (Suc k, upd x k m))
(tol S) (0, emp))

lemma *gen-enumerate*[*autoref-rules-raw*]:

assumes PRIO-TAG-GEN-ALGO

assumes to-list: SIDE-GEN-ALGO (is-set-to-list A Rs tol)

assumes empty: GEN-OP emp op-map-empty ($\langle A, \text{nat-rel} \rangle$ Rm)

assumes update: GEN-OP upd op-map-update (A \rightarrow nat-rel \rightarrow $\langle A, \text{nat-rel} \rangle$
Rm \rightarrow $\langle A, \text{nat-rel} \rangle$ Rm)

shows (λ S. RETURN (*gen-enumerate* tol upd emp S), *op-set-enumerate*) \in
 $\langle A \rangle$ Rs \rightarrow $\langle \langle A, \text{nat-rel} \rangle$ Rm \rangle nres-rel

\langle proof \rangle

lemma *gen-enumerate-it-to-list*[*refine-transfer-post-simp*]:

gen-enumerate (it-to-list it) =

(λ upd emp S. snd (foldli (it-to-list it) S) (λ -. True)

(λ x s. case s of (k, m) \Rightarrow (Suc k, upd x k m)) (0, emp)))

\langle proof \rangle

definition *gen-build* **where**

gen-build tol upd emp f X \equiv fold (λ x. upd x (f x)) (tol X) emp

lemma *gen-build*[*autoref-rules*]:

assumes PRIO-TAG-GEN-ALGO

assumes to-list: SIDE-GEN-ALGO (is-set-to-list A Rs tol)

assumes empty: GEN-OP emp op-map-empty ($\langle A, B \rangle$ Rm)

assumes update: GEN-OP upd op-map-update (A \rightarrow B \rightarrow $\langle A, B \rangle$ Rm \rightarrow $\langle A,$
B \rangle Rm)

shows (λ f X. *gen-build* tol upd emp f X, λ f X. (Some \circ f) |' X) \in

(A \rightarrow B) \rightarrow $\langle A \rangle$ Rs \rightarrow $\langle A, B \rangle$ Rm

\langle proof \rangle

definition to-list it s \equiv it s top Cons Nil

lemma *map2set-to-list*:

assumes GEN-ALGO-tag (is-map-to-list Rk unit-rel R it)

shows is-set-to-list Rk (map2set-rel R) (to-list (map-iterator-dom \circ (foldli \circ
it)))

\langle proof \rangle

lemma *CAST-to-list*[*autoref-rules-raw*]:
assumes *PRIO-TAG-GEN-ALGO*
assumes *SIDE-GEN-ALGO* (*is-set-to-list* *A Rs tol*)
shows (*tol, CAST*) $\in \langle A \rangle Rs \rightarrow \langle A \rangle list\text{-set-rel}$
 $\langle proof \rangle$

lemma *param-foldli*:

assumes (*xs, ys*) $\in \langle Ra \rangle list\text{-rel}$
assumes (*c, d*) $\in Rs \rightarrow bool\text{-rel}$
assumes $\bigwedge x y. (x, y) \in Ra \implies x \in set\ xs \implies y \in set\ ys \implies (f\ x, g\ y) \in$
 $Rs \rightarrow Rs$

assumes (*a, b*) $\in Rs$
shows (*foldli xs c f a, foldli ys d g b*) $\in Rs$

$\langle proof \rangle$

lemma *det-fold-sorted-set*:

assumes 1: *det-fold-set ordR c' f' σ' result*
assumes 2: *is-set-to-sorted-list ordR Rk Rs tsl*
assumes *SREF*[*param*]: (*s, s'*) $\in \langle Rk \rangle Rs$
assumes [*param*]: (*c, c'*) $\in R\sigma \rightarrow Id$
assumes [*param*]: $\bigwedge x y. (x, y) \in Rk \implies y \in s' \implies (f\ x, f'\ y) \in R\sigma \rightarrow R\sigma$
assumes [*param*]: (σ, σ') $\in R\sigma$
shows (*foldli (tsl s) c f σ , result s'*) $\in R\sigma$

$\langle proof \rangle$

lemma *det-fold-set*:

assumes *det-fold-set* ($\lambda - . True$) *c' f' σ' result*
assumes *is-set-to-list* *Rk Rs tsl*
assumes (*s, s'*) $\in \langle Rk \rangle Rs$
assumes (*c, c'*) $\in R\sigma \rightarrow Id$
assumes $\bigwedge x y. (x, y) \in Rk \implies y \in s' \implies (f\ x, f'\ y) \in R\sigma \rightarrow R\sigma$
assumes (σ, σ') $\in R\sigma$
shows (*foldli (tsl s) c f σ , result s'*) $\in R\sigma$

$\langle proof \rangle$

lemma *gen-image*[*autoref-rules-raw*]:

assumes *PRIO-TAG-GEN-ALGO*
assumes *IT*: *SIDE-GEN-ALGO* (*is-set-to-list* *Rk Rs1 it1*)
assumes *INS*: *GEN-OP ins2 Set.insert* (*Rk' \rightarrow $\langle Rk' \rangle Rs2 \rightarrow $\langle Rk' \rangle Rs2$$*)
assumes *EMPTY*: *GEN-OP empty2* $\{ \}$ ($\langle Rk' \rangle Rs2$)
assumes $\bigwedge xi\ x. (xi, x) \in Rk \implies x \in s \implies (fi\ xi, f\ \$\ x) \in Rk'$
assumes (*l, s*) $\in \langle Rk \rangle Rs1$
shows (*gen-image* ($\lambda x. foldli (it1\ x) empty2 ins2 fi\ l,$
 $(OP\ image\ ::\ (Rk \rightarrow Rk') \rightarrow (\langle Rk \rangle Rs1) \rightarrow (\langle Rk' \rangle Rs2))\ \$\ f\ \$\ s$) $\in (\langle Rk' \rangle Rs2)$

$\langle proof \rangle$

end

end

29 Implementation of Deterministic Rabin Automata

theory *DRA-Implement*

imports

DRA-Refine

../Basic/Implement

begin

datatype (*'label*, *'state*) *drai* = *drai*
 (*alphabeti*: *'label list*)
 (*initiali*: *'state*)
 (*transizioni*: *'label* \Rightarrow *'state* \Rightarrow *'state*)
 (*conditioni*: *'state rabin gen*)

definition *drai-rel* :: (*'label*₁ \times *'label*₂) *set* \Rightarrow (*'state*₁ \times *'state*₂) *set* \Rightarrow
 ((*'label*₁, *'state*₁) *drai* \times (*'label*₂, *'state*₂) *drai*) *set* **where**
 [*to-relAPP*]: *drai-rel* *L S* \equiv {(*A*₁, *A*₂).
 (*alphabeti* *A*₁, *alphabeti* *A*₂) \in $\langle L \rangle$ *list-rel* \wedge
 (*initiali* *A*₁, *initiali* *A*₂) \in *S* \wedge
 (*transizioni* *A*₁, *transizioni* *A*₂) \in *L* \rightarrow *S* \rightarrow *S* \wedge
 (*conditioni* *A*₁, *conditioni* *A*₂) \in \langle *rabin-rel S* \rangle *list-rel*}

lemma *drai-param*[*param*]:

(*drai*, *drai*) \in $\langle L \rangle$ *list-rel* \rightarrow *S* \rightarrow (*L* \rightarrow *S* \rightarrow *S*) \rightarrow
 \langle *rabin-rel S* \rangle *list-rel* \rightarrow $\langle L, S \rangle$ *drai-rel*
 (*alphabeti*, *alphabeti*) \in $\langle L, S \rangle$ *drai-rel* \rightarrow $\langle L \rangle$ *list-rel*
 (*initiali*, *initiali*) \in $\langle L, S \rangle$ *drai-rel* \rightarrow *S*
 (*transizioni*, *transizioni*) \in $\langle L, S \rangle$ *drai-rel* \rightarrow *L* \rightarrow *S* \rightarrow *S*
 (*conditioni*, *conditioni*) \in $\langle L, S \rangle$ *drai-rel* \rightarrow \langle *rabin-rel S* \rangle *list-rel*
 \langle *proof* \rangle

definition *drai-dra-rel* :: (*'label*₁ \times *'label*₂) *set* \Rightarrow (*'state*₁ \times *'state*₂) *set* \Rightarrow
 ((*'label*₁, *'state*₁) *drai* \times (*'label*₂, *'state*₂) *dra*) *set* **where**
 [*to-relAPP*]: *drai-dra-rel* *L S* \equiv {(*A*₁, *A*₂).
 (*alphabeti* *A*₁, *alphabet* *A*₂) \in $\langle L \rangle$ *list-set-rel* \wedge
 (*initiali* *A*₁, *initial* *A*₂) \in *S* \wedge
 (*transizioni* *A*₁, *transition* *A*₂) \in *L* \rightarrow *S* \rightarrow *S* \wedge
 (*conditioni* *A*₁, *condition* *A*₂) \in \langle *rabin-rel S* \rangle *list-rel*}

lemma *drai-dra-param*[*param*, *autoref-rules*]:

(*drai*, *dra*) \in $\langle L \rangle$ *list-set-rel* \rightarrow *S* \rightarrow (*L* \rightarrow *S* \rightarrow *S*) \rightarrow
 \langle *rabin-rel S* \rangle *list-rel* \rightarrow $\langle L, S \rangle$ *drai-dra-rel*
 (*alphabeti*, *alphabet*) \in $\langle L, S \rangle$ *drai-dra-rel* \rightarrow $\langle L \rangle$ *list-set-rel*
 (*initiali*, *initial*) \in $\langle L, S \rangle$ *drai-dra-rel* \rightarrow *S*
 (*transizioni*, *transition*) \in $\langle L, S \rangle$ *drai-dra-rel* \rightarrow *L* \rightarrow *S* \rightarrow *S*
 (*conditioni*, *condition*) \in $\langle L, S \rangle$ *drai-dra-rel* \rightarrow \langle *rabin-rel S* \rangle *list-rel*
 \langle *proof* \rangle

definition *drai-dra* :: (*'label*, *'state*) *drai* \Rightarrow (*'label*, *'state*) *dra* **where**

$drai-dra\ A \equiv dra\ (set\ (alphabeti\ A))\ (initiali\ A)\ (transizioni\ A)\ (conditioni\ A)$
definition $drai-invar :: ('label, 'state)\ drai \Rightarrow bool$ **where**
 $drai-invar\ A \equiv distinct\ (alphabeti\ A)$

lemma $drai-dra-id-param[param]: (drai-dra, id) \in \langle L, S \rangle\ drai-dra-rel \rightarrow \langle L, S \rangle$
 $dra-rel$
 $\langle proof \rangle$

lemma $drai-dra-br: \langle Id, Id \rangle\ drai-dra-rel = br\ drai-dra\ drai-invar$
 $\langle proof \rangle$

end

30 Exploration of Deterministic Rabin Automata

theory $DRA-Nodes$

imports

$DFS-Framework.Reachable-Nodes$

$DRA-Implement$

begin

definition $dra-G :: ('label, 'state)\ dra \Rightarrow 'state\ graph-rec$ **where**
 $dra-G\ A \equiv (\mid\ g-V = UNIV, g-E = E-of-succ\ (successors\ A), g-V0 = \{initial\ A\}\ \mid)$

lemma $dra-G-graph[simp]: graph\ (dra-G\ A)\ \langle proof \rangle$

lemma $dra-G-reachable-nodes: op-reachable\ (dra-G\ A) = nodes\ A$
 $\langle proof \rangle$

context

begin

interpretation $autoref-syn\ \langle proof \rangle$

lemma $dra-G-ahs: dra-G\ A = (\mid\ g-V = UNIV, g-E = E-of-succ\ (\lambda\ p.\ CAST$
 $((\lambda\ a.\ transition\ A\ a\ p :: S)\ 'alphabet\ A :: \langle S \rangle\ ahs-rel\ bhc), g-V0 = \{initial\ A\}\ \mid)$
 $\langle proof \rangle$

schematic-goal $drai-Gi:$

notes $map2set-to-list[autoref-ga-rules]$

fixes $S :: ('statei \times 'state)\ set$

assumes $[autoref-ga-rules]: is-bounded-hashcode\ S\ seq\ bhc$

assumes $[autoref-ga-rules]: is-valid-def-hm-size\ TYPE('statei)\ hms$

assumes $[autoref-rules]: (seq, HOL.eq) \in S \rightarrow S \rightarrow bool-rel$

assumes $[autoref-rules]: (Ai, A) \in \langle L, S \rangle\ drai-dra-rel$

shows $(?f :: ?'a, RETURN\ (dra-G\ A)) \in ?A$

$\langle proof \rangle$

concrete-definition $drai-Gi$ **uses** $drai-Gi$

```

lemma drai-Gi-refine[autoref-rules]:
  fixes S :: ('statei × 'state) set
  assumes SIDE-GEN-ALGO (is-bounded-hashcode S seq bhc)
  assumes SIDE-GEN-ALGO (is-valid-def-hm-size TYPE('statei) hms)
  assumes GEN-OP seq HOL.eq (S → S → bool-rel)
  shows (DRA-Nodes.drai-Gi seq bhc hms, dra-G) ∈ ⟨L, S⟩ drai-dra-rel →
  ⟨unit-rel, S⟩ g-impl-rel-ext
  ⟨proof⟩

```

```

schematic-goal dra-nodes:
  fixes S :: ('statei × 'state) set
  assumes [simp]: finite ((g-E (dra-G A))* “ g-V0 (dra-G A))
  assumes [autoref-ga-rules]: is-bounded-hashcode S seq bhc
  assumes [autoref-ga-rules]: is-valid-def-hm-size TYPE('statei) hms
  assumes [autoref-rules]: (seq, HOL.eq) ∈ S → S → bool-rel
  assumes [autoref-rules]: (Ai, A) ∈ ⟨L, S⟩ drai-dra-rel
  shows (?f :: ?'a, op-reachable (dra-G A)) ∈ ?R ⟨proof⟩

```

concrete-definition dra-nodes **uses** dra-nodes

```

lemma dra-nodes-refine[autoref-rules]:
  fixes S :: ('statei × 'state) set
  assumes SIDE-PRECOND (finite (nodes A))
  assumes SIDE-GEN-ALGO (is-bounded-hashcode S seq bhc)
  assumes SIDE-GEN-ALGO (is-valid-def-hm-size TYPE('statei) hms)
  assumes GEN-OP seq HOL.eq (S → S → bool-rel)
  assumes (Ai, A) ∈ ⟨L, S⟩ drai-dra-rel
  shows (DRA-Nodes.dra-nodes seq bhc hms Ai,
  (OP nodes ∷ ⟨L, S⟩ drai-dra-rel → ⟨S⟩ ahs-rel bhc) $ A) ∈ ⟨S⟩ ahs-rel bhc
  ⟨proof⟩

```

end

end

31 Explicit Deterministic Rabin Automata

```

theory DRA-Explicit
imports DRA-Nodes
begin

```

```

datatype ('label, 'state) drae = drae
  (alphabet: 'label set)
  (initiale: 'state)
  (transitione: ('state × 'label × 'state) set)
  (conditione: ('state set × 'state set) list)

```

definition drae-rel **where**

```

[to-relAPP]: drae-rel L S ≡ {(A1, A2).
  (alphabet A1, alphabet A2) ∈ ⟨L⟩ set-rel ∧

```


$(\text{initiale } A_1, \text{initiale } A_2) \in S \wedge$
 $(\text{transitione } A_1, \text{transitione } A_2) \in \langle S \times_r L \times_r S \rangle \text{ set-rel} \wedge$
 $(\text{conditione } A_1, \text{conditione } A_2) \in \langle \langle S \rangle \text{ set-rel} \times_r \langle S \rangle \text{ set-rel} \rangle \text{ list-rel}$

lemma *drae-param*[*param, autoref-rules*]:

$(\text{drae}, \text{drae}) \in \langle L \rangle \text{ set-rel} \rightarrow S \rightarrow \langle S \times_r L \times_r S \rangle \text{ set-rel} \rightarrow$
 $\langle \langle S \rangle \text{ set-rel} \times_r \langle S \rangle \text{ set-rel} \rangle \text{ list-rel} \rightarrow \langle L, S \rangle \text{ drae-rel}$
 $(\text{alphabetete}, \text{alphabetete}) \in \langle L, S \rangle \text{ drae-rel} \rightarrow \langle L \rangle \text{ set-rel}$
 $(\text{initiale}, \text{initiale}) \in \langle L, S \rangle \text{ drae-rel} \rightarrow S$
 $(\text{transitione}, \text{transitione}) \in \langle L, S \rangle \text{ drae-rel} \rightarrow \langle S \times_r L \times_r S \rangle \text{ set-rel}$
 $(\text{conditione}, \text{conditione}) \in \langle L, S \rangle \text{ drae-rel} \rightarrow \langle \langle S \rangle \text{ set-rel} \times_r \langle S \rangle \text{ set-rel} \rangle \text{ list-rel}$
 $\langle \text{proof} \rangle$

lemma *drae-rel-id*[*simp*]: $\langle \text{Id}, \text{Id} \rangle \text{ drae-rel} = \text{Id} \langle \text{proof} \rangle$

lemma *drae-rel-comp*[*simp*]: $\langle L_1 \ O \ L_2, S_1 \ O \ S_2 \rangle \text{ drae-rel} = \langle L_1, S_1 \rangle \text{ drae-rel} \ O$
 $\langle L_2, S_2 \rangle \text{ drae-rel}$
 $\langle \text{proof} \rangle$

consts *i-drae-scheme* :: *interface* \Rightarrow *interface* \Rightarrow *interface*

context

begin

interpretation *autoref-syn* $\langle \text{proof} \rangle$

lemma *drae-scheme-itype*[*autoref-itype*]:

$\text{drae} ::_i \langle L \rangle_i \text{ i-set} \rightarrow_i S \rightarrow_i \langle \langle S, \langle L, S \rangle_i \text{ i-prod} \rangle_i \text{ i-prod} \rangle_i \text{ i-set} \rightarrow_i$
 $\langle \langle \langle S \rangle_i \text{ i-set}, \langle S \rangle_i \text{ i-set} \rangle_i \text{ i-prod} \rangle_i \text{ i-list} \rightarrow_i \langle L, S \rangle_i \text{ i-drae-scheme}$
 $\text{alphabetete} ::_i \langle L, S \rangle_i \text{ i-drae-scheme} \rightarrow_i \langle L \rangle_i \text{ i-set}$
 $\text{initiale} ::_i \langle L, S \rangle_i \text{ i-drae-scheme} \rightarrow_i S$
 $\text{transitione} ::_i \langle L, S \rangle_i \text{ i-drae-scheme} \rightarrow_i \langle \langle S, \langle L, S \rangle_i \text{ i-prod} \rangle_i \text{ i-prod} \rangle_i \text{ i-set}$
 $\text{conditione} ::_i \langle L, S \rangle_i \text{ i-drae-scheme} \rightarrow_i \langle \langle \langle S \rangle_i \text{ i-set}, \langle S \rangle_i \text{ i-set} \rangle_i \text{ i-prod} \rangle_i \text{ i-list}$
 $\langle \text{proof} \rangle$

end

datatype (*'label, 'state*) *draei* = *draei*

$(\text{alphabetetei}: \text{'label list})$
 $(\text{initialei}: \text{'state})$
 $(\text{transitionei}: (\text{'state} \times \text{'label} \times \text{'state}) \text{ list})$
 $(\text{conditionei}: (\text{'state list} \times \text{'state list}) \text{ list})$

definition *draei-rel* **where**

$[\text{to-relAPP}]: \text{draei-rel } L \ S \equiv \{(A_1, A_2).$
 $(\text{alphabetetei } A_1, \text{alphabetetei } A_2) \in \langle L \rangle \text{ list-rel} \wedge$
 $(\text{initialei } A_1, \text{initialei } A_2) \in S \wedge$
 $(\text{transitionei } A_1, \text{transitionei } A_2) \in \langle S \times_r L \times_r S \rangle \text{ list-rel} \wedge$
 $(\text{conditionei } A_1, \text{conditionei } A_2) \in \langle \langle S \rangle \text{ list-rel} \times_r \langle S \rangle \text{ list-rel} \rangle \text{ list-rel}$

lemma *draei-param*[*param*, *autoref-rules*]:

$(draei, draei) \in \langle L \rangle \text{ list-rel} \rightarrow S \rightarrow \langle S \times_r L \times_r S \rangle \text{ list-rel} \rightarrow$
 $\langle \langle S \rangle \text{ list-rel} \times_r \langle S \rangle \text{ list-rel} \rangle \text{ list-rel} \rightarrow \langle L, S \rangle \text{ draei-rel}$
 $(alphabetei, alphabetei) \in \langle L, S \rangle \text{ draei-rel} \rightarrow \langle L \rangle \text{ list-rel}$
 $(initialei, initialei) \in \langle L, S \rangle \text{ draei-rel} \rightarrow S$
 $(transitionei, transitionei) \in \langle L, S \rangle \text{ draei-rel} \rightarrow \langle S \times_r L \times_r S \rangle \text{ list-rel}$
 $(conditionei, conditionei) \in \langle L, S \rangle \text{ draei-rel} \rightarrow \langle \langle S \rangle \text{ list-rel} \times_r \langle S \rangle \text{ list-rel} \rangle$
 list-rel
 $\langle \text{proof} \rangle$

definition *draei-drae-rel* **where**

$[to\text{-rel}APP]: \text{draei-drae-rel } L \ S \equiv \{(A_1, A_2).$
 $(alphabetei \ A_1, \ alphabete \ A_2) \in \langle L \rangle \text{ list-set-rel} \wedge$
 $(initialei \ A_1, \ initiale \ A_2) \in S \wedge$
 $(transitionei \ A_1, \ transitione \ A_2) \in \langle S \times_r L \times_r S \rangle \text{ list-set-rel} \wedge$
 $(conditionei \ A_1, \ conditione \ A_2) \in \langle \langle S \rangle \text{ list-set-rel} \times_r \langle S \rangle \text{ list-set-rel} \rangle \text{ list-rel}\}$

lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of draei-drae-rel i-drae-scheme*]

lemma *draei-drae-param*[*param*, *autoref-rules*]:

$(draei, drae) \in \langle L \rangle \text{ list-set-rel} \rightarrow S \rightarrow \langle S \times_r L \times_r S \rangle \text{ list-set-rel} \rightarrow$
 $\langle \langle S \rangle \text{ list-set-rel} \times_r \langle S \rangle \text{ list-set-rel} \rangle \text{ list-rel} \rightarrow \langle L, S \rangle \text{ draei-drae-rel}$
 $(alphabetei, alphabete) \in \langle L, S \rangle \text{ draei-drae-rel} \rightarrow \langle L \rangle \text{ list-set-rel}$
 $(initialei, initiale) \in \langle L, S \rangle \text{ draei-drae-rel} \rightarrow S$
 $(transitionei, transitione) \in \langle L, S \rangle \text{ draei-drae-rel} \rightarrow \langle S \times_r L \times_r S \rangle \text{ list-set-rel}$
 $(conditionei, conditione) \in \langle L, S \rangle \text{ draei-drae-rel} \rightarrow \langle \langle S \rangle \text{ list-set-rel} \times_r \langle S \rangle$
 $\text{list-set-rel} \rangle \text{ list-rel}$
 $\langle \text{proof} \rangle$

definition *draei-drae* **where**

$\text{draei-drae } A \equiv \text{drae } (\text{set } (\text{alphabetei } A)) \ (\text{initialei } A)$
 $(\text{set } (\text{transitionei } A)) \ (\text{map } (\text{map-prod } \text{set } \text{set}) \ (\text{conditionei } A))$

lemma *draei-drae-id-param*[*param*]: $(draei-drae, id) \in \langle L, S \rangle \text{ draei-drae-rel} \rightarrow$
 $\langle L, S \rangle \text{ drae-rel}$
 $\langle \text{proof} \rangle$

abbreviation *transitions* $L \ S \ s \equiv \bigcup a \in L. \bigcup p \in S. \{p\} \times \{a\} \times \{s \ a \ p\}$

abbreviation *succs* $T \ a \ p \equiv \text{the-elem } ((T \ \{\{p\}\} \ \{\{a\}\})$

definition *wft* :: $'label \ set \Rightarrow 'state \ set \Rightarrow ('state \times 'label \times 'state) \ set \Rightarrow \text{bool}$
where

$wft \ L \ S \ T \equiv \forall a \in L. \forall p \in S. \text{is-singleton } ((T \ \{\{p\}\} \ \{\{a\}\})$

lemma *wft-param*[*param*]:

assumes *bijective* S *bijective* L

shows $(wft, wft) \in \langle L \rangle \text{ set-rel} \rightarrow \langle S \rangle \text{ set-rel} \rightarrow \langle S \times_r L \times_r S \rangle \text{ set-rel} \rightarrow \text{bool-rel}$
 $\langle \text{proof} \rangle$

lemma *wft-transitions*: $wft\ L\ S\ (transitions\ L\ S\ s)\ \langle proof \rangle$

definition *dra-drae* **where** $dra-drae\ A \equiv drae\ (alphabet\ A)\ (initial\ A)$
 $(transitions\ (alphabet\ A)\ (nodes\ A)\ (transition\ A))$
 $(map\ (\lambda\ (P,\ Q).\ (Set.filter\ P\ (nodes\ A),\ Set.filter\ Q\ (nodes\ A)))\ (condition\ A))$

definition *drae-dra* **where** $drae-dra\ A \equiv dra\ (alphabet\ A)\ (initiale\ A)$
 $(succs\ (transitione\ A))\ (map\ (\lambda\ (I,\ F).\ (\lambda\ p.\ p \in I,\ \lambda\ p.\ p \in F))\ (conditione\ A))$

lemma *set-rel-Domain-Range*[*intro!*, *simp*]: $(Domain\ A,\ Range\ A) \in \langle A \rangle\ set-rel\ \langle proof \rangle$

lemma *dra-drae-param*[*param*]: $(dra-drae,\ dra-drae) \in \langle L,\ S \rangle\ dra-rel \rightarrow \langle L,\ S \rangle\ drae-rel\ \langle proof \rangle$

lemma *drae-dra-param*[*param*]:
assumes *bijjective L* *bijjective S*
assumes *wft (Range L) (Range S) (transitione B)*
assumes [*param*]: $(A,\ B) \in \langle L,\ S \rangle\ drae-rel$
shows $(drae-dra\ A,\ drae-dra\ B) \in \langle L,\ S \rangle\ dra-rel\ \langle proof \rangle$

lemma *succs-transitions-param*[*param*]:
 $(succs \circ transitions\ L\ S,\ id) \in (Id-on\ L \rightarrow Id-on\ S \rightarrow Id-on\ S) \rightarrow (Id-on\ L \rightarrow Id-on\ S \rightarrow Id-on\ S)\ \langle proof \rangle$

lemma *drae-dra-dra-drae-param*[*param*]:
 $((drae-dra \circ dra-drae)\ A,\ id\ A) \in \langle Id-on\ (alphabet\ A),\ Id-on\ (nodes\ A) \rangle\ dra-rel\ \langle proof \rangle$

definition *draei-dra-rel* **where**
 $[to-relAPP]: draei-dra-rel\ L\ S \equiv \{(Ae,\ A).\ (drae-dra\ (draei-drae\ Ae),\ A) \in \langle L,\ S \rangle\ dra-rel\}$

lemma *draei-dra-id*[*param*]: $(drae-dra \circ draei-drae,\ id) \in \langle L,\ S \rangle\ draei-dra-rel \rightarrow \langle L,\ S \rangle\ dra-rel\ \langle proof \rangle$

end

32 Explore and Enumerate Nodes of Deterministic Rabin Automata

```
theory DRA-Translate
imports DRA-Explicit
begin
```

32.1 Syntax

no-syntax *-do-let* :: [pttrn, 'a] ⇒ do-bind ((2let - =/ -) [1000, 13] 13)
syntax *-do-let* :: [pttrn, 'a] ⇒ do-bind ((2let - =/ -) 13)

33 Image on Explicit Automata

definition *drae-image* **where** *drae-image* f $A \equiv$ *drae* (*alphabet* A) (f (*initiale* A))
 $((\lambda (p, a, q). (f\ p, a, f\ q)) \text{ 'transitione } A)$ (*map* (*map-prod* (*image* f) (*image* f))) (*conditione* A)

lemma *drae-image-param*[*param*]: (*drae-image*, *drae-image*) ∈ ($S \rightarrow T$) → $\langle L, S \rangle$ *drae-rel* → $\langle L, T \rangle$ *drae-rel*
 \langle proof \rangle

lemma *drae-image-id*[*simp*]: *drae-image* *id* = *id* \langle proof \rangle

lemma *drae-image-dra-drae*: *drae-image* f (*dra-drae* A) = *drae* (*alphabet* A) (f (*initial* A))
 $(\bigcup p \in \text{nodes } A. \bigcup a \in \text{alphabet } A. f \text{ ' } \{p\} \times \{a\} \times f \text{ ' } \{\text{transition } A\ a\ p\})$
 $(\text{map } (\lambda (P, Q). (f \text{ ' } \{p \in \text{nodes } A. P\ p\}, f \text{ ' } \{p \in \text{nodes } A. Q\ p\})))$ (*condition* A)
 \langle proof \rangle

34 Exploration and Translation

definition *trans-spec* **where**

trans-spec A $f \equiv \bigcup p \in \text{nodes } A. \bigcup a \in \text{alphabet } A. f \text{ ' } \{p\} \times \{a\} \times f \text{ ' } \{\text{transition } A\ a\ p\}$

definition *trans-algo* **where**

trans-algo N L S $f \equiv$
 FOREACH N (λp T . do {
 ASSERT ($p \in N$);
 FOREACH L (λa T . do {
 ASSERT ($a \in L$);
 let $q = S\ a\ p$;
 ASSERT ($(f\ p, a, f\ q) \notin T$);
 RETURN (*insert* ($f\ p, a, f\ q$) T) }
) T }
) {}

lemma *trans-algo-refine*:

assumes *finite* (*nodes* A) *finite* (*alphabet* A) *inj-on* f (*nodes* A)

assumes $N = \text{nodes } A$ $L = \text{alphabet } A$ $S = \text{transition } A$

shows (*trans-algo* N L S f , *SPEC* (*HOL.eq* (*trans-spec* A f))) ∈ $\langle Id \rangle$ *nres-rel*
 \langle proof \rangle

definition *to-draei* :: ('state, 'label) *dra* ⇒ ('state, 'label) *dra*

where $to\text{-draei} \equiv id$

schematic-goal $to\text{-draei}\text{-impl}$:

```

fixes  $S :: ('statei \times 'state) \text{ set}$ 
assumes  $[simp]: \text{finite } (nodes\ A)$ 
assumes  $[autoref\text{-ga}\text{-rules}]: \text{is-bounded-hashcode } S\ seq\ bhc$ 
assumes  $[autoref\text{-ga}\text{-rules}]: \text{is-valid-def-hm-size } TYPE('statei)\ hms$ 
assumes  $[autoref\text{-rules}]: (seq, HOL.eq) \in S \rightarrow S \rightarrow \text{bool-rel}$ 
assumes  $[autoref\text{-rules}]: (Ai, A) \in \langle L, S \rangle \text{ drai-dra-rel}$ 
shows  $(?f :: ?'a, \text{do } \{$ 
   $\text{let } N = nodes\ A;$ 
   $f \leftarrow \text{op-set-enumerate } N;$ 
   $ASSERT\ (dom\ f = N);$ 
   $ASSERT\ (f\ (initial\ A) \neq None);$ 
   $ASSERT\ (\forall a \in alphabet\ A. \forall p \in dom\ f. f\ (transition\ A\ a\ p) \neq None);$ 
   $T \leftarrow \text{trans-algo } N\ (alphabet\ A)\ (transition\ A)\ (\lambda x. \text{the } (f\ x));$ 
   $RETURN\ (\text{drae } (alphabet\ A)\ ((\lambda x. \text{the } (f\ x))\ (initial\ A))\ T$ 
     $(\text{map } (\lambda (P, Q). ((\lambda x. \text{the } (f\ x))\ ' \{p \in N. P\ p\}, (\lambda x. \text{the } (f\ x))\ ' \{p \in$ 
 $N. Q\ p\}))\ (\text{condition } A)))$ 
   $\}) \in ?R$ 
 $\langle \text{proof} \rangle$ 

```

concrete-definition $to\text{-draei}\text{-impl}$ **uses** $to\text{-draei}\text{-impl}$

lemma $to\text{-draei}\text{-impl}\text{-refine''}$:

```

fixes  $S :: ('statei \times 'state) \text{ set}$ 
assumes  $\text{finite } (nodes\ A)$ 
assumes  $\text{is-bounded-hashcode } S\ seq\ bhc$ 
assumes  $\text{is-valid-def-hm-size } TYPE('statei)\ hms$ 
assumes  $(seq, HOL.eq) \in S \rightarrow S \rightarrow \text{bool-rel}$ 
assumes  $(Ai, A) \in \langle L, S \rangle \text{ drai-dra-rel}$ 
shows  $(RETURN\ (to\text{-draei}\text{-impl } seq\ bhc\ hms\ Ai), \text{do } \{$ 
   $f \leftarrow \text{op-set-enumerate } (nodes\ A);$ 
   $RETURN\ (\text{drae-image } (\text{the } \circ f)\ (\text{dra-drae } A))$ 
   $\}) \in \langle \langle L, \text{nat-rel} \rangle \text{ draei-drae-rel} \rangle \text{ nres-rel}$ 
 $\langle \text{proof} \rangle$ 

```

context

```

fixes  $Ai\ A$ 
fixes  $seq\ bhc\ hms$ 
fixes  $S :: ('statei \times 'state) \text{ set}$ 
assumes  $a: \text{finite } (nodes\ A)$ 
assumes  $b: \text{is-bounded-hashcode } S\ seq\ bhc$ 
assumes  $c: \text{is-valid-def-hm-size } TYPE('statei)\ hms$ 
assumes  $d: (seq, HOL.eq) \in S \rightarrow S \rightarrow \text{bool-rel}$ 
assumes  $e: (Ai, A) \in \langle Id, S \rangle \text{ drai-dra-rel}$ 

```

begin

definition f' **where** $f' \equiv \text{SOME } f'$.

$(to\text{-draei}\text{-impl}\ seq\ bhc\ hms\ Ai, drae\text{-image}\ (the \circ f')\ (dra\text{-drae}\ A)) \in \langle Id, nat\text{-rel} \rangle draei\text{-drae}\text{-rel} \wedge$
 $dom\ f' = nodes\ A \wedge inj\text{-on}\ f'\ (nodes\ A)$

lemma 1: $\exists f'. (to\text{-draei}\text{-impl}\ seq\ bhc\ hms\ Ai, drae\text{-image}\ (the \circ f')\ (dra\text{-drae}\ A)) \in$
 $\langle Id, nat\text{-rel} \rangle draei\text{-drae}\text{-rel} \wedge dom\ f' = nodes\ A \wedge inj\text{-on}\ f'\ (nodes\ A)$
 $\langle proof \rangle$

lemma f' -refine: $(to\text{-draei}\text{-impl}\ seq\ bhc\ hms\ Ai, drae\text{-image}\ (the \circ f')\ (dra\text{-drae}\ A)) \in$
 $\langle Id, nat\text{-rel} \rangle draei\text{-drae}\text{-rel} \langle proof \rangle$

lemma f' -dom: $dom\ f' = nodes\ A \langle proof \rangle$

lemma f' -inj: $inj\text{-on}\ f'\ (nodes\ A) \langle proof \rangle$

definition f where $f \equiv the \circ f'$

definition g where $g = inv\text{-into}\ (nodes\ A)\ f$

lemma $inj\text{-}f$ [intro!, simp]: $inj\text{-on}\ f\ (nodes\ A)$
 $\langle proof \rangle$

lemma $inj\text{-}g$ [intro!, simp]: $inj\text{-on}\ g\ (f\ ' nodes\ A)$
 $\langle proof \rangle$

definition rel where $rel \equiv \{(f\ p, p) \mid p. p \in nodes\ A\}$

lemma $rel\text{-alt}\text{-def}$: $rel = (br\ f\ (\lambda p. p \in nodes\ A))^{-1}$
 $\langle proof \rangle$

lemma $rel\text{-inv}\text{-def}$: $rel = br\ g\ (\lambda k. k \in f\ ' nodes\ A)$
 $\langle proof \rangle$

lemma $rel\text{-domain}$ [simp]: $Domain\ rel = f\ ' nodes\ A \langle proof \rangle$

lemma $rel\text{-range}$ [simp]: $Range\ rel = nodes\ A \langle proof \rangle$

lemma [intro!, simp]: $bijjective\ rel \langle proof \rangle$

lemma [simp]: $Id\text{-on}\ (f\ ' nodes\ A)\ O\ rel = rel \langle proof \rangle$

lemma [simp]: $rel\ O\ Id\text{-on}\ (nodes\ A) = rel \langle proof \rangle$

lemma [param]: $(f, f) \in Id\text{-on}\ (Range\ rel) \rightarrow Id\text{-on}\ (Domain\ rel) \langle proof \rangle$

lemma [param]: $(g, g) \in Id\text{-on}\ (Domain\ rel) \rightarrow Id\text{-on}\ (Range\ rel) \langle proof \rangle$

lemma [param]: $(id, f) \in rel \rightarrow Id\text{-on}\ (Domain\ rel) \langle proof \rangle$

lemma [param]: $(f, id) \in rel \rightarrow Id\text{-on}\ (Range\ rel) \langle proof \rangle$

lemma [param]: $(id, g) \in rel \rightarrow Id\text{-on}\ (Domain\ rel) \langle proof \rangle$

lemma [param]: $(g, id) \in rel \rightarrow Id\text{-on}\ (Range\ rel) \langle proof \rangle$

lemma $to\text{-draei}\text{-impl}\text{-refine}'$:

$(to\text{-draei}\text{-impl}\ seq\ bhc\ hms\ Ai, to\text{-draei}\ A) \in \langle Id\text{-on}\ (alphabet\ A), rel \rangle draei\text{-drae}\text{-rel}$
 $\langle proof \rangle$

end

context

begin

interpretation *autoref-syn* $\langle proof \rangle$

lemma *to-draei-impl-refine*[*autoref-rules*]:

fixes $S :: ('statei \times 'state) \text{ set}$

assumes *SIDE-PRECOND* (*finite* (*nodes A*))

assumes *SIDE-GEN-ALGO* (*is-bounded-hashcode* $S \text{ seq } \text{bhc}$)

assumes *SIDE-GEN-ALGO* (*is-valid-def-hm-size* $TYPE('statei) \text{ hms}$)

assumes *GEN-OP seq HOL.eq* ($S \rightarrow S \rightarrow \text{bool-rel}$)

assumes $(Ai, A) \in \langle Id, S \rangle \text{ drai-dra-rel}$

shows (*to-draei-impl seq bhc hms Ai*,

$(OP \text{ to-draei} :: \langle Id, S \rangle \text{ drai-dra-rel} \rightarrow$

$\langle Id\text{-on } (\text{alphabet } A), \text{ rel } Ai \ A \ \text{seq } \text{bhc } \text{hms} \rangle \text{ draei-dra-rel}$) $\$ A \in$

$\langle Id\text{-on } (\text{alphabet } A), \text{ rel } Ai \ A \ \text{seq } \text{bhc } \text{hms} \rangle \text{ draei-dra-rel}$

$\langle proof \rangle$

end

end

35 Nondeterministic Büchi Automata

theory *NBA*

imports *../Nondeterministic*

begin

datatype (*'label, 'state*) *nba = nba*

(*alphabet: 'label set*)

(*initial: 'state set*)

(*transition: 'label \Rightarrow 'state \Rightarrow 'state set*)

(*accepting: 'state pred*)

global-interpretation *nba: automaton nba alphabet initial transition accepting*

defines *path = nba.path and run = nba.run and reachable = nba.reachable*

and *nodes = nba.nodes*

$\langle proof \rangle$

global-interpretation *nba: automaton-run nba alphabet initial transition accepting $\lambda P \ w \ r \ p. \ \text{infs } P \ (p \ \#\# \ r)$*

defines *language = nba.language*

$\langle proof \rangle$

abbreviation *target where target \equiv nba.target*

abbreviation *states where states \equiv nba.states*

abbreviation *trace where trace \equiv nba.trace*

abbreviation *successors where successors \equiv nba.successors TYPE('label)*

instantiation *nba :: (type, type) order*

begin

definition *less-eq-nba :: ('a, 'b) nba \Rightarrow ('a, 'b) nba \Rightarrow bool where*

$A \leq B \equiv \text{alphabet } A \leq \text{alphabet } B \wedge \text{initial } A \leq \text{initial } B \wedge$
 $\text{transition } A \leq \text{transition } B \wedge \text{accepting } A \leq \text{accepting } B$
definition *less-nba* :: ('a, 'b) nba \Rightarrow ('a, 'b) nba \Rightarrow bool **where**
less-nba A B $\equiv A \leq B \wedge A \neq B$

instance *<proof>*

end

lemma *nodes-mono: mono nodes*
<proof>

lemma *language-mono: mono language*
<proof>

lemma *simulation-language:*
assumes *alphabet* A \subseteq *alphabet* B
assumes $\bigwedge p. p \in \text{initial } A \Rightarrow \exists q \in \text{initial } B. (p, q) \in R$
assumes $\bigwedge a p p' q. p' \in \text{transition } A \ a \ p \Rightarrow (p, q) \in R \Rightarrow \exists q' \in \text{transition } B \ a \ q. (p', q') \in R$
assumes $\bigwedge p q. (p, q) \in R \Rightarrow \text{accepting } A \ p \Rightarrow \text{accepting } B \ q$
shows *language* A \subseteq *language* B
<proof>

end

36 Nondeterministic Generalized Büchi Automata

theory *NGBA*

imports ../Nondeterministic

begin

datatype ('label, 'state) *ngba* = *ngba*
(alphabet: 'label set)
(initial: 'state set)
(transition: 'label \Rightarrow 'state \Rightarrow 'state set)
(accepting: 'state pred gen)

global-interpretation *ngba: automaton ngba alphabet initial transition accepting*
defines *path* = *ngba.path* **and** *run* = *ngba.run* **and** *reachable* = *ngba.reachable*
and *nodes* = *ngba.nodes*
<proof>

global-interpretation *ngba: automaton-run ngba alphabet initial transition accepting* $\lambda P w r p. \text{gen infs } P \ (p \ \#\# \ r)$
defines *language* = *ngba.language*
<proof>

abbreviation *target* **where** *target* \equiv *ngba.target*

abbreviation *states* **where** *states* \equiv *ngba.states*

abbreviation *trace* **where** *trace* \equiv *ngba.trace*
abbreviation *successors* **where** *successors* \equiv *ngba.successors* *TYPE('label')*

end

37 Nondeterministic Büchi Automata Combinations

theory *NBA-Combine*
imports *NBA NGBA*
begin

global-interpretation *degeneralization: automaton-degeneralization-run*
ngba ngba.alphabet ngba.initial ngba.transition ngba.accepting $\lambda P w r p.$ *gen*
infs P (p ## r)
nba nba.alphabet nba.initial nba.transition nba.accepting $\lambda P w r p.$ *infs P (p*
r)
fst id
defines *degeneralize* = *degeneralization.degeneralize*
 \langle *proof* \rangle

lemmas *degeneralize-language[simp]* = *degeneralization.degeneralize-language[folded*
NBA.language-def]

lemmas *degeneralize-nodes-finite[iff]* = *degeneralization.degeneralize-nodes-finite[folded*
NBA.nodes-def]

global-interpretation *intersection: automaton-intersection-run*
nba nba.alphabet nba.initial nba.transition nba.accepting $\lambda P w r p.$ *infs P (p*
r)
nba nba.alphabet nba.initial nba.transition nba.accepting $\lambda P w r p.$ *infs P (p*
r)
ngba ngba.alphabet ngba.initial ngba.transition ngba.accepting $\lambda P w r p.$ *gen*
infs P (p ## r)
 $\lambda c_1 c_2.$ [*c*₁ \circ *fst*, *c*₂ \circ *snd*]
defines *intersect'* = *intersection.product*
 \langle *proof* \rangle

lemmas *intersect'-language[simp]* = *intersection.product-language[folded NGBA.language-def]*

lemmas *intersect'-nodes-finite[intro]* = *intersection.product-nodes-finite[folded*
NGBA.nodes-def]

global-interpretation *union: automaton-union-run*
nba nba.alphabet nba.initial nba.transition nba.accepting $\lambda P w r p.$ *infs P (p*
r)
nba nba.alphabet nba.initial nba.transition nba.accepting $\lambda P w r p.$ *infs P (p*
r)
nba nba.alphabet nba.initial nba.transition nba.accepting $\lambda P w r p.$ *infs P (p*
r)

case-sum
defines *union* = *union.sum*
 ⟨*proof*⟩

lemmas *union-language* = *union.sum-language*
lemmas *union-nodes-finite* = *union.sum-nodes-finite*

global-interpretation *intersection-list: automaton-intersection-list-run*
nba nba.alphabet nba.initial nba.transition nba.accepting $\lambda P w r p. \text{infs } P (p \text{ \#\# } r)$
ngba ngba.alphabet ngba.initial ngba.transition ngba.accepting $\lambda P w r p. \text{gen infs } P (p \text{ \#\# } r)$
 $\lambda cs. \text{map } (\lambda k ps. (cs ! k) (ps ! k)) [0 ..< \text{length } cs]$
defines *intersect-list'* = *intersection-list.product*
 ⟨*proof*⟩

lemmas *intersect-list'-language*[*simp*] = *intersection-list.product-language*[*folded NGBA.language-def*]
lemmas *intersect-list'-nodes-finite*[*intro*] = *intersection-list.product-nodes-finite*[*folded NGBA.nodes-def*]

global-interpretation *union-list: automaton-union-list-run*
nba nba.alphabet nba.initial nba.transition nba.accepting $\lambda P w r p. \text{infs } P (p \text{ \#\# } r)$
nba nba.alphabet nba.initial nba.transition nba.accepting $\lambda P w r p. \text{infs } P (p \text{ \#\# } r)$
 $\lambda cs (k, p). (cs ! k) p$
defines *union-list* = *union-list.sum*
 ⟨*proof*⟩

lemmas *union-list-language* = *union-list.sum-language*
lemmas *union-list-nodes-finite* = *union-list.sum-nodes-finite*

abbreviation *intersect* **where** *intersect A B* $\equiv \text{degeneralize } (\text{intersect}' A B)$

lemma *intersect-language*[*simp*]: *NBA.language* (*intersect A B*) = *NBA.language* $A \cap \text{NBA.language } B$
 ⟨*proof*⟩

lemma *intersect-nodes-finite*[*intro*]:
assumes *finite* (*NBA.nodes A*) *finite* (*NBA.nodes B*)
shows *finite* (*NBA.nodes* (*intersect A B*))
 ⟨*proof*⟩

abbreviation *intersect-list* **where** *intersect-list AA* $\equiv \text{degeneralize } (\text{intersect-list}' AA)$

lemma *intersect-list-language*[*simp*]: *NBA.language* (*intersect-list AA*) = $\bigcap (\text{NBA.language } \text{'set } AA)$
 ⟨*proof*⟩

lemma *intersect-list-nodes-finite*[intro]:
assumes *list-all (finite o NBA.nodes) AA*
shows *finite (NBA.nodes (intersect-list AA))*
 ⟨*proof*⟩

end

38 Connecting Nondeterministic Büchi Automata to CAVA Automata Structures

theory *NBA-Graphs*
imports
NBA
CAVA-Automata.Automata-Impl
begin

no-notation *build* (infixr ## 65)

38.1 Regular Graphs

definition *nba-g* :: ('label, 'state) nba ⇒ 'state graph-rec **where**
nba-g A ≡ (| *g-V* = UNIV, *g-E* = *E-of-succ (successors A)*, *g-V0* = *initial A* |)

lemma *nba-g-graph*[simp]: *graph (nba-g A)* ⟨*proof*⟩

lemma *nba-g-V0*: *g-V0 (nba-g A) = initial A* ⟨*proof*⟩

lemma *nba-g-E-rtrancl*: $(g-E (nba-g A))^* = \{(p, q). q \in \text{reachable } A \ p\}$
 ⟨*proof*⟩

lemma *nba-g-rtrancl-path*: $(g-E (nba-g A))^* = \{(p, \text{target } r \ p) \mid r \ p. \text{NBA.path } A \ r \ p\}$
 ⟨*proof*⟩

lemma *nba-g-trancl-path*: $(g-E (nba-g A))^+ = \{(p, \text{target } r \ p) \mid r \ p. \text{NBA.path } A \ r \ p \wedge r \neq []\}$
 ⟨*proof*⟩

lemma *nba-g-ipath-run*:
assumes *ipath (g-E (nba-g A)) r*
obtains *w*
where *run A (w ||| smap (r o Suc) nats) (r 0)*
 ⟨*proof*⟩

lemma *nba-g-run-ipath*:
assumes *run A (w ||| r) p*
shows *ipath (g-E (nba-g A)) (snth (p ## r))*
 ⟨*proof*⟩

38.2 Indexed Generalized Büchi Graphs

definition *nba-igbg* :: ('label, 'state) nba ⇒ 'state igb-graph-rec **where**

$nba\text{-igbg } A \equiv \text{graph-rec.extend } (nba\text{-g } A)$
 $(\mid \text{igbg-num-acc} = 1, \text{igbg-acc} = \lambda p. \text{if accepting } A \text{ p then } \{0\} \text{ else } \{\}) \mid)$

lemma *acc-run-language*:
assumes *igbg-graph* ($nba\text{-igbg } A$)
shows $\text{Ex } (\text{igbg-graph.is-acc-run } (nba\text{-igbg } A)) \longleftrightarrow \text{language } A \neq \{\}$
 $\langle \text{proof} \rangle$

end

39 Relations on Nondeterministic Büchi Automata

theory *NBA-Refine*

imports

NBA

$\dots/Transition\text{-Systems}/Transition\text{-System-Refine}$

begin

definition *nba-rel* :: $(\text{'label}_1 \times \text{'label}_2) \text{ set} \Rightarrow (\text{'state}_1 \times \text{'state}_2) \text{ set} \Rightarrow$
 $((\text{'label}_1, \text{'state}_1) \text{ nba} \times (\text{'label}_2, \text{'state}_2) \text{ nba}) \text{ set where}$
 $[\text{to-relAPP}]: \text{nba-rel } L \ S \equiv \{(A_1, A_2).$
 $(\text{alphabet } A_1, \text{alphabet } A_2) \in \langle L \rangle \text{ set-rel} \wedge$
 $(\text{initial } A_1, \text{initial } A_2) \in \langle S \rangle \text{ set-rel} \wedge$
 $(\text{transition } A_1, \text{transition } A_2) \in L \rightarrow S \rightarrow \langle S \rangle \text{ set-rel} \wedge$
 $(\text{accepting } A_1, \text{accepting } A_2) \in S \rightarrow \text{bool-rel}\}$

lemma *nba-param*[*param*]:
 $(\text{nba}, \text{nba}) \in \langle L \rangle \text{ set-rel} \rightarrow \langle S \rangle \text{ set-rel} \rightarrow (L \rightarrow S \rightarrow \langle S \rangle \text{ set-rel}) \rightarrow (S \rightarrow$
 $\text{bool-rel}) \rightarrow$
 $\langle L, S \rangle \text{ nba-rel}$
 $(\text{alphabet}, \text{alphabet}) \in \langle L, S \rangle \text{ nba-rel} \rightarrow \langle L \rangle \text{ set-rel}$
 $(\text{initial}, \text{initial}) \in \langle L, S \rangle \text{ nba-rel} \rightarrow \langle S \rangle \text{ set-rel}$
 $(\text{transition}, \text{transition}) \in \langle L, S \rangle \text{ nba-rel} \rightarrow L \rightarrow S \rightarrow \langle S \rangle \text{ set-rel}$
 $(\text{accepting}, \text{accepting}) \in \langle L, S \rangle \text{ nba-rel} \rightarrow S \rightarrow \text{bool-rel}$
 $\langle \text{proof} \rangle$

lemma *nba-rel-id*[*simp*]: $\langle \text{Id}, \text{Id} \rangle \text{ nba-rel} = \text{Id} \langle \text{proof} \rangle$

lemma *nba-rel-comp*[*trans*]:

assumes [*param*]: $(A, B) \in \langle L_1, S_1 \rangle \text{ nba-rel } (B, C) \in \langle L_2, S_2 \rangle \text{ nba-rel}$

shows $(A, C) \in \langle L_1 \ O \ L_2, S_1 \ O \ S_2 \rangle \text{ nba-rel}$

$\langle \text{proof} \rangle$

lemma *nba-rel-converse*[*simp*]: $(\langle L, S \rangle \text{ nba-rel})^{-1} = \langle L^{-1}, S^{-1} \rangle \text{ nba-rel}$

$\langle \text{proof} \rangle$

lemma *nba-rel-eq*: $(A, A) \in \langle \text{Id-on } (\text{alphabet } A), \text{Id-on } (\text{nodes } A) \rangle \text{ nba-rel}$

$\langle \text{proof} \rangle$

lemma *enableds-param*[*param*]: $(\text{nba.enableds}, \text{nba.enableds}) \in \langle L, S \rangle \text{ nba-rel} \rightarrow$
 $S \rightarrow \langle L \times_r S \rangle \text{ set-rel}$

```

    <proof>
lemma paths-param[param]: (nba.paths, nba.paths) ∈ ⟨L, S⟩ nba-rel → S → ⟨⟨L
×r S⟩ list-rel⟩ set-rel
    <proof>
lemma runs-param[param]: (nba.runs, nba.runs) ∈ ⟨L, S⟩ nba-rel → S → ⟨⟨L
×r S⟩ stream-rel⟩ set-rel
    <proof>

lemma reachable-param[param]: (reachable, reachable) ∈ ⟨L, S⟩ nba-rel → S →
⟨S⟩ set-rel
    <proof>
lemma nodes-param[param]: (nodes, nodes) ∈ ⟨L, S⟩ nba-rel → ⟨S⟩ set-rel
    <proof>

lemma language-param[param]: (language, language) ∈ ⟨L, S⟩ nba-rel → ⟨⟨L
stream-rel⟩ set-rel⟩
    <proof>
end

```

40 Implementation of Nondeterministic Büchi Automata

```

theory NBA-Implement
imports
  NBA-Refine
  ../Basic/Implement
begin

```

```

consts i-nba-scheme :: interface ⇒ interface ⇒ interface

```

```

context
begin

```

```

interpretation autoref-syn <proof>

```

```

lemma nba-scheme-itype[autoref-itype]:
  nba ::i ⟨L⟩i i-set →i ⟨S⟩i i-set →i (L →i S →i ⟨S⟩i i-set) →i ⟨S⟩i i-set →i
  ⟨L, S⟩i i-nba-scheme
  alphabet ::i ⟨L, S⟩i i-nba-scheme →i ⟨L⟩i i-set
  initial ::i ⟨L, S⟩i i-nba-scheme →i ⟨S⟩i i-set
  transition ::i ⟨L, S⟩i i-nba-scheme →i L →i S →i ⟨S⟩i i-set
  accepting ::i ⟨L, S⟩i i-nba-scheme →i ⟨S⟩i i-set
  <proof>

```

```

end

```

```

datatype ('label, 'state) nbai = nbai

```

(*alphabeti*: 'label list)
 (*initiali*: 'state list)
 (*transizioni*: 'label \Rightarrow 'state \Rightarrow 'state list)
 (*acceptingi*: 'state \Rightarrow bool)

definition *nbai-rel* :: ('label₁ \times 'label₂) set \Rightarrow ('state₁ \times 'state₂) set \Rightarrow
 (('label₁, 'state₁) nbai \times ('label₂, 'state₂) nbai) set **where**
 [*to-relAPP*]: *nbai-rel* *L S* \equiv {(*A*₁, *A*₂).
 (*alphabeti* *A*₁, *alphabeti* *A*₂) \in $\langle L \rangle$ list-rel \wedge
 (*initiali* *A*₁, *initiali* *A*₂) \in $\langle S \rangle$ list-rel \wedge
 (*transizioni* *A*₁, *transizioni* *A*₂) \in $L \rightarrow S \rightarrow \langle S \rangle$ list-rel \wedge
 (*acceptingi* *A*₁, *acceptingi* *A*₂) \in $S \rightarrow$ bool-rel}

lemma *nbai-param*[*param*, *autoref-rules*]:
 (*nbai*, *nbai*) \in $\langle L \rangle$ list-rel \rightarrow $\langle S \rangle$ list-rel \rightarrow ($L \rightarrow S \rightarrow \langle S \rangle$ list-rel) \rightarrow
 ($S \rightarrow$ bool-rel) \rightarrow $\langle L, S \rangle$ nbai-rel
 (*alphabeti*, *alphabeti*) \in $\langle L, S \rangle$ nbai-rel \rightarrow $\langle L \rangle$ list-rel
 (*initiali*, *initiali*) \in $\langle L, S \rangle$ nbai-rel \rightarrow $\langle S \rangle$ list-rel
 (*transizioni*, *transizioni*) \in $\langle L, S \rangle$ nbai-rel \rightarrow $L \rightarrow S \rightarrow \langle S \rangle$ list-rel
 (*acceptingi*, *acceptingi*) \in $\langle L, S \rangle$ nbai-rel \rightarrow ($S \rightarrow$ bool-rel)
 \langle proof \rangle

definition *nbai-nba-rel* :: ('label₁ \times 'label₂) set \Rightarrow ('state₁ \times 'state₂) set \Rightarrow
 (('label₁, 'state₁) nbai \times ('label₂, 'state₂) nba) set **where**
 [*to-relAPP*]: *nbai-nba-rel* *L S* \equiv {(*A*₁, *A*₂).
 (*alphabeti* *A*₁, *alphabet* *A*₂) \in $\langle L \rangle$ list-set-rel \wedge
 (*initiali* *A*₁, *initial* *A*₂) \in $\langle S \rangle$ list-set-rel \wedge
 (*transizioni* *A*₁, *transition* *A*₂) \in $L \rightarrow S \rightarrow \langle S \rangle$ list-set-rel \wedge
 (*acceptingi* *A*₁, *accepting* *A*₂) \in $S \rightarrow$ bool-rel}

lemmas [*autoref-rel-intf*] = *REL-INTFI*[of *nbai-nba-rel* *i-nba-scheme*]

lemma *nbai-nba-param*[*param*, *autoref-rules*]:
 (*nbai*, *nba*) \in $\langle L \rangle$ list-set-rel \rightarrow $\langle S \rangle$ list-set-rel \rightarrow ($L \rightarrow S \rightarrow \langle S \rangle$ list-set-rel) \rightarrow
 ($S \rightarrow$ bool-rel) \rightarrow $\langle L, S \rangle$ nbai-nba-rel
 (*alphabeti*, *alphabet*) \in $\langle L, S \rangle$ nbai-nba-rel \rightarrow $\langle L \rangle$ list-set-rel
 (*initiali*, *initial*) \in $\langle L, S \rangle$ nbai-nba-rel \rightarrow $\langle S \rangle$ list-set-rel
 (*transizioni*, *transition*) \in $\langle L, S \rangle$ nbai-nba-rel \rightarrow $L \rightarrow S \rightarrow \langle S \rangle$ list-set-rel
 (*acceptingi*, *accepting*) \in $\langle L, S \rangle$ nbai-nba-rel \rightarrow $S \rightarrow$ bool-rel
 \langle proof \rangle

definition *nbai-nba* :: ('label, 'state) nbai \Rightarrow ('label, 'state) nba **where**
nbai-nba *A* \equiv nba (set (*alphabeti* *A*)) (set (*initiali* *A*)) (λ *a p*. set (*transizioni*
A a p)) (*acceptingi* *A*)

definition *nbai-invar* :: ('label, 'state) nbai \Rightarrow bool **where**
nbai-invar *A* \equiv distinct (*alphabeti* *A*) \wedge distinct (*initiali* *A*) \wedge (\forall *a p*. distinct
 (*transizioni* *A a p*))

lemma *nbai-nba-id-param*[*param*]: $(nbai-nba, id) \in \langle L, S \rangle nbai-nba-rel \rightarrow \langle L, S \rangle$
nba-rel
 $\langle proof \rangle$

lemma *nbai-nba-br*: $\langle Id, Id \rangle nbai-nba-rel = br\ nbai-nba\ nbai-invar$
 $\langle proof \rangle$

end

41 Algorithms on Nondeterministic Büchi Automata

theory *NBA-Algorithms*

imports

NBA-Graphs

NBA-Implement

DFS-Framework.Reachable-Nodes

Gabow-SCC.Gabow-GBG-Code

begin

41.1 Miscellaneous Amendments

lemma (in *igb-fr-graph*) *acc-run-lasso-prpl*: $Ex\ is-acc-run \implies Ex\ is-lasso-prpl$
 $\langle proof \rangle$

lemma (in *igb-fr-graph*) *lasso-prpl-acc-run-iff*: $Ex\ is-lasso-prpl \longleftrightarrow Ex\ is-acc-run$
 $\langle proof \rangle$

lemma [*autoref-rel-intf*]: *REL-INTF igbg-impl-rel-ext i-igbg* $\langle proof \rangle$

41.2 Operations

definition *op-language-empty* **where** [*simp*]: *op-language-empty* $A \equiv language\ A = \{\}$

lemmas [*autoref-op-pat*] = *op-language-empty-def*[*symmetric*]

41.3 Implementations

context

begin

interpretation *autoref-syn* $\langle proof \rangle$

lemma *nba-g-ahs*: $nba-g\ A = (\mid g-V = UNIV, g-E = E-of-succ\ (\lambda p.\ CAST\ ((\bigcup a \in alphabet\ A.\ transition\ A\ a\ p) :: \langle S \rangle\ list-set-rel) :: \langle S \rangle\ ahs-rel\ bhc)),\ g-V0 = initial\ A\ \mid)$
 $\langle proof \rangle$

schematic-goal *nbai-gi*:

notes [*autoref-ga-rules*] = *map2set-to-list*

fixes $S :: ('state_i \times 'state) \text{ set}$
assumes $[autoref-ga-rules]: is-bounded-hashcode\ S\ seq\ bhc$
assumes $[autoref-ga-rules]: is-valid-def-hm-size\ TYPE('state_i)\ hms$
assumes $[autoref-rules]: (seq, HOL.eq) \in S \rightarrow S \rightarrow bool-rel$
assumes $[autoref-rules]: (Ai, A) \in \langle L, S \rangle\ nbai-nba-rel$
shows $(?f :: ?'a, RETURN\ (nba-g\ A)) \in ?A$
 $\langle proof \rangle$

concrete-definition $nbai-gi$ **uses** $nbai-gi$

lemma $nbai-gi-refine[autoref-rules]:$

fixes $S :: ('state_i \times 'state) \text{ set}$
assumes $SIDE-GEN-ALGO\ (is-bounded-hashcode\ S\ seq\ bhc)$
assumes $SIDE-GEN-ALGO\ (is-valid-def-hm-size\ TYPE('state_i)\ hms)$
assumes $GEN-OP\ seq\ HOL.eq\ (S \rightarrow S \rightarrow bool-rel)$
shows $(NBA-Algorithms.nbai-gi\ seq\ bhc\ hms, nba-g) \in$
 $\langle L, S \rangle\ nbai-nba-rel \rightarrow \langle unit-rel, S \rangle\ g-impl-rel-ext$
 $\langle proof \rangle$

schematic-goal $nba-nodes:$

fixes $S :: ('state_i \times 'state) \text{ set}$
assumes $[simp]: finite\ ((g-E\ (nba-g\ A))^* \text{ “ } g-V0\ (nba-g\ A))$
assumes $[autoref-ga-rules]: is-bounded-hashcode\ S\ seq\ bhc$
assumes $[autoref-ga-rules]: is-valid-def-hm-size\ TYPE('state_i)\ hms$
assumes $[autoref-rules]: (seq, HOL.eq) \in S \rightarrow S \rightarrow bool-rel$
assumes $[autoref-rules]: (Ai, A) \in \langle L, S \rangle\ nbai-nba-rel$
shows $(?f :: ?'a, op-reachable\ (nba-g\ A)) \in ?R\ \langle proof \rangle$

concrete-definition $nba-nodes$ **uses** $nba-nodes$

lemma $nba-nodes-refine[autoref-rules]:$

fixes $S :: ('state_i \times 'state) \text{ set}$
assumes $SIDE-PRECOND\ (finite\ (nodes\ A))$
assumes $SIDE-GEN-ALGO\ (is-bounded-hashcode\ S\ seq\ bhc)$
assumes $SIDE-GEN-ALGO\ (is-valid-def-hm-size\ TYPE('state_i)\ hms)$
assumes $GEN-OP\ seq\ HOL.eq\ (S \rightarrow S \rightarrow bool-rel)$
assumes $(Ai, A) \in \langle L, S \rangle\ nbai-nba-rel$
shows $(NBA-Algorithms.nba-nodes\ seq\ bhc\ hms\ Ai,$
 $(OP\ nodes :: \langle L, S \rangle\ nbai-nba-rel \rightarrow \langle S \rangle\ ahs-rel\ bhc)\ \$\ A) \in \langle S \rangle\ ahs-rel\ bhc$
 $\langle proof \rangle$

lemma $nba-igbg-ahs: nba-igbg\ A = (\mid g-V = UNIV, g-E = E-of-succ\ (\lambda\ p.$
 $CAST$

$((\bigcup a \in alphabet\ A.\ transition\ A\ a\ p :: \langle S \rangle\ list-set-rel) :: \langle S \rangle\ ahs-rel\ bhc),$
 $g-V0 = initial\ A,$

$igbg-num-acc = 1, igbg-acc = \lambda\ p.\ if\ accepting\ A\ p\ then\ \{0\}\ else\ \{\}\ \mid)$
 $\langle proof \rangle$

schematic-goal $nbai-igbgi:$

notes $[autoref-ga-rules] = map2set-to-list$

fixes $S :: ('state_i \times 'state) \text{ set}$

assumes $[autoref-ga-rules]: is-bounded-hashcode\ S\ seq\ bhc$

assumes $[autoref-ga-rules]: is-valid-def-hm-size\ TYPE('state_i)\ hms$


```

assumes [autoref-rules]: (seq, HOL.eq) ∈ S → S → bool-rel
assumes [autoref-rules]: (Ai, A) ∈ ⟨L, S⟩ nbai-nba-rel
shows (?f :: ?'a, RETURN (nba-igbg A)) ∈ ?A
  ⟨proof⟩
concrete-definition nbai-igbgi uses nbai-igbgi
lemma nbai-igbgi-refine[autoref-rules]:
  fixes S :: ('statei × 'state) set
  assumes SIDE-GEN-ALGO (is-bounded-hashcode S seq bhc)
  assumes SIDE-GEN-ALGO (is-valid-def-hm-size TYPE('statei) hms)
  assumes GEN-OP seq HOL.eq (S → S → bool-rel)
  shows (NBA-Algorithms.nbai-igbgi seq bhc hms, nba-igbg) ∈
    ⟨L, S⟩ nbai-nba-rel → igbg-impl-rel-ext unit-rel S
  ⟨proof⟩

schematic-goal nba-language-empty:
  fixes S :: ('statei × 'state) set
  assumes [simp]: igb-fr-graph (nba-igbg A)
  assumes [autoref-ga-rules]: is-bounded-hashcode S seq bhs
  assumes [autoref-ga-rules]: is-valid-def-hm-size TYPE('statei) hms
  assumes [autoref-rules]: (seq, HOL.eq) ∈ S → S → bool-rel
  assumes [autoref-rules]: (Ai, A) ∈ ⟨L, S⟩ nbai-nba-rel
  shows (?f :: ?'a, do { r ← op-find-lasso-spec (nba-igbg A); RETURN (r =
None)}) ∈ ?A
  ⟨proof⟩
concrete-definition nba-language-empty uses nba-language-empty
lemma nba-language-empty-refine[autoref-rules]:
  fixes S :: ('statei × 'state) set
  assumes SIDE-PRECOND (finite (nodes A))
  assumes SIDE-GEN-ALGO (is-bounded-hashcode S seq bhc)
  assumes SIDE-GEN-ALGO (is-valid-def-hm-size TYPE('statei) hms)
  assumes GEN-OP seq HOL.eq (S → S → bool-rel)
  assumes (Ai, A) ∈ ⟨L, S⟩ nbai-nba-rel
  shows (NBA-Algorithms.nba-language-empty seq bhc hms Ai,
    (OP op-language-empty ::: ⟨L, S⟩ nbai-nba-rel → bool-rel) $ A) ∈ bool-rel
  ⟨proof⟩

end

end

```

42 Explicit Nondeterministic Büchi Automata

```

theory NBA-Explicit
imports NBA-Algorithms
begin

```

```

datatype ('label, 'state) nbae = nbae
  (alphabet: 'label set)
  (initiale: 'state set)

```

(*transitione*: ('state × 'label × 'state) set)
 (*acceptinge*: 'state set)

definition *nbae-rel* **where**

[*to-relAPP*]: *nbae-rel* L S \equiv $\{(A_1, A_2).$
 (*alphabetete* A_1 , *alphabetete* A_2) \in $\langle L \rangle$ *set-rel* \wedge
 (*initiale* A_1 , *initiale* A_2) \in $\langle S \rangle$ *set-rel* \wedge
 (*transitione* A_1 , *transitione* A_2) \in $\langle S \times_r L \times_r S \rangle$ *set-rel* \wedge
 (*acceptinge* A_1 , *acceptinge* A_2) \in $\langle S \rangle$ *set-rel $\}$*

lemma *nbae-param*[*param*, *autoref-rules*]:

(*nbae*, *nbae*) \in $\langle L \rangle$ *set-rel* \rightarrow $\langle S \rangle$ *set-rel* \rightarrow $\langle S \times_r L \times_r S \rangle$ *set-rel* \rightarrow
 $\langle S \rangle$ *set-rel* \rightarrow $\langle L, S \rangle$ *nbae-rel*
 (*alphabetete*, *alphabetete*) \in $\langle L, S \rangle$ *nbae-rel* \rightarrow $\langle L \rangle$ *set-rel*
 (*initiale*, *initiale*) \in $\langle L, S \rangle$ *nbae-rel* \rightarrow $\langle S \rangle$ *set-rel*
 (*transitione*, *transitione*) \in $\langle L, S \rangle$ *nbae-rel* \rightarrow $\langle S \times_r L \times_r S \rangle$ *set-rel*
 (*acceptinge*, *acceptinge*) \in $\langle L, S \rangle$ *nbae-rel* \rightarrow $\langle S \rangle$ *set-rel*
 \langle *proof* \rangle

lemma *nbae-rel-id*[*simp*]: $\langle Id, Id \rangle$ *nbae-rel* = Id \langle *proof* \rangle

lemma *nbae-rel-comp*[*simp*]: $\langle L_1 \ O \ L_2, S_1 \ O \ S_2 \rangle$ *nbae-rel* = $\langle L_1, S_1 \rangle$ *nbae-rel* O
 $\langle L_2, S_2 \rangle$ *nbae-rel*
 \langle *proof* \rangle

consts *i-nbae-scheme* :: *interface* \Rightarrow *interface* \Rightarrow *interface*

context

begin

interpretation *autoref-syn* \langle *proof* \rangle

lemma *nbae-scheme-itype*[*autoref-itype*]:

nbae ::_{*i*} $\langle L \rangle_i$ *i-set* \rightarrow_i $\langle S \rangle_i$ *i-set* \rightarrow_i $\langle \langle S, \langle L, S \rangle_i \ i\text{-prod} \rangle_i \ i\text{-prod} \rangle_i$ *i-set* \rightarrow_i $\langle S \rangle_i$
i-set \rightarrow_i
 $\langle L, S \rangle_i$ *i-nbae-scheme*
alphabetete ::_{*i*} $\langle L, S \rangle_i$ *i-nbae-scheme* \rightarrow_i $\langle L \rangle_i$ *i-set*
initiale ::_{*i*} $\langle L, S \rangle_i$ *i-nbae-scheme* \rightarrow_i $\langle S \rangle_i$ *i-set*
transitione ::_{*i*} $\langle L, S \rangle_i$ *i-nbae-scheme* \rightarrow_i $\langle \langle S, \langle L, S \rangle_i \ i\text{-prod} \rangle_i \ i\text{-prod} \rangle_i$ *i-set*
acceptinge ::_{*i*} $\langle L, S \rangle_i$ *i-nbae-scheme* \rightarrow_i $\langle S \rangle_i$ *i-set*
 \langle *proof* \rangle

end

datatype ('label, 'state) *nbaei* = *nbaei*

(*alphabetei*: 'label list)
 (*initialei*: 'state list)
 (*transitionei*: ('state × 'label × 'state) list)
 (*acceptingei*: 'state list)

definition *nbaei-rel* **where**

[*to-relAPP*]: $nbaei-rel\ L\ S \equiv \{(A_1, A_2).$
 $(alphabet\ A_1, alphabet\ A_2) \in \langle L \rangle\ list-rel \wedge$
 $(initiale\ A_1, initiale\ A_2) \in \langle S \rangle\ list-rel \wedge$
 $(transition\ A_1, transition\ A_2) \in \langle S \times_r L \times_r S \rangle\ list-rel \wedge$
 $(accepting\ A_1, accepting\ A_2) \in \langle S \rangle\ list-rel\}$

lemma *nbaei-param*[*param, autoref-rules*]:

$(nbaei, nbaei) \in \langle L \rangle\ list-rel \rightarrow \langle S \rangle\ list-rel \rightarrow \langle S \times_r L \times_r S \rangle\ list-rel \rightarrow$
 $\langle S \rangle\ list-rel \rightarrow \langle L, S \rangle\ nbaei-rel$
 $(alphabet\ A_1, alphabet\ A_2) \in \langle L, S \rangle\ nbaei-rel \rightarrow \langle L \rangle\ list-rel$
 $(initiale\ A_1, initiale\ A_2) \in \langle L, S \rangle\ nbaei-rel \rightarrow \langle S \rangle\ list-rel$
 $(transition\ A_1, transition\ A_2) \in \langle L, S \rangle\ nbaei-rel \rightarrow \langle S \times_r L \times_r S \rangle\ list-rel$
 $(accepting\ A_1, accepting\ A_2) \in \langle L, S \rangle\ nbaei-rel \rightarrow \langle S \rangle\ list-rel$
 $\langle proof \rangle$

definition *nbaei-nbae-rel* **where**

[*to-relAPP*]: $nbaei-nbae-rel\ L\ S \equiv \{(A_1, A_2).$
 $(alphabet\ A_1, alphabet\ A_2) \in \langle L \rangle\ list-set-rel \wedge$
 $(initiale\ A_1, initiale\ A_2) \in \langle S \rangle\ list-set-rel \wedge$
 $(transition\ A_1, transition\ A_2) \in \langle S \times_r L \times_r S \rangle\ list-set-rel \wedge$
 $(accepting\ A_1, accepting\ A_2) \in \langle S \rangle\ list-set-rel\}$

lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of nbaei-nbae-rel i-nbae-scheme*]

lemma *nbaei-nbae-param*[*param, autoref-rules*]:

$(nbaei, nbae) \in \langle L \rangle\ list-set-rel \rightarrow \langle S \rangle\ list-set-rel \rightarrow \langle S \times_r L \times_r S \rangle\ list-set-rel$
 \rightarrow
 $\langle S \rangle\ list-set-rel \rightarrow \langle L, S \rangle\ nbaei-nbae-rel$
 $(alphabet\ A_1, alphabet\ A_2) \in \langle L, S \rangle\ nbaei-nbae-rel \rightarrow \langle L \rangle\ list-set-rel$
 $(initiale\ A_1, initiale\ A_2) \in \langle L, S \rangle\ nbaei-nbae-rel \rightarrow \langle S \rangle\ list-set-rel$
 $(transition\ A_1, transition\ A_2) \in \langle L, S \rangle\ nbaei-nbae-rel \rightarrow \langle S \times_r L \times_r S \rangle\ list-set-rel$
 $(accepting\ A_1, accepting\ A_2) \in \langle L, S \rangle\ nbaei-nbae-rel \rightarrow \langle S \rangle\ list-set-rel$
 $\langle proof \rangle$

definition *nbaei-nbae* **where**

$nbaei-nbae\ A \equiv nbae\ (set\ (alphabet\ A))\ (set\ (initiale\ A))$
 $(set\ (transition\ A))\ (set\ (accepting\ A))$

lemma *nbaei-nbae-id-param*[*param*]: $(nbaei-nbae, id) \in \langle L, S \rangle\ nbaei-nbae-rel \rightarrow$
 $\langle L, S \rangle\ nbae-rel$
 $\langle proof \rangle$

abbreviation *transitions* $L\ S\ s \equiv \bigcup a \in L. \bigcup p \in S. \{p\} \times \{a\} \times s\ a\ p$

abbreviation *succs* $T\ a\ p \equiv (T\ \{\{p\}\}\ \{\{a\}\})$

definition *nba-nbae* **where** $nba-nbae\ A \equiv nbae\ (alphabet\ A)\ (initial\ A)$

$(transitions\ (alphabet\ A))\ (nodes\ A)\ (transition\ A)\ (Set.filter\ (accepting\ A))$

(*nodes A*)

definition *nbae-nba* **where** *nbae-nba A* \equiv *nba* (*alphabet A*) (*initiale A*)
(*succs* (*transition A*)) ($\lambda p. p \in$ *accepting A*)

lemma *nba-nbae-param*[*param*]: (*nba-nbae*, *nba-nbae*) \in $\langle L, S \rangle$ *nba-rel* \rightarrow $\langle L, S \rangle$
nbae-rel
 \langle *proof* \rangle

lemma *nbae-nba-param*[*param*]:
assumes *bijective L bijective S*
shows (*nbae-nba*, *nbae-nba*) \in $\langle L, S \rangle$ *nbae-rel* \rightarrow $\langle L, S \rangle$ *nba-rel*
 \langle *proof* \rangle

lemma *nbae-nba-nba-nbae-param*[*param*]:
(*nbae-nba* \circ *nba-nbae*) *A*, *id A*) \in \langle *Id-on* (*alphabet A*), *Id-on* (*nodes A*) \rangle *nba-rel*
 \langle *proof* \rangle

definition *nbaei-nba-rel* **where**
[*to-relAPP*]: *nbaei-nba-rel L S* \equiv $\{(Ae, A). (nbae-nba (nbaei-nbae Ae), A) \in \langle L, S \rangle nba-rel\}$

lemma *nbaei-nba-id*[*param*]: (*nbae-nba* \circ *nbaei-nbae*, *id*) \in $\langle L, S \rangle$ *nbaei-nba-rel*
 \rightarrow $\langle L, S \rangle$ *nba-rel*
 \langle *proof* \rangle

schematic-goal *nbae-nba-impl*:
assumes [*autoref-rules*]: (*leq*, *HOL.eq*) \in $L \rightarrow L \rightarrow$ *bool-rel*
assumes [*autoref-rules*]: (*seq*, *HOL.eq*) \in $S \rightarrow S \rightarrow$ *bool-rel*
shows (*?f*, *nbae-nba*) \in $\langle L, S \rangle$ *nbaei-nbae-rel* \rightarrow $\langle L, S \rangle$ *nbae-nba-rel*
 \langle *proof* \rangle

concrete-definition *nbae-nba-impl* **uses** *nbae-nba-impl*

lemma *nbae-nba-impl-refine*[*autoref-rules*]:
assumes *GEN-OP leq HOL.eq* ($L \rightarrow L \rightarrow$ *bool-rel*)
assumes *GEN-OP seq HOL.eq* ($S \rightarrow S \rightarrow$ *bool-rel*)
shows (*nbae-nba-impl leq seq*, *nbae-nba*) \in $\langle L, S \rangle$ *nbaei-nbae-rel* \rightarrow $\langle L, S \rangle$
nbae-nba-rel
 \langle *proof* \rangle

end

43 Explore and Enumerate Nodes of Nondeterministic Büchi Automata

theory *NBA-Translate*
imports *NBA-Explicit*
begin

43.1 Syntax

no-syntax *-do-let* :: [*pstrn*, 'a] \Rightarrow *do-bind* ((*2let* - =/ -) [*1000*, *13*] *13*)

syntax *-do-let* :: [pttrn, 'a] ⇒ do-bind ((2let - =/ -) 13)

44 Image on Explicit Automata

definition *nbae-image* **where** $nbae\text{-image } f A \equiv nbae \text{ (alphabet } A) \text{ (} f \text{ ' initiale } A)$

$((\lambda (p, a, q). (f p, a, f q)) \text{ ' transitione } A) \text{ (} f \text{ ' acceptinge } A)$

lemma *nbae-image-param*[param]: $(nbae\text{-image}, nbae\text{-image}) \in (S \rightarrow T) \rightarrow \langle L, S \rangle nbae\text{-rel} \rightarrow \langle L, T \rangle nbae\text{-rel}$

$\langle proof \rangle$

lemma *nbae-image-id*[simp]: $nbae\text{-image } id = id \text{ } \langle proof \rangle$

lemma *nbae-image-nba-nbae*: $nbae\text{-image } f \text{ (nba-nbae } A) = nbae \text{ (alphabet } A) \text{ (} f \text{ ' initial } A)$

$(\bigcup p \in nodes A. \bigcup a \in alphabet A. f \text{ ' } \{p\} \times \{a\} \times f \text{ ' transition } A a p)$

$(f \text{ ' } \{p \in nodes A. accepting A p\})$

$\langle proof \rangle$

45 Exploration and Translation

definition *trans-spec* **where**

$trans\text{-spec } A f \equiv \bigcup p \in nodes A. \bigcup a \in alphabet A. f \text{ ' } \{p\} \times \{a\} \times f \text{ ' transition } A a p$

definition *trans-algo* **where**

$trans\text{-algo } N L S f \equiv$

```

FOREACH N (λ p T. do {
  ASSERT (p ∈ N);
  FOREACH L (λ a T. do {
    ASSERT (a ∈ L);
    FOREACH (S a p) (λ q T. do {
      ASSERT (q ∈ S a p);
      ASSERT ((f p, a, f q) ∉ T);
      RETURN (insert (f p, a, f q) T) }
    ) T }
  ) T }
) {}

```

lemma *trans-algo-refine*:

assumes $finite \text{ (nodes } A) \text{ finite (alphabet } A) inj\text{-on } f \text{ (nodes } A)$

assumes $N = nodes A \ L = alphabet A \ S = transition A$

shows $(trans\text{-algo } N L S f, SPEC (HOL.eq (trans\text{-spec } A f))) \in \langle Id \rangle nres\text{-rel}$

$\langle proof \rangle$

definition *nba-image* :: $(state_1 \Rightarrow state_2) \Rightarrow (label, state_1) nba \Rightarrow (label, state_2) nba$ **where**

nba-image $f A \equiv nba$
 (alphabet A)
 (f 'initial A)
 ($\lambda a p. f$ 'transition $A a$ (*inv-into* (*nodes* A) $f p$))
 ($\lambda p. accepting A$ (*inv-into* (*nodes* A) $f p$))

lemma *nba-image-rel*[*param*]:
assumes *inj-on* f (*nodes* A)
shows $(A, nba-image f A) \in \langle Id-on (alphabet A), br f (\lambda p. p \in nodes A) \rangle$
nba-rel
 <*proof*>

lemma *nba-image-nodes*[*simp*]:
assumes *inj-on* f (*nodes* A)
shows *nodes* (*nba-image* $f A$) = f ' *nodes* A
 <*proof*>

lemma *nba-image-language*[*simp*]:
assumes *inj-on* f (*nodes* A)
shows *language* (*nba-image* $f A$) = *language* A
 <*proof*>

lemma *nba-image-nbae*:
assumes *inj-on* f (*nodes* A)
shows *nbae-image* f (*nba-nbae* A) = *nba-nbae* (*nba-image* $f A$)
 <*proof*>

definition *op-translate* :: (*'label*, *'state*) *nba* \Rightarrow (*'label*, *nat*) *nbae nres* **where**
op-translate $A \equiv SPEC (\lambda B. \exists f. inj-on f (nodes A) \wedge B = nba-nbae (nba-image f A))$

lemma *op-translate-language*:
assumes (*RETURN* A_i , *op-translate* A) $\in \langle \langle Id, nat-rel \rangle nbaei-nbae-rel \rangle nres-rel$
shows *language* (*nbae-nba* (*nbaei-nbae* A_i)) = *language* A
 <*proof*>

schematic-goal *to-nbaei-impl*:
fixes $S :: ('state_i \times 'state) set$
assumes [*simp*]: *finite* (*nodes* A)
assumes [*autoref-ga-rules*]: *is-bounded-hashcode* $S seq bhc$
assumes [*autoref-ga-rules*]: *is-valid-def-hm-size* $TYPE('state_i) hms$
assumes [*autoref-rules*]: $(seq, HOL.eq) \in S \rightarrow S \rightarrow bool-rel$
assumes [*autoref-rules*]: $(A_i, A) \in \langle L, S \rangle nba_i-nba-rel$
shows ($?f :: ?'a$, do {
 let $N = nodes A$;
 $f \leftarrow op-set-enumerate N$;
 })

```

    ASSERT (dom f = N);
    ASSERT (∀ p ∈ initial A. f p ≠ None);
    ASSERT (∀ a ∈ alphabet A. ∀ p ∈ dom f. ∀ q ∈ transition A a p. f q ≠
None);
    T ← trans-algo N (alphabet A) (transition A) (λ x. the (f x));
    RETURN (nbae (alphabet A) ((λ x. the (f x)) ' initial A) T
      ((λ x. the (f x)) ' {p ∈ N. accepting A p}))
  }) ∈ ?R
  <proof>
concrete-definition to-nbaei-impl uses to-nbaei-impl

```

```

context
begin

```

```

  interpretation autoref-syn <proof>

```

```

lemma to-nbaei-impl-refine[autoref-rules]:

```

```

  fixes S :: ('statei × 'state) set
  assumes SIDE-PRECOND (finite (nodes A))
  assumes SIDE-GEN-ALGO (is-bounded-hashcode S seq bhc)
  assumes SIDE-GEN-ALGO (is-valid-def-hm-size TYPE('statei) hms)
  assumes GEN-OP seq HOL.eq (S → S → bool-rel)
  assumes (Ai, A) ∈ ⟨L, S⟩ nbai-nba-rel
  shows (RETURN (to-nbaei-impl seq bhc hms Ai),
    (OP op-translate ::: ⟨L, S⟩ nbai-nba-rel → ⟨⟨L, nat-rel⟩ nbaei-nbae-rel
nres-rel) $ A) ∈
    ⟨⟨L, nat-rel⟩ nbaei-nbae-rel⟩ nres-rel
  <proof>

```

```

end

```

```

end

```

46 Connecting Nondeterministic Generalized Büchi Automata to CAVA Automata Structures

```

theory NGBA-Graphs
imports
  NGBA
  CAVA-Automata.Automata-Impl
begin

```

```

  no-notation build (infixr ## 65)

```

46.1 Regular Graphs

```

definition ngba-g :: ('label, 'state) ngba ⇒ 'state graph-rec where
  ngba-g A ≡ (| g-V = UNIV, g-E = E-of-succ (successors A), g-V0 = initial A

```

)

lemma *ngba-g-graph[simp]*: *graph (ngba-g A) <proof>*

lemma *ngba-g-V0*: *g-V0 (ngba-g A) = initial A <proof>*

lemma *ngba-g-E-rtrancl*: *(g-E (ngba-g A))* = {(p, q). q ∈ reachable A p}*
<proof>

lemma *ngba-g-rtrancl-path*: *(g-E (ngba-g A))* = {(p, target r p) | r p. NGBA.path A r p}*
<proof>

lemma *ngba-g-trancl-path*: *(g-E (ngba-g A))^+ = {(p, target r p) | r p. NGBA.path A r p ∧ r ≠ []}*
<proof>

lemma *ngba-g-ipath-run*:

assumes *ipath (g-E (ngba-g A)) r*

obtains *w*

where *run A (w ||| smap (r ∘ Suc) nats) (r 0)*

<proof>

lemma *ngba-g-run-ipath*:

assumes *run A (w ||| r) p*

shows *ipath (g-E (ngba-g A)) (snth (p ## r))*

<proof>

46.2 Indexed Generalized Büchi Graphs

definition *ngba-acc* :: *'state pred gen ⇒ 'state ⇒ nat set where*

ngba-acc cs p ≡ {k ∈ {0 ..< length cs}. (cs ! k) p}

lemma *ngba-acc-param[param]*: *(ngba-acc, ngba-acc) ∈ (S → bool-rel) list-rel → S → (nat-rel) set-rel*
<proof>

definition *ngba-igbg* :: *('label, 'state) ngba ⇒ 'state igb-graph-rec where*

ngba-igbg A ≡ graph-rec.extend (ngba-g A) (| igbg-num-acc = length (accepting A), igbg-acc = ngba-acc (accepting A) |)

lemma *acc-run-language*:

assumes *igbg-graph (ngba-igbg A)*

shows *Ex (igbg-graph.is-acc-run (ngba-igbg A)) ↔ language A ≠ {}*

<proof>

end

47 Relations on Nondeterministic Generalized Büchi Automata

theory *NGBA-Refine*

imports

NGBA

../Transition-Systems/Transition-System-Refine

begin

definition *ngba-rel* :: ('label₁ × 'label₂) set ⇒ ('state₁ × 'state₂) set ⇒
 (('label₁, 'state₁) ngba × ('label₂, 'state₂) ngba) set **where**
 [*to-relAPP*]: *ngba-rel* L S ≡ {(A₁, A₂).
 (alphabet A₁, alphabet A₂) ∈ ⟨L⟩ set-rel ∧
 (initial A₁, initial A₂) ∈ ⟨S⟩ set-rel ∧
 (transition A₁, transition A₂) ∈ L → S → ⟨S⟩ set-rel ∧
 (accepting A₁, accepting A₂) ∈ ⟨S → bool-rel⟩ list-rel}

lemma *ngba-param*[*param*]:

(ngba, ngba) ∈ ⟨L⟩ set-rel → ⟨S⟩ set-rel → (L → S → ⟨S⟩ set-rel) → ⟨S →
 bool-rel⟩ list-rel →
 ⟨L, S⟩ ngba-rel
 (alphabet, alphabet) ∈ ⟨L, S⟩ ngba-rel → ⟨L⟩ set-rel
 (initial, initial) ∈ ⟨L, S⟩ ngba-rel → ⟨S⟩ set-rel
 (transition, transition) ∈ ⟨L, S⟩ ngba-rel → L → S → ⟨S⟩ set-rel
 (accepting, accepting) ∈ ⟨L, S⟩ ngba-rel → ⟨S → bool-rel⟩ list-rel
 ⟨proof⟩

lemma *ngba-rel-id*[*simp*]: ⟨Id, Id⟩ ngba-rel = Id ⟨proof⟩

lemma *enableds-param*[*param*]: (ngba.enableds, ngba.enableds) ∈ ⟨L, S⟩ ngba-rel
 → S → ⟨L ×_r S⟩ set-rel
 ⟨proof⟩

lemma *paths-param*[*param*]: (ngba.paths, ngba.paths) ∈ ⟨L, S⟩ ngba-rel → S →
 ⟨⟨L ×_r S⟩ list-rel⟩ set-rel
 ⟨proof⟩

lemma *runs-param*[*param*]: (ngba.runs, ngba.runs) ∈ ⟨L, S⟩ ngba-rel → S → ⟨⟨L
 ×_r S⟩ stream-rel⟩ set-rel
 ⟨proof⟩

lemma *reachable-param*[*param*]: (reachable, reachable) ∈ ⟨L, S⟩ ngba-rel → S →
 ⟨S⟩ set-rel
 ⟨proof⟩

lemma *nodes-param*[*param*]: (nodes, nodes) ∈ ⟨L, S⟩ ngba-rel → ⟨S⟩ set-rel
 ⟨proof⟩

lemma *gen-param*[*param*]: (gen, gen) ∈ (A → B → bool-rel) → ⟨A⟩ list-rel → B
 → bool-rel
 ⟨proof⟩

lemma *language-param*[*param*]: (*language*, *language*) ∈ ⟨*L*, *S*⟩ *ngba-rel* → ⟨⟨*L*⟩
stream-rel⟩ *set-rel*
 ⟨*proof*⟩

end

48 Implementation of Nondeterministic Generalized Büchi Automata

theory *NGBA-Implement*

imports

NGBA-Refine

../Basic/Implement

begin

consts *i-ngba-scheme* :: *interface* ⇒ *interface* ⇒ *interface*

context

begin

interpretation *autoref-syn* ⟨*proof*⟩

lemma *ngba-scheme-itype*[*autoref-itype*]:

ngba ::_{*i*} ⟨*L*⟩_{*i*} *i-set* →_{*i*} ⟨*S*⟩_{*i*} *i-set* →_{*i*} (*L* →_{*i*} *S* →_{*i*} ⟨*S*⟩_{*i*} *i-set*) →_{*i*} ⟨⟨*S*⟩_{*i*} *i-set*⟩_{*i*}
i-list →_{*i*}

⟨*L*, *S*⟩_{*i*} *i-ngba-scheme*

alphabet ::_{*i*} ⟨*L*, *S*⟩_{*i*} *i-ngba-scheme* →_{*i*} ⟨*L*⟩_{*i*} *i-set*

initial ::_{*i*} ⟨*L*, *S*⟩_{*i*} *i-ngba-scheme* →_{*i*} ⟨*S*⟩_{*i*} *i-set*

transition ::_{*i*} ⟨*L*, *S*⟩_{*i*} *i-ngba-scheme* →_{*i*} *L* →_{*i*} *S* →_{*i*} ⟨*S*⟩_{*i*} *i-set*

accepting ::_{*i*} ⟨*L*, *S*⟩_{*i*} *i-ngba-scheme* →_{*i*} ⟨⟨*S*⟩_{*i*} *i-set*⟩_{*i*} *i-list*

⟨*proof*⟩

end

datatype (*'label*, *'state*) *ngbai* = *ngbai*

(*alphabeti*: *'label list*)

(*initiali*: *'state list*)

(*transitioni*: *'label* ⇒ *'state* ⇒ *'state list*)

(*acceptingi*: (*'state* ⇒ *bool*) *list*)

definition *ngbai-rel* :: (*'label*₁ × *'label*₂) *set* ⇒ (*'state*₁ × *'state*₂) *set* ⇒

((*'label*₁, *'state*₁) *ngbai* × (*'label*₂, *'state*₂) *ngbai*) *set* **where**

[*to-relAPP*]: *ngbai-rel* *L S* ≡ {(*A*₁, *A*₂).

(*alphabeti* *A*₁, *alphabeti* *A*₂) ∈ ⟨*L*⟩ *list-rel* ∧

(*initiali* *A*₁, *initiali* *A*₂) ∈ ⟨*S*⟩ *list-rel* ∧

(*transitioni* *A*₁, *transitioni* *A*₂) ∈ *L* → *S* → ⟨*S*⟩ *list-rel* ∧

(*acceptingi* *A*₁, *acceptingi* *A*₂) ∈ ⟨*S* → *bool-rel*⟩ *list-rel*}

lemma *ngbai-param*[*param*]:

$(ngbai, ngbai) \in \langle L \rangle \text{ list-rel} \rightarrow \langle S \rangle \text{ list-rel} \rightarrow (L \rightarrow S \rightarrow \langle S \rangle \text{ list-rel}) \rightarrow$
 $\langle S \rightarrow \text{bool-rel} \rangle \text{ list-rel} \rightarrow \langle L, S \rangle \text{ ngbai-rel}$
 $(\text{alphabeti}, \text{alphabeti}) \in \langle L, S \rangle \text{ ngbai-rel} \rightarrow \langle L \rangle \text{ list-rel}$
 $(\text{initiali}, \text{initiali}) \in \langle L, S \rangle \text{ ngbai-rel} \rightarrow \langle S \rangle \text{ list-rel}$
 $(\text{transitioni}, \text{transitioni}) \in \langle L, S \rangle \text{ ngbai-rel} \rightarrow L \rightarrow S \rightarrow \langle S \rangle \text{ list-rel}$
 $(\text{acceptingi}, \text{acceptingi}) \in \langle L, S \rangle \text{ ngbai-rel} \rightarrow \langle S \rightarrow \text{bool-rel} \rangle \text{ list-rel}$
 $\langle \text{proof} \rangle$

definition *ngbai-ngba-rel* :: $(\text{'label}_1 \times \text{'label}_2) \text{ set} \Rightarrow (\text{'state}_1 \times \text{'state}_2) \text{ set} \Rightarrow$
 $((\text{'label}_1, \text{'state}_1) \text{ ngbai} \times (\text{'label}_2, \text{'state}_2) \text{ ngba}) \text{ set}$ **where**
 $[to\text{-relAPP}]: \text{ngbai-ngba-rel } L \ S \equiv \{(A_1, A_2).$
 $(\text{alphabeti } A_1, \text{alphabet } A_2) \in \langle L \rangle \text{ list-set-rel} \wedge$
 $(\text{initiali } A_1, \text{initial } A_2) \in \langle S \rangle \text{ list-set-rel} \wedge$
 $(\text{transitioni } A_1, \text{transition } A_2) \in L \rightarrow S \rightarrow \langle S \rangle \text{ list-set-rel} \wedge$
 $(\text{acceptingi } A_1, \text{accepting } A_2) \in \langle S \rightarrow \text{bool-rel} \rangle \text{ list-rel}\}$

lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of ngbai-ngba-rel i-ngba-scheme*]

lemma *ngbai-ngba-param*[*param, autoref-rules*]:

$(ngbai, ngba) \in \langle L \rangle \text{ list-set-rel} \rightarrow \langle S \rangle \text{ list-set-rel} \rightarrow (L \rightarrow S \rightarrow \langle S \rangle \text{ list-set-rel})$
 \rightarrow
 $\langle S \rightarrow \text{bool-rel} \rangle \text{ list-rel} \rightarrow \langle L, S \rangle \text{ ngbai-ngba-rel}$
 $(\text{alphabeti}, \text{alphabet}) \in \langle L, S \rangle \text{ ngbai-ngba-rel} \rightarrow \langle L \rangle \text{ list-set-rel}$
 $(\text{initiali}, \text{initial}) \in \langle L, S \rangle \text{ ngbai-ngba-rel} \rightarrow \langle S \rangle \text{ list-set-rel}$
 $(\text{transitioni}, \text{transition}) \in \langle L, S \rangle \text{ ngbai-ngba-rel} \rightarrow L \rightarrow S \rightarrow \langle S \rangle \text{ list-set-rel}$
 $(\text{acceptingi}, \text{accepting}) \in \langle L, S \rangle \text{ ngbai-ngba-rel} \rightarrow \langle S \rightarrow \text{bool-rel} \rangle \text{ list-rel}$
 $\langle \text{proof} \rangle$

definition *ngbai-ngba* :: $(\text{'label}, \text{'state}) \text{ ngbai} \Rightarrow (\text{'label}, \text{'state}) \text{ ngba}$ **where**
 $\text{ngbai-ngba } A \equiv \text{ngba} (\text{set } (\text{alphabeti } A)) (\text{set } (\text{initiali } A)) (\lambda a \text{ p. set } (\text{transitioni } A \ a \ p)) (\text{acceptingi } A)$

definition *ngbai-invar* :: $(\text{'label}, \text{'state}) \text{ ngbai} \Rightarrow \text{bool}$ **where**
 $\text{ngbai-invar } A \equiv \text{distinct } (\text{alphabeti } A) \wedge \text{distinct } (\text{initiali } A) \wedge (\forall a \text{ p. distinct } (\text{transitioni } A \ a \ p))$

lemma *ngbai-ngba-id-param*[*param*]: $(\text{ngbai-ngba}, \text{id}) \in \langle L, S \rangle \text{ ngbai-ngba-rel} \rightarrow$
 $\langle L, S \rangle \text{ ngba-rel}$
 $\langle \text{proof} \rangle$

lemma *ngbai-ngba-br*: $\langle \text{Id}, \text{Id} \rangle \text{ ngbai-ngba-rel} = \text{br ngbai-ngba ngbai-invar}$
 $\langle \text{proof} \rangle$

end

theory *Degeneralization-Refine*

imports *Degeneralization Refine*

begin

lemma *degen-param*[*param*]: (*degen*, *degen*) ∈ ⟨*S* → *bool-rel*⟩ *list-rel* → *S* ×_{*r*}
nat-rel → *bool-rel*
 ⟨*proof*⟩

lemma *count-param*[*param*]: (*Degeneralization.count*, *Degeneralization.count*) ∈
 ⟨*A* → *bool-rel*⟩ *list-rel* → *A* → *nat-rel* → *nat-rel*
 ⟨*proof*⟩

end

49 Algorithms on Nondeterministic Generalized Büchi Automata

theory *NGBA-Algorithms*

imports

NGBA-Graphs
NGBA-Implement
NBA-Combine
NBA-Algorithms
Degeneralization-Refine

begin

49.1 Operations

definition *op-language-empty* **where** [*simp*]: *op-language-empty* *A* ≡ *NGBA.language*
A = {}

lemmas [*autoref-op-pat*] = *op-language-empty-def*[*symmetric*]

49.2 Implementations

context

begin

interpretation *autoref-syn* ⟨*proof*⟩

lemma *ngba-g-ahs*: *ngba-g* *A* = (| *g-V* = *UNIV*, *g-E* = *E-of-succ* (λ *p*. *CAST*
 ((∪ *a* ∈ *ngba.alphabet* *A*. *ngba.transition* *A* *a* *p* :: ⟨*S*⟩ *list-set-rel*) :: ⟨*S*⟩
ahs-rel *bhc*)),
g-V0 = *ngba.initial* *A* |)
 ⟨*proof*⟩

schematic-goal *ngbai-gi*:

notes [*autoref-ga-rules*] = *map2set-to-list*

fixes *S* :: ('*statei* × '*state*) *set*

assumes [*autoref-ga-rules*]: *is-bounded-hashcode* *S* *seq* *bhc*

assumes [*autoref-ga-rules*]: *is-valid-def-hm-size* *TYPE*(''*statei*) *hms*

assumes [*autoref-rules*]: (*seq*, *HOL.eq*) ∈ *S* → *S* → *bool-rel*

assumes [*autoref-rules*]: (*Ai*, *A*) ∈ ⟨*L*, *S*⟩ *ngbai-ngba-rel*

shows ($?f :: ?'a, RETURN (ngba-g A) \in ?A$)
 $\langle proof \rangle$

concrete-definition *ngbai-gi uses ngbai-gi*

lemma *ngbai-gi-refine*[*autoref-rules*]:

fixes $S :: ('state_i \times 'state) set$

assumes *SIDE-GEN-ALGO* (*is-bounded-hashcode* $S seq bhc$)

assumes *SIDE-GEN-ALGO* (*is-valid-def-hm-size* $TYPE('state_i) hms$)

assumes *GEN-OP seq HOL.eq* ($S \rightarrow S \rightarrow bool-rel$)

shows (*NGBA-Algorithms.ngbai-gi seq bhc hms, ngba-g*) \in

$\langle L, S \rangle ngbai-ngba-rel \rightarrow \langle unit-rel, S \rangle g-impl-rel-ext$

$\langle proof \rangle$

schematic-goal *ngba-nodes*:

fixes $S :: ('state_i \times 'state) set$

assumes [*simp*]: *finite* ($((g-E (ngba-g A))^* \text{“} g-V0 (ngba-g A))$)

assumes [*autoref-ga-rules*]: *is-bounded-hashcode* $S seq bhc$

assumes [*autoref-ga-rules*]: *is-valid-def-hm-size* $TYPE('state_i) hms$

assumes [*autoref-rules*]: (*seq, HOL.eq*) $\in S \rightarrow S \rightarrow bool-rel$

assumes [*autoref-rules*]: (A_i, A) $\in \langle L, S \rangle ngbai-ngba-rel$

shows ($?f :: ?'a, op-reachable (ngba-g A) \in ?R$) $\langle proof \rangle$

concrete-definition *ngba-nodes uses ngba-nodes*

lemma *ngba-nodes-refine*[*autoref-rules*]:

fixes $S :: ('state_i \times 'state) set$

assumes *SIDE-PRECOND* (*finite* ($NGBA.nodes A$))

assumes *SIDE-GEN-ALGO* (*is-bounded-hashcode* $S seq bhc$)

assumes *SIDE-GEN-ALGO* (*is-valid-def-hm-size* $TYPE('state_i) hms$)

assumes *GEN-OP seq HOL.eq* ($S \rightarrow S \rightarrow bool-rel$)

assumes (A_i, A) $\in \langle L, S \rangle ngbai-ngba-rel$

shows (*NGBA-Algorithms.ngba-nodes seq bhc hms* $A_i,$

$(OP\ NGBA.nodes :: \langle L, S \rangle ngbai-ngba-rel \rightarrow \langle S \rangle ahs-rel\ bhc) \$ A) \in \langle S \rangle$

ahs-rel bhc

$\langle proof \rangle$

lemma *ngba-igbg-ahs*: $ngba-igbg\ A = (\mid g-V = UNIV, g-E = E-of-succ\ (\lambda p.$
CAST

$((\bigcup a \in NGBA.alphabet\ A. NGBA.transition\ A\ a\ p :: \langle S \rangle list-set-rel) :: \langle S \rangle$

ahs-rel bhc), $g-V0 = NGBA.initial\ A,$

$igbg-num-acc = length\ (NGBA.accepting\ A), igbg-acc = ngba-acc\ (NGBA.accepting$
 $A) \mid)$

$\langle proof \rangle$

definition *ngba-acc-bs* $cs\ p \equiv fold\ (\lambda (k, c)\ bs. if\ c\ p\ then\ bs-insert\ k\ bs\ else$
 $bs) (List.enumerate\ 0\ cs) (bs-empty\ ())$

lemma *ngba-acc-bs-empty*[*simp*]: $ngba-acc-bs\ []\ p = bs-empty\ ()$ $\langle proof \rangle$

lemma *ngba-acc-bs-insert*[*simp*]:

assumes $c\ p$

shows $ngba-acc-bs\ (cs\ @\ [c])\ p = bs-insert\ (length\ cs)\ (ngba-acc-bs\ cs\ p)$

$\langle proof \rangle$

lemma *ngba-acc-bs-skip*[simp]:

assumes $\neg c\ p$

shows *ngba-acc-bs* (*cs* @ [*c*]) *p* = *ngba-acc-bs cs p*

<proof>

lemma *ngba-acc-bs-correct*[simp]: *bs- α* (*ngba-acc-bs cs p*) = *ngba-acc cs p*

<proof>

lemma *ngba-acc-impl-bs*[*autoref-rules*]: (*ngba-acc-bs, ngba-acc*) \in (*S* \rightarrow *bool-rel*)
list-rel \rightarrow *S* \rightarrow *<nat-rel>* *bs-set-rel*

<proof>

schematic-goal *ngbai-igbgi*:

notes [*autoref-ga-rules*] = *map2set-to-list*

fixes *S* :: ('*statei* \times '*state*) *set*

assumes [*autoref-ga-rules*]: *is-bounded-hashcode S seq bhc*

assumes [*autoref-ga-rules*]: *is-valid-def-hm-size TYPE('statei) hms*

assumes [*autoref-rules*]: (*seq, HOL.eq*) \in *S* \rightarrow *S* \rightarrow *bool-rel*

assumes [*autoref-rules*]: (*Ai, A*) \in *<L, S>* *ngbai-ngba-rel*

shows (*?f* :: '*a*, *RETURN (ngbai-igbg A)*) \in *?A*

<proof>

concrete-definition *ngbai-igbgi uses ngbai-igbgi*

lemma *ngbai-igbgi-refine*[*autoref-rules*]:

fixes *S* :: ('*statei* \times '*state*) *set*

assumes *SIDE-GEN-ALGO (is-bounded-hashcode S seq bhc)*

assumes *SIDE-GEN-ALGO (is-valid-def-hm-size TYPE('statei) hms)*

assumes *GEN-OP seq HOL.eq (S* \rightarrow *S* \rightarrow *bool-rel)*

shows (*NGBA-Algorithms.ngbai-igbgi seq bhc hms, ngbai-igbg*) \in

<L, S> *ngbai-ngba-rel* \rightarrow *igbg-impl-rel-ext unit-rel S*

<proof>

schematic-goal *ngba-language-empty*:

fixes *S* :: ('*statei* \times '*state*) *set*

assumes [*simp*]: *igb-fr-graph (ngba-igbg A)*

assumes [*autoref-ga-rules*]: *is-bounded-hashcode S seq bhs*

assumes [*autoref-ga-rules*]: *is-valid-def-hm-size TYPE('statei) hms*

assumes [*autoref-rules*]: (*seq, HOL.eq*) \in *S* \rightarrow *S* \rightarrow *bool-rel*

assumes [*autoref-rules*]: (*Ai, A*) \in *<L, S>* *ngbai-ngba-rel*

shows (*?f* :: '*a*, *do { r* \leftarrow *op-find-lasso-spec (ngba-igbg A); RETURN (r =*
None)}) \in *?A*

<proof>

concrete-definition *ngba-language-empty uses ngba-language-empty*

lemma *nba-language-empty-refine*[*autoref-rules*]:

fixes *S* :: ('*statei* \times '*state*) *set*

assumes *SIDE-PRECOND (finite (NGBA.nodes A))*

assumes *SIDE-GEN-ALGO (is-bounded-hashcode S seq bhc)*

assumes *SIDE-GEN-ALGO (is-valid-def-hm-size TYPE('statei) hms)*

assumes *GEN-OP seq HOL.eq (S* \rightarrow *S* \rightarrow *bool-rel)*

assumes (*Ai, A*) \in *<L, S>* *ngbai-ngba-rel*

shows (*NGBA-Algorithms.ngba-language-empty seq bhc hms Ai*,
 (*OP op-language-empty* :: $\langle L, S \rangle$ *ngbai-ngba-rel* \rightarrow *bool-rel*) \$ *A*) \in *bool-rel*
 \langle *proof* \rangle

lemma *degeneralize-alt-def: degeneralize A = nba*
 (*ngba.alphabet A*)
 ((λ *p*. (*p*, 0)) ‘ *ngba.initial A*)
 (λ *a* (*p*, *k*). (λ *q*. (*q*, *Degeneralization.count* (*ngba.accepting A*) *p k*)) ‘
ngba.transition A a p)
 (*degen* (*ngba.accepting A*))
 \langle *proof* \rangle

schematic-goal *ngba-degeneralize: (?f :: ?'a, degeneralize) \in ?R*
 \langle *proof* \rangle

concrete-definition *ngba-degeneralize uses ngba-degeneralize*

lemmas *ngba-degeneralize-refine[autoref-rules] = ngba-degeneralize.refine*

schematic-goal *nba-intersect'*:

assumes [*autoref-rules*]: (*seq, HOL.eq*) \in $L \rightarrow L \rightarrow$ *bool-rel*

shows (λ *f*, *intersect'*) \in $\langle L, S \rangle$ *nbai-nba-rel* \rightarrow $\langle L, T \rangle$ *nbai-nba-rel* \rightarrow $\langle L, S$
 $\times_r T \rangle$ *ngbai-ngba-rel*
 \langle *proof* \rangle

concrete-definition *nba-intersect' uses nba-intersect'*

lemma *nba-intersect'-refine[autoref-rules]*:

assumes *GEN-OP seq HOL.eq* ($L \rightarrow L \rightarrow$ *bool-rel*)

shows (*nba-intersect' seq, intersect'*) \in

$\langle L, S \rangle$ *nbai-nba-rel* \rightarrow $\langle L, T \rangle$ *nbai-nba-rel* \rightarrow $\langle L, S \times_r T \rangle$ *ngbai-ngba-rel*

\langle *proof* \rangle

end

end

50 Nondeterministic Büchi Transition Automata

theory *NBTA*

imports *../Nondeterministic*

begin

datatype (*'label, 'state*) *nbta = nbta*
 (*alphabet: 'label set*)
 (*initial: 'state set*)
 (*transition: 'label \Rightarrow 'state \Rightarrow 'state set*)
 (*accepting: ('state \times 'label \times 'state) pred*)

global-interpretation *nbta: automaton nbta alphabet initial transition accepting*

defines *path = nbta.path and run = nbta.run and reachable = nbta.reachable*

and *nodes = nbta.nodes*

\langle *proof* \rangle

```

global-interpretation nbta: automaton-run nbta alphabet initial transition ac-
cepting
   $\lambda P w r p. \text{infs } P (p \#\# r \|\| w \|\| r)$ 
  defines language = nbta.language
   $\langle \text{proof} \rangle$ 

abbreviation target where target  $\equiv$  nbta.target
abbreviation states where states  $\equiv$  nbta.states
abbreviation trace where trace  $\equiv$  nbta.trace
abbreviation successors where successors  $\equiv$  nbta.successors TYPE('label)

end

```

51 Nondeterministic Generalized Büchi Transition Automata

```

theory NGBTA
imports ../Nondeterministic
begin

  datatype ('label, 'state) ngbta = ngbta
    (alphabet: 'label set)
    (initial: 'state set)
    (transition: 'label  $\Rightarrow$  'state  $\Rightarrow$  'state set)
    (accepting: ('state  $\times$  'label  $\times$  'state) pred gen)

  global-interpretation ngbta: automaton ngbta alphabet initial transition accept-
ing
  defines path = ngbta.path and run = ngbta.run and reachable = ngbta.reachable
and nodes = ngbta.nodes
   $\langle \text{proof} \rangle$ 
  global-interpretation ngbta: automaton-run ngbta alphabet initial transition
accepting
   $\lambda P w r p. \text{gen infs } P (p \#\# r \|\| w \|\| r)$ 
  defines language = ngbta.language
   $\langle \text{proof} \rangle$ 

  abbreviation target where target  $\equiv$  ngbta.target
  abbreviation states where states  $\equiv$  ngbta.states
  abbreviation trace where trace  $\equiv$  ngbta.trace
  abbreviation successors where successors  $\equiv$  ngbta.successors TYPE('label)

end

```


52 Nondeterministic Büchi Transition Automata Combinations

theory *NBTA-Combine*
imports *NBTA NGBTA*
begin

global-interpretation *degeneralization: automaton-degeneralization-run*
 $ngbta\ ngbta.alphabet\ ngbta.initial\ ngbta.transition\ ngbta.accepting\ \lambda\ P\ w\ r\ p.$
gen infs $P\ (p\ \#\#\ r\ ||| w\ ||| r)$
 $nbta\ nbta.alphabet\ nbta.initial\ nbta.transition\ nbta.accepting\ \lambda\ P\ w\ r\ p.\ infs\ P$
 $(p\ \#\#\ r\ ||| w\ ||| r)$
 $id\ \lambda\ ((p,\ k),\ a,\ (q,\ l)).\ ((p,\ a,\ q),\ k)$
defines *degeneralize* = *degeneralization.degeneralize*
 $\langle proof \rangle$

lemmas *degeneralize-language[simp]* = *degeneralization.degeneralize-language[folded NBTA.language-def]*

lemmas *degeneralize-nodes-finite[iff]* = *degeneralization.degeneralize-nodes-finite[folded NBTA.nodes-def]*

global-interpretation *intersection: automaton-intersection-run*
 $nbta\ nbta.alphabet\ nbta.initial\ nbta.transition\ nbta.accepting\ \lambda\ P\ w\ r\ p.\ infs\ P$
 $(p\ \#\#\ r\ ||| w\ ||| r)$
 $nbta\ nbta.alphabet\ nbta.initial\ nbta.transition\ nbta.accepting\ \lambda\ P\ w\ r\ p.\ infs\ P$
 $(p\ \#\#\ r\ ||| w\ ||| r)$
 $ngbta\ ngbta.alphabet\ ngbta.initial\ ngbta.transition\ ngbta.accepting\ \lambda\ P\ w\ r\ p.$
gen infs $P\ (p\ \#\#\ r\ ||| w\ ||| r)$
 $\lambda\ c_1\ c_2.\ [c_1\ \circ\ (\lambda\ ((p_1,\ p_2),\ a,\ (q_1,\ q_2)).\ (p_1,\ a,\ q_1)),\ c_2\ \circ\ (\lambda\ ((p_1,\ p_2),\ a,\ (q_1,\ q_2)).\ (p_2,\ a,\ q_2))]$
defines *intersect'* = *intersection.product*
 $\langle proof \rangle$

lemmas *intersect'-language[simp]* = *intersection.product-language[folded NGBTA.language-def]*

lemmas *intersect'-nodes-finite[intro]* = *intersection.product-nodes-finite[folded NGBTA.nodes-def]*

global-interpretation *union: automaton-union-run*
 $nbta\ nbta.alphabet\ nbta.initial\ nbta.transition\ nbta.accepting\ \lambda\ P\ w\ r\ p.\ infs\ P$
 $(p\ \#\#\ r\ ||| w\ ||| r)$
 $nbta\ nbta.alphabet\ nbta.initial\ nbta.transition\ nbta.accepting\ \lambda\ P\ w\ r\ p.\ infs\ P$
 $(p\ \#\#\ r\ ||| w\ ||| r)$
 $nbta\ nbta.alphabet\ nbta.initial\ nbta.transition\ nbta.accepting\ \lambda\ P\ w\ r\ p.\ infs\ P$
 $(p\ \#\#\ r\ ||| w\ ||| r)$
 $\lambda\ c_1\ c_2\ m.\ case\ m\ of\ (Inl\ p,\ a,\ Inl\ q) \Rightarrow c_1\ (p,\ a,\ q) \mid (Inr\ p,\ a,\ Inr\ q) \Rightarrow c_2\ (p,\ a,\ q)$
defines *union* = *union.sum*
 $\langle proof \rangle$

lemmas *union-language* = *union.sum-language*
lemmas *union-nodes-finite* = *union.sum-nodes-finite*

abbreviation *intersect* **where** *intersect A B* \equiv *degeneralize (intersect' A B)*

lemma *intersect-language[simp]*: *NBTA.language (intersect A B) = NBTA.language A* \cap *NBTA.language B*
<proof>

lemma *intersect-nodes-finite[intro]*:
assumes *finite (NBTA.nodes A)* *finite (NBTA.nodes B)*
shows *finite (NBTA.nodes (intersect A B))*
<proof>

end