The Transcendence of Certain Infinite Series

Angeliki Koutsoukou-Argyraki and Wenda Li

June 11, 2019

Abstract

We formalize the proofs of two transcendence criteria by J. Hančl and P. Rucki that assert the transcendence of the sums of certain infinite series built up by sequences that fulfil certain properties. Both proofs make use of Roth’s celebrated theorem on diophantine approximations to algebraic numbers from 1955 which we implement as an assumption without having formalised its proof.

Contents

1 The transcendence of certain infinite series
1.1 Misc ................................................................. 2
1.2 Main proofs ...................................................... 3
1.3 Acknowledgements .............................................. 3

1 The transcendence of certain infinite series

theory Transcendence-Series imports
  HOL−Analysis.Analysis
  HOL−Computational-Algebra.Polynomial
  Prime-Number-Theorem.Prime-Number-Theorem-Library
begin

We formalise the proofs of two transcendence criteria by J. Hančl and P. Rucki that assert the transcendence of the sums of certain infinite series built up by sequences that fulfil certain properties (Theorems 2.1 and 2.2 in [1], HanclRucki1 and HanclRucki2 here respectively). Both proofs make use of Roth’s celebrated theorem on diophantine approximations to algebraic numbers from 1955 [2] which we assume and implement within the locale RothsTheorem.

A small mistake was detected in the original proof of Theorem 2.1, and the authors suggested to us a fix for the problem (in communication by email). Our formalised proof incorporates this correction (see the Remark in the proof of HanclRucki1).
1.1 Misc

**Lemma** `powr-less-inverse-iff`:
fixes $x, y, z :: \mathbb{R}$
assumes $x > 0$, $y > 0$, $z > 0$
shows $x^{y} < z \iff x < z^{\text{inverse}(y)}$
⟨proof⟩

**Lemma** `powr-less-inverse-iff'`:
fixes $x, y, z :: \mathbb{R}$
assumes $x > 0$, $y > 0$, $z > 0$
shows $z < x^{y} \iff z^{\text{inverse}(y)} < x$
⟨proof⟩

**Lemma** `powr-less-eq-inverse-iff`:
fixes $x, y, z :: \mathbb{R}$
assumes $x > 0$, $y > 0$, $z > 0$
shows $x^{y} \leq z \iff x \leq z^{\text{inverse}(y)}$
⟨proof⟩

**Lemma** `powr-less-eq-inverse-iff'`:
fixes $x, y, z :: \mathbb{R}$
assumes $x > 0$, $y > 0$, $z > 0$
shows $z \leq x^{y} \iff z^{\text{inverse}(y)} \leq x$
⟨proof⟩

**Lemma** `tendsto-PInfy-mono`:
assumes $(\text{ereal } o f) \to \infty$ for all $x$
such that $f x \leq g x$
shows $(\text{ereal } o g) \to \infty$
⟨proof⟩

**Lemma** `limsup-infinity-imp-Inf-many`:
assumes $\limsup f = \infty$
shows $(\forall m. (\exists i > m. f i > \text{ereal } m))$
⟨proof⟩

**Lemma** `snd-quotient-plus-leq`:
defines $d e = (\text{snd } o \text{quotient-of})$
shows $d e (x + y) \leq d e x \cdot d e y$
⟨proof⟩

**Lemma** `quotient-of-inj: inj quotient-of`:
⟨proof⟩

**Lemma** `infinite-inj-imageE`:
assumes infinite $A$ inj-on $f A f$ is $A \subseteq B$
shows infinite $B$
⟨proof⟩

**Lemma** `incseq-tendsto-limsup`:
fixes $f :: \mathbb{N} \Rightarrow 'a :: \{\text{complete-linorder}, \text{linorder-topology}\}$
assumes incseq f
shows f −→ limsup f
⟨proof⟩

1.2 Main proofs
Since the proof of Roths theorem has not been formalized yet, we implement it into a locale and used it as an assumption.

locale RothsTheorem =
  assumes RothsTheorem:∀ ξ κ. algebraic ξ ∧ ξ ∉ Q ∧ infinite {p.q. q>0 ∧ coprime p q ∧ |ξ − of-int p / of-int q| < 1 / q powr κ} −→ κ ≤ 2

theorem (in RothsTheorem) HanclRucki1:
  fixes a b ::nat⇒int and δ ::real
  defines aa≡(λn. real-of-int (a n)) and bb≡(λn. real-of-int (b n))
  assumes a-pos:∀ k. a k >0 and b-pos:∀ k. b k >0 and δ >0
  and limsup-inf:limsup (λ k. aa (k+1)/(∏ i = 0..k. aa i)) powr (2+δ)*(1/bb (k+1))) = ∞
  and liminf-1:liminf (λk. aa (k+1) / aa k * bb k / bb (k+1)) > 1
  shows ¬ algebraic(suminf (λ k. bb k / aa k))
⟨proof⟩

theorem (in RothsTheorem) HanclRucki2:
  fixes a b ::nat⇒int and δ ·ε ::real
  defines aa≡(λn. real-of-int (a n)) and bb≡(λn. real-of-int (b n))
  assumes a-pos:∀ k. a k >0 and b-pos:∀ k. b k >0 and δ >0
  and ε >0
  and limsup-inf:limsup (λ k.(aa (k+1)/(∏ i = 0..k. aa i)) powr (2+(2/ε) + δ))
  * (1/(bb (k+1)))) = ∞
  and ratio-large:∀ k. ( k ≥ t −→ (( aa(k+1)/bb (k+1)) ) powr (1/(1+ε))
  ≥ (( aa k/bb k ) powr(1/(1+ε)))+1)
  shows ¬ algebraic(suminf (λ k. bb k / aa k))
⟨proof⟩

1.3 Acknowledgements
A.K.-A. and W.L. were supported by the ERC Advanced Grant ALEXANDRIA (Project 742178) funded by the European Research Council and led by Professor Lawrence Paulson at the University of Cambridge, UK. Thanks to Iosif Pinelis for his clarification on MathOverflow regarding the summability of the series in RothsTheorem.HanclRucki2 https://mathoverflow.net/questions/323069/why-is-this-series-summable and to Manuel Eberl for his helpful comments.
end
References
