

The Transcendence of Certain Infinite Series

Angeliki Koutsoukou-Argraki and Wenda Li

September 13, 2023

Abstract

We formalize the proofs of two transcendence criteria by J. Hančl and P. Rucki that assert the transcendence of the sums of certain infinite series built up by sequences that fulfil certain properties. Both proofs make use of Roth's celebrated theorem on diophantine approximations to algebraic numbers from 1955 which we implement as an assumption without having formalised its proof.

Contents

1	The transcendence of certain infinite series	1
1.1	Misc	2
1.2	Main proofs	3
1.3	Acknowledgements	3

1 The transcendence of certain infinite series

```
theory Transcendence-Series imports  
  HOL-Analysis.Multivariate-Analysis  
  HOL-Computational-Algebra.Polynomial  
  Prime-Number-Theorem.Prime-Number-Theorem-Library  
begin
```

We formalise the proofs of two transcendence criteria by J. Hančl and P. Rucki that assert the transcendence of the sums of certain infinite series built up by sequences that fulfil certain properties (Theorems 2.1 and 2.2 in [1], `HanclRucki1` and `HanclRucki2` here respectively). Both proofs make use of Roth's celebrated theorem on diophantine approximations to algebraic numbers from 1955 [2] which we assume and implement within the locale `RothsTheorem`.

A small mistake was detected in the original proof of Theorem 2.1, and the authors gave us a fix for the problem (by email). Our formalised proof incorporates this correction (see the Remark in the proof of `HanclRucki1`).

1.1 Misc

lemma *powr-less-inverse-iff*:

fixes $x\ y\ z::\text{real}$

assumes $x>0\ y>0\ z>0$

shows $x\ \text{powr}\ y < z \longleftrightarrow x < z\ \text{powr}\ (\text{inverse}\ y)$

<proof>

lemma *powr-less-inverse-iff'*:

fixes $x\ y\ z::\text{real}$

assumes $x>0\ y>0\ z>0$

shows $z < x\ \text{powr}\ y \longleftrightarrow z\ \text{powr}\ (\text{inverse}\ y) < x$

<proof>

lemma *powr-less-eq-inverse-iff*:

fixes $x\ y\ z::\text{real}$

assumes $x>0\ y>0\ z>0$

shows $x\ \text{powr}\ y \leq z \longleftrightarrow x \leq z\ \text{powr}\ (\text{inverse}\ y)$

<proof>

lemma *powr-less-eq-inverse-iff'*:

fixes $x\ y\ z::\text{real}$

assumes $x>0\ y>0\ z>0$

shows $z \leq x\ \text{powr}\ y \longleftrightarrow z\ \text{powr}\ (\text{inverse}\ y) \leq x$

<proof>

lemma *tendsto-PInfty-mono*:

assumes $(\text{ereal}\ o\ f) \longrightarrow \infty \ \forall_F\ x\ \text{in}\ \text{sequentially.}\ f\ x \leq g\ x$

shows $(\text{ereal}\ o\ g) \longrightarrow \infty$

<proof>

lemma *limsup-infinity-imp-Inf-many*:

assumes $\text{limsup}\ f = \infty$

shows $(\forall\ m. (\exists\ \infty\ i. f\ i > \text{ereal}\ m))$ *<proof>*

lemma *snd-quotient-plus-leq*:

defines $de \equiv (\text{snd}\ o\ \text{quotient-of})$

shows $de\ (x+y) \leq de\ x * de\ y$

<proof>

lemma *quotient-of-inj: inj quotient-of*

<proof>

lemma *infinite-inj-imageE*:

assumes $\text{infinite}\ A\ \text{inj-on}\ f\ A\ f\ 'A \subseteq B$

shows $\text{infinite}\ B$

<proof>

lemma *incseq-tendsto-limsup*:

fixes $f::\text{nat} \Rightarrow 'a::\{\text{complete-linorder}, \text{linorder-topology}\}$

assumes *incseq* *f*
shows $f \longrightarrow \text{limsup } f$
 $\langle \text{proof} \rangle$

1.2 Main proofs

Since the proof of Roth's theorem has not been formalized yet, we formalize the statement in a locale and use it as an assumption.

locale *RothsTheorem* =

assumes *RothsTheorem*: $\forall \xi \kappa. \text{algebraic } \xi \wedge \xi \notin \mathbf{Q} \wedge \text{infinite } \{(p,q). q > 0 \wedge \text{coprime } p \ q \wedge |\xi - \text{of-int } p / \text{of-int } q| < 1 / q \text{ powr } \kappa\} \longrightarrow \kappa \leq 2$

theorem (in *RothsTheorem*) *HanclRucki1*:

fixes $a \ b :: \text{nat} \Rightarrow \text{int}$ **and** $\delta :: \text{real}$

defines $aa \equiv (\lambda n. \text{real-of-int } (a \ n))$ **and** $bb \equiv (\lambda n. \text{real-of-int } (b \ n))$

assumes *a-pos*: $\forall k. a \ k > 0$ **and** *b-pos*: $\forall k. b \ k > 0$ **and** $\delta > 0$

and *limsup-infy*: $\text{limsup } (\lambda k. aa \ (k+1) / (\prod_{i=0..k} aa \ i) \text{ powr } (2+\delta) * (1 / bb \ (k+1))) = \infty$

and *liminf-1*: $\text{liminf } (\lambda k. aa \ (k+1) / aa \ k * bb \ k / bb \ (k+1)) > 1$

shows $\neg \text{algebraic}(\text{suminf } (\lambda k. bb \ k / aa \ k))$

$\langle \text{proof} \rangle$

theorem (in *RothsTheorem*) *HanclRucki2*:

fixes $a \ b :: \text{nat} \Rightarrow \text{int}$ **and** $\delta \ \varepsilon :: \text{real}$

defines $aa \equiv (\lambda n. \text{real-of-int } (a \ n))$ **and** $bb \equiv (\lambda n. \text{real-of-int } (b \ n))$

assumes *a-pos*: $\forall k. a \ k > 0$ **and** *b-pos*: $\forall k. b \ k > 0$ **and** $\delta > 0$

and $\varepsilon > 0$

and *limsup-infi*: $\text{limsup } (\lambda k. (aa \ (k+1) / (\prod_{i=0..k} aa \ i) \text{ powr } (2+(2/\varepsilon) + \delta)) * (1 / (bb \ (k+1)))) = \infty$

and *ratio-large*: $\forall k. (k \geq t \longrightarrow ((aa \ (k+1) / bb \ (k+1)) \text{ powr } (1/(1+\varepsilon))) \geq ((aa \ k / bb \ k) \text{ powr } (1/(1+\varepsilon))) + 1)$

shows $\neg \text{algebraic}(\text{suminf } (\lambda k. bb \ k / aa \ k))$

$\langle \text{proof} \rangle$

1.3 Acknowledgements

A.K.-A. and W.L. were supported by the ERC Advanced Grant ALEXANDRIA (Project 742178) funded by the European Research Council and led by Professor Lawrence Paulson at the University of Cambridge, UK. Thanks to Iosif Pinelis for his clarification on MathOverflow regarding the summability of the series in *RothsTheorem.HanclRucki2* <https://mathoverflow.net/questions/323069/why-is-this-series-summable> and to Manuel Eberl for his helpful comments.

end

References

- [1] J. Hančl and P. Rucki. The transcendence of certain infinite series. *Rocky Mountain Journal of Mathematics*, 35(2):531–537, 2005.
- [2] K. F. Roth. Rational approximations to algebraic numbers. *Mathematika*, 2(3):1–20, 1955.