The Transcendence of Certain Infinite Series

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Abstract

We formalize the proofs of two transcendence criteria by J. Hančl and P. Rucki that assert the transcendence of the sums of certain infinite series built up by sequences that fulfil certain properties. Both proofs make use of Roth's celebrated theorem on diophantine approximations to algebraic numbers from 1955 which we implement as an assumption without having formalised its proof.

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1 The transcendence of certain infinite series

theory *Transcendence-Series* **imports**
HOL−Analysis.Multivariate-Analysis
HOL−Computational-Algebra.Polynomial
Prime-Number-Theorem.Prime-Number-Theorem-Library
begin

We formalise the proofs of two transcendence criteria by J. Hančl and P. Rucki that assert the transcendence of the sums of certain infinite series built up by sequences that fulfil certain properties (Theorems 2.1 and 2.2 in [1], HanclRucki1 and HanclRucki2 here respectively). Both proofs make use of Roth's celebrated theorem on diophantine approximations to algebraic numbers from 1955 [2] which we assume and implement within the locale RothsTheorem.

A small mistake was detected in the original proof of Theorem 2.1, and the authors gave us a fix for the problem (by email). Our formalised proof incorporates this correction (see the Remark in the proof of HanclRucki1).

1.1 Misc

lemma *powr-less-inverse-iff*:
  fixes $x\ y\ z$::*real*
  assumes $x>0\ y>0\ z>0$
  shows x *powr* $y < z \longleftrightarrow x < z$ *powr* (*inverse* y)
⟨*proof*⟩

lemma *powr-less-inverse-iff'*:
  fixes $x\ y\ z$::*real*
  assumes $x>0\ y>0\ z>0$
  shows $z< x$ *powr* $y \longleftrightarrow z$ *powr* (*inverse* y) $< x$
  ⟨*proof*⟩

lemma *powr-less-eq-inverse-iff*:
  fixes $x\ y\ z$::*real*
  assumes $x>0\ y>0\ z>0$
  shows x *powr* $y \leq z \longleftrightarrow x \leq z$ *powr* (*inverse* y)
  ⟨*proof*⟩

lemma *powr-less-eq-inverse-iff'*:
  fixes $x\ y\ z$::*real*
  assumes $x>0\ y>0\ z>0$
  shows $z \leq x$ *powr* $y \longleftrightarrow z$ *powr* (*inverse* y) $\leq x$
  ⟨*proof*⟩

lemma *tendsto-PInfty-mono*:
  assumes (*ereal o f*) $\longrightarrow \infty$ $\forall_F$ *x in sequentially*. $f\ x \leq g\ x$
  shows (*ereal o g*) $\longrightarrow \infty$
  ⟨*proof*⟩

lemma *limsup-infinity-imp-Inf-many*:
  assumes *limsup* $f = \infty$
  shows ($\forall$ *m*. ($\exists_\infty i$. $f\ i >$ *ereal m*)) ⟨*proof*⟩

lemma *snd-quotient-plus-leq*:
  defines *de*≡(*snd o quotient-of*)
  shows *de* $(x+y) \leq$ *de* $x *$ *de* y
⟨*proof*⟩

lemma *quotient-of-inj*: *inj quotient-of*
  ⟨*proof*⟩

lemma *infinite-inj-imageE*:
  assumes *infinite A inj-on f A f ‘ A* $\subseteq B$
  shows *infinite B*
  ⟨*proof*⟩

lemma *incseq-tendsto-limsup*:
  fixes f::*nat* $\Rightarrow$ *'a*::{*complete-linorder*,*linorder-topology*}

assumes *incseq f*
shows $f \longrightarrow \mathit{limsup}\ f$
⟨*proof*⟩

1.2 Main proofs

Since the proof of Roth's theorem has not been formalized yet, we formalize the statement in a locale and use it as an assumption.

locale *RothsTheorem* =
assumes *RothsTheorem*:$\forall \xi\ \kappa.\ \mathit{algebraic}\ \xi \wedge \xi \notin \mathbb{Q} \wedge \mathit{infinite}\ \{(p,q).\ q>0 \wedge \mathit{coprime}\ p\ q \wedge |\xi - \mathit{of\text{-}int}\ p\ /\ \mathit{of\text{-}int}\ q| < 1\ /\ q\ \mathit{powr}\ \kappa\} \longrightarrow \kappa \leq 2$

theorem (**in** *RothsTheorem*) *HanclRucki1*:
fixes *a b* ::*nat*⇒*int* **and** δ ::*real*
defines $aa \equiv (\lambda n.\ \mathit{real\text{-}of\text{-}int}\ (a\ n))$ **and** $bb \equiv (\lambda n.\ \mathit{real\text{-}of\text{-}int}\ (b\ n))$
assumes *a-pos*:$\forall\ k.\ a\ k > 0$ **and** *b-pos*:$\forall\ k.\ b\ k > 0$ **and** $\delta > 0$
and *limsup-infy*:$\mathit{limsup}\ (\lambda\ k.\ aa\ (k+1)/(\prod i = 0..k.\ aa\ i)\mathit{powr}(2+\delta)*(1/bb\ (k+1))) = \infty$
and *liminf-1*:$\mathit{liminf}\ (\lambda k.\ aa\ (k+1)\ /\ aa\ k * bb\ k\ /\ bb\ (k+1)) > 1$
shows $\neg\ \mathit{algebraic}(\mathit{suminf}\ (\lambda\ k.\ bb\ k\ /\ aa\ k))$
⟨*proof*⟩

theorem (**in** *RothsTheorem*) *HanclRucki2*:
fixes *a b* ::*nat*⇒*int* **and** δ ε ::*real*
defines $aa \equiv (\lambda n.\ \mathit{real\text{-}of\text{-}int}\ (a\ n))$ **and** $bb \equiv (\lambda n.\ \mathit{real\text{-}of\text{-}int}\ (b\ n))$
assumes *a-pos*:$\forall\ k.\ a\ k > 0$ **and** *b-pos*:$\forall\ k.\ b\ k > 0$ **and** $\delta > 0$
and $\varepsilon > 0$
and *limsup-infi*:$\mathit{limsup}\ (\lambda\ k.(aa\ (k+1)/(\prod i = 0..k.\ aa\ i)\mathit{powr}(2+(2/\varepsilon) + \delta)) * (1/(bb\ (k+1)))) = \infty$
and *ratio-large*:$\forall\ k.\ (\ k \geq t \longrightarrow\ ((\ aa(k+1)/bb(\ k+1))\ )\ \mathit{powr}(1/(1+\varepsilon)) \geq\ ((\ aa\ k/bb\ k)\ \mathit{powr}(1/(1+\varepsilon)))+1)$
shows $\neg\ \mathit{algebraic}(\mathit{suminf}\ (\lambda\ k.\ bb\ k\ /aa\ k))$
⟨*proof*⟩

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end

References

[1] J. Hančl and P. Rucki. The transcendence of certain infinite series. *Rocky Mountain Journal of Mathematics*, 35(2):531–537, 2005.

[2] K. F. Roth. Rational approximations to algebraic numbers. *Mathematika*, 2(3):1–20, 1955.