

The Tortoise and the Hare Algorithm

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Abstract

We formalize the [Tortoise and Hare cycle-finding algorithm](#) ascribed to Floyd by [Knuth \(1981, p7, exercise 6\)](#), and an improved version due to [Brent \(1980\)](#).

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1 Introduction

[Knuth \(1981, p7, exercise 6\)](#) frames the problem like so: given a finite set X , an initial value $x_0 \in X$, and a function $f : X \rightarrow X$, define the infinite sequence x by recursion: $x_{i+1} = f(x_i)$. Show that the sequence is ultimately periodic, i.e., that there exist λ and μ where

$$x_0, x_1, \dots, x_\mu, \dots, x_{\mu+\lambda-1}$$

are distinct, but $x_{n+\lambda} = x_n$ when $n \geq \mu$.

Secondly (and he ascribes this to Robert W. Floyd), show that there is an $\nu > 0$ such that $x_\nu = x_{2\nu}$.

These facts are supposed to yield the insight required to develop the Tortoise and Hare algorithm, which calculates λ and μ for any f and x_0 using only $O(\lambda + \mu)$ steps and a bounded number of memory locations. We fill in the details in §5.

We also show the correctness of [Brent \(1980\)](#)’s algorithm in §6, which satisfies the same resource bounds and is more efficient in practice.

These algorithms have been used to analyze random number generators (Knuth 1981, op. cit.) and factor large numbers (Brent 1980). See Nivasch (2004) for further discussion, and an algorithm that is not constant-space but is more efficient in some situations. Wang and Zhang (2012) also survey these algorithms and present a new one.

2 Point-free notation

We adopt point-free notation for our assertions over program states.

abbreviation (*input*)

$pred_K :: 'b \Rightarrow 'a \Rightarrow 'b (\langle _ \rangle)$ **where**
 $\langle f \rangle \equiv \lambda s. f$

abbreviation (*input*)

$pred_not :: ('a \Rightarrow bool) \Rightarrow 'a \Rightarrow bool (\neg)$ **where**
 $\neg a \equiv \lambda s. \neg a s$

abbreviation (*input*)

$pred_conj :: ('a \Rightarrow bool) \Rightarrow ('a \Rightarrow bool) \Rightarrow 'a \Rightarrow bool$ (**infixr** \wedge 35) **where**
 $a \wedge b \equiv \lambda s. a s \wedge b s$

abbreviation (*input*)

$pred_implies :: ('a \Rightarrow bool) \Rightarrow ('a \Rightarrow bool) \Rightarrow 'a \Rightarrow bool$ (**infixr** \longrightarrow 25) **where**
 $a \longrightarrow b \equiv \lambda s. a s \longrightarrow b s$

abbreviation (*input*)

$pred_eq :: ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow bool$ (**infix** $=$ 40) **where**
 $a = b \equiv \lambda s. a s = b s$

abbreviation (*input*)

$pred_member :: ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b \text{ set}) \Rightarrow 'a \Rightarrow bool$ (**infix** \in 40) **where**
 $a \in b \equiv \lambda s. a s \in b s$

abbreviation (*input*)

$pred_neq :: ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow bool$ (**infix** \neq 40) **where**
 $a \neq b \equiv \lambda s. a s \neq b s$

abbreviation (*input*)

$pred_If :: ('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ (**if** $(_)$ / **then** $(_)$ / **else** $(_)$ [0, 0, 10] 10) **where**
if P **then** x **else** $y \equiv \lambda s. \text{if } P s \text{ then } x s \text{ else } y s$

abbreviation (*input*)

$pred_less :: ('a \Rightarrow 'b::ord) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow bool$ (**infix** $<$ 40) **where**
 $a < b \equiv \lambda s. a s < b s$

abbreviation (*input*)

$pred_le :: ('a \Rightarrow 'b::ord) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow bool$ (**infix** \leq 40) **where**
 $a \leq b \equiv \lambda s. a s \leq b s$

abbreviation (*input*)

$pred_plus :: ('a \Rightarrow 'b::plus) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ (**infixl** $+$ 65) **where**
 $a + b \equiv \lambda s. a s + b s$

abbreviation (*input*)

$pred_minus :: ('a \Rightarrow 'b::minus) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ (**infixl** $-$ 65) **where**
 $a - b \equiv \lambda s. a s - b s$

abbreviation (*input*)

$fun_fanout :: ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'c) \Rightarrow 'a \Rightarrow 'b \times 'c$ (**infix** \bowtie 35) **where**
 $f \bowtie g \equiv \lambda x. (f x, g x)$

abbreviation (*input*)

$pred_all :: ('b \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow bool$ (**binder** \forall 10) **where**
 $\forall x. P x \equiv \lambda s. \forall x. P x s$

abbreviation (*input*)

$pred_ex :: ('b \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow bool$ (**binder** \exists 10) **where**
 $\exists x. P x \equiv \lambda s. \exists x. P x s$

3 “Monoidal” Hoare logic

In the absence of a general-purpose development of Hoare Logic for total correctness in Isabelle/HOL¹, we adopt the following syntactic contrivance that eases making multiple assertions about function results. “Programs” consist of the state-transformer semantics of statements.

definition $valid :: ('s \Rightarrow bool) \Rightarrow ('s \Rightarrow 's) \Rightarrow ('s \Rightarrow bool) \Rightarrow bool$ ($\{_\}/_/\{_\}$) **where**
 $\{P\} c \{Q\} \equiv \forall s. P s \longrightarrow Q (c s)$

notation $(input)$ id (*SKIP*)

notation $fcomp$ (**infixl** $::$ 60)

named_theorems wp_intro *weakest precondition intro rules*

lemma $seqI[wp_intro]$:
assumes $\{Q\} d \{R\}$
assumes $\{P\} c \{Q\}$
shows $\{P\} c ;; d \{R\}$
using $assms$ **by** (*simp add: valid_def*)

lemma $iteI[wp_intro]$:
assumes $\{P'\} x \{Q\}$
assumes $\{P''\} y \{Q\}$
shows $\{if\ b\ then\ P'\ else\ P''\} if\ b\ then\ x\ else\ y \{Q\}$
using $assms$ **by** (*simp add: valid_def*)

lemma $assignI[wp_intro]$:
shows $\{Q \circ f\} f \{Q\}$
by (*simp add: valid_def*)

lemma $whileI$:
assumes $\{I'\} c \{I\}$
assumes $\bigwedge s. I s \Longrightarrow if\ b\ s\ then\ I'\ s\ else\ Q\ s$
assumes $wf\ r$
assumes $\bigwedge s. [I\ s; b\ s] \Longrightarrow (c\ s, s) \in r$
shows $\{I\} while\ b\ c \{Q\}$
using $assms$ **by** (*simp add: while_rule valid_def*)

lemma $hoare_pre$:
assumes $\{R\} f \{Q\}$
assumes $\bigwedge s. P\ s \Longrightarrow R\ s$
shows $\{P\} f \{Q\}$
using $assms$ **by** (*simp add: valid_def*)

lemma $hoare_post_imp$:
assumes $\{P\} a \{Q\}$
assumes $\bigwedge s. Q\ s \Longrightarrow R\ s$
shows $\{P\} a \{R\}$
using $assms$ **by** (*simp add: valid_def*)

Note that the $assignI$ rule applies to all state transformers, and therefore the order in which we attempt to use the wp_intro rules matters.

4 Properties of iterated functions on finite sets

We begin by fixing the f and $x0$ under consideration in a locale, and establishing Knuth’s properties.

The sequence is modelled as a function $seq :: nat \Rightarrow 'a$ in the obvious way.

¹At the time of writing the distribution contains several for partial correctness, and one for total correctness over a language with restricted expressions. SIMPL (Schirmer (2008)) is overkill for our present purposes.

```

locale fx0 =
  fixes f :: 'a::finite  $\Rightarrow$  'a
  fixes x0 :: 'a
begin

```

```

definition seq' :: 'a  $\Rightarrow$  nat  $\Rightarrow$  'a where
  seq' x i  $\equiv$  (f  $\widehat{\widehat{}}$  i) x

```

```

abbreviation seq  $\equiv$  seq' x0

```

The parameters *lambda* and *mu* must exist by the pigeonhole principle.

```

lemma seq'_not_inj_on_card_UNIV:
  shows  $\neg$ inj_on (seq' x) {0 .. card (UNIV::'a set)}
by (simp add: inj_on_iff_eq_card)
  (metis UNIV_I card_mono finite lessI not_less subsetI)

```

```

definition properties :: nat  $\Rightarrow$  nat  $\Rightarrow$  bool where
  properties lambda mu  $\equiv$ 
    0 < lambda
     $\wedge$  inj_on seq {0 ..< mu + lambda}
     $\wedge$  ( $\forall i \geq \text{mu}. \forall j. \text{seq} (i + j * \text{lambda}) = \text{seq } i$ )

```

```

lemma properties_existence:

```

```

  obtains lambda mu
  where properties lambda mu

```

```

proof -

```

```

  obtain l where l: inj_on seq {0..l}  $\wedge$   $\neg$ inj_on seq {0..Suc l}
  using ex_least_nat_less[where P= $\lambda$ ub. \neg inj_on seq {0..ub} and n=card (UNIV :: 'a set)]
  seq'_not_inj_on_card_UNIV

```

```

  by fastforce

```

```

moreover

```

```

from l obtain mu where mu: mu  $\leq$  l  $\wedge$  seq (Suc l) = seq mu
  by (fastforce simp: atLeastAtMostSuc_conv)

```

```

moreover

```

```

define lambda where lambda = l - mu + 1
have seq (i + j * lambda) = seq i if mu  $\leq$  i for i j
using that proof (induct j)
  case (Suc j)
  from l mu have F: seq (l + j + 1) = seq (mu + j) for j
  by (fastforce elim: seq_inj)
  from mu Suc F[where j=i + j * lambda - mu] show ?case
  by (simp add: lambda_def field_simps)

```

```

qed simp

```

```

ultimately have properties lambda mu

```

```

  by (auto simp: properties_def lambda_def atLeastLessThanSuc_atLeastAtMost)

```

```

then show thesis ..

```

```

qed

```

```

end

```

To ease further reasoning, we define a new locale that fixes *lambda* and *mu*, and assume these properties hold. We then derive further rules that are easy to apply.

```

locale properties = fx0 +
  fixes lambda mu :: nat
  assumes P: properties lambda mu
begin

```

```

lemma properties_lambda_gt_0:
  shows 0 < lambda
using P by (simp add: properties_def)

```

```

lemma properties_loop:
  assumes mu  $\leq$  i
  shows seq (i + j * lambda) = seq i
using P assms by (simp add: properties_def)

```

```

lemma properties_mod_lambda:
  assumes  $\mu \leq i$ 
  shows  $\text{seq } i = \text{seq } (\mu + (i - \mu) \bmod \text{lambda})$ 
using properties_loop[where  $i = \mu + (i - \mu) \bmod \text{lambda}$  and  $j = (i - \mu) \text{ div } \text{lambda}$ ] assms
by simp

```

```

lemma properties_distinct:
  assumes  $j \in \{0 <..< \text{lambda}\}$ 
  shows  $\text{seq } (i + j) \neq \text{seq } i$ 
proof(cases  $\mu \leq i$ )
  case True
  from assms have A:  $(i + j) \bmod \text{lambda} \neq i \bmod \text{lambda}$  for  $i$ 
  by (auto simp add: mod_eq_dvd_iff_nat)
  from  $\langle \mu \leq i \rangle$ 
  have  $\text{seq } (i + j) = \text{seq } (\mu + (i + j - \mu) \bmod \text{lambda})$ 
  and  $\text{seq } i = \text{seq } (\mu + (i - \mu) \bmod \text{lambda})$ 
  by (auto intro: properties_mod_lambda)
  with P  $\langle \mu \leq i \rangle$  assms A[where  $i = i - \mu$ ] show ?thesis
  by (clarsimp simp: properties_def inj_on_eq_iff)
next
  case False with P assms show ?thesis
  by (clarsimp simp: properties_def inj_on_eq_iff)
qed

```

```

lemma properties_distinct_contrapos:
  assumes  $\text{seq } (i + j) = \text{seq } i$ 
  shows  $j \notin \{0 <..< \text{lambda}\}$ 
using assms by (rule contrapos_pp) (simp add: properties_distinct)

```

```

lemma properties_loops_ge_mu:
  assumes  $\text{seq } (i + j) = \text{seq } i$ 
  assumes  $0 < j$ 
  shows  $\mu \leq i$ 
proof(rule classical)
  assume X:  $\neg ?thesis$  show ?thesis
  proof(cases  $\mu \leq i + j$ )
    case True with P X assms show ?thesis
    by (fastforce simp: properties_def inj_on_eq_iff
        dest: properties_mod_lambda)
  next
    case False with P assms show ?thesis
    by (fastforce simp add: properties_def inj_on_eq_iff)
  qed
qed
end

```

5 The Tortoise and the Hare

The key to the Tortoise and Hare algorithm is that any nu such that $\text{seq } (nu + nu) = \text{seq } nu$ must be divisible by lambda . Intuitively the first nu steps get us into the loop. If the second nu steps return us to the same value of the sequence, then we must have gone around the loop one or more times.

```

lemma (in properties) lambda_dvd_nu:
  assumes  $\text{seq } (i + i) = \text{seq } i$ 
  shows  $\text{lambda} \text{ dvd } i$ 
proof(cases  $i = 0$ )
  case False
  with assms have  $\mu \leq i$  by (auto simp: properties_loops_ge_mu)
  with assms have  $\text{seq } (i + i \bmod \text{lambda}) = \text{seq } i$ 
  using properties_loop[where  $i = i + i \bmod \text{lambda}$  and  $j = i \text{ div } \text{lambda}$ ] by simp
  from properties_distinct_contrapos[OF this] show ?thesis
  by simp (meson dvd_eq_mod_eq_0 mod_less_divisor not_less properties_lambda_gt_0)
qed simp

```

The program is split into three loops; we find nu , mu and $lambda$ in that order.

5.1 Finding nu

The state space of the program tracks each of the variables we wish to discover, and the current positions of the Tortoise and Hare.

```
record 'a state =
  nu :: nat —  $\nu$ 
  m  :: nat —  $\mu$ 
  l  :: nat —  $\lambda$ 
  hare :: 'a
  tortoise :: 'a
```

```
context properties
begin
```

The Hare proceeds at twice the speed of the Tortoise. The program tracks how many steps the Tortoise has taken in nu .

```
definition (in fx0) find_nu :: 'a state  $\Rightarrow$  'a state where
  find_nu  $\equiv$ 
    ( $\lambda s. s(| nu := 1, tortoise := f(x0), hare := f(f(x0)) |)$ ) ;;
    while (hare  $\neq$  tortoise)
      ( $\lambda s. s(| nu := nu s + 1, tortoise := f(tortoise s), hare := f(f(hare s)) |)$ )
```

If this program terminates, we expect $seq \circ (nu + nu) = seq \circ nu$ to hold in the final state.

The simplest approach to showing termination is to define a suitable nu in terms of $lambda$ and mu , which also gives us an upper bound on the number of calls to f .

```
definition nu_witness :: nat where
  nu_witness  $\equiv mu + lambda - mu \text{ mod } lambda$ 
```

This constant has the following useful properties:

```
lemma nu_witness_properties:
```

```
  mu < nu_witness
  nu_witness  $\leq lambda + mu$ 
  lambda dvd nu_witness
  mu = 0  $\implies nu_witness = lambda$ 
```

```
unfolding nu_witness_def
```

```
using properties_lambda_gt_0
```

```
apply (simp_all add: less_diff_conv divide_simps)
```

```
apply (metis minus_mod_eq_div_mult [symmetric] dvd_def mod_add_self2 mult.commute)
```

```
done
```

These demonstrate that $nu_witness$ has the key property:

```
lemma nu_witness:
```

```
  shows seq (nu_witness + nu_witness) = seq nu_witness
```

```
using nu_witness_properties properties_loop
```

```
by (clarsimp simp: dvd_def field_simps)
```

Termination amounts to showing that the Tortoise gets closer to $nu_witness$ on each iteration of the loop.

```
definition find_nu_measure :: (nat  $\times$  nat) set where
```

```
  find_nu_measure  $\equiv measure (\lambda \nu. nu_witness - \nu)$ 
```

```
lemma find_nu_measure_wellfounded:
```

```
  wf find_nu_measure
```

```
by (simp add: find_nu_measure_def)
```

```
lemma find_nu_measure_decreases:
```

```
  assumes seq ( $\nu + \nu$ )  $\neq seq \nu$ 
```

```
  assumes  $\nu \leq nu_witness$ 
```

```
  shows (Suc  $\nu, \nu$ )  $\in find_nu_measure$ 
```

```
using nu_witness_properties nu_witness assms
```

```
by (auto simp: find_nu_measure_def le_eq_less_or_eq)
```

The remainder of the Hoare proof is straightforward.

lemma *find_nu*:

```

  {⟨True⟩} find_nu {nu ∈ {0 <.. lambda + mu}} ∧ seq ∘ (nu + nu) = seq ∘ nu ∧ hare = seq ∘ nu}
apply (simp add: find_nu_def)
apply (rule hoare_pre)
apply (rule whileI[where I=nu ∈ {0 <.. nu_witness} ∧ (∀ i. ⟨0 < i⟩ ∧ ⟨i⟩ < nu ⟶ ⟨seq (i + i) ≠ seq i⟩)
  ∧ tortoise = seq ∘ nu ∧ hare = seq ∘ (nu + nu)
  and r=inv_image find_nu_measure nu]
  wp_intro)+
using nu_witness_properties nu_witness
apply (fastforce simp: le_eq_less_or_eq elim: less_SucE)
apply (simp add: find_nu_measure_wellfounded)
apply (simp add: find_nu_measure_decreases)
apply (rule wp_intro)
using nu_witness_properties
apply auto
done

```

5.1.1 Side observations

We can also show termination ala [Filliâtre \(2007\)](#).

definition *find_nu_measures* :: (nat × nat) set **where**

```

find_nu_measures ≡
measures [λν. mu - ν, λν. LEAST i. seq (ν + ν + i) = seq ν]

```

lemma *find_nu_measures_wellfounded*:

```

wf find_nu_measures
by (simp add: find_nu_measures_def)

```

lemma *find_nu_measures_existence*:

```

assumes ν: mu ≤ ν
shows ∃ i. seq (ν + ν + i) = seq ν
proof(cases seq (ν + ν) = seq ν)
case False
from properties_lambda_gt_0 obtain k where k: ν ≤ k * lambda
by (metis One_nat_def Suc_leI mult.right_neutral mult_le_mono order_refl)
from ν k have seq (ν + ν + (k * lambda - ν)) = seq (mu + (ν - mu) + k * lambda) by (simp add: field_simps)
also from ν properties_loop have ... = seq ν by simp
finally show ?thesis by blast
qed (simp add: exI[where x=0])

```

lemma *find_nu_measures_decreases*:

```

assumes ν: seq (ν + ν) ≠ seq ν
shows (Suc ν, ν) ∈ find_nu_measures
proof(cases mu ≤ ν)
case True
then have mu ≤ Suc ν by simp
have (LEAST i. seq (Suc ν + Suc ν + i) = seq (Suc ν)) < (LEAST i. seq (ν + ν + i) = seq ν)
proof(rule LeastI2_wellorder_ex[OF find_nu_measures_existence[OF ⟨mu ≤ Suc ν⟩]],
  rule LeastI2_wellorder_ex[OF find_nu_measures_existence[OF ⟨mu ≤ ν⟩]])
fix x y
assume x: seq (Suc ν + Suc ν + x) = seq (Suc ν)
  ∀ z. seq (Suc ν + Suc ν + z) = seq (Suc ν) ⟶ x ≤ z
assume y: seq (ν + ν + y) = seq ν
from ν ⟨mu ≤ ν⟩ y have 0 < y by (cases y) simp_all
with y have seq (Suc ν + Suc ν + (y - 1)) = seq (Suc ν) by (auto elim: seq_inj)
with ⟨0 < y⟩ spec[OF x(2), where x=y - 1] y show x < y by simp
qed
with True ν show ?thesis by (simp add: find_nu_measures_def)
qed (auto simp: find_nu_measures_def)

```

lemma *find_nu_Filliâtre*:

```

{⟨True⟩} find_nu {⟨0⟩ < nu ∧ seq ∘ (nu + nu) = seq ∘ nu ∧ hare = seq ∘ nu}
apply (simp add: find_nu_def)

```

```

apply (rule hoare_pre)
apply (rule whileI[where  $I = \langle 0 \rangle < nu \wedge tortoise = seq \circ nu \wedge hare = seq \circ (nu + nu)$ 
and  $r = inv\_image\ find\_nu\_measures\ nu]$ 
      wp_intro)+
apply clarsimp
apply (simp add: find_nu_measures_wellfounded)
apply (simp add: find_nu_measures_decreases)
apply (rule wp_intro)
apply (simp add: properties_lambda_gt_0)
done

```

This approach does not provide an upper bound on nu however.

Harper (2011) observes (in his §13.5.2) that if mu is zero then $nu = lambda$.

```

lemma Harper:
  assumes  $mu = 0$ 
  shows  $\{\langle True \rangle\}\ find\_nu\ \{\nu = \langle lambda \rangle\}$ 
by (rule hoare_post_imp[OF find_nu]) (fastforce simp: assms dvd_def dest: lambda_dvd_nu)

```

5.2 Finding mu

We recover mu from nu by exploiting the fact that $lambda$ divides nu : the Tortoise, reset to $x0$ and the Hare, both now moving at the same speed, will meet at mu .

```

lemma mu_nu:
  assumes  $si: seq\ (i + i) = seq\ i$ 
  assumes  $j: mu \leq j$ 
  shows  $seq\ (j + i) = seq\ j$ 
using lambda_dvd_nu[OF si] properties_loop[OF j]
by (clarsimp simp: dvd_def field_simps)

```

```

definition (in fx0) find_mu :: 'a state  $\Rightarrow$  'a state where
  find_mu  $\equiv$ 
    ( $\lambda s. s\ \{\ m := 0, tortoise := x0\ \}$ ) ;;
    while (hare  $\neq$  tortoise)
      ( $\lambda s. s\ \{\ tortoise := f\ (tortoise\ s), hare := f\ (hare\ s), m := m\ s + 1\ \}$ )

```

```

lemma find_mu:
   $\{\nu \in \{\langle 0 \dots lambda + mu \rangle\} \wedge seq \circ (nu + nu) = seq \circ nu \wedge hare = seq \circ nu\}$ 
  find_mu
   $\{\nu \in \{\langle 0 \dots lambda + mu \rangle\} \wedge tortoise = \langle seq\ mu \rangle \wedge m = \langle mu \rangle\}$ 
apply (simp add: find_mu_def)
apply (rule hoare_pre)
apply (rule whileI[where  $I = \nu \in \{\langle 0 \dots lambda + mu \rangle\} \wedge seq \circ (nu + nu) = seq \circ nu \wedge m \leq \langle mu \rangle$ 
   $\wedge tortoise = seq \circ m \wedge hare = seq \circ (m + nu)$ 
and  $r = measure\ (\langle mu \rangle - m)$ ]
      wp_intro)+
  using properties_loops_ge_mu
  apply (force dest: mu_nu simp: less_eq_Suc_le[symmetric])
  apply simp
  apply (force dest: mu_nu simp: le_eq_less_or_eq)
  apply (rule wp_intro)
apply simp
done

```

5.3 Finding $lambda$

With the Tortoise parked at mu , we find $lambda$ by walking the Hare around the loop.

```

definition (in fx0) find_lambda :: 'a state  $\Rightarrow$  'a state where
  find_lambda  $\equiv$ 
    ( $\lambda s. s\ \{\ l := 1, hare := f\ (tortoise\ s)\ \}$ ) ;;
    while (hare  $\neq$  tortoise)
      ( $\lambda s. s\ \{\ hare := f\ (hare\ s), l := l\ s + 1\ \}$ )

```

```

lemma find_lambda:

```



```

  {nu ∈ {0<..lambda + mu}} ∧ tortoise = ⟨seq mu⟩ ∧ m = ⟨mu⟩}
  find_lambda
  {nu ∈ {0<..lambda + mu}} ∧ l = ⟨lambda⟩ ∧ m = ⟨mu⟩}
apply (simp add: find_lambda_def)
apply (rule hoare_pre)
apply (rule whileI[where I=nu ∈ {0<..lambda + mu} ∧ l ∈ {0<..lambda}
  ∧ tortoise = ⟨seq mu⟩ ∧ hare = seq ∘ (⟨mu⟩ + l) ∧ m = ⟨mu⟩
  and r=measure (⟨lambda⟩ - l)]
  wp_intro)+
using properties_lambda_gt_0 properties_mod_lambda[where i=mu + lambda] properties_distinct[where i=mu]
apply (fastforce simp: less_eq_Suc_le[symmetric])
apply simp
using properties_mod_lambda[where i=mu + lambda]
apply (fastforce simp: le_eq_less_or_eq)
apply (rule wp_intro)
using properties_lambda_gt_0
apply simp
done

```

5.4 Top level

The complete program is simply the steps composed in order.

definition (in *fx0*) *tortoise_hare* :: 'a state ⇒ 'a state **where**
tortoise_hare ≡ *find_nu* ;; *find_mu* ;; *find_lambda*

theorem *tortoise_hare*:

{⟨True⟩} *tortoise_hare* {nu ∈ {0<..lambda + mu}} ∧ l = ⟨lambda⟩ ∧ m = ⟨mu⟩}

unfolding *tortoise_hare_def*

by (rule *find_nu find_mu find_lambda wp_intro*)+

end

corollary *tortoise_hare_correct*:

assumes *s'*: *s'* = *fx0.tortoise_hare f x arbitrary*

shows *fx0.properties f x (l s') (m s')*

using *assms properties.tortoise_hare*[**where** *f=f* **and** *?x0.0=x*]

by (fastforce *intro: fx0.properties_existence*[**where** *f=f* **and** *?x0.0=x*]
simp: Basis.properties_def valid_def)

Isabelle can generate code from these definitions.

schematic_goal *tortoise_hare_code*[*code*]:

fx0.tortoise_hare f x = ?code

unfolding *fx0.tortoise_hare_def fx0.find_nu_def fx0.find_mu_def fx0.find_lambda_def fcomp_assoc[symmetric] fcomp_comp*

by (rule *refl*)

export_code *fx0.tortoise_hare* **in** *SML*

6 Brent's algorithm

[Brent \(1980\)](#) improved on the Tortoise and Hare algorithm and used it to factor large primes. In practice it makes significantly fewer calls to the function *f* before detecting a loop.

We begin by defining the base-2 logarithm.

fun *lg* :: *nat* ⇒ *nat* **where**

[*simp del*]: *lg x* = (if *x* ≤ 1 then 0 else 1 + *lg (x div 2)*)

lemma *lg_safe*:

lg 0 = 0

lg (Suc 0) = 0

lg (Suc (Suc 0)) = 1

0 < *x* ⇒ *lg (x + x)* = 1 + *lg x*

by (*simp_all add: lg_simps*)

lemma *lg_inv*:

```

0 < x  $\implies$  lg (2 ^ x) = x
proof(induct x)
  case (Suc x) then show ?case
    by (cases x, simp_all add: lg_simps Suc_lessI not_le)
qed simp

```

```

lemma lg_inv2:
  <2 ^ lg x = x> if <2 ^ i = x> for x
proof -
  have <2 ^ lg (2 ^ i) = (2::nat) ^ i>
    by (induction i) (simp_all add: lg_safe mult_2)
  with that show ?thesis
    by simp
qed

```

```

lemmas lg_simps = lg_safe lg_inv lg_inv2

```

6.1 Finding *lambda*

Imagine now that the Tortoise carries an unbounded number of carrots, which he passes to the Hare when they meet, and the Hare has a teleporter. The Hare eats a carrot each time she waits for the function f to execute, and initially has just one. If she runs out of carrots before meeting the Tortoise again, she teleports him to her position, and he gives her twice as many carrots as the last time they met (tracked by the variable *carrots*). By counting how many carrots she has eaten from when she last teleported the Tortoise (recorded in *l*) until she finally has surplus carrots when she meets him again, the Hare directly discovers *lambda*.

```

record 'a state =
  m :: nat —  $\mu$ 
  l :: nat —  $\lambda$ 
  carrots :: nat
  hare :: 'a
  tortoise :: 'a

```

```

context properties
begin

```

```

definition (in fx0) find_lambda :: 'a state  $\Rightarrow$  'a state where
  find_lambda  $\equiv$ 
    ( $\lambda$ s. s(| carrots := 1, l := 1, tortoise := x0, hare := f x0 |)) ;;
    while (hare  $\neq$  tortoise)
      ( ( if carrots = l then ( $\lambda$ s. s(| tortoise := hare s, carrots := 2 * carrots s, l := 0 |))
        else SKIP ) ;;
        ( $\lambda$ s. s(| hare := f (hare s), l := l s + 1 |)) )

```

The termination argument goes intuitively as follows. The Hare eats as many carrots as it takes to teleport the Tortoise into the loop. Afterwards she continues the teleportation dance until the Tortoise has given her enough carrots to make it all the way around the loop and back to him.

We can calculate the Tortoise's position as a function of *carrots*.

```

definition carrots_total :: nat  $\Rightarrow$  nat where
  carrots_total c  $\equiv$   $\sum$  i < lg c. 2 ^ i

```

```

lemma carrots_total_simps:
  carrots_total (Suc 0) = 0
  carrots_total (Suc (Suc 0)) = 1
  2 ^ i = c  $\implies$  carrots_total (c + c) = c + carrots_total c
by (auto simp: carrots_total_def lg_simps)

```

```

definition find_lambda_measures :: ( (nat  $\times$  nat)  $\times$  (nat  $\times$  nat) ) set where
  find_lambda_measures  $\equiv$ 
    measures [ $\lambda$ (l, c). mu - carrots_total c,
               $\lambda$ (l, c). LEAST i. lambda  $\leq$  c * 2 ^ i,
               $\lambda$ (l, c). c - l]

```

```

lemma find_lambda_measures_wellfounded:

```

wf find_lambda_measures
by (*simp add: find_lambda_measures_def*)

lemma *find_lambda_measures_decreases1*:

assumes $c = 2^i$
assumes $\mu \leq \text{carrots_total } c \longrightarrow c \leq \text{lambda}$
assumes $\text{seq } (\text{carrots_total } c) \neq \text{seq } (\text{carrots_total } c + c)$
shows $((c', 2 * c), (c, c)) \in \text{find_lambda_measures}$
proof(*cases* $\mu \leq \text{carrots_total } c$)
case *False* **with** *assms* **show** *?thesis*
by (*auto simp: find_lambda_measures_def carrots_total_simps mult_2 field_simps diff_less_mono2*)
next
case *True*
{ fix x **assume** $x: (0::\text{nat}) < x$ **have** $\exists n. \text{lambda} \leq x * 2^n$
proof(*induct* *lambda*)
case (*Suc i*)
then obtain n **where** $i \leq x * 2^n$ **by** *blast*
with x **show** *?case*
by (*clarsimp intro!: exI[where x=Suc n] simp: field_simps mult_2*)
(metis Nat.add_0_right Suc_leI linorder_neqE_nat mult_eq_0_iff add_left_cancel not_le numeral_2_eq_2
old.nat.distinct(2) power_not_zero trans_le_add2)
qed *simp* } **note** $ex = \text{this}$
have (*LEAST j. lambda* $\leq 2^{(i+1)} * 2^j$) < (*LEAST j. lambda* $\leq 2^i * 2^j$)
proof(*rule LeastI2_wellorder_ex[OF ex, rotated], rule LeastI2_wellorder_ex[OF ex, rotated]*)
fix $x y$
assume $\text{lambda} \leq 2^i * 2^y$
 $\text{lambda} \leq 2^{(i+1)} * 2^x$
 $\forall z. \text{lambda} \leq 2^{(i+1)} * 2^z \longrightarrow x \leq z$
with *True* *assms* *properties_loop* **where** $i = \text{carrots_total } c$ **and** $j = 1$
show $x < y$ **by** (*cases y, auto simp: less_Suc_eq_le*)
qed *simp_all*
with *True* $\langle c = 2^i \rangle$ **show** *?thesis*
by (*clarsimp simp: find_lambda_measures_def mult_2 carrots_total_simps field_simps power_add*)
qed

lemma *find_lambda_measures_decreases2*:

assumes $ls < c$
shows $((\text{Suc } ls, c), (ls, c)) \in \text{find_lambda_measures}$
using *assms* **by** (*simp add: find_lambda_measures_def*)

lemma *find_lambda*:

$\{\langle \text{True} \rangle\} \text{find_lambda } \{l = \langle \text{lambda} \rangle\}$
apply (*simp add: find_lambda_def*)
apply (*rule hoare_pre*)
apply (*rule whileI[where I= $\langle 0 \rangle < l \wedge l \leq \text{carrots} \wedge (\langle \mu \rangle \leq \text{carrots_total} \circ \text{carrots} \longrightarrow l \leq \langle \text{lambda} \rangle) \wedge (\exists i. \text{carrots} = 2^i)$*)
 $\wedge \text{tortoise} = \text{seq} \circ \text{carrots_total} \circ \text{carrots} \wedge \text{hare} = \text{seq} \circ (l + (\text{carrots_total} \circ \text{carrots}))$
and $r = \text{inv_image find_lambda_measures } (l \boxtimes \text{carrots})$
wp_intro)
using *properties_lambda_gt_0*
apply (*clarsimp simp: field_simps mult_2_right carrots_total_simps*)
apply (*intro conjI impI*)
apply (*metis mult_2 power_Suc*)
apply (*case_tac* $\mu \leq \text{carrots_total } (l s)$)
apply (*cut_tac* $i = \text{carrots_total } (l s)$ **and** $j = l s$ **in** *properties_distinct_contrapos, simp_all add: field_simps*)[1]
apply (*cut_tac* $i = \text{carrots_total } (l s)$ **and** $j = l s$ **in** *properties_loops_ge_mu, simp_all add: field_simps*)[1]
apply (*cut_tac* $i = \text{carrots_total } (2^x)$ **and** $j = 1$ **in** *properties_loop, simp*)
apply (*fastforce simp: le_eq_less_or_eq field_simps*)
apply (*cut_tac* $i = \text{carrots_total } (2^x)$ **and** $j = l s$ **in** *properties_loops_ge_mu, simp_all add: field_simps*)[1]
apply (*cut_tac* $i = \text{carrots_total } (2^x)$ **and** $j = l s$ **in** *properties_distinct_contrapos, simp_all add: field_simps*)[1]
apply (*simp add: find_lambda_measures_wellfounded*)
apply (*clarsimp simp: add commute find_lambda_measures_decreases1 find_lambda_measures_decreases2*)
apply (*rule wp_intro*)
using *properties_lambda_gt_0*
apply (*simp add: carrots_total_simps exI[where x=0]*)

done

6.2 Finding μ

With λ in hand, we can find μ using the same approach as for the Tortoise and Hare (§5.2), after we first move the Hare to λ .

definition (in $fx0$) $find_mu :: 'a\ state \Rightarrow 'a\ state$ **where**

```
 $find\_mu \equiv$   
  ( $\lambda s. s \langle m := 0, tortoise := x0, hare := seq\ (l\ s) \rangle$ ) ;;  
  while ( $hare \neq tortoise$ )  
    ( $\lambda s. s \langle tortoise := f\ (tortoise\ s), hare := f\ (hare\ s), m := m\ s + 1 \rangle$ )
```

lemma $find_mu$:

```
 $\langle l = \langle \lambda \rangle \rangle find\_mu \langle l = \langle \lambda \rangle \wedge m = \langle \mu \rangle \rangle$   
apply ( $simp\ add: find\_mu\_def$ )  
apply ( $rule\ hoare\_pre$ )  
apply ( $rule\ whileI[\mathbf{where}\ I=l = \langle \lambda \rangle \wedge m \leq \langle \mu \rangle \wedge tortoise = seq \circ m \wedge hare = seq \circ (m + l)$   
      and  $r=measure\ (\langle \mu \rangle - m)]$   
       $wp\_intro$ )  
  using  $properties\_lambda\_gt\_0\ properties\_loop[\mathbf{where}\ i=\mu\ \mathbf{and}\ j=1]$   
  apply ( $fastforce\ simp: le\_less\ dest: properties\_loops\_ge\_mu$ )  
  apply  $simp$   
  using  $properties\_loop[\mathbf{where}\ i=\mu\ \mathbf{and}\ j=1, simplified]$   
  apply ( $fastforce\ simp: le\_eq\_less\_or\_eq$ )  
apply ( $rule\ wp\_intro$ )  
apply  $simp$   
done
```

6.3 Top level

definition (in $fx0$) $brent :: 'a\ state \Rightarrow 'a\ state$ **where**

```
 $brent \equiv find\_lambda ;; find\_mu$ 
```

theorem $brent$:

```
 $\langle \langle True \rangle \rangle brent \langle l = \langle \lambda \rangle \wedge m = \langle \mu \rangle \rangle$   
unfolding  $brent\_def$   
by ( $rule\ find\_lambda\ find\_mu\ wp\_intro$ )  
end
```

end

corollary $brent_correct$:

```
assumes  $s': s' = fx0.brent\ f\ x\ arbitrary$   
shows  $fx0.properties\ f\ x\ (l\ s')\ (m\ s')$   
using  $assms\ properties.brent[\mathbf{where}\ f=f\ \mathbf{and}\ ?x0.0=x]$   
by ( $fastforce\ intro: fx0.properties\_existence[\mathbf{where}\ f=f\ \mathbf{and}\ ?x0.0=x]$   
       $simp: Basis.properties\_def\ valid\_def$ )
```

schematic_goal $brent_code[code]$:

```
 $fx0.brent\ f\ x = ?code$   
unfolding  $fx0.brent\_def\ fx0.find\_lambda\_def\ fx0.find\_mu\_def\ fcomp\_assoc[symmetric]\ fcomp\_comp$   
by ( $rule\ refl$ )
```

export_code $fx0.brent$ **in** SML

7 Concluding remarks

Leino (2012) uses an SMT solver to verify a Tortoise-and-Hare cycle-finder. He finds the parameters λ and μ initially by using a “ghost” depth-first search, while we use more economical non-constructive methods.

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