Topological Groups

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March 17, 2025

Abstract

Topological groups are blends of groups and topological spaces with the property that the multiplication and inversion operations are continuous functions. They frequently occur in mathematics and physics, e.g. in the form of Lie groups. We formalize the theory of topological groups on top of HOL-Algebra and HOL-Analysis. Topological groups are defined via a locale. We also introduce a set-based notion of uniform spaces in order to define the uniform structures of topological groups. The most notable formalized result is the Birkhoff-Kakutani theorem which characterizes metrizable topological groups. Our formalization also defines the important matrix groups $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, O_n, SO_n and proves them to be topological groups.

The formalized results and proofs have been taken from the textbooks of Arhangelskii and Tkachenko [1], Bump [2] and James [4]. These lecture notes [5] have also been helpful.

Contents

1 Uniform spaces

theory *Uniform-Structure*
imports *HOL−Analysis.Abstract-Topology HOL−Analysis.Abstract-Metric-Spaces*
begin

Summary This section introduces a set-based notion of uniformities and connects it to the *uniform-space* type class.

1.1 Definitions and basic results

definition *uniformity-on* :: $'a\ set \Rightarrow (('a \times 'a)\ set \Rightarrow bool) \Rightarrow bool$ **where**
uniformity-on $X\ \mathcal{E} \longleftrightarrow$
$(\exists E.\ \mathcal{E}\ E) \wedge$
$(\forall E.\ \mathcal{E}\ E \longrightarrow E \subseteq X \times X \wedge Id\text{-}on\ X \subseteq E \wedge \mathcal{E}\ (E^{-1}) \wedge (\exists F.\ \mathcal{E}\ F \wedge F\ O\ F \subseteq E) \wedge$
$(\forall F.\ E \subseteq F \wedge F \subseteq X \times X \longrightarrow \mathcal{E}\ F)) \wedge$
$(\forall E\ F.\ \mathcal{E}\ E \longrightarrow \mathcal{E}\ F \longrightarrow \mathcal{E}\ (E \cap F))$

typedef $'a$ *uniformity* $= \{(X :: 'a\ set, \mathcal{E}).\ uniformity\text{-}on\ X\ \mathcal{E}\}$
morphisms *uniformity-rep uniformity*
⟨*proof*⟩

definition *uspace* :: $'a\ uniformity \Rightarrow 'a\ set$ **where**
uspace $\Phi = (let\ (X, \mathcal{E}) = uniformity\text{-}rep\ \Phi\ in\ X)$

definition *entourage-in* :: $'a\ uniformity \Rightarrow ('a \times 'a)\ set \Rightarrow bool$ **where**
entourage-in $\Phi = (let\ (X, \mathcal{E}) = uniformity\text{-}rep\ \Phi\ in\ \mathcal{E})$

lemma *uniformity-inverse'*[*simp*]:
assumes *uniformity-on* $X\ \mathcal{E}$
shows *uspace* $(uniformity\ (X, \mathcal{E})) = X \wedge entourage\text{-}in\ (uniformity\ (X, \mathcal{E})) = \mathcal{E}$
⟨*proof*⟩

lemma *uniformity-entourages*:
shows *uniformity-on* (*uspace* Φ) (*entourage-in* Φ)
⟨*proof*⟩

lemma *entourages-exist*: $\exists E.\ entourage\text{-}in\ \Phi\ E$
⟨*proof*⟩

lemma *entourage-in-space*[*elim*]: $entourage\text{-}in\ \Phi\ E \Longrightarrow E \subseteq uspace\ \Phi \times uspace\ \Phi$
⟨*proof*⟩

lemma *entourage-superset*[*intro*]:
$entourage\text{-}in\ \Phi\ E \Longrightarrow E \subseteq F \Longrightarrow F \subseteq uspace\ \Phi \times uspace\ \Phi \Longrightarrow entourage\text{-}in\ \Phi\ F$
⟨*proof*⟩

lemma *entourage-intersection*[*intro*]: $entourage\text{-}in\ \Phi\ E \Longrightarrow entourage\text{-}in\ \Phi\ F \Longrightarrow entourage\text{-}in\ \Phi\ (E \cap F)$
⟨*proof*⟩

lemma *entourage-converse*[*intro*]: $entourage\text{-}in\ \Phi\ E \Longrightarrow entourage\text{-}in\ \Phi\ (E^{-1})$
⟨*proof*⟩

lemma *entourage-diagonal*[*dest*]:
assumes *entourage*: $entourage\text{-}in\ \Phi\ E$ **and** *in-space*: $x \in uspace\ \Phi$
shows $(x,x) \in E$
⟨*proof*⟩

lemma *smaller-entourage*:
assumes *entourage*: $entourage\text{-}in\ \Phi\ E$
shows $\exists F.\ entourage\text{-}in\ \Phi\ F \wedge (\forall x\ y\ z.\ (x,y) \in F \wedge (y,z) \in F \longrightarrow (x,z) \in E)$
⟨*proof*⟩

lemma *entire-space-entourage*: $entourage\text{-}in\ \Phi\ (uspace\ \Phi \times uspace\ \Phi)$
⟨*proof*⟩

definition *utopology* :: $'a\ uniformity \Rightarrow 'a\ topology$ **where**
$utopology\ \Phi = topology\ (\lambda U.\ U \subseteq uspace\ \Phi \wedge (\forall x{\in}U.\ \exists E.\ entourage\text{-}in\ \Phi\ E \wedge E``\{x\} \subseteq U))$

lemma *openin-utopology* [*iff*]:
fixes Φ :: $'a\ uniformity$
defines $uopen\ U \equiv U \subseteq uspace\ \Phi \wedge (\forall x{\in}U.\ \exists E.\ entourage\text{-}in\ \Phi\ E \wedge E``\{x\} \subseteq U)$
shows $openin\ (utopology\ \Phi) = uopen$
⟨*proof*⟩

lemma *topspace-utopology*[*simp*]:
shows $topspace\ (utopology\ \Phi) = uspace\ \Phi$
⟨*proof*⟩

definition *ucontinuous* :: $'a\ uniformity \Rightarrow 'b\ uniformity \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool$ **where**

ucontinuous Φ Ψ *f* ⟷
  f ∈ *uspace* Φ → *uspace* Ψ ∧
  (∀ *E*. *entourage-in* Ψ *E* ⟶ *entourage-in* Φ {(*x*, *y*) ∈ *uspace* Φ × *uspace* Φ. (*f x*, *f y*) ∈ *E*})

lemma *ucontinuous-image-subset* [*dest*]: *ucontinuous* Φ Ψ *f* ⟹ *f*‘(*uspace* Φ) ⊆ *uspace* Ψ
  ⟨*proof*⟩

lemma *entourage-preimage-ucontinuous* [*dest*]:
  assumes *ucontinuous* Φ Ψ *f* **and** *entourage-in* Ψ *E*
  shows *entourage-in* Φ {(*x*, *y*) ∈ *uspace* Φ × *uspace* Φ. (*f x*, *f y*) ∈ *E*}
  ⟨*proof*⟩

lemma *ucontinuous-imp-continuous*:
  assumes *ucontinuous* Φ Ψ *f*
  shows *continuous-map* (*utopology* Φ) (*utopology* Ψ) *f*
⟨*proof*⟩

1.2 Metric spaces as uniform spaces

context *Metric-space*
begin

abbreviation *mentourage* :: *real* ⇒ (′*a* × ′*a*) *set* **where**
mentourage ε ≡ {(*x*,*y*) ∈ *M* × *M*. *d x y* < ε}

definition *muniformity* :: ′*a uniformity* **where**
muniformity = *uniformity* (*M*, λ*E*. *E* ⊆ *M* × *M* ∧ (∃ε > *0*. *mentourage* ε ⊆ *E*))

lemma
  uspace-muniformity[*simp*]: *uspace muniformity* = *M* **and**
  entourage-muniformity: *entourage-in muniformity* = (λ*E*. *E* ⊆ *M* × *M* ∧ (∃ε > *0*. *mentourage* ε ⊆ *E*))
⟨*proof*⟩

lemma *uniformity-induces-mtopology* [*simp*]: *utopology muniformity* = *mtopology*
⟨*proof*⟩

1.3 Connection to type class

end

The following connects the *uniform-space* class to the set based notion *Uniform-Structure.uniformity-on.*

Given a type ′*a* which is an instance of the class *uniform-space*, it is possible to introduce an ′*a uniformity* on the entire universe: *UNIV*:

definition *uniformity-of-space* :: (′*a* :: *uniform-space*) *uniformity* **where**

uniformity-of-space = *uniformity* (*UNIV* :: *'a set*, (λS. $\forall_F$ *x in uniformity-class.uniformity*. *x*∈*S*))

The induced uniformity fulfills the required conditions, i.e., the class based notion implies the set-based notion.

lemma *uniformity-on-uniformity-of-space-aux*:
uniformity-on (*UNIV* :: (*'a* :: *uniform-space*) *set*) (λS. $\forall_F$ *x in uniformity-class.uniformity*. *x*∈*S*)
⟨*proof*⟩

lemma *uniformity-rep-uniformity-of-space*:
uniformity-rep uniformity-of-space = (*UNIV*, (λS. $\forall_F$ *x in uniformity-class.uniformity*. *x* ∈ *S*))
⟨*proof*⟩

lemma *uspace-uniformity-space* [*simp*, *iff*]:
uspace uniformity-of-space = *UNIV*
⟨*proof*⟩

lemma *entourage-in-uniformity-space*:
entourage-in uniformity-of-space S =($\forall_F$ *x in uniformity-class.uniformity*. *x* ∈ *S*)
⟨*proof*⟩

Compatibility of the *Metric-space.muniformity* with the uniformity based on the class based hierarchy.

lemma (*uniformity-of-space* :: (*'a* :: *metric-space*) *uniformity*) = *Met-TC.muniformity*
⟨*proof*⟩

end

2 General theory of Topological Groups

theory *Topological-Group*
imports
HOL−Algebra.Group
HOL−Algebra.Coset
HOL−Analysis.Abstract-Topology
HOL−Analysis.Product-Topology
HOL−Analysis.T1-Spaces
HOL−Analysis.Abstract-Metric-Spaces
Uniform-Structure
begin

Summary In this section we define topological groups and prove basic results about them. We also introduce the left and right uniform structures of topological groups and prove the Birkhoff-Kakutani theorem.

2.1 Auxiliary definitions and results

2.1.1 Miscellaneous

lemma *connected-components-homeo*:
assumes *homeo*: *homeomorphic-map* T_1 T_2 φ **and** *in-space*: $x \in$ *topspace* T_1
shows φ‘(*connected-component-of-set* T_1 x) = *connected-component-of-set* T_2 (φ x)
⟨*proof*⟩

lemma *open-map-prod-top*:
assumes *open-map* T_1 T_3 f **and** *open-map* T_2 T_4 g
shows *open-map* (*prod-topology* T_1 T_2) (*prod-topology* T_3 T_4) ($\lambda(x, y)$. (f x, g y))
⟨*proof*⟩

lemma *injective-quotient-map-homeo*:
assumes *quotient-map T1 T2 q* **and** *inj*: *inj-on q* (*topspace T1*)
shows *homeomorphic-map T1 T2 q* ⟨*proof*⟩

lemma (**in** *group*) *subgroupI-alt*:
assumes *subset*: $H \subseteq$ *carrier* G **and** *nonempty*: $H \neq \{\}$ **and**
closed: $\bigwedge\sigma\ \tau.\ \sigma \in H \wedge \tau \in H \Longrightarrow \sigma \otimes$ *inv* $\tau \in H$
shows *subgroup H G*
⟨*proof*⟩

lemma *subgroup-intersection*:
assumes *subgroup H G* **and** *subgroup* H' G
shows *subgroup* $(H \cap H')$ G
⟨*proof*⟩

2.1.2 Quotient topology

definition *quot-topology* :: *'a topology* ⇒ (*'a* ⇒ *'b*) ⇒ *'b topology* **where**
quot-topology T q = *topology* (λU. $U \subseteq q$‘(*topspace T*) ∧ *openin T* $\{x \in$ *topspace* $T.\ q\ x \in U\}$)

lemma *quot-topology-open*:
fixes T :: *'a topology* **and** q :: *'a* ⇒ *'b*
defines *openin-quot* $U \equiv U \subseteq q$‘(*topspace T*) ∧ *openin T* $\{x \in$ *topspace* $T.\ q\ x \in U\}$
shows *openin* (*quot-topology T q*) = *openin-quot*
⟨*proof*⟩

lemma *projection-quotient-map*: *quotient-map T* (*quot-topology T q*) *q*
⟨*proof*⟩

corollary *topspace-quot-topology* [*simp*]: *topspace* (*quot-topology T q*) = q‘(*topspace* T)
⟨*proof*⟩

corollary *projection-continuous*: *continuous-map T* (*quot-topology T q*) *q*
⟨*proof*⟩

2.2 Definition and basic results

locale *topological-group* = *group* +
fixes *T* :: *'g topology*
assumes *group-is-space* [*simp*]: *topspace T* = *carrier G*
assumes *inv-continuous*: *continuous-map T T* (λσ. *inv* σ)
assumes *mul-continuous*: *continuous-map* (*prod-topology T T*) *T* (λ(σ,τ). σ⊗τ)
begin

lemma *in-space-iff-in-group* [*iff*]: σ ∈ *topspace T* ⟷ σ ∈ *carrier G*
⟨*proof*⟩

lemma *translations-continuous* [*intro*]:
assumes *in-group*: σ ∈ *carrier G*
shows *continuous-map T T* (λτ. σ⊗τ) **and** *continuous-map T T* (λτ. τ⊗σ)
⟨*proof*⟩

lemma *translations-homeos*:
assumes *in-group*: σ ∈ *carrier G*
shows *homeomorphic-map T T* (λτ. σ⊗τ) **and** *homeomorphic-map T T* (λτ. τ⊗σ)
⟨*proof*⟩

abbreviation *conjugation* :: *'g* ⇒ *'g* ⇒ *'g* **where**
conjugation σ τ ≡ σ ⊗ τ ⊗ *inv* σ

corollary *conjugation-homeo*:
assumes *in-group*: σ ∈ *carrier G*
shows *homeomorphic-map T T* (*conjugation* σ)
⟨*proof*⟩

corollary *open-set-translations*:
assumes *open-set*: *openin T U* **and** *in-group*: σ ∈ *carrier G*
shows *openin T* (σ <# *U*) **and** *openin T* (*U* #> σ)
⟨*proof*⟩

corollary *closed-set-translations*:
assumes *closed-set*: *closedin T U* **and** *in-group*: σ ∈ *carrier G*
shows *closedin T* (σ <# *U*) **and** *closedin T* (*U* #> σ)
⟨*proof*⟩

lemma *inverse-homeo*: *homeomorphic-map T T* (λσ. *inv* σ)
⟨*proof*⟩

2.3 Subspaces and quotient spaces

abbreviation *connected-component-1* :: *'g set* **where**
connected-component-1 ≡ *connected-component-of-set T* **1**

lemma *connected-component-1-props*:
shows *connected-component-1* ⊲ *G* **and** *closedin T connected-component-1*
⟨*proof*⟩

lemma *group-prod-space* [*simp*]: *topspace* (*prod-topology T T*) = (*carrier G*) × (*carrier G*)
⟨*proof*⟩

no-notation *eq-closure-of* (‹*closure'-of*1›)

lemma *subgroup-closure*:
assumes *H-subgroup*: *subgroup H G*
shows *subgroup* (*T closure-of H*) *G*
⟨*proof*⟩

lemma *normal-subgroup-closure*:
assumes *normal-subgroup*: *N* ⊲ *G*
shows (*T closure-of N*) ⊲ *G*
⟨*proof*⟩

lemma *topological-subgroup*:
assumes *subgroup H G*
shows *topological-group* (*G* (|*carrier* := *H*|)) (*subtopology T H*)
⟨*proof*⟩

Topology on the set of cosets of some subgroup

abbreviation *coset-topology* :: *'g set* ⇒ *'g set topology* **where**
coset-topology H ≡ *quot-topology T* (*r-coset G H*)

lemma *coset-topology-topspace*[*simp*]:
shows *topspace* (*coset-topology H*) = (*r-coset G H*)‘(*carrier G*)
⟨*proof*⟩

lemma *projection-open-map*:
assumes *subgroup*: *subgroup H G*
shows *open-map T* (*coset-topology H*) (*r-coset G H*)
⟨*proof*⟩

lemma *topological-quotient-group*:
assumes *normal-subgroup*: *N* ⊲ *G*
shows *topological-group* (*G Mod N*) (*coset-topology N*)
⟨*proof*⟩

See [3] for our approach to proving that quotient groups of topological groups are topological.

abbreviation *neighborhood* :: $'g \Rightarrow 'g$ *set* $\Rightarrow$ *bool* **where**
neighborhood σ *U* $\equiv$ *openin T U* $\wedge$ $\sigma \in U$

abbreviation *symmetric* :: $'g$ *set* $\Rightarrow$ *bool* **where**
symmetric S $\equiv$ $\{$*inv* $\sigma \mid \sigma.\ \sigma \in S\} \subseteq S$

Note that this implies the other inclusion, so symmetric subsets are equal to their image under inversion.

lemma *neighborhoods-of-1*:
  assumes *neighborhood* **1** *U*
  shows $\exists V.$ *neighborhood* **1** *V* $\wedge$ *symmetric V* $\wedge$ $V <\#> V \subseteq U$
⟨*proof*⟩

lemma *Hausdorff-coset-space*:
  assumes *subgroup*: *subgroup H G* **and** *H-closed*: *closedin T H*
  shows *Hausdorff-space* (*coset-topology H*)
⟨*proof*⟩

lemma *Hausdorff-coset-space-converse*:
  assumes *subgroup*: *subgroup H G*
  assumes *Hausdorff*: *Hausdorff-space* (*coset-topology H*)
  shows *closedin T H*
⟨*proof*⟩

corollary *Hausdorff-coset-space-iff*:
  assumes *subgroup*: *subgroup H G*
  shows *Hausdorff-space* (*coset-topology H*) $\longleftrightarrow$ *closedin T H*
  ⟨*proof*⟩

corollary *topological-group-hausdorff-iff-one-closed*:
  shows *Hausdorff-space T* $\longleftrightarrow$ *closedin T* $\{\mathbf{1}\}$
⟨*proof*⟩

lemma *set-mult-one-subset*:
  assumes $A \subseteq$ *carrier G* $\wedge$ $B \subseteq$ *carrier G* **and** $\mathbf{1} \in B$
  shows $A \subseteq A <\#> B$
  ⟨*proof*⟩

lemma *open-set-mult-open*:
  assumes *openin T U* $\wedge$ $S \subseteq$ *carrier G*
  shows *openin T* $(S <\#> U)$
⟨*proof*⟩

lemma *open-set-inv-open*:
  assumes *openin T U*
  shows *openin T* (*set-inv U*)
⟨*proof*⟩

lemma *open-set-in-carrier*[*elim*]:

assumes *openin T U*
shows *U* ⊆ *carrier G*
⟨*proof*⟩

2.4 Uniform structures

abbreviation *left-entourage* :: *'g set* ⇒ (*'g* × *'g*) *set* **where**
left-entourage U ≡ {(σ,τ) ∈ *carrier G* × *carrier G*. *inv* σ ⊗ τ ∈ *U*}

abbreviation *right-entourage* :: *'g set* ⇒ (*'g* × *'g*) *set* **where**
right-entourage U ≡ {(σ,τ) ∈ *carrier G* × *carrier G*. σ ⊗ *inv* τ ∈ *U*}

definition *left-uniformity* :: *'g uniformity* **where** *left-uniformity* =
uniformity (*carrier G*, λ*E*. *E* ⊆ *carrier G* × *carrier G* ∧ (∃ *U*. *neighborhood* **1** *U* ∧ *left-entourage U* ⊆ *E*))

definition *right-uniformity* :: *'g uniformity* **where** *right-uniformity* =
uniformity (*carrier G*, λ*E*. *E* ⊆ *carrier G* × *carrier G* ∧ (∃ *U*. *neighborhood* **1** *U* ∧ *right-entourage U* ⊆ *E*))

lemma
uspace-left-uniformity[*simp*]: *uspace left-uniformity* = *carrier G* (**is** *?space-def*)
and
entourage-left-uniformity: *entourage-in left-uniformity* =
(λ*E*. *E* ⊆ *carrier G* × *carrier G* ∧ (∃ *U*. *neighborhood* **1** *U* ∧ *left-entourage U* ⊆ *E*)) (**is** *?entourage-def*)
⟨*proof*⟩

lemma
uspace-right-uniformity[*simp*]: *uspace right-uniformity* = *carrier G* (**is** *?space-def*)
and
entourage-right-uniformity: *entourage-in right-uniformity* =
(λ*E*. *E* ⊆ *carrier G* × *carrier G* ∧ (∃ *U*. *neighborhood* **1** *U* ∧ *right-entourage U* ⊆ *E*)) (**is** *?entourage-def*)
⟨*proof*⟩

lemma *left-uniformity-induces-group-topology* [*simp*]:
shows *utopology left-uniformity* = *T*
⟨*proof*⟩

lemma *right-uniformity-induces-group-topology* [*simp*]:
shows *utopology right-uniformity* = *T*
⟨*proof*⟩

lemma *translations-ucontinuous*:
assumes *in-group*: σ ∈ *carrier G*
shows *ucontinuous left-uniformity left-uniformity* (λτ. σ ⊗ τ) **and**
ucontinuous right-uniformity right-uniformity (λτ. τ ⊗ σ)
⟨*proof*⟩

2.5 The Birkhoff-Kakutani theorem

2.5.1 Prenorms on groups

definition *group-prenorm* :: $('g \Rightarrow real) \Rightarrow bool$ **where**
group-prenorm $N \longleftrightarrow$
  $N\ \mathbf{1} = 0 \wedge$
  $(\forall \sigma\ \tau.\ \sigma \in carrier\ G \wedge \tau \in carrier\ G \longrightarrow N\ (\sigma \otimes \tau) \leq N\ \sigma + N\ \tau) \wedge$
  $(\forall \sigma \in carrier\ G.\ N\ (inv\ \sigma) = N\ \sigma)$

lemma *group-prenorm-clauses*[*elim*]:
  assumes *group-prenorm N*
  obtains
    $N\ \mathbf{1} = 0$ **and**
    $\bigwedge\sigma\ \tau.\ \sigma \in carrier\ G \Longrightarrow \tau \in carrier\ G \Longrightarrow N\ (\sigma \otimes \tau) \leq N\ \sigma + N\ \tau$ **and**
    $\bigwedge\sigma.\ \sigma \in carrier\ G \Longrightarrow N\ (inv\ \sigma) = N\ \sigma$
  ⟨*proof*⟩

proposition *group-prenorm-nonnegative*:
  assumes *prenorm*: *group-prenorm N*
  shows $\forall \sigma \in carrier\ G.\ N\ \sigma \geq 0$
⟨*proof*⟩

proposition *group-prenorm-reverse-triangle-ineq*:
  assumes *prenorm*: *group-prenorm N* **and** *in-group*: $\sigma \in carrier\ G \wedge \tau \in carrier$
G
  shows $|N\ \sigma - N\ \tau| \leq N\ (\sigma \otimes inv\ \tau)$
⟨*proof*⟩

definition *induced-group-prenorm* :: $('g \Rightarrow real) \Rightarrow 'g \Rightarrow real$ **where**
induced-group-prenorm $f\ \sigma = (SUP\ \tau \in carrier\ G.\ |f\ (\tau \otimes \sigma) - f\ \tau|)$

lemma *induced-group-prenorm-welldefined*:
  fixes $f :: 'g \Rightarrow real$
  assumes *f-bounded*: $\exists\, c.\forall \tau \in carrier\ G.\ |f\ \tau| \leq c$ **and** *in-group*: $\sigma \in carrier\ G$
  shows *bdd-above* $((\lambda\tau.\ |f\ (\tau \otimes \sigma) - f\ \tau|)\ `(carrier\ G))$
⟨*proof*⟩

lemma *bounded-function-induces-group-prenorm*:
  fixes $f :: 'g \Rightarrow real$
  assumes *f-bounded*: $\exists\, c.\forall \sigma \in carrier\ G.\ |f\ \sigma| \leq c$
  shows *group-prenorm* (*induced-group-prenorm f*)
⟨*proof*⟩

lemma *neighborhood-1-translation*:
  assumes *neighborhood* $\mathbf{1}$ U **and** $\sigma \in carrier\ G \vee \sigma \in topspace\ T$
  shows *neighborhood* $\sigma\ (\sigma <\#\ U)$
⟨*proof*⟩

proposition *group-prenorm-continuous-if-continuous-at-1*:

assumes *prenorm*: *group-prenorm N* **and**
continuous-at-1: ∀ε>0.∃U. *neighborhood* **1** U ∧ (∀σ∈U. N σ < ε)
shows *continuous-map T euclideanreal N*
⟨*proof*⟩

2.5.2 A prenorm respecting the group topology

context
fixes U :: *nat* ⇒ ′*g set*
assumes *U-neighborhood*: ∀n. *neighborhood* **1** (U n)
assumes *U-props*: ∀n. *symmetric* (U n) ∧ (U (n + 1)) <#> (U (n + 1)) ⊆ (U n)
begin

private fun V :: *nat* ⇒ *nat* ⇒ ′*g set* **where**
V m n = (
if m = 0 then {} *else*
if m = 1 then U n else
if m > 2^n then carrier G else
if even m then V (m div 2) (n − 1) else
V ((m − 1) div 2) (n − 1) <#> *U n*
)

private lemma *U-in-group*: *U k* ⊆ *carrier G* ⟨*proof*⟩ **lemma** *V-in-group*:
shows *V m n* ⊆ *carrier G*
⟨*proof*⟩ **lemma** *V-mult*:
shows *m* ≥ *1* ⟹ *V m n* <#> *U n* ⊆ *V (m + 1) n*
⟨*proof*⟩ **lemma** *V-mono*:
assumes *smaller*: (*real* m_1)/2^n_1 ≤ (*real* m_2)/2^n_2 **and** *not-zero*: m_1 ≥ *1* ∧ m_2 ≥ *1*
shows V m_1 n_1 ⊆ V m_2 n_2
⟨*proof*⟩ **lemma** *approx-number-by-multiples*:
assumes *hx*: *x* ≥ *0* **and** *hc*: *c* > *0*
shows ∃k :: *nat* ≥ *1*. (*real* (k−1))/c ≤ x ∧ x < (*real* k)/c
⟨*proof*⟩

lemma *construction-of-prenorm-respecting-topology*:
shows ∃N. *group-prenorm* N ∧
(∀n. {σ ∈ *carrier G*. N σ < 1/2^n} ⊆ U n) ∧
(∀n. U n ⊆ {σ ∈ *carrier G*. N σ ≤ 2/2^n})
⟨*proof*⟩
end

2.5.3 Proof of Birkhoff-Kakutani

lemma *first-countable-neighborhoods-of-1-sequence*:
assumes *first-countable T*
shows ∃U :: *nat* ⇒ ′*g set*.
(∀n. *neighborhood* **1** (U n) ∧ *symmetric* (U n) ∧ U (n + 1) <#> U (n + 1) ⊆ U n) ∧

$(\forall\, W.\ neighborhood\ \mathbf{1}\ W \longrightarrow (\exists\, n.\ U\ n \subseteq W))$
⟨*proof*⟩

definition *left-invariant-metric* $\Delta \longleftrightarrow$ *Metric-space* (*carrier G*) $\Delta \wedge$
$(\forall \sigma\ \tau\ \varrho.\ \sigma \in carrier\ G \wedge \tau \in carrier\ G \wedge \varrho \in carrier\ G \longrightarrow \Delta\ (\varrho \otimes \sigma)\ (\varrho \otimes \tau) = \Delta\ \sigma\ \tau)$

definition *right-invariant-metric* $\Delta \longleftrightarrow$ *Metric-space* (*carrier G*) $\Delta \wedge$
$(\forall \sigma\ \tau\ \varrho.\ \sigma \in carrier\ G \wedge \tau \in carrier\ G \wedge \varrho \in carrier\ G \longrightarrow \Delta\ (\sigma \otimes \varrho)\ (\tau \otimes \varrho) = \Delta\ \sigma\ \tau)$

lemma *left-invariant-metricE*:
assumes *left-invariant-metric* Δ $\sigma \in carrier\ G$ $\tau \in carrier\ G$ $\varrho \in carrier\ G$
shows $\Delta\ (\varrho \otimes \sigma)\ (\varrho \otimes \tau) = \Delta\ \sigma\ \tau$
⟨*proof*⟩

lemma *right-invariant-metricE*:
assumes *right-invariant-metric* Δ $\sigma \in carrier\ G$ $\tau \in carrier\ G$ $\varrho \in carrier\ G$
shows $\Delta\ (\sigma \otimes \varrho)\ (\tau \otimes \varrho) = \Delta\ \sigma\ \tau$
⟨*proof*⟩

theorem *Birkhoff-Kakutani-left*:
assumes *Hausdorff*: *Hausdorff-space T* **and** *first-countable*: *first-countable T*
shows $\exists\,\Delta.$ *left-invariant-metric* $\Delta \wedge$ *Metric-space.mtopology* (*carrier G*) $\Delta = T$
⟨*proof*⟩

theorem *Birkhoff-Kakutani-right*:
assumes *Hausdorff*: *Hausdorff-space T* **and** *first-countable*: *first-countable T*
shows $\exists\,\Delta.$ *right-invariant-metric* $\Delta \wedge$ *Metric-space.mtopology* (*carrier G*) $\Delta = T$
⟨*proof*⟩

corollary *Birkhoff-Kakutani-iff*:
shows *metrizable-space T* $\longleftrightarrow$ *Hausdorff-space T* $\wedge$ *first-countable T*
⟨*proof*⟩

end

end

3 Examples of Topological Groups

theory *Topological-Group-Examples*
imports *Topological-Group*
begin

Summary This section gives examples of topological groups.

lemma (**in** *group*) *discrete-topological-group*:

shows *topological-group G* (*discrete-topology* (*carrier G*))
⟨*proof*⟩

lemma *topological-group-real-power-space*:
defines 𝔑 :: (*real*^′*n*) *monoid* ≡ (| *carrier* = *UNIV*, *mult* = (+), *one* = *0* |)
defines *T* :: (*real*^′*n*) *topology* ≡ *euclidean*
shows *topological-group* 𝔑 *T*
⟨*proof*⟩

definition *unit-group* :: (′*a* :: *field*) *monoid* **where**
unit-group = (| *carrier* = *UNIV* − {*0*}, *mult* = (∗), *one* = *1* |)

lemma
group-unit-group: *group unit-group* **and**
inv-unit-group: *x* ∈ *carrier unit-group* ⟹ $inv_{unit\text{-}group}$ *x* = *inverse x*
⟨*proof*⟩

lemma *topological-group-real-unit-group*:
defines *T* :: *real topology* ≡ *subtopology euclidean* (*UNIV* − {*0*})
shows *topological-group unit-group T*
⟨*proof*⟩

end

4 Matrix groups

theory *Matrix-Group*
imports
Topological-Group
Topological-Group-Examples
HOL−Analysis.Determinants
begin

Summary In this section we define the general linear group and some of its subgroups. We also introduce topologies on vector types and use them to prove the aforementioned groups to be topological groups.

4.1 Topologies on vector types

definition *vec-topology* :: ′*a topology* ⇒ (′*a*^′*n*) *topology* **where**
vec-topology T = *quot-topology* (*product-topology* (λ*i*. *T*) *UNIV*) *vec-lambda*

lemma *producttop-vectop-homeo*:
shows *homeomorphic-map* (*product-topology* (λ*i*. *T*) *UNIV*) (*vec-topology T*) *vec-lambda*
⟨*proof*⟩

lemma *homeo-inverse-homeo*:

assumes *homeo*: *homeomorphic-map X Y f* **and** *fg-id*: $\forall y \in$ *topspace* $Y.\ f\ (g\ y)$ $= y$ **and**
g-image: $\forall y \in$ *topspace* $Y.\ g\ y \in$ *topspace* X
shows *homeomorphic-map Y X g*
⟨*proof*⟩

lemma *vectop-producttop-homeo*:
shows *homeomorphic-map* (*vec-topology T*) (*product-topology* ($\lambda i.\ T$) *UNIV*) *vec-nth*
⟨*proof*⟩

lemma *vec-topology-euclidean* [*simp*]:
defines T :: ('*a* :: *topological-space*) *topology* ≡ *euclidean*
defines T_{vec} :: ('*a*^'*n*) *topology* ≡ *euclidean*
shows *vec-topology* $T = T_{vec}$
⟨*proof*⟩

lemma *vec-projection-continuous*:
shows *continuous-map* (*vec-topology T*) T ($\lambda v.\ v\$i$)
⟨*proof*⟩

lemma *vec-components-continuous-imp-continuous*:
fixes f :: '$x \Rightarrow$ '*a*^'*n*
assumes $\forall i.$ *continuous-map X T* ($\lambda x.$ ($f\ x$) \$ i)
shows *continuous-map X* (*vec-topology T*) f
⟨*proof*⟩

definition *matrix-topology* :: '*a topology* ⇒ ('*a*^'*n*^'*m*) *topology* **where**
matrix-topology T = *vec-topology* (*vec-topology T*)

lemma *matrix-topology-euclidean*[*simp*]:
shows *matrix-topology euclidean* = *euclidean*
⟨*proof*⟩

lemma *matrix-projection-continuous*:
shows *continuous-map* (*matrix-topology T*) T ($\lambda A.\ A\$i\$j$)
⟨*proof*⟩

lemma *matrix-components-continuous-imp-continuous*:
fixes f :: '$x \Rightarrow$ '*a*^'*n*^'*m*
assumes $\bigwedge i\ j.$ *continuous-map X T* ($\lambda x.$ ($f\ x$) \$ i \$ j)
shows *continuous-map X* (*matrix-topology T*) f
⟨*proof*⟩

4.2 The general linear group as a topological group

definition *GL* :: (('*a* :: *field*)^'*n*^'*n*) *monoid* **where**
GL = ⦇*carrier* = {*A*. *invertible A*}, *monoid.mult* = (∗∗), *one* = *mat 1*⦈

definition *GL-topology* :: (*real^'n^'n*) *topology* **where**
GL-topology = *subtopology euclidean* (*carrier GL*)

lemma *topspace-GL*: *topspace GL-topology* = {*A*. *invertible A*}
⟨*proof*⟩

4.2.1 Continuity of matrix operations

lemma *det-continuous*:
defines *T* :: (*real^'n^'n*) *topology* ≡ *euclidean*
shows *continuous-map T euclideanreal det*
⟨*proof*⟩

lemma *matrix-mul-continuous*:
defines *T1* :: (*real^'n^'m*) *topology* ≡ *euclidean*
defines *T2* :: (*real^'r^'n*) *topology* ≡ *euclidean*
defines *T3* :: (*real^'r^'m*) *topology* ≡ *euclidean*
shows *continuous-map* (*prod-topology T1 T2*) *T3* (λ(*A*,*B*). *A* ∗∗ *B*)
⟨*proof*⟩

lemma *transpose-continuous*:
shows *continuous-map* (*euclidean* :: ((*'a* :: *topological-space*)^*'n*^*'m*) *topology*) *euclidean transpose*
⟨*proof*⟩

4.2.2 Continuity of matrix inversion

lemma *matrix-mul-columns*:
fixes *A* :: (*'a* :: *semiring-1*)^*'n*^*'m* **and** *B* :: *'a^'k^'n*
shows *column j* (*A* ∗∗ *B*) = *A* ∗*v* (*column j B*)
⟨*proof*⟩

lemma *matrix-columns-unique*:
assumes ∀ *j*. *column j A* = *column j B*
shows *A* = *B*
⟨*proof*⟩

lemma *matrix-inv-is-inv*:
assumes *invertible A*
shows *A* ∗∗ (*matrix-inv A*) = *mat 1* **and** (*matrix-inv A*) ∗∗ *A* = *mat 1*
⟨*proof*⟩

lemma *invertible-imp-right-inverse-is-inverse*:
assumes *invertible*: *invertible A* **and** *A* ∗∗ *B* = *mat 1*
shows *matrix-inv A* = *B*
⟨*proof*⟩

lemma *matrix-inv-invertible*:
assumes *invertible A*
shows *invertible* (*matrix-inv A*)

⟨*proof*⟩

lemma *det-inv*:
  fixes $A :: ('a :: field)^{\frown}'n^{\frown}'n$
  assumes $det\ A \neq 0$
  shows $det\ (matrix\text{-}inv\ A) = 1\ /\ det\ A$
⟨*proof*⟩

See proposition "cramer" from HOL-Analysis.Determinants

definition $cramer\text{-}inv :: ('a :: field)^{\frown}'n^{\frown}'n \Rightarrow 'a^{\frown}'n^{\frown}'n$ **where**
$cramer\text{-}inv\ A = (\chi\ i\ j.\ det(\chi\ k\ l.\ if\ l = i\ then\ (axis\ j\ 1)\ \$\ k\ else\ A\$k\$l)\ /\ det\ A)$

lemma *cramer-inv-is-inverse*:
  assumes *invertible*: $invertible\ (A :: ('a :: field)^{\frown}'n^{\frown}'n)$
  shows $matrix\text{-}inv\ A = cramer\text{-}inv\ A$
⟨*proof*⟩

lemma *matrix-inv-continuous*:
  shows $continuous\text{-}map\ (GL\text{-}topology :: (real^{\frown}'n^{\frown}'n)\ topology)\ GL\text{-}topology\ matrix\text{-}inv$
⟨*proof*⟩

4.2.3 The general linear group is topological

lemma
  GL-group: *group GL* **and**
  GL-carrier [*simp*]: $carrier\ GL = \{A.\ invertible\ A\}$ **and**
  GL-inv [*simp*]: $A \in carrier\ GL \Longrightarrow inv_{GL}\ A = matrix\text{-}inv\ A$
⟨*proof*⟩

lemma
  GL-topological-group: *topological-group GL GL-topology* **and**
  GL-open: $openin\ (euclidean :: (real^{\frown}'n^{\frown}'n)\ topology)\ (carrier\ GL)$
⟨*proof*⟩

4.3 Subgroups of the general linear group

definition $SL :: (('a :: field)^{\frown}'n^{\frown}'n)\ monoid$ **where**
$SL = GL\ (\!|carrier := \{A.\ det\ A = 1\}|\!)$

lemma *det-homomorphism*: *group-hom GL unit-group det*
⟨*proof*⟩

lemma
  SL-kernel-det: $carrier\ (SL :: (('a :: field)^{\frown}'n^{\frown}'n)\ monoid) = kernel\ GL\ unit\text{-}group\ det$ **and**
  SL-subgroup: $subgroup\ (carrier\ SL)\ (GL :: ('a^{\frown}'n^{\frown}'n)\ monoid)$ **and**
  SL-carrier [*simp*]: $carrier\ SL = \{A.\ det\ A = 1\}$
⟨*proof*⟩

lemma
SL-topological-group: *topological-group SL* (*subtopology GL-topology* (*carrier SL*))
and
SL-closed: *closedin GL-topology* (*carrier SL*)
⟨*proof*⟩

definition *GO* :: (*real*^'*n*^'*n*) *monoid* **where**
GO = *GL* (|*carrier* := {*A*. *orthogonal-matrix A*}|)

lemma
GO-subgroup: *subgroup* {*A* :: *real*^'*n*^'*n*. *orthogonal-matrix A*} *GL* **and**
GO-carrier [*simp*]: *carrier GO* = {*A*. *orthogonal-matrix A*}
⟨*proof*⟩

lemma
GO-topological-group: *topological-group GO* (*subtopology GL-topology* (*carrier GO*))
and
GO-closed: *closedin* (*GL-topology* :: (*real*^'*n*^'*n*) *topology*) (*carrier GO*)
⟨*proof*⟩

definition *SO* :: (*real*^'*n*^'*n*) *monoid* **where**
SO = *GL* (|*carrier* := {*A*. *orthogonal-matrix A* ∧ *det A* = *1*}|)

lemma
SO-carrier [*simp*]: *carrier SO* = {*A*. *orthogonal-matrix A* ∧ *det A* = *1*} **and**
SO-subgroup: *subgroup* {*A* :: *real*^'*n*^'*n*. *orthogonal-matrix A* ∧ *det A* = *1*} *GL*
⟨*proof*⟩

lemma
SO-topological-group: *topological-group SO* (*subtopology GL-topology* (*carrier SO*))
and
SO-closed: *closedin GL-topology* (*carrier SO*)
⟨*proof*⟩

end

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