

# The independence of Tarski's Euclidean axiom

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## Abstract

Tarski's axioms of plane geometry are formalized and, using the standard real Cartesian model, shown to be consistent. A substantial theory of the projective plane is developed. Building on this theory, the Klein–Beltrami model of the hyperbolic plane is defined and shown to satisfy all of Tarski's axioms except his Euclidean axiom; thus Tarski's Euclidean axiom is shown to be independent of his other axioms of plane geometry.

An earlier version of this work was the subject of the author's MSc thesis [2], which contains natural-language explanations of some of the more interesting proofs.

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## 1 Metric and semimetric spaces

```

theory Metric
imports HOL-Analysis.Analysis
begin

locale semimetric =
  fixes dist :: 'p ⇒ 'p ⇒ real
  assumes nonneg [simp]: dist x y ≥ 0
  and eq-0 [simp]: dist x y = 0 ⟷ x = y
  and symm: dist x y = dist y x
begin

```

**lemma** *refl [simp]: dist x x = 0*  
 ⟨proof⟩  
**end**

**locale** *metric =*  
**fixes** *dist :: 'p ⇒ 'p ⇒ real*  
**assumes** *[simp]: dist x y = 0 ⟷ x = y*  
**and** *triangle: dist x z ≤ dist y x + dist y z*

**sublocale** *metric < semimetric*  
 ⟨proof⟩

**definition** *norm-dist :: ('a::real-normed-vector) ⇒ 'a ⇒ real* **where**  
*[simp]: norm-dist x y ≜ norm (x - y)*

**interpretation** *norm-metric: metric norm-dist*  
 ⟨proof⟩

**end**

## 2 Miscellaneous results

**theory** *Miscellany*  
**imports** *Metric*  
**begin**

**lemma** *unordered-pair-element-equality:*  
**assumes**  $\{p, q\} = \{r, s\}$  **and**  $p = r$   
**shows**  $q = s$   
 ⟨proof⟩

**lemma** *unordered-pair-equality:*  $\{p, q\} = \{q, p\}$   
 ⟨proof⟩

**lemma** *cosine-rule:*  
**fixes**  $a\ b\ c :: \text{real} \wedge ('n::\text{finite})$   
**shows**  $(\text{norm-dist } a\ c)^2 =$   
 $(\text{norm-dist } a\ b)^2 + (\text{norm-dist } b\ c)^2 + 2 * ((a - b) \cdot (b - c))$   
 ⟨proof⟩

**lemma** *scalar-equiv:*  $r * s\ x = r *_R\ x$   
 ⟨proof⟩

**lemma** *norm-dist-dot:*  $(\text{norm-dist } x\ y)^2 = (x - y) \cdot (x - y)$   
 ⟨proof⟩

**definition** *dep2 :: 'a::real-vector ⇒ 'a ⇒ bool* **where**  
 $\text{dep2 } u\ v \triangleq \exists w\ r\ s. u = r *_R\ w \wedge v = s *_R\ w$

**lemma** *real2-eq*:

**fixes**  $u\ v :: \text{real}^2$

**assumes**  $u\$1 = v\$1$  **and**  $u\$2 = v\$2$

**shows**  $u = v$

*<proof>*

**definition** *rotate2*  $:: \text{real}^2 \Rightarrow \text{real}^2$  **where**

*rotate2*  $x \triangleq \text{vector} [-x\$2, x\$1]$

**declare** *vector-2* [*simp*]

**lemma** *rotate2* [*simp*]:

$(\text{rotate2 } x)\$1 = -x\$2$

$(\text{rotate2 } x)\$2 = x\$1$

*<proof>*

**lemma** *rotate2-rotate2* [*simp*]:  $\text{rotate2 } (\text{rotate2 } x) = -x$

*<proof>*

**lemma** *rotate2-dot* [*simp*]:  $(\text{rotate2 } u) \cdot (\text{rotate2 } v) = u \cdot v$

*<proof>*

**lemma** *rotate2-scaleR* [*simp*]:  $\text{rotate2 } (k *_R x) = k *_R (\text{rotate2 } x)$

*<proof>*

**lemma** *rotate2-uminus* [*simp*]:  $\text{rotate2 } (-x) = -(\text{rotate2 } x)$

*<proof>*

**lemma** *rotate2-eq* [*iff*]:  $\text{rotate2 } x = \text{rotate2 } y \longleftrightarrow x = y$

*<proof>*

**lemma** *dot2-rearrange-1*:

**fixes**  $u\ x :: \text{real}^2$

**assumes**  $u \cdot x = 0$  **and**  $x\$1 \neq 0$

**shows**  $u = (u\$2 / x\$1) *_R (\text{rotate2 } x)$  (**is**  $u = ?u'$ )

*<proof>*

**lemma** *dot2-rearrange-2*:

**fixes**  $u\ x :: \text{real}^2$

**assumes**  $u \cdot x = 0$  **and**  $x\$2 \neq 0$

**shows**  $u = -(u\$1 / x\$2) *_R (\text{rotate2 } x)$  (**is**  $u = ?u'$ )

*<proof>*

**lemma** *dot2-rearrange*:

**fixes**  $u\ x :: \text{real}^2$

**assumes**  $u \cdot x = 0$  **and**  $x \neq 0$

**shows**  $\exists k. u = k *_R (\text{rotate2 } x)$

*<proof>*

**lemma** *real2-orthogonal-dep2*:  
**fixes**  $u\ v\ x :: \text{real}^2$   
**assumes**  $x \neq 0$  **and**  $u \cdot x = 0$  **and**  $v \cdot x = 0$   
**shows**  $\text{dep2}\ u\ v$   
 $\langle \text{proof} \rangle$

**lemma** *dot-left-diff-distrib*:  
**fixes**  $u\ v\ x :: \text{real}^n$   
**shows**  $(u - v) \cdot x = (u \cdot x) - (v \cdot x)$   
 $\langle \text{proof} \rangle$

**lemma** *dot-right-diff-distrib*:  
**fixes**  $u\ v\ x :: \text{real}^n$   
**shows**  $x \cdot (u - v) = (x \cdot u) - (x \cdot v)$   
 $\langle \text{proof} \rangle$

**lemma** *am-gm2*:  
**fixes**  $a\ b :: \text{real}$   
**assumes**  $a \geq 0$  **and**  $b \geq 0$   
**shows**  $\text{sqrt}\ (a * b) \leq (a + b) / 2$   
**and**  $\text{sqrt}\ (a * b) = (a + b) / 2 \iff a = b$   
 $\langle \text{proof} \rangle$

**lemma** *refl-on-allrel*:  $\text{refl-on}\ A\ (A \times A)$   
 $\langle \text{proof} \rangle$

**lemma** *refl-on-restrict*:  
**assumes**  $\text{refl-on}\ A\ r$   
**shows**  $\text{refl-on}\ (A \cap B)\ (r \cap B \times B)$   
 $\langle \text{proof} \rangle$

**lemma** *sym-allrel*:  $\text{sym}\ (A \times A)$   
 $\langle \text{proof} \rangle$

**lemma** *sym-restrict*:  
**assumes**  $\text{sym}\ r$   
**shows**  $\text{sym}\ (r \cap A \times A)$   
 $\langle \text{proof} \rangle$

**lemma** *trans-allrel*:  $\text{trans}\ (A \times A)$   
 $\langle \text{proof} \rangle$

**lemma** *equiv-Int*:  
**assumes**  $\text{equiv}\ A\ r$  **and**  $\text{equiv}\ B\ s$   
**shows**  $\text{equiv}\ (A \cap B)\ (r \cap s)$   
 $\langle \text{proof} \rangle$

**lemma** *equiv-allrel*:  $\text{equiv}\ A\ (A \times A)$   
 $\langle \text{proof} \rangle$

**lemma** *equiv-restrict*:  
**assumes** *equiv A r*  
**shows** *equiv (A ∩ B) (r ∩ B × B)*  
*<proof>*

**lemma** *invertible-times-eq-zero*:  
**fixes** *x :: real<sup>'n</sup> and A :: real<sup>'n</sup><sup>'n</sup>*  
**assumes** *invertible A and A \*v x = 0*  
**shows** *x = 0*  
*<proof>*

**lemma** *times-invertible-eq-zero*:  
**fixes** *x :: real<sup>'n</sup> and A :: real<sup>'n</sup><sup>'n</sup>*  
**assumes** *invertible A and x v\* A = 0*  
**shows** *x = 0*  
*<proof>*

**lemma** *matrix-id-invertible*:  
*invertible (mat 1 :: ('a::semiring-1)<sup>'n</sup><sup>'n</sup>)*  
*<proof>*

**lemma** *Image-refl-on-nonempty*:  
**assumes** *refl-on A r and x ∈ A*  
**shows** *x ∈ r<sup>''</sup>{x}*  
*<proof>*

**lemma** *quotient-element-nonempty*:  
**assumes** *equiv A r and X ∈ A//r*  
**shows**  $\exists x. x \in X$   
*<proof>*

**lemma** *zero-3*:  $(3::3) = 0$   
*<proof>*

**lemma** *card-suc-ge-insert*:  
**fixes** *A and x*  
**shows**  $\text{card } A + 1 \geq \text{card } (\text{insert } x \ A)$   
*<proof>*

**lemma** *card-le-UNIV*:  
**fixes** *A :: ('n::finite) set*  
**shows**  $\text{card } A \leq \text{CARD}('n)$   
*<proof>*

**lemma** *partition-Image-element*:  
**assumes** *equiv A r and X ∈ A//r and x ∈ X*  
**shows**  $r<sup>''</sup>\{x\} = X$   
*<proof>*

**lemma** *card-insert-ge*:  $\text{card} (\text{insert } x \ A) \geq \text{card } A$   
*<proof>*

**lemma** *choose-1*:  
 **assumes**  $\text{card } S = 1$   
 **shows**  $\exists x. S = \{x\}$   
*<proof>*

**lemma** *choose-2*:  
 **assumes**  $\text{card } S = 2$   
 **shows**  $\exists x \ y. S = \{x,y\}$   
*<proof>*

**lemma** *choose-3*:  
 **assumes**  $\text{card } S = 3$   
 **shows**  $\exists x \ y \ z. S = \{x,y,z\}$   
*<proof>*

**lemma** *card-gt-0-diff-singleton*:  
 **assumes**  $\text{card } S > 0$  **and**  $x \in S$   
 **shows**  $\text{card} (S - \{x\}) = \text{card } S - 1$   
*<proof>*

**lemma** *eq-3-or-of-3*:  
 **fixes**  $j :: 4$   
 **shows**  $j = 3 \vee (\exists j'::3. j = \text{of-int } (\text{Rep-bit1 } j'))$   
*<proof>*

**lemma** *sgn-plus*:  
 **fixes**  $x \ y :: 'a::\text{linordered-idom}$   
 **assumes**  $\text{sgn } x = \text{sgn } y$   
 **shows**  $\text{sgn } (x + y) = \text{sgn } x$   
*<proof>*

**lemma** *sgn-div*:  
 **fixes**  $x \ y :: 'a::\text{linordered-field}$   
 **assumes**  $y \neq 0$  **and**  $\text{sgn } x = \text{sgn } y$   
 **shows**  $x / y > 0$   
*<proof>*

**lemma** *abs-plus*:  
 **fixes**  $x \ y :: 'a::\text{linordered-idom}$   
 **assumes**  $\text{sgn } x = \text{sgn } y$   
 **shows**  $|x + y| = |x| + |y|$   
*<proof>*

**lemma** *sgn-plus-abs*:  
 **fixes**  $x \ y :: 'a::\text{linordered-idom}$

**assumes**  $|x| > |y|$   
**shows**  $\text{sgn}(x + y) = \text{sgn } x$   
 ⟨*proof*⟩

**end**

### 3 Tarski's geometry

**theory** *Tarski*  
**imports** *Complex-Main Miscellany Metric*  
**begin**

#### 3.1 The axioms

The axioms, and all theorems beginning with *th* followed by a number, are based on corresponding axioms and theorems in [3].

**locale** *tarski-first3* =  
**fixes**  $C :: 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow \text{bool}$  ( $- \equiv -$  [99,99,99,99] 50)  
**assumes**  $A1: \forall a b. a b \equiv b a$   
**and**  $A2: \forall a b p q r s. a b \equiv p q \wedge a b \equiv r s \longrightarrow p q \equiv r s$   
**and**  $A3: \forall a b c. a b \equiv c c \longrightarrow a = b$

**locale** *tarski-first5* = *tarski-first3* +  
**fixes**  $B :: 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow \text{bool}$   
**assumes**  $A4: \forall q a b c. \exists x. B q a x \wedge a x \equiv b c$   
**and**  $A5: \forall a b c d a' b' c' d'. a \neq b \wedge B a b c \wedge B a' b' c'$   
 $\wedge a b \equiv a' b' \wedge b c \equiv b' c' \wedge a d \equiv a' d' \wedge$   
 $b d \equiv b' d' \longrightarrow c d \equiv c' d'$

**locale** *tarski-absolute-space* = *tarski-first5* +  
**assumes**  $A6: \forall a b. B a b a \longrightarrow a = b$   
**and**  $A7: \forall a b c p q. B a p c \wedge B b q c \longrightarrow (\exists x. B p x b \wedge B q x a)$   
**and**  $A11: \forall X Y. (\exists a. \forall x y. x \in X \wedge y \in Y \longrightarrow B a x y)$   
 $\longrightarrow (\exists b. \forall x y. x \in X \wedge y \in Y \longrightarrow B x b y)$

**locale** *tarski-absolute* = *tarski-absolute-space* +  
**assumes**  $A8: \exists a b c. \neg B a b c \wedge \neg B b c a \wedge \neg B c a b$   
**and**  $A9: \forall p q a b c. p \neq q \wedge a p \equiv a q \wedge b p \equiv b q \wedge c p \equiv c q$   
 $\longrightarrow B a b c \vee B b c a \vee B c a b$

**locale** *tarski-space* = *tarski-absolute-space* +  
**assumes**  $A10: \forall a b c d t. B a d t \wedge B b d c \wedge a \neq d$   
 $\longrightarrow (\exists x y. B a b x \wedge B a c y \wedge B x t y)$

**locale** *tarski* = *tarski-absolute* + *tarski-space*



### 3.2 Semimetric spaces satisfy the first three axioms

**context** *semimetric*

**begin**

**definition** *smC* :: '*p* ⇒ '*p* ⇒ '*p* ⇒ '*p* ⇒ *bool* (- - ≡<sub>sm</sub> - - [99,99,99,99] 50)

**where** [*smp*]: *a b* ≡<sub>sm</sub> *c d* ≜ *dist a b* = *dist c d*

**end**

**sublocale** *semimetric* < *tarski-first3 smC*

⟨*proof*⟩

### 3.3 Some consequences of the first three axioms

**context** *tarski-first3*

**begin**

**lemma** *A1'*: *a b* ≡ *b a*

⟨*proof*⟩

**lemma** *A2'*:  $\llbracket a b \equiv p q; a b \equiv r s \rrbracket \implies p q \equiv r s$

⟨*proof*⟩

**lemma** *A3'*: *a b* ≡ *c c* ⇒ *a* = *b*

⟨*proof*⟩

**theorem** *th2-1*: *a b* ≡ *a b*

⟨*proof*⟩

**theorem** *th2-2*: *a b* ≡ *c d* ⇒ *c d* ≡ *a b*

⟨*proof*⟩

**theorem** *th2-3*:  $\llbracket a b \equiv c d; c d \equiv e f \rrbracket \implies a b \equiv e f$

⟨*proof*⟩

**theorem** *th2-4*: *a b* ≡ *c d* ⇒ *b a* ≡ *c d*

⟨*proof*⟩

**theorem** *th2-5*: *a b* ≡ *c d* ⇒ *a b* ≡ *d c*

⟨*proof*⟩

**definition** *is-segment* :: '*p set* ⇒ *bool* **where**

*is-segment X* ≜ ∃ *x y*. *X* = {*x*, *y*}

**definition** *segments* :: '*p set set* **where**

*segments* = {*X*. *is-segment X*}

**definition** *SC* :: '*p set* ⇒ '*p set* ⇒ *bool* **where**

*SC X Y* ≜ ∃ *w x y z*. *X* = {*w*, *x*} ∧ *Y* = {*y*, *z*} ∧ *w x* ≡ *y z*

**definition** *SC-rel* :: ('*p set* × '*p set*) *set* **where**

*SC-rel* = {(*X*, *Y*) | *X Y*. *SC X Y*}

**lemma** *left-segment-congruence*:

**assumes**  $\{a, b\} = \{p, q\}$  **and**  $p \equiv c$   $q \equiv d$   
**shows**  $a \equiv c$   $b \equiv d$

*<proof>*

**lemma** *right-segment-congruence*:

**assumes**  $\{c, d\} = \{p, q\}$  **and**  $a \equiv p$   $b \equiv q$   
**shows**  $a \equiv c$   $b \equiv d$

*<proof>*

**lemma** *C-SC-equiv*:  $a \equiv c$   $b \equiv d = SC \{a, b\} \{c, d\}$

*<proof>*

**lemmas** *SC-refl* = *th2-1* [*simplified*]

**lemma** *SC-rel-refl*: *refl-on segments SC-rel*

*<proof>*

**lemma** *SC-sym*:

**assumes**  $SC \ X \ Y$   
**shows**  $SC \ Y \ X$

*<proof>*

**lemma** *SC-sym'*:  $SC \ X \ Y = SC \ Y \ X$

*<proof>*

**lemma** *SC-rel-sym*: *sym SC-rel*

*<proof>*

**lemma** *SC-trans*:

**assumes**  $SC \ X \ Y$  **and**  $SC \ Y \ Z$   
**shows**  $SC \ X \ Z$

*<proof>*

**lemma** *SC-rel-trans*: *trans SC-rel*

*<proof>*

**lemma** *A3-reversed*:

**assumes**  $a \equiv b$   $b \equiv c$   
**shows**  $b = c$

*<proof>*

**lemma** *equiv-segments-SC-rel*: *equiv segments SC-rel*

*<proof>*

**end**

### 3.4 Some consequences of the first five axioms

context *tarski-first5*

begin

lemma *A4'*:  $\exists x. B\ q\ a\ x \wedge a\ x \equiv b\ c$   
*<proof>*

theorem *th2-8*:  $a\ a \equiv b\ b$   
*<proof>*

definition *OFS* ::  $[p, p, p, p, p, p, p, p] \Rightarrow bool$  where  
 $OFS\ a\ b\ c\ d\ a'\ b' c' d' \triangleq$   
 $B\ a\ b\ c \wedge B\ a' b' c' \wedge a\ b \equiv a' b' \wedge b\ c \equiv b' c' \wedge a\ d \equiv a' d' \wedge b\ d \equiv b' d'$

lemma *A5'*:  $\llbracket OFS\ a\ b\ c\ d\ a' b' c' d'; a \neq b \rrbracket \Longrightarrow c\ d \equiv c' d'$   
*<proof>*

theorem *th2-11*:

assumes *hypotheses*:

$B\ a\ b\ c$

$B\ a' b' c'$

$a\ b \equiv a' b'$

$b\ c \equiv b' c'$

shows  $a\ c \equiv a' c'$   
*<proof>*

lemma *A4-unique*:

assumes  $q \neq a$  and  $B\ q\ a\ x$  and  $a\ x \equiv b\ c$

and  $B\ q\ a\ x'$  and  $a\ x' \equiv b\ c$

shows  $x = x'$

*<proof>*

theorem *th2-12*:

assumes  $q \neq a$

shows  $\exists!x. B\ q\ a\ x \wedge a\ x \equiv b\ c$

*<proof>*

end

### 3.5 Simple theorems about betweenness

theorem (in *tarski-first5*) *th3-1*:  $B\ a\ b\ b$   
*<proof>*

context *tarski-absolute-space*

begin

lemma *A6'*:  
assumes  $B\ a\ b\ a$   
shows  $a = b$   
*<proof>*

**lemma A7':**  
assumes  $B a p c$  and  $B b q c$   
shows  $\exists x. B p x b \wedge B q x a$   
(proof)

**lemma A11':**  
assumes  $\forall x y. x \in X \wedge y \in Y \longrightarrow B a x y$   
shows  $\exists b. \forall x y. x \in X \wedge y \in Y \longrightarrow B x b y$   
(proof)

**theorem th3-2:**  
assumes  $B a b c$   
shows  $B c b a$   
(proof)

**theorem th3-4:**  
assumes  $B a b c$  and  $B b a c$   
shows  $a = b$   
(proof)

**theorem th3-5-1:**  
assumes  $B a b d$  and  $B b c d$   
shows  $B a b c$   
(proof)

**theorem th3-6-1:**  
assumes  $B a b c$  and  $B a c d$   
shows  $B b c d$   
(proof)

**theorem th3-7-1:**  
assumes  $b \neq c$  and  $B a b c$  and  $B b c d$   
shows  $B a c d$   
(proof)

**theorem th3-7-2:**  
assumes  $b \neq c$  and  $B a b c$  and  $B b c d$   
shows  $B a b d$   
(proof)

end

### 3.6 Simple theorems about congruence and betweenness

**definition (in tarski-first5)**  $Col :: 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$  where  
 $Col a b c \triangleq B a b c \vee B b c a \vee B c a b$

end

## 4 Real Euclidean space and Tarski's axioms

```
theory Euclid-Tarski
imports Tarski
begin
```

### 4.1 Real Euclidean space satisfies the first five axioms

**abbreviation**

```
real-euclid-C :: [real^('n::finite), real^('n), real^('n), real^('n)] ⇒ bool
(- - ≡ℝ - - [99,99,99,99] 50) where
  real-euclid-C ≜ norm-metric.smC
```

**definition** *real-euclid-B* :: [real^('n::finite), real^('n), real^('n)] ⇒ bool

```
(Bℝ - - - [99,99,99] 50) where
  Bℝ a b c ≜ ∃ l. 0 ≤ l ∧ l ≤ 1 ∧ b - a = l *R (c - a)
```

**interpretation** *real-euclid*: tarski-first5 *real-euclid-C* *real-euclid-B*  
{proof}

### 4.2 Real Euclidean space also satisfies axioms 6, 7, and 11

**lemma** *rearrange-real-euclid-B*:

```
fixes w y z :: real^('n) and h
shows y - w = h *R (z - w) ⟷ y = h *R z + (1 - h) *R w
{proof}
```

**interpretation** *real-euclid*: tarski-absolute-space *real-euclid-C* *real-euclid-B*  
{proof}

### 4.3 Real Euclidean space satisfies the Euclidean axiom

**lemma** *rearrange-real-euclid-B-2*:

```
fixes a b c :: real^('n::finite)
assumes l ≠ 0
shows b - a = l *R (c - a) ⟷ c = (1/l) *R b + (1 - 1/l) *R a
{proof}
```

**interpretation** *real-euclid*: tarski-space *real-euclid-C* *real-euclid-B*  
{proof}

### 4.4 The real Euclidean plane

**lemma** *Col-dep2*:

```
real-euclid.Col a b c ⟷ dep2 (b - a) (c - a)
{proof}
```

**lemma** *non-Col-example*:

```
¬(real-euclid.Col 0 (vector [1/2,0] :: real^2) (vector [0,1/2]))
(is ¬ (real-euclid.Col ?a ?b ?c))
```

*<proof>*

**interpretation** *real-euclid*:

*tarski real-euclid-C::([real^2, real^2, real^2, real^2] ⇒ bool) real-euclid-B*  
*<proof>*

## 4.5 Special cases of theorems of Tarski's geometry

**lemma** *real-euclid-B-disjunction*:

**assumes**  $l \geq 0$  **and**  $b - a = l *_{\mathbb{R}} (c - a)$

**shows**  $B_{\mathbb{R}} a b c \vee B_{\mathbb{R}} a c b$

*<proof>*

The following are true in Tarski's geometry, but to prove this would require much more development of it, so only the Euclidean case is proven here.

**theorem** *real-euclid-th5-1*:

**assumes**  $a \neq b$  **and**  $B_{\mathbb{R}} a b c$  **and**  $B_{\mathbb{R}} a b d$

**shows**  $B_{\mathbb{R}} a c d \vee B_{\mathbb{R}} a d c$

*<proof>*

**theorem** *real-euclid-th5-3*:

**assumes**  $B_{\mathbb{R}} a b d$  **and**  $B_{\mathbb{R}} a c d$

**shows**  $B_{\mathbb{R}} a b c \vee B_{\mathbb{R}} a c b$

*<proof>*

**end**

## 5 Linear algebra

**theory** *Linear-Algebra2*

**imports** *Miscellany*

**begin**

**lemma** *exhaust-4*:

**fixes**  $x :: 4$

**shows**  $x = 1 \vee x = 2 \vee x = 3 \vee x = 4$

*<proof>*

**lemma** *forall-4*:  $(\forall i::4. P i) \longleftrightarrow P 1 \wedge P 2 \wedge P 3 \wedge P 4$

*<proof>*

**lemma** *UNIV-4*:  $(UNIV::(4 \text{ set})) = \{1, 2, 3, 4\}$

*<proof>*

**lemma** *vector-4*:

**fixes**  $w :: 'a::zero$

**shows**  $(\text{vector } [w, x, y, z] :: 'a^4)\$1 = w$

**and**  $(\text{vector } [w, x, y, z] :: 'a^4)\$2 = x$

**and** (vector  $[w, x, y, z] :: 'a^4$ )\$3 = y  
**and** (vector  $[w, x, y, z] :: 'a^4$ )\$4 = z  
 <proof>

**definition**

*is-basis* :: (real<sup>'n</sup>) set ⇒ bool **where**  
*is-basis* S ≜ independent S ∧ span S = UNIV

**lemma** *card-finite*:

**assumes** card S = CARD('n::finite)  
**shows** finite S  
 <proof>

**lemma** *independent-is-basis*:

**fixes** B :: (real<sup>'n</sup>) set  
**shows** independent B ∧ card B = CARD('n) ⟷ *is-basis* B  
 <proof>

**lemma** *basis-finite*:

**fixes** B :: (real<sup>'n</sup>) set  
**assumes** *is-basis* B  
**shows** finite B  
 <proof>

**lemma** *basis-expand*:

**assumes** *is-basis* B  
**shows** ∃ c. v = (∑ w∈B. (c w) \*<sub>R</sub> w)  
 <proof>

**lemma** *not-span-independent-insert*:

**fixes** v :: ('a::real-vector)<sup>'n</sup>  
**assumes** independent S **and** v ∉ span S  
**shows** independent (insert v S)  
 <proof>

**lemma** *orthogonal-sum*:

**fixes** v :: real<sup>'n</sup>  
**assumes** ∧ w. w∈S ⇒ orthogonal v w  
**shows** orthogonal v (∑ w∈S. c w \*<sub>s</sub> w)  
 <proof>

**lemma** *orthogonal-self-eq-0*:

**fixes** v :: ('a::real-inner)<sup>'n</sup>  
**assumes** orthogonal v v  
**shows** v = 0  
 <proof>

**lemma** *orthogonal-in-span-eq-0*:

**fixes** v :: real<sup>'n</sup>

**assumes**  $v \in \text{span } S$  **and**  $\bigwedge w. w \in S \implies \text{orthogonal } v w$   
**shows**  $v = 0$   
 $\langle \text{proof} \rangle$

**lemma** *orthogonal-independent*:

**fixes**  $v :: \text{real}^n$   
**assumes** *independent*  $S$  **and**  $v \neq 0$  **and**  $\bigwedge w. w \in S \implies \text{orthogonal } v w$   
**shows** *independent* (*insert*  $v S$ )  
 $\langle \text{proof} \rangle$

**lemma** *dot-scaleR-mult*:

**shows**  $(k *_R a) \cdot b = k * (a \cdot b)$  **and**  $a \cdot (k *_R b) = k * (a \cdot b)$   
 $\langle \text{proof} \rangle$

**lemma** *dependent-explicit-finite*:

**fixes**  $S :: ('a :: \{\text{real-vector, field}\})^n$  *set*  
**assumes** *finite*  $S$   
**shows** *dependent*  $S \longleftrightarrow (\exists u. (\exists v \in S. u v \neq 0) \wedge (\sum v \in S. u v *_R v) = 0)$   
 $\langle \text{proof} \rangle$

**lemma** *dependent-explicit-2*:

**fixes**  $v w :: ('a :: \{\text{field, real-vector}\})^n$   
**assumes**  $v \neq w$   
**shows** *dependent*  $\{v, w\} \longleftrightarrow (\exists i j. (i \neq 0 \vee j \neq 0) \wedge i *_R v + j *_R w = 0)$   
 $\langle \text{proof} \rangle$

## 5.1 Matrices

**lemma** *zero-not-invertible*:

$\neg (\text{invertible } (0 :: \text{real}^n{}^n))$   
 $\langle \text{proof} \rangle$

Based on matrix-vector-column in HOL/Multivariate\_Analysis/Euclidean\_Space.thy in Isabelle 2009-1:

**lemma** *vector-matrix-row*:

**fixes**  $x :: ('a :: \text{comm-semiring-1})^m$  **and**  $A :: ('a^m{}^n)$   
**shows**  $x v * A = (\sum i \in UNIV. (x \$ i) * s (A \$ i))$   
 $\langle \text{proof} \rangle$

**lemma** *matrix-inv*:

**assumes** *invertible*  $M$   
**shows** *matrix-inv*  $M ** M = \text{mat } 1$   
**and**  $M ** \text{matrix-inv } M = \text{mat } 1$   
 $\langle \text{proof} \rangle$

**lemma** *matrix-inv-invertible*:

**assumes** *invertible*  $M$   
**shows** *invertible* (*matrix-inv*  $M$ )  
 $\langle \text{proof} \rangle$



**lemma** *invertible-times-non-zero*:

**fixes**  $M :: \text{real}^{\prime n} \wedge \prime n$   
**assumes** *invertible*  $M$  **and**  $v \neq 0$   
**shows**  $M * v \neq 0$   
*<proof>*

**lemma** *matrix-right-invertible-ker*:

**fixes**  $M :: \text{real}^{\prime m :: \text{finite}} \wedge \prime n$   
**shows**  $(\exists M'. M ** M' = \text{mat } 1) \longleftrightarrow (\forall x. x v * M = 0 \longrightarrow x = 0)$   
*<proof>*

**lemma** *left-invertible-iff-invertible*:

**fixes**  $M :: \text{real}^{\prime n} \wedge \prime n$   
**shows**  $(\exists N. N ** M = \text{mat } 1) \longleftrightarrow \text{invertible } M$   
*<proof>*

**lemma** *right-invertible-iff-invertible*:

**fixes**  $M :: \text{real}^{\prime n} \wedge \prime n$   
**shows**  $(\exists N. M ** N = \text{mat } 1) \longleftrightarrow \text{invertible } M$   
*<proof>*

**definition** *symmatrix* ::  $\prime a \wedge \prime n \wedge \prime n \Rightarrow \text{bool}$  **where**

*symmatrix*  $M \triangleq \text{transpose } M = M$

**lemma** *symmatrix-preserve*:

**fixes**  $M N :: (\prime a :: \text{comm-semiring-1}) \wedge \prime n \wedge \prime n$   
**assumes** *symmatrix*  $M$   
**shows** *symmatrix*  $(N ** M ** \text{transpose } N)$   
*<proof>*

**lemma** *non-zero-mult-invertible-non-zero*:

**fixes**  $M :: \text{real}^{\prime n} \wedge \prime n$   
**assumes**  $v \neq 0$  **and** *invertible*  $M$   
**shows**  $v v * M \neq 0$   
*<proof>*

**end**

## 6 Right group actions

**theory** *Action*

**imports** *HOL-Algebra.Group*

**begin**

**locale** *action = group +*

**fixes**  $act :: \prime b \Rightarrow \prime a \Rightarrow \prime b$  (**infixl**  $< o \ 69$ )  
**assumes** *id-act* [*simp*]:  $b < o \ \mathbf{1} = b$   
**and** *act-act'*:

$g \in \text{carrier } G \wedge h \in \text{carrier } G \longrightarrow (b <_o g) <_o h = b <_o (g \otimes h)$   
**begin**

**lemma** *act-act*:

**assumes**  $g \in \text{carrier } G$  **and**  $h \in \text{carrier } G$   
**shows**  $(b <_o g) <_o h = b <_o (g \otimes h)$   
 $\langle \text{proof} \rangle$

**lemma** *act-act-inv [simp]*:

**assumes**  $g \in \text{carrier } G$   
**shows**  $b <_o g <_o \text{inv } g = b$   
 $\langle \text{proof} \rangle$

**lemma** *act-inv-act [simp]*:

**assumes**  $g \in \text{carrier } G$   
**shows**  $b <_o \text{inv } g <_o g = b$   
 $\langle \text{proof} \rangle$

**lemma** *act-inv-iff*:

**assumes**  $g \in \text{carrier } G$   
**shows**  $b <_o \text{inv } g = c \longleftrightarrow b = c <_o g$   
 $\langle \text{proof} \rangle$

**end**

**end**

## 7 Projective geometry

**theory** *Projective*

**imports** *Linear-Algebra2*  
*Euclid-Tarski*  
*Action*  
**begin**

### 7.1 Proportionality on non-zero vectors

**context** *vector-space*

**begin**

**definition** *proportionality* ::  $('b \times 'b)$  set **where**

$\text{proportionality} \triangleq \{(x, y). x \neq 0 \wedge y \neq 0 \wedge (\exists k. x = \text{scale } k \ y)\}$

**definition** *non-zero-vectors* ::  $'b$  set **where**

$\text{non-zero-vectors} \triangleq \{x. x \neq 0\}$

**lemma** *proportionality-refl-on*: *refl-on local.non-zero-vectors local.proportionality*

$\langle \text{proof} \rangle$

**lemma** *proportionality-sym*: *sym local.proportionality*  
⟨*proof*⟩

**lemma** *proportionality-trans*: *trans local.proportionality*  
⟨*proof*⟩

**theorem** *proportionality-equiv*: *equiv local.non-zero-vectors local.proportionality*  
⟨*proof*⟩

**end**

**definition** *invertible-proportionality* ::  
((*real*^(*n*::*finite*)^*n*) × (*real*^*n*^*n*)) **set** **where**  
*invertible-proportionality* ≜  
*real-vector.proportionality* ∩ (*Collect invertible* × *Collect invertible*)

**lemma** *invertible-proportionality-equiv*:  
*equiv (Collect invertible :: (real*^(*n*::*finite*)^*n*) **set**)  
*invertible-proportionality*  
(**is** *equiv ?invs -*)  
⟨*proof*⟩

## 7.2 Points of the real projective plane

**typedef** *proj2* = (*real-vector.non-zero-vectors :: (real*^3) **set**)//*real-vector.proportionality*  
⟨*proof*⟩

**definition** *proj2-rep* :: *proj2* ⇒ *real*^3 **where**  
*proj2-rep* *x* ≜ *ε v. v ∈ Rep-proj2 x*

**definition** *proj2-abs* :: *real*^3 ⇒ *proj2* **where**  
*proj2-abs* *v* ≜ *Abs-proj2 (real-vector.proportionality “ {v}*)

**lemma** *proj2-rep-in*: *proj2-rep* *x* ∈ *Rep-proj2* *x*  
⟨*proof*⟩

**lemma** *proj2-rep-non-zero*: *proj2-rep* *x* ≠ 0  
⟨*proof*⟩

**lemma** *proj2-rep-abs*:  
**fixes** *v :: real*^3  
**assumes** *v* ∈ *real-vector.non-zero-vectors*  
**shows** (*v*, *proj2-rep (proj2-abs v)*) ∈ *real-vector.proportionality*  
⟨*proof*⟩

**lemma** *proj2-abs-rep*: *proj2-abs (proj2-rep* *x*) = *x*  
⟨*proof*⟩

**lemma** *proj2-abs-mult*:

**assumes**  $c \neq 0$   
**shows**  $\text{proj2-abs } (c *_{\mathbb{R}} v) = \text{proj2-abs } v$   
 $\langle \text{proof} \rangle$

**lemma** *proj2-abs-mult-rep*:  
**assumes**  $c \neq 0$   
**shows**  $\text{proj2-abs } (c *_{\mathbb{R}} \text{proj2-rep } x) = x$   
 $\langle \text{proof} \rangle$

**lemma** *proj2-rep-inj*: *inj proj2-rep*  
 $\langle \text{proof} \rangle$

**lemma** *proj2-rep-abs2*:  
**assumes**  $v \neq 0$   
**shows**  $\exists k. k \neq 0 \wedge \text{proj2-rep } (\text{proj2-abs } v) = k *_{\mathbb{R}} v$   
 $\langle \text{proof} \rangle$

**lemma** *proj2-abs-abs-mult*:  
**assumes**  $\text{proj2-abs } v = \text{proj2-abs } w$  **and**  $w \neq 0$   
**shows**  $\exists c. v = c *_{\mathbb{R}} w$   
 $\langle \text{proof} \rangle$

**lemma** *dependent-proj2-abs*:  
**assumes**  $p \neq 0$  **and**  $q \neq 0$  **and**  $i \neq 0 \vee j \neq 0$  **and**  $i *_{\mathbb{R}} p + j *_{\mathbb{R}} q = 0$   
**shows**  $\text{proj2-abs } p = \text{proj2-abs } q$   
 $\langle \text{proof} \rangle$

**lemma** *proj2-rep-dependent*:  
**assumes**  $i *_{\mathbb{R}} \text{proj2-rep } v + j *_{\mathbb{R}} \text{proj2-rep } w = 0$   
**(is**  $i *_{\mathbb{R}} ?p + j *_{\mathbb{R}} ?q = 0$ **)**  
**and**  $i \neq 0 \vee j \neq 0$   
**shows**  $v = w$   
 $\langle \text{proof} \rangle$

**lemma** *proj2-rep-independent*:  
**assumes**  $p \neq q$   
**shows** *independent*  $\{\text{proj2-rep } p, \text{proj2-rep } q\}$   
 $\langle \text{proof} \rangle$

### 7.3 Lines of the real projective plane

**definition** *proj2-Col* ::  $[\text{proj2}, \text{proj2}, \text{proj2}] \Rightarrow \text{bool}$  **where**  
 $\text{proj2-Col } p \ q \ r \triangleq$   
 $(\exists i \ j \ k. i *_{\mathbb{R}} \text{proj2-rep } p + j *_{\mathbb{R}} \text{proj2-rep } q + k *_{\mathbb{R}} \text{proj2-rep } r = 0$   
 $\wedge (i \neq 0 \vee j \neq 0 \vee k \neq 0))$

**lemma** *proj2-Col-abs*:  
**assumes**  $p \neq 0$  **and**  $q \neq 0$  **and**  $r \neq 0$  **and**  $i \neq 0 \vee j \neq 0 \vee k \neq 0$   
**and**  $i *_{\mathbb{R}} p + j *_{\mathbb{R}} q + k *_{\mathbb{R}} r = 0$

**shows**  $\text{proj2-Col } (\text{proj2-abs } p) (\text{proj2-abs } q) (\text{proj2-abs } r)$   
**(is**  $\text{proj2-Col } ?pp ?pq ?pr)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{proj2-Col-permute}$ :  
**assumes**  $\text{proj2-Col } a b c$   
**shows**  $\text{proj2-Col } a c b$   
**and**  $\text{proj2-Col } b a c$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{proj2-Col-coincide}$ :  $\text{proj2-Col } a a c$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{proj2-Col-iff}$ :  
**assumes**  $a \neq r$   
**shows**  $\text{proj2-Col } a r t \longleftrightarrow$   
 $t = a \vee (\exists i. t = \text{proj2-abs } (i *_{\mathbb{R}} (\text{proj2-rep } a) + (\text{proj2-rep } r)))$   
 $\langle \text{proof} \rangle$

**definition**  $\text{proj2-Col-coeff} :: \text{proj2} \Rightarrow \text{proj2} \Rightarrow \text{proj2} \Rightarrow \text{real}$  **where**  
 $\text{proj2-Col-coeff } a r t \triangleq \epsilon i. t = \text{proj2-abs } (i *_{\mathbb{R}} \text{proj2-rep } a + \text{proj2-rep } r)$

**lemma**  $\text{proj2-Col-coeff}$ :  
**assumes**  $\text{proj2-Col } a r t$  **and**  $a \neq r$  **and**  $t \neq a$   
**shows**  $t = \text{proj2-abs } ((\text{proj2-Col-coeff } a r t) *_{\mathbb{R}} \text{proj2-rep } a + \text{proj2-rep } r)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{proj2-Col-coeff-unique}'$ :  
**assumes**  $a \neq 0$  **and**  $r \neq 0$  **and**  $\text{proj2-abs } a \neq \text{proj2-abs } r$   
**and**  $\text{proj2-abs } (i *_{\mathbb{R}} a + r) = \text{proj2-abs } (j *_{\mathbb{R}} a + r)$   
**shows**  $i = j$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{proj2-Col-coeff-unique}$ :  
**assumes**  $a \neq r$   
**and**  $\text{proj2-abs } (i *_{\mathbb{R}} \text{proj2-rep } a + \text{proj2-rep } r)$   
 $= \text{proj2-abs } (j *_{\mathbb{R}} \text{proj2-rep } a + \text{proj2-rep } r)$   
**shows**  $i = j$   
 $\langle \text{proof} \rangle$

**datatype**  $\text{proj2-line} = P2L \text{proj2}$

**definition**  $L2P :: \text{proj2-line} \Rightarrow \text{proj2}$  **where**  
 $L2P l \triangleq \text{case } l \text{ of } P2L p \Rightarrow p$

**lemma**  $L2P$ - $P2L$  [*simp*]:  $L2P (P2L p) = p$   
 $\langle \text{proof} \rangle$

**lemma**  $P2L$ - $L2P$  [*simp*]:  $P2L (L2P l) = l$

$\langle \text{proof} \rangle$

**lemma** *L2P-inj* [*simp*]:  
assumes  $L2P\ l = L2P\ m$   
shows  $l = m$   
 $\langle \text{proof} \rangle$

**lemma** *P2L-to-L2P*:  $P2L\ p = l \longleftrightarrow p = L2P\ l$   
 $\langle \text{proof} \rangle$

**definition** *proj2-line-abs* ::  $real^3 \Rightarrow \text{proj2-line}$  **where**  
 $\text{proj2-line-abs}\ v \triangleq P2L\ (\text{proj2-abs}\ v)$

**definition** *proj2-line-rep* ::  $\text{proj2-line} \Rightarrow real^3$  **where**  
 $\text{proj2-line-rep}\ l \triangleq \text{proj2-rep}\ (L2P\ l)$

**lemma** *proj2-line-rep-abs*:  
assumes  $v \neq 0$   
shows  $\exists k. k \neq 0 \wedge \text{proj2-line-rep}\ (\text{proj2-line-abs}\ v) = k *_{\mathbb{R}} v$   
 $\langle \text{proof} \rangle$

**lemma** *proj2-line-abs-rep* [*simp*]:  $\text{proj2-line-abs}\ (\text{proj2-line-rep}\ l) = l$   
 $\langle \text{proof} \rangle$

**lemma** *proj2-line-rep-non-zero*:  $\text{proj2-line-rep}\ l \neq 0$   
 $\langle \text{proof} \rangle$

**lemma** *proj2-line-rep-dependent*:  
assumes  $i *_{\mathbb{R}} \text{proj2-line-rep}\ l + j *_{\mathbb{R}} \text{proj2-line-rep}\ m = 0$   
and  $i \neq 0 \vee j \neq 0$   
shows  $l = m$   
 $\langle \text{proof} \rangle$

**lemma** *proj2-line-abs-mult*:  
assumes  $k \neq 0$   
shows  $\text{proj2-line-abs}\ (k *_{\mathbb{R}} v) = \text{proj2-line-abs}\ v$   
 $\langle \text{proof} \rangle$

**lemma** *proj2-line-abs-abs-mult*:  
assumes  $\text{proj2-line-abs}\ v = \text{proj2-line-abs}\ w$  and  $w \neq 0$   
shows  $\exists k. v = k *_{\mathbb{R}} w$   
 $\langle \text{proof} \rangle$

**definition** *proj2-incident* ::  $\text{proj2} \Rightarrow \text{proj2-line} \Rightarrow \text{bool}$  **where**  
 $\text{proj2-incident}\ p\ l \triangleq (\text{proj2-rep}\ p) \cdot (\text{proj2-line-rep}\ l) = 0$

**lemma** *proj2-points-define-line*:  
shows  $\exists l. \text{proj2-incident}\ p\ l \wedge \text{proj2-incident}\ q\ l$   
 $\langle \text{proof} \rangle$

**definition** *proj2-line-through* :: *proj2*  $\Rightarrow$  *proj2*  $\Rightarrow$  *proj2-line* **where**  
*proj2-line-through* *p q*  $\triangleq \epsilon$  *l*. *proj2-incident* *p l*  $\wedge$  *proj2-incident* *q l*

**lemma** *proj2-line-through-incident*:  
**shows** *proj2-incident* *p* (*proj2-line-through* *p q*)  
**and** *proj2-incident* *q* (*proj2-line-through* *p q*)  
(*proof*)

**lemma** *proj2-line-through-unique*:  
**assumes** *p*  $\neq$  *q* **and** *proj2-incident* *p l* **and** *proj2-incident* *q l*  
**shows** *l* = *proj2-line-through* *p q*  
(*proof*)

**lemma** *proj2-incident-unique*:  
**assumes** *proj2-incident* *p l*  
**and** *proj2-incident* *q l*  
**and** *proj2-incident* *p m*  
**and** *proj2-incident* *q m*  
**shows** *p* = *q*  $\vee$  *l* = *m*  
(*proof*)

**lemma** *proj2-lines-define-point*:  $\exists$  *p*. *proj2-incident* *p l*  $\wedge$  *proj2-incident* *p m*  
(*proof*)

**definition** *proj2-intersection* :: *proj2-line*  $\Rightarrow$  *proj2-line*  $\Rightarrow$  *proj2* **where**  
*proj2-intersection* *l m*  $\triangleq$  *L2P* (*proj2-line-through* (*L2P* *l*) (*L2P* *m*))

**lemma** *proj2-incident-switch*:  
**assumes** *proj2-incident* *p l*  
**shows** *proj2-incident* (*L2P* *l*) (*P2L* *p*)  
(*proof*)

**lemma** *proj2-intersection-incident*:  
**shows** *proj2-incident* (*proj2-intersection* *l m*) *l*  
**and** *proj2-incident* (*proj2-intersection* *l m*) *m*  
(*proof*)

**lemma** *proj2-intersection-unique*:  
**assumes** *l*  $\neq$  *m* **and** *proj2-incident* *p l* **and** *proj2-incident* *p m*  
**shows** *p* = *proj2-intersection* *l m*  
(*proof*)

**lemma** *proj2-not-self-incident*:  
 $\neg$  (*proj2-incident* *p* (*P2L* *p*))  
(*proof*)

**lemma** *proj2-another-point-on-line*:  
 $\exists$  *q*. *q*  $\neq$  *p*  $\wedge$  *proj2-incident* *q l*

*<proof>*

**lemma** *proj2-another-line-through-point:*

$\exists m. m \neq l \wedge \text{proj2-incident } p \ m$

*<proof>*

**lemma** *proj2-incident-abs:*

**assumes**  $v \neq 0$  **and**  $w \neq 0$

**shows**  $\text{proj2-incident } (\text{proj2-abs } v) (\text{proj2-line-abs } w) \longleftrightarrow v \cdot w = 0$

*<proof>*

**lemma** *proj2-incident-left-abs:*

**assumes**  $v \neq 0$

**shows**  $\text{proj2-incident } (\text{proj2-abs } v) \ l \longleftrightarrow v \cdot (\text{proj2-line-rep } l) = 0$

*<proof>*

**lemma** *proj2-incident-right-abs:*

**assumes**  $v \neq 0$

**shows**  $\text{proj2-incident } p (\text{proj2-line-abs } v) \longleftrightarrow (\text{proj2-rep } p) \cdot v = 0$

*<proof>*

**definition** *proj2-set-Col* ::  $\text{proj2 set} \Rightarrow \text{bool}$  **where**

$\text{proj2-set-Col } S \triangleq \exists l. \forall p \in S. \text{proj2-incident } p \ l$

**lemma** *proj2-subset-Col:*

**assumes**  $T \subseteq S$  **and**  $\text{proj2-set-Col } S$

**shows**  $\text{proj2-set-Col } T$

*<proof>*

**definition** *proj2-no-3-Col* ::  $\text{proj2 set} \Rightarrow \text{bool}$  **where**

$\text{proj2-no-3-Col } S \triangleq \text{card } S = 4 \wedge (\forall p \in S. \neg \text{proj2-set-Col } (S - \{p\}))$

**lemma** *proj2-Col-iff-not-invertible:*

$\text{proj2-Col } p \ q \ r$

$\longleftrightarrow \neg \text{invertible } (\text{vector } [\text{proj2-rep } p, \text{proj2-rep } q, \text{proj2-rep } r] :: \text{real}^3)$

(**is**  $\longleftrightarrow \neg \text{invertible } (\text{vector } [?u, ?v, ?w])$ )

*<proof>*

**lemma** *not-invertible-iff-proj2-set-Col:*

$\neg \text{invertible } (\text{vector } [\text{proj2-rep } p, \text{proj2-rep } q, \text{proj2-rep } r] :: \text{real}^3)$

$\longleftrightarrow \text{proj2-set-Col } \{p, q, r\}$

(**is**  $\neg \text{invertible } ?M \longleftrightarrow -$ )

*<proof>*

**lemma** *proj2-Col-iff-set-Col:*

$\text{proj2-Col } p \ q \ r \longleftrightarrow \text{proj2-set-Col } \{p, q, r\}$

*<proof>*

**lemma** *proj2-incident-Col:*



**assumes** *proj2-incident p l and proj2-incident q l and proj2-incident r l*  
**shows** *proj2-Col p q r*  
 ⟨*proof*⟩

**lemma** *proj2-incident-iff-Col*:  
**assumes**  $p \neq q$  **and** *proj2-incident p l and proj2-incident q l*  
**shows** *proj2-incident r l*  $\longleftrightarrow$  *proj2-Col p q r*  
 ⟨*proof*⟩

**lemma** *proj2-incident-iff*:  
**assumes**  $p \neq q$  **and** *proj2-incident p l and proj2-incident q l*  
**shows** *proj2-incident r l*  
 $\longleftrightarrow r = p \vee (\exists k. r = \text{proj2-abs } (k *_R \text{proj2-rep } p + \text{proj2-rep } q))$   
 ⟨*proof*⟩

**lemma** *not-proj2-set-Col-iff-span*:  
**assumes**  $\text{card } S = 3$   
**shows**  $\neg \text{proj2-set-Col } S \longleftrightarrow \text{span } (\text{proj2-rep } ` S) = \text{UNIV}$   
 ⟨*proof*⟩

**lemma** *proj2-no-3-Col-span*:  
**assumes** *proj2-no-3-Col S and*  $p \in S$   
**shows**  $\text{span } (\text{proj2-rep } ` (S - \{p\})) = \text{UNIV}$   
 ⟨*proof*⟩

**lemma** *fourth-proj2-no-3-Col*:  
**assumes**  $\neg \text{proj2-Col } p q r$   
**shows**  $\exists s. \text{proj2-no-3-Col } \{s, r, p, q\}$   
 ⟨*proof*⟩

**lemma** *proj2-set-Col-expand*:  
**assumes** *proj2-set-Col S and*  $\{p, q, r\} \subseteq S$  **and**  $p \neq q$  **and**  $r \neq p$   
**shows**  $\exists k. r = \text{proj2-abs } (k *_R \text{proj2-rep } p + \text{proj2-rep } q)$   
 ⟨*proof*⟩

## 7.4 Collineations of the real projective plane

**typedef** *cltn2* =  
 (*Collect invertible* ::  $(\text{real}^3)^3$  set) // *invertible-proportionality*  
 ⟨*proof*⟩

**definition** *cltn2-rep* :: *cltn2*  $\Rightarrow$   $\text{real}^3^3$  **where**  
*cltn2-rep*  $A \triangleq \epsilon B. B \in \text{Rep-cltn2 } A$

**definition** *cltn2-abs* ::  $\text{real}^3^3 \Rightarrow$  *cltn2* **where**  
*cltn2-abs*  $B \triangleq \text{Abs-cltn2 } (\text{invertible-proportionality } ` \{B\})$

**definition** *cltn2-independent* :: *cltn2 set*  $\Rightarrow$  *bool* **where**  
*cltn2-independent*  $X \triangleq \text{independent } \{\text{cltn2-rep } A \mid A. A \in X\}$

**definition** *apply-cltn2* :: *proj2*  $\Rightarrow$  *cltn2*  $\Rightarrow$  *proj2* **where**  
*apply-cltn2* *x A*  $\triangleq$  *proj2-abs* (*proj2-rep* *x v\* cltn2-rep A*)

**lemma** *cltn2-rep-in*: *cltn2-rep B*  $\in$  *Rep-cltn2 B*  
 $\langle$ *proof* $\rangle$

**lemma** *cltn2-rep-invertible*: *invertible (cltn2-rep A)*  
 $\langle$ *proof* $\rangle$

**lemma** *cltn2-rep-abs*:  
**fixes** *A* :: *real^3^3*  
**assumes** *invertible A*  
**shows** (*A*, *cltn2-rep (cltn2-abs A)*)  $\in$  *invertible-proportionality*  
 $\langle$ *proof* $\rangle$

**lemma** *cltn2-rep-abs2*:  
**assumes** *invertible A*  
**shows**  $\exists k. k \neq 0 \wedge \text{cltn2-rep (cltn2-abs A)} = k *_R A$   
 $\langle$ *proof* $\rangle$

**lemma** *cltn2-abs-rep*: *cltn2-abs (cltn2-rep A)* = *A*  
 $\langle$ *proof* $\rangle$

**lemma** *cltn2-abs-mult*:  
**assumes** *k*  $\neq$  0 **and** *invertible A*  
**shows** *cltn2-abs (k \*\_R A)* = *cltn2-abs A*  
 $\langle$ *proof* $\rangle$

**lemma** *cltn2-abs-mult-rep*:  
**assumes** *k*  $\neq$  0  
**shows** *cltn2-abs (k \*\_R cltn2-rep A)* = *A*  
 $\langle$ *proof* $\rangle$

**lemma** *apply-cltn2-abs*:  
**assumes** *x*  $\neq$  0 **and** *invertible A*  
**shows** *apply-cltn2 (proj2-abs x) (cltn2-abs A)* = *proj2-abs (x v\* A)*  
 $\langle$ *proof* $\rangle$

**lemma** *apply-cltn2-left-abs*:  
**assumes** *v*  $\neq$  0  
**shows** *apply-cltn2 (proj2-abs v) C* = *proj2-abs (v v\* cltn2-rep C)*  
 $\langle$ *proof* $\rangle$

**lemma** *apply-cltn2-right-abs*:  
**assumes** *invertible M*  
**shows** *apply-cltn2 p (cltn2-abs M)* = *proj2-abs (proj2-rep p v\* M)*  
 $\langle$ *proof* $\rangle$

**lemma** *non-zero-mult-rep-non-zero*:

**assumes**  $v \neq 0$

**shows**  $v v^* \text{cltn2-rep } C \neq 0$

*<proof>*

**lemma** *rep-mult-rep-non-zero*:  $\text{proj2-rep } p v^* \text{cltn2-rep } A \neq 0$

*<proof>*

**definition** *cltn2-image* ::  $\text{proj2 set} \Rightarrow \text{cltn2} \Rightarrow \text{proj2 set}$  **where**

$\text{cltn2-image } P A \triangleq \{\text{apply-cltn2 } p A \mid p. p \in P\}$

### 7.4.1 As a group

**definition** *cltn2-id* ::  $\text{cltn2}$  **where**

$\text{cltn2-id} \triangleq \text{cltn2-abs } (\text{mat } 1)$

**definition** *cltn2-compose* ::  $\text{cltn2} \Rightarrow \text{cltn2} \Rightarrow \text{cltn2}$  **where**

$\text{cltn2-compose } A B \triangleq \text{cltn2-abs } (\text{cltn2-rep } A ** \text{cltn2-rep } B)$

**definition** *cltn2-inverse* ::  $\text{cltn2} \Rightarrow \text{cltn2}$  **where**

$\text{cltn2-inverse } A \triangleq \text{cltn2-abs } (\text{matrix-inv } (\text{cltn2-rep } A))$

**lemma** *cltn2-compose-abs*:

**assumes** *invertible*  $M$  **and** *invertible*  $N$

**shows**  $\text{cltn2-compose } (\text{cltn2-abs } M) (\text{cltn2-abs } N) = \text{cltn2-abs } (M ** N)$

*<proof>*

**lemma** *cltn2-compose-left-abs*:

**assumes** *invertible*  $M$

**shows**  $\text{cltn2-compose } (\text{cltn2-abs } M) A = \text{cltn2-abs } (M ** \text{cltn2-rep } A)$

*<proof>*

**lemma** *cltn2-compose-right-abs*:

**assumes** *invertible*  $M$

**shows**  $\text{cltn2-compose } A (\text{cltn2-abs } M) = \text{cltn2-abs } (\text{cltn2-rep } A ** M)$

*<proof>*

**lemma** *cltn2-abs-rep-abs-mult*:

**assumes** *invertible*  $M$  **and** *invertible*  $N$

**shows**  $\text{cltn2-abs } (\text{cltn2-rep } (\text{cltn2-abs } M) ** N) = \text{cltn2-abs } (M ** N)$

*<proof>*

**lemma** *cltn2-assoc*:

$\text{cltn2-compose } (\text{cltn2-compose } A B) C = \text{cltn2-compose } A (\text{cltn2-compose } B C)$

*<proof>*

**lemma** *cltn2-left-id*:  $\text{cltn2-compose } \text{cltn2-id } A = A$

*<proof>*

**lemma** *cltn2-left-inverse*:  $\text{cltn2-compose } (\text{cltn2-inverse } A) A = \text{cltn2-id}$   
 ⟨proof⟩

**lemma** *cltn2-left-inverse-ex*:  
 $\exists B. \text{cltn2-compose } B A = \text{cltn2-id}$   
 ⟨proof⟩

**interpretation** *cltn2*:  
 $\text{group } (|\text{carrier} = \text{UNIV}, \text{mult} = \text{cltn2-compose}, \text{one} = \text{cltn2-id}|)$   
 ⟨proof⟩

**lemma** *cltn2-inverse-inv* [simp]:  
 $\text{inv } (|\text{carrier} = \text{UNIV}, \text{mult} = \text{cltn2-compose}, \text{one} = \text{cltn2-id}|) A$   
 $= \text{cltn2-inverse } A$   
 ⟨proof⟩

**lemmas** *cltn2-inverse-id* [simp] = *cltn2.inv-one* [simplified]  
 and *cltn2-inverse-compose* = *cltn2.inv-mult-group* [simplified]

## 7.4.2 As a group action

**lemma** *apply-cltn2-id* [simp]:  $\text{apply-cltn2 } p \text{ cltn2-id} = p$   
 ⟨proof⟩

**lemma** *apply-cltn2-compose*:  
 $\text{apply-cltn2 } (\text{apply-cltn2 } p A) B = \text{apply-cltn2 } p (\text{cltn2-compose } A B)$   
 ⟨proof⟩

**interpretation** *cltn2*:  
 $\text{action } (|\text{carrier} = \text{UNIV}, \text{mult} = \text{cltn2-compose}, \text{one} = \text{cltn2-id}|) \text{ apply-cltn2}$   
 ⟨proof⟩

**definition** *cltn2-transpose* ::  $\text{cltn2} \Rightarrow \text{cltn2}$  **where**  
 $\text{cltn2-transpose } A \triangleq \text{cltn2-abs } (\text{transpose } (\text{cltn2-rep } A))$

**definition** *apply-cltn2-line* ::  $\text{proj2-line} \Rightarrow \text{cltn2} \Rightarrow \text{proj2-line}$  **where**  
 $\text{apply-cltn2-line } l A$   
 $\triangleq \text{P2L } (\text{apply-cltn2 } (\text{L2P } l) (\text{cltn2-transpose } (\text{cltn2-inverse } A)))$

**lemma** *cltn2-transpose-abs*:  
**assumes** *invertible*  $M$   
**shows**  $\text{cltn2-transpose } (\text{cltn2-abs } M) = \text{cltn2-abs } (\text{transpose } M)$   
 ⟨proof⟩

**lemma** *cltn2-transpose-compose*:  
 $\text{cltn2-transpose } (\text{cltn2-compose } A B)$   
 $= \text{cltn2-compose } (\text{cltn2-transpose } B) (\text{cltn2-transpose } A)$   
 ⟨proof⟩

**lemma** *cltn2-transpose-transpose*:  $\text{cltn2-transpose} (\text{cltn2-transpose } A) = A$   
(proof)

**lemma** *cltn2-transpose-id* [simp]:  $\text{cltn2-transpose } \text{cltn2-id} = \text{cltn2-id}$   
(proof)

**lemma** *apply-cltn2-line-id* [simp]:  $\text{apply-cltn2-line } l \text{ cltn2-id} = l$   
(proof)

**lemma** *apply-cltn2-line-compose*:  
 $\text{apply-cltn2-line} (\text{apply-cltn2-line } l A) B$   
 $= \text{apply-cltn2-line } l (\text{cltn2-compose } A B)$   
(proof)

**interpretation** *cltn2-line*:  
*action*  
(|carrier = UNIV, mult = cltn2-compose, one = cltn2-id|)  
*apply-cltn2-line*  
(proof)

**lemmas** *apply-cltn2-inv* [simp] = *cltn2.act-act-inv* [simplified]  
**lemmas** *apply-cltn2-line-inv* [simp] = *cltn2-line.act-act-inv* [simplified]

**lemma** *apply-cltn2-line-alt-def*:  
 $\text{apply-cltn2-line } l A$   
 $= \text{proj2-line-abs} (\text{cltn2-rep} (\text{cltn2-inverse } A) *v \text{proj2-line-rep } l)$   
(proof)

**lemma** *rep-mult-line-rep-non-zero*:  $\text{cltn2-rep } A *v \text{proj2-line-rep } l \neq 0$   
(proof)

**lemma** *apply-cltn2-incident*:  
 $\text{proj2-incident } p (\text{apply-cltn2-line } l A)$   
 $\longleftrightarrow \text{proj2-incident} (\text{apply-cltn2 } p (\text{cltn2-inverse } A)) l$   
(proof)

**lemma** *apply-cltn2-preserve-incident* [iff]:  
 $\text{proj2-incident} (\text{apply-cltn2 } p A) (\text{apply-cltn2-line } l A)$   
 $\longleftrightarrow \text{proj2-incident } p l$   
(proof)

**lemma** *apply-cltn2-preserve-set-Col*:  
**assumes** *proj2-set-Col*  $S$   
**shows** *proj2-set-Col*  $\{\text{apply-cltn2 } p C \mid p. p \in S\}$   
(proof)

**lemma** *apply-cltn2-injective*:  
**assumes**  $\text{apply-cltn2 } p C = \text{apply-cltn2 } q C$   
**shows**  $p = q$

*<proof>*

**lemma** *apply-cltn2-line-injective*:

**assumes** *apply-cltn2-line l C = apply-cltn2-line m C*

**shows**  $l = m$

*<proof>*

**lemma** *apply-cltn2-line-unique*:

**assumes**  $p \neq q$  **and** *proj2-incident p l and proj2-incident q l*

**and** *proj2-incident (apply-cltn2 p C) m*

**and** *proj2-incident (apply-cltn2 q C) m*

**shows** *apply-cltn2-line l C = m*

*<proof>*

**lemma** *apply-cltn2-unique*:

**assumes**  $l \neq m$  **and** *proj2-incident p l and proj2-incident p m*

**and** *proj2-incident q (apply-cltn2-line l C)*

**and** *proj2-incident q (apply-cltn2-line m C)*

**shows** *apply-cltn2 p C = q*

*<proof>*

### 7.4.3 Parts of some Statements from [1]

All theorems with names beginning with *statement* are based on corresponding theorems in [1].

**lemma** *statement52-existence*:

**fixes**  $a :: \text{proj2}^3$  **and**  $a3 :: \text{proj2}$

**assumes** *proj2-no-3-Col (insert a3 (range (( $\$$ ) a)))*

**shows**  $\exists A. \text{apply-cltn2} (\text{proj2-abs} (\text{vector } [1,1,1])) A = a3 \wedge$

$(\forall j. \text{apply-cltn2} (\text{proj2-abs} (\text{axis } j \ 1)) A = a\$j)$

*<proof>*

**lemma** *statement53-existence*:

**fixes**  $p :: \text{proj2}^4^2$

**assumes**  $\forall i. \text{proj2-no-3-Col} (\text{range } ((\text{\$}) (p\$i)))$

**shows**  $\exists C. \forall j. \text{apply-cltn2} (p\$0\$j) C = p\$1\$j$

*<proof>*

**lemma** *apply-cltn2-linear*:

**assumes**  $j *_R v + k *_R w \neq 0$

**shows**  $j *_R (v \text{ v* } \text{cltn2-rep } C) + k *_R (w \text{ v* } \text{cltn2-rep } C) \neq 0$

(**is**  $?u \neq 0$ )

**and** *apply-cltn2 (proj2-abs (j \*\_R v + k \*\_R w)) C*

$= \text{proj2-abs} (j *_R (v \text{ v* } \text{cltn2-rep } C) + k *_R (w \text{ v* } \text{cltn2-rep } C))$

*<proof>*

**lemma** *apply-cltn2-imp-mult*:

**assumes** *apply-cltn2 p C = q*

**shows**  $\exists k. k \neq 0 \wedge \text{proj2-rep } p \text{ v* } \text{cltn2-rep } C = k *_R \text{proj2-rep } q$

*<proof>*

**lemma** *statement55*:

**assumes**  $p \neq q$   
**and** *apply-cltn2*  $p C = q$   
**and** *apply-cltn2*  $q C = p$   
**and** *proj2-incident*  $p l$   
**and** *proj2-incident*  $q l$   
**and** *proj2-incident*  $r l$   
**shows** *apply-cltn2* (*apply-cltn2*  $r C$ )  $C = r$

*<proof>*

## 7.5 Cross ratios

**definition** *cross-ratio* :: *proj2*  $\Rightarrow$  *proj2*  $\Rightarrow$  *proj2*  $\Rightarrow$  *proj2*  $\Rightarrow$  *real* **where**  
*cross-ratio*  $p q r s \triangleq$  *proj2-Col-coeff*  $p q s /$  *proj2-Col-coeff*  $p q r$

**definition** *cross-ratio-correct* :: *proj2*  $\Rightarrow$  *proj2*  $\Rightarrow$  *proj2*  $\Rightarrow$  *proj2*  $\Rightarrow$  *bool* **where**  
*cross-ratio-correct*  $p q r s \triangleq$   
*proj2-set-Col*  $\{p,q,r,s\} \wedge p \neq q \wedge r \neq p \wedge s \neq p \wedge r \neq q$

**lemma** *proj2-Col-coeff-abs*:

**assumes**  $p \neq q$  **and**  $j \neq 0$   
**shows** *proj2-Col-coeff*  $p q$  (*proj2-abs* ( $i *_R$  *proj2-rep*  $p + j *_R$  *proj2-rep*  $q$ ))  
 $= i/j$   
(**is** *proj2-Col-coeff*  $p q$  ? $r = i/j$ )

*<proof>*

**lemma** *proj2-set-Col-coeff*:

**assumes** *proj2-set-Col*  $S$  **and**  $\{p,q,r\} \subseteq S$  **and**  $p \neq q$  **and**  $r \neq p$   
**shows**  $r =$  *proj2-abs* (*proj2-Col-coeff*  $p q r *_R$  *proj2-rep*  $p +$  *proj2-rep*  $q$ )  
(**is**  $r =$  *proj2-abs* ( $?i *_R ?u + ?v$ ))

*<proof>*

**lemma** *cross-ratio-abs*:

**fixes**  $u v :: real^3$  **and**  $i j k l :: real$   
**assumes**  $u \neq 0$  **and**  $v \neq 0$  **and** *proj2-abs*  $u \neq$  *proj2-abs*  $v$   
**and**  $j \neq 0$  **and**  $l \neq 0$   
**shows** *cross-ratio* (*proj2-abs*  $u$ ) (*proj2-abs*  $v$ )  
(*proj2-abs* ( $i *_R u + j *_R v$ ))  
(*proj2-abs* ( $k *_R u + l *_R v$ ))  
 $= j * k / (i * l)$   
(**is** *cross-ratio* ? $p$  ? $q$  ? $r$  ? $s = -$ )

*<proof>*

**lemma** *cross-ratio-abs2*:

**assumes**  $p \neq q$   
**shows** *cross-ratio*  $p q$   
(*proj2-abs* ( $i *_R$  *proj2-rep*  $p +$  *proj2-rep*  $q$ ))

$(\text{proj2-abs } (j *_R \text{proj2-rep } p + \text{proj2-rep } q))$   
 $= j/i$   
**(is cross-ratio  $p q ?r ?s = -$ )**  
 <proof>

**lemma cross-ratio-correct-cltn2:**  
**assumes** *cross-ratio-correct*  $p q r s$   
**shows** *cross-ratio-correct* (*apply-cltn2*  $p C$ ) (*apply-cltn2*  $q C$ )  
 (*apply-cltn2*  $r C$ ) (*apply-cltn2*  $s C$ )  
**(is cross-ratio-correct**  $?pC ?qC ?rC ?sC$ **)**  
 <proof>

**lemma cross-ratio-cltn2:**  
**assumes** *proj2-set-Col*  $\{p,q,r,s\}$  **and**  $p \neq q$  **and**  $r \neq p$  **and**  $s \neq p$   
**shows** *cross-ratio* (*apply-cltn2*  $p C$ ) (*apply-cltn2*  $q C$ )  
 (*apply-cltn2*  $r C$ ) (*apply-cltn2*  $s C$ )  
 $= \text{cross-ratio } p q r s$   
**(is cross-ratio**  $?pC ?qC ?rC ?sC = -$ **)**  
 <proof>

**lemma cross-ratio-unique:**  
**assumes** *cross-ratio-correct*  $p q r s$  **and** *cross-ratio-correct*  $p q r t$   
**and** *cross-ratio*  $p q r s = \text{cross-ratio } p q r t$   
**shows**  $s = t$   
 <proof>

**lemma cltn2-three-point-line:**  
**assumes**  $p \neq q$  **and**  $r \neq p$  **and**  $r \neq q$   
**and** *proj2-incident*  $p l$  **and** *proj2-incident*  $q l$  **and** *proj2-incident*  $r l$   
**and** *apply-cltn2*  $p C = p$  **and** *apply-cltn2*  $q C = q$  **and** *apply-cltn2*  $r C = r$   
**and** *proj2-incident*  $s l$   
**shows** *apply-cltn2*  $s C = s$  **(is**  $?sC = s$ **)**  
 <proof>

**lemma cross-ratio-equal-cltn2:**  
**assumes** *cross-ratio-correct*  $p q r s$   
**and** *cross-ratio-correct* (*apply-cltn2*  $p C$ ) (*apply-cltn2*  $q C$ )  
 (*apply-cltn2*  $r C$ )  $t$   
**(is cross-ratio-correct**  $?pC ?qC ?rC t$ **)**  
**and** *cross-ratio* (*apply-cltn2*  $p C$ ) (*apply-cltn2*  $q C$ ) (*apply-cltn2*  $r C$ )  $t$   
 $= \text{cross-ratio } p q r s$   
**shows**  $t = \text{apply-cltn2 } s C$  **(is**  $t = ?sC$ **)**  
 <proof>

**lemma proj2-Col-distinct-coeff-non-zero:**  
**assumes** *proj2-Col*  $p q r$  **and**  $p \neq q$  **and**  $r \neq p$  **and**  $r \neq q$   
**shows** *proj2-Col-coeff*  $p q r \neq 0$   
 <proof>



**lemma** *cross-ratio-product*:

**assumes** *proj2-Col*  $p\ q\ s$  **and**  $p \neq q$  **and**  $s \neq p$  **and**  $s \neq q$

**shows**  $\text{cross-ratio } p\ q\ r\ s * \text{cross-ratio } p\ q\ s\ t = \text{cross-ratio } p\ q\ r\ t$

*<proof>*

**lemma** *cross-ratio-equal-1*:

**assumes** *proj2-Col*  $p\ q\ r$  **and**  $p \neq q$  **and**  $r \neq p$  **and**  $r \neq q$

**shows**  $\text{cross-ratio } p\ q\ r\ r = 1$

*<proof>*

**lemma** *cross-ratio-1-equal*:

**assumes** *cross-ratio-correct*  $p\ q\ r\ s$  **and**  $\text{cross-ratio } p\ q\ r\ s = 1$

**shows**  $r = s$

*<proof>*

**lemma** *cross-ratio-swap-34*:

**shows**  $\text{cross-ratio } p\ q\ s\ r = 1 / (\text{cross-ratio } p\ q\ r\ s)$

*<proof>*

**lemma** *cross-ratio-swap-13-24*:

**assumes** *cross-ratio-correct*  $p\ q\ r\ s$  **and**  $r \neq s$

**shows**  $\text{cross-ratio } r\ s\ p\ q = \text{cross-ratio } p\ q\ r\ s$

*<proof>*

**lemma** *cross-ratio-swap-12*:

**assumes** *cross-ratio-correct*  $p\ q\ r\ s$  **and** *cross-ratio-correct*  $q\ p\ r\ s$

**shows**  $\text{cross-ratio } q\ p\ r\ s = 1 / (\text{cross-ratio } p\ q\ r\ s)$

*<proof>*

## 7.6 Cartesian subspace of the real projective plane

**definition** *vector2-append1* ::  $\text{real}^2 \Rightarrow \text{real}^3$  **where**

*vector2-append1*  $v = \text{vector } [v\$1, v\$2, 1]$

**lemma** *vector2-append1-non-zero*: *vector2-append1*  $v \neq 0$

*<proof>*

**definition** *proj2-pt* ::  $\text{real}^2 \Rightarrow \text{proj2}$  **where**

*proj2-pt*  $v \triangleq \text{proj2-abs } (\text{vector2-append1 } v)$

**lemma** *proj2-pt-scalar*:

$\exists c. c \neq 0 \wedge \text{proj2-rep } (\text{proj2-pt } v) = c *_R \text{vector2-append1 } v$

*<proof>*

**abbreviation** *z-non-zero* ::  $\text{proj2} \Rightarrow \text{bool}$  **where**

*z-non-zero*  $p \triangleq (\text{proj2-rep } p)\$3 \neq 0$

**definition** *cart2-pt* ::  $\text{proj2} \Rightarrow \text{real}^2$  **where**

*cart2-pt*  $p \triangleq$

$vector [(proj2-rep p)\$1 / (proj2-rep p)\$3, (proj2-rep p)\$2 / (proj2-rep p)\$3]$

**definition** *cart2-append1* :: *proj2*  $\Rightarrow$  *real*<sup>3</sup> **where**  
*cart2-append1* *p*  $\triangleq (1 / ((proj2-rep p)\$3)) *_R proj2-rep p$

**lemma** *cart2-append1-z*:  
**assumes** *z-non-zero p*  
**shows** (*cart2-append1 p*)\\$3 = 1  
*<proof>*

**lemma** *cart2-append1-non-zero*:  
**assumes** *z-non-zero p*  
**shows** *cart2-append1 p*  $\neq 0$   
*<proof>*

**lemma** *proj2-rep-cart2-append1*:  
**assumes** *z-non-zero p*  
**shows** *proj2-rep p* = ((*proj2-rep p*)\\$3) \*\_R *cart2-append1 p*  
*<proof>*

**lemma** *proj2-abs-cart2-append1*:  
**assumes** *z-non-zero p*  
**shows** *proj2-abs (cart2-append1 p)* = *p*  
*<proof>*

**lemma** *cart2-append1-inj*:  
**assumes** *z-non-zero p* **and** *cart2-append1 p* = *cart2-append1 q*  
**shows** *p* = *q*  
*<proof>*

**lemma** *cart2-append1*:  
**assumes** *z-non-zero p*  
**shows** *vector2-append1 (cart2-pt p)* = *cart2-append1 p*  
*<proof>*

**lemma** *cart2-proj2*: *cart2-pt (proj2-pt v)* = *v*  
*<proof>*

**lemma** *z-non-zero-proj2-pt*: *z-non-zero (proj2-pt v)*  
*<proof>*

**lemma** *cart2-append1-proj2*: *cart2-append1 (proj2-pt v)* = *vector2-append1 v*  
*<proof>*

**lemma** *proj2-pt-inj*: *inj proj2-pt*  
*<proof>*

**lemma** *proj2-cart2*:  
**assumes** *z-non-zero p*

**shows**  $\text{proj2-pt } (\text{cart2-pt } p) = p$   
 ⟨proof⟩

**lemma** *cart2-injective*:

**assumes**  $z\text{-non-zero } p$  **and**  $z\text{-non-zero } q$  **and**  $\text{cart2-pt } p = \text{cart2-pt } q$   
**shows**  $p = q$   
 ⟨proof⟩

**lemma** *proj2-Col-iff-euclid*:

$\text{proj2-Col } (\text{proj2-pt } a) (\text{proj2-pt } b) (\text{proj2-pt } c) \longleftrightarrow \text{real-euclid.Col } a \ b \ c$   
**(is**  $\text{proj2-Col } ?p \ ?q \ ?r \longleftrightarrow -$   
 ⟨proof⟩

**lemma** *proj2-Col-iff-euclid-cart2*:

**assumes**  $z\text{-non-zero } p$  **and**  $z\text{-non-zero } q$  **and**  $z\text{-non-zero } r$   
**shows**  
 $\text{proj2-Col } p \ q \ r \longleftrightarrow \text{real-euclid.Col } (\text{cart2-pt } p) (\text{cart2-pt } q) (\text{cart2-pt } r)$   
**(is**  $- \longleftrightarrow \text{real-euclid.Col } ?a \ ?b \ ?c$   
 ⟨proof⟩

**lemma** *euclid-Col-cart2-incident*:

**assumes**  $z\text{-non-zero } p$  **and**  $z\text{-non-zero } q$  **and**  $z\text{-non-zero } r$  **and**  $p \neq q$   
**and**  $\text{proj2-incident } p \ l$  **and**  $\text{proj2-incident } q \ l$   
**and**  $\text{real-euclid.Col } (\text{cart2-pt } p) (\text{cart2-pt } q) (\text{cart2-pt } r)$   
**(is**  $\text{real-euclid.Col } ?cp \ ?cq \ ?cr$   
**shows**  $\text{proj2-incident } r \ l$   
 ⟨proof⟩

**lemma** *euclid-B-cart2-common-line*:

**assumes**  $z\text{-non-zero } p$  **and**  $z\text{-non-zero } q$  **and**  $z\text{-non-zero } r$   
**and**  $B_{\mathbb{R}} (\text{cart2-pt } p) (\text{cart2-pt } q) (\text{cart2-pt } r)$   
**(is**  $B_{\mathbb{R}} \ ?cp \ ?cq \ ?cr$   
**shows**  $\exists \ l. \ \text{proj2-incident } p \ l \wedge \ \text{proj2-incident } q \ l \wedge \ \text{proj2-incident } r \ l$   
 ⟨proof⟩

**lemma** *cart2-append1-between*:

**assumes**  $z\text{-non-zero } p$  **and**  $z\text{-non-zero } q$  **and**  $z\text{-non-zero } r$   
**shows**  $B_{\mathbb{R}} (\text{cart2-pt } p) (\text{cart2-pt } q) (\text{cart2-pt } r)$   
 $\longleftrightarrow (\exists \ k \geq 0. \ k \leq 1$   
 $\wedge \text{cart2-append1 } q = k *_R \text{cart2-append1 } r + (1 - k) *_R \text{cart2-append1 } p)$   
 ⟨proof⟩

**lemma** *cart2-append1-between-right-strict*:

**assumes**  $z\text{-non-zero } p$  **and**  $z\text{-non-zero } q$  **and**  $z\text{-non-zero } r$   
**and**  $B_{\mathbb{R}} (\text{cart2-pt } p) (\text{cart2-pt } q) (\text{cart2-pt } r)$  **and**  $q \neq r$   
**shows**  $\exists \ k \geq 0. \ k < 1$   
 $\wedge \text{cart2-append1 } q = k *_R \text{cart2-append1 } r + (1 - k) *_R \text{cart2-append1 } p$   
 ⟨proof⟩

**lemma** *cart2-append1-between-strict*:  
**assumes** *z-non-zero p* **and** *z-non-zero q* **and** *z-non-zero r*  
**and**  $B_{\mathbb{R}}$  (*cart2-pt p*) (*cart2-pt q*) (*cart2-pt r*) **and**  $q \neq p$  **and**  $q \neq r$   
**shows**  $\exists k > 0. k < 1$   
 $\wedge$  *cart2-append1 q* =  $k *_R$  *cart2-append1 r* +  $(1 - k) *_R$  *cart2-append1 p*  
⟨*proof*⟩

**end**

## 8 The hyperbolic plane and Tarski's axioms

**theory** *Hyperbolic-Tarski*  
**imports** *Euclid-Tarski*  
*Projective*  
*HOL-Library.Quadratic-Discriminant*  
**begin**

### 8.1 Characterizing a specific conic in the projective plane

**definition** *M* ::  $\text{real}^3$  where

$M \triangleq$  *vector* [  
*vector* [1, 0, 0],  
*vector* [0, 1, 0],  
*vector* [0, 0, -1]]

**lemma** *M-symmatrix: symmatrix M*  
⟨*proof*⟩

**lemma** *M-self-inverse: M \*\* M = mat 1*  
⟨*proof*⟩

**lemma** *M-invertible: invertible M*  
⟨*proof*⟩

**definition** *polar* :: *proj2*  $\Rightarrow$  *proj2-line* where  
*polar p*  $\triangleq$  *proj2-line-abs* ( $M *_v$  *proj2-rep p*)

**definition** *pole* :: *proj2-line*  $\Rightarrow$  *proj2* where  
*pole l*  $\triangleq$  *proj2-abs* ( $M *_v$  *proj2-line-rep l*)

**lemma** *polar-abs*:  
**assumes**  $v \neq 0$   
**shows** *polar* (*proj2-abs v*) = *proj2-line-abs* ( $M *_v v$ )  
⟨*proof*⟩

**lemma** *pole-abs*:  
**assumes**  $v \neq 0$   
**shows** *pole* (*proj2-line-abs v*) = *proj2-abs* ( $M *_v v$ )  
⟨*proof*⟩

**lemma** *polar-rep-non-zero*:  $M *v \text{proj2-rep } p \neq 0$   
*<proof>*

**lemma** *pole-polar*:  $\text{pole } (\text{polar } p) = p$   
*<proof>*

**lemma** *pole-rep-non-zero*:  $M *v \text{proj2-line-rep } l \neq 0$   
*<proof>*

**lemma** *polar-pole*:  $\text{polar } (\text{pole } l) = l$   
*<proof>*

**lemma** *polar-inj*:  
  **assumes**  $\text{polar } p = \text{polar } q$   
  **shows**  $p = q$   
*<proof>*

**definition** *conic-sgn* ::  $\text{proj2} \Rightarrow \text{real}$  **where**  
   $\text{conic-sgn } p \triangleq \text{sgn } (\text{proj2-rep } p \cdot (M *v \text{proj2-rep } p))$

**lemma** *conic-sgn-abs*:  
  **assumes**  $v \neq 0$   
  **shows**  $\text{conic-sgn } (\text{proj2-abs } v) = \text{sgn } (v \cdot (M *v v))$   
*<proof>*

**lemma** *sgn-conic-sgn*:  $\text{sgn } (\text{conic-sgn } p) = \text{conic-sgn } p$   
*<proof>*

**definition** *S* ::  $\text{proj2}$  **set where**  
   $S \triangleq \{p. \text{conic-sgn } p = 0\}$

**definition** *K2* ::  $\text{proj2}$  **set where**  
   $K2 \triangleq \{p. \text{conic-sgn } p < 0\}$

**lemma** *S-K2-empty*:  $S \cap K2 = \{\}$   
*<proof>*

**lemma** *K2-abs*:  
  **assumes**  $v \neq 0$   
  **shows**  $\text{proj2-abs } v \in K2 \iff v \cdot (M *v v) < 0$   
*<proof>*

**definition** *K2-centre* =  $\text{proj2-abs } (\text{vector } [0,0,1])$

**lemma** *K2-centre-non-zero*:  $\text{vector } [0,0,1] \neq (0 :: \text{real}^3)$   
*<proof>*

**lemma** *K2-centre-in-K2*:  $K2\text{-centre} \in K2$

*<proof>*

**lemma** *K2-imp-M-neg*:

**assumes**  $v \neq 0$  **and**  $\text{proj2-abs } v \in K2$

**shows**  $v \cdot (M *v v) < 0$

*<proof>*

**lemma** *M-neg-imp-z-squared-big*:

**assumes**  $v \cdot (M *v v) < 0$

**shows**  $(v\$3)^2 > (v\$1)^2 + (v\$2)^2$

*<proof>*

**lemma** *M-neg-imp-z-non-zero*:

**assumes**  $v \cdot (M *v v) < 0$

**shows**  $v\$3 \neq 0$

*<proof>*

**lemma** *M-neg-imp-K2*:

**assumes**  $v \cdot (M *v v) < 0$

**shows**  $\text{proj2-abs } v \in K2$

*<proof>*

**lemma** *M-reverse*:  $a \cdot (M *v b) = b \cdot (M *v a)$

*<proof>*

**lemma** *S-abs*:

**assumes**  $v \neq 0$

**shows**  $\text{proj2-abs } v \in S \iff v \cdot (M *v v) = 0$

*<proof>*

**lemma** *S-alt-def*:  $p \in S \iff \text{proj2-rep } p \cdot (M *v \text{proj2-rep } p) = 0$

*<proof>*

**lemma** *incident-polar*:

$\text{proj2-incident } p \text{ (polar } q) \iff \text{proj2-rep } p \cdot (M *v \text{proj2-rep } q) = 0$

*<proof>*

**lemma** *incident-own-polar-in-S*:  $\text{proj2-incident } p \text{ (polar } p) \iff p \in S$

*<proof>*

**lemma** *incident-polar-swap*:

**assumes**  $\text{proj2-incident } p \text{ (polar } q)$

**shows**  $\text{proj2-incident } q \text{ (polar } p)$

*<proof>*

**lemma** *incident-pole-polar*:

**assumes**  $\text{proj2-incident } p \ l$

**shows**  $\text{proj2-incident } (\text{pole } l) \text{ (polar } p)$

*<proof>*

**definition** *z-zero* :: *proj2-line* **where**

*z-zero*  $\triangleq$  *proj2-line-abs* (vector [0,0,1])

**lemma** *z-zero*:

**assumes** (*proj2-rep* p)\$3 = 0

**shows** *proj2-incident* p *z-zero*

*<proof>*

**lemma** *z-zero-conic-sgn-1*:

**assumes** *proj2-incident* p *z-zero*

**shows** *conic-sgn* p = 1

*<proof>*

**lemma** *conic-sgn-not-1-z-non-zero*:

**assumes** *conic-sgn* p  $\neq$  1

**shows** *z-non-zero* p

*<proof>*

**lemma** *z-zero-not-in-S*:

**assumes** *proj2-incident* p *z-zero*

**shows** p  $\notin$  S

*<proof>*

**lemma** *line-incident-point-not-in-S*:  $\exists$  p. p  $\notin$  S  $\wedge$  *proj2-incident* p l

*<proof>*

**lemma** *apply-cltn2-abs-abs-in-S*:

**assumes** v  $\neq$  0 **and** *invertible* J

**shows** *apply-cltn2* (*proj2-abs* v) (*cltn2-abs* J)  $\in$  S

$\longleftrightarrow$  v  $\cdot$  (J \*\* M \*\* *transpose* J \*v v) = 0

*<proof>*

**lemma** *apply-cltn2-right-abs-in-S*:

**assumes** *invertible* J

**shows** *apply-cltn2* p (*cltn2-abs* J)  $\in$  S

$\longleftrightarrow$  (*proj2-rep* p)  $\cdot$  (J \*\* M \*\* *transpose* J \*v (*proj2-rep* p)) = 0

*<proof>*

**lemma** *apply-cltn2-abs-in-S*:

**assumes** v  $\neq$  0

**shows** *apply-cltn2* (*proj2-abs* v) C  $\in$  S

$\longleftrightarrow$  v  $\cdot$  (*cltn2-rep* C \*\* M \*\* *transpose* (*cltn2-rep* C) \*v v) = 0

*<proof>*

**lemma** *apply-cltn2-in-S*:

*apply-cltn2* p C  $\in$  S

$\longleftrightarrow$  *proj2-rep* p  $\cdot$  (*cltn2-rep* C \*\* M \*\* *transpose* (*cltn2-rep* C) \*v *proj2-rep* p)  
= 0

*<proof>*

**lemma** *norm-M*:  $(\text{vector2-append1 } v) \cdot (M *v \text{vector2-append1 } v) = (\text{norm } v)^2 - 1$   
*<proof>*

## 8.2 Some specific points and lines of the projective plane

**definition** *east* = *proj2-abs* (*vector* [1,0,1])

**definition** *west* = *proj2-abs* (*vector* [-1,0,1])

**definition** *north* = *proj2-abs* (*vector* [0,1,1])

**definition** *south* = *proj2-abs* (*vector* [0,-1,1])

**definition** *far-north* = *proj2-abs* (*vector* [0,1,0])

**lemmas** *compass-defs* = *east-def west-def north-def south-def*

**lemma** *compass-non-zero*:

**shows** *vector* [1,0,1]  $\neq (0 :: \text{real}^3)$

**and** *vector* [-1,0,1]  $\neq (0 :: \text{real}^3)$

**and** *vector* [0,1,1]  $\neq (0 :: \text{real}^3)$

**and** *vector* [0,-1,1]  $\neq (0 :: \text{real}^3)$

**and** *vector* [0,1,0]  $\neq (0 :: \text{real}^3)$

**and** *vector* [1,0,0]  $\neq (0 :: \text{real}^3)$

*<proof>*

**lemma** *east-west-distinct*: *east*  $\neq$  *west*

*<proof>*

**lemma** *north-south-distinct*: *north*  $\neq$  *south*

*<proof>*

**lemma** *north-not-east-or-west*: *north*  $\notin \{\text{east}, \text{west}\}$

*<proof>*

**lemma** *compass-in-S*:

**shows** *east*  $\in S$  **and** *west*  $\in S$  **and** *north*  $\in S$  **and** *south*  $\in S$

*<proof>*

**lemma** *east-west-tangents*:

**shows** *polar east* = *proj2-line-abs* (*vector* [-1,0,1])

**and** *polar west* = *proj2-line-abs* (*vector* [1,0,1])

*<proof>*

**lemma** *east-west-tangents-distinct*: *polar east*  $\neq$  *polar west*

*<proof>*

**lemma** *east-west-tangents-incident-far-north*:

**shows** *proj2-incident far-north* (*polar east*)

**and** *proj2-incident far-north* (*polar west*)



*<proof>*

**lemma** *east-west-tangents-far-north:*

*proj2-intersection (polar east) (polar west) = far-north*

*<proof>*

**instantiation** *proj2 :: zero*

**begin**

**definition** *proj2-zero-def: 0 = proj2-pt 0*

**instance** *<proof>*

**end**

**definition** *equator  $\triangleq$  proj2-line-abs (vector [0,1,0])*

**definition** *meridian  $\triangleq$  proj2-line-abs (vector [1,0,0])*

**lemma** *equator-meridian-distinct: equator  $\neq$  meridian*

*<proof>*

**lemma** *east-west-on-equator:*

**shows** *proj2-incident east equator and proj2-incident west equator*

*<proof>*

**lemma** *north-far-north-distinct: north  $\neq$  far-north*

*<proof>*

**lemma** *north-south-far-north-on-meridian:*

**shows** *proj2-incident north meridian and proj2-incident south meridian*

**and** *proj2-incident far-north meridian*

*<proof>*

**lemma** *K2-centre-on-equator-meridian:*

**shows** *proj2-incident K2-centre equator*

**and** *proj2-incident K2-centre meridian*

*<proof>*

**lemma** *on-equator-meridian-is-K2-centre:*

**assumes** *proj2-incident a equator and proj2-incident a meridian*

**shows** *a = K2-centre*

*<proof>*

**definition** *rep-equator-reflect  $\triangleq$  vector [*

*vector [1, 0,0],*

*vector [0,-1,0],*

*vector [0, 0,1]] :: real^3^3*

**definition** *rep-meridian-reflect  $\triangleq$  vector [*

*vector [-1,0,0],*

*vector [ 0,1,0],*

*vector [ 0,0,1]] :: real^3^3*

**definition** *equator-reflect  $\triangleq$  cltn2-abs rep-equator-reflect*

**definition** *meridian-reflect*  $\triangleq$  *cltn2-abs rep-meridian-reflect*

**lemmas** *compass-reflect-defs* = *equator-reflect-def meridian-reflect-def rep-equator-reflect-def rep-meridian-reflect-def*

**lemma** *compass-reflect-self-inverse*:

**shows** *rep-equator-reflect* \*\* *rep-equator-reflect* = *mat 1*  
**and** *rep-meridian-reflect* \*\* *rep-meridian-reflect* = *mat 1*  
(*proof*)

**lemma** *compass-reflect-invertible*:

**shows** *invertible rep-equator-reflect* **and** *invertible rep-meridian-reflect*  
(*proof*)

**lemma** *compass-reflect-compass*:

**shows** *apply-cltn2 east meridian-reflect* = *west*  
**and** *apply-cltn2 west meridian-reflect* = *east*  
**and** *apply-cltn2 north meridian-reflect* = *north*  
**and** *apply-cltn2 south meridian-reflect* = *south*  
**and** *apply-cltn2 K2-centre meridian-reflect* = *K2-centre*  
**and** *apply-cltn2 east equator-reflect* = *east*  
**and** *apply-cltn2 west equator-reflect* = *west*  
**and** *apply-cltn2 north equator-reflect* = *south*  
**and** *apply-cltn2 south equator-reflect* = *north*  
**and** *apply-cltn2 K2-centre equator-reflect* = *K2-centre*  
(*proof*)

**lemma** *on-equator-rep*:

**assumes** *z-non-zero a* **and** *proj2-incident a equator*  
**shows**  $\exists x. a = \text{proj2-abs } (\text{vector } [x,0,1])$   
(*proof*)

**lemma** *on-meridian-rep*:

**assumes** *z-non-zero a* **and** *proj2-incident a meridian*  
**shows**  $\exists y. a = \text{proj2-abs } (\text{vector } [0,y,1])$   
(*proof*)

### 8.3 Definition of the Klein–Beltrami model of the hyperbolic plane

**abbreviation** *hyp2* == *K2*

**typedef** *hyp2* = *K2*  
(*proof*)

**definition** *hyp2-rep* :: *hyp2*  $\Rightarrow$  *real^2* **where**  
*hyp2-rep p*  $\triangleq$  *cart2-pt (Rep-hyp2 p)*

**definition** *hyp2-abs* :: *real^2*  $\Rightarrow$  *hyp2* **where**

$hyp2-abs\ v = Abs-hyp2\ (proj2-pt\ v)$

**lemma** *norm-lt-1-iff-in-hyp2*:

**shows**  $norm\ v < 1 \longleftrightarrow proj2-pt\ v \in hyp2$   
*<proof>*

**lemma** *norm-eq-1-iff-in-S*:

**shows**  $norm\ v = 1 \longleftrightarrow proj2-pt\ v \in S$   
*<proof>*

**lemma** *norm-le-1-iff-in-hyp2-S*:

$norm\ v \leq 1 \longleftrightarrow proj2-pt\ v \in hyp2 \cup S$   
*<proof>*

**lemma** *proj2-pt-hyp2-rep*:  $proj2-pt\ (hyp2-rep\ p) = Rep-hyp2\ p$

*<proof>*

**lemma** *hyp2-rep-abs*:

**assumes**  $norm\ v < 1$   
**shows**  $hyp2-rep\ (hyp2-abs\ v) = v$   
*<proof>*

**lemma** *hyp2-abs-rep*:  $hyp2-abs\ (hyp2-rep\ p) = p$

*<proof>*

**lemma** *norm-hyp2-rep-lt-1*:  $norm\ (hyp2-rep\ p) < 1$

*<proof>*

**lemma** *hyp2-S-z-non-zero*:

**assumes**  $p \in hyp2 \cup S$   
**shows**  $z-non-zero\ p$   
*<proof>*

**lemma** *hyp2-S-not-equal*:

**assumes**  $a \in hyp2$  **and**  $p \in S$   
**shows**  $a \neq p$   
*<proof>*

**lemma** *hyp2-S-cart2-inj*:

**assumes**  $p \in hyp2 \cup S$  **and**  $q \in hyp2 \cup S$  **and**  $cart2-pt\ p = cart2-pt\ q$   
**shows**  $p = q$   
*<proof>*

**lemma** *on-equator-in-hyp2-rep*:

**assumes**  $a \in hyp2$  **and**  $proj2-incident\ a\ equator$   
**shows**  $\exists x. |x| < 1 \wedge a = proj2-abs\ (vector\ [x,0,1])$   
*<proof>*

**lemma** *on-meridian-in-hyp2-rep*:

**assumes**  $a \in \text{hyp2}$  **and** *proj2-incident a meridian*  
**shows**  $\exists y. |y| < 1 \wedge a = \text{proj2-abs (vector [0,y,1])}$   
 ⟨proof⟩

**definition**  $\text{hyp2-cltn2} :: \text{hyp2} \Rightarrow \text{cltn2} \Rightarrow \text{hyp2}$  **where**  
 $\text{hyp2-cltn2 } p \ A \triangleq \text{Abs-hyp2 (apply-cltn2 (Rep-hyp2 } p) \ A)$

**definition**  $\text{is-K2-isometry} :: \text{cltn2} \Rightarrow \text{bool}$  **where**  
 $\text{is-K2-isometry } J \triangleq (\forall p. \text{apply-cltn2 } p \ J \in S \longleftrightarrow p \in S)$

**lemma**  $\text{cltn2-id-is-K2-isometry}$ :  $\text{is-K2-isometry cltn2-id}$   
 ⟨proof⟩

**lemma**  $J\text{-}M\text{-}J\text{-transpose-K2-isometry}$ :  
**assumes**  $k \neq 0$   
**and**  $\text{rep}J \ ** \ M \ ** \ \text{transpose rep}J = k \ *_R \ M$  (**is**  $?N = -$ )  
**shows**  $\text{is-K2-isometry (cltn2-abs rep}J)$  (**is**  $\text{is-K2-isometry } ?J$ )  
 ⟨proof⟩

**lemma**  $\text{equator-reflect-K2-isometry}$ :  
**shows**  $\text{is-K2-isometry equator-reflect}$   
 ⟨proof⟩

**lemma**  $\text{meridian-reflect-K2-isometry}$ :  
**shows**  $\text{is-K2-isometry meridian-reflect}$   
 ⟨proof⟩

**lemma**  $\text{cltn2-compose-is-K2-isometry}$ :  
**assumes**  $\text{is-K2-isometry } H$  **and**  $\text{is-K2-isometry } J$   
**shows**  $\text{is-K2-isometry (cltn2-compose } H \ J)$   
 ⟨proof⟩

**lemma**  $\text{cltn2-inverse-is-K2-isometry}$ :  
**assumes**  $\text{is-K2-isometry } J$   
**shows**  $\text{is-K2-isometry (cltn2-inverse } J)$   
 ⟨proof⟩

**interpretation**  $K2\text{-isometry-subgroup}$ : *subgroup*  
 Collect  $\text{is-K2-isometry}$   
 ( $|\text{carrier} = \text{UNIV}$ ,  $\text{mult} = \text{cltn2-compose}$ ,  $\text{one} = \text{cltn2-id}$ )  
 ⟨proof⟩

**interpretation**  $K2\text{-isometry}$ : *group*  
 ( $|\text{carrier} = \text{Collect is-K2-isometry}$ ,  $\text{mult} = \text{cltn2-compose}$ ,  $\text{one} = \text{cltn2-id}$ )  
 ⟨proof⟩

**lemma**  $K2\text{-isometry-inverse-inv [simp]}$ :  
**assumes**  $\text{is-K2-isometry } J$   
**shows**  $\text{inv}(|\text{carrier} = \text{Collect is-K2-isometry}$ ,  $\text{mult} = \text{cltn2-compose}$ ,  $\text{one} = \text{cltn2-id}|)$

$J$   
 $= \text{cltn2-inverse } J$   
 $\langle \text{proof} \rangle$

**definition** *real-hyp2-C* ::  $[\text{hyp2}, \text{hyp2}, \text{hyp2}, \text{hyp2}] \Rightarrow \text{bool}$   
 $(- \equiv_K - - [99,99,99,99] 50)$  **where**  
 $p \ q \equiv_K \ r \ s \triangleq$   
 $(\exists A. \text{is-K2-isometry } A \wedge \text{hyp2-cltn2 } p \ A = r \wedge \text{hyp2-cltn2 } q \ A = s)$

**definition** *real-hyp2-B* ::  $[\text{hyp2}, \text{hyp2}, \text{hyp2}] \Rightarrow \text{bool}$   
 $(B_K - - - [99,99,99] 50)$  **where**  
 $B_K \ p \ q \ r \triangleq B_{\mathbb{R}} (\text{hyp2-rep } p) (\text{hyp2-rep } q) (\text{hyp2-rep } r)$

## 8.4 $K$ -isometries map the interior of the conic to itself

**lemma** *collinear-quadratic*:  
**assumes**  $t = i *_R a + r$   
**shows**  $t \cdot (M *v t) =$   
 $(a \cdot (M *v a)) * i^2 + 2 * (a \cdot (M *v r)) * i + r \cdot (M *v r)$   
 $\langle \text{proof} \rangle$

**lemma** *S-quadratic'*:  
**assumes**  $p \neq 0$  **and**  $q \neq 0$  **and**  $\text{proj2-abs } p \neq \text{proj2-abs } q$   
**shows**  $\text{proj2-abs } (k *_R p + q) \in S$   
 $\longleftrightarrow p \cdot (M *v p) * k^2 + p \cdot (M *v q) * 2 * k + q \cdot (M *v q) = 0$   
 $\langle \text{proof} \rangle$

**lemma** *S-quadratic*:  
**assumes**  $p \neq q$  **and**  $r = \text{proj2-abs } (k *_R \text{proj2-rep } p + \text{proj2-rep } q)$   
**shows**  $r \in S$   
 $\longleftrightarrow \text{proj2-rep } p \cdot (M *v \text{proj2-rep } p) * k^2$   
 $+ \text{proj2-rep } p \cdot (M *v \text{proj2-rep } q) * 2 * k$   
 $+ \text{proj2-rep } q \cdot (M *v \text{proj2-rep } q)$   
 $= 0$   
 $\langle \text{proof} \rangle$

**definition** *quarter-discrim* ::  $\text{real}^3 \Rightarrow \text{real}^3 \Rightarrow \text{real}$  **where**  
 $\text{quarter-discrim } p \ q \triangleq (p \cdot (M *v q))^2 - p \cdot (M *v p) * (q \cdot (M *v q))$

**lemma** *quarter-discrim-invariant*:  
**assumes**  $t = i *_R a + r$   
**shows**  $\text{quarter-discrim } a \ t = \text{quarter-discrim } a \ r$   
 $\langle \text{proof} \rangle$

**lemma** *quarter-discrim-positive*:  
**assumes**  $p \neq 0$  **and**  $q \neq 0$  **and**  $\text{proj2-abs } p \neq \text{proj2-abs } q$  (**is**  $?pp \neq ?pq$ )  
**and**  $\text{proj2-abs } p \in K2$   
**shows**  $\text{quarter-discrim } p \ q > 0$   
 $\langle \text{proof} \rangle$

**lemma** *quarter-discrim-self-zero*:

**assumes** *proj2-abs a = proj2-abs b*

**shows** *quarter-discrim a b = 0*

*<proof>*

**definition** *S-intersection-coeff1* ::  $\text{real}^3 \Rightarrow \text{real}^3 \Rightarrow \text{real}$  **where**

*S-intersection-coeff1 p q*

$\triangleq (-p \cdot (M *v q) + \text{sqrt} (\text{quarter-discrim } p q)) / (p \cdot (M *v p))$

**definition** *S-intersection-coeff2* ::  $\text{real}^3 \Rightarrow \text{real}^3 \Rightarrow \text{real}$  **where**

*S-intersection-coeff2 p q*

$\triangleq (-p \cdot (M *v q) - \text{sqrt} (\text{quarter-discrim } p q)) / (p \cdot (M *v p))$

**definition** *S-intersection1-rep* ::  $\text{real}^3 \Rightarrow \text{real}^3 \Rightarrow \text{real}^3$  **where**

*S-intersection1-rep p q*  $\triangleq (S\text{-intersection-coeff1 } p q) *_{\mathbb{R}} p + q$

**definition** *S-intersection2-rep* ::  $\text{real}^3 \Rightarrow \text{real}^3 \Rightarrow \text{real}^3$  **where**

*S-intersection2-rep p q*  $\triangleq (S\text{-intersection-coeff2 } p q) *_{\mathbb{R}} p + q$

**definition** *S-intersection1* ::  $\text{real}^3 \Rightarrow \text{real}^3 \Rightarrow \text{proj2}$  **where**

*S-intersection1 p q*  $\triangleq \text{proj2-abs } (S\text{-intersection1-rep } p q)$

**definition** *S-intersection2* ::  $\text{real}^3 \Rightarrow \text{real}^3 \Rightarrow \text{proj2}$  **where**

*S-intersection2 p q*  $\triangleq \text{proj2-abs } (S\text{-intersection2-rep } p q)$

**lemmas** *S-intersection-coeffs-defs* =

*S-intersection-coeff1-def S-intersection-coeff2-def*

**lemmas** *S-intersections-defs* =

*S-intersection1-def S-intersection2-def*

*S-intersection1-rep-def S-intersection2-rep-def*

**lemma** *S-intersection-coeffs-distinct*:

**assumes**  $p \neq 0$  **and**  $q \neq 0$  **and**  $\text{proj2-abs } p \neq \text{proj2-abs } q$  (**is**  $?pp \neq ?pq$ )

**and**  $\text{proj2-abs } p \in K2$

**shows**  $S\text{-intersection-coeff1 } p q \neq S\text{-intersection-coeff2 } p q$

*<proof>*

**lemma** *S-intersections-distinct*:

**assumes**  $p \neq 0$  **and**  $q \neq 0$  **and**  $\text{proj2-abs } p \neq \text{proj2-abs } q$  (**is**  $?pp \neq ?pq$ )

**and**  $\text{proj2-abs } p \in K2$

**shows**  $S\text{-intersection1 } p q \neq S\text{-intersection2 } p q$

*<proof>*

**lemma** *S-intersections-in-S*:

**assumes**  $p \neq 0$  **and**  $q \neq 0$  **and**  $\text{proj2-abs } p \neq \text{proj2-abs } q$  (**is**  $?pp \neq ?pq$ )

**and**  $\text{proj2-abs } p \in K2$

**shows**  $S\text{-intersection1 } p q \in S$  **and**  $S\text{-intersection2 } p q \in S$

$\langle$ proof $\rangle$

**lemma** *S-intersections-Col:*

**assumes**  $p \neq 0$  **and**  $q \neq 0$

**shows**  $\text{proj2-Col } (\text{proj2-abs } p) (\text{proj2-abs } q) (\text{S-intersection1 } p \ q)$

(**is**  $\text{proj2-Col } ?pp \ ?pq \ ?pr$ )

**and**  $\text{proj2-Col } (\text{proj2-abs } p) (\text{proj2-abs } q) (\text{S-intersection2 } p \ q)$

(**is**  $\text{proj2-Col } ?pp \ ?pq \ ?ps$ )

$\langle$ proof $\rangle$

**lemma** *S-intersections-incident:*

**assumes**  $p \neq 0$  **and**  $q \neq 0$  **and**  $\text{proj2-abs } p \neq \text{proj2-abs } q$  (**is**  $?pp \neq ?pq$ )

**and**  $\text{proj2-incident } (\text{proj2-abs } p) \ l$  **and**  $\text{proj2-incident } (\text{proj2-abs } q) \ l$

**shows**  $\text{proj2-incident } (\text{S-intersection1 } p \ q) \ l$  (**is**  $\text{proj2-incident } ?pr \ l$ )

**and**  $\text{proj2-incident } (\text{S-intersection2 } p \ q) \ l$  (**is**  $\text{proj2-incident } ?ps \ l$ )

$\langle$ proof $\rangle$

**lemma** *K2-line-intersect-twice:*

**assumes**  $a \in K2$  **and**  $a \neq r$

**shows**  $\exists \ s \ u. \ s \neq u \wedge s \in S \wedge u \in S \wedge \text{proj2-Col } a \ r \ s \wedge \text{proj2-Col } a \ r \ u$

$\langle$ proof $\rangle$

**lemma** *point-in-S-polar-is-tangent:*

**assumes**  $p \in S$  **and**  $q \in S$  **and**  $\text{proj2-incident } q \ (\text{polar } p)$

**shows**  $q = p$

$\langle$ proof $\rangle$

**lemma** *line-through-K2-intersect-S-twice:*

**assumes**  $p \in K2$  **and**  $\text{proj2-incident } p \ l$

**shows**  $\exists \ q \ r. \ q \neq r \wedge q \in S \wedge r \in S \wedge \text{proj2-incident } q \ l \wedge \text{proj2-incident } r \ l$

$\langle$ proof $\rangle$

**lemma** *line-through-K2-intersect-S-again:*

**assumes**  $p \in K2$  **and**  $\text{proj2-incident } p \ l$

**shows**  $\exists \ r. \ r \neq p \wedge r \in S \wedge \text{proj2-incident } r \ l$

$\langle$ proof $\rangle$

**lemma** *line-through-K2-intersect-S:*

**assumes**  $p \in K2$  **and**  $\text{proj2-incident } p \ l$

**shows**  $\exists \ r. \ r \in S \wedge \text{proj2-incident } r \ l$

$\langle$ proof $\rangle$

**lemma** *line-intersect-S-at-most-twice:*

$\exists \ p \ q. \ \forall \ r \in S. \ \text{proj2-incident } r \ l \longrightarrow r = p \vee r = q$

$\langle$ proof $\rangle$

**lemma** *card-line-intersect-S:*

**assumes**  $T \subseteq S$  **and**  $\text{proj2-set-Col } T$

**shows**  $\text{card } T \leq 2$

*<proof>*

**lemma** *line-S-two-intersections-only:*

**assumes**  $p \neq q$  **and**  $p \in S$  **and**  $q \in S$  **and**  $r \in S$   
**and** *proj2-incident*  $p l$  **and** *proj2-incident*  $q l$  **and** *proj2-incident*  $r l$   
**shows**  $r = p \vee r = q$

*<proof>*

**lemma** *line-through-K2-intersect-S-exactly-twice:*

**assumes**  $p \in K2$  **and** *proj2-incident*  $p l$   
**shows**  $\exists q r. q \neq r \wedge q \in S \wedge r \in S \wedge$  *proj2-incident*  $q l \wedge$  *proj2-incident*  $r l$   
 $\wedge (\forall s \in S. \text{proj2-incident } s l \longrightarrow s = q \vee s = r)$

*<proof>*

**lemma** *tangent-not-through-K2:*

**assumes**  $p \in S$  **and**  $q \in K2$   
**shows**  $\neg$  *proj2-incident*  $q$  (*polar*  $p$ )

*<proof>*

**lemma** *outside-exists-line-not-intersect-S:*

**assumes** *conic-sgn*  $p = 1$   
**shows**  $\exists l. \text{proj2-incident } p l \wedge (\forall q. \text{proj2-incident } q l \longrightarrow q \notin S)$

*<proof>*

**lemma** *lines-through-intersect-S-twice-in-K2:*

**assumes**  $\forall l. \text{proj2-incident } p l$   
 $\longrightarrow (\exists q r. q \neq r \wedge q \in S \wedge r \in S \wedge \text{proj2-incident } q l \wedge \text{proj2-incident } r l)$   
**shows**  $p \in K2$

*<proof>*

**lemma** *line-through-hyp2-pole-not-in-hyp2:*

**assumes**  $a \in \text{hyp2}$  **and** *proj2-incident*  $a l$   
**shows** *pole*  $l \notin \text{hyp2}$

*<proof>*

**lemma** *statement60-one-way:*

**assumes** *is-K2-isometry*  $J$  **and**  $p \in K2$   
**shows** *apply-cltn2*  $p J \in K2$  (**is**  $?p' \in K2$ )

*<proof>*

**lemma** *is-K2-isometry-hyp2-S:*

**assumes**  $p \in \text{hyp2} \cup S$  **and** *is-K2-isometry*  $J$   
**shows** *apply-cltn2*  $p J \in \text{hyp2} \cup S$

*<proof>*

**lemma** *is-K2-isometry-z-non-zero:*

**assumes**  $p \in \text{hyp2} \cup S$  **and** *is-K2-isometry*  $J$   
**shows** *z-non-zero* (*apply-cltn2*  $p J$ )

*<proof>*



**lemma** *cart2-append1-apply-cltn2*:  
**assumes**  $p \in \text{hyp2} \cup S$  **and** *is-K2-isometry*  $J$   
**shows**  $\exists k. k \neq 0$   
 $\wedge \text{cart2-append1 } p \ v* \ \text{cltn2-rep } J = k \ *_R \ \text{cart2-append1 } (\text{apply-cltn2 } p \ J)$   
 $\langle \text{proof} \rangle$

## 8.5 The $K$ -isometries form a group action

**lemma** *hyp2-cltn2-id [simp]*: *hyp2-cltn2*  $p$  *cltn2-id* =  $p$   
 $\langle \text{proof} \rangle$

**lemma** *apply-cltn2-Rep-hyp2*:  
**assumes** *is-K2-isometry*  $J$   
**shows** *apply-cltn2* (*Rep-hyp2*  $p$ )  $J \in \text{hyp2}$   
 $\langle \text{proof} \rangle$

**lemma** *Rep-hyp2-cltn2*:  
**assumes** *is-K2-isometry*  $J$   
**shows** *Rep-hyp2* (*hyp2-cltn2*  $p$   $J$ ) = *apply-cltn2* (*Rep-hyp2*  $p$ )  $J$   
 $\langle \text{proof} \rangle$

**lemma** *hyp2-cltn2-compose*:  
**assumes** *is-K2-isometry*  $H$   
**shows** *hyp2-cltn2* (*hyp2-cltn2*  $p$   $H$ )  $J$  = *hyp2-cltn2*  $p$  (*cltn2-compose*  $H$   $J$ )  
 $\langle \text{proof} \rangle$

**interpretation** *K2-isometry: action*  
(|*carrier* = *Collect is-K2-isometry*, *mult* = *cltn2-compose*, *one* = *cltn2-id*|)  
*hyp2-cltn2*  
 $\langle \text{proof} \rangle$

## 8.6 The Klein–Beltrami model satisfies Tarski’s first three axioms

**lemma** *three-in-S-tangent-intersection-no-3-Col*:  
**assumes**  $p \in S$  **and**  $q \in S$  **and**  $r \in S$   
**and**  $p \neq q$  **and**  $r \notin \{p, q\}$   
**shows** *proj2-no-3-Col* {*proj2-intersection* (*polar*  $p$ ) (*polar*  $q$ ),  $r, p, q$ }  
**(is** *proj2-no-3-Col* { $?s, r, p, q$ })  
 $\langle \text{proof} \rangle$

**lemma** *statement65-special-case*:  
**assumes**  $p \in S$  **and**  $q \in S$  **and**  $r \in S$  **and**  $p \neq q$  **and**  $r \notin \{p, q\}$   
**shows**  $\exists J. \text{is-K2-isometry } J$   
 $\wedge \text{apply-cltn2 east } J = p$   
 $\wedge \text{apply-cltn2 west } J = q$   
 $\wedge \text{apply-cltn2 north } J = r$   
 $\wedge \text{apply-cltn2 far-north } J = \text{proj2-intersection } (\text{polar } p) (\text{polar } q)$

*<proof>*

**lemma** *statement66-existence*:

**assumes**  $a1 \in K2$  **and**  $a2 \in K2$  **and**  $p1 \in S$  **and**  $p2 \in S$

**shows**  $\exists J. \text{is-}K2\text{-isometry } J \wedge \text{apply-cltn2 } a1 \ J = a2 \wedge \text{apply-cltn2 } p1 \ J = p2$

*<proof>*

**lemma** *K2-isometry-swap*:

**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$

**shows**  $\exists J. \text{is-}K2\text{-isometry } J \wedge \text{apply-cltn2 } a \ J = b \wedge \text{apply-cltn2 } b \ J = a$

*<proof>*

**theorem** *hyp2-axiom1*:  $\forall a \ b. a \ b \equiv_K b \ a$

*<proof>*

**theorem** *hyp2-axiom2*:  $\forall a \ b \ p \ q \ r \ s. a \ b \equiv_K p \ q \wedge a \ b \equiv_K r \ s \longrightarrow p \ q \equiv_K r \ s$

*<proof>*

**theorem** *hyp2-axiom3*:  $\forall a \ b \ c. a \ b \equiv_K c \ c \longrightarrow a = b$

*<proof>*

**interpretation** *hyp2*: *tarski-first3 real-hyp2-C*

*<proof>*

## 8.7 Some lemmas about betweenness

**lemma** *S-at-edge*:

**assumes**  $p \in S$  **and**  $q \in \text{hyp2} \cup S$  **and**  $r \in \text{hyp2} \cup S$  **and** *proj2-Col*  $p \ q \ r$

**shows**  $B_{\mathbb{R}} (\text{cart2-pt } p) (\text{cart2-pt } q) (\text{cart2-pt } r)$

$\vee B_{\mathbb{R}} (\text{cart2-pt } p) (\text{cart2-pt } r) (\text{cart2-pt } q)$

(**is**  $B_{\mathbb{R}} \ ?cp \ ?cq \ ?cr \ \vee \ -$ )

*<proof>*

**lemma** *hyp2-in-middle*:

**assumes**  $p \in S$  **and**  $q \in S$  **and**  $r \in \text{hyp2} \cup S$  **and** *proj2-Col*  $p \ q \ r$

**and**  $p \neq q$

**shows**  $B_{\mathbb{R}} (\text{cart2-pt } p) (\text{cart2-pt } r) (\text{cart2-pt } q)$  (**is**  $B_{\mathbb{R}} \ ?cp \ ?cr \ ?cq$ )

*<proof>*

**lemma** *hyp2-incident-in-middle*:

**assumes**  $p \neq q$  **and**  $p \in S$  **and**  $q \in S$  **and**  $a \in \text{hyp2} \cup S$

**and** *proj2-incident*  $p \ l$  **and** *proj2-incident*  $q \ l$  **and** *proj2-incident*  $a \ l$

**shows**  $B_{\mathbb{R}} (\text{cart2-pt } p) (\text{cart2-pt } a) (\text{cart2-pt } q)$

*<proof>*

**lemma** *extend-to-S*:

**assumes**  $p \in \text{hyp2} \cup S$  **and**  $q \in \text{hyp2} \cup S$

**shows**  $\exists r \in S. B_{\mathbb{R}} (\text{cart2-pt } p) (\text{cart2-pt } q) (\text{cart2-pt } r)$

(**is**  $\exists r \in S. B_{\mathbb{R}} \ ?cp \ ?cq (\text{cart2-pt } r)$ )

*<proof>*

**definition** *endpoint-in-S* :: *proj2*  $\Rightarrow$  *proj2*  $\Rightarrow$  *proj2* **where**  
*endpoint-in-S* *a b*  
 $\triangleq \epsilon p. p \in S \wedge B_{\mathbb{R}} (\text{cart2-pt } a) (\text{cart2-pt } b) (\text{cart2-pt } p)$

**lemma** *endpoint-in-S*:

**assumes** *a*  $\in$  *hyp2*  $\cup$  *S* **and** *b*  $\in$  *hyp2*  $\cup$  *S*  
**shows** *endpoint-in-S* *a b*  $\in$  *S* (**is** *?p*  $\in$  *S*)  
**and**  $B_{\mathbb{R}} (\text{cart2-pt } a) (\text{cart2-pt } b) (\text{cart2-pt } (\text{endpoint-in-S } a b))$   
(**is**  $B_{\mathbb{R}}$  *?ca* *?cb* *?cp*)

*<proof>*

**lemma** *endpoint-in-S-swap*:

**assumes** *a*  $\neq$  *b* **and** *a*  $\in$  *hyp2*  $\cup$  *S* **and** *b*  $\in$  *hyp2*  $\cup$  *S*  
**shows** *endpoint-in-S* *a b*  $\neq$  *endpoint-in-S* *b a* (**is** *?p*  $\neq$  *?q*)

*<proof>*

**lemma** *endpoint-in-S-incident*:

**assumes** *a*  $\neq$  *b* **and** *a*  $\in$  *hyp2*  $\cup$  *S* **and** *b*  $\in$  *hyp2*  $\cup$  *S*  
**and** *proj2-incident* *a l* **and** *proj2-incident* *b l*  
**shows** *proj2-incident* (*endpoint-in-S* *a b*) *l* (**is** *proj2-incident* *?p l*)

*<proof>*

**lemma** *endpoints-in-S-incident-unique*:

**assumes** *a*  $\neq$  *b* **and** *a*  $\in$  *hyp2*  $\cup$  *S* **and** *b*  $\in$  *hyp2*  $\cup$  *S* **and** *p*  $\in$  *S*  
**and** *proj2-incident* *a l* **and** *proj2-incident* *b l* **and** *proj2-incident* *p l*  
**shows** *p* = *endpoint-in-S* *a b*  $\vee$  *p* = *endpoint-in-S* *b a*  
(**is** *p* = *?q*  $\vee$  *p* = *?r*)

*<proof>*

**lemma** *endpoint-in-S-unique*:

**assumes** *a*  $\neq$  *b* **and** *a*  $\in$  *hyp2*  $\cup$  *S* **and** *b*  $\in$  *hyp2*  $\cup$  *S* **and** *p*  $\in$  *S*  
**and**  $B_{\mathbb{R}} (\text{cart2-pt } a) (\text{cart2-pt } b) (\text{cart2-pt } p)$  (**is**  $B_{\mathbb{R}}$  *?ca* *?cb* *?cp*)  
**shows** *p* = *endpoint-in-S* *a b* (**is** *p* = *?q*)

*<proof>*

**lemma** *between-hyp2-S*:

**assumes** *p*  $\in$  *hyp2*  $\cup$  *S* **and** *r*  $\in$  *hyp2*  $\cup$  *S* **and** *k*  $\geq$  0 **and** *k*  $\leq$  1  
**shows** *proj2-pt* (*k*  $*_{\mathbb{R}}$  (*cart2-pt* *r*) + (1 - *k*)  $*_{\mathbb{R}}$  (*cart2-pt* *p*))  $\in$  *hyp2*  $\cup$  *S*  
(**is** *proj2-pt* *?cq*  $\in$  -)

*<proof>*

## 8.8 The Klein–Beltrami model satisfies axiom 4

**definition** *expansion-factor* :: *proj2*  $\Rightarrow$  *cltn2*  $\Rightarrow$  *real* **where**  
*expansion-factor* *p J*  $\triangleq$  (*cart2-append1* *p v*  $*_{\mathbb{R}}$  (*cltn2-rep* *J*))\$3

**lemma** *expansion-factor*:

**assumes**  $p \in \text{hyp2} \cup S$  **and** *is-K2-isometry*  $J$   
**shows** *expansion-factor*  $p J \neq 0$   
**and** *cart2-append1*  $p v * \text{cltn2-rep } J$   
 $= \text{expansion-factor } p J *_R \text{cart2-append1 } (\text{apply-cltn2 } p J)$   
 <proof>

**lemma** *expansion-factor-linear-apply-cltn2*:  
**assumes**  $p \in \text{hyp2} \cup S$  **and**  $q \in \text{hyp2} \cup S$  **and**  $r \in \text{hyp2} \cup S$   
**and** *is-K2-isometry*  $J$   
**and** *cart2-pt*  $r = k *_R \text{cart2-pt } p + (1 - k) *_R \text{cart2-pt } q$   
**shows** *expansion-factor*  $r J *_R \text{cart2-append1 } (\text{apply-cltn2 } r J)$   
 $= (k * \text{expansion-factor } p J) *_R \text{cart2-append1 } (\text{apply-cltn2 } p J)$   
 $+ ((1 - k) * \text{expansion-factor } q J) *_R \text{cart2-append1 } (\text{apply-cltn2 } q J)$   
 (is  $?er *_R - = (k * ?ep) *_R - + ((1 - k) * ?eq) *_R -$ )  
 <proof>

**lemma** *expansion-factor-linear*:  
**assumes**  $p \in \text{hyp2} \cup S$  **and**  $q \in \text{hyp2} \cup S$  **and**  $r \in \text{hyp2} \cup S$   
**and** *is-K2-isometry*  $J$   
**and** *cart2-pt*  $r = k *_R \text{cart2-pt } p + (1 - k) *_R \text{cart2-pt } q$   
**shows** *expansion-factor*  $r J$   
 $= k * \text{expansion-factor } p J + (1 - k) * \text{expansion-factor } q J$   
 (is  $?er = k * ?ep + (1 - k) * ?eq$ )  
 <proof>

**lemma** *expansion-factor-sgn-invariant*:  
**assumes**  $p \in \text{hyp2} \cup S$  **and**  $q \in \text{hyp2} \cup S$  **and** *is-K2-isometry*  $J$   
**shows** *sgn* (*expansion-factor*  $p J$ ) = *sgn* (*expansion-factor*  $q J$ )  
 (is *sgn*  $?ep = \text{sgn } ?eq$ )  
 <proof>

**lemma** *statement-63*:  
**assumes**  $p \in \text{hyp2} \cup S$  **and**  $q \in \text{hyp2} \cup S$  **and**  $r \in \text{hyp2} \cup S$   
**and** *is-K2-isometry*  $J$  **and**  $B_{\mathbb{R}}$  (*cart2-pt*  $p$ ) (*cart2-pt*  $q$ ) (*cart2-pt*  $r$ )  
**shows**  $B_{\mathbb{R}}$   
 (*cart2-pt* (*apply-cltn2*  $p J$ ))  
 (*cart2-pt* (*apply-cltn2*  $q J$ ))  
 (*cart2-pt* (*apply-cltn2*  $r J$ ))  
 <proof>

**theorem** *hyp2-axiom4*:  $\forall q a b c. \exists x. B_K q a x \wedge a x \equiv_K b c$   
 <proof>

## 8.9 More betweenness theorems

**lemma** *hyp2-S-points-fix-line*:  
**assumes**  $a \in \text{hyp2}$  **and**  $p \in S$  **and** *is-K2-isometry*  $J$   
**and** *apply-cltn2*  $a J = a$  (is  $?aJ = a$ )  
**and** *apply-cltn2*  $p J = p$  (is  $?pJ = p$ )

**and** *proj2-incident*  $a$   $l$  **and** *proj2-incident*  $p$   $l$  **and** *proj2-incident*  $b$   $l$   
**shows** *apply-cltn2*  $b$   $J = b$  (**is**  $?bJ = b$ )  
 $\langle$ *proof* $\rangle$

**lemma** *K2-isometry-endpoint-in-S*:  
**assumes**  $a \neq b$  **and**  $a \in \text{hyp2} \cup S$  **and**  $b \in \text{hyp2} \cup S$  **and** *is-K2-isometry*  $J$   
**shows** *apply-cltn2* (*endpoint-in-S*  $a$   $b$ )  $J$   
 $=$  *endpoint-in-S* (*apply-cltn2*  $a$   $J$ ) (*apply-cltn2*  $b$   $J$ )  
(**is**  $?pJ = \text{endpoint-in-S } ?aJ ?bJ$ )  
 $\langle$ *proof* $\rangle$

**lemma** *between-endpoint-in-S*:  
**assumes**  $a \neq b$  **and**  $b \neq c$   
**and**  $a \in \text{hyp2} \cup S$  **and**  $b \in \text{hyp2} \cup S$  **and**  $c \in \text{hyp2} \cup S$   
**and**  $B_{\mathbb{R}}$  (*cart2-pt*  $a$ ) (*cart2-pt*  $b$ ) (*cart2-pt*  $c$ ) (**is**  $B_{\mathbb{R}}$   $?ca$   $?cb$   $?cc$ )  
**shows** *endpoint-in-S*  $a$   $b = \text{endpoint-in-S } b$   $c$  (**is**  $?p = ?q$ )  
 $\langle$ *proof* $\rangle$

**lemma** *hyp2-extend-segment-unique*:  
**assumes**  $a \neq b$  **and**  $B_K$   $a$   $b$   $c$  **and**  $B_K$   $a$   $b$   $d$  **and**  $b$   $c \equiv_K$   $b$   $d$   
**shows**  $c = d$   
 $\langle$ *proof* $\rangle$

**lemma** *line-S-match-intersections*:  
**assumes**  $p \neq q$  **and**  $r \neq s$  **and**  $p \in S$  **and**  $q \in S$  **and**  $r \in S$  **and**  $s \in S$   
**and** *proj2-set-Col*  $\{p, q, r, s\}$   
**shows**  $(p = r \wedge q = s) \vee (q = r \wedge p = s)$   
 $\langle$ *proof* $\rangle$

**definition** *are-endpoints-in-S* ::  $[\text{proj2}, \text{proj2}, \text{proj2}, \text{proj2}] \Rightarrow \text{bool}$  **where**  
*are-endpoints-in-S*  $p$   $q$   $a$   $b$   
 $\triangleq p \neq q \wedge p \in S \wedge q \in S \wedge a \in \text{hyp2} \wedge b \in \text{hyp2} \wedge \text{proj2-set-Col } \{p, q, a, b\}$

**lemma** *are-endpoints-in-S'*:  
**assumes**  $p \neq q$  **and**  $a \neq b$  **and**  $p \in S$  **and**  $q \in S$  **and**  $a \in \text{hyp2} \cup S$   
**and**  $b \in \text{hyp2} \cup S$  **and** *proj2-set-Col*  $\{p, q, a, b\}$   
**shows**  $(p = \text{endpoint-in-S } a$   $b \wedge q = \text{endpoint-in-S } b$   $a)$   
 $\vee (q = \text{endpoint-in-S } a$   $b \wedge p = \text{endpoint-in-S } b$   $a)$   
(**is**  $(p = ?r \wedge q = ?s) \vee (q = ?r \wedge p = ?s)$ )  
 $\langle$ *proof* $\rangle$

**lemma** *are-endpoints-in-S*:  
**assumes**  $a \neq b$  **and** *are-endpoints-in-S*  $p$   $q$   $a$   $b$   
**shows**  $(p = \text{endpoint-in-S } a$   $b \wedge q = \text{endpoint-in-S } b$   $a)$   
 $\vee (q = \text{endpoint-in-S } a$   $b \wedge p = \text{endpoint-in-S } b$   $a)$   
 $\langle$ *proof* $\rangle$

**lemma** *S-intersections-endpoints-in-S*:  
**assumes**  $a \neq 0$  **and**  $b \neq 0$  **and** *proj2-abs*  $a \neq \text{proj2-abs } b$  (**is**  $?pa \neq ?pb$ )

**and**  $\text{proj2-abs } a \in \text{hyp2}$  **and**  $\text{proj2-abs } b \in \text{hyp2} \cup S$   
**shows**  $(S\text{-intersection1 } a \ b = \text{endpoint-in-S } ?pa \ ?pb$   
 $\wedge S\text{-intersection2 } a \ b = \text{endpoint-in-S } ?pb \ ?pa)$   
 $\vee (S\text{-intersection2 } a \ b = \text{endpoint-in-S } ?pa \ ?pb$   
 $\wedge S\text{-intersection1 } a \ b = \text{endpoint-in-S } ?pb \ ?pa)$   
**(is**  $(?pp = ?pr \wedge ?pq = ?ps) \vee (?pq = ?pr \wedge ?pp = ?ps))$   
 $\langle \text{proof} \rangle$

**lemma** *between-endpoints-in-S*:  
**assumes**  $a \neq b$  **and**  $a \in \text{hyp2} \cup S$  **and**  $b \in \text{hyp2} \cup S$   
**shows**  $B_{\mathbb{R}}$   
 $(\text{cart2-pt } (\text{endpoint-in-S } a \ b)) (\text{cart2-pt } a) (\text{cart2-pt } (\text{endpoint-in-S } b \ a))$   
**(is**  $B_{\mathbb{R}} \ ?cp \ ?ca \ ?cq)$   
 $\langle \text{proof} \rangle$

**lemma** *S-hyp2-S-cart2-append1*:  
**assumes**  $p \neq q$  **and**  $p \in S$  **and**  $q \in S$  **and**  $a \in \text{hyp2}$   
**and**  $\text{proj2-incident } p \ l$  **and**  $\text{proj2-incident } q \ l$  **and**  $\text{proj2-incident } a \ l$   
**shows**  $\exists k. k > 0 \wedge k < 1$   
 $\wedge \text{cart2-append1 } a = k *_{\mathbb{R}} \text{cart2-append1 } q + (1 - k) *_{\mathbb{R}} \text{cart2-append1 } p$   
 $\langle \text{proof} \rangle$

**lemma** *are-endpoints-in-S-swap-34*:  
**assumes**  $\text{are-endpoints-in-S } p \ q \ a \ b$   
**shows**  $\text{are-endpoints-in-S } p \ q \ b \ a$   
 $\langle \text{proof} \rangle$

**lemma** *proj2-set-Col-endpoints-in-S*:  
**assumes**  $a \neq b$  **and**  $a \in \text{hyp2} \cup S$  **and**  $b \in \text{hyp2} \cup S$   
**shows**  $\text{proj2-set-Col } \{\text{endpoint-in-S } a \ b, \text{endpoint-in-S } b \ a, a, b\}$   
**(is**  $\text{proj2-set-Col } \{?p, ?q, a, b\}$   
 $\langle \text{proof} \rangle$

**lemma** *endpoints-in-S-are-endpoints-in-S*:  
**assumes**  $a \neq b$  **and**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$   
**shows**  $\text{are-endpoints-in-S } (\text{endpoint-in-S } a \ b) (\text{endpoint-in-S } b \ a) \ a \ b$   
**(is**  $\text{are-endpoints-in-S } ?p \ ?q \ a \ b)$   
 $\langle \text{proof} \rangle$

**lemma** *endpoint-in-S-S-hyp2-distinct*:  
**assumes**  $p \in S$  **and**  $a \in \text{hyp2} \cup S$  **and**  $p \neq a$   
**shows**  $\text{endpoint-in-S } p \ a \neq p$   
 $\langle \text{proof} \rangle$

**lemma** *endpoint-in-S-S-strict-hyp2-distinct*:  
**assumes**  $p \in S$  **and**  $a \in \text{hyp2}$   
**shows**  $\text{endpoint-in-S } p \ a \neq p$   
 $\langle \text{proof} \rangle$

**lemma** *end-and-opposite-are-endpoints-in-S*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $p \in S$   
**and** *proj2-incident*  $a$   $l$  **and** *proj2-incident*  $b$   $l$  **and** *proj2-incident*  $p$   $l$   
**shows** *are-endpoints-in-S*  $p$  (*endpoint-in-S*  $p$   $b$ )  $a$   $b$   
*(is are-endpoints-in-S*  $p$   $?q$   $a$   $b$ )  
 $\langle \text{proof} \rangle$

**lemma** *real-hyp2-B-hyp2-cltn2*:  
**assumes** *is-K2-isometry*  $J$  **and**  $B_K$   $a$   $b$   $c$   
**shows**  $B_K$  (*hyp2-cltn2*  $a$   $J$ ) (*hyp2-cltn2*  $b$   $J$ ) (*hyp2-cltn2*  $c$   $J$ )  
*(is*  $B_K$   $?aJ$   $?bJ$   $?cJ$ )  
 $\langle \text{proof} \rangle$

**lemma** *real-hyp2-C-hyp2-cltn2*:  
**assumes** *is-K2-isometry*  $J$   
**shows**  $a$   $b \equiv_K$  (*hyp2-cltn2*  $a$   $J$ ) (*hyp2-cltn2*  $b$   $J$ ) **(is**  $a$   $b \equiv_K$   $?aJ$   $?bJ$ )  
 $\langle \text{proof} \rangle$

## 8.10 Perpendicularity

**definition** *M-perp* :: *proj2-line*  $\Rightarrow$  *proj2-line*  $\Rightarrow$  *bool* **where**  
*M-perp*  $l$   $m \triangleq$  *proj2-incident* (*pole*  $l$ )  $m$

**lemma** *M-perp-sym*:  
**assumes** *M-perp*  $l$   $m$   
**shows** *M-perp*  $m$   $l$   
 $\langle \text{proof} \rangle$

**lemma** *M-perp-to-compass*:  
**assumes** *M-perp*  $l$   $m$  **and**  $a \in \text{hyp2}$  **and** *proj2-incident*  $a$   $l$   
**and**  $b \in \text{hyp2}$  **and** *proj2-incident*  $b$   $m$   
**shows**  $\exists J. \text{is-K2-isometry } J$   
 $\wedge$  *apply-cltn2-line equator*  $J = l \wedge$  *apply-cltn2-line meridian*  $J = m$   
 $\langle \text{proof} \rangle$

**definition** *drop-perp* :: *proj2*  $\Rightarrow$  *proj2-line*  $\Rightarrow$  *proj2-line* **where**  
*drop-perp*  $p$   $l \triangleq$  *proj2-line-through*  $p$  (*pole*  $l$ )

**lemma** *drop-perp-incident*: *proj2-incident*  $p$  (*drop-perp*  $p$   $l$ )  
 $\langle \text{proof} \rangle$

**lemma** *drop-perp-perp*: *M-perp*  $l$  (*drop-perp*  $p$   $l$ )  
 $\langle \text{proof} \rangle$

**definition** *perp-foot* :: *proj2*  $\Rightarrow$  *proj2-line*  $\Rightarrow$  *proj2* **where**  
*perp-foot*  $p$   $l \triangleq$  *proj2-intersection*  $l$  (*drop-perp*  $p$   $l$ )

**lemma** *perp-foot-incident*:  
**shows** *proj2-incident* (*perp-foot*  $p$   $l$ )  $l$

**and** *proj2-incident* (*perp-foot* *p l*) (*drop-perp* *p l*)  
<proof>

**lemma** *M-perp-hyp2*:

**assumes** *M-perp* *l m* **and** *a*  $\in$  *hyp2* **and** *proj2-incident* *a l* **and** *b*  $\in$  *hyp2*  
**and** *proj2-incident* *b m* **and** *proj2-incident* *c l* **and** *proj2-incident* *c m*  
**shows** *c*  $\in$  *hyp2*

<proof>

**lemma** *perp-foot-hyp2*:

**assumes** *a*  $\in$  *hyp2* **and** *proj2-incident* *a l* **and** *b*  $\in$  *hyp2*  
**shows** *perp-foot* *b l*  $\in$  *hyp2*

<proof>

**definition** *perp-up* :: *proj2*  $\Rightarrow$  *proj2-line*  $\Rightarrow$  *proj2* **where**

*perp-up* *a l*

$\triangleq$  if *proj2-incident* *a l* then  $\epsilon$  *p*. *p*  $\in$  *S*  $\wedge$  *proj2-incident* *p* (*drop-perp* *a l*)  
else *endpoint-in-S* (*perp-foot* *a l*) *a*

**lemma** *perp-up-degenerate-in-S-incident*:

**assumes** *a*  $\in$  *hyp2* **and** *proj2-incident* *a l*  
**shows** *perp-up* *a l*  $\in$  *S* (**is** ?*p*  $\in$  *S*)  
**and** *proj2-incident* (*perp-up* *a l*) (*drop-perp* *a l*)

<proof>

**lemma** *perp-up-non-degenerate-in-S-at-end*:

**assumes** *a*  $\in$  *hyp2* **and** *b*  $\in$  *hyp2* **and** *proj2-incident* *b l*  
**and**  $\neg$  *proj2-incident* *a l*  
**shows** *perp-up* *a l*  $\in$  *S*  
**and**  $B_{\mathbb{R}}$  (*cart2-pt* (*perp-foot* *a l*)) (*cart2-pt* *a*) (*cart2-pt* (*perp-up* *a l*))

<proof>

**lemma** *perp-up-in-S*:

**assumes** *a*  $\in$  *hyp2* **and** *b*  $\in$  *hyp2* **and** *proj2-incident* *b l*  
**shows** *perp-up* *a l*  $\in$  *S*

<proof>

**lemma** *perp-up-incident*:

**assumes** *a*  $\in$  *hyp2* **and** *b*  $\in$  *hyp2* **and** *proj2-incident* *b l*  
**shows** *proj2-incident* (*perp-up* *a l*) (*drop-perp* *a l*)  
(**is** *proj2-incident* ?*p* ?*m*)

<proof>

**lemma** *drop-perp-same-line-pole-in-S*:

**assumes** *drop-perp* *p l* = *l*  
**shows** *pole* *l*  $\in$  *S*

<proof>

**lemma** *hyp2-drop-perp-not-same-line*:



**assumes**  $a \in \text{hyp2}$   
**shows**  $\text{drop-perp } a \ l \neq l$   
 <proof>

**lemma** *hyp2-incident-perp-foot-same-point*:  
**assumes**  $a \in \text{hyp2}$  **and**  $\text{proj2-incident } a \ l$   
**shows**  $\text{perp-foot } a \ l = a$   
 <proof>

**lemma** *perp-up-at-end*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $\text{proj2-incident } b \ l$   
**shows**  $B_{\mathbb{R}} (\text{cart2-pt } (\text{perp-foot } a \ l)) (\text{cart2-pt } a) (\text{cart2-pt } (\text{perp-up } a \ l))$   
 <proof>

**definition** *perp-down* ::  $\text{proj2} \Rightarrow \text{proj2-line} \Rightarrow \text{proj2}$  **where**  
 $\text{perp-down } a \ l \triangleq \text{endpoint-in-S } (\text{perp-up } a \ l) \ a$

**lemma** *perp-down-in-S*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $\text{proj2-incident } b \ l$   
**shows**  $\text{perp-down } a \ l \in S$   
 <proof>

**lemma** *perp-down-incident*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $\text{proj2-incident } b \ l$   
**shows**  $\text{proj2-incident } (\text{perp-down } a \ l) (\text{drop-perp } a \ l)$   
 <proof>

**lemma** *perp-up-down-distinct*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $\text{proj2-incident } b \ l$   
**shows**  $\text{perp-up } a \ l \neq \text{perp-down } a \ l$   
 <proof>

**lemma** *perp-up-down-foot-are-endpoints-in-S*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $\text{proj2-incident } b \ l$   
**shows**  $\text{are-endpoints-in-S } (\text{perp-up } a \ l) (\text{perp-down } a \ l) (\text{perp-foot } a \ l) \ a$   
 <proof>

**lemma** *perp-foot-opposite-endpoint-in-S*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $c \in \text{hyp2}$  **and**  $a \neq b$   
**shows**  
 $\text{endpoint-in-S } (\text{endpoint-in-S } a \ b) (\text{perp-foot } c (\text{proj2-line-through } a \ b))$   
 $= \text{endpoint-in-S } b \ a$   
 (is  $\text{endpoint-in-S } ?p \ ?d = \text{endpoint-in-S } b \ a$ )  
 <proof>

**lemma** *endpoints-in-S-perp-foot-are-endpoints-in-S*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $c \in \text{hyp2}$  **and**  $a \neq b$   
**and**  $\text{proj2-incident } a \ l$  **and**  $\text{proj2-incident } b \ l$   
**shows**  $\text{are-endpoints-in-S}$

(*endpoint-in-S a b*) (*endpoint-in-S b a*) *a* (*perp-foot c l*)  
 ⟨*proof*⟩

**definition** *right-angle* :: *proj2* ⇒ *proj2* ⇒ *proj2* ⇒ *bool* **where**  
*right-angle p a q*  
 ≙  $p \in S \wedge q \in S \wedge a \in \text{hyp2}$   
 ∧ *M-perp (proj2-line-through p a) (proj2-line-through a q)*

**lemma** *perp-foot-up-right-angle*:  
**assumes**  $p \in S$  **and**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and** *proj2-incident p l*  
**and** *proj2-incident b l*  
**shows** *right-angle p (perp-foot a l) (perp-up a l)*  
 ⟨*proof*⟩

**lemma** *M-perp-unique*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and** *proj2-incident a l*  
**and** *proj2-incident b m* **and** *proj2-incident b n* **and** *M-perp l m*  
**and** *M-perp l n*  
**shows**  $m = n$   
 ⟨*proof*⟩

**lemma** *perp-foot-eq-implies-drop-perp-eq*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and** *proj2-incident a l*  
**and** *perp-foot b l = perp-foot c l*  
**shows** *drop-perp b l = drop-perp c l*  
 ⟨*proof*⟩

**lemma** *right-angle-to-compass*:  
**assumes** *right-angle p a q*  
**shows**  $\exists J. \text{is-K2-isometry } J \wedge \text{apply-cltn2 } p \ J = \text{east}$   
 ∧  $\text{apply-cltn2 } a \ J = \text{K2-centre} \wedge \text{apply-cltn2 } q \ J = \text{north}$   
 ⟨*proof*⟩

**lemma** *right-angle-to-right-angle*:  
**assumes** *right-angle p a q* **and** *right-angle r b s*  
**shows**  $\exists J. \text{is-K2-isometry } J$   
 ∧  $\text{apply-cltn2 } p \ J = r \wedge \text{apply-cltn2 } a \ J = b \wedge \text{apply-cltn2 } q \ J = s$   
 ⟨*proof*⟩

## 8.11 Functions of distance

**definition** *exp-2dist* :: *proj2* ⇒ *proj2* ⇒ *real* **where**  
*exp-2dist a b*  
 ≙ if  $a = b$   
 then 1  
 else *cross-ratio (endpoint-in-S a b) (endpoint-in-S b a) a b*

**definition** *cosh-dist* :: *proj2* ⇒ *proj2* ⇒ *real* **where**  
*cosh-dist a b* ≙  $(\text{sqrt } (\text{exp-2dist } a \ b) + \text{sqrt } (1 / (\text{exp-2dist } a \ b))) / 2$

**lemma** *exp-2dist-formula*:

**assumes**  $a \neq 0$  **and**  $b \neq 0$  **and**  $\text{proj2-abs } a \in \text{hyp2}$  (**is**  $?pa \in \text{hyp2}$ )  
**and**  $\text{proj2-abs } b \in \text{hyp2}$  (**is**  $?pb \in \text{hyp2}$ )  
**shows**  $\text{exp-2dist } (\text{proj2-abs } a) (\text{proj2-abs } b)$   
 $= (a \cdot (M *v b) + \text{sqrt } (\text{quarter-discrim } a b))$   
 $\quad / (a \cdot (M *v b) - \text{sqrt } (\text{quarter-discrim } a b))$   
 $\vee \text{exp-2dist } (\text{proj2-abs } a) (\text{proj2-abs } b)$   
 $= (a \cdot (M *v b) - \text{sqrt } (\text{quarter-discrim } a b))$   
 $\quad / (a \cdot (M *v b) + \text{sqrt } (\text{quarter-discrim } a b))$   
**(is**  $?e2d = (?aMb + ?sqd) / (?aMb - ?sqd)$   
 $\vee ?e2d = (?aMb - ?sqd) / (?aMb + ?sqd)$ )  
 $\langle \text{proof} \rangle$

**lemma** *cosh-dist-formula*:

**assumes**  $a \neq 0$  **and**  $b \neq 0$  **and**  $\text{proj2-abs } a \in \text{hyp2}$  (**is**  $?pa \in \text{hyp2}$ )  
**and**  $\text{proj2-abs } b \in \text{hyp2}$  (**is**  $?pb \in \text{hyp2}$ )  
**shows**  $\text{cosh-dist } (\text{proj2-abs } a) (\text{proj2-abs } b)$   
 $= |a \cdot (M *v b)| / \text{sqrt } (a \cdot (M *v a) * (b \cdot (M *v b)))$   
**(is**  $\text{cosh-dist } ?pa ?pb = |?aMb| / \text{sqrt } (?aMa * ?bMb)$ )  
 $\langle \text{proof} \rangle$

**lemma** *cosh-dist-perp-special-case*:

**assumes**  $|x| < 1$  **and**  $|y| < 1$   
**shows**  $\text{cosh-dist } (\text{proj2-abs } (\text{vector } [x,0,1])) (\text{proj2-abs } (\text{vector } [0,y,1]))$   
 $= (\text{cosh-dist } K2\text{-centre } (\text{proj2-abs } (\text{vector } [x,0,1])))$   
 $\quad * (\text{cosh-dist } K2\text{-centre } (\text{proj2-abs } (\text{vector } [0,y,1])))$   
**(is**  $\text{cosh-dist } ?pa ?pb = (\text{cosh-dist } ?po ?pa) * (\text{cosh-dist } ?po ?pb)$ )  
 $\langle \text{proof} \rangle$

**lemma** *K2-isometry-cross-ratio-endpoints-in-S*:

**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $\text{is-K2-isometry } J$  **and**  $a \neq b$   
**shows**  $\text{cross-ratio } (\text{apply-cltn2 } (\text{endpoint-in-S } a b) J)$   
 $(\text{apply-cltn2 } (\text{endpoint-in-S } b a) J) (\text{apply-cltn2 } a J) (\text{apply-cltn2 } b J)$   
 $= \text{cross-ratio } (\text{endpoint-in-S } a b) (\text{endpoint-in-S } b a) a b$   
**(is**  $\text{cross-ratio } ?pJ ?qJ ?aJ ?bJ = \text{cross-ratio } ?p ?q a b$ )  
 $\langle \text{proof} \rangle$

**lemma** *K2-isometry-exp-2dist*:

**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $\text{is-K2-isometry } J$   
**shows**  $\text{exp-2dist } (\text{apply-cltn2 } a J) (\text{apply-cltn2 } b J) = \text{exp-2dist } a b$   
**(is**  $\text{exp-2dist } ?aJ ?bJ = -$ )  
 $\langle \text{proof} \rangle$

**lemma** *K2-isometry-cosh-dist*:

**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $\text{is-K2-isometry } J$   
**shows**  $\text{cosh-dist } (\text{apply-cltn2 } a J) (\text{apply-cltn2 } b J) = \text{cosh-dist } a b$   
 $\langle \text{proof} \rangle$

**lemma** *cosh-dist-perp*:

**assumes**  $M\text{-perp } l m$  **and**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $c \in \text{hyp2}$   
**and**  $\text{proj2-incident } a l$  **and**  $\text{proj2-incident } b l$   
**and**  $\text{proj2-incident } b m$  **and**  $\text{proj2-incident } c m$   
**shows**  $\text{cosh-dist } a c = \text{cosh-dist } b a * \text{cosh-dist } b c$

$\langle \text{proof} \rangle$

**lemma** *are-endpoints-in-S-ordered-cross-ratio*:

**assumes**  $\text{are-endpoints-in-S } p q a b$   
**and**  $B_{\mathbb{R}} (\text{cart2-pt } a) (\text{cart2-pt } b) (\text{cart2-pt } p)$  **(is**  $B_{\mathbb{R}} ?ca ?cb ?cp)$   
**shows**  $\text{cross-ratio } p q a b \geq 1$

$\langle \text{proof} \rangle$

**lemma** *cross-ratio-S-S-hyp2-hyp2-positive*:

**assumes**  $\text{are-endpoints-in-S } p q a b$   
**shows**  $\text{cross-ratio } p q a b > 0$

$\langle \text{proof} \rangle$

**lemma** *cosh-dist-general*:

**assumes**  $\text{are-endpoints-in-S } p q a b$   
**shows**  $\text{cosh-dist } a b$   
 $= (\text{sqrt } (\text{cross-ratio } p q a b) + 1 / \text{sqrt } (\text{cross-ratio } p q a b)) / 2$

$\langle \text{proof} \rangle$

**lemma** *exp-2dist-positive*:

**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$   
**shows**  $\text{exp-2dist } a b > 0$

$\langle \text{proof} \rangle$

**lemma** *cosh-dist-at-least-1*:

**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$   
**shows**  $\text{cosh-dist } a b \geq 1$

$\langle \text{proof} \rangle$

**lemma** *cosh-dist-positive*:

**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$   
**shows**  $\text{cosh-dist } a b > 0$

$\langle \text{proof} \rangle$

**lemma** *cosh-dist-perp-divide*:

**assumes**  $M\text{-perp } l m$  **and**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $c \in \text{hyp2}$   
**and**  $\text{proj2-incident } a l$  **and**  $\text{proj2-incident } b l$  **and**  $\text{proj2-incident } b m$   
**and**  $\text{proj2-incident } c m$   
**shows**  $\text{cosh-dist } b c = \text{cosh-dist } a c / \text{cosh-dist } b a$

$\langle \text{proof} \rangle$

**lemma** *real-hyp2-C-cross-ratio-endpoints-in-S*:

**assumes**  $a \neq b$  **and**  $a b \equiv_K c d$   
**shows**  $\text{cross-ratio } (\text{endpoint-in-S } (\text{Rep-hyp2 } a) (\text{Rep-hyp2 } b))$

$(\text{endpoint-in-}S \text{ (Rep-hyp2 } b) \text{ (Rep-hyp2 } a)) \text{ (Rep-hyp2 } a) \text{ (Rep-hyp2 } b)$   
 $= \text{cross-ratio } (\text{endpoint-in-}S \text{ (Rep-hyp2 } c) \text{ (Rep-hyp2 } d))$   
 $(\text{endpoint-in-}S \text{ (Rep-hyp2 } d) \text{ (Rep-hyp2 } c)) \text{ (Rep-hyp2 } c) \text{ (Rep-hyp2 } d)$   
 $(\text{is cross-ratio } ?p \ ?q \ ?a' \ ?b' = \text{cross-ratio } ?r \ ?s \ ?c' \ ?d')$   
 <proof>

**lemma** *real-hyp2-C-exp-2dist*:  
**assumes**  $a \ b \equiv_K \ c \ d$   
**shows**  $\text{exp-2dist } (\text{Rep-hyp2 } a) \text{ (Rep-hyp2 } b)$   
 $= \text{exp-2dist } (\text{Rep-hyp2 } c) \text{ (Rep-hyp2 } d)$   
 $(\text{is exp-2dist } ?a' \ ?b' = \text{exp-2dist } ?c' \ ?d')$   
 <proof>

**lemma** *real-hyp2-C-cosh-dist*:  
**assumes**  $a \ b \equiv_K \ c \ d$   
**shows**  $\text{cosh-dist } (\text{Rep-hyp2 } a) \text{ (Rep-hyp2 } b)$   
 $= \text{cosh-dist } (\text{Rep-hyp2 } c) \text{ (Rep-hyp2 } d)$   
 <proof>

**lemma** *cross-ratio-in-terms-of-cosh-dist*:  
**assumes**  $\text{are-endpoints-in-}S \ p \ q \ a \ b$   
**and**  $B_{\mathbb{R}} \text{ (cart2-pt } a) \text{ (cart2-pt } b) \text{ (cart2-pt } p)$   
**shows**  $\text{cross-ratio } p \ q \ a \ b$   
 $= 2 * (\text{cosh-dist } a \ b)^2 + 2 * \text{cosh-dist } a \ b * \text{sqrt } ((\text{cosh-dist } a \ b)^2 - 1) - 1$   
 $(\text{is } ?pqab = 2 * ?ab^2 + 2 * ?ab * \text{sqrt } (?ab^2 - 1) - 1)$   
 <proof>

**lemma** *are-endpoints-in-S-cross-ratio-correct*:  
**assumes**  $\text{are-endpoints-in-}S \ p \ q \ a \ b$   
**shows**  $\text{cross-ratio-correct } p \ q \ a \ b$   
 <proof>

**lemma** *endpoints-in-S-cross-ratio-correct*:  
**assumes**  $a \neq b$  **and**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$   
**shows**  $\text{cross-ratio-correct } (\text{endpoint-in-}S \ a \ b) \text{ (endpoint-in-}S \ b \ a) \ a \ b$   
 <proof>

**lemma** *endpoints-in-S-perp-foot-cross-ratio-correct*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $c \in \text{hyp2}$  **and**  $a \neq b$   
**and**  $\text{proj2-incident } a \ l$  **and**  $\text{proj2-incident } b \ l$   
**shows**  $\text{cross-ratio-correct}$   
 $(\text{endpoint-in-}S \ a \ b) \text{ (endpoint-in-}S \ b \ a) \ a \text{ (perp-foot } c \ l)$   
 $(\text{is cross-ratio-correct } ?p \ ?q \ a \ ?d)$   
 <proof>

**lemma** *cosh-dist-unique*:  
**assumes**  $a \in \text{hyp2}$  **and**  $b \in \text{hyp2}$  **and**  $c \in \text{hyp2}$  **and**  $p \in S$   
**and**  $B_{\mathbb{R}} \text{ (cart2-pt } a) \text{ (cart2-pt } b) \text{ (cart2-pt } p)$  **(is**  $B_{\mathbb{R}} \ ?ca \ ?cb \ ?cp)$   
**and**  $B_{\mathbb{R}} \text{ (cart2-pt } a) \text{ (cart2-pt } c) \text{ (cart2-pt } p)$  **(is**  $B_{\mathbb{R}} \ ?ca \ ?cc \ ?cp)$

and  $\text{cosh-dist } a \ b = \text{cosh-dist } a \ c$  (is  $?ab = ?ac$ )  
 shows  $b = c$   
 ⟨proof⟩

**lemma** *cosh-dist-swap*:  
 assumes  $a \in \text{hyp2}$  and  $b \in \text{hyp2}$   
 shows  $\text{cosh-dist } a \ b = \text{cosh-dist } b \ a$   
 ⟨proof⟩

**lemma** *exp-2dist-1-equal*:  
 assumes  $a \in \text{hyp2}$  and  $b \in \text{hyp2}$  and  $\text{exp-2dist } a \ b = 1$   
 shows  $a = b$   
 ⟨proof⟩

### 8.11.1 A formula for a cross ratio involving a perpendicular foot

**lemma** *described-perp-foot-cross-ratio-formula*:  
 assumes  $a \neq b$  and  $c \in \text{hyp2}$  and *are-endpoints-in-S*  $p \ q \ a \ b$   
 and *proj2-incident*  $p \ l$  and *proj2-incident*  $q \ l$  and *M-perp*  $l \ m$   
 and *proj2-incident*  $d \ l$  and *proj2-incident*  $d \ m$  and *proj2-incident*  $c \ m$   
 shows  $\text{cross-ratio } p \ q \ d \ a$   
 $= (\text{cosh-dist } b \ c * \text{sqrt } (\text{cross-ratio } p \ q \ a \ b) - \text{cosh-dist } a \ c)$   
 $/ (\text{cosh-dist } a \ c * \text{cross-ratio } p \ q \ a \ b$   
 $- \text{cosh-dist } b \ c * \text{sqrt } (\text{cross-ratio } p \ q \ a \ b))$   
 (is  $?pqda = (?bc * \text{sqrt } ?pqab - ?ac) / (?ac * ?pqab - ?bc * \text{sqrt } ?pqab)$ )  
 ⟨proof⟩

**lemma** *perp-foot-cross-ratio-formula*:  
 assumes  $a \in \text{hyp2}$  and  $b \in \text{hyp2}$  and  $c \in \text{hyp2}$  and  $a \neq b$   
 shows  $\text{cross-ratio } (\text{endpoint-in-S } a \ b) (\text{endpoint-in-S } b \ a)$   
 $(\text{perp-foot } c (\text{proj2-line-through } a \ b)) \ a$   
 $= (\text{cosh-dist } b \ c * \text{sqrt } (\text{exp-2dist } a \ b) - \text{cosh-dist } a \ c)$   
 $/ (\text{cosh-dist } a \ c * \text{exp-2dist } a \ b - \text{cosh-dist } b \ c * \text{sqrt } (\text{exp-2dist } a \ b))$   
 (is  $\text{cross-ratio } ?p \ ?q \ ?d \ a$   
 $= (?bc * \text{sqrt } ?pqab - ?ac) / (?ac * ?pqab - ?bc * \text{sqrt } ?pqab)$ )  
 ⟨proof⟩

## 8.12 The Klein–Beltrami model satisfies axiom 5

**lemma** *statement69*:  
 assumes  $a \ b \equiv_K \ a' \ b'$  and  $b \ c \equiv_K \ b' \ c'$  and  $a \ c \equiv_K \ a' \ c'$   
 shows  $\exists J. \text{is-K2-isometry } J$   
 $\wedge \text{hyp2-cltn2 } a \ J = a' \wedge \text{hyp2-cltn2 } b \ J = b' \wedge \text{hyp2-cltn2 } c \ J = c'$   
 ⟨proof⟩

**theorem** *hyp2-axiom5*:  
 $\forall a \ b \ c \ d \ a' \ b' \ c' \ d'.$   
 $a \neq b \wedge B_K \ a \ b \ c \wedge B_K \ a' \ b' \ c' \wedge a \ b \equiv_K \ a' \ b' \wedge b \ c \equiv_K \ b' \ c'$   
 $\wedge a \ d \equiv_K \ a' \ d' \wedge b \ d \equiv_K \ b' \ d'$   
 $\longrightarrow c \ d \equiv_K \ c' \ d'$

*<proof>*

**interpretation** *hyp2: tarski-first5 real-hyp2-C real-hyp2-B*  
*<proof>*

### 8.13 The Klein–Beltrami model satisfies axioms 6, 7, and 11

**theorem** *hyp2-axiom6*:  $\forall a b. B_K a b a \longrightarrow a = b$   
*<proof>*

**lemma** *between-inverse*:

**assumes**  $B_{\mathbb{R}} (\text{hyp2-rep } p) v (\text{hyp2-rep } q)$   
**shows**  $\text{hyp2-rep } (\text{hyp2-abs } v) = v$

*<proof>*

**lemma** *between-switch*:

**assumes**  $B_{\mathbb{R}} (\text{hyp2-rep } p) v (\text{hyp2-rep } q)$   
**shows**  $B_K p (\text{hyp2-abs } v) q$

*<proof>*

**theorem** *hyp2-axiom7*:

$\forall a b c p q. B_K a p c \wedge B_K b q c \longrightarrow (\exists x. B_K p x b \wedge B_K q x a)$

*<proof>*

**theorem** *hyp2-axiom11*:

$\forall X Y. (\exists a. \forall x y. x \in X \wedge y \in Y \longrightarrow B_K a x y)$   
 $\longrightarrow (\exists b. \forall x y. x \in X \wedge y \in Y \longrightarrow B_K x b y)$

*<proof>*

**interpretation** *tarski-absolute-space real-hyp2-C real-hyp2-B*  
*<proof>*

### 8.14 The Klein–Beltrami model satisfies the dimension-specific axioms

**lemma** *hyp2-rep-abs-examples*:

**shows**  $\text{hyp2-rep } (\text{hyp2-abs } 0) = 0$  (**is**  $\text{hyp2-rep } ?a = ?ca$ )  
**and**  $\text{hyp2-rep } (\text{hyp2-abs } (\text{vector } [1/2,0])) = \text{vector } [1/2,0]$   
(**is**  $\text{hyp2-rep } ?b = ?cb$ )  
**and**  $\text{hyp2-rep } (\text{hyp2-abs } (\text{vector } [0,1/2])) = \text{vector } [0,1/2]$   
(**is**  $\text{hyp2-rep } ?c = ?cc$ )  
**and**  $\text{hyp2-rep } (\text{hyp2-abs } (\text{vector } [1/4,1/4])) = \text{vector } [1/4,1/4]$   
(**is**  $\text{hyp2-rep } ?d = ?cd$ )  
**and**  $\text{hyp2-rep } (\text{hyp2-abs } (\text{vector } [1/2,1/2])) = \text{vector } [1/2,1/2]$   
(**is**  $\text{hyp2-rep } ?t = ?ct$ )

*<proof>*

**theorem** *hyp2-axiom8*:  $\exists a b c. \neg B_K a b c \wedge \neg B_K b c a \wedge \neg B_K c a b$   
*<proof>*

**theorem** *hyp2-axiom9*:

$\forall p q a b c. p \neq q \wedge a p \equiv_K a q \wedge b p \equiv_K b q \wedge c p \equiv_K c q$   
 $\longrightarrow B_K a b c \vee B_K b c a \vee B_K c a b$   
(proof)

**interpretation** *hyp2*: *tarski-absolute real-hyp2-C real-hyp2-B*  
(proof)

## 8.15 The Klein–Beltrami model violates the Euclidean axiom

**theorem** *hyp2-axiom10-false*:

**shows**  $\neg (\forall a b c d t. B_K a d t \wedge B_K b d c \wedge a \neq d$   
 $\longrightarrow (\exists x y. B_K a b x \wedge B_K a c y \wedge B_K x t y))$   
(proof)

**theorem** *hyp2-not-tarski*:  $\neg (\textit{tarski real-hyp2-C real-hyp2-B})$   
(proof)

Therefore axiom 10 is independent.

**end**

## References

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