Order Extension and Szpilrajn's Theorem

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We formalize a more general version of Szpilrajn's extension theorem [3], employing the terminology of Bossert and Suzumura [2]. We also formalize Theorem 2.7 of their book. Our extension theorem states that any preorder can be extended to a total preorder while maintaining its structure. The proof of the extension theorem follows the proof presented in the Wikipedia article [1].

1 Definitions

1.1 Symmetric and asymmetric factor of a relation

According to Bossert and Suzumura, every relation can be partitioned into its symmetric and asymmetric factor. The symmetric factor of a relation r contains all pairs $(x, y) \in r$ where $(y, x) \in r$. Conversely, the asymmetric factor contains all pairs where this is not the case. In terms of an order $(\le)$, the asymmetric factor contains all $(x, y) \in \{(x, y) \mid x\ y.\ x \le y\}$ where $x < y$.

definition *sym-factor* :: $'a\ rel \Rightarrow\ 'a\ rel$
  where *sym-factor* $r \equiv \{(x, y) \in r.\ (y, x) \in r\}$

lemma *sym-factor-def'*: *sym-factor* $r = r \cap r^{-1}$
  ⟨*proof*⟩

definition *asym-factor* :: $'a\ rel \Rightarrow\ 'a\ rel$
  where *asym-factor* $r = \{(x, y) \in r.\ (y, x) \notin r\}$

1.1.1 Properties of the symmetric factor

lemma *sym-factorI*[*intro*]: $(x, y) \in r \Longrightarrow (y, x) \in r \Longrightarrow (x, y) \in$ *sym-factor* r
  ⟨*proof*⟩

lemma *sym-factorE*[*elim?*]:
  assumes $(x, y) \in$ *sym-factor* r **obtains** $(x, y) \in r\ (y, x) \in r$
  ⟨*proof*⟩

lemma *sym-sym-factor*[*simp*]: *sym* (*sym-factor r*)
⟨*proof*⟩

lemma *trans-sym-factor*[*simp*]: *trans r* $\Longrightarrow$ *trans* (*sym-factor r*)
⟨*proof*⟩

lemma *refl-on-sym-factor*[*simp*]: *refl-on A r* $\Longrightarrow$ *refl-on A* (*sym-factor r*)
⟨*proof*⟩

lemma *sym-factor-absorb-if-sym*[*simp*]: *sym r* $\Longrightarrow$ *sym-factor r* = *r*
⟨*proof*⟩

lemma *sym-factor-idem*[*simp*]: *sym-factor* (*sym-factor r*) = *sym-factor r*
⟨*proof*⟩

lemma *sym-factor-reflc*[*simp*]: *sym-factor* ($r^=$) = (*sym-factor r*)$^=$
⟨*proof*⟩

lemma *sym-factor-Restr*[*simp*]: *sym-factor* (*Restr r A*) = *Restr* (*sym-factor r*) *A*
⟨*proof*⟩

In contrast to *asym-factor*, the *sym-factor* is monotone.

lemma *sym-factor-mono*: $r \subseteq s \Longrightarrow$ *sym-factor r* $\subseteq$ *sym-factor s*
⟨*proof*⟩

1.1.2 Properties of the asymmetric factor

lemma *asym-factorI*[*intro*]: $(x, y) \in r \Longrightarrow (y, x) \notin r \Longrightarrow (x, y) \in$ *asym-factor r*
⟨*proof*⟩

lemma *asym-factorE*[*elim?*]:
assumes $(x, y) \in$ *asym-factor r* **obtains** $(x, y) \in r$
⟨*proof*⟩

lemma *refl-not-in-asym-factor*[*simp*]: $(x, x) \notin$ *asym-factor r*
⟨*proof*⟩

lemma *irrefl-asym-factor*[*simp*]: *irrefl* (*asym-factor r*)
⟨*proof*⟩

lemma *asym-asym-factor*[*simp*]: *asym* (*asym-factor r*)
⟨*proof*⟩

lemma *trans-asym-factor*[*simp*]: *trans r* $\Longrightarrow$ *trans* (*asym-factor r*)
⟨*proof*⟩

lemma *asym-if-irrefl-trans*: *irrefl r* $\Longrightarrow$ *trans r* $\Longrightarrow$ *asym r*
⟨*proof*⟩

lemma *antisym-if-irrefl-trans*: *irrefl r* $\Longrightarrow$ *trans r* $\Longrightarrow$ *antisym r*

⟨*proof*⟩

lemma *asym-factor-asym-rel*[*simp*]: *asym r* $\Longrightarrow$ *asym-factor r* = *r*
⟨*proof*⟩

lemma *irrefl-trans-asym-factor-id*[*simp*]: *irrefl r* $\Longrightarrow$ *trans r* $\Longrightarrow$ *asym-factor r* = *r*
⟨*proof*⟩

lemma *asym-factor-id*[*simp*]: *asym-factor* (*asym-factor r*) = *asym-factor r*
⟨*proof*⟩

lemma *asym-factor-rtrancl*: *asym-factor* (r^*) = *asym-factor* (r^+)
⟨*proof*⟩

lemma *asym-factor-Restr*[*simp*]: *asym-factor* (*Restr r A*) = *Restr* (*asym-factor r*) *A*
⟨*proof*⟩

lemma *acyclic-asym-factor*[*simp*]: *acyclic r* $\Longrightarrow$ *acyclic* (*asym-factor r*)
⟨*proof*⟩

1.1.3 Relations between symmetric and asymmetric factor

We prove that *sym-factor* and *asym-factor* partition the input relation.

lemma *sym-asym-factor-Un*: *sym-factor r* ∪ *asym-factor r* = *r*
⟨*proof*⟩

lemma *disjnt-sym-asym-factor*[*simp*]: *disjnt* (*sym-factor r*) (*asym-factor r*)
⟨*proof*⟩

lemma *Field-sym-asym-factor-Un*:
Field (*sym-factor r*) ∪ *Field* (*asym-factor r*) = *Field r*
⟨*proof*⟩

lemma *asym-factor-tranclE*:
assumes (*a*, *b*) ∈ (*asym-factor r*)$^+$ **shows** (*a*, *b*) ∈ r^+
⟨*proof*⟩

1.2 Extension of Orders

We use the definition of Bossert and Suzumura for *extends*. The requirement $r \subseteq R$ is obvious. The second requirement *asym-factor r* ⊆ *asym-factor R* enforces that the extension R maintains all strict preferences of r (viewing r as a preference relation).

definition *extends* :: *'a rel* ⇒ *'a rel* ⇒ *bool*
where *extends R r* ≡ *r* ⊆ *R* ∧ *asym-factor r* ⊆ *asym-factor R*

We define a stronger notion of *extends* where we also demand that *sym-factor* $R \subseteq$ (*sym-factor* $r)^=$. This enforces that the extension does not introduce preference cycles between previously unrelated pairs $(x, y) \in R - r$.

definition *strict-extends* :: *'a rel* ⇒ *'a rel* ⇒ *bool*
where *strict-extends R r* ≡ *extends R r* ∧ *sym-factor R* ⊆ (*sym-factor r*)$^=$

lemma *extendsI*[*intro*]: *r* ⊆ *R* ⟹ *asym-factor r* ⊆ *asym-factor R* ⟹ *extends R r*
⟨*proof*⟩

lemma *extendsE*:
assumes *extends R r*
obtains *r* ⊆ *R* *asym-factor r* ⊆ *asym-factor R*
⟨*proof*⟩

lemma *trancl-subs-extends-if-trans*: *extends r-ext r* ⟹ *trans r-ext* ⟹ r^+ ⊆ *r-ext*
⟨*proof*⟩

lemma *extends-if-strict-extends*: *strict-extends r-ext ext* ⟹ *extends r-ext ext*
⟨*proof*⟩

lemma *strict-extendsI*[*intro*]:
assumes *r* ⊆ *R* *asym-factor r* ⊆ *asym-factor R* *sym-factor R* ⊆ (*sym-factor r*)$^=$
shows *strict-extends R r*
⟨*proof*⟩

lemma *strict-extendsE*:
assumes *strict-extends R r*
obtains *r* ⊆ *R* *asym-factor r* ⊆ *asym-factor R* *sym-factor R* ⊆ (*sym-factor r*)$^=$
⟨*proof*⟩

lemma *strict-extends-antisym-Restr*:
assumes *strict-extends R r*
assumes *antisym* (*Restr r A*)
shows *antisym* ((*R* − *r*) ∪ *Restr r A*)
⟨*proof*⟩

Here we prove that we have no preference cycles between previously unrelated pairs.

lemma *antisym-Diff-if-strict-extends*:
assumes *strict-extends R r*
shows *antisym* (*R* − *r*)
⟨*proof*⟩

lemma *strict-extends-antisym*:
assumes *strict-extends R r*
assumes *antisym r*
shows *antisym R*

⟨*proof*⟩

lemma *strict-extends-if-strict-extends-reflc*:
assumes *strict-extends r-ext* $(r^=)$
shows *strict-extends r-ext r*
⟨*proof*⟩

lemma *strict-extends-diff-Id*:
assumes *irrefl r trans r*
assumes *strict-extends r-ext* $(r^=)$
shows *strict-extends* (*r-ext* − *Id*) *r*
⟨*proof*⟩

Both *extends* and *strict-extends* form a partial order since they are reflexive, transitive, and antisymmetric.

lemma shows
reflp-extends: *reflp extends* **and**
transp-extends: *transp extends* **and**
antisymp-extends: *antisymp extends*
⟨*proof*⟩

lemma shows
reflp-strict-extends: *reflp strict-extends* **and**
transp-strict-extends: *transp strict-extends* **and**
antisymp-strict-extends: *antisymp strict-extends*
⟨*proof*⟩

1.3 Missing order definitions

lemma *preorder-onD*[*dest?*]:
assumes *preorder-on A r*
shows *refl-on A r trans r*
⟨*proof*⟩

lemma *preorder-onI*[*intro*]: *refl-on A r* ⟹ *trans r* ⟹ *preorder-on A r*
⟨*proof*⟩

abbreviation *preorder* ≡ *preorder-on UNIV*

lemma *preorder-rtrancl*: *preorder* (r^*)
⟨*proof*⟩

definition *total-preorder-on A r* ≡ *preorder-on A r* ∧ *total-on A r*

abbreviation *total-preorder r* ≡ *total-preorder-on UNIV r*

lemma *total-preorder-onI*[*intro*]:
refl-on A r ⟹ *trans r* ⟹ *total-on A r* ⟹ *total-preorder-on A r*
⟨*proof*⟩

lemma *total-preorder-onD*[*dest?*]:
assumes *total-preorder-on A r*
shows *refl-on A r trans r total-on A r*
⟨*proof*⟩

definition *strict-partial-order r* ≡ *trans r* ∧ *irrefl r*

lemma *strict-partial-orderI*[*intro*]:
trans r ⟹ *irrefl r* ⟹ *strict-partial-order r*
⟨*proof*⟩

lemma *strict-partial-orderD*[*dest?*]:
assumes *strict-partial-order r*
shows *trans r irrefl r*
⟨*proof*⟩

lemma *strict-partial-order-acyclic*:
assumes *strict-partial-order r*
shows *acyclic r*
⟨*proof*⟩

abbreviation *partial-order* ≡ *partial-order-on UNIV*

lemma *partial-order-onI*[*intro*]:
refl-on A r ⟹ *trans r* ⟹ *antisym r* ⟹ *partial-order-on A r*
⟨*proof*⟩

lemma *linear-order-onI*[*intro*]:
refl-on A r ⟹ *trans r* ⟹ *antisym r* ⟹ *total-on A r* ⟹ *linear-order-on A r*
⟨*proof*⟩

lemma *linear-order-onD*[*dest?*]:
assumes *linear-order-on A r*
shows *refl-on A r trans r antisym r total-on A r*
⟨*proof*⟩

A typical example is (⊂) on sets:

lemma *strict-partial-order-subset*:
strict-partial-order {(*x*,*y*). *x* ⊂ *y*}
⟨*proof*⟩

We already have a definition of a strict linear order in *strict-linear-order*.

2 Extending preorders to total preorders

We start by proving that a preorder with two incomparable elements x and y can be strictly extended to a preorder where $x < y$.

lemma *can-extend-preorder*:
  assumes *preorder-on A r*
    and $y \in A$ $x \in A$ $(y, x) \notin r$
  shows
    preorder-on A $((insert\ (x, y)\ r)^+)$ *strict-extends* $((insert\ (x, y)\ r)^+)$ *r*
⟨*proof*⟩

With this, we can start the proof of our main extension theorem. For this we will use a variant of Zorns Lemma, which only considers nonempty chains:

lemma *Zorns-po-lemma-nonempty*:
  assumes *po*: *Partial-order r*
    and *u*: $\bigwedge C.\ [\![C \in Chains\ r;\ C \neq \{\}]\!] \Longrightarrow \exists\, u \in Field\ r.\ \forall\, a \in C.\ (a, u) \in r$
    and $r \neq \{\}$
  shows $\exists\, m \in Field\ r.\ \forall\, a \in Field\ r.\ (m, a) \in r \longrightarrow a = m$
⟨*proof*⟩

theorem *strict-extends-preorder-on*:
  assumes *preorder-on A base-r*
  shows $\exists\, r.$ *total-preorder-on A r* $\wedge$ *strict-extends r base-r*
⟨*proof*⟩

With this extension theorem, we can easily prove Szpilrajn's theorem and its equivalent for partial orders.

corollary *partial-order-extension*:
  assumes *partial-order-on A r*
  shows $\exists$ *r-ext. linear-order-on A r-ext* $\wedge$ $r \subseteq$ *r-ext*
⟨*proof*⟩

corollary *Szpilrajn*:
  assumes *strict-partial-order r*
  shows $\exists$ *r-ext. strict-linear-order r-ext* $\wedge$ $r \subseteq$ *r-ext*
⟨*proof*⟩

corollary *acyclic-order-extension*:
  assumes *acyclic r*
  shows $\exists$ *r-ext. strict-linear-order r-ext* $\wedge$ $r \subseteq$ *r-ext*
⟨*proof*⟩

3 Consistency

As a weakening of transitivity, Suzumura introduces the notion of consistency which rules out all preference cycles that contain at least one strict preference. Consistency characterises those order relations which can be extended (in terms of *extends*) to a total order relation.

definition *consistent* :: *'a rel* $\Rightarrow$ *bool*

where *consistent* $r = (\forall (x, y) \in r^+. (y, x) \notin$ *asym-factor* $r)$

lemma *consistentI*: $(\bigwedge x\ y.\ (x, y) \in r^+ \Longrightarrow (y, x) \notin$ *asym-factor* $r) \Longrightarrow$ *consistent* r
⟨*proof*⟩

lemma *consistent-if-preorder-on*[*simp*]:
preorder-on $A\ r \Longrightarrow$ *consistent* r
⟨*proof*⟩

lemma *consistent-asym-factor*[*simp*]: *consistent* $r \Longrightarrow$ *consistent* (*asym-factor* r)
⟨*proof*⟩

lemma *acyclic-asym-factor-if-consistent*[*simp*]: *consistent* $r \Longrightarrow$ *acyclic* (*asym-factor* r)
⟨*proof*⟩

lemma *consistent-Restr*[*simp*]: *consistent* $r \Longrightarrow$ *consistent* (*Restr* $r\ A$)
⟨*proof*⟩

This corresponds to Theorem 2.2 [2].

theorem *trans-if-refl-total-consistent*:
assumes *refl* r *total* r **and** *consistent* r
shows *trans* r
⟨*proof*⟩

lemma *order-extension-if-consistent*:
assumes *consistent* r
obtains *r-ext* **where** *extends r-ext r total-preorder r-ext*
⟨*proof*⟩

lemma *consistent-if-extends-trans*:
assumes *extends r-ext r trans r-ext*
shows *consistent* r
⟨*proof*⟩

With Theorem 2.6 [2], we show that *consistent* characterises the existence of order extensions.

corollary *order-extension-iff-consistent*:
($\exists$ *r-ext. extends r-ext r* $\wedge$ *total-preorder r-ext*) $\longleftrightarrow$ *consistent* r
⟨*proof*⟩

The following theorem corresponds to Theorem 2.7 [2]. Bossert and Suzumura claim that this theorem generalises Szpilrajn's theorem; however, we cannot use the theorem to strictly extend a given order Q. Therefore, it is not strong enough to extend a strict partial order to a strict linear order. It works for total preorders (called orderings by Bossert and Suzumura). Unfortunately, we were not able to generalise the theorem to allow for strict extensions.

theorem *general-order-extension-iff-consistent*:
  assumes $\bigwedge x\ y.\ [\![\ x \in S;\ y \in S;\ x \neq y\ ]\!] \Longrightarrow (x,\ y) \notin Q^+$
  assumes *total-preorder-on S Ord*
  shows ($\exists$ *Ext. extends Ext Q* $\wedge$ *total-preorder Ext* $\wedge$ *Restr Ext S* = *Ord*)
    $\longleftrightarrow$ *consistent Q* (**is** *?ExExt* $\longleftrightarrow$ -)
⟨*proof*⟩

4 Strong consistency

We define a stronger version of *consistent* which requires that the relation does not contain hidden preference cycles, i.e. if there is a preference cycle then all the elements in the cycle should already be related (in both directions). In contrast to consistency which characterises relations that can be extended, strong consistency characterises relations that can be extended strictly (cf. *strict-extends*).

definition *strongly-consistent r* $\equiv$ *sym-factor* $(r^+) \subseteq$ *sym-factor* $(r^=)$

lemma *consistent-if-strongly-consistent*: *strongly-consistent r* $\Longrightarrow$ *consistent r*
  ⟨*proof*⟩

lemma *strongly-consistentI*: *sym-factor* $(r^+) \subseteq$ *sym-factor* $(r^=) \Longrightarrow$ *strongly-consistent r*
  ⟨*proof*⟩

lemma *strongly-consistent-if-trans-strict-extension*:
  assumes *strict-extends r-ext r*
  assumes *trans r-ext*
  shows *strongly-consistent r*
⟨*proof*⟩

lemma *strict-order-extension-if-consistent*:
  assumes *strongly-consistent r*
  obtains *r-ext* **where** *strict-extends r-ext r total-preorder r-ext*
⟨*proof*⟩

experiment begin

We can instantiate the above theorem to get Szpilrajn's theorem.

lemma
  assumes *strict-partial-order r*
  shows $\exists$ *r-ext. strict-linear-order r-ext* $\wedge$ $r \subseteq$ *r-ext*
⟨*proof*⟩

end

References

[1] Wikipedia: Szpilrajn extension theorem. https://en.wikipedia.org/wiki/Szpilrajn_extension_theorem. Accessed: 2019-07-27.

[2] W. Bossert and K. Suzumura. *Consistency, Choice, and Rationality*. Harvard University Press, 2010.

[3] E. Szpilrajn. Sur l'extension de l'ordre partiel. *Fundamenta Mathematicae*, 16:386–389, 1930.