Synthetic Completeness

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Abstract

In this work, I provide an abstract framework for proving the completeness of a logical calculus using the synthetic method. The synthetic method is based on maximal consistent saturated sets (MCSs). A set of formulas is consistent (with respect to the calculus) when we cannot derive a contradiction from it. It is maximally consistent when it contains every formula that is consistent with it. For logics where it is relevant, it is saturated when it contains a witness for every existential formula. To prove completeness using these maximal consistent saturated sets, we prove a truth lemma: every formula in an MCS has a satisfying model. Here, Hintikka sets provide a useful stepping stone. These can be seen as characterizations of the MCSs based on simple subformula conditions rather than via the calculus. We then prove that every Hintikka set gives rise to a satisfying model and that MCSs are Hintikka sets. Now, assume a valid formula cannot be derived. Then its negation must be consistent and therefore satisfiable. This contradicts validity and the original formula must be derivable.

To start, I build maximal consistent saturated sets for any logic that satisfies a small set of assumptions. I do this using a transfinite version of Lindenbaum's lemma, which allows me to support languages of any cardinality. I then prove useful abstract results about derivations and refutations as they relate to MCSs. Finally, I show how Hintikka sets can be derived from the logic's semantics, outlining one way to prove the required truth lemma.

To demonstrate the versatility of the framework, I instantiate it with five different examples. The formalization contains soundness and completeness results for: a propositional tableau calculus, a propositional sequent calculus, an axiomatic system for modal logic, a labelled natural deduction system for hybrid logic and a natural deduction system for first-order logic. The tableau example uses custom Hintikka sets based on the calculus, but the other four examples derive them from the semantics in the style of the framework. The hybrid and first-order logic examples rely on saturated MCSs. This places requirements on the cardinalities of their languages to ensure that there are enough witnesses available. In both cases, the type of witnesses must be infinite and have cardinality at least that of the type of propositional/predicate symbols.

Contents

Chapter 1

Maximal Consistent Sets

theory *Maximal-Consistent-Sets* **imports** *HOL−Cardinals.Cardinal-Order-Relation* **begin**

1.1 Utility

lemma *Set-Diff-Un*: ‹$X - (Y \cup Z) = X - Y - Z$›
⟨*proof*⟩

lemma *infinite-Diff-fin-Un*: ‹*infinite* $(X - Y) \Longrightarrow$ *finite* $Z \Longrightarrow$ *infinite* $(X - (Z \cup Y))$›
⟨*proof*⟩

lemma *infinite-Diff-subset*: ‹*infinite* $(X - A) \Longrightarrow B \subseteq A \Longrightarrow$ *infinite* $(X - B)$›
⟨*proof*⟩

lemma *finite-bound*:
fixes X :: ‹$('a$ :: *size*$)$ *set*›
assumes ‹*finite* X› ‹$X \neq \{\}$›
shows ‹$\exists\, x \in X.\ \forall\, y \in X.$ *size* $y \leq$ *size* x›
⟨*proof*⟩

lemma *infinite-UNIV-size*:
fixes f :: ‹$('a$ :: *size*$) \Rightarrow 'a$›
assumes ‹$\bigwedge x.$ *size* $x <$ *size* $(f\, x)$›
shows ‹*infinite* $(UNIV$ :: $'a$ *set*$)$›
⟨*proof*⟩

lemma *split-finite-sets*:
assumes ‹*finite* A› ‹*finite* B›
assumes ‹$A \subseteq B \cup S$›
shows ‹$\exists\, B'\ C.$ *finite* $C \wedge (B' \cup C = A) \wedge B' \subseteq B \wedge C \subseteq S$›
⟨*proof*⟩

lemma *split-list*:
assumes ‹*set* $A \subseteq$ *set* $B \cup S$›
shows ‹$\exists\, B'\ C.$ *set* $(B'$ @ $C) =$ *set* $A \wedge$ *set* $B' \subseteq$ *set* $B \wedge$ *set* $C \subseteq S$›
⟨*proof*⟩

lemma *struct-split*:
assumes ‹$\bigwedge A\ B.\ P\ A \Longrightarrow$ *set* $A \subseteq$ *set* $B \Longrightarrow P\ B$› ‹$P\ A$› ‹*set* $A \subseteq$ *set* $B \cup X$›
shows ‹$\exists\, C.$ *set* $C \subseteq X \wedge P\ (B$ @ $C)$›
⟨*proof*⟩

context *wo-rel* **begin**

lemma *underS-bound*: ‹$a \in underS\ n \Longrightarrow b \in underS\ n \Longrightarrow a \in under\ b \vee b \in under\ a$›
⟨*proof*⟩

lemma *finite-underS-bound*:
assumes ‹*finite* X› ‹$X \subseteq underS\ n$› ‹$X \neq \{\}$›
shows ‹$\exists\, a \in X.\ \forall\, b \in X.\ b \in under\ a$›
⟨*proof*⟩

lemma *finite-bound-under*:
assumes ‹*finite* p› ‹$p \subseteq (\bigcup n \in Field\ r.\ f\ n)$›
shows ‹$\exists\, m.\ p \subseteq (\bigcup n \in under\ m.\ f\ n)$›
⟨*proof*⟩

lemma *underS-trans*: ‹$a \in underS\ b \Longrightarrow b \in underS\ c \Longrightarrow a \in underS\ c$›
⟨*proof*⟩

end

lemma *card-of-infinite-smaller-Union*:
assumes ‹$\forall\, x.\ |f\ x| <o |X|$› ‹*infinite* X›
shows ‹$|\bigcup x \in X.\ f\ x| \leq o\ |X|$›
⟨*proof*⟩

lemma *card-of-params-marker-lists*:
assumes ‹*infinite* (*UNIV* :: *'i set*)› ‹$|UNIV :: 'm\ set| \leq o\ |UNIV :: nat\ set|$›
shows ‹$|UNIV :: ('i + 'm \times nat)\ list\ set| \leq o\ |UNIV :: 'i\ set|$›
⟨*proof*⟩

1.2 Base Locale

locale *MCS-Base* =
fixes *consistent* :: ‹*'a set* ⇒ *bool*›
assumes *consistent-hereditary*: ‹$\bigwedge S\ S'.\ consistent\ S \Longrightarrow S' \subseteq S \Longrightarrow consistent\ S'$›
and *inconsistent-finite*: ‹$\bigwedge S.\ \neg\ consistent\ S \Longrightarrow \exists\, S' \subseteq S.\ finite\ S' \wedge \neg\ consistent\ S'$›
begin

definition *maximal* :: ‹*'a set* ⇒ *bool*› **where**
‹$maximal\ S \equiv \forall\, p.\ consistent\ (\{p\} \cup S) \longrightarrow p \in S$›

end

1.3 Ordinal Locale

locale *MCS-Lim-Ord* = *MCS-Base* +
fixes *r* :: ‹*'a rel*›
assumes *WELL*: ‹*Well-order r*›
and *Cinfinite-r*: ‹*Cinfinite r*›
fixes *params* :: ‹*'a* ⇒ *'i set*›
and *witness* :: ‹*'a* ⇒ *'a set* ⇒ *'a set*›
assumes *finite-params*: ‹$\bigwedge p.\ finite\ (params\ p)$›
and *finite-witness-params*: ‹$\bigwedge p\ S.\ finite\ (\bigcup q \in witness\ p\ S.\ params\ q)$›
and *consistent-witness*: ‹$\bigwedge p\ S.\ consistent\ (\{p\} \cup S)$›

$\Longrightarrow$ *infinite* (*UNIV* − ($\bigcup q \in S.$ *params* q))
$\Longrightarrow$ *consistent* (*witness* p $S \cup \{p\} \cup S$)›
begin

lemma *wo-rel-r*: ‹*wo-rel r*›
⟨*proof*⟩

lemma *isLimOrd-r*: ‹*isLimOrd r*›
⟨*proof*⟩

1.3.1 Lindenbaum Extension

abbreviation *paramss* :: ‹'*a set* ⇒ '*i set*› **where**
‹*paramss S* ≡ $\bigcup p \in S.$ *params p*›

definition *extendS* :: ‹'*a* ⇒ '*a set* ⇒ '*a set*› **where**
‹*extendS n prev* ≡ *if consistent* ({*n*} ∪ *prev*) *then witness n prev* ∪ {*n*} ∪ *prev else prev*›

definition *extendL* :: ‹('*a* ⇒ '*a set*) ⇒ '*a* ⇒ '*a set*› **where**
‹*extendL rec n* ≡ $\bigcup m \in$ *underS r n. rec m*›

definition *extend* :: ‹'*a set* ⇒ '*a* ⇒ '*a set*› **where**
‹*extend S n* ≡ *worecZSL r S extendS extendL n*›

lemma *adm-woL-extendL*: ‹*adm-woL r extendL*›
⟨*proof*⟩

definition *Extend* :: ‹'*a set* ⇒ '*a set*› **where**
‹*Extend S* ≡ $\bigcup n \in$ *Field r. extend S n*›

lemma *extend-subset*: ‹$n \in$ *Field* $r \Longrightarrow S \subseteq$ *extend S n*›
⟨*proof*⟩

lemma *Extend-subset'*: ‹*Field* $r \neq \{\} \Longrightarrow S \subseteq$ *Extend S*›
⟨*proof*⟩

lemma *extend-underS*: ‹$m \in$ *underS r n* $\Longrightarrow$ *extend S m* $\subseteq$ *extend S n*›
⟨*proof*⟩

lemma *extend-under*: ‹$m \in$ *under r n* $\Longrightarrow$ *extend S m* $\subseteq$ *extend S n*›
⟨*proof*⟩

1.3.2 Consistency

lemma *params-origin*:
assumes ‹$a \in$ *paramss* (*extend S n*)›
shows ‹$a \in$ *paramss S* ∨ (∃ $m \in$ *underS r n.* $a \in$ *paramss* (*witness m* (*extend S m*) ∪ {*m*}))›
⟨*proof*⟩

lemma *consistent-extend*:
assumes ‹*consistent S*› ‹$r \leq o$ |*UNIV* − *paramss S*|›
shows ‹*consistent* (*extend S n*)›
⟨*proof*⟩

lemma *consistent-Extend*:
assumes ‹*consistent S*› ‹$r \leq o$ |*UNIV* − *paramss S*|›

shows ‹*consistent* (*Extend S*)›
⟨*proof*⟩

lemma *Extend-bound*: ‹$n \in \mathit{Field}\ r \Longrightarrow \mathit{extend}\ S\ n \subseteq \mathit{Extend}\ S$›
⟨*proof*⟩

1.3.3 Maximality

definition *maximal'* :: ‹'*a set* ⇒ *bool*› **where**
‹$\mathit{maximal'}\ S \equiv \forall\, p \in \mathit{Field}\ r.\ \mathit{consistent}\ (\{p\} \cup S) \longrightarrow p \in S$›

lemma *maximal'-Extend*: ‹*maximal'* (*Extend S*)›
⟨*proof*⟩

1.3.4 Saturation

definition *saturated'* :: ‹'*a set* ⇒ *bool*› **where**
‹$\mathit{saturated'}\ S \equiv \forall\, p \in S.\ p \in \mathit{Field}\ r \longrightarrow (\exists\, S'.\ \mathit{witness}\ p\ S' \subseteq S)$›

lemma *saturated'-Extend*:
assumes ‹*consistent* (*Extend S*)›
shows ‹*saturated'* (*Extend S*)›
⟨*proof*⟩

end

1.4 Locale with Saturation

locale *MCS-Saturation* = *MCS-Base* +
assumes *infinite-UNIV*: ‹*infinite* (*UNIV* :: '*a set*)›
fixes *params* :: ‹'*a* ⇒ '*i set*›
and *witness* :: ‹'*a* ⇒ '*a set* ⇒ '*a set*›
assumes ‹$\bigwedge p.$ *finite* (*params p*)›
and ‹$\bigwedge p\ S.$ *finite* $(\bigcup q \in \mathit{witness}\ p\ S.\ \mathit{params}\ q)$›
and ‹$\bigwedge p\ S.$ *consistent* $(\{p\} \cup S) \Longrightarrow$ *infinite* $(\mathit{UNIV} - (\bigcup q \in S.\ \mathit{params}\ q))$
$\Longrightarrow$ *consistent* (*witness p S* ∪ {*p*} ∪ *S*)›

sublocale *MCS-Saturation* ⊆ *MCS-Lim-Ord* - ‹|*UNIV*|›
⟨*proof*⟩

context *MCS-Saturation* **begin**

theorem *Extend-subset*: ‹$S \subseteq \mathit{Extend}\ S$›
⟨*proof*⟩

lemma *maximal-maximal'*: ‹$\mathit{maximal}\ S \longleftrightarrow \mathit{maximal'}\ S$›
⟨*proof*⟩

lemma *maximal-Extend*: ‹*maximal* (*Extend S*)›
⟨*proof*⟩

definition *saturated* :: ‹'*a set* ⇒ *bool*› **where**
‹$\mathit{saturated}\ S \equiv \forall\, p \in S.\ \exists\, S'.\ \mathit{witness}\ p\ S' \subseteq S$›

lemma *saturated-saturated'*: ‹$\mathit{saturated}\ S \longleftrightarrow \mathit{saturated'}\ S$›

⟨*proof*⟩

lemma *saturated-Extend*:
assumes ‹*consistent* (*Extend S*)›
shows ‹*saturated* (*Extend S*)›
⟨*proof*⟩

theorem *MCS-Extend*:
assumes ‹*consistent S*› ‹|*UNIV* :: *'a set*| ≤*o* |*UNIV* − *paramss S*|›
shows ‹*consistent* (*Extend S*)› ‹*maximal* (*Extend S*)› ‹*saturated* (*Extend S*)›
⟨*proof*⟩

end

1.5 Locale without Saturation

locale *MCS-No-Saturation* = *MCS-Base* +
assumes ‹*infinite* (*UNIV* :: *'a set*)›

sublocale *MCS-No-Saturation* ⊆ *MCS-Saturation consistent* ‹λ-. {} :: *'a set*› ‹λ- -. {}›
⟨*proof*⟩

context *MCS-No-Saturation* **begin**

lemma *always-saturated* [*simp*]: ‹*saturated H*›
⟨*proof*⟩

theorem *MCS-Extend'*:
assumes ‹*consistent S*›
shows ‹*consistent* (*Extend S*)› ‹*maximal* (*Extend S*)›
⟨*proof*⟩

end

1.6 Truth Lemma

locale *Truth-Base* =
fixes *semics* :: ‹*'model* ⇒ (*'model* ⇒ *'fm* ⇒ *bool*) ⇒ *'fm* ⇒ *bool*›
and *semantics* :: ‹*'model* ⇒ *'fm* ⇒ *bool*›
and *models-from* :: ‹*'a set* ⇒ *'model set*›
and *rel* :: ‹*'a set* ⇒ *'model* ⇒ *'fm* ⇒ *bool*›
assumes *semics-semantics*: ‹*semantics M p* ⟷ *semics M semantics p*›
and *Hintikka-model*: ‹⋀*H M p*. ∀ *M* ∈ *models-from H*. ∀ *p*. *semics M* (*rel H*) *p* ⟷ *rel H M p* ⟹
M ∈ *models-from H* ⟹ *semantics M p* ⟷ *rel H M p*›

locale *Truth-Saturation* = *MCS-Saturation* + *Truth-Base* +
assumes *MCS-Hintikka*: ‹⋀*H*. *consistent H* ⟹ *maximal H* ⟹ *saturated H* ⟹
∀ *M* ∈ *models-from H*. ∀ *p*. *semics M* (*rel H*) *p* ⟷ *rel H M p*›
begin

theorem *truth-lemma-saturation*:
assumes ‹*consistent H*› ‹*maximal H*› ‹*saturated H*› ‹*M* ∈ *models-from H*›
shows ‹*semantics M p* ⟷ *rel H M p*›
⟨*proof*⟩

end

locale *Truth-No-Saturation* = *MCS-No-Saturation* + *Truth-Base* +
  assumes *MCS-Hintikka*: ‹$\bigwedge H$. *consistent* $H \Longrightarrow$ *maximal* $H \Longrightarrow$
    $\forall M \in$ *models-from* H. $\forall p$. *semics* M (*rel* H) $p \longleftrightarrow$ *rel* H M p›
begin

theorem *truth-lemma-no-saturation*:
  assumes ‹*consistent* H› ‹*maximal* H› ‹$M \in$ *models-from* H›
  shows ‹*semantics* M $p \longleftrightarrow$ *rel* H M p›
  ⟨*proof*⟩

end

end

Chapter 2

Derivations

theory *Derivations* **imports** *Maximal-Consistent-Sets* **begin**

2.1 Rearranging Assumptions

locale *Derivations* =
  fixes *derive* :: ‹'*a list* ⇒ '*a* ⇒ *bool*›
  assumes *derive-struct*: ‹⋀*A B p*. *derive A p* ⟹ *set A* ⊆ *set B* ⟹ *derive B p*›
begin

theorem *derive-split*:
  assumes ‹*set A* ⊆ *set B* ∪ *X*› ‹*derive A p*›
  shows ‹∃ *C*. *set C* ⊆ *X* ∧ *derive* (*B* @ *C*) *p*›
  ⟨*proof*⟩

corollary *derive-split1*:
  assumes ‹*set A* ⊆ {*q*} ∪ *X*› ‹*derive A p*›
  shows ‹∃ *C*. *set C* ⊆ *X* ∧ *derive* (*q* # *C*) *p*›
  ⟨*proof*⟩

end

2.2 MCSs and Deriving Falsity

locale *Derivations-MCS* = *Derivations* + *MCS-Base* +
  fixes *fls*
  assumes *consistent-derive-fls*: ‹⋀*S*. *consistent S* = (∄ *S'*. *set S'* ⊆ *S* ∧ *derive S' fls*)›
begin

theorem *MCS-derive-fls*:
  assumes ‹*consistent S*› ‹*maximal S*›
  shows ‹*p* ∉ *S* ⟷ (∃ *S'*. *set S'* ⊆ *S* ∧ *derive* (*p* # *S'*) *fls*)›
⟨*proof*⟩

end

2.3 MCSs and Derivability

locale *Derivations-MCS-Cut* = *Derivations-MCS* +
  assumes *derive-assm*: ‹⋀*A p*. *p* ∈ *set A* ⟹ *derive A p*›

and *derive-cut*: ‹$\bigwedge A\ B\ p\ q.\ derive\ A\ p \Longrightarrow derive\ (p\ \#\ B)\ q \Longrightarrow derive\ (A\ @\ B)\ q$›
begin

theorem *MCS-derive*:
assumes ‹*consistent S*› ‹*maximal S*›
shows ‹$p \in S \longleftrightarrow (\exists S'.\ set\ S' \subseteq S \wedge derive\ S'\ p)$›
⟨*proof*⟩

end

end

Chapter 3

Refutations

theory *Refutations* **imports** *Maximal-Consistent-Sets* **begin**

3.1 Rearranging Refutations

locale *Refutations* =
  fixes *refute* :: ‹*'a list* ⇒ *bool*›
  assumes *refute-struct*: ‹⋀*A B*. *refute A* ⟹ *set A* ⊆ *set B* ⟹ *refute B*›
begin

theorem *refute-split*:
  assumes ‹*set A* ⊆ *set B* ∪ *X*› ‹*refute A*›
  shows ‹∃ *C*. *set C* ⊆ *X* ∧ *refute* (*B* @ *C*)›
  ⟨*proof*⟩

corollary *refute-split1*:
  assumes ‹*set A* ⊆ {*q*} ∪ *X*› ‹*refute A*›
  shows ‹∃ *C*. *set C* ⊆ *X* ∧ *refute* (*q* # *C*)›
  ⟨*proof*⟩

end

3.2 MCSs and Refutability

locale *Refutations-MCS* = *Refutations* + *MCS-Base* +
  assumes *consistent-refute*: ‹⋀*S*. *consistent S* = (∄ *S'*. *set S'* ⊆ *S* ∧ *refute S'*)›
begin

theorem *MCS-refute*:
  assumes ‹*consistent S*› ‹*maximal S*›
  shows ‹*p* ∉ *S* ⟷ (∃ *S'*. *set S'* ⊆ *S* ∧ *refute* (*p* # *S'*))›
⟨*proof*⟩

end

end

Chapter 4

Example: Propositional Tableau Calculus

theory *Example-Propositional-Tableau* **imports** *Refutations* **begin**

4.1 Syntax

datatype *'p fm*
= *Pro 'p* (‹‡›)
| *Neg* ‹*'p fm*› (‹¬ -› [*70*] *70*)
| *Imp* ‹*'p fm*› ‹*'p fm*› (**infixr** ‹⟶› *55*)

4.2 Semantics

type-synonym *'p model* = ‹*'p* ⇒ *bool*›

fun *semantics* :: ‹*'p model* ⇒ *'p fm* ⇒ *bool*› (‹⟦-⟧›) **where**
‹⟦*I*⟧ (‡*P*) ⟷ *I P*›
| ‹⟦*I*⟧ (¬ *p*) ⟷ ¬ ⟦*I*⟧ *p*›
| ‹⟦*I*⟧ (*p* ⟶ *q*) ⟷ ⟦*I*⟧ *p* ⟶ ⟦*I*⟧ *q*›

4.3 Calculus

inductive *Calculus* :: ‹*'p fm list* ⇒ *bool*› (‹⊢$_T$ -› [*50*] *50*) **where**
Axiom [*intro*]: ‹⊢$_T$ ‡*P* # ¬ ‡*P* # *A*›
| *NegI* [*intro*]: ‹⊢$_T$ *p* # *A* ⟹ ⊢$_T$ ¬ ¬ *p* # *A*›
| *ImpP* [*intro*]: ‹⊢$_T$ ¬ *p* # *A* ⟹ ⊢$_T$ *q* # *A* ⟹ ⊢$_T$ (*p* ⟶ *q*) # *A*›
| *ImpN* [*intro*]: ‹⊢$_T$ *p* # ¬ *q* # *A* ⟹ ⊢$_T$ ¬ (*p* ⟶ *q*) # *A*›
| *Weaken*: ‹⊢$_T$ *A* ⟹ *set A* ⊆ *set B* ⟹ ⊢$_T$ *B*›

lemma *Weaken2*:
assumes ‹⊢$_T$ *p* # *A*› ‹⊢$_T$ *q* # *B*›
shows ‹⊢$_T$ *p* # *A* @ *B* ∧ ⊢$_T$ *q* # *A* @ *B*›
⟨*proof*⟩

4.4 Soundness

theorem *soundness*: ‹⊢$_T$ *A* ⟹ ∃ *p* ∈ *set A*. ¬ ⟦*I*⟧ *p*›
⟨*proof*⟩

corollary *soundness'*: ‹$\vdash_T [\neg\ p] \Longrightarrow [\![I]\!]\ p$›
⟨*proof*⟩

corollary ‹$\neg\ (\vdash_T [])$›
⟨*proof*⟩

4.5 Maximal Consistent Sets

definition *consistent* :: ‹*'p fm set* ⇒ *bool*› **where**
‹*consistent S* ≡ ∄*S'*. *set S'* ⊆ *S* ∧ $\vdash_T$ *S'*›

interpretation *MCS-No-Saturation consistent*
⟨*proof*⟩

interpretation *Refutations-MCS Calculus consistent*
⟨*proof*⟩

4.6 Truth Lemma

abbreviation (*input*) *hmodel* :: ‹*'p fm set* ⇒ *'p model*› **where**
‹*hmodel H* ≡ λ*P*. ‡*P* ∈ *H*›

locale *Hintikka* =
fixes *H* :: ‹*'a fm set*›
assumes *AxiomH*: ‹⋀*P*. ‡*P* ∈ *H* ⟹ ¬ ‡*P* ∈ *H* ⟹ *False*›
and *NegIH*: ‹⋀*p*. ¬ ¬ *p* ∈ *H* ⟹ *p* ∈ *H*›
and *ImpPH*: ‹⋀*p q*. *p* ⟶ *q* ∈ *H* ⟹ ¬ *p* ∈ *H* ∨ *q* ∈ *H*›
and *ImpNH*: ‹⋀*p q*. ¬ (*p* ⟶ *q*) ∈ *H* ⟹ *p* ∈ *H* ∧ ¬ *q* ∈ *H*›

lemma *Hintikka-model*:
assumes ‹*Hintikka H*›
shows ‹(*p* ∈ *H* ⟶ [![*hmodel H*]!] *p*) ∧ (¬ *p* ∈ *H* ⟶ ¬ [![*hmodel H*]!] *p*)›
⟨*proof*⟩

lemma *MCS-Hintikka*:
assumes ‹*consistent H*› ‹*maximal H*›
shows ‹*Hintikka H*›
⟨*proof*⟩

lemma *truth-lemma*:
assumes ‹*consistent H*› ‹*maximal H*› ‹*p* ∈ *H*›
shows ‹[![*hmodel H*]!] *p*›
⟨*proof*⟩

4.7 Completeness

theorem *strong-completeness*:
assumes ‹∀*M* :: *'p model*. (∀*q* ∈ *X*. [![*M*]!] *q*) ⟶ [![*M*]!] *p*›
shows ‹∃*A*. *set A* ⊆ *X* ∧ $\vdash_T$ ¬ *p* # *A*›
⟨*proof*⟩

abbreviation *valid* :: ‹*'p fm* ⇒ *bool*› **where**
‹*valid p* ≡ ∀*M*. [![*M*]!] *p*›

theorem *completeness*:
  assumes ‹*valid p*›
  shows ‹$\vdash_T$ [$\neg$ *p*]›
  ⟨*proof*⟩

theorem *main*: ‹*valid p* $\longleftrightarrow$ $\vdash_T$ [$\neg$ *p*]›
  ⟨*proof*⟩

end

Chapter 5

Example: Propositional Sequent Calculus

theory *Example-Propositional-SC* **imports** *Derivations* **begin**

5.1 Syntax

datatype *'p fm*
= *Fls* (‹⊥›)
| *Pro 'p* (‹‡›)
| *Imp* ‹*'p fm*› ‹*'p fm*› (**infixr** ‹⟶› *55*)

abbreviation *Neg* (‹¬ -› [*70*] *70*) **where** ‹¬ *p* ≡ *p* ⟶ ⊥›

5.2 Semantics

type-synonym *'p model* = ‹*'p* ⇒ *bool*›

fun *semantics* :: ‹*'p model* ⇒ *'p fm* ⇒ *bool*› (‹⟦-⟧›) **where**
‹⟦-⟧ ⊥ ⟷ *False*›
| ‹⟦*I*⟧ (‡*P*) ⟷ *I P*›
| ‹⟦*I*⟧ (*p* ⟶ *q*) ⟷ ⟦*I*⟧ *p* ⟶ ⟦*I*⟧ *q*›

5.3 Calculus

inductive *Calculus* :: ‹*'p fm list* ⇒ *'p fm list* ⇒ *bool*› (‹- ⊢$_S$ -› [*50*, *50*] *50*) **where**
Axiom [*intro*]: ‹*p* # *A* ⊢$_S$ *p* # *B*›
| *FlsL* [*intro*]: ‹⊥ # *A* ⊢$_S$ *B*›
| *FlsR* [*dest*]: ‹*A* ⊢$_S$ ⊥ # *B* ⟹ *A* ⊢$_S$ *B*›
| *ImpL* [*intro*]: ‹*A* ⊢$_S$ *p* # *B* ⟹ *q* # *A* ⊢$_S$ *B* ⟹ (*p* ⟶ *q*) # *A* ⊢$_S$ *B*›
| *ImpR* [*intro*]: ‹*p* # *A* ⊢$_S$ *q* # *B* ⟹ *A* ⊢$_S$ (*p* ⟶ *q*) # *B*›
| *Cut* [*elim*]: ‹*A* ⊢$_S$ *p* # *B* ⟹ *p* # *C* ⊢$_S$ *D* ⟹ *A* @ *C* ⊢$_S$ *B* @ *D*›
| *WeakenL*: ‹*A* ⊢$_S$ *B* ⟹ *set A* ⊆ *set A'* ⟹ *A'* ⊢$_S$ *B*›
| *WeakenR*: ‹*A* ⊢$_S$ *B* ⟹ *set B* ⊆ *set B'* ⟹ *A* ⊢$_S$ *B'*›

lemma *Boole*: ‹¬ *p* # *A* ⊢$_S$ [] ⟹ *A* ⊢$_S$ [*p*]›
⟨*proof*⟩

5.4 Soundness

theorem *soundness*: ‹$A \vdash_S B \Longrightarrow \forall\, q \in \mathit{set}\ A.\ [\![I]\!]\ q \Longrightarrow \exists\, p \in \mathit{set}\ B.\ [\![I]\!]\ p$›
⟨*proof*⟩

corollary *soundness'*: ‹$[] \vdash_S [p] \Longrightarrow [\![I]\!]\ p$›
⟨*proof*⟩

corollary ‹$\neg\ ([] \vdash_S [])$›
⟨*proof*⟩

5.5 Maximal Consistent Sets

definition *consistent* :: ‹$'p\ \mathit{fm}\ \mathit{set} \Rightarrow \mathit{bool}$› **where**
‹$\mathit{consistent}\ S \equiv \nexists S'.\ \mathit{set}\ S' \subseteq S \wedge S' \vdash_S [\bot]$›

interpretation *MCS-No-Saturation consistent*
⟨*proof*⟩

interpretation *Derivations-MCS-Cut* ‹$\lambda A\ p.\ A \vdash_S [p]$› *consistent* ‹$\bot$›
⟨*proof*⟩

5.6 Truth Lemma

abbreviation *hmodel* :: ‹$'p\ \mathit{fm}\ \mathit{set} \Rightarrow 'p\ \mathit{model}$› **where**
‹$\mathit{hmodel}\ H \equiv \lambda P.\ \ddagger P \in H$›

fun *semics* :: ‹$'p\ \mathit{model} \Rightarrow ('p\ \mathit{model} \Rightarrow 'p\ \mathit{fm} \Rightarrow \mathit{bool}) \Rightarrow 'p\ \mathit{fm} \Rightarrow \mathit{bool}$› **where**
‹$\mathit{semics}\ \text{-}\ \text{-}\ \bot \longleftrightarrow \mathit{False}$›
| ‹$\mathit{semics}\ I\ \text{-}\ (\ddagger P) \longleftrightarrow I\ P$›
| ‹$\mathit{semics}\ I\ \mathit{rel}\ (p \longrightarrow q) \longleftrightarrow \mathit{rel}\ I\ p \longrightarrow \mathit{rel}\ I\ q$›

fun *rel* :: ‹$'p\ \mathit{fm}\ \mathit{set} \Rightarrow 'p\ \mathit{model} \Rightarrow 'p\ \mathit{fm} \Rightarrow \mathit{bool}$› **where**
‹$\mathit{rel}\ H\ \text{-}\ p = (p \in H)$›

theorem *Hintikka-model'*:
assumes ‹$\bigwedge p.\ \mathit{semics}\ (\mathit{hmodel}\ H)\ (\mathit{rel}\ H)\ p \longleftrightarrow p \in H$›
shows ‹$p \in H \longleftrightarrow [\![\mathit{hmodel}\ H]\!]\ p$›
⟨*proof*⟩

lemma *Hintikka-Extend*:
assumes ‹$\mathit{consistent}\ H$› ‹$\mathit{maximal}\ H$›
shows ‹$\mathit{semics}\ (\mathit{hmodel}\ H)\ (\mathit{rel}\ H)\ p \longleftrightarrow p \in H$›
⟨*proof*⟩

interpretation *Truth-No-Saturation consistent semics semantics* ‹$\lambda H.\ \{\mathit{hmodel}\ H\}$› *rel*
⟨*proof*⟩

5.7 Completeness

theorem *strong-completeness*:
assumes ‹$\forall\, M :: 'p\ \mathit{model}.\ (\forall\, q \in X.\ [\![M]\!]\ q) \longrightarrow [\![M]\!]\ p$›
shows ‹$\exists\, A.\ \mathit{set}\ A \subseteq X \wedge A \vdash_S [p]$›
⟨*proof*⟩

abbreviation *valid* :: ‹*'p fm* ⇒ *bool*› **where**
‹*valid p* ≡ ∀ *M*. ⟦*M*⟧ *p*›

theorem *completeness*:
assumes ‹*valid p*›
shows ‹[] ⊢$_S$ [*p*]›
⟨*proof*⟩

theorem *main*: ‹*valid p* ⟷ [] ⊢$_S$ [*p*]›
⟨*proof*⟩

end

Chapter 6

Example: Modal Logic

theory *Example-Modal-Logic* **imports** *Derivations* **begin**

6.1 Syntax

datatype $('i, 'p)$ *fm*
  = *Fls* (‹$\bot$›)
  | *Pro* $'p$ (‹$\ddagger$›)
  | *Imp* ‹$('i, 'p)$ *fm*› ‹$('i, 'p)$ *fm*› (**infixr** ‹$\longrightarrow$› *55*)
  | *Box* $'i$ ‹$('i, 'p)$ *fm*› (‹$\Box$›)

abbreviation *Neg* (‹$\neg$ -› [*70*] *70*) **where**
  ‹$\neg\ p \equiv p \longrightarrow \bot$›

6.2 Semantics

datatype $('i, 'p, 'w)$ *model* =
  Model ($\mathcal{W}$: ‹$'w$ *set*›) ($\mathcal{R}$: ‹$'i \Rightarrow 'w \Rightarrow 'w$ *set*›) ($\mathcal{V}$: ‹$'w \Rightarrow 'p \Rightarrow$ *bool*›)

type-synonym $('i, 'p, 'w)$ *ctx* = ‹$('i, 'p, 'w)$ *model* $\times$ $'w$›

fun *semantics* :: ‹$('i, 'p, 'w)$ *ctx* $\Rightarrow$ $('i, 'p)$ *fm* $\Rightarrow$ *bool*› (‹- $\models$ -› [*50*, *50*] *50*) **where**
  ‹- $\models \bot \longleftrightarrow$ *False*›
| ‹$(M, w) \models \ddagger P \longleftrightarrow \mathcal{V}\ M\ w\ P$›
| ‹$(M, w) \models p \longrightarrow q \longleftrightarrow (M, w) \models p \longrightarrow (M, w) \models q$›
| ‹$(M, w) \models \Box\ i\ p \longleftrightarrow (\forall v \in \mathcal{W}\ M \cap \mathcal{R}\ M\ i\ w.\ (M, v) \models p)$›

6.3 Calculus

primrec *eval* :: ‹$('p \Rightarrow$ *bool*$) \Rightarrow (('i, 'p)$ *fm* $\Rightarrow$ *bool*$) \Rightarrow ('i, 'p)$ *fm* $\Rightarrow$ *bool*› **where**
  ‹*eval* - - $\bot$ = *False*›
| ‹*eval* *g* - ($\ddagger P$) = *g* *P*›
| ‹*eval* *g* *h* (*p* $\longrightarrow$ *q*) = (*eval* *g* *h* *p* $\longrightarrow$ *eval* *g* *h* *q*)›
| ‹*eval* - *h* ($\Box$ *i* *p*) = *h* ($\Box$ *i* *p*)›

abbreviation ‹*tautology* *p* $\equiv \forall$ *g* *h*. *eval* *g* *h* *p*›

inductive *Calculus* :: ‹$('i, 'p)$ *fm* $\Rightarrow$ *bool*› (‹$\vdash_\Box$ -› [*50*] *50*) **where**
  A1: ‹*tautology* *p* $\Longrightarrow \vdash_\Box$ *p*›
| *A2*: ‹$\vdash_\Box \Box$ *i* (*p* $\longrightarrow$ *q*) $\longrightarrow \Box$ *i* *p* $\longrightarrow \Box$ *i* *q*›

| *R1*: ‹$\vdash_\Box p \Longrightarrow \vdash_\Box p \longrightarrow q \Longrightarrow \vdash_\Box q$›
| *R2*: ‹$\vdash_\Box p \Longrightarrow \vdash_\Box \Box\, i\, p$›

primrec *imply* :: ‹$('i, 'p)\ fm\ list \Rightarrow ('i, 'p)\ fm \Rightarrow ('i, 'p)\ fm$› (**infixr** ‹$\rightsquigarrow$› *56*) **where**
‹$([] \rightsquigarrow p) = p$›
| ‹$(q \# A \rightsquigarrow p) = (q \longrightarrow A \rightsquigarrow p)$›

abbreviation *Calculus-assms* (‹- $\vdash_\Box$ ->› [*50*, *50*] *50*) **where**
‹$A \vdash_\Box p \equiv \vdash_\Box A \rightsquigarrow p$›

6.4 Soundness

lemma *eval-semantics*: ‹$eval\ (g\ w)\ (\lambda q.\ (Model\ W\ r\ g,\ w) \models q)\ p = ((Model\ W\ r\ g,\ w) \models p)$›
⟨*proof*⟩

lemma *tautology*:
assumes ‹$tautology\ p$›
shows ‹$(M,\ w) \models p$›
⟨*proof*⟩

theorem *soundness*:
assumes ‹$\bigwedge M\ w\ p.\ A\ p \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow (M,\ w) \models p$›
shows ‹$\vdash_\Box p \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow (M,\ w) \models p$›
⟨*proof*⟩

6.5 Admissible rules

lemma *K-imply-head*: ‹$p \# A \vdash_\Box p$›
⟨*proof*⟩

lemma *K-imply-Cons*:
assumes ‹$A \vdash_\Box q$›
shows ‹$p \# A \vdash_\Box q$›
⟨*proof*⟩

lemma *K-right-mp*:
assumes ‹$A \vdash_\Box p$› ‹$A \vdash_\Box p \longrightarrow q$›
shows ‹$A \vdash_\Box q$›
⟨*proof*⟩

lemma *deduct1*: ‹$A \vdash_\Box p \longrightarrow q \Longrightarrow p \# A \vdash_\Box q$›
⟨*proof*⟩

lemma *imply-append* [*iff*]: ‹$(A\ @\ B \rightsquigarrow r) = (A \rightsquigarrow B \rightsquigarrow r)$›
⟨*proof*⟩

lemma *imply-swap-append*: ‹$A\ @\ B \vdash_\Box r \Longrightarrow B\ @\ A \vdash_\Box r$›
⟨*proof*⟩

lemma *K-ImpI*: ‹$p \# A \vdash_\Box q \Longrightarrow A \vdash_\Box p \longrightarrow q$›
⟨*proof*⟩

lemma *imply-mem* [*simp*]: ‹$p \in set\ A \Longrightarrow A \vdash_\Box p$›
⟨*proof*⟩

lemma *add-imply* [*simp*]: ‹$\vdash_\Box q \Longrightarrow A \vdash_\Box q$›
⟨*proof*⟩

lemma *K-imply-weaken*: ‹$A \vdash_\Box q \Longrightarrow set\ A \subseteq set\ A' \Longrightarrow A' \vdash_\Box q$›
⟨*proof*⟩

lemma *K-Boole*:
assumes ‹$(\neg p)\ \#\ A \vdash_\Box \bot$›
shows ‹$A \vdash_\Box p$›
⟨*proof*⟩

lemma *K-distrib-K-imp*:
assumes ‹$\vdash_\Box \Box\ i\ (A \rightsquigarrow q)$›
shows ‹$map\ (\Box\ i)\ A \vdash_\Box \Box\ i\ q$›
⟨*proof*⟩

interpretation *Derivations Calculus-assms*
⟨*proof*⟩

6.6 Maximal Consistent Sets

definition *consistent* :: ‹('*i*, '*p*) *fm set* ⇒ *bool*› **where**
‹*consistent* $S \equiv \nexists S'.\ set\ S' \subseteq S \wedge S' \vdash_\Box \bot$›

interpretation *MCS-No-Saturation consistent*
⟨*proof*⟩

interpretation *Derivations-MCS-Cut Calculus-assms consistent* ‹⊥›
⟨*proof*⟩

lemma *exists-finite-inconsistent*:
assumes ‹¬ *consistent* $(\{\neg p\} \cup V)$›
obtains *W* **where** ‹$\{\neg p\} \cup W \subseteq \{\neg p\} \cup V$› ‹$(\neg p) \notin W$› ‹*finite W*› ‹¬ *consistent* $(\{\neg p\} \cup W)$›
⟨*proof*⟩

lemma *MCS-consequent*:
assumes ‹*consistent V*› ‹*maximal V*› ‹$p \longrightarrow q \in V$› ‹$p \in V$›
shows ‹$q \in V$›
⟨*proof*⟩

theorem *deriv-in-maximal*:
assumes ‹*consistent V*› ‹*maximal V*› ‹$\vdash_\Box p$›
shows ‹$p \in V$›
⟨*proof*⟩

theorem *exactly-one-in-maximal*:
assumes ‹*consistent V*› ‹*maximal V*›
shows ‹$p \in V \longleftrightarrow (\neg p) \notin V$›
⟨*proof*⟩

6.7 Truth Lemma

abbreviation *val* :: ‹('*i*, '*p*) *fm set* ⇒ '*p* ⇒ *bool*› **where**
‹*val V P* ≡ ‡*P* ∈ *V*›

abbreviation *known* :: ‹('i, 'p) fm set ⇒ 'i ⇒ ('i, 'p) fm set› **where**
‹known V i ≡ {p. □ i p ∈ V}›

abbreviation *reach* :: ‹'i ⇒ ('i, 'p) fm set ⇒ ('i, 'p) fm set set› **where**
‹reach i V ≡ {W. known V i ⊆ W}›

abbreviation *mcss* :: ‹('i, 'p) fm set set› **where**
‹mcss ≡ {W. consistent W ∧ maximal W}›

abbreviation *canonical* :: ‹('i, 'p, ('i, 'p) fm set) model› **where**
‹canonical ≡ Model mcss reach val›

fun *semics* ::
‹('i, 'p, 'w) ctx ⇒ (('i, 'p, 'w) ctx ⇒ ('i, 'p) fm ⇒ bool) ⇒ ('i, 'p) fm ⇒ bool› **where**
‹semics - - ⊥ ⟷ False›
| ‹semics (M, w) - (‡P) ⟷ 𝒱 M w P›
| ‹semics (M, w) rel (p ⟶ q) ⟷ rel (M, w) p ⟶ rel (M, w) q›
| ‹semics (M, w) rel (□ i p) ⟷ (∀ v ∈ 𝒲 M ∩ ℛ M i w. rel (M, v) p)›

fun *rel* :: ‹('i, 'p) fm set ⇒ ('i, 'p, ('i, 'p) fm set) ctx ⇒ ('i, 'p) fm ⇒ bool› **where**
‹rel - (-, w) p = (p ∈ w)›

lemma *Hintikka-model'*:
fixes V :: ‹('i, 'p) fm set›
assumes ‹⋀(V :: ('i, 'p) fm set) p. V ∈ mcss ⟹ semics (canonical, V) (rel H) p ⟷ p ∈ V›
shows ‹V ∈ mcss ⟹ (canonical, V) ⊨ p ⟷ p ∈ V›
⟨*proof*⟩

lemma *maximal-extension*:
assumes ‹consistent V›
shows ‹∃ W. V ⊆ W ∧ consistent W ∧ maximal W›
⟨*proof*⟩

lemma *Hintikka-canonical*:
assumes ‹V ∈ mcss›
shows ‹semics (canonical, V) (rel H) p ⟷ rel H (canonical, V) p›
⟨*proof*⟩

interpretation *Truth-No-Saturation consistent semics semantics*
‹λ-. {(canonical, V) | V. V ∈ mcss}› rel
⟨*proof*⟩

lemma *Truth-lemma*:
assumes ‹consistent V› ‹maximal V›
shows ‹(canonical, V) ⊨ p ⟷ p ∈ V›
⟨*proof*⟩

lemma *canonical-model*:
assumes ‹consistent S› ‹p ∈ S›
defines ‹V ≡ Extend S› **and** ‹M ≡ canonical›
shows ‹(M, V) ⊨ p› ‹consistent V› ‹maximal V›
⟨*proof*⟩

6.8 Completeness

theorem *strong-completeness*:
  assumes ‹$\forall\, M :: ('i,\ 'p,\ ('i,\ 'p)\ fm\ set)\ model.\ \forall\, w \in \mathcal{W}\ M.$
    $(\forall\, q \in X.\ (M,\ w) \models q) \longrightarrow (M,\ w) \models p$›
  shows ‹$\exists\, A.\ set\ A \subseteq X \wedge A \vdash_{\Box} p$›
⟨*proof*⟩

abbreviation *valid* :: ‹$('i,\ 'p)\ fm \Rightarrow bool$› **where**
  ‹$valid\ p \equiv \forall\, (M :: ('i,\ 'p,\ ('i,\ 'p)\ fm\ set)\ model).\ \forall\, w \in \mathcal{W}\ M.\ (M,\ w) \models p$›

corollary *completeness*: ‹$valid\ p \Longrightarrow \vdash_{\Box} p$›
  ⟨*proof*⟩

theorem *main*: ‹$valid\ p \longleftrightarrow \vdash_{\Box} p$›
  ⟨*proof*⟩

end

Chapter 7

Example: Hybrid Logic

theory *Example-Hybrid-Logic* **imports** *Derivations* **begin**

7.1 Syntax

datatype (*nominals-fm*: *'i*, *'p*) *fm*
= *Fls* (‹⊥›)
| *Pro 'p* (‹‡›)
| *Nom 'i* (‹•›)
| *Imp* ‹(*'i*, *'p*) *fm*› ‹(*'i*, *'p*) *fm*› (**infixr** ‹⟶› *55*)
| *Dia* ‹(*'i*, *'p*) *fm*› (‹◇›)
| *Sat 'i* ‹(*'i*, *'p*) *fm*› (‹@›)

abbreviation *Neg* (‹¬ -› [*70*] *70*) **where** ‹¬ *p* ≡ *p* ⟶ ⊥›

type-synonym (*'i*, *'p*) *lbd* = ‹*'i* × (*'i*, *'p*) *fm*›

primrec *nominals-lbd* :: ‹(*'i*, *'p*) *lbd* ⇒ *'i set*› **where**
‹*nominals-lbd* (*i*, *p*) = {*i*} ∪ *nominals-fm p*›

abbreviation *nominals* :: ‹(*'i*, *'p*) *lbd set* ⇒ *'i set*› **where**
‹*nominals S* ≡ ⋃ *ip* ∈ *S*. *nominals-lbd ip*›

lemma *finite-nominals-fm*: ‹*finite* (*nominals-fm p*)›
⟨*proof*⟩

lemma *finite-nominals-lbd*: ‹*finite* (*nominals-lbd p*)›
⟨*proof*⟩

7.2 Semantics

datatype (*'w*, *'p*) *model* =
Model (*R*: ‹*'w* ⇒ *'w set*›) (*V*: ‹*'w* ⇒ *'p* ⇒ *bool*›)

type-synonym (*'i*, *'p*, *'w*) *ctx* = ‹(*'w*, *'p*) *model* × (*'i* ⇒ *'w*) × *'w*›

fun *semantics* :: ‹(*'i*, *'p*, *'w*) *ctx* ⇒ (*'i*, *'p*) *fm* ⇒ *bool*› (‹- ⊨ -› [*50*, *50*] *50*) **where**
‹(*M*, *g*, *w*) ⊨ ⊥ ⟷ *False*›
| ‹(*M*, -, *w*) ⊨ ‡*P* ⟷ *V M w P*›
| ‹(-, *g*, *w*) ⊨ •*i* ⟷ *w* = *g i*›
| ‹(*M*, *g*, *w*) ⊨ (*p* ⟶ *q*) ⟷ (*M*, *g*, *w*) ⊨ *p* ⟶ (*M*, *g*, *w*) ⊨ *q*›

| ‹$(M,\ g,\ w) \models \Diamond\ p \longleftrightarrow (\exists\, v \in R\ M\ w.\ (M,\ g,\ v) \models p)$›
| ‹$(M,\ g,\ \text{-}) \models @i\ p \longleftrightarrow (M,\ g,\ g\ i) \models p$›

lemma *semantics-fresh*: ‹$i \notin$ *nominals-fm* $p \Longrightarrow ((M,\ g,\ w) \models p) = ((M,\ g(i := v),\ w) \models p)$›
⟨*proof*⟩

lemma *semantics-fresh-lbd*:
‹$k \notin$ *nominals-lbd* $(i,\ p) \Longrightarrow ((M,\ g,\ w) \models p) = ((M,\ g(k := v),\ w) \models p)$›
⟨*proof*⟩

7.3 Calculus

inductive *Calculus* :: ‹$('i,\ 'p)$ *lbd list* ⇒ $('i,\ 'p)$ *lbd* ⇒ *bool*› (‹- ⊢@ -› [*50*, *50*] *50*) **where**
Assm [*intro*]: ‹$(i,\ p) \in$ *set* $A \Longrightarrow A \vdash_@ (i,\ p)$›
| *Ref* [*intro*]: ‹$A \vdash_@ (i,\ \cdot i)$›
| *Nom* [*intro*]: ‹$A \vdash_@ (i,\ \cdot k) \Longrightarrow A \vdash_@ (i,\ p) \Longrightarrow A \vdash_@ (k,\ p)$›
| *FlsE* [*elim*]: ‹$A \vdash_@ (i,\ \bot) \Longrightarrow A \vdash_@ (k,\ p)$›
| *ImpI* [*intro*]: ‹$(i,\ p)\ \#\ A \vdash_@ (i,\ q) \Longrightarrow A \vdash_@ (i,\ p \longrightarrow q)$›
| *ImpE* [*elim*]: ‹$A \vdash_@ (i,\ p \longrightarrow q) \Longrightarrow A \vdash_@ (i,\ p) \Longrightarrow A \vdash_@ (i,\ q)$›
| *SatI* [*intro*]: ‹$A \vdash_@ (i,\ p) \Longrightarrow A \vdash_@ (k,\ @i\ p)$›
| *SatE* [*elim*]: ‹$A \vdash_@ (i,\ @k\ p) \Longrightarrow A \vdash_@ (k,\ p)$›
| *DiaI* [*intro*]: ‹$A \vdash_@ (i,\ \Diamond\ (\cdot k)) \Longrightarrow A \vdash_@ (k,\ p) \Longrightarrow A \vdash_@ (i,\ \Diamond\ p)$›
| *DiaE* [*elim*]: ‹$A \vdash_@ (i,\ \Diamond\ p) \Longrightarrow k \notin$ *nominals* $(\{(i,\ p),\ (j,\ q)\} \cup$ *set* $A) \Longrightarrow$
$(k,\ p)\ \#\ (i,\ \Diamond\ (\cdot k))\ \#\ A \vdash_@ (j,\ q) \Longrightarrow A \vdash_@ (j,\ q)$›
| *Clas*: ‹$(i,\ p \longrightarrow q)\ \#\ A \vdash_@ (i,\ p) \Longrightarrow A \vdash_@ (i,\ p)$›
| *Weak*: ‹$A \vdash_@ (i,\ p) \Longrightarrow (k,\ q)\ \#\ A \vdash_@ (i,\ p)$›

7.4 Soundness

theorem *soundness*: ‹$A \vdash_@ (i,\ p) \Longrightarrow$ *list-all* $(\lambda(i,\ p).\ (M,\ g,\ g\ i) \models p)\ A \Longrightarrow (M,\ g,\ g\ i) \models p$›
⟨*proof*⟩

corollary *soundness'*: ‹$[] \vdash_@ (i,\ p) \Longrightarrow i \notin$ *nominals-fm* $p \Longrightarrow (M,\ g,\ w) \models p$›
⟨*proof*⟩

corollary ‹$\neg\ ([] \vdash_@ (i,\ \bot))$›
⟨*proof*⟩

7.5 Admissible Rules

lemma *Assm-head*: ‹$(p,\ i)\ \#\ A \vdash_@ (p,\ i)$›
⟨*proof*⟩

lemma *deduct1*: ‹$A \vdash_@ (i,\ p \longrightarrow q) \Longrightarrow (i,\ p)\ \#\ A \vdash_@ (i,\ q)$›
⟨*proof*⟩

lemma *Boole*: ‹$(i,\ \neg\ p)\ \#\ A \vdash_@ (i,\ \bot) \Longrightarrow A \vdash_@ (i,\ p)$›
⟨*proof*⟩

lemma *Weak'*: ‹$A \vdash_@ (i,\ p) \Longrightarrow B\ @\ A \vdash_@ (i,\ p)$›
⟨*proof*⟩

lemma *ImpI'*:
assumes ‹$(k,\ q)\ \#\ A \vdash_@ (i,\ p)$›

shows ‹$A \vdash_@ (i, (@k\ q) \longrightarrow p)$›
⟨*proof*⟩

lemma *Weaken*: ‹$A \vdash_@ (i, p) \Longrightarrow set\ A \subseteq set\ B \Longrightarrow B \vdash_@ (i, p)$›
⟨*proof*⟩

interpretation *Derivations Calculus*
⟨*proof*⟩

7.6 Maximal Consistent Sets

definition *consistent* :: ‹$('i, 'p)\ lbd\ set \Rightarrow bool$› **where**
‹$consistent\ S \equiv \nexists S'\ a.\ set\ S' \subseteq S \wedge S' \vdash_@ (a, \bot)$›

lemma *consistent-add-witness*:
assumes ‹$consistent\ S$› ‹$(i, \Diamond\ p) \in S$› ‹$k \notin nominals\ S$›
shows ‹$consistent\ (\{(k, p), (i, \Diamond\ (\cdot k))\} \cup S)$›
⟨*proof*⟩

fun *witness* :: ‹$('i, 'p)\ lbd \Rightarrow ('i, 'p)\ lbd\ set \Rightarrow ('i, 'p)\ lbd\ set$› **where**
‹$witness\ (i, \Diamond\ p)\ S = (let\ k = (SOME\ k.\ k \notin nominals\ (\{(i, p)\} \cup S))\ in\ \{(k, p), (i, \Diamond\ (\cdot k))\})$›
| ‹$witness\ (\text{-}, \text{-})\ \text{-} = \{\}$›

lemma *consistent-witness'*:
assumes ‹$consistent\ (\{(i, p)\} \cup S)$› ‹$infinite\ (UNIV - nominals\ S)$›
shows ‹$consistent\ (witness\ (i, p)\ S \cup \{(i, p)\} \cup S)$›
⟨*proof*⟩

interpretation *MCS-Saturation consistent nominals-lbd witness*
⟨*proof*⟩

interpretation *Derivations-MCS-Cut Calculus consistent* ‹$(undefined, \bot)$›
⟨*proof*⟩

lemma *saturated*: ‹$saturated\ H \Longrightarrow (i, \Diamond p) \in H \Longrightarrow \exists k.\ (i, \Diamond\ (\cdot k)) \in H \wedge (k, p) \in H$›
⟨*proof*⟩

7.7 Nominals

lemma *MCS-Nom-refl*:
assumes ‹$consistent\ S$› ‹$maximal\ S$›
shows ‹$(i, \cdot i) \in S$›
⟨*proof*⟩

lemma *MCS-Nom-sym*:
assumes ‹$consistent\ S$› ‹$maximal\ S$› ‹$(i, \cdot k) \in S$›
shows ‹$(k, \cdot i) \in S$›
⟨*proof*⟩

lemma *MCS-Nom-trans*:
assumes ‹$consistent\ S$› ‹$maximal\ S$› ‹$(i, \cdot j) \in S$› ‹$(j, \cdot k) \in S$›
shows ‹$(i, \cdot k) \in S$›
⟨*proof*⟩

7.8 Truth Lemma

fun *semics* :: ‹('*i*, '*p*, '*w*) *ctx* ⇒ (('*i*, '*p*, '*w*) *ctx* ⇒ ('*i*, '*p*) *fm* ⇒ *bool*) ⇒ ('*i*, '*p*) *fm* ⇒ *bool*›
where
‹*semics* - - ⊥ ⟷ *False*›
| ‹*semics* (*M*, -, *w*) - (‡*P*) ⟷ *V M w P*›
| ‹*semics* (-, *g*, *w*) - (•*i*) ⟷ *w* = *g i*›
| ‹*semics* (*M*, *g*, *w*) *rel* (*p* ⟶ *q*) ⟷ *rel* (*M*, *g*, *w*) *p* ⟶ *rel* (*M*, *g*, *w*) *q*›
| ‹*semics* (*M*, *g*, *w*) *rel* (◇ *p*) ⟷ (∃ *v* ∈ *R M w*. *rel* (*M*, *g*, *v*) *p*)›
| ‹*semics* (*M*, *g*, -) *rel* (@ *i p*) ⟷ *rel* (*M*, *g*, *g i*) *p*›

fun *rel* :: ‹('*i*, '*p*) *lbd set* ⇒ ('*i*, '*p*, '*i*) *ctx* ⇒ ('*i*, '*p*) *fm* ⇒ *bool*› **where**
‹*rel H* (-, -, *i*) *p* = ((*i*, *p*) ∈ *H*)›

definition *val* :: ‹('*i*, '*p*) *lbd set* ⇒ '*i* ⇒ '*p* ⇒ *bool*› **where**
‹*val H i P* ≡ (*i*, ‡*P*) ∈ *H*›

definition *hequiv* :: ‹('*i*, '*p*) *lbd set* ⇒ '*i* ⇒ '*i* ⇒ *bool*› **where**
‹*hequiv H i k* ≡ (*i*, •*k*) ∈ *H*›

lemma *hequiv-reflp*:
assumes ‹*consistent H*› ‹*maximal H*›
shows ‹*reflp* (*hequiv H*)›
⟨*proof*⟩

lemma *hequiv-symp*:
assumes ‹*consistent H*› ‹*maximal H*›
shows ‹*symp* (*hequiv H*)›
⟨*proof*⟩

lemma *hequiv-transp*:
assumes ‹*consistent H*› ‹*maximal H*›
shows ‹*transp* (*hequiv H*)›
⟨*proof*⟩

lemma *hequiv-equivp*:
assumes ‹*consistent H*› ‹*maximal H*›
shows ‹*equivp* (*hequiv H*)›
⟨*proof*⟩

definition *assign* :: ‹('*i*, '*p*) *lbd set* ⇒ '*i* ⇒ '*i*› **where**
‹*assign H i* ≡ *minim* (|*UNIV*|) {*k*. *hequiv H i k*}›

lemma *hequiv-ne*:
assumes ‹*consistent H*› ‹*maximal H*›
shows ‹{*k*. *hequiv H i k*} ≠ {}›
⟨*proof*⟩

lemma *hequiv-assign*:
assumes ‹*consistent H*› ‹*maximal H*›
shows ‹*hequiv H i* (*assign H i*)›
⟨*proof*⟩

lemma *hequiv-Nom*:
assumes ‹*consistent H*› ‹*maximal H*› ‹*hequiv H i k*› ‹(*i*, *p*) ∈ *H*›
shows ‹(*k*, *p*) ∈ *H*›

⟨*proof*⟩

definition *reach* :: ‹('*i*, '*p*) *lbd set* ⇒ '*i* ⇒ '*i set*› **where**
‹*reach H i* ≡ {*assign H k* |*k*. (*i*, ◇ (•*k*)) ∈ *H*}›

abbreviation *canonical* :: ‹('*i* × ('*i*, '*p*) *fm*) *set* ⇒ '*i* ⇒ ('*i*, '*p*, '*i*) *ctx*› **where**
‹*canonical H i* ≡ (*Model* (*reach H*) (*val H*), *assign H*, *assign H i*)›

lemma *Hintikka-model'*:
assumes ‹⋀*i p*. *semics* (*canonical H i*) (*rel H*) *p* ⟷ *rel H* (*canonical H i*) *p*›
shows ‹(*canonical H i* ⊨ *p*) ⟷ *rel H* (*canonical H i*) *p*›
⟨*proof*⟩

lemma *Hintikka-Extend'*:
assumes ‹*consistent H*› ‹*maximal H*› ‹*saturated H*›
shows ‹*semics* (*canonical H i*) (*rel H*) *p* ⟷ *rel H* (*canonical H i*) *p*›
⟨*proof*⟩

interpretation *Truth-Saturation*
consistent nominals-lbd witness semics semantics ‹λ*H*. {*canonical H i* |*i*. *True*}› *rel*
⟨*proof*⟩

lemma *Truth-lemma*:
assumes ‹*consistent H*› ‹*maximal H*› ‹*saturated H*›
shows ‹(*canonical H i* ⊨ *p*) ⟷ (*i*, *p*) ∈ *H*›
⟨*proof*⟩

7.9 Cardinalities

datatype *marker* = *FlsM* | *ImpM* | *DiaM* | *SatM*

type-synonym ('*i*, '*p*) *enc* = ‹('*i* + '*p*) + *marker* × *nat*›

abbreviation ‹*NOM i* ≡ *Inl* (*Inl i*)›
abbreviation ‹*PRO x* ≡ *Inl* (*Inr x*)›
abbreviation ‹*FLS* ≡ *Inr* (*FlsM*, *0*)›
abbreviation ‹*IMP n* ≡ *Inr* (*FlsM*, *n*)›
abbreviation ‹*DIA* ≡ *Inr* (*DiaM*, *0*)›
abbreviation ‹*SAT* ≡ *Inr* (*SatM*, *0*)›

primrec *encode* :: ‹('*i*, '*p*) *fm* ⇒ ('*i*, '*p*) *enc list*› **where**
‹*encode* ⊥ = [*FLS*]›
| ‹*encode* (‡*P*) = [*PRO P*]›
| ‹*encode* (•*i*) = [*NOM i*]›
| ‹*encode* (*p* ⟶ *q*) = *IMP* (*length* (*encode p*)) # *encode p* @ *encode q*›
| ‹*encode* (◇ *p*) = *DIA* # *encode p*›
| ‹*encode* (@ *i p*) = *SAT* # *NOM i* # *encode p*›

lemma *encode-ne* [*simp*]: ‹*encode p* ≠ []›
⟨*proof*⟩

lemma *inj-encode'*: ‹*encode p* = *encode q* ⟹ *p* = *q*›
⟨*proof*⟩

lemma *inj-encode*: ‹*inj encode*›

⟨*proof*⟩

primrec *encode-lbd* :: ‹('*i*, '*p*) *lbd* ⇒ ('*i*, '*p*) *enc list*› **where**
‹*encode-lbd* (*i*, *p*) = *NOM i* # *encode p*›

lemma *inj-encode-lbd'*: ‹*encode-lbd* (*i*, *p*) = *encode-lbd* (*k*, *q*) ⟹ *i* = *k* ∧ *p* = *q*›
⟨*proof*⟩

lemma *inj-encode-lbd*: ‹*inj encode-lbd*›
⟨*proof*⟩

lemma *finite-marker*: ‹*finite* (*UNIV* :: *marker set*)›
⟨*proof*⟩

lemma *card-of-lbd*:
assumes ‹*infinite* (*UNIV* :: '*i set*)›
shows ‹|*UNIV* :: ('*i*, '*p*) *lbd set*| ≤*o* |*UNIV* :: '*i set*| +*c* |*UNIV* :: '*p set*|›
⟨*proof*⟩

7.10 Completeness

theorem *strong-completeness*:
fixes *p* :: ‹('*i*, '*p*) *fm*›
assumes ‹∀ *M* :: ('*i*, '*p*) *model*. ∀ *g w*.
(∀ (*k*, *q*) ∈ *X*. (*M*, *g*, *g k*) ⊨ *q*) ⟶ (*M*, *g*, *w*) ⊨ *p*›
‹*infinite* (*UNIV* :: '*i set*)›
‹|*UNIV* :: '*i set*| +*c* |*UNIV* :: '*p set*| ≤*o* |*UNIV* − *nominals X*|›
shows ‹∃ *A*. *set A* ⊆ *X* ∧ *A* ⊢@ (*i*, *p*)›
⟨*proof*⟩

abbreviation *valid* :: ‹('*i*, '*p*) *fm* ⇒ *bool*› **where**
‹*valid p* ≡ ∀ (*M* :: ('*i*, '*p*) *model*) *g w*. (*M*, *g*, *w*) ⊨ *p*›

theorem *completeness*:
fixes *p* :: ‹('*i*, '*p*) *fm*›
assumes ‹*valid p*› ‹*infinite* (*UNIV* :: '*i set*)› ‹|*UNIV* :: '*p set*| ≤*o* |*UNIV* :: '*i set*|›
shows ‹[] ⊢@ (*i*, *p*)›
⟨*proof*⟩

corollary *completeness'*:
fixes *p* :: ‹('*i*, '*i*) *fm*›
assumes ‹*valid p*› ‹*infinite* (*UNIV* :: '*i set*)›
shows ‹[] ⊢@ (*i*, *p*)›
⟨*proof*⟩

theorem *main*:
fixes *p* :: ‹('*i*, '*p*) *fm*›
assumes ‹*i* ∉ *nominals-fm p*› ‹*infinite* (*UNIV* :: '*i set*)› ‹|*UNIV* :: '*p set*| ≤*o* |*UNIV* :: '*i set*|›
shows ‹*valid p* ⟷ [] ⊢@ (*i*, *p*)›
⟨*proof*⟩

corollary *main'*:
fixes *p* :: ‹('*i*, '*i*) *fm*›
assumes ‹*i* ∉ *nominals-fm p*› ‹*infinite* (*UNIV* :: '*i set*)›
shows ‹*valid p* ⟷ [] ⊢@ (*i*, *p*)›

⟨*proof*⟩

end

Chapter 8

Example: First-Order Logic

theory *Example-First-Order-Logic* **imports** *Derivations* **begin**

8.1 Syntax

datatype (*params-tm*: *'f*) *tm*
  = *Var nat* (‹#›)
  | *Fun 'f* ‹*'f tm list*› (‹†›)

abbreviation *Const* (‹⋆›) **where** ‹⋆*a* ≡ †*a* []›

datatype (*params-fm*: *'f*, *'p*) *fm*
  = *Fls* (‹⊥›)
  | *Pre 'p* ‹*'f tm list*› (‹‡›)
  | *Imp* ‹(*'f*, *'p*) *fm*› ‹(*'f*, *'p*) *fm*› (**infixr** ‹⟶› *55*)
  | *Uni* ‹(*'f*, *'p*) *fm*› (‹∀ ›)

abbreviation *Neg* (‹¬ -› [*70*] *70*) **where** ‹¬ *p* ≡ *p* ⟶ ⊥›

8.2 Semantics

type-synonym (*'a*, *'f*, *'p*) *model* = ‹(*nat* ⇒ *'a*) × (*'f* ⇒ *'a list* ⇒ *'a*) × (*'p* ⇒ *'a list* ⇒ *bool*)›

fun *semantics-tm* :: ‹(*nat* ⇒ *'a*) × (*'f* ⇒ *'a list* ⇒ *'a*) ⇒ *'f tm* ⇒ *'a*› (‹⦇-⦈›) **where**
  ‹⦇(*E*, -)⦈ (#*n*) = *E n*›
| ‹⦇(*E*, *F*)⦈ (†*f ts*) = *F f* (*map* ⦇(*E*, *F*)⦈ *ts*)›

primrec *add-env* :: ‹*'a* ⇒ (*nat* ⇒ *'a*) ⇒ *nat* ⇒ *'a*› (**infix** ‹⨟› *0*) **where**
  ‹(*t* ⨟ *s*) *0* = *t*›
| ‹(*t* ⨟ *s*) (*Suc n*) = *s n*›

fun *semantics-fm* :: ‹(*'a*, *'f*, *'p*) *model* ⇒ (*'f*, *'p*) *fm* ⇒ *bool*› (‹⟦-⟧›) **where**
  ‹⟦-⟧ ⊥ ⟷ *False*›
| ‹⟦(*E*, *F*, *G*)⟧ (‡*P ts*) ⟷ *G P* (*map* ⦇(*E*, *F*)⦈ *ts*)›
| ‹⟦(*E*, *F*, *G*)⟧ (*p* ⟶ *q*) ⟷ ⟦(*E*, *F*, *G*)⟧ *p* ⟶ ⟦(*E*, *F*, *G*)⟧ *q*›
| ‹⟦(*E*, *F*, *G*)⟧ (∀ *p*) ⟷ (∀ *x*. ⟦(*x* ⨟ *E*, *F*, *G*)⟧ *p*)›

8.3 Operations

primrec *lift-tm* :: ‹*'f tm* ⇒ *'f tm*› **where**

‹*lift-tm* (#*n*) = #(*n*+*1*)›
| ‹*lift-tm* (†*f ts*) = †*f* (*map lift-tm ts*)›

primrec *sub-tm* :: ‹(*nat* ⇒ '*f tm*) ⇒ '*f tm* ⇒ '*f tm*› **where**
‹*sub-tm s* (#*n*) = *s n*›
| ‹*sub-tm s* (†*f ts*) = †*f* (*map* (*sub-tm s*) *ts*)›

primrec *sub-fm* :: ‹(*nat* ⇒ '*f tm*) ⇒ ('*f*, '*p*) *fm* ⇒ ('*f*, '*p*) *fm*› **where**
‹*sub-fm - ⊥* = *⊥*›
| ‹*sub-fm s* (‡*P ts*) = ‡*P* (*map* (*sub-tm s*) *ts*)›
| ‹*sub-fm s* (*p* ⟶ *q*) = *sub-fm s p* ⟶ *sub-fm s q*›
| ‹*sub-fm s* (∀ *p*) = ∀ (*sub-fm* (#*0* ⨟ λ*n*. *lift-tm* (*s n*)) *p*)›

abbreviation *inst-single* :: ‹'*f tm* ⇒ ('*f*, '*p*) *fm* ⇒ ('*f*, '*p*) *fm*› (‹⟨-⟩›) **where**
‹⟨*t*⟩ ≡ *sub-fm* (*t* ⨟ #)›

abbreviation ‹*params'* *S* ≡ ⋃ *p* ∈ *S*. *params-fm p*›

abbreviation ‹*params l* ≡ *params'* (*set l*)›

lemma *upd-params-tm* [*simp*]: ‹*f* ∉ *params-tm t* ⟹ (|(*E*, *F*(*f* := *x*))|) *t* = (|(*E*, *F*)|) *t*›
⟨*proof*⟩

lemma *upd-params-fm* [*simp*]: ‹*f* ∉ *params-fm p* ⟹ ⟦(*E*, *F*(*f* := *x*), *G*)⟧ *p* = ⟦(*E*, *F*, *G*)⟧ *p*›
⟨*proof*⟩

lemma *finite-params-tm* [*simp*]: ‹*finite* (*params-tm t*)›
⟨*proof*⟩

lemma *finite-params-fm* [*simp*]: ‹*finite* (*params-fm p*)›
⟨*proof*⟩

lemma *env* [*simp*]: ‹*P* ((*x* ⨟ *E*) *n*) = (*P x* ⨟ λ*n*. *P* (*E n*)) *n*›
⟨*proof*⟩

lemma *lift-lemma*: ‹(|(*x* ⨟ *E*, *F*)|) (*lift-tm t*) = (|(*E*, *F*)|) *t*›
⟨*proof*⟩

lemma *sub-tm-semantics*: ‹(|(*E*, *F*)|) (*sub-tm s t*) = (|(λ*n*. (|(*E*, *F*)|) (*s n*), *F*)|) *t*›
⟨*proof*⟩

lemma *sub-fm-semantics* [*simp*]: ‹⟦(*E*, *F*, *G*)⟧ (*sub-fm s p*) = ⟦(λ*n*. (|(*E*, *F*)|) (*s n*), *F*, *G*)⟧ *p*›
⟨*proof*⟩

lemma *sub-tm-Var*: ‹*sub-tm* # *t* = *t*›
⟨*proof*⟩

lemma *reduce-Var* [*simp*]: ‹(# *0* ⨟ λ*n*. # (*Suc n*)) = #›
⟨*proof*⟩

lemma *sub-fm-Var* [*simp*]:
fixes *p* :: ‹('*f*, '*p*) *fm*›
shows ‹*sub-fm* # *p* = *p*›
⟨*proof*⟩

lemma *semantics-tm-id* [*simp*]: ‹(|(#, †)|) *t* = *t*›

⟨*proof*⟩

lemma *semantics-tm-id-map* [*simp*]: ‹*map* (⦇(#, †)⦈) *ts* = *ts*›
⟨*proof*⟩

The built-in *size* is not invariant under substitution.

primrec *size-fm* :: ‹('*f*, '*p*) *fm* ⇒ *nat*› **where**
‹*size-fm* ⊥ = *1*›
| ‹*size-fm* (‡- -) = *1*›
| ‹*size-fm* (*p* ⟶ *q*) = *1* + *size-fm p* + *size-fm q*›
| ‹*size-fm* (∀ *p*) = *1* + *size-fm p*›

lemma *size-sub-fm* [*simp*]: ‹*size-fm* (*sub-fm s p*) = *size-fm p*›
⟨*proof*⟩

8.4 Calculus

inductive *Calculus* :: ‹('*f*, '*p*) *fm list* ⇒ ('*f*, '*p*) *fm* ⇒ *bool*› (‹- ⊢∀ -› [*50*, *50*] *50*) **where**
Assm [*intro*]: ‹*p* ∈ *set A* ⟹ *A* ⊢∀ *p*›
| *FlsE* [*elim*]: ‹*A* ⊢∀ ⊥ ⟹ *A* ⊢∀ *p*›
| *ImpI* [*intro*]: ‹*p* # *A* ⊢∀ *q* ⟹ *A* ⊢∀ *p* ⟶ *q*›
| *ImpE* [*elim*]: ‹*A* ⊢∀ *p* ⟶ *q* ⟹ *A* ⊢∀ *p* ⟹ *A* ⊢∀ *q*›
| *UniI* [*intro*]: ‹*A* ⊢∀ ⟨⋆*a*⟩*p* ⟹ *a* ∉ *params* (*p* # *A*) ⟹ *A* ⊢∀ ∀ *p*›
| *UniE* [*elim*]: ‹*A* ⊢∀ ∀ *p* ⟹ *A* ⊢∀ ⟨*t*⟩*p*›
| *Clas*: ‹(*p* ⟶ *q*) # *A* ⊢∀ *p* ⟹ *A* ⊢∀ *p*›
| *Weak*: ‹*A* ⊢∀ *p* ⟹ *q* # *A* ⊢∀ *p*›

8.5 Soundness

theorem *soundness*: ‹*A* ⊢∀ *p* ⟹ *list-all* ⟦(*E*, *F*, *G*)⟧ *A* ⟹ ⟦(*E*, *F*, *G*)⟧ *p*›
⟨*proof*⟩

corollary *soundness'*: ‹[] ⊢∀ *p* ⟹ ⟦*M*⟧ *p*›
⟨*proof*⟩

corollary ‹¬ ([] ⊢∀ ⊥)›
⟨*proof*⟩

8.6 Admissible Rules

lemma *Assm-head*: ‹*p* # *A* ⊢∀ *p*›
⟨*proof*⟩

lemma *deduct1*: ‹*A* ⊢∀ *p* ⟶ *q* ⟹ *p* # *A* ⊢∀ *q*›
⟨*proof*⟩

lemma *Boole*: ‹(¬ *p*) # *A* ⊢∀ ⊥ ⟹ *A* ⊢∀ *p*›
⟨*proof*⟩

lemma *Weak'*: ‹*A* ⊢∀ *p* ⟹ *B* @ *A* ⊢∀ *p*›
⟨*proof*⟩

lemma *Weaken*: ‹*A* ⊢∀ *p* ⟹ *set A* ⊆ *set B* ⟹ *B* ⊢∀ *p*›
⟨*proof*⟩

interpretation *Derivations Calculus*
⟨*proof*⟩

8.7 Maximal Consistent Sets

definition *consistent* :: ‹('f, 'p) *fm set* ⇒ *bool*› **where**
‹*consistent S* ≡ ∄*S'*. *set S'* ⊆ *S* ∧ *S'* ⊢∀ ⊥›

fun *witness* :: ‹('f, 'p) *fm* ⇒ ('f, 'p) *fm set* ⇒ ('f, 'p) *fm set*› **where**
‹*witness* (¬ (∀ *p*)) *S* = {¬ ⟨⋆(*SOME a*. *a* ∉ *params'* ({*p*} ∪ *S*))⟩*p*}›
| ‹*witness* - - = {}›

lemma *consistent-add-instance*:
assumes ‹*consistent S*› ‹∀ *p* ∈ *S*›
shows ‹*consistent* ({⟨*t*⟩*p*} ∪ *S*)›
⟨*proof*⟩

lemma *consistent-add-witness*:
assumes ‹*consistent S*› ‹¬ (∀ *p*) ∈ *S*› ‹*a* ∉ *params'* *S*›
shows ‹*consistent* ({¬ ⟨⋆*a*⟩*p*} ∪ *S*)›
⟨*proof*⟩

lemma *consistent-witness'*:
assumes ‹*consistent* ({*p*} ∪ *S*)› ‹*infinite* (*UNIV* − *params'* *S*)›
shows ‹*consistent* (*witness p S* ∪ {*p*} ∪ *S*)›
⟨*proof*⟩

interpretation *MCS-Saturation consistent params-fm witness*
⟨*proof*⟩

interpretation *Derivations-MCS-Cut Calculus consistent* ‹⊥›
⟨*proof*⟩

8.8 Truth Lemma

abbreviation *hmodel* :: ‹('f, 'p) *fm set* ⇒ ('f *tm*, 'f, 'p) *model*› **where**
‹*hmodel H* ≡ (#, †, λ*P ts*. ‡*P ts* ∈ *H*)›

fun *semics* ::
‹('a, 'f, 'p) *model* ⇒ (('a, 'f, 'p) *model* ⇒ ('f, 'p) *fm* ⇒ *bool*) ⇒ ('f, 'p) *fm* ⇒ *bool*› **where**
‹*semics* - - ⊥ ⟷ *False*›
| ‹*semics* (*E*, *F*, *G*) - (‡*P ts*) ⟷ *G P* (*map* (⟦(*E*, *F*)⟧) *ts*)›
| ‹*semics* (*E*, *F*, *G*) *rel* (*p* ⟶ *q*) ⟷ *rel* (*E*, *F*, *G*) *p* ⟶ *rel* (*E*, *F*, *G*) *q*›
| ‹*semics* (*E*, *F*, *G*) *rel* (∀ *p*) ⟷ (∀ *x*. *rel* (*x* ⨾ *E*, *F*, *G*) *p*)›

fun *rel* :: ‹('f, 'p) *fm set* ⇒ ('f *tm*, 'f, 'p) *model* ⇒ ('f, 'p) *fm* ⇒ *bool*› **where**
‹*rel H* (*E*, -, -) *p* = (*sub-fm E p* ∈ *H*)›

theorem *Hintikka-model'*:
assumes ‹⋀*p*. *semics* (*hmodel H*) (*rel H*) *p* ⟷ *p* ∈ *H*›
shows ‹*p* ∈ *H* ⟷ ⟦*hmodel H*⟧ *p*›
⟨*proof*⟩

lemma *Hintikka-Extend*:

assumes ‹*consistent H*› ‹*maximal H*› ‹*saturated H*›
shows ‹*semics* (*hmodel H*) (*rel H*) *p* ⟷ *p* ∈ *H*›
⟨*proof*⟩

interpretation *Truth-Saturation*
consistent params-fm witness semics semantics-fm ‹λ*H*. {*hmodel H*}› *rel*
⟨*proof*⟩

8.9 Cardinalities

datatype *marker* = *VarM* | *FunM* | *TmM* | *FlsM* | *PreM* | *ImpM* | *UniM*

type-synonym ('*f*, '*p*) *enc* = ‹('*f* + '*p*) + *marker* × *nat*›

abbreviation ‹*FUNS f* ≡ *Inl* (*Inl f*)›
abbreviation ‹*PRES p* ≡ *Inl* (*Inr p*)›

abbreviation ‹*VAR n* ≡ *Inr* (*VarM*, *n*)›
abbreviation ‹*FUN n* ≡ *Inr* (*FunM*, *n*)›
abbreviation ‹*TM n* ≡ *Inr* (*TmM*, *n*)›

abbreviation ‹*PRE n* ≡ *Inr* (*PreM*, *n*)›
abbreviation ‹*FLS* ≡ *Inr* (*FlsM*, *0*)›
abbreviation ‹*IMP n* ≡ *Inr* (*FlsM*, *n*)›
abbreviation ‹*UNI* ≡ *Inr* (*UniM*, *0*)›

primrec
encode-tm :: ‹'*f tm* ⇒ ('*f*, '*p*) *enc list*› **and**
encode-tms :: ‹'*f tm list* ⇒ ('*f*, '*p*) *enc list*› **where**
‹*encode-tm* (#*n*) = [*VAR n*]›
| ‹*encode-tm* (†*f ts*) = *FUN* (*length ts*) # *FUNS f* # *encode-tms ts*›
| ‹*encode-tms* [] = []›
| ‹*encode-tms* (*t* # *ts*) = *TM* (*length* (*encode-tm t*)) # *encode-tm t* @ *encode-tms ts*›

lemma *encode-tm-ne* [*simp*]: ‹*encode-tm t* ≠ []›
⟨*proof*⟩

lemma *inj-encode-tm'*:
‹(*encode-tm t* :: ('*f*, '*p*) *enc list*) = *encode-tm s* ⟹ *t* = *s*›
‹(*encode-tms ts* :: ('*f*, '*p*) *enc list*) = *encode-tms ss* ⟹ *ts* = *ss*›
⟨*proof*⟩

lemma *inj-encode-tm*: ‹*inj encode-tm*›
⟨*proof*⟩

primrec *encode-fm* :: ‹('*f*, '*p*) *fm* ⇒ ('*f*, '*p*) *enc list*› **where**
‹*encode-fm* ⊥ = [*FLS*]›
| ‹*encode-fm* (‡*P ts*) = *PRE* (*length ts*) # *PRES P* # *encode-tms ts*›
| ‹*encode-fm* (*p* ⟶ *q*) = *IMP* (*length* (*encode-fm p*)) # *encode-fm p* @ *encode-fm q*›
| ‹*encode-fm* (∀ *p*) = *UNI* # *encode-fm p*›

lemma *encode-fm-ne* [*simp*]: ‹*encode-fm p* ≠ []›
⟨*proof*⟩

lemma *inj-encode-fm'*: ‹*encode-fm p* = *encode-fm q* ⟹ *p* = *q*›

⟨*proof*⟩

lemma *inj-encode-fm*: ‹*inj encode-fm*›
⟨*proof*⟩

lemma *finite-marker*: ‹*finite* (*UNIV* :: *marker set*)›
⟨*proof*⟩

lemma *card-of-fm*:
assumes ‹*infinite* (*UNIV* :: *'f set*)›
shows ‹|*UNIV* :: (*'f*, *'p*) *fm set*| ≤*o* |*UNIV* :: *'f set*| +*c* |*UNIV* :: *'p set*|›
⟨*proof*⟩

8.10 Completeness

theorem *strong-completeness*:
assumes ‹∀ *M* :: (*'f tm*, *'f*, *'p*) *model*. (∀ *q* ∈ *X*. ⟦*M*⟧ *q*) ⟶ ⟦*M*⟧ *p*›
‹*infinite* (*UNIV* :: *'f set*)›
‹|*UNIV* :: *'f set*| +*c* |*UNIV* :: *'p set*| ≤*o* |*UNIV* − *params'* *X*|›
shows ‹∃ *A*. *set A* ⊆ *X* ∧ *A* ⊢∀ *p*›
⟨*proof*⟩

abbreviation *valid* :: ‹(*'f*, *'p*) *fm* ⇒ *bool*› **where**
‹*valid p* ≡ ∀ *M* :: (*'f tm*, -, -) *model*. ⟦*M*⟧ *p*›

theorem *completeness*:
fixes *p* :: ‹(*'f*, *'p*) *fm*›
assumes ‹*valid p*› ‹*infinite* (*UNIV* :: *'f set*)› ‹|*UNIV* :: *'p set*| ≤*o* |*UNIV* :: *'f set*|›
shows ‹[] ⊢∀ *p*›
⟨*proof*⟩

corollary *completeness'*:
fixes *p* :: ‹(*'f*, *'f*) *fm*›
assumes ‹*valid p*› ‹*infinite* (*UNIV* :: *'f set*)›
shows ‹[] ⊢∀ *p*›
⟨*proof*⟩

theorem *main*:
fixes *p* :: ‹(*'f*, *'p*) *fm*›
assumes ‹*infinite* (*UNIV* :: *'f set*)› ‹|*UNIV* :: *'p set*| ≤*o* |*UNIV* :: *'f set*|›
shows ‹*valid p* ⟷ [] ⊢∀ *p*›
⟨*proof*⟩

corollary *main'*:
fixes *p* :: ‹(*'f*, *'f*) *fm*›
assumes ‹*infinite* (*UNIV* :: *'f set*)›
shows ‹*valid p* ⟷ [] ⊢∀ *p*›
⟨*proof*⟩

end

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