

Symmetric Polynomials

Manuel Eberl

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Abstract

A symmetric polynomial is a polynomial in variables X_1, \dots, X_n that does not discriminate between its variables, i. e. it is invariant under any permutation of them. These polynomials are important in the study of the relationship between the coefficients of a univariate polynomial and its roots in its algebraic closure.

This article provides a definition of symmetric polynomials and the elementary symmetric polynomials e_1, \dots, e_n and proofs of their basic properties, including three notable ones:

- Vieta's formula, which gives an explicit expression for the k -th coefficient of a univariate monic polynomial in terms of its roots x_1, \dots, x_n , namely $c_k = (-1)^{n-k} e_{n-k}(x_1, \dots, x_n)$.
- Second, the Fundamental Theorem of Symmetric Polynomials, which states that any symmetric polynomial is itself a uniquely determined polynomial combination of the elementary symmetric polynomials.
- Third, as a corollary of the previous two, that given a polynomial over some ring R , any symmetric polynomial combination of its roots is also in R even when the roots are not.

Both the symmetry property itself and the witness for the Fundamental Theorem are executable.

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1 Vieta's Formulas

```
theory Vieta
imports
  HOL-Library.FuncSet
  HOL-Computational-Algebra.Computational-Algebra
begin
```

1.1 Auxiliary material

lemma *card-vimage-inter*:

assumes *inj*: *inj-on* *f* *A* **and** *subset*: $X \subseteq f^{-1} A$
shows $\text{card } (f^{-1} X \cap A) = \text{card } X$

proof -

have $\text{card } (f^{-1} X \cap A) = \text{card } (f^{-1} (f^{-1} X \cap A))$
by (*subst card-image*) (*auto intro!*: *inj-on-subset*[*OF inj*])
also have $f^{-1} (f^{-1} X \cap A) = X$
using *assms* **by** *auto*
finally show *?thesis* .

qed

lemma *bij-betw-image-fixed-card-subset*:

assumes *inj-on* *f* *A*
shows $\text{bij-betw } (\lambda X. f^{-1} X) \{X. X \subseteq A \wedge \text{card } X = k\} \{X. X \subseteq f^{-1} A \wedge \text{card } X = k\}$
using *assms inj-on-subset*[*OF assms*]
by (*intro bij-betwI*[*of - -* $\lambda X. f^{-1} X \cap A$]) (*auto simp: card-image card-vimage-inter*)

lemma *image-image-fixed-card-subset*:

assumes *inj-on* *f* *A*
shows $(\lambda X. f^{-1} X)^{-1} \{X. X \subseteq A \wedge \text{card } X = k\} = \{X. X \subseteq f^{-1} A \wedge \text{card } X = k\}$
using *bij-betw-imp-surj-on*[*OF bij-betw-image-fixed-card-subset*[*OF assms, of k*]]
.

lemma *prod-uminus*: $(\prod_{x \in A}. -f\ x :: 'a :: \text{comm-ring-1}) = (-1)^{\wedge \text{card } A} *$
 $(\prod_{x \in A}. f\ x)$

by (*induction A rule: infinite-finite-induct*) (*auto simp: algebra-simps*)

theorem *prod-sum-PiE*:

fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c :: \text{comm-semiring-1}$
assumes *finite*: *finite* *A* **and** *finite*: $\bigwedge x. x \in A \implies \text{finite } (B\ x)$
shows $(\prod_{x \in A}. \sum_{y \in B\ x}. f\ x\ y) = (\sum_{g \in \text{PiE } A\ B}. \prod_{x \in A}. f\ x\ (g\ x))$
using *assms*

proof (*induction A rule: finite-induct*)

case *empty*
thus *?case* **by** *auto*

next

case (*insert* *x* *A*)
have $(\sum_{g \in \text{PiE } (\text{insert } x\ A)\ B}. \prod_{x \in \text{insert } x\ A}. f\ x\ (g\ x)) =$

$(\sum_{g \in Pi_E} (insert\ x\ A)\ B.\ f\ x\ (g\ x) * (\prod_{x' \in A} f\ x'\ (g\ x')))$
using *insert by simp*
also have $(\lambda g. \prod_{x' \in A} f\ x'\ (g\ x')) = (\lambda g. \prod_{x' \in A} f\ x'\ (if\ x' = x\ then\ undefined\ else\ g\ x'))$
using *insert by (intro ext prod.cong) auto*
also have $(\sum_{g \in Pi_E} (insert\ x\ A)\ B.\ f\ x\ (g\ x) * \dots\ g) =$
 $(\sum_{(y,g) \in B\ x \times Pi_E\ A\ B} f\ x\ y * (\prod_{x' \in A} f\ x'\ (g\ x')))$
using *insert.premis insert.hyps*
by *(intro sum.reindex-bij-witness[of - $\lambda(y,g). g(x := y)$ $\lambda g. (g\ x, g(x := unde- fined))$]))*
(auto simp: PiE-def extensional-def)
also have $\dots = (\sum_{y \in B\ x} \sum_{g \in Pi_E\ A\ B} f\ x\ y * (\prod_{x' \in A} f\ x'\ (g\ x')))$
by *(subst sum.cartesian-product) auto*
also have $\dots = (\sum_{y \in B\ x} f\ x\ y) * (\sum_{g \in Pi_E\ A\ B} \prod_{x' \in A} f\ x'\ (g\ x'))$
using *insert by (subst sum.swap) (simp add: sum-distrib-left sum-distrib-right)*
also have $(\sum_{g \in Pi_E\ A\ B} \prod_{x' \in A} f\ x'\ (g\ x')) = (\prod_{x \in A} \sum_{y \in B\ x} f\ x\ y)$
using *insert.premis by (intro insert.IH [symmetric]) auto*
also have $(\sum_{y \in B\ x} f\ x\ y) * \dots = (\prod_{x \in insert\ x\ A} \sum_{y \in B\ x} f\ x\ y)$
using *insert.hyps by simp*
finally show *?case ..*
qed

corollary *prod-add:*

fixes *f1 f2 :: 'a \Rightarrow 'c :: comm-semiring-1*
assumes *finite: finite A*
shows $(\prod_{x \in A} f1\ x + f2\ x) = (\sum_{X \in Pow\ A} (\prod_{x \in X} f1\ x) * (\prod_{x \in A-X} f2\ x))$
proof *-*
have $(\prod_{x \in A} f1\ x + f2\ x) = (\sum_{g \in A \rightarrow_E\ UNIV} \prod_{x \in A} if\ g\ x\ then\ f1\ x\ else\ f2\ x)$
using *prod-sum-PiE[of A $\lambda\cdot$. UNIV :: bool set $\lambda x\ y. if\ y\ then\ f1\ x\ else\ f2\ x$]*
assms
by *(simp-all add: UNIV-bool add-ac)*
also have $\dots = (\sum_{X \in Pow\ A} \prod_{x \in A} if\ x \in X\ then\ f1\ x\ else\ f2\ x)$
by *(intro sum.reindex-bij-witness*
 $[of - \lambda X\ x. if\ x \in A\ then\ x \in X\ else\ undefined\ \lambda P. \{x \in A. P\ x\}])$ *auto*
also have $\dots = (\sum_{X \in Pow\ A} (\prod_{x \in X} f1\ x) * (\prod_{x \in A-X} f2\ x))$
proof *(intro sum.cong refl, goal-cases)*
case *(1 X)*
let *?f = $\lambda x. if\ x \in X\ then\ f1\ x\ else\ f2\ x$*
have $prod\ f1\ X * prod\ f2\ (A - X) = prod\ ?f\ X * prod\ ?f\ (A - X)$
by *(intro arg-cong2[of - - - (*)] prod.cong) auto*
also have $\dots = prod\ ?f\ (X \cup (A - X))$
using *1 by (subst prod.union-disjoint) (auto intro: finite-subset[OF - finite])*
also have $X \cup (A - X) = A$ **using** *1 by auto*
finally show *?case ..*
qed
finally show *?thesis .*
qed

corollary *prod-diff1*:

fixes $f1\ f2 :: 'a \Rightarrow 'c :: \text{comm-ring-1}$
assumes *finite*: $\text{finite } A$
shows $(\prod_{x \in A}. f1\ x - f2\ x) = (\sum_{X \in \text{Pow } A}. (-1) \wedge \text{card } X * (\prod_{x \in X}. f2\ x) * (\prod_{x \in A-X}. f1\ x))$
proof –
have $(\prod_{x \in A}. f1\ x - f2\ x) = (\prod_{x \in A}. -f2\ x + f1\ x)$
by *simp*
also have $\dots = (\sum_{X \in \text{Pow } A}. (\prod_{x \in X}. -f2\ x) * \text{prod } f1\ (A - X))$
by *(rule prod-add) fact+*
also have $\dots = (\sum_{X \in \text{Pow } A}. (-1) \wedge \text{card } X * (\prod_{x \in X}. f2\ x) * \text{prod } f1\ (A - X))$
by *(simp add: prod-uminus)*
finally show *?thesis* .
qed

corollary *prod-diff2*:

fixes $f1\ f2 :: 'a \Rightarrow 'c :: \text{comm-ring-1}$
assumes *finite*: $\text{finite } A$
shows $(\prod_{x \in A}. f1\ x - f2\ x) = (\sum_{X \in \text{Pow } A}. (-1) \wedge (\text{card } A - \text{card } X) * (\prod_{x \in X}. f1\ x) * (\prod_{x \in A-X}. f2\ x))$
proof –
have $(\prod_{x \in A}. f1\ x - f2\ x) = (\prod_{x \in A}. f1\ x + (-f2\ x))$
by *simp*
also have $\dots = (\sum_{X \in \text{Pow } A}. (\prod_{x \in X}. f1\ x) * (\prod_{x \in A-X}. -f2\ x))$
by *(rule prod-add) fact+*
also have $\dots = (\sum_{X \in \text{Pow } A}. (-1) \wedge \text{card } (A - X) * (\prod_{x \in X}. f1\ x) * (\prod_{x \in A-X}. f2\ x))$
by *(simp add: prod-uminus mult-ac)*
also have $\dots = (\sum_{X \in \text{Pow } A}. (-1) \wedge (\text{card } A - \text{card } X) * (\prod_{x \in X}. f1\ x) * (\prod_{x \in A-X}. f2\ x))$
using *finite-subset[OF - assms]* **by** *(intro sum.cong refl, subst card-Diff-subset) auto*
finally show *?thesis* .
qed

1.2 Main proofs

Our goal is to determine the coefficients of some fully factored polynomial $p(X) = c(X - x_1) \dots (X - x_n)$ in terms of the x_i . It is clear that it is sufficient to consider monic polynomials (i.e. $c = 1$), since the general case follows easily from this one.

We start off by expanding the product over the linear factors:

lemma *poly-from-roots*:

fixes $f :: 'a \Rightarrow 'b :: \text{comm-ring-1}$ **assumes** *fin*: $\text{finite } A$
shows $(\prod_{x \in A}. [-f\ x, 1:]) = (\sum_{X \in \text{Pow } A}. \text{monom } ((-1) \wedge \text{card } X * (\prod_{x \in X}. f\ x)) (\text{card } (A - X)))$

proof –
have $(\prod x \in A. [-f x, 1:]) = (\prod x \in A. [:0, 1:] - [:f x:])$
by *simp*
also have $\dots = (\sum X \in Pow A. (-1) \wedge card X * (\prod x \in X. [:f x:]) * monom 1$
 $(card (A - X)))$
using *fin* **by** (*subst prod-diff1*) (*auto simp: monom-altdef mult-ac*)
also have $\dots = (\sum X \in Pow A. monom ((-1) \wedge card X * (\prod x \in X. f x)) (card$
 $(A - X)))$
proof (*intro sum.cong refl, goal-cases*)
case (1 X)
have $(-1 :: 'b poly) \wedge card X = [:(-1) \wedge card X:]$
by (*induction X rule: infinite-finite-induct*) (*auto simp: one-pCons alge-*
bra-simps)
moreover have $(\prod x \in X. [:f x:]) = [:\prod x \in X. f x:]$
by (*induction X rule: infinite-finite-induct*) *auto*
ultimately show ?*case* **by** (*simp add: smult-monom*)
qed
finally show ?*thesis* .
qed

Comparing coefficients yields Vieta's formula:

theorem *coeff-poly-from-roots*:
fixes $f :: 'a \Rightarrow 'b :: comm-ring-1$
assumes *fin*: *finite A* **and** $k: k \leq card A$
shows $coeff (\prod x \in A. [-f x, 1:]) k =$
 $(-1) \wedge (card A - k) * (\sum X \mid X \subseteq A \wedge card X = card A - k. (\prod x \in X.$
 $f x))$
proof –
have $(\prod x \in A. [-f x, 1:]) = (\sum X \in Pow A. monom ((-1) \wedge card X * (\prod x \in X.$
 $f x)) (card (A - X)))$
by (*intro poly-from-roots fin*)
also have $coeff \dots k = (\sum X \mid X \subseteq A \wedge card X = card A - k. (-1) \wedge (card$
 $A - k) * (\prod x \in X. f x))$
unfolding *coeff-sum coeff-monom* **using** *finite-subset[OF - fin] k card-mono[OF*
fin]
by (*intro sum.mono-neutral-cong-right*) (*auto simp: card-Diff-subset*)
also have $\dots = (-1) \wedge (card A - k) * (\sum X \mid X \subseteq A \wedge card X = card A - k.$
 $(\prod x \in X. f x))$
by (*simp add: sum-distrib-left*)
finally show ?*thesis* .
qed

If the roots are all distinct, we can get the following alternative representation:

corollary *coeff-poly-from-roots'*:
fixes $f :: 'a \Rightarrow 'b :: comm-ring-1$
assumes *fin*: *finite A* **and** *inj*: *inj-on f A* **and** $k: k \leq card A$
shows $coeff (\prod x \in A. [-f x, 1:]) k =$
 $(-1) \wedge (card A - k) * (\sum X \mid X \subseteq f' A \wedge card X = card A - k. \prod X)$

proof –
have $\text{coeff} (\prod_{x \in A}. [-f\ x, 1:])\ k =$
 $(-1)^{\wedge}(\text{card } A - k) * (\sum X \mid X \subseteq A \wedge \text{card } X = \text{card } A - k. (\prod_{x \in X}. f\ x))$
by (*intro coeff-poly-from-roots assms*)
also have $(\sum X \mid X \subseteq A \wedge \text{card } X = \text{card } A - k. (\prod_{x \in X}. f\ x)) =$
 $(\sum X \mid X \subseteq A \wedge \text{card } X = \text{card } A - k. \prod (f'X))$
by (*intro sum.cong refl, subst prod.reindex*) (*auto intro: inj-on-subset[OF inj]*)
also have $\dots = (\sum X \in (\lambda X. f'X) \cdot \{X. X \subseteq A \wedge \text{card } X = \text{card } A - k\}. \prod X)$
by (*subst sum.reindex*) (*auto intro!: inj-on-image inj-on-subset[OF inj]*)
also have $(\lambda X. f'X) \cdot \{X. X \subseteq A \wedge \text{card } X = \text{card } A - k\} = \{X. X \subseteq f' A$
 $\wedge \text{card } X = \text{card } A - k\}$
by (*intro image-image-fixed-card-subset inj*)
finally show *?thesis* .
qed
end

2 Symmetric Polynomials

theory *Symmetric-Polynomials*
imports
Vieta
Polynomials.More-MPoly-Type
HOL-Combinatorics.Permutations
begin

2.1 Auxiliary facts

An infinite set has infinitely many infinite subsets.

lemma *infinite-infinite-subsets*:

assumes *infinite A*
shows *infinite {X. X ⊆ A ∧ infinite X}*
proof –
have $\forall k. \exists X. X \subseteq A \wedge \text{infinite } X \wedge \text{card } (A - X) = k$ **for** $k :: \text{nat}$
proof
fix $k :: \text{nat}$ **obtain** Y **where** *finite Y card Y = k Y ⊆ A*
using *infinite-arbitrarily-large[of A k] assms* **by** *auto*
moreover from this have $A - (A - Y) = Y$ **by** *auto*
ultimately show $\exists X. X \subseteq A \wedge \text{infinite } X \wedge \text{card } (A - X) = k$
using *assms* **by** (*intro exI[of - A - Y]*) *auto*
qed
from *choice[OF this]* **obtain** f
where $f: \lambda k. f\ k \subseteq A \wedge \text{infinite } (f\ k) \wedge \text{card } (A - f\ k) = k$ **by** *blast*
have $k = l$ **if** $f\ k = f\ l$ **for** $k\ l$
proof (*rule ccontr*)
assume $k \neq l$
hence $\text{card } (A - f\ k) \neq \text{card } (A - f\ l)$

```

    using f[of k] f[of l] by auto
    with ⟨f k = f l⟩ show False by simp
qed
hence inj f by (auto intro: injI)
moreover have range f ⊆ {X. X ⊆ A ∧ infinite X}
  using f by auto
ultimately show ?thesis
  by (subst infinite-iff-countable-subset) auto
qed

```

An infinite set contains infinitely many finite subsets of any fixed nonzero cardinality.

```

lemma infinite-card-subsets:
  assumes infinite A k > 0
  shows infinite {X. X ⊆ A ∧ finite X ∧ card X = k}
proof -
  obtain B where B: B ⊆ A finite B card B = k - 1
    using infinite-arbitrarily-large[OF assms(1), of k - 1] by blast
  define f where f = (λx. insert x B)
  have f '(A - B) ⊆ {X. X ⊆ A ∧ finite X ∧ card X = k}
    using assms B by (auto simp: f-def)
  moreover have inj-on f (A - B)
    by (auto intro!: inj-onI simp: f-def)
  hence infinite (f '(A - B))
    using assms B by (subst finite-image-iff) auto
  ultimately show ?thesis
    by (rule infinite-super)
qed

```

```

lemma comp-bij-eq-iff:
  assumes bij f
  shows g ∘ f = h ∘ f ⟷ g = h
proof
  assume *: g ∘ f = h ∘ f
  show g = h
  proof
    fix x
    obtain y where [simp]: x = f y using bij-is-surj[OF assms] by auto
    have (g ∘ f) y = (h ∘ f) y by (simp only: *)
    thus g x = h x by simp
  qed
qed
qed auto

```

```

lemma sum-list-replicate [simp]:
  sum-list (replicate n x) = of-nat n * (x :: 'a :: semiring-1)
  by (induction n) (auto simp: algebra-simps)

```

```

lemma ex-subset-of-card:
  assumes finite A card A ≥ k

```



```

shows  $\exists B. B \subseteq A \wedge \text{card } B = k$ 
using assms
proof (induction arbitrary: k rule: finite-induct)
  case empty
  thus ?case by auto
next
  case (insert x A k)
  show ?case
  proof (cases k = 0)
    case True
    thus ?thesis by (intro exI[of - {}]) auto
  next
  case False
  from insert have  $\exists B \subseteq A. \text{card } B = k - 1$  by (intro insert.IH) auto
  then obtain B where  $B: B \subseteq A \text{ card } B = k - 1$  by auto
  with insert have [simp]:  $x \notin B$  by auto
  have insert x B  $\subseteq$  insert x A
  using B insert by auto
  moreover have  $\text{card } (\text{insert } x B) = k$ 
  using insert B finite-subset[of B A] False by (subst card.insert-remove) auto
  ultimately show ?thesis by blast
qed
qed

```

```

lemma length-sorted-list-of-set [simp]:  $\text{length } (\text{sorted-list-of-set } A) = \text{card } A$ 
using distinct-card[of sorted-list-of-set A] by (cases finite A) simp-all

```

```

lemma upt-add-eq-append':  $i \leq j \implies j \leq k \implies [i..<k] = [i..<j] @ [j..<k]$ 
using upt-add-eq-append[of i j k - j] by simp

```

2.2 Subrings and ring homomorphisms

```

locale ring-closed =
  fixes  $A :: 'a :: \text{comm-ring-1 set}$ 
  assumes zero-closed [simp]:  $0 \in A$ 
  assumes one-closed [simp]:  $1 \in A$ 
  assumes add-closed [simp]:  $x \in A \implies y \in A \implies (x + y) \in A$ 
  assumes mult-closed [simp]:  $x \in A \implies y \in A \implies (x * y) \in A$ 
  assumes uminus-closed [simp]:  $x \in A \implies -x \in A$ 
begin

```

```

lemma minus-closed [simp]:  $x \in A \implies y \in A \implies x - y \in A$ 
using add-closed[of x -y] uminus-closed[of y] by auto

```

```

lemma sum-closed [intro]:  $(\bigwedge x. x \in X \implies f x \in A) \implies \text{sum } f X \in A$ 
by (induction X rule: infinite-finite-induct) auto

```

```

lemma power-closed [intro]:  $x \in A \implies x \wedge n \in A$ 
by (induction n) auto

```

lemma *Sum-any-closed* [intro]: $(\bigwedge x. f x \in A) \implies \text{Sum-any } f \in A$
unfolding *Sum-any.expand-set* **by** (rule *sum-closed*)

lemma *prod-closed* [intro]: $(\bigwedge x. x \in X \implies f x \in A) \implies \text{prod } f X \in A$
by (induction *X* rule: *infinite-finite-induct*) *auto*

lemma *Prod-any-closed* [intro]: $(\bigwedge x. f x \in A) \implies \text{Prod-any } f \in A$
unfolding *Prod-any.expand-set* **by** (rule *prod-closed*)

lemma *prod-fun-closed* [intro]: $(\bigwedge x. f x \in A) \implies (\bigwedge x. g x \in A) \implies \text{prod-fun } f g$
 $x \in A$
by (*auto simp: prod-fun-def when-def intro!*: *Sum-any-closed mult-closed*)

lemma *of-nat-closed* [simp, intro]: *of-nat* $n \in A$
by (induction *n*) *auto*

lemma *of-int-closed* [simp, intro]: *of-int* $n \in A$
by (induction *n*) *auto*

end

locale *ring-homomorphism* =
fixes $f :: 'a :: \text{comm-ring-1} \Rightarrow 'b :: \text{comm-ring-1}$
assumes *add[simp]*: $f (x + y) = f x + f y$
assumes *uminus[simp]*: $f (-x) = -f x$
assumes *mult[simp]*: $f (x * y) = f x * f y$
assumes *zero[simp]*: $f 0 = 0$
assumes *one [simp]*: $f 1 = 1$
begin

lemma *diff* [simp]: $f (x - y) = f x - f y$
using *add[of x -y]* **by** (*simp del: add*)

lemma *power* [simp]: $f (x \wedge n) = f x \wedge n$
by (induction *n*) *auto*

lemma *sum* [simp]: $f (\text{sum } g A) = (\sum x \in A. f (g x))$
by (induction *A* rule: *infinite-finite-induct*) *auto*

lemma *prod* [simp]: $f (\text{prod } g A) = (\prod x \in A. f (g x))$
by (induction *A* rule: *infinite-finite-induct*) *auto*

end

lemma *ring-homomorphism-id* [intro]: *ring-homomorphism* *id*
by *standard auto*

lemma *ring-homomorphism-id'* [intro]: *ring-homomorphism* $(\lambda x. x)$

by *standard auto*

lemma *ring-homomorphism-of-int* [intro]: *ring-homomorphism of-int*
by *standard auto*

2.3 Various facts about multivariate polynomials

lemma *poly-mapping-nat-ge-0* [simp]: $(m :: \text{nat} \Rightarrow_0 \text{nat}) \geq 0$

proof (*cases* $m = 0$)

case *False*

hence *Poly-Mapping.lookup* $m \neq \text{Poly-Mapping.lookup } 0$ by *transfer auto*

hence $\exists k. \text{Poly-Mapping.lookup } m \ k \neq 0$ by (*auto simp: fun-eq-iff*)

from *LeastI-ex*[OF *this*] *Least-le*[of $\lambda k. \text{Poly-Mapping.lookup } m \ k \neq 0$] **show**
?thesis

by (*force simp: less-eq-poly-mapping-def less-fun-def*)

qed *auto*

lemma *poly-mapping-nat-le-0* [simp]: $(m :: \text{nat} \Rightarrow_0 \text{nat}) \leq 0 \longleftrightarrow m = 0$

unfolding *less-eq-poly-mapping-def poly-mapping-eq-iff less-fun-def* by *auto*

lemma *of-nat-diff-poly-mapping-nat*:

assumes $m \geq n$

shows $\text{of-nat } (m - n) = (\text{of-nat } m - \text{of-nat } n :: 'a :: \text{monoid-add} \Rightarrow_0 \text{nat})$

by (*auto intro!: poly-mapping-eqI simp: lookup-of-nat lookup-minus when-def*)

lemma *mpoly-coeff-transfer* [transfer-rule]:

rel-fun cr-mpoly (=) *poly-mapping.lookup MPoly-Type.coeff*

unfolding *MPoly-Type.coeff-def* by *transfer-prover*

lemma *mapping-of-sum*: $(\sum x \in A. \text{mapping-of } (f \ x)) = \text{mapping-of } (\text{sum } f \ A)$

by (*induction A rule: infinite-finite-induct*) (*auto simp: plus-mpoly.rep-eq zero-mpoly.rep-eq*)

lemma *mapping-of-eq-0-iff* [simp]: $\text{mapping-of } p = 0 \longleftrightarrow p = 0$

by *transfer auto*

lemma *Sum-any-mapping-of*: $\text{Sum-any } (\lambda x. \text{mapping-of } (f \ x)) = \text{mapping-of } (\text{Sum-any } f)$

by (*simp add: Sum-any.expand-set mapping-of-sum*)

lemma *Sum-any-parametric-cr-mpoly* [transfer-rule]:

(*rel-fun* (*rel-fun* (=) *cr-mpoly*) *cr-mpoly*) *Sum-any Sum-any*

by (*auto simp: rel-fun-def cr-mpoly-def Sum-any-mapping-of*)

lemma *lookup-mult-of-nat* [simp]: $\text{lookup } (\text{of-nat } n * m) \ k = n * \text{lookup } m \ k$

proof –

have $\text{of-nat } n * m = (\sum i < n. m)$ by *simp*

also have $\text{lookup } \dots \ k = (\sum i < n. \text{lookup } m \ k)$

by (*simp only: lookup-sum*)

also have $\dots = n * \text{lookup } m \ k$

by *simp*
finally show *?thesis* .
qed

lemma *mpoly-eqI*:
assumes $\bigwedge mon. MPoly-Type.coeff\ p\ mon = MPoly-Type.coeff\ q\ mon$
shows $p = q$
using *assms* **by** (*transfer*, *transfer*) (*auto simp: fun-eq-iff*)

lemma *coeff-mpoly-times*:
 $MPoly-Type.coeff\ (p * q)\ mon = prod-fun\ (MPoly-Type.coeff\ p)\ (MPoly-Type.coeff\ q)\ mon$
by (*transfer'*, *transfer'*) *auto*

lemma (**in** *ring-closed*) *coeff-mult-closed* [*intro*]:
 $(\bigwedge x. coeff\ p\ x \in A) \implies (\bigwedge x. coeff\ q\ x \in A) \implies coeff\ (p * q)\ x \in A$
by (*auto simp: coeff-mpoly-times prod-fun-closed*)

lemma *coeff-notin-vars*:
assumes $\neg(keys\ m \subseteq vars\ p)$
shows $coeff\ p\ m = 0$
using *assms* **unfolding** *vars-def* **by** *transfer'* (*auto simp: in-keys-iff*)

lemma *finite-coeff-support* [*intro*]: $finite\ \{m. coeff\ p\ m \neq 0\}$
by *transfer simp*

lemma *insertion-altdef*:
 $insertion\ f\ p = Sum-any\ (\lambda m. coeff\ p\ m * Prod-any\ (\lambda i. f\ i \wedge lookup\ m\ i))$
by (*transfer'*, *transfer'*) (*simp add: insertion-fun-def*)

lemma *mpoly-coeff-uminus* [*simp*]: $coeff\ (-p)\ m = -coeff\ p\ m$
by *transfer auto*

lemma *Sum-any-uminus*: $Sum-any\ (\lambda x. -f\ x :: 'a :: ab-group-add) = -Sum-any\ f$
by (*simp add: Sum-any.expand-set sum-negf*)

lemma *insertion-uminus* [*simp*]: $insertion\ f\ (-p :: 'a :: comm-ring-1\ mpoly) = -insertion\ f\ p$
by (*simp add: insertion-altdef Sum-any-uminus*)

lemma *Sum-any-lookup*: $finite\ \{x. g\ x \neq 0\} \implies Sum-any\ (\lambda x. lookup\ (g\ x)\ y) = lookup\ (Sum-any\ g)\ y$
by (*auto simp: Sum-any.expand-set lookup-sum intro!: sum.mono-neutral-left*)

lemma *Sum-any-diff*:
assumes $finite\ \{x. f\ x \neq 0\}$
assumes $finite\ \{x. g\ x \neq 0\}$
shows $Sum-any\ (\lambda x. f\ x - g\ x :: 'a :: ab-group-add) = Sum-any\ f - Sum-any\ g$

proof –
have $\{x. f x - g x \neq 0\} \subseteq \{x. f x \neq 0\} \cup \{x. g x \neq 0\}$ **by** *auto*
moreover have *finite* $(\{x. f x \neq 0\} \cup \{x. g x \neq 0\})$
by (*subst finite-Un*) (*insert assms, auto*)
ultimately have *finite* $\{x. f x - g x \neq 0\}$
by (*rule finite-subset*)
with *assms* **show** *?thesis*
by (*simp add: algebra-simps Sum-any.distrib [symmetric]*)
qed

lemma *insertion-diff*:
 $insertion f (p - q :: 'a :: comm-ring-1 mpoly) = insertion f p - insertion f q$
proof (*transfer, transfer*)
fix $f :: nat \Rightarrow 'a$ **and** $p q :: (nat \Rightarrow_0 nat) \Rightarrow 'a$
assume *fin*: *finite* $\{x. p x \neq 0\}$ *finite* $\{x. q x \neq 0\}$
have $insertion\text{-}fun f (\lambda x. p x - q x) =$
 $(\sum m. p m * (\prod v. f v \wedge lookup m v) - q m * (\prod v. f v \wedge lookup m v))$
by (*simp add: insertion-fun-def algebra-simps Sum-any-diff*)
also have $\dots = (\sum m. p m * (\prod v. f v \wedge lookup m v)) - (\sum m. q m * (\prod v. f v \wedge lookup m v))$
by (*subst Sum-any-diff*) (*auto intro: finite-subset[OF - fin(1)] finite-subset[OF - fin(2)]*)
also have $\dots = insertion\text{-}fun f p - insertion\text{-}fun f q$
by (*simp add: insertion-fun-def*)
finally show $insertion\text{-}fun f (\lambda x. p x - q x) = \dots$
qed

lemma *insertion-power*: $insertion f (p \wedge n) = insertion f p \wedge n$
by (*induction n*) (*simp-all add: insertion-mult*)

lemma *insertion-sum*: $insertion f (sum g A) = (\sum x \in A. insertion f (g x))$
by (*induction A rule: infinite-finite-induct*) (*auto simp: insertion-add*)

lemma *insertion-prod*: $insertion f (prod g A) = (\prod x \in A. insertion f (g x))$
by (*induction A rule: infinite-finite-induct*) (*auto simp: insertion-mult*)

lemma *coeff-Var*: $coeff (Var i) m = (1 \text{ when } m = Poly\text{-}Mapping.single i 1)$
by *transfer'* (*auto simp: Var₀-def lookup-single when-def*)

lemma *vars-Var*: $vars (Var i :: 'a :: \{one, zero\} mpoly) = (if (0 :: 'a) = 1 \text{ then } \{\} \text{ else } \{i\})$
unfolding *vars-def* **by** (*auto simp: Var.rep-eq Var₀-def*)

lemma *insertion-Var* [*simp*]: $insertion f (Var i) = f i$

proof –
have $insertion f (Var i) = (\sum m. (1 \text{ when } m = Poly\text{-}Mapping.single i 1) * (\prod i. f i \wedge lookup m i))$
by (*simp add: insertion-altdef coeff-Var*)
also have $\dots = (\prod j. f j \wedge lookup (Poly\text{-}Mapping.single i 1) j)$

by (*subst Sum-any.expand-superset*[of {*Poly-Mapping.single i 1*}]) (*auto simp: when-def*)
also have $\dots = f i$
by (*subst Prod-any.expand-superset*[of {*i*}]) (*auto simp: when-def lookup-single*)
finally show *?thesis* .
qed

lemma *insertion-Sum-any*:
assumes *finite* {*x. g x* $\neq 0$ }
shows *insertion f (Sum-any g) = Sum-any* ($\lambda x.$ *insertion f (g x)*)
unfolding *Sum-any.expand-set insertion-sum*
by (*intro sum.mono-neutral-right*) (*auto intro!: finite-subset[OF - assms]*)

lemma *keys-diff-subset*:
 $keys (f - g) \subseteq keys f \cup keys g$
by *transfer auto*

lemma *keys-empty-iff* [*simp*]: $keys p = \{\}$ $\longleftrightarrow p = 0$
by *transfer auto*

lemma *mpoly-coeff-0* [*simp*]: *MPoly-Type.coeff* 0 *m* = 0
by *transfer auto*

lemma *lookup-1*: *lookup 1 m* = (if *m* = 0 then 1 else 0)
by *transfer (simp add: when-def)*

lemma *mpoly-coeff-1*: *MPoly-Type.coeff* 1 *m* = (if *m* = 0 then 1 else 0)
by (*simp add: MPoly-Type.coeff-def one-mpoly.rep-eq lookup-1*)

lemma *lookup-Const₀*: *lookup (Const₀ c) m* = (if *m* = 0 then *c* else 0)
unfolding *Const₀-def* **by** (*simp add: lookup-single when-def*)

lemma *mpoly-coeff-Const*: *MPoly-Type.coeff* (*Const c*) *m* = (if *m* = 0 then *c* else 0)
by (*simp add: MPoly-Type.coeff-def Const.rep-eq lookup-Const₀*)

lemma *coeff-smult* [*simp*]: *coeff (smult c p) m* = (*c* :: 'a :: *mult-zero*) * *coeff p m*
by *transfer (auto simp: map-lookup)*

lemma *in-keys-mapI*: $x \in keys m \implies f (lookup m x) \neq 0 \implies x \in keys (Poly-Mapping.map f m)$
by *transfer auto*

lemma *keys-uminus* [*simp*]: $keys (-m) = keys m$
by *transfer auto*

lemma *vars-uminus* [*simp*]: $vars (-p) = vars p$
unfolding *vars-def* **by** *transfer' auto*

lemma *vars-smult*: $\text{vars } (\text{smult } c \ p) \subseteq \text{vars } p$
unfolding *vars-def* **by** (*transfer'*, *transfer'*) *auto*

lemma *vars-0* [*simp*]: $\text{vars } 0 = \{\}$
unfolding *vars-def* **by** *transfer'* *simp*

lemma *vars-1* [*simp*]: $\text{vars } 1 = \{\}$
unfolding *vars-def* **by** *transfer'* *simp*

lemma *vars-sum*: $\text{vars } (\text{sum } f \ A) \subseteq (\bigcup x \in A. \text{vars } (f \ x))$
using *vars-add* **by** (*induction* *A* *rule*: *infinite-finite-induct*) *auto*

lemma *vars-prod*: $\text{vars } (\text{prod } f \ A) \subseteq (\bigcup x \in A. \text{vars } (f \ x))$
using *vars-mult* **by** (*induction* *A* *rule*: *infinite-finite-induct*) *auto*

lemma *vars-Sum-any*: $\text{vars } (\text{Sum-any } h) \subseteq (\bigcup i. \text{vars } (h \ i))$
unfolding *Sum-any.expand-set* **by** (*intro* *order.trans*[*OF vars-sum*]) *auto*

lemma *vars-Prod-any*: $\text{vars } (\text{Prod-any } h) \subseteq (\bigcup i. \text{vars } (h \ i))$
unfolding *Prod-any.expand-set* **by** (*intro* *order.trans*[*OF vars-prod*]) *auto*

lemma *vars-power*: $\text{vars } (p \ ^n) \subseteq \text{vars } p$
using *vars-mult* **by** (*induction* *n*) *auto*

lemma *vars-diff*: $\text{vars } (p1 - p2) \subseteq \text{vars } p1 \cup \text{vars } p2$
unfolding *vars-def*
proof *transfer'*
fix *p1 p2* :: (*nat* \Rightarrow_0 *nat*) \Rightarrow_0 'a
show $\bigcup (\text{keys } ' \ \text{keys } (p1 - p2)) \subseteq \bigcup (\text{keys } ' \ (\text{keys } p1)) \cup \bigcup (\text{keys } ' \ (\text{keys } p2))$
using *keys-diff-subset*[*of p1 p2*] **by** (*auto* *simp* *flip*: *not-in-keys-iff-lookup-eq-zero*)
qed

lemma *insertion-smult* [*simp*]: $\text{insertion } f \ (\text{smult } c \ p) = c * \text{insertion } f \ p$
unfolding *insertion-altdef*
by (*subst* *Sum-any-right-distrib*)
(auto *intro*: *finite-subset*[*OF - finite-coeff-support*[*of p*]] *simp*: *mult.assoc*)

lemma *coeff-add* [*simp*]: $\text{coeff } (p + q) \ m = \text{coeff } p \ m + \text{coeff } q \ m$
by *transfer'* (*simp* *add*: *lookup-add*)

lemma *coeff-diff* [*simp*]: $\text{coeff } (p - q) \ m = \text{coeff } p \ m - \text{coeff } q \ m$
by *transfer'* (*simp* *add*: *lookup-minus*)

lemma *insertion-monom* [*simp*]:
 $\text{insertion } f \ (\text{monom } m \ c) = c * \text{Prod-any } (\lambda x. f \ x \ ^{\text{lookup } m \ x})$
proof –
have $\text{insertion } f \ (\text{monom } m \ c) =$
 $(\sum m'. (c \ \text{when } m = m') * (\prod v. f \ v \ ^{\text{lookup } m' \ v}))$
by (*simp* *add*: *insertion-def* *insertion-aux-def* *insertion-fun-def* *lookup-single*)

also have $\dots = c * (\prod v. f v \hat{\text{lookup}} m v)$
by (*subst Sum-any.expand-superset[of {m}]*) (*auto simp: when-def*)
finally show *?thesis* .
qed

lemma *insertion-aux-Const₀ [simp]: insertion-aux f (Const₀ c) = c*
proof –
have *insertion-aux f (Const₀ c) = ($\sum m. (c \text{ when } m = 0) * (\prod v. f v \hat{\text{lookup}} m v)$)*
by (*simp add: Const₀-def insertion-aux-def insertion-fun-def lookup-single*)
also have $\dots = (\sum m \in \{0\}. (c \text{ when } m = 0) * (\prod v. f v \hat{\text{lookup}} m v))$
by (*intro Sum-any.expand-superset*) (*auto simp: when-def*)
also have $\dots = c$ **by** *simp*
finally show *?thesis* .
qed

lemma *insertion-Const [simp]: insertion f (Const c) = c*
by (*simp add: insertion-def Const.rep-eq*)

lemma *coeffs-0 [simp]: coeffs 0 = {}*
by *transfer auto*

lemma *coeffs-1 [simp]: coeffs 1 = {1}*
by *transfer auto*

lemma *coeffs-Const: coeffs (Const c) = (if c = 0 then {} else {c})*
unfolding *Const-def Const₀-def* **by** *transfer' auto*

lemma *coeffs-subset: coeffs (Const c) \subseteq {c}*
by (*auto simp: coeffs-Const*)

lemma *keys-Const₀: keys (Const₀ c) = (if c = 0 then {} else {0})*
unfolding *Const₀-def* **by** *transfer' auto*

lemma *vars-Const [simp]: vars (Const c) = {}*
unfolding *vars-def* **by** *transfer' (auto simp: keys-Const₀)*

lemma *prod-fun-compose-bij:*
assumes *bij f and f: $\bigwedge x y. f (x + y) = f x + f y$*
shows *prod-fun m1 m2 (f x) = prod-fun (m1 \circ f) (m2 \circ f) x*
proof –
have [*simp*]: $f x = f y \iff x = y$ **for** $x y$
using $\langle \text{bij } f \rangle$ **by** (*auto dest!: bij-is-inj inj-onD*)
have *prod-fun (m1 \circ f) (m2 \circ f) x =*
 $\text{Sum-any } ((\lambda l. m1 l * \text{Sum-any } ((\lambda q. m2 q \text{ when } f x = l + q) \circ f)) \circ f)$
by (*simp add: prod-fun-def f(1) [symmetric] o-def*)
also have $\dots = \text{Sum-any } ((\lambda l. m1 l * \text{Sum-any } ((\lambda q. m2 q \text{ when } f x = l + q))))$
by (*simp only: Sum-any.reindex-cong[OF assms(1) refl, symmetric]*)
also have $\dots = \text{prod-fun } m1 m2 (f x)$

by (*simp add: prod-fun-def*)
finally show *?thesis ..*
qed

lemma *add-nat-poly-mapping-zero-iff* [*simp*]:
 $(a + b :: 'a \Rightarrow_0 \text{nat}) = 0 \iff a = 0 \wedge b = 0$
by *transfer (auto simp: fun-eq-iff)*

lemma *prod-fun-nat-0*:
fixes $f g :: ('a \Rightarrow_0 \text{nat}) \Rightarrow 'b::\text{semiring-0}$
shows $\text{prod-fun } f g 0 = f 0 * g 0$
proof –

have $\text{prod-fun } f g 0 = (\sum l. f l * (\sum q. g q \text{ when } 0 = l + q))$
unfolding *prod-fun-def ..*
also have $(\lambda l. \sum q. g q \text{ when } 0 = l + q) = (\lambda l. \sum q \in \{0\}. g q \text{ when } 0 = l + q)$
by (*intro ext Sum-any.expand-superset*) (*auto simp: when-def*)
also have $(\sum l. f l * \dots l) = (\sum l \in \{0\}. f l * \dots l)$
by (*intro ext Sum-any.expand-superset*) (*auto simp: when-def*)
finally show *?thesis by simp*

qed

lemma *mpoly-coeff-times-0*: $\text{coeff } (p * q) 0 = \text{coeff } p 0 * \text{coeff } q 0$
by (*simp add: coeff-mpoly-times prod-fun-nat-0*)

lemma *mpoly-coeff-prod-0*: $\text{coeff } (\prod x \in A. f x) 0 = (\prod x \in A. \text{coeff } (f x) 0)$
by (*induction A rule: infinite-finite-induct*) (*auto simp: mpoly-coeff-times-0 mpoly-coeff-1*)

lemma *mpoly-coeff-power-0*: $\text{coeff } (p \wedge^n) 0 = \text{coeff } p 0 \wedge^n$
by (*induction n*) (*auto simp: mpoly-coeff-times-0 mpoly-coeff-1*)

lemma *prod-fun-max*:

fixes $f g :: 'a::\{\text{linorder, ordered-cancel-comm-monoid-add}\} \Rightarrow 'b::\text{semiring-0}$
assumes *zero*: $\bigwedge m. m > a \implies f m = 0 \wedge m > b \implies g m = 0$
assumes *fin*: $\text{finite } \{m. f m \neq 0\} \text{ finite } \{m. g m \neq 0\}$
shows $\text{prod-fun } f g (a + b) = f a * g b$

proof –

note $\text{fin}' = \text{finite-subset}[OF - \text{fin}(1)] \text{ finite-subset}[OF - \text{fin}(2)]$
have $\text{prod-fun } f g (a + b) = (\sum l. f l * (\sum q. g q \text{ when } a + b = l + q))$
by (*simp add: prod-fun-def Sum-any-right-distrib*)
also have $\dots = (\sum l. \sum q. f l * g q \text{ when } a + b = l + q)$
by (*subst Sum-any-right-distrib*) (*auto intro!: Sum-any.cong fin'(2) simp: when-def*)
also {
fix $l q$ **assume** $lq: a + b = l + q$ ($a, b \neq (l, q)$) **and** $nz: f l * g q \neq 0$
from *nz* **and** *zero* **have** $l \leq a \leq b$ **by** (*auto intro: leI*)
moreover from *this* **and** $lq(2)$ **have** $l < a \vee q < b$ **by** *auto*
ultimately have $l + q < a + b$
by (*auto intro: add-less-le-mono add-le-less-mono*)
with $lq(1)$ **have** *False* **by** *simp*
}

hence $(\sum l. \sum q. f l * g q \text{ when } a + b = l + q) = (\sum l. \sum q. f l * g q \text{ when } (a, b) = (l, q))$
by *(intro Sum-any.cong refl) (auto simp: when-def)*
also have $\dots = (\sum (l,q). f l * g q \text{ when } (a, b) = (l, q))$
by *(intro Sum-any.cartesian-product[of {(a, b)}]) auto*
also have $\dots = (\sum (l,q) \in \{(a,b)\}. f l * g q \text{ when } (a, b) = (l, q))$
by *(intro Sum-any.expand-superset) auto*
also have $\dots = f a * g b$ **by** *simp*
finally show *?thesis .*
qed

lemma *prod-fun-gt-max-eq-zero:*

fixes $f g :: 'a :: \{\text{linorder, ordered-cancel-comm-monoid-add}\} \Rightarrow 'b :: \text{semiring-0}$
assumes $m > a + b$
assumes zero: $\bigwedge m. m > a \implies f m = 0 \ \bigwedge m. m > b \implies g m = 0$
assumes fin: $\text{finite } \{m. f m \neq 0\} \ \text{finite } \{m. g m \neq 0\}$
shows $\text{prod-fun } f g m = 0$

proof –

note $\text{fin}' = \text{finite-subset}[OF - \text{fin}(1)] \ \text{finite-subset}[OF - \text{fin}(2)]$
have $\text{prod-fun } f g m = (\sum l. f l * (\sum q. g q \text{ when } m = l + q))$
by *(simp add: prod-fun-def Sum-any-right-distrib)*
also have $\dots = (\sum l. \sum q. f l * g q \text{ when } m = l + q)$
by *(subst Sum-any-right-distrib) (auto intro!: Sum-any.cong fin'(2) simp: when-def)*
also {
fix $l q$ **assume** $lq: m = l + q$ **and** $nz: f l * g q \neq 0$
from nz **and zero** **have** $l \leq a \ \ q \leq b$ **by** *(auto intro: leI)*
hence $l + q \leq a + b$ **by** *(intro add-mono)*
also have $\dots < m$ **by** *fact*
finally have $l + q < m$.
}
hence $(\sum l. \sum q. f l * g q \text{ when } m = l + q) = (\sum l. \sum q. f l * g q \text{ when } \text{False})$
by *(intro Sum-any.cong refl) (auto simp: when-def)*
also have $\dots = 0$ **by** *simp*
finally show *?thesis .*

qed

2.4 Restricting a monomial to a subset of variables

lift-definition *restrictpm* $:: 'a \text{ set} \Rightarrow ('a \Rightarrow_0 'b :: \text{zero}) \Rightarrow ('a \Rightarrow_0 'b) \text{ is}$

$\lambda A f x. \text{if } x \in A \text{ then } f x \text{ else } 0$
by *(erule finite-subset[rotated]) auto*

lemma *lookup-restrictpm:* $\text{lookup } (\text{restrictpm } A m) x = (\text{if } x \in A \text{ then } \text{lookup } m x \text{ else } 0)$

by *transfer auto*

lemma *lookup-restrictpm-in* $[\text{simp}]: x \in A \implies \text{lookup } (\text{restrictpm } A m) x = \text{lookup } m x$

and *lookup-restrict-pm-not-in* $[\text{simp}]: x \notin A \implies \text{lookup } (\text{restrictpm } A m) x = 0$

by (simp-all add: lookup-restrictpm)

lemma keys-restrictpm [simp]: keys (restrictpm A m) = keys m \cap A
by transfer auto

lemma restrictpm-add: restrictpm X (m1 + m2) = restrictpm X m1 + restrictpm X m2
by transfer auto

lemma restrictpm-id [simp]: keys m \subseteq X \implies restrictpm X m = m
by transfer (auto simp: fun-eq-iff)

lemma restrictpm-orthogonal [simp]: keys m \subseteq -X \implies restrictpm X m = 0
by transfer (auto simp: fun-eq-iff)

lemma restrictpm-add-disjoint:
X \cap Y = {} \implies restrictpm X m + restrictpm Y m = restrictpm (X \cup Y) m
by transfer (auto simp: fun-eq-iff)

lemma restrictpm-add-complements:
restrictpm X m + restrictpm (-X) m = m restrictpm (-X) m + restrictpm X m = m
by (subst restrictpm-add-disjoint; force)+

2.5 Mapping over a polynomial

lift-definition map-mpoly :: ('a :: zero \Rightarrow 'b :: zero) \Rightarrow 'a mpoly \Rightarrow 'b mpoly is
 $\lambda(f :: 'a \Rightarrow 'b) (p :: (nat \Rightarrow_0 nat) \Rightarrow_0 'a). \text{Poly-Mapping.map } f \ p .$

lift-definition mapm-mpoly :: ((nat \Rightarrow_0 nat) \Rightarrow 'a :: zero \Rightarrow 'b :: zero) \Rightarrow 'a mpoly
 \Rightarrow 'b mpoly is
 $\lambda(f :: (nat \Rightarrow_0 nat) \Rightarrow 'a \Rightarrow 'b) (p :: (nat \Rightarrow_0 nat) \Rightarrow_0 'a). \text{Poly-Mapping.mapp } f \ p .$

lemma poly-mapping-map-conv-mapp: Poly-Mapping.map f = Poly-Mapping.mapp
($\lambda\cdot. f$)
by (auto simp: Poly-Mapping.mapp-def Poly-Mapping.map-def map-fun-def
o-def fun-eq-iff when-def in-keys-iff cong: if-cong)

lemma map-mpoly-conv-mapm-mpoly: map-mpoly f = mapm-mpoly ($\lambda\cdot. f$)
by transfer' (auto simp: poly-mapping-map-conv-mapp)

lemma map-mpoly-comp: f 0 = 0 \implies map-mpoly f (map-mpoly g p) = map-mpoly
(f \circ g) p
by (transfer', transfer') (auto simp: when-def fun-eq-iff)

lemma mapp-mapp:
($\bigwedge x. f \ x \ 0 = 0$) \implies Poly-Mapping.mapp f (Poly-Mapping.mapp g m) =
Poly-Mapping.mapp ($\lambda x \ y. f \ x \ (g \ x \ y)$) m

by *transfer'* (*auto simp: fun-eq-iff lookup-mapp in-keys-iff*)

lemma *mapm-mpoly-comp*:

$(\bigwedge x. f x 0 = 0) \implies \text{mapm-mpoly } f (\text{mapm-mpoly } g p) = \text{mapm-mpoly } (\lambda m c. f m (g m c)) p$

by *transfer'* (*simp add: mapp-mapp*)

lemma *coeff-map-mpoly*:

$\text{coeff } (\text{map-mpoly } f p) m = (\text{if } \text{coeff } p m = 0 \text{ then } 0 \text{ else } f (\text{coeff } p m))$

by (*transfer, transfer'*) *auto*

lemma *coeff-map-mpoly'* [*simp*]: $f 0 = 0 \implies \text{coeff } (\text{map-mpoly } f p) m = f (\text{coeff } p m)$

by (*subst coeff-map-mpoly*) *auto*

lemma *coeff-mapm-mpoly*: $\text{coeff } (\text{mapm-mpoly } f p) m = (\text{if } \text{coeff } p m = 0 \text{ then } 0 \text{ else } f m (\text{coeff } p m))$

by (*transfer, transfer'*) (*auto simp: in-keys-iff*)

lemma *coeff-mapm-mpoly'* [*simp*]: $(\bigwedge m. f m 0 = 0) \implies \text{coeff } (\text{mapm-mpoly } f p) m = f m (\text{coeff } p m)$

by (*subst coeff-mapm-mpoly*) *auto*

lemma *vars-map-mpoly-subset*: $\text{vars } (\text{map-mpoly } f p) \subseteq \text{vars } p$

unfolding *vars-def* **by** (*transfer', transfer'*) (*auto simp: map-mpoly.rep-eq*)

lemma *coeff-sum* [*simp*]: $\text{coeff } (\text{sum } f A) m = (\sum x \in A. \text{coeff } (f x) m)$

by (*induction A rule: infinite-finite-induct*) *auto*

lemma *coeff-Sum-any*: $\text{finite } \{x. f x \neq 0\} \implies \text{coeff } (\text{Sum-any } f) m = \text{Sum-any } (\lambda x. \text{coeff } (f x) m)$

by (*auto simp add: Sum-any.expand-set intro!: sum.mono-neutral-right*)

lemma *Sum-any-zeroI*: $(\bigwedge x. f x = 0) \implies \text{Sum-any } f = 0$

by (*auto simp: Sum-any.expand-set*)

lemma *insertion-Prod-any*:

$\text{finite } \{x. g x \neq 1\} \implies \text{insertion } f (\text{Prod-any } g) = \text{Prod-any } (\lambda x. \text{insertion } f (g x))$

by (*auto simp: Prod-any.expand-set insertion-prod intro!: prod.mono-neutral-right*)

lemma *insertion-insertion*:

$\text{insertion } g (\text{insertion } k p) =$

$\text{insertion } (\lambda x. \text{insertion } g (k x)) (\text{map-mpoly } (\text{insertion } g) p)$ (**is** *?lhs = ?rhs*)

proof –

have $\text{insertion } g (\text{insertion } k p) =$

$(\sum x. \text{insertion } g (\text{coeff } p x) * \text{insertion } g (\prod i. k i \wedge \text{lookup } x i))$

unfolding *insertion-altdef*[*of k p*]

by (*subst insertion-Sum-any*)

$(\text{auto intro: finite-subset}[OF - \text{finite-coeff-support}[of p]] \text{ simp: insertion-mult})$
also have $\dots = (\sum x. \text{insertion } g (\text{coeff } p x) * (\prod i. \text{insertion } g (k i) \wedge \text{lookup } x i))$
proof $(\text{intro Sum-any.cong})$
fix x **show** $\text{insertion } g (\text{coeff } p x) * \text{insertion } g (\prod i. k i \wedge \text{lookup } x i) =$
 $\text{insertion } g (\text{coeff } p x) * (\prod i. \text{insertion } g (k i) \wedge \text{lookup } x i)$
by $(\text{subst insertion-Prod-any})$
 $(\text{auto simp: insertion-power intro!: finite-subset}[OF - \text{finite-lookup}[of x]])$
 Nat.gr0I
qed
also have $\dots = \text{insertion } (\lambda x. \text{insertion } g (k x)) (\text{map-mpoly } (\text{insertion } g) p)$
unfolding $\text{insertion-altdef}[of - \text{map-mpoly } f p \text{ for } f]$ **by** auto
finally show $?thesis$.
qed

lemma $\text{insertion-substitute-linear}$:
 $\text{insertion } (\lambda i. c i * f i) p =$
 $\text{insertion } f (\text{mapm-mpoly } (\lambda m d. \text{Prod-any } (\lambda i. c i \wedge \text{lookup } m i) * d) p)$
unfolding insertion-altdef
proof $(\text{intro Sum-any.cong, goal-cases})$
case $(1 m)$
have $\text{coeff } (\text{mapm-mpoly } (\lambda m. (*) (\prod i. c i \wedge \text{lookup } m i)) p) m * (\prod i. f i \wedge$
 $\text{lookup } m i) =$
 $\text{MPoly-Type.coeff } p m * ((\prod i. c i \wedge \text{lookup } m i) * (\prod i. f i \wedge \text{lookup } m i))$
by $(\text{simp add: mult-ac})$
also have $(\prod i. c i \wedge \text{lookup } m i) * (\prod i. f i \wedge \text{lookup } m i) =$
 $(\prod i. (c i * f i) \wedge \text{lookup } m i)$
by $(\text{subst Prod-any.distrib } [\text{symmetric}])$
 $(\text{auto simp: power-mult-distrib intro!: finite-subset}[OF - \text{finite-lookup}[of m]])$
 Nat.gr0I
finally show $?case$ **by** simp
qed

lemma $\text{vars-mapm-mpoly-subset}$: $\text{vars } (\text{mapm-mpoly } f p) \subseteq \text{vars } p$
unfolding vars-def **using** $\text{keys-mapp-subset}[of f]$ **by** $(\text{auto simp: mapm-mpoly.rep-eq})$

lemma map-mpoly-cong :
assumes $\bigwedge m. f (\text{coeff } p m) = g (\text{coeff } p m) \quad p = q$
shows $\text{map-mpoly } f p = \text{map-mpoly } g q$
using assms **by** $(\text{intro mpoly-eqI } (\text{auto simp: coeff-map-mpoly}))$

2.6 The leading monomial and leading coefficient

The leading monomial of a multivariate polynomial is the one with the largest monomial w.r.t. the monomial ordering induced by the standard variable ordering. The leading coefficient is the coefficient of the leading monomial.

As a convention, the leading monomial of the zero polynomial is defined to

be the same as that of any non-constant zero polynomial, i. e. the monomial $X_1^0 \dots X_n^0$.

lift-definition *lead-monom* :: 'a :: zero mpoly \Rightarrow (nat \Rightarrow_0 nat) is
 $\lambda f :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a. \text{Max} (\text{insert } 0 (\text{keys } f))$.

lemma *lead-monom-geI* [intro]:
assumes *coeff p m \neq 0*
shows $m \leq \text{lead-monom } p$
using *assms* **by** (auto simp: *lead-monom-def coeff-def in-keys-iff*)

lemma *coeff-gt-lead-monom-zero* [simp]:
assumes $m > \text{lead-monom } p$
shows $\text{coeff } p \ m = 0$
using *lead-monom-geI*[of $p \ m$] *assms* **by** force

lemma *lead-monom-nonzero-eq*:
assumes $p \neq 0$
shows $\text{lead-monom } p = \text{Max} (\text{keys} (\text{mapping-of } p))$
using *assms* **by** transfer (simp add: *max-def*)

lemma *lead-monom-0* [simp]: $\text{lead-monom } 0 = 0$
by (simp add: *lead-monom-def zero-mpoly.rep-eq*)

lemma *lead-monom-1* [simp]: $\text{lead-monom } 1 = 0$
by (simp add: *lead-monom-def one-mpoly.rep-eq*)

lemma *lead-monom-Const* [simp]: $\text{lead-monom} (\text{Const } c) = 0$
by (simp add: *lead-monom-def Const.rep-eq Const₀-def*)

lemma *lead-monom-uminus* [simp]: $\text{lead-monom} (-p) = \text{lead-monom } p$
by (simp add: *lead-monom-def uminus-mpoly.rep-eq*)

lemma *keys-mult-const* [simp]:
fixes $c :: 'a :: \{\text{semiring-0}, \text{semiring-no-zero-divisors}\}$
assumes $c \neq 0$
shows $\text{keys} (\text{Poly-Mapping.map } ((*) \ c) \ p) = \text{keys } p$
using *assms* **by** transfer auto

lemma *lead-monom-eq-0-iff*: $\text{lead-monom } p = 0 \iff \text{vars } p = \{\}$
unfolding *vars-def* **by** transfer' (auto simp: *Max-eq-iff*)

lemma *lead-monom-monom*: $\text{lead-monom} (\text{monom } m \ c) = (\text{if } c = 0 \text{ then } 0 \text{ else } m)$
by (auto simp add: *lead-monom-def monom.rep-eq Const₀-def max-def*)

lemma *lead-monom-monom'* [simp]: $c \neq 0 \implies \text{lead-monom} (\text{monom } m \ c) = m$
by (simp add: *lead-monom-monom*)

lemma *lead-monom-numeral* [simp]: $\text{lead-monom} (\text{numeral } n) = 0$

unfolding *monom-numeral[symmetric]* **by** (*subst lead-monom-monom*) *auto*

lemma *lead-monom-add*: $\text{lead-monom } (p + q) \leq \max (\text{lead-monom } p) (\text{lead-monom } q)$

proof *transfer*

fix $p\ q :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a$

show $\text{Max } (\text{insert } 0 (\text{keys } (p + q))) \leq \max (\text{Max } (\text{insert } 0 (\text{keys } p))) (\text{Max } (\text{insert } 0 (\text{keys } q)))$

proof (*rule Max.boundedI*)

fix m **assume** $m: m \in \text{insert } 0 (\text{keys } (p + q))$

thus $m \leq \max (\text{Max } (\text{insert } 0 (\text{keys } p))) (\text{Max } (\text{insert } 0 (\text{keys } q)))$

proof

assume $m \in \text{keys } (p + q)$

with *keys-add[of p q]* **have** $m \in \text{keys } p \vee m \in \text{keys } q$

by (*auto simp: in-keys-iff plus-poly-mapping.rep-eq*)

thus *?thesis* **by** (*auto simp: le-max-iff-disj*)

qed *auto*

qed *auto*

qed

lemma *lead-monom-diff*: $\text{lead-monom } (p - q) \leq \max (\text{lead-monom } p) (\text{lead-monom } q)$

proof *transfer*

fix $p\ q :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a$

show $\text{Max } (\text{insert } 0 (\text{keys } (p - q))) \leq \max (\text{Max } (\text{insert } 0 (\text{keys } p))) (\text{Max } (\text{insert } 0 (\text{keys } q)))$

proof (*rule Max.boundedI*)

fix m **assume** $m: m \in \text{insert } 0 (\text{keys } (p - q))$

thus $m \leq \max (\text{Max } (\text{insert } 0 (\text{keys } p))) (\text{Max } (\text{insert } 0 (\text{keys } q)))$

proof

assume $m \in \text{keys } (p - q)$

with *keys-diff-subset[of p q]* **have** $m \in \text{keys } p \vee m \in \text{keys } q$ **by** *auto*

thus *?thesis* **by** (*auto simp: le-max-iff-disj*)

qed *auto*

qed *auto*

qed

definition *lead-coeff* **where** $\text{lead-coeff } p = \text{coeff } p (\text{lead-monom } p)$

lemma *vars-empty-iff*: $\text{vars } p = \{\} \iff p = \text{Const } (\text{lead-coeff } p)$

proof

assume $\text{vars } p = \{\}$

hence [*simp*]: $\text{lead-monom } p = 0$

by (*simp add: lead-monom-eq-0-iff*)

have [*simp*]: $\text{mon} \neq 0 \iff (\text{mon} > (0 :: \text{nat} \Rightarrow_0 \text{nat}))$ **for** mon

by (*auto simp: order.strict-iff-order*)

thus $p = \text{Const } (\text{lead-coeff } p)$

by (*intro mpoly-eqI*) (*auto simp: mpoly-coeff-Const lead-coeff-def*)

next

assume $p = \text{Const } (\text{lead-coeff } p)$
also have $\text{vars } \dots = \{\}$ **by** simp
finally show $\text{vars } p = \{\}$.

qed

lemma $\text{lead-coeff-0 } [\text{simp}]: \text{lead-coeff } 0 = 0$
by $(\text{simp add: lead-coeff-def})$

lemma $\text{lead-coeff-1 } [\text{simp}]: \text{lead-coeff } 1 = 1$
by $(\text{simp add: lead-coeff-def mpoly-coeff-1})$

lemma $\text{lead-coeff-Const } [\text{simp}]: \text{lead-coeff } (\text{Const } c) = c$
by $(\text{simp add: lead-coeff-def mpoly-coeff-Const})$

lemma $\text{lead-coeff-monom } [\text{simp}]: \text{lead-coeff } (\text{monom } p \ c) = c$
by $(\text{simp add: lead-coeff-def coeff-monom when-def lead-monom-monom})$

lemma $\text{lead-coeff-nonzero } [\text{simp}]: p \neq 0 \implies \text{lead-coeff } p \neq 0$
unfolding $\text{lead-coeff-def lead-monom-def}$
by $(\text{cases keys } (\text{mapping-of } p) = \{\}) (\text{auto simp: coeff-def max-def})$

lemma

fixes $c :: 'a :: \text{semiring-0}$
assumes $c * \text{lead-coeff } p \neq 0$
shows $\text{lead-monom-smult } [\text{simp}]: \text{lead-monom } (\text{smult } c \ p) = \text{lead-monom } p$
and $\text{lead-coeff-smult } [\text{simp}]: \text{lead-coeff } (\text{smult } c \ p) = c * \text{lead-coeff } p$

proof –

from $\text{assms have } *: \text{keys } (\text{mapping-of } p) \neq \{\}$
by auto
from $\text{assms have } \text{coeff } (\text{MPoly-Type.smult } c \ p) (\text{lead-monom } p) \neq 0$
by $(\text{simp add: lead-coeff-def})$
hence $\text{smult-nz}: \text{MPoly-Type.smult } c \ p \neq 0$ **by** $(\text{auto simp del: coeff-smult})$
with $\text{assms have } **: \text{keys } (\text{mapping-of } (\text{smult } c \ p)) \neq \{\}$
by simp

have $\text{Max } (\text{keys } (\text{mapping-of } (\text{smult } c \ p))) = \text{Max } (\text{keys } (\text{mapping-of } p))$

proof $(\text{safe intro!}: \text{antisym Max.coboundedI})$

have $\text{lookup } (\text{mapping-of } p) (\text{Max } (\text{keys } (\text{mapping-of } p))) = \text{lead-coeff } p$
using * **by** $(\text{simp add: lead-coeff-def lead-monom-def max-def coeff-def})$
with $\text{assms show } \text{Max } (\text{keys } (\text{mapping-of } p)) \in \text{keys } (\text{mapping-of } (\text{smult } c \ p))$
using * **by** $(\text{auto simp: smult.rep-eq intro!: in-keys-mapI})$
from $\text{smult-nz have } \text{lead-coeff } (\text{smult } c \ p) \neq 0$
by $(\text{intro lead-coeff-nonzero}) \text{ auto}$
hence $\text{coeff } p (\text{Max } (\text{keys } (\text{mapping-of } (\text{smult } c \ p)))) \neq 0$
using $\text{assms} * **$ **by** $(\text{auto simp: lead-coeff-def lead-monom-def max-def})$
thus $\text{Max } (\text{keys } (\text{mapping-of } (\text{smult } c \ p))) \in \text{keys } (\text{mapping-of } p)$
by $(\text{auto simp: smult.rep-eq coeff-def in-keys-iff})$

qed auto

with * ** show $\text{lead-monom } (\text{smult } c \ p) = \text{lead-monom } p$
by (*simp add: lead-monom-def max-def*)
thus $\text{lead-coeff } (\text{smult } c \ p) = c * \text{lead-coeff } p$
by (*simp add: lead-coeff-def*)
qed

lemma *lead-coeff-mult-aux*:
 $\text{coeff } (p * q) (\text{lead-monom } p + \text{lead-monom } q) = \text{lead-coeff } p * \text{lead-coeff } q$
proof (*cases p = 0 ∨ q = 0*)
case *False*
define *a b* **where** $a = \text{lead-monom } p$ **and** $b = \text{lead-monom } q$
have $\text{coeff } (p * q) (a + b) = \text{coeff } p \ a * \text{coeff } q \ b$
unfolding *coeff-mpoly-times*
by (*rule prod-fun-max*) (*insert False, auto simp: a-def b-def*)
thus *?thesis* **by** (*simp add: a-def b-def lead-coeff-def*)
qed *auto*

lemma *lead-monom-mult-le*: $\text{lead-monom } (p * q) \leq \text{lead-monom } p + \text{lead-monom } q$
proof (*cases p * q = 0*)
case *False*
show *?thesis*
proof (*intro leI notI*)
assume $\text{lead-monom } p + \text{lead-monom } q < \text{lead-monom } (p * q)$
hence $\text{lead-coeff } (p * q) = 0$
unfolding *lead-coeff-def coeff-mpoly-times* **by** (*rule prod-fun-gt-max-eq-zero*)
auto
with *False* **show** *False* **by** *simp*
qed
qed *auto*

lemma *lead-monom-mult*:
assumes $\text{lead-coeff } p * \text{lead-coeff } q \neq 0$
shows $\text{lead-monom } (p * q) = \text{lead-monom } p + \text{lead-monom } q$
by (*intro antisym lead-monom-mult-le lead-monom-geI*)
(insert assms, auto simp: lead-coeff-mult-aux)

lemma *lead-coeff-mult*:
assumes $\text{lead-coeff } p * \text{lead-coeff } q \neq 0$
shows $\text{lead-coeff } (p * q) = \text{lead-coeff } p * \text{lead-coeff } q$
using *assms* **by** (*simp add: lead-monom-mult lead-coeff-mult-aux lead-coeff-def*)

lemma *keys-lead-monom-subset*: $\text{keys } (\text{lead-monom } p) \subseteq \text{vars } p$
proof (*cases p = 0*)
case *False*
hence $\text{lead-coeff } p \neq 0$ **by** *simp*
hence $\text{coeff } p (\text{lead-monom } p) \neq 0$ **unfolding** *lead-coeff-def* .
thus *?thesis* **unfolding** *vars-def* **by** *transfer' (auto simp: max-def in-keys-iff)*
qed *auto*

lemma
assumes $(\prod i \in A. \text{lead-coeff } (f i)) \neq 0$
shows *lead-monom-prod*: $\text{lead-monom } (\prod i \in A. f i) = (\sum i \in A. \text{lead-monom } (f i))$ (**is** *?th1*)
and *lead-coeff-prod*: $\text{lead-coeff } (\prod i \in A. f i) = (\prod i \in A. \text{lead-coeff } (f i))$ (**is** *?th2*)
proof –
have *?th1* \wedge *?th2* **using** *assms*
proof (*induction A rule: infinite-finite-induct*)
case (*insert x A*)
from *insert* **have** *nz*: $\text{lead-coeff } (f x) \neq 0$ $(\prod i \in A. \text{lead-coeff } (f i)) \neq 0$ **by** *auto*
note *IH* = *insert.IH[OF this(2)]*
from *insert* **have** *nz'*: $\text{lead-coeff } (f x) * \text{lead-coeff } (\prod i \in A. f i) \neq 0$
by (*subst IH*) *auto*
from *insert.prem*s *insert.hyps* *nz* *nz'* **show** *?case*
by (*auto simp: lead-monom-mult lead-coeff-mult IH*)
qed *auto*
thus *?th1* *?th2* **by** *blast+*
qed

lemma *lead-monom-sum-le*: $(\bigwedge x. x \in X \implies \text{lead-monom } (h x) \leq ub) \implies \text{lead-monom } (\text{sum } h X) \leq ub$
by (*induction X rule: infinite-finite-induct*) (*auto intro!: order.trans[OF lead-monom-add]*)

The leading monomial of a sum where the leading monomial the summands are distinct is simply the maximum of the leading monomials.

lemma *lead-monom-sum*:
assumes *inj-on* $(\text{lead-monom} \circ h)$ *X* **and** *finite* *X* **and** $X \neq \{\}$ **and** $\bigwedge x. x \in X \implies h x \neq 0$
defines $m \equiv \text{Max } ((\text{lead-monom} \circ h) \text{ ` } X)$
shows $\text{lead-monom } (\sum x \in X. h x) = m$
proof (*rule antisym*)
show $\text{lead-monom } (\text{sum } h X) \leq m$ **unfolding** *m-def* **using** *assms*
by (*intro lead-monom-sum-le Max-ge finite-imageI*) *auto*
next
from *assms* **have** $m \in (\text{lead-monom} \circ h) \text{ ` } X$
unfolding *m-def* **by** (*intro Max-in finite-imageI*) *auto*
then obtain *x* **where** $x \in X$ $m = \text{lead-monom } (h x)$ **by** *auto*
have $\text{coeff } (\sum x \in X. h x) m = (\sum x \in X. \text{coeff } (h x) m)$
by *simp*
also have $\dots = (\sum x \in \{x\}. \text{coeff } (h x) m)$
proof (*intro sum.mono-neutral-right ballI*)
fix *y* **assume** $y: y \in X - \{x\}$
hence $(\text{lead-monom} \circ h) y \leq m$
using *assms* **unfolding** *m-def* **by** (*intro Max-ge finite-imageI*) *auto*
moreover have $(\text{lead-monom} \circ h) y \neq (\text{lead-monom} \circ h) x$
using $\langle x \in X \rangle$ *y inj-onD[OF assms(1), of x y]* **by** *auto*

ultimately have $\text{lead-monom } (h\ y) < m$
using x **by** *auto*
thus $\text{coeff } (h\ y)\ m = 0$ **by** *simp*
qed (*insert x assms, auto*)
also have $\dots = \text{coeff } (h\ x)\ m$ **by** *simp*
also have $\dots = \text{lead-coeff } (h\ x)$ **using** x **by** (*simp add: lead-coeff-def*)
also have $\dots \neq 0$ **using** *assms x* **by** *auto*
finally show $\text{lead-monom } (\text{sum } h\ X) \geq m$ **by** (*intro lead-monom-geI*)
qed

lemma *lead-coeff-eq-0-iff* [*simp*]: $\text{lead-coeff } p = 0 \longleftrightarrow p = 0$
by (*cases p = 0*) *auto*

lemma
fixes $f :: - \Rightarrow 'a :: \text{semidom mpoly}$
assumes $\bigwedge i. i \in A \implies f\ i \neq 0$
shows $\text{lead-monom-prod}'$ [*simp*]: $\text{lead-monom } (\prod_{i \in A}. f\ i) = (\sum_{i \in A}. \text{lead-monom } (f\ i))$ (*is ?th1*)
and $\text{lead-coeff-prod}'$ [*simp*]: $\text{lead-coeff } (\prod_{i \in A}. f\ i) = (\prod_{i \in A}. \text{lead-coeff } (f\ i))$ (*is ?th2*)
proof –
from *assms* **have** $(\prod_{i \in A}. \text{lead-coeff } (f\ i)) \neq 0$
by (*cases finite A*) *auto*
thus *?th1 ?th2* **by** (*simp-all add: lead-monom-prod lead-coeff-prod*)
qed

lemma
fixes $p :: 'a :: \text{comm-semiring-1 mpoly}$
assumes $\text{lead-coeff } p \wedge n \neq 0$
shows *lead-monom-power*: $\text{lead-monom } (p \wedge n) = \text{of-nat } n * \text{lead-monom } p$
and *lead-coeff-power*: $\text{lead-coeff } (p \wedge n) = \text{lead-coeff } p \wedge n$
using *assms* lead-monom-prod [*of* $\lambda-. p \{..<n\}$] lead-coeff-prod [*of* $\lambda-. p \{..<n\}$]
by *simp-all*

lemma
fixes $p :: 'a :: \text{semidom mpoly}$
assumes $p \neq 0$
shows $\text{lead-monom-power}'$ [*simp*]: $\text{lead-monom } (p \wedge n) = \text{of-nat } n * \text{lead-monom } p$
and $\text{lead-coeff-power}'$ [*simp*]: $\text{lead-coeff } (p \wedge n) = \text{lead-coeff } p \wedge n$
using *assms* $\text{lead-monom-prod}'$ [*of* $\{..<n\}$ $\lambda-. p$] $\text{lead-coeff-prod}'$ [*of* $\{..<n\}$ $\lambda-. p$]
by *simp-all*

2.7 Turning a set of variables into a monomial

Given a finite set $\{X_1, \dots, X_n\}$ of variables, the following is the monomial $X_1 \dots X_n$:

lift-definition *monom-of-set* :: $\text{nat set} \Rightarrow (\text{nat} \Rightarrow_0 \text{nat})$ **is**
 $\lambda X\ x. \text{if finite } X \wedge x \in X \text{ then } 1 \text{ else } 0$

by *auto*

lemma *lookup-monom-of-set*:

Poly-Mapping.lookup (monom-of-set X) i = (if finite X \wedge $i \in X$ then 1 else 0)

by *transfer auto*

lemma *lookup-monom-of-set-1* [*simp*]:

*finite X \implies $i \in X \implies$ *Poly-Mapping.lookup (monom-of-set X) i = 1**

and *lookup-monom-of-set-0* [*simp*]:

*$i \notin X \implies$ *Poly-Mapping.lookup (monom-of-set X) i = 0**

by (*simp-all add: lookup-monom-of-set*)

lemma *keys-monom-of-set*: *keys (monom-of-set X) = (if finite X then X else {})*

by *transfer auto*

lemma *keys-monom-of-set-finite* [*simp*]: *finite X \implies keys (monom-of-set X) = X*

by (*simp add: keys-monom-of-set*)

lemma *monom-of-set-eq-iff* [*simp*]: *finite X \implies finite Y \implies monom-of-set X = monom-of-set Y \iff X = Y*

by *transfer (auto simp: fun-eq-iff)*

lemma *monom-of-set-empty* [*simp*]: *monom-of-set {} = 0*

by *transfer auto*

lemma *monom-of-set-eq-zero-iff* [*simp*]: *monom-of-set X = 0 \iff infinite X \vee X = {}*

by *transfer (auto simp: fun-eq-iff)*

lemma *zero-eq-monom-of-set-iff* [*simp*]: *0 = monom-of-set X \iff infinite X \vee X = {}*

by *transfer (auto simp: fun-eq-iff)*

2.8 Permuting the variables of a polynomial

Next, we define the operation of permuting the variables of a monomial and polynomial.

lift-definition *permutep* :: *('a \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b :: zero) is*

*$\lambda f p.$ if *bij* *f* then $p \circ f$ else p*

proof –

fix *f* :: 'a \Rightarrow 'a **and** *g* :: 'a \Rightarrow 'b

assume *: *finite {x. g x \neq 0}*

show *finite {x. (if *bij* *f* then $g \circ f$ else g) x \neq 0}*

proof (*cases bij f*)

case *True*

with * **have** *finite (f - ' {x. g x \neq 0})*

by (*intro finite-vimageI*) (*auto dest: bij-is-inj*)

with *True* **show** *?thesis* **by** *auto*

qed (*use * in auto*)

qed

lift-definition *mpoly-map-vars* :: (nat \Rightarrow nat) \Rightarrow 'a :: zero *mpoly* \Rightarrow 'a *mpoly* **is**
 $\lambda f p.$ *permutep* (*permutep* f) p .

lemma *keys-permutep*: *bij* f \Longrightarrow *keys* (*permutep* f m) = f - ' *keys* m
by *transfer auto*

lemma *permutep-id''* [*simp*]: *permutep* id = id
by *transfer' (auto simp: fun-eq-iff)*

lemma *permutep-id'''* [*simp*]: *permutep* ($\lambda x. x$) = id
by *transfer' (auto simp: fun-eq-iff)*

lemma *permutep-0* [*simp*]: *permutep* f 0 = 0
by *transfer auto*

lemma *permutep-single*:
bij f \Longrightarrow *permutep* f (*Poly-Mapping.single* a b) = *Poly-Mapping.single* (*inv-into*
UNIV f a) b
by *transfer (auto simp: fun-eq-iff when-def inv-f-f surj-f-inv-f bij-is-inj bij-is-surj)*

lemma *mpoly-map-vars-id* [*simp*]: *mpoly-map-vars* id = id
by *transfer auto*

lemma *mpoly-map-vars-id'* [*simp*]: *mpoly-map-vars* ($\lambda x. x$) = id
by *transfer auto*

lemma *lookup-permutep*:
Poly-Mapping.lookup (*permutep* f m) x = (*if* *bij* f *then* *Poly-Mapping.lookup* m (f
x) *else* *Poly-Mapping.lookup* m x)
by *transfer auto*

lemma *inj-permutep* [*intro*]: *inj* (*permutep* (f :: 'a \Rightarrow 'a) :: - \Rightarrow 'a \Rightarrow_0 'b :: zero)
unfolding *inj-def*

proof (*transfer, safe*)

fix f :: 'a \Rightarrow 'a **and** x y :: 'a \Rightarrow 'b

assume eq: (*if* *bij* f *then* x \circ f *else* x) = (*if* *bij* f *then* y \circ f *else* y)

show x = y

proof (*cases* *bij* f)

case *True*

show ?thesis

proof

fix t :: 'a

from <*bij* f> **obtain** s **where** t = f s

by (*auto dest!: bij-is-surj*)

with eq **and** *True* **show** x t = y t

by (*auto simp: fun-eq-iff*)

qed

qed (use eq in auto)
qed

lemma *surj-permutep* [intro]: *surj* (*permutep* ($f :: 'a \Rightarrow 'a$) :: $- \Rightarrow 'a \Rightarrow_0 'b :: \text{zero}$)
unfolding *surj-def*
proof (*transfer, safe*)
fix $f :: 'a \Rightarrow 'a$ **and** $y :: 'a \Rightarrow 'b$
assume $fin: \text{finite } \{t. y t \neq 0\}$
show $\exists x \in \{f. \text{finite } \{x. f x \neq 0\}\}. y = (\text{if } \text{bij } f \text{ then } x \circ f \text{ else } x)$
proof (*cases bij f*)
case *True*
with fin **have** *finite* (*the-inv f - ' {t. y t ≠ 0}*)
by (*intro finite-vimageI*) (*auto simp: bij-is-inj bij-betw-the-inv-into*)
moreover **have** $y \circ \text{the-inv } f \circ f = y$
using *True* **by** (*simp add: fun-eq-iff the-inv-f-f bij-is-inj*)
ultimately **show** *?thesis* **by** (*intro beXI[of - y ∘ the-inv f]*) (*auto simp: True*)
qed (use *fin* in *auto*)
qed

lemma *bij-permutep* [intro]: *bij* (*permutep f*)
using *inj-permutep[of f] surj-permutep[of f]* **by** (*simp add: bij-def*)

lemma *mpoly-map-vars-map-mpoly*:
 $\text{mpoly-map-vars } f (\text{map-mpoly } g p) = \text{map-mpoly } g (\text{mpoly-map-vars } f p)$
by (*transfer', transfer'*) (*auto simp: fun-eq-iff*)

lemma *coeff-mpoly-map-vars*:
fixes $f :: \text{nat} \Rightarrow \text{nat}$ **and** $p :: 'a :: \text{zero mpolynomial}$
assumes *bij f*
shows $\text{MPolynomial.Type.coeff } (\text{mpoly-map-vars } f p) \text{ mon} =$
 $\text{MPolynomial.Type.coeff } p (\text{permutep } f \text{ mon})$
using *assms* **by** *transfer'* (*simp add: lookup-permutep bij-permutep*)

lemma *permutep-monom-of-set*:
assumes *bij f*
shows $\text{permutep } f (\text{monom-of-set } A) = \text{monom-of-set } (f - ' A)$
using *assms* **by** *transfer* (*auto simp: fun-eq-iff bij-is-inj finite-vimage-iff*)

lemma *permutep-comp*: $\text{bij } f \Longrightarrow \text{bij } g \Longrightarrow \text{permutep } (f \circ g) = \text{permutep } g \circ$
 $\text{permutep } f$
by *transfer'* (*auto simp: fun-eq-iff bij-comp*)

lemma *permutep-comp'*: $\text{bij } f \Longrightarrow \text{bij } g \Longrightarrow \text{permutep } (f \circ g) \text{ mon} = \text{permutep } g$
 $(\text{permutep } f \text{ mon})$
by *transfer* (*auto simp: fun-eq-iff bij-comp*)

lemma *mpoly-map-vars-comp*:
 $\text{bij } f \Longrightarrow \text{bij } g \Longrightarrow \text{mpoly-map-vars } f (\text{mpoly-map-vars } g p) = \text{mpoly-map-vars } (f$
 $\circ g) p$

by transfer (auto simp: bij-permutep permutep-comp)

lemma permutep-id [simp]: permutep id mon = mon
by transfer auto

lemma permutep-id' [simp]: permutep ($\lambda x. x$) mon = mon
by transfer auto

lemma inv-permutep [simp]:
fixes $f :: 'a \Rightarrow 'a$
assumes *bij f*
shows $\text{inv-into UNIV (permutep f) = permutep (inv-into UNIV f)}$
proof
fix $m :: 'a \Rightarrow_0 'b$
show $\text{inv-into UNIV (permutep f) m = permutep (inv-into UNIV f) m}$
using *permutep-comp'[of inv-into UNIV f f m] assms inj-iff[of f]*
by (intro inv-f-eq) (auto simp: bij-imp-bij-inv bij-is-inj)
qed

lemma mpoly-map-vars-monom:
 $\text{bij f} \implies \text{mpoly-map-vars f (monom m c) = monom (permutep (inv-into UNIV f) m) c}$
by transfer' (simp add: permutep-single bij-permutep)

lemma vars-mpoly-map-vars:
fixes $f :: \text{nat} \Rightarrow \text{nat}$ and $p :: 'a :: \text{zero mpoly}$
assumes *bij f*
shows $\text{vars (mpoly-map-vars f p) = f ' vars p}$
using *assms unfolding vars-def*
proof transfer'
fix $f :: \text{nat} \Rightarrow \text{nat}$ and $p :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a$
assume *f: bij f*
have *eq: f (inv-into UNIV f x) = x for x*
using *f by (subst surj-f-inv-f[of f]) (auto simp: bij-is-surj)*
show $\bigcup (\text{keys ' keys (permutep (permutep f) p)}) = f ' \bigcup (\text{keys ' keys p})$
proof safe
fix $m x$ assume *mx: m ∈ keys (permutep (permutep f) p) x ∈ keys m*
from *mx* have $\text{permutep f m} \in \text{keys p}$
by (auto simp: keys-permutep bij-permutep f)
with *mx* have $f (\text{inv-into UNIV f x}) \in f ' (\bigcup_{m \in \text{keys p. keys m})$
by (intro imageI) (auto intro!: bexI[of - permutep f m] simp: keys-permutep f
eq)
with *eq* show $x \in f ' (\bigcup_{m \in \text{keys p. keys m})$ by simp
next
fix $m x$ assume *mx: m ∈ keys p x ∈ keys m*
from *mx* have $\text{permutep id m} \in \text{keys p}$ by simp
also have $\text{id} = \text{inv-into UNIV f} \circ f$ using *f* by (intro ext) (auto simp: bij-is-inj
inv-f-f)
also have $\text{permutep} \dots m = \text{permutep f (permutep (inv-into UNIV f) m)}$

by (*simp add: permutep-comp f bij-imp-bij-inv*)
finally have **: *permutep f (permutep (inv-into UNIV f) m) ∈ keys p* .
moreover from *f mx have f x ∈ keys (permutep (inv-into UNIV f) m)*
 by (*auto simp: keys-permutep bij-imp-bij-inv inv-f-f bij-is-inj*)
ultimately show $f x \in \bigcup (\text{keys } \text{' } \text{keys (permutep (permutep f) p)})$ **using** *f*
 by (*auto simp: keys-permutep bij-permutep*)
qed
qed

lemma *permutep-eq-monom-of-set-iff [simp]*:
 assumes *bij f*
 shows $\text{permutep } f \text{ mon} = \text{monom-of-set } A \longleftrightarrow \text{mon} = \text{monom-of-set } (f \text{' } A)$
proof
 assume *eq: permutep f mon = monom-of-set A*
have *permutep (inv-into UNIV f) (permutep f mon) = monom-of-set (inv-into UNIV f -' A)*
 using *assms by (simp add: eq bij-imp-bij-inv assms permutep-monom-of-set)*
also have *inv-into UNIV f -' A = f ' A*
 using *assms by (force simp: bij-is-surj image-iff inv-f-f bij-is-inj surj-f-inv-f)*
also have *permutep (inv-into UNIV f) (permutep f mon) = permutep (f ∘ inv-into UNIV f) mon*
 using *assms by (simp add: permutep-comp bij-imp-bij-inv)*
also have *f ∘ inv-into UNIV f = id*
 by (*subst surj-iff [symmetric]*) (*insert assms, auto simp: bij-is-surj*)
finally show $\text{mon} = \text{monom-of-set } (f \text{' } A)$ **by** *simp*
qed (*insert assms, auto simp: permutep-monom-of-set inj-vimage-image-eq bij-is-inj*)

lemma *permutep-monom-of-set-permutes [simp]*:
 assumes $\pi \text{ permutes } A$
 shows $\text{permutep } \pi (\text{monom-of-set } A) = \text{monom-of-set } A$
 using *assms*
 by (*transfer (auto simp: if-splits fun-eq-iff permutes-in-image)*)

lemma *mpoly-map-vars-0 [simp]*: $\text{mpoly-map-vars } f \ 0 = 0$
 by (*transfer, transfer'*) (*simp add: o-def*)

lemma *permutep-eq-0-iff [simp]*: $\text{permutep } f \ m = 0 \longleftrightarrow m = 0$
proof *transfer*
fix $f :: 'a \Rightarrow 'a$ **and** $m :: 'a \Rightarrow 'b$ **assume** *finite {x. m x ≠ 0}*
show $((\text{if } \text{bij } f \text{ then } m \circ f \text{ else } m) = (\lambda k. 0)) = (m = (\lambda k. 0))$
proof (*cases bij f*)
 case *True*
hence $(\forall x. m (f x) = 0) \longleftrightarrow (\forall x. m x = 0)$
 using *bij-iff[of f]* **by** *metis*
with True show *?thesis* **by** (*auto simp: fun-eq-iff*)
qed (*auto simp: fun-eq-iff*)
qed

lemma *mpoly-map-vars-1 [simp]*: $\text{mpoly-map-vars } f \ 1 = 1$

by (*transfer*, *transfer'*) (*auto simp: o-def fun-eq-iff when-def*)

lemma *permutep-Const₀* [*simp*]: $(\bigwedge x. f x = 0 \longleftrightarrow x = 0) \implies \text{permutep } f (\text{Const}_0 c) = \text{Const}_0 c$

unfolding *Const₀-def* **by** *transfer'* (*auto simp: when-def fun-eq-iff*)

lemma *permutep-add* [*simp*]: $\text{permutep } f (m1 + m2) = \text{permutep } f m1 + \text{permutep } f m2$

unfolding *Const₀-def* **by** *transfer'* (*auto simp: when-def fun-eq-iff*)

lemma *permutep-diff* [*simp*]: $\text{permutep } f (m1 - m2) = \text{permutep } f m1 - \text{permutep } f m2$

unfolding *Const₀-def* **by** *transfer'* (*auto simp: when-def fun-eq-iff*)

lemma *permutep-uminus* [*simp*]: $\text{permutep } f (-m) = -\text{permutep } f m$

unfolding *Const₀-def* **by** *transfer'* (*auto simp: when-def fun-eq-iff*)

lemma *permutep-mult* [*simp*]:

$(\bigwedge x y. f (x + y) = f x + f y) \implies \text{permutep } f (m1 * m2) = \text{permutep } f m1 * \text{permutep } f m2$

unfolding *Const₀-def* **by** *transfer'* (*auto simp: when-def fun-eq-iff prod-fun-compose-bij*)

lemma *mpoly-map-vars-Const* [*simp*]: $\text{mpoly-map-vars } f (\text{Const } c) = \text{Const } c$

by *transfer* (*auto simp: o-def fun-eq-iff when-def*)

lemma *mpoly-map-vars-add* [*simp*]: $\text{mpoly-map-vars } f (p + q) = \text{mpoly-map-vars } f p + \text{mpoly-map-vars } f q$

by *transfer simp*

lemma *mpoly-map-vars-diff* [*simp*]: $\text{mpoly-map-vars } f (p - q) = \text{mpoly-map-vars } f p - \text{mpoly-map-vars } f q$

by *transfer simp*

lemma *mpoly-map-vars-uminus* [*simp*]: $\text{mpoly-map-vars } f (-p) = -\text{mpoly-map-vars } f p$

by *transfer simp*

lemma *permutep-smult*:

$\text{permutep } (\text{permutep } f) (\text{Poly-Mapping.map } ((*) c) p) = \text{Poly-Mapping.map } ((*) c) (\text{permutep } (\text{permutep } f) p)$

by *transfer'* (*auto split: if-splits simp: fun-eq-iff*)

lemma *mpoly-map-vars-smult* [*simp*]: $\text{mpoly-map-vars } f (\text{smult } c p) = \text{smult } c (\text{mpoly-map-vars } f p)$

by *transfer (simp add: permutep-smult)*

lemma *mpoly-map-vars-mult* [*simp*]: $\text{mpoly-map-vars } f (p * q) = \text{mpoly-map-vars } f p * \text{mpoly-map-vars } f q$

by *transfer simp*

lemma *mpoly-map-vars-sum* [*simp*]: $mpoly\text{-map}\text{-vars } f (sum\ g\ A) = (\sum x \in A. mpoly\text{-map}\text{-vars } f (g\ x))$

by (*induction A rule: infinite-finite-induct*) *auto*

lemma *mpoly-map-vars-prod* [*simp*]: $mpoly\text{-map}\text{-vars } f (prod\ g\ A) = (\prod x \in A. mpoly\text{-map}\text{-vars } f (g\ x))$

by (*induction A rule: infinite-finite-induct*) *auto*

lemma *mpoly-map-vars-eq-0-iff* [*simp*]: $mpoly\text{-map}\text{-vars } f\ p = 0 \longleftrightarrow p = 0$

by *transfer auto*

lemma *permutep-eq-iff* [*simp*]: $permutep\ f\ p = permutep\ f\ q \longleftrightarrow p = q$

by *transfer (auto simp: comp-bij-eq-iff)*

lemma *mpoly-map-vars-Sum-any* [*simp*]:

$mpoly\text{-map}\text{-vars } f (Sum\text{-any } g) = Sum\text{-any } (\lambda x. mpoly\text{-map}\text{-vars } f (g\ x))$

by (*simp add: Sum-any.expand-set*)

lemma *mpoly-map-vars-power* [*simp*]: $mpoly\text{-map}\text{-vars } f (p \wedge^n) = mpoly\text{-map}\text{-vars } f\ p \wedge^n$

by (*induction n*) *auto*

lemma *mpoly-map-vars-monom-single* [*simp*]:

assumes *bij f*

shows $mpoly\text{-map}\text{-vars } f (monom (Poly\text{-Mapping}\text{-single } i\ n)\ c) = monom (Poly\text{-Mapping}\text{-single } (f\ i)\ n)\ c$

using *assms* **by** (*simp add: mpoly-map-vars-monom permutep-single bij-imp-bij-inv inv-inv-eq*)

lemma *insertion-mpoly-map-vars*:

assumes *bij f*

shows $insertion\ g (mpoly\text{-map}\text{-vars } f\ p) = insertion\ (g \circ f)\ p$

proof –

have $insertion\ g (mpoly\text{-map}\text{-vars } f\ p) =$

$(\sum m. coeff\ p (permutep\ f\ m) * (\prod i. g\ i \wedge lookup\ m\ i))$

using *assms* **by** (*simp add: insertion-altdef coeff-mpoly-map-vars*)

also have $\dots = Sum\text{-any } (\lambda m. coeff\ p (permutep\ f\ m) *$

$Prod\text{-any } (\lambda i. g\ (f\ i) \wedge lookup\ m\ (f\ i)))$

by (*intro Sum-any.cong arg-cong[where ?f = $\lambda y. x * y$ for x]*

Prod-any.reindex-cong[OF assms]) (*auto simp: o-def*)

also have $\dots = Sum\text{-any } (\lambda m. coeff\ p\ m * (\prod i. g\ (f\ i) \wedge lookup\ m\ i))$

by (*intro Sum-any.reindex-cong [OF bij-permutep[of f], symmetric]*)

(*auto simp: o-def lookup-permutep assms*)

also have $\dots = insertion\ (g \circ f)\ p$

by (*simp add: insertion-altdef*)

finally show *?thesis* .

qed

```

lemma permutep-cong:
  assumes f permutes (-keys p) g permutes (-keys p) p = q
  shows permutep f p = permutep g q
proof (intro poly-mapping-eqI)
  fix k :: 'a
  show lookup (permutep f p) k = lookup (permutep g q) k
  proof (cases k ∈ keys p)
    case False
    with assms have f k ∉ keys p g k ∉ keys p
    using permutes-in-image[of - -keys p k] by auto
    thus ?thesis using assms by (auto simp: lookup-permutep permutes-bij in-keys-iff)
  qed (insert assms, auto simp: lookup-permutep permutes-bij permutes-not-in)
qed

```

```

lemma mpoly-map-vars-cong:
  assumes f permutes (-vars p) g permutes (-vars q) p = q
  shows mpoly-map-vars f p = mpoly-map-vars g (q :: 'a :: zero mpoly)
proof (intro mpoly-eqI)
  fix mon :: nat  $\Rightarrow_0$  nat
  show coeff (mpoly-map-vars f p) mon = coeff (mpoly-map-vars g q) mon
  proof (cases keys mon ⊆ vars p)
    case True
    with assms have permutep f mon = permutep g mon
    by (intro permutep-cong assms(1,2)[THEN permutes-subset]) auto
    thus ?thesis using assms by (simp add: coeff-mpoly-map-vars permutes-bij)
  next
  case False
  hence  $\neg(\text{keys } mon \subseteq f \text{ ' vars } q) \neg(\text{keys } mon \subseteq g \text{ ' vars } q)$ 
  using assms by (auto simp: subset-iff permutes-not-in)
  thus ?thesis using assms
    by (subst (1 2) coeff-notin-vars)
    (auto simp: coeff-notin-vars vars-mpoly-map-vars permutes-bij)
  qed
qed

```

2.9 Symmetric polynomials

A polynomial is symmetric on a set of variables if it is invariant under any permutation of that set.

definition *symmetric-mpoly* :: *nat set* \Rightarrow 'a :: *zero mpoly* \Rightarrow *bool* **where**
symmetric-mpoly A p = ($\forall \pi. \pi \text{ permutes } A \longrightarrow \text{mpoly-map-vars } \pi p = p$)

lemma *symmetric-mpoly-empty* [*simp, intro*]: *symmetric-mpoly* {} *p*
by (*simp add: symmetric-mpoly-def*)

A polynomial is trivially symmetric on any set of variables that do not occur in it.

lemma *symmetric-mpoly-orthogonal*:

```

assumes vars p  $\cap$  A = {}
shows symmetric-mpoly A p
unfolding symmetric-mpoly-def
proof safe
  fix  $\pi$  assume  $\pi$ :  $\pi$  permutes A
  with assms have  $\pi$  x = x if x  $\in$  vars p for x
    using that permutes-not-in[of  $\pi$  A x] by auto
  from assms have mpoly-map-vars  $\pi$  p = mpoly-map-vars id p
    by (intro mpoly-map-vars-cong permutes-subset[OF  $\pi$ ] permutes-id) auto
  also have ... = p by simp
  finally show mpoly-map-vars  $\pi$  p = p .
qed

```

```

lemma symmetric-mpoly-monom [intro]:
  assumes keys m  $\cap$  A = {}
  shows symmetric-mpoly A (monom m c)
  using assms vars-monom-subset[of m c] by (intro symmetric-mpoly-orthogonal)
  auto

```

```

lemma symmetric-mpoly-subset:
  assumes symmetric-mpoly A p B  $\subseteq$  A
  shows symmetric-mpoly B p
  unfolding symmetric-mpoly-def
proof safe
  fix  $\pi$  assume  $\pi$  permutes B
  with assms have  $\pi$  permutes A using permutes-subset by blast
  with assms show mpoly-map-vars  $\pi$  p = p
    by (auto simp: symmetric-mpoly-def)
qed

```

If a polynomial is symmetric over some set of variables, that set must either be a subset of the variables occurring in the polynomial or disjoint from it.

```

lemma symmetric-mpoly-imp-orthogonal-or-subset:
  assumes symmetric-mpoly A p
  shows vars p  $\cap$  A = {}  $\vee$  A  $\subseteq$  vars p
proof (rule ccontr)
  assume  $\neg$ (vars p  $\cap$  A = {}  $\vee$  A  $\subseteq$  vars p)
  then obtain x y where xy: x  $\in$  vars p  $\cap$  A y  $\in$  A - vars p by auto
  define  $\pi$  where  $\pi$  = transpose x y
  from xy have  $\pi$ :  $\pi$  permutes A
    unfolding  $\pi$ -def by (intro permutes-swap-id) auto
  from xy have y  $\in$   $\pi$  ' vars p by (auto simp:  $\pi$ -def transpose-def)
  also from  $\pi$  have  $\pi$  ' vars p = vars (mpoly-map-vars  $\pi$  p)
    by (auto simp: vars-mpoly-map-vars permutes-bij)
  also have mpoly-map-vars  $\pi$  p = p
    using assms  $\pi$  by (simp add: symmetric-mpoly-def)
  finally show False using xy by auto
qed

```

Symmetric polynomials are closed under ring operations.

lemma *symmetric-mpoly-add* [intro]:

symmetric-mpoly A p \implies *symmetric-mpoly A q* \implies *symmetric-mpoly A (p + q)*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-diff* [intro]:

symmetric-mpoly A p \implies *symmetric-mpoly A q* \implies *symmetric-mpoly A (p - q)*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-uminus* [intro]: *symmetric-mpoly A p* \implies *symmetric-mpoly A (-p)*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-uminus-iff* [simp]: *symmetric-mpoly A (-p)* \longleftrightarrow *symmetric-mpoly A p*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-smult* [intro]: *symmetric-mpoly A p* \implies *symmetric-mpoly A (smult c p)*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-mult* [intro]:

symmetric-mpoly A p \implies *symmetric-mpoly A q* \implies *symmetric-mpoly A (p * q)*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-0* [simp, intro]: *symmetric-mpoly A 0*

and *symmetric-mpoly-1* [simp, intro]: *symmetric-mpoly A 1*

and *symmetric-mpoly-Const* [simp, intro]: *symmetric-mpoly A (Const c)*

by (*simp-all add: symmetric-mpoly-def*)

lemma *symmetric-mpoly-power* [intro]:

symmetric-mpoly A p \implies *symmetric-mpoly A (p ^ n)*

by (*induction n*) (*auto intro!: symmetric-mpoly-mult*)

lemma *symmetric-mpoly-sum* [intro]:

($\bigwedge i. i \in B \implies$ *symmetric-mpoly A (f i)*) \implies *symmetric-mpoly A (sum f B)*

by (*induction B rule: infinite-finite-induct*) (*auto intro!: symmetric-mpoly-add*)

lemma *symmetric-mpoly-prod* [intro]:

($\bigwedge i. i \in B \implies$ *symmetric-mpoly A (f i)*) \implies *symmetric-mpoly A (prod f B)*

by (*induction B rule: infinite-finite-induct*) (*auto intro!: symmetric-mpoly-mult*)

An symmetric sum or product over polynomials yields a symmetric polynomial:

lemma *symmetric-mpoly-symmetric-sum*:

assumes *g permutes X*

assumes $\bigwedge x \pi. x \in X \implies \pi$ *permutes A* \implies *mpoly-map-vars* $\pi (f x) = f (g x)$

shows *symmetric-mpoly A* ($\sum_{x \in X}. f x$)

unfolding *symmetric-mpoly-def*

proof *safe*
fix π **assume** π : π *permutes* A
have $\text{mpoly-map-vars } \pi (\text{sum } f X) = (\sum x \in X. \text{mpoly-map-vars } \pi (f x))$
by *simp*
also have $\dots = (\sum x \in X. f (g x))$
by (*intro sum.cong assms* π *refl*)
also have $\dots = (\sum x \in g'X. f x)$
using *assms* **by** (*subst sum.reindex*) (*auto simp: permutes-inj-on*)
also have $g' X = X$
using *assms* **by** (*simp add: permutes-image*)
finally show $\text{mpoly-map-vars } \pi (\text{sum } f X) = \text{sum } f X$.
qed

lemma *symmetric-mpoly-symmetric-prod*:
assumes g *permutes* X
assumes $\bigwedge x \pi. x \in X \implies \pi$ *permutes* $A \implies \text{mpoly-map-vars } \pi (f x) = f (g x)$
shows *symmetric-mpoly* $A (\prod x \in X. f x)$
unfolding *symmetric-mpoly-def*
proof *safe*
fix π **assume** π : π *permutes* A
have $\text{mpoly-map-vars } \pi (\text{prod } f X) = (\prod x \in X. \text{mpoly-map-vars } \pi (f x))$
by *simp*
also have $\dots = (\prod x \in X. f (g x))$
by (*intro prod.cong assms* π *refl*)
also have $\dots = (\prod x \in g'X. f x)$
using *assms* **by** (*subst prod.reindex*) (*auto simp: permutes-inj-on*)
also have $g' X = X$
using *assms* **by** (*simp add: permutes-image*)
finally show $\text{mpoly-map-vars } \pi (\text{prod } f X) = \text{prod } f X$.
qed

If p is a polynomial that is symmetric on some subset of variables A , then for the leading monomial of p , the exponents of these variables are decreasing w. r. t. the variable ordering.

theorem *lookup-lead-monom-decreasing*:
assumes *symmetric-mpoly* $A p$
defines $m \equiv \text{lead-monom } p$
assumes $i \in A j \in A i \leq j$
shows $\text{lookup } m i \geq \text{lookup } m j$
proof (*cases* $p = 0$)
case [*simp*]: *False*
show *?thesis*
proof (*intro leI notI*)
assume *less*: $\text{lookup } m i < \text{lookup } m j$
define π **where** $\pi = \text{transpose } i j$
from *assms* **have** π : π *permutes* A
unfolding π -*def* **by** (*intro permutes-swap-id*) *auto*
have [*simp*]: $\pi \circ \pi = \text{id } \pi i = j \pi j = i \bigwedge k. k \neq i \implies k \neq j \implies \pi k = k$
by (*auto simp:* π -*def Fun.swap-def fun-eq-iff*)

have $0 \neq \text{lead-coeff } p$ **by** *simp*
also have $\text{lead-coeff } p = \text{MPoly-Type.coeff } (\text{mpoly-map-vars } \pi \ p)$ (*permutep* π
 m)
using π **by** (*simp add: lead-coeff-def m-def coeff-mpoly-map-vars*
permutes-bij permutep-comp' [symmetric])
also have $\text{mpoly-map-vars } \pi \ p = p$
using π *assms* **by** (*simp add: symmetric-mpoly-def*)
finally have $\text{permutep } \pi \ m \leq m$ **by** (*auto simp: m-def*)

moreover have $\text{lookup } m \ i < \text{lookup } (\text{permutep } \pi \ m) \ i$
and $(\forall k < i. \text{lookup } m \ k = \text{lookup } (\text{permutep } \pi \ m) \ k)$
using *assms* π *less* **by** (*auto simp: lookup-permutep permutes-bij*)
hence $m < \text{permutep } \pi \ m$
by (*auto simp: less-poly-mapping-def less-fun-def*)
ultimately show *False* **by** *simp*
qed
qed (*auto simp: m-def*)

2.10 The elementary symmetric polynomials

The k -th elementary symmetric polynomial for a finite set of variables A , with k ranging between 1 and $|A|$, is the sum of the product of all subsets of A with cardinality k :

lift-definition $\text{sym-mpoly-aux} :: \text{nat set} \Rightarrow \text{nat} \Rightarrow (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a :: \{\text{zero-neq-one}\}$
is

$\lambda X \ k \ \text{mon. if finite } X \wedge (\exists Y. Y \subseteq X \wedge \text{card } Y = k \wedge \text{mon} = \text{monom-of-set } Y)$
 $\text{then } 1 \ \text{else } 0$

proof –

fix $k :: \text{nat}$ **and** $X :: \text{nat set}$

show $\text{finite } \{x. (\text{if finite } X \wedge (\exists Y \subseteq X. \text{card } Y = k \wedge x = \text{monom-of-set } Y) \text{ then } 1 \ \text{else } 0) \neq$

$(0 :: 'a)\}$ (**is** *finite* $?A$)

proof (*cases finite* X)

case *True*

have $?A \subseteq \text{monom-of-set } ' \text{Pow } X$ **by** *auto*

moreover from *True* **have** *finite* (*monom-of-set* $' \text{Pow } X$) **by** *simp*

ultimately show $?thesis$ **by** (*rule finite-subset*)

qed *auto*

qed

lemma *lookup-sym-mpoly-aux*:

Poly-Mapping.lookup (*sym-mpoly-aux* $X \ k$) *mon* =

(*if finite* $X \wedge (\exists Y. Y \subseteq X \wedge \text{card } Y = k \wedge \text{mon} = \text{monom-of-set } Y)$ *then* 1
else 0)

by *transfer' simp*

lemma *lookup-sym-mpoly-aux-monom-of-set* [*simp*]:

assumes *finite* $X \ Y \subseteq X \ \text{card } Y = k$

shows $Poly\text{-Mapping.lookup } (sym\text{-mpoly-aux } X \ k) \ (monom\text{-of-set } Y) = 1$
using *assms* **by** (*auto simp: lookup-sym-mpoly-aux*)

lemma *keys-sym-mpoly-aux*: $m \in keys \ (sym\text{-mpoly-aux } A \ k) \implies keys \ m \subseteq A$
by *transfer'* (*auto split: if-splits simp: keys-monom-of-set*)

lift-definition *sym-mpoly* :: $nat \ set \Rightarrow nat \Rightarrow 'a :: \{zero\text{-neq-one}\} \ mpoly$ **is**
sym-mpoly-aux .

lemma *vars-sym-mpoly-subset*: $vars \ (sym\text{-mpoly } A \ k) \subseteq A$
using *keys-sym-mpoly-aux* **by** (*auto simp: vars-def sym-mpoly.rep-eq*)

lemma *coeff-sym-mpoly*:
 $MPoly\text{-Type.coeff } (sym\text{-mpoly } X \ k) \ mon =$
 (*if finite X* $\wedge (\exists Y. Y \subseteq X \wedge card \ Y = k \wedge mon = monom\text{-of-set } Y)$ *then 1*
else 0)
by *transfer'* (*simp add: lookup-sym-mpoly-aux*)

lemma *sym-mpoly-infinite*: $\neg finite \ A \implies sym\text{-mpoly } A \ k = 0$
by (*transfer, transfer*) *auto*

lemma *sym-mpoly-altdef*: $sym\text{-mpoly } A \ k = (\sum X \mid X \subseteq A \wedge card \ X = k. monom$
 (*monom-of-set X*) $1)$

proof (*cases finite A*)

case *False*

hence *: *infinite* $\{X. X \subseteq A \wedge infinite \ X\}$
by (*rule infinite-infinite-subsets*)

have *infinite* $\{X. X \subseteq A \wedge card \ X = 0\}$
by (*rule infinite-super[OF - *]*) *auto*

moreover **have** **: *infinite* $\{X. X \subseteq A \wedge finite \ X \wedge card \ X = k\}$ **if** $k \neq 0$
using *that infinite-card-subsets[of A k] False* **by** *auto*

have *infinite* $\{X. X \subseteq A \wedge card \ X = k\}$ **if** $k \neq 0$
by (*rule infinite-super[OF - **[OF that]]*) *auto*

ultimately **show** *?thesis* **using** *False*

by (*cases k = 0*) (*simp-all add: sym-mpoly-infinite*)

next

case *True*

show *?thesis*

proof (*intro mpoly-eqI, goal-cases*)

case ($1 \ m$)

show *?case*

proof (*cases* $\exists X. X \subseteq A \wedge card \ X = k \wedge m = monom\text{-of-set } X$)

case *False*

thus *?thesis* **by** (*auto simp: coeff-sym-mpoly coeff-sum coeff-monom*)

next

case *True*

then **obtain** X **where** $X: X \subseteq A \ card \ X = k \ m = monom\text{-of-set } X$
by *blast*

have *coeff* $(\sum X \mid X \subseteq A \wedge card \ X = k.$

$\text{monom } (\text{monom-of-set } X) \ 1) \ m = (\sum X \in \{X\}. \ 1)$ **unfolding** *coeff-sum*
proof (*intro sum.mono-neutral-cong-right ballI*)
fix Y **assume** $Y: Y \in \{X. X \subseteq A \wedge \text{card } X = k\} - \{X\}$
hence $X = Y$ **if** *monom-of-set* $X = \text{monom-of-set } Y$
using *that finite-subset[OF X(1)] finite-subset[of Y A] <finite A>* **by** *auto*
thus *coeff* (*monom (monom-of-set Y) 1*) $m = 0$
using $X \ Y$ **by** (*auto simp: coeff-monom when-def*)
qed (*insert X <finite A>, auto simp: coeff-monom*)
thus *?thesis using <finite A>* **by** (*auto simp: coeff-sym-mpoly coeff-sum coeff-monom*)
qed
qed
qed

lemma *coeff-sym-mpoly-monom-of-set [simp]*:
assumes *finite X Y* $Y \subseteq X$ *card Y = k*
shows *MPoly-Type.coeff (sym-mpoly X k) (monom-of-set Y) = 1*
using *assms* **by** (*auto simp: coeff-sym-mpoly*)

lemma *coeff-sym-mpoly-0*: *coeff (sym-mpoly X k) 0 = (if finite X \wedge $k = 0$ then 1 else 0)*
proof –
consider *finite X k = 0 | finite X k \neq 0 | infinite X* **by** *blast*
thus *?thesis*
proof *cases*
assume *finite X k = 0*
hence *coeff (sym-mpoly X k) (monom-of-set {})* $= 1$
by (*subst coeff-sym-mpoly-monom-of-set*) *auto*
thus *?thesis unfolding monom-of-set-empty using <finite X> <k = 0>* **by** *simp*
next
assume *finite X k \neq 0*
hence $\neg(\exists Y. \text{finite } Y \wedge Y \subseteq X \wedge \text{card } Y = k \wedge \text{monom-of-set } Y = 0)$
by *auto*
thus *?thesis using <k \neq 0>*
by (*auto simp: coeff-sym-mpoly*)
next
assume *infinite X*
thus *?thesis by (simp add: coeff-sym-mpoly)*
qed
qed

lemma *symmetric-sym-mpoly [intro]*:
assumes $A \subseteq B$
shows *symmetric-mpoly A (sym-mpoly B k :: 'a :: zero-neq-one mpoly)*
unfolding *symmetric-mpoly-def*
proof (*safe intro!: mpoly-eqI*)
fix π **and** *mon :: nat \Rightarrow_0 nat* **assume** $\pi: \pi$ *permutes A*
from π **have** π' : π *permutes B* **by** (*rule permutes-subset*) *fact*
from π **have** *MPoly-Type.coeff (mpoly-map-vars π (sym-mpoly B k :: 'a mpoly))*

$mon =$
 $MPoly-Type.coeff (sym-mpoly B k :: 'a mpoly) (permutep \pi mon)$
by (*simp add: coeff-mpoly-map-vars permutes-bij*)
also have $\dots = 1 \longleftrightarrow MPoly-Type.coeff (sym-mpoly B k :: 'a mpoly) mon = 1$
(is ?lhs = 1 \longleftrightarrow ?rhs = 1)
proof
assume $?rhs = 1$
then obtain Y **where** *finite B and* $Y: Y \subseteq B \text{ card } Y = k \text{ mon} = \text{monom-of-set } Y$
by (*auto simp: coeff-sym-mpoly split: if-splits*)
with π' **have** $\pi - ' Y \subseteq B \text{ card } (\pi - ' Y) = k \text{ permutep } \pi \text{ mon} = \text{monom-of-set } (\pi - ' Y)$
by (*auto simp: permutes-in-image card-vimage-inj permutep-monom-of-set permutes-bij permutes-inj permutes-surj*)
thus $?lhs = 1$ **using** $\langle \text{finite } B \rangle$ **by** (*auto simp: coeff-sym-mpoly*)
next
assume $?lhs = 1$
then obtain Y **where** *finite B and* $Y: Y \subseteq B \text{ card } Y = k \text{ permutep } \pi \text{ mon} = \text{monom-of-set } Y$
by (*auto simp: coeff-sym-mpoly split: if-splits*)
from $Y(1)$ **have** *inj-on* πY **using** *inj-on-subset[of $\pi UNIV Y$]* π'
by (*auto simp: permutes-inj*)
with $Y \pi'$ **have** $\pi - ' Y \subseteq B \text{ card } (\pi - ' Y) = k \text{ mon} = \text{monom-of-set } (\pi - ' Y)$
by (*auto simp: permutes-in-image card-image permutep-monom-of-set permutes-bij permutes-inj permutes-surj*)
thus $?rhs = 1$ **using** $\langle \text{finite } B \rangle$ **by** (*auto simp: coeff-sym-mpoly*)
qed
hence $?lhs = ?rhs$
by (*auto simp: coeff-sym-mpoly split: if-splits*)
finally show $MPoly-Type.coeff (mpoly-map-vars \pi (sym-mpoly B k :: 'a mpoly))$
 $mon =$
 $MPoly-Type.coeff (sym-mpoly B k :: 'a mpoly) mon .$
qed

lemma *insertion-sym-mpoly:*

assumes *finite X*
shows $\text{insertion } f (sym-mpoly X k) = (\sum Y \mid Y \subseteq X \wedge \text{card } Y = k. \text{prod } f Y)$
using *assms*
proof (*transfer, transfer*)
fix $f :: \text{nat} \Rightarrow 'a$ **and** $k :: \text{nat}$ **and** $X :: \text{nat set}$
assume $X: \text{finite } X$
have *insertion-fun* $f (\lambda mon.$
 $\text{if } \text{finite } X \wedge (\exists Y \subseteq X. \text{card } Y = k \wedge \text{mon} = \text{monom-of-set } Y) \text{ then } 1$
 $\text{else } 0) =$
 $(\sum m. (\prod v. f v \wedge \text{poly-mapping.lookup } m v) \text{ when } (\exists Y \subseteq X. \text{card } Y = k \wedge$
 $m = \text{monom-of-set } Y))$
by (*auto simp add: insertion-fun-def X when-def intro!: Sum-any.cong*)
also have $\dots = (\sum m \mid \exists Y \in \text{Pow } X. \text{card } Y = k \wedge m = \text{monom-of-set } Y. (\prod v.$
 $f v \wedge \text{poly-mapping.lookup } m v) \text{ when } (\exists Y \subseteq X. \text{card } Y = k \wedge m = \text{monom-of-set } Y))$

$Y)$
by (*rule Sum-any.expand-superset*) (*use X in auto*)
also have $\dots = (\sum m \mid \exists Y \in \text{Pow } X. \text{card } Y = k \wedge m = \text{monom-of-set } Y. (\prod v. f v \hat{\ } \text{poly-mapping.lookup } m v))$
by (*intro sum.cong*) (*auto simp: when-def*)
also have $\dots = (\sum Y \mid Y \subseteq X \wedge \text{card } Y = k. (\prod v. f v \hat{\ } \text{poly-mapping.lookup } (\text{monom-of-set } Y) v))$
by (*rule sum.reindex-bij-witness[of - monom-of-set keys]*) (*auto simp: finite-subset[OF - X]*)
also have $\dots = (\sum Y \mid Y \subseteq X \wedge \text{card } Y = k. \prod v \in Y. f v)$
proof (*intro sum.cong when-cong refl, goal-cases*)
case (1 Y)
hence *finite Y* **by** (*auto dest: finite-subset[OF - X]*)
with 1 **have** $(\prod v. f v \hat{\ } \text{poly-mapping.lookup } (\text{monom-of-set } Y) v) = (\prod v::\text{nat. if } v \in Y \text{ then } f v \text{ else } 1)$
by (*intro Prod-any.cong*) (*auto simp: lookup-monom-of-set*)
also have $\dots = (\prod v \in Y. f v)$
by (*rule Prod-any.conditionalize [symmetric]*) *fact+*
finally show ?*case* .
qed
finally show *insertion-fun f*
 $(\lambda \text{mon. if finite } X \wedge (\exists Y \subseteq X. \text{card } Y = k \wedge \text{mon} = \text{monom-of-set } Y) \text{ then } 1 \text{ else } 0) = (\sum Y \mid Y \subseteq X \wedge \text{card } Y = k. \text{prod } f Y) .$

qed

lemma *sym-mpoly-nz* [*simp*]:
assumes *finite A k ≤ card A*
shows *sym-mpoly A k ≠ (0 :: 'a :: zero-neq-one mpoly)*
proof –
from *assms* **obtain** B **where** $B \subseteq A$ *card B = k*
using *ex-subset-of-card* **by** *blast*
with *assms* **have** *coeff (sym-mpoly A k :: 'a mpoly) (monom-of-set B) = 1*
by (*intro coeff-sym-mpoly-monom-of-set*)
thus ?*thesis* **by** *auto*

qed

lemma *coeff-sym-mpoly-0-or-1*: *coeff (sym-mpoly A k) m ∈ {0, 1}*
by (*transfer, transfer*) *auto*

lemma *lead-coeff-sym-mpoly* [*simp*]:
assumes *finite A k ≤ card A*
shows *lead-coeff (sym-mpoly A k) = 1*
proof –
from *assms* **have** *lead-coeff (sym-mpoly A k) ≠ 0* **by** *simp*
thus ?*thesis* **using** *coeff-sym-mpoly-0-or-1* [*of A k lead-monom (sym-mpoly A k)*]
unfolding *lead-coeff-def* **by** *blast*
qed

lemma *lead-monom-sym-mpoly*:
assumes *sorted xs distinct xs k ≤ length xs*
shows *lead-monom (sym-mpoly (set xs) k :: 'a :: zero-neq-one mpoly) = monom-of-set (set (take k xs)) (is lead-monom ?p = -)*

proof –
let *?m = lead-monom ?p*
have *sym: symmetric-mpoly (set xs) (sym-mpoly (set xs) k)*
by (*intro symmetric-sym-mpoly auto*)
from *assms* **have** [*simp*]: *card (set xs) = length xs*
by (*subst distinct-card auto*)
from *assms* **have** *lead-coeff ?p = 1*
by (*subst lead-coeff-sym-mpoly auto*)
then obtain *X* **where** *X: X ⊆ set xs card X = k ?m = monom-of-set X*
unfolding *lead-coeff-def* **by** (*subst (asm) coeff-sym-mpoly (auto split: if-splits)*)
define *ys* **where** *ys = map (λx. if x ∈ X then 1 else 0 :: nat) xs*
have [*simp*]: *length ys = length xs* **by** (*simp add: ys-def*)

have *ys-altdef: ys = map (lookup ?m) xs*
unfolding *ys-def* **using** *X finite-subset[OF X(1)]*
by (*intro map-cong (auto simp: lookup-monom-of-set)*)

define *i* **where** *i = Min (insert (length xs) {i. i < length xs ∧ ys ! i = 0})*
have *i ≤ length xs* **by** (*auto simp: i-def*)
have *in-X: xs ! j ∈ X if j < i for j*
using *that* **unfolding** *i-def* **by** (*auto simp: ys-def*)
have *not-in-X: xs ! j ∉ X if i ≤ j j < length xs for j*
proof –
have *ne: {i. i < length xs ∧ ys ! i = 0} ≠ {}*
proof
assume [*simp*]: *{i. i < length xs ∧ ys ! i = 0} = {}*
from *that* **show** *False* **by** (*simp add: i-def*)
qed
hence *Min {i. i < length xs ∧ ys ! i = 0} ∈ {i. i < length xs ∧ ys ! i = 0}*
using *that* **by** (*intro Min-in auto*)
also have *Min {i. i < length xs ∧ ys ! i = 0} = i*
unfolding *i-def* **using** *ne* **by** (*subst Min-insert (auto simp: min-def)*)
finally have *i: ys ! i = 0 i < length xs* **by** *simp-all*

have *lookup ?m (xs ! j) ≤ lookup ?m (xs ! i)* **using** *that assms*
by (*intro lookup-lead-monom-decreasing[OF sym]*)
(auto intro!: sorted-nth-mono simp: set-conv-nth)
also have *... = 0* **using** *i* **by** (*simp add: ys-altdef*)
finally show *?thesis* **using** *that X finite-subset[OF X(1)]* **by** (*auto simp: lookup-monom-of-set*)
qed

from *X* **have** *k = card X*
by *simp*
also have *X = (λi. xs ! i) ‘ {i. i < length xs ∧ xs ! i ∈ X}*

```

    using X by (auto simp: set-conv-nth)
  also have card ... = (∑ i | i < length xs ∧ xs ! i ∈ X. 1)
    using assms by (subst card-image) (auto intro!: inj-on-nth)
  also have ... = (∑ i | i < length xs. if xs ! i ∈ X then 1 else 0)
    by (intro sum.mono-neutral-cong-left) auto
  also have ... = sum-list ys
    by (auto simp: sum-list-sum-nth ys-def intro!: sum.cong)
  also have ys = take i ys @ drop i ys by simp
  also have sum-list ... = sum-list (take i ys) + sum-list (drop i ys)
    by (subst sum-list-append) auto
  also have take i ys = replicate i 1 using ‹i ≤ length xs› in-X
    by (intro replicate-eqI) (auto simp: ys-def set-conv-nth)
  also have sum-list ... = i by simp
  also have drop i ys = replicate (length ys - i) 0 using ‹i ≤ length xs› not-in-X
    by (intro replicate-eqI) (auto simp: ys-def set-conv-nth)
  also have sum-list ... = 0 by simp
  finally have i = k by simp

  have X = set (filter (λx. x ∈ X) xs)
    using X by auto
  also have xs = take i xs @ drop i xs by simp
  also note filter-append
  also have filter (λx. x ∈ X) (take i xs) = take i xs
    using in-X by (intro filter-True) (auto simp: set-conv-nth)
  also have filter (λx. x ∈ X) (drop i xs) = []
    using not-in-X by (intro filter-False) (auto simp: set-conv-nth)
  finally have X = set (take i xs) by simp
  with ‹i = k› and X show ?thesis by simp
qed

```

2.11 Induction on the leading monomial

We show that the monomial ordering for a fixed set of variables is well-founded, so we can perform induction on the leading monomial of a polynomial.

definition *monom-less-on* **where**

$$\text{monom-less-on } A = \{(m1, m2). m1 < m2 \wedge \text{keys } m1 \subseteq A \wedge \text{keys } m2 \subseteq A\}$$

lemma *wf-monom-less-on*:

assumes *finite A*

shows *wf (monom-less-on A :: ((nat ⇒₀ 'b :: {zero, wellorder}) × -) set)*

proof (*rule wf-subset*)

define *n* **where** *n = Suc (Max (insert 0 A))*

have *less-n: k < n if k ∈ A for k*

using *that assms* **by** (*auto simp: n-def less-Suc-eq-le Max-ge-iff*)

define *f* :: *(nat ⇒₀ 'b) ⇒ 'b list* **where** *f = (λm. map (lookup m) [0..<n])*

show *wf (inv-image (lexn {(x,y). x < y} n) f)*

```

  by (intro wf-inv-image wf-learn wellorder-class.wf)
show monom-less-on A ⊆ inv-image (lexn {(x, y). x < y} n) f
proof safe
  fix m1 m2 :: nat ⇒₀ 'b assume (m1, m2) ∈ monom-less-on A
  hence m12: m1 < m2 keys m1 ⊆ A keys m2 ⊆ A
  by (auto simp: monom-less-on-def)
  then obtain k where k: lookup m1 k < lookup m2 k ∀ i < k. lookup m1 i =
lookup m2 i
  by (auto simp: less-poly-mapping-def less-fun-def)
  have ¬(lookup m1 k = 0 ∧ lookup m2 k = 0)
  proof (intro notI)
    assume lookup m1 k = 0 ∧ lookup m2 k = 0
    hence [simp]: lookup m1 k = 0 lookup m2 k = 0 by blast+
    from k(1) show False by simp
  qed
  hence k ∈ A using m12 by (auto simp: in-keys-iff)
  hence k < n by (simp add: less-n)

define as where as = map (lookup m1) [0..

```

```

  show  $P p'$ 
  by (rule 1) (insert * 1.premis keys-lead-monom-subset, auto simp: monom-less-on-def)
qed (insert 1, auto)
qed

```

```

lemma lead-monom-induct' [case-names less]:
  fixes  $p :: 'a :: zero mpoly$ 
  assumes  $IH: \bigwedge p. (\bigwedge p'. \text{vars } p' \subseteq \text{vars } p \implies \text{lead-monom } p' < \text{lead-monom } p \implies P p') \implies P p$ 
  shows  $P p$ 
proof -
  have finite (vars  $p$ ) vars  $p \subseteq \text{vars } p$  by (auto simp: vars-finite)
  thus ?thesis
    by (induction rule: lead-monom-induct) (use IH in blast)
qed

```

2.12 The fundamental theorem of symmetric polynomials

```

lemma lead-coeff-sym-mpoly-powerprod:
  assumes finite  $A \bigwedge x. x \in X \implies f x \in \{1..card A\}$ 
  shows lead-coeff  $(\prod_{x \in X}. \text{sym-mpoly } A (f (x::'a)) \wedge g x) = 1$ 
proof -
  have eq: lead-coeff  $(\text{sym-mpoly } A (f x) \wedge g x :: 'b mpoly) = 1$  if  $x \in X$  for  $x$ 
    using that assms by (subst lead-coeff-power) (auto simp: lead-coeff-sym-mpoly assms)
  hence  $(\prod_{x \in X}. \text{lead-coeff } (\text{sym-mpoly } A (f x) \wedge g x :: 'b mpoly)) = (\prod_{x \in X}. 1)$ 
    by (intro prod.cong eq refl)
  also have  $\dots = 1$  by simp
  finally have eq':  $(\prod_{x \in X}. \text{lead-coeff } (\text{sym-mpoly } A (f x) \wedge g x :: 'b mpoly)) = 1$ 
  .
  show ?thesis by (subst lead-coeff-prod) (auto simp: eq eq')
qed

```

```

context
  fixes  $A :: nat set$  and  $xs n f$  and  $decr :: 'a :: comm-ring-1 mpoly \Rightarrow bool$ 
  defines  $xs \equiv \text{sorted-list-of-set } A$ 
  defines  $n \equiv card A$ 
  defines  $f \equiv (\lambda i. \text{if } i < n \text{ then } xs ! i \text{ else } 0)$ 
  defines  $decr \equiv (\lambda p. \forall i \in A. \forall j \in A. i \leq j \longrightarrow \text{lookup } (\text{lead-monom } p) i \geq \text{lookup } (\text{lead-monom } p) j)$ 
begin

```

The computation of the witness for the fundamental theorem works like this: Given some polynomial p (that is assumed to be symmetric in the variables in A), we inspect its leading monomial, which is of the form $cX_1^{i_1} \dots X_n^{i_n}$ where the $A = \{X_1, \dots, X_n\}$, c contains only variables not in A , and the sequence i_j is decreasing. The latter holds because p is symmetric.

Now, we form the polynomial $q := ce_1^{i_1-i_2} e_2^{i_2-i_3} \dots e_n^{i_n}$, which has the same leading term as p . Then $p - q$ has a smaller leading monomial, so by induc-

tion, we can assume it to be of the required form and obtain a witness for $p - q$.

Now, we only need to add $cY_1^{i_1-i_2} \dots Y_n^{i_n}$ to that witness and we obtain a witness for p .

definition *fund-sym-step-coeff* :: 'a mpoly \Rightarrow 'a mpoly **where**
fund-sym-step-coeff p = monom (restrictpm (-A) (lead-monom p)) (lead-coeff p)

definition *fund-sym-step-monom* :: 'a mpoly \Rightarrow (nat \Rightarrow_0 nat) **where**
fund-sym-step-monom p = (
 let g = (λi . if $i < n$ then lookup (lead-monom p) (f i) else 0)
 in ($\sum_{i < n}$. Poly-Mapping.single (Suc i) (g i - g (Suc i))))

definition *fund-sym-step-poly* :: 'a mpoly \Rightarrow 'a mpoly **where**
fund-sym-step-poly p = (
 let g = (λi . if $i < n$ then lookup (lead-monom p) (f i) else 0)
 in *fund-sym-step-coeff* p * ($\prod_{i < n}$. sym-mpoly A (Suc i) \wedge (g i - g (Suc i))))

The following function computes the witness, with the convention that it returns a constant polynomial if the input was not symmetric:

function (*domintros*) *fund-sym-poly-wit* :: 'a :: comm-ring-1 mpoly \Rightarrow 'a mpoly
mpoly where
fund-sym-poly-wit p =
 (if \neg symmetric-mpoly A p \vee lead-monom p = 0 \vee vars p \cap A = {} then Const
 p else
fund-sym-poly-wit (p - *fund-sym-step-poly* p) +
 monom (*fund-sym-step-monom* p) (*fund-sym-step-coeff* p))
by *auto*

lemma *coeff-fund-sym-step-coeff*: *coeff* (*fund-sym-step-coeff* p) m \in {lead-coeff p, 0}
by (*auto simp: fund-sym-step-coeff-def coeff-monom when-def*)

lemma *vars-fund-sym-step-coeff*: vars (*fund-sym-step-coeff* p) \subseteq vars p - A
unfolding *fund-sym-step-coeff-def* **using** *keys-lead-monom-subset*[of p]
by (*intro order.trans[OF vars-monom-subset]*) *auto*

lemma *keys-fund-sym-step-monom*: keys (*fund-sym-step-monom* p) \subseteq {1..n}
unfolding *fund-sym-step-monom-def* *Let-def*
by (*intro order.trans[OF keys-sum] UN-least, subst keys-single*) *auto*

lemma *coeff-fund-sym-step-poly*:
assumes C: $\forall m$. *coeff* p m \in C **and** *ring-closed* C
shows *coeff* (*fund-sym-step-poly* p) m \in C
proof -
interpret *ring-closed* C **by** *fact*
have *: $\bigwedge m$. *coeff* (p \wedge x) m \in C **if** $\bigwedge m$. *coeff* p m \in C **for** p x
using *that* **by** (*induction* x)
 (*auto simp: coeff-mpoly-times mpoly-coeff-1 intro!: prod-fun-closed*)


```

have **:  $\bigwedge m. \text{coeff} (\text{prod } f \ X) \ m \in C \text{ if } \bigwedge i \ m. \ i \in X \implies \text{coeff} (f \ i) \ m \in C$ 
for  $X$  and  $f :: \text{nat} \Rightarrow -$ 
using that by (induction  $X$  rule: infinite-finite-induct)
      (auto simp: coeff-mpoly-times mpoly-coeff-1 intro!: prod-fun-closed)
show ?thesis using  $C$ 
unfolding fund-sym-step-poly-def Let-def fund-sym-step-coeff-def coeff-mpoly-times
by (intro prod-fun-closed)
      (auto simp: coeff-monom when-def lead-coeff-def coeff-sym-mpoly intro!: * **)
qed

```

We now show various relevant properties of the subtracted polynomial:

1. Its leading term is the same as that of the input polynomial.
2. It contains now new variables.
3. It is symmetric in the variables in A .

lemma *fund-sym-step-poly*:

```

shows  $\text{finite } A \implies p \neq 0 \implies \text{decr } p \implies \text{lead-monom} (\text{fund-sym-step-poly } p) = \text{lead-monom } p$ 
and  $\text{finite } A \implies p \neq 0 \implies \text{decr } p \implies \text{lead-coeff} (\text{fund-sym-step-poly } p) = \text{lead-coeff } p$ 
and  $\text{finite } A \implies p \neq 0 \implies \text{decr } p \implies \text{fund-sym-step-poly } p = \text{fund-sym-step-coeff } p * (\prod x. \text{sym-mpoly } A \ x \ ^{\text{lookup} (\text{fund-sym-step-monom } p) \ x})$ 
and  $\text{vars} (\text{fund-sym-step-poly } p) \subseteq \text{vars } p \cup A$ 
and  $\text{symmetric-mpoly } A (\text{fund-sym-step-poly } p)$ 

```

proof –

```

define  $g$  where  $g = (\lambda i. \text{if } i < n \text{ then } \text{lookup} (\text{lead-monom } p) (f \ i) \text{ else } 0)$ 
define  $q$  where  $q = (\prod i < n. \text{sym-mpoly } A (\text{Suc } i) \ ^{(g \ i - g (\text{Suc } i))} :: 'a \ \text{mpoly})$ 
define  $c$  where  $c = \text{monom} (\text{restrictpm } (-A) (\text{lead-monom } p)) (\text{lead-coeff } p)$ 
have [simp]:  $\text{fund-sym-step-poly } p = c * q$ 
by (simp add: fund-sym-step-poly-def fund-sym-step-coeff-def c-def q-def f-def g-def)

```

```

have  $\text{vars} (c * q) \subseteq \text{vars } p \cup A$ 
using keys-lead-monom-subset[of  $p$ ]
      vars-monom-subset[of restrictpm  $(-A) (\text{lead-monom } p) \text{ lead-coeff } p$ ]
unfolding c-def q-def
by (intro order.trans[OF vars-mult] order.trans[OF vars-prod] order.trans[OF vars-power]
      Un-least UN-least order.trans[OF vars-sym-mpoly-subset]) auto
thus  $\text{vars} (\text{fund-sym-step-poly } p) \subseteq \text{vars } p \cup A$ 
by simp
have  $\text{symmetric-mpoly } A (c * q)$  unfolding c-def q-def
by (intro symmetric-mpoly-mult symmetric-mpoly-monom symmetric-mpoly-prod symmetric-mpoly-power symmetric-sym-mpoly) auto

```

thus *symmetric-mpoly A (fund-sym-step-poly p)* **by** *simp*

assume *finite: finite A* **and** [*simp*]: $p \neq 0$ **and** *decr p*
have *set xs = A distinct xs* **and** [*simp*]: *length xs = n*
using *finite* **by** (*auto simp: xs-def n-def*)
have [*simp*]: *lead-coeff c = lead-coeff p lead-monom c = restrictpm (- A) (lead-monom p)*
by (*simp-all add: c-def lead-monom-monom*)
hence *f-range [simp]: f i ∈ A if i < n* **for** *i*
using *that ⟨set xs = A⟩* **by** (*auto simp: f-def set-conv-nth*)
have *sorted xs* **by** (*simp add: xs-def*)
hence *f-mono: f i ≤ f j if i ≤ j j < n* **for** *i j* **using** *that*
by (*auto simp: f-def n-def intro: sorted-nth-mono*)
hence *g-mono: g i ≥ g j if i ≤ j* **for** *i j*
unfolding *g-def* **using** *that using ⟨decr p⟩* **by** (*auto simp: decr-def*)

have *: $(\prod_{i < n}. \text{lead-coeff } (\text{sym-mpoly } A \text{ (Suc } i) \wedge (g \ i - g \ (\text{Suc } i)) :: 'a \ \text{mpoly}))$
 $=$
 $(\prod_{i < \text{card } A}. 1)$
using *⟨finite A⟩* **by** (*intro prod.cong*) (*auto simp: n-def lead-coeff-power*)
hence *lead-coeff q = $(\prod_{i < n}. \text{lead-coeff } (\text{sym-mpoly } A \text{ (Suc } i) \wedge (g \ i - g \ (\text{Suc } i)) :: 'a \ \text{mpoly}))$*
by (*simp add: lead-coeff-prod lead-coeff-power n-def q-def*)
also have $\dots = (\prod_{i < n}. 1)$
using *⟨finite A⟩* **by** (*intro prod.cong*) (*auto simp: lead-coeff-power n-def*)
finally have [*simp*]: *lead-coeff q = 1* **by** *simp*

have *lead-monom q = $(\sum_{i < n}. \text{lead-monom } (\text{sym-mpoly } A \text{ (Suc } i) \wedge (g \ i - g \ (\text{Suc } i)) :: 'a \ \text{mpoly}))$*
using * **by** (*simp add: q-def lead-monom-prod lead-coeff-power n-def*)
also have $\dots = (\sum_{i < n}. \text{of-nat } (g \ i - g \ (\text{Suc } i)) * \text{lead-monom } (\text{sym-mpoly } A \text{ (Suc } i) :: 'a \ \text{mpoly}))$
using *⟨finite A⟩* **by** (*intro sum.cong*) (*auto simp: lead-monom-power n-def*)
also have $\dots = (\sum_{i < n}. \text{of-nat } (g \ i - g \ (\text{Suc } i)) * \text{monom-of-set } (\text{set } (\text{take } (\text{Suc } i) \ xs)))$
proof (*intro sum.cong refl, goal-cases*)
case (1 *i*)
have *lead-monom (sym-mpoly A (Suc i) :: 'a mpoly) =*
lead-monom (sym-mpoly (set xs) (Suc i) :: 'a mpoly)
by (*simp add: ⟨set xs = A⟩*)
also from 1 **have** $\dots = \text{monom-of-set } (\text{set } (\text{take } (\text{Suc } i) \ xs))$
by (*subst lead-monom-sym-mpoly*) (*auto simp: xs-def n-def*)
finally show ?*case* **by** *simp*

qed
finally have *lead-monom-q:*
*lead-monom q = $(\sum_{i < n}. \text{of-nat } (g \ i - g \ (\text{Suc } i)) * \text{monom-of-set } (\text{set } (\text{take } (\text{Suc } i) \ xs)))$.*

have *lead-monom (c * q) = lead-monom c + lead-monom q*

```

    by (simp add: lead-monom-mult)
  also have ... = lead-monom p (is ?S = -)
  proof (intro poly-mapping-eqI)
    fix i :: nat
    show lookup (lead-monom c + lead-monom q) i = lookup (lead-monom p) i
    proof (cases i ∈ A)
      case False
      hence lookup (lead-monom c + lead-monom q) i = lookup (lead-monom p) i
+
      (∑ j < n. (g j - g (Suc j)) * lookup (monom-of-set (set (take (Suc j)
xs))) i)
      (is - = - + ?S) by (simp add: lookup-add lead-monom-q lookup-sum)
    also from False have ?S = 0
      by (intro sum.neutral) (auto simp: lookup-monom-of-set ⟨set xs = A⟩ dest!:
in-set-takeD)
    finally show ?thesis by simp
  next
  case True
  with ⟨set xs = A⟩ obtain m where m: i = xs ! m m < n
    by (auto simp: set-conv-nth)
  have lookup (lead-monom c + lead-monom q) i =
    (∑ j < n. (g j - g (Suc j)) * lookup (monom-of-set (set (take (Suc j)
xs))) i)
    using True by (simp add: lookup-add lookup-sum lead-monom-q)
  also have ... = (∑ j | j < n ∧ i ∈ set (take (Suc j) xs). g j - g (Suc j))
    by (intro sum.mono-neutral-cong-right) auto
  also have {j. j < n ∧ i ∈ set (take (Suc j) xs)} = {m..

```

```

  have *: lookup (fund-sym-step-monom p) k = (if k ∈ {1..n} then g (k - 1) - g
k else 0) for k

```

```

proof –
  have lookup (fund-sym-step-monom p) k =
    ( $\sum x \in (\text{if } k \in \{1..n\} \text{ then } \{k - 1\} \text{ else } \{\}) . g(k - 1) - g\ k$ )
  unfolding fund-sym-step-monom-def lookup-sum Let-def
  by (intro sum.mono-neutral-cong-right)
    (auto simp: g-def lookup-single when-def split: if-splits)
  thus ?thesis by simp
qed
hence ( $\prod x . \text{sym-mpoly } A\ x \hat{\text{lookup}} (\text{fund-sym-step-monom } p)\ x :: 'a\ \text{mpoly}$ ) =
  ( $\prod x \in \{1..n\} . \text{sym-mpoly } A\ x \hat{\text{lookup}} (\text{fund-sym-step-monom } p)\ x$ )
  by (intro Prod-any.expand-superset) auto
also have  $\dots = (\prod x < n . \text{sym-mpoly } A\ (\text{Suc } x) \hat{\text{lookup}} (\text{fund-sym-step-monom } p)\ (\text{Suc } x))$ 
  by (intro prod.reindex-bij-witness[of - Suc  $\lambda i . i - 1$ ]) auto
also have  $\dots = q$ 
  unfolding q-def by (intro prod.cong) (auto simp: *)
finally show fund-sym-step-poly p =
  fund-sym-step-coeff p * ( $\prod x . \text{sym-mpoly } A\ x \hat{\text{lookup}} (\text{fund-sym-step-monom } p)\ x$ )
  by (simp add: c-def q-def f-def g-def fund-sym-step-monom-def fund-sym-step-coeff-def)
qed

```

If the input is well-formed, a single step of the procedure always decreases the leading monomial.

lemma *lead-monom-fund-sym-step-poly-less*:

```

assumes finite A and lead-monom p  $\neq 0$  and decr p
shows lead-monom (p - fund-sym-step-poly p) < lead-monom p
proof (cases p = fund-sym-step-poly p)
  case True
    thus ?thesis using assms by (auto simp: order.strict-iff-order)
  next
    case False
      from assms have [simp]: p  $\neq 0$  by auto
      let ?q = fund-sym-step-poly p and ?m = lead-monom p
      have coeff (p - ?q) ?m = 0
        using fund-sym-step-poly[of p] assms by (simp add: lead-coeff-def)
      moreover have lead-coeff (p - ?q)  $\neq 0$  using False by auto
      ultimately have lead-monom (p - ?q)  $\neq$  ?m
        unfolding lead-coeff-def by auto
      moreover have lead-monom (p - ?q)  $\leq$  ?m
        using fund-sym-step-poly[of p] assms
        by (intro order.trans[OF lead-monom-diff] max.boundedI) auto
      ultimately show ?thesis by (auto simp: order.strict-iff-order)
qed

```

Finally, we prove that the witness is indeed well-defined for all inputs.

lemma *fund-sym-poly-wit-dom-aux*:

```

assumes finite B vars p  $\subseteq B$  A  $\subseteq B$ 
shows fund-sym-poly-wit-dom p

```

```

using assms(1-3)
proof (induction p rule: lead-monom-induct)
  case (less p)
  have [simp]: finite A by (rule finite-subset[of - B]) fact+
  show ?case
  proof (cases lead-monom p = 0 ∨ ¬symmetric-mpoly A p)
    case False
    hence [simp]: p ≠ 0 by auto
    note decr = lookup-lead-monom-decreasing[of A p]
    have vars (p - fund-sym-step-poly p) ⊆ B
      using fund-sym-step-poly[of p] decr False less.premis less.hyps ⟨A ⊆ B⟩
      by (intro order.trans[OF vars-diff]) auto
    hence fund-sym-poly-wit-dom (p - local.fund-sym-step-poly p)
      using False less.premis less.hyps decr
      by (intro less.IH fund-sym-step-poly symmetric-mpoly-diff
        lead-monom-fund-sym-step-poly-less) (auto simp: decr-def)
    thus ?thesis using fund-sym-poly-wit.domintros by blast
  qed (auto intro: fund-sym-poly-wit.domintros)
qed

```

```

lemma fund-sym-poly-wit-dom [intro]: fund-sym-poly-wit-dom p
proof -
  consider ¬symmetric-mpoly A p | vars p ∩ A = {} | symmetric-mpoly A p A
   $\subseteq$  vars p
  using symmetric-mpoly-imp-orthogonal-or-subset[of A p] by blast
  thus ?thesis
  proof cases
    assume symmetric-mpoly A p A ⊆ vars p
    thus ?thesis using fund-sym-poly-wit-dom-aux[of vars p p] by (auto simp:
vars-finite)
  qed (auto intro: fund-sym-poly-wit.domintros)
qed

```

```

termination fund-sym-poly-wit
by (intro allI fund-sym-poly-wit-dom)

```

Next, we prove that our witness indeed fulfils all the properties stated by the fundamental theorem:

1. If the original polynomial was in $R[X_1, \dots, X_n, \dots, X_m]$ where the X_1 to X_n are the symmetric variables, then the witness is a polynomial in $R[X_{n+1}, \dots, X_m][Y_1, \dots, Y_n]$. This means that its coefficients are polynomials in the variables of the original polynomial, minus the symmetric ones, and the (new and independent) variables of the witness polynomial range from 1 to n .
2. Substituting the i -th symmetric polynomial $e_i(X_1, \dots, X_n)$ for the Y_i variable for every i yields the original polynomial.

3. The coefficient ring R need not be the entire type; if the coefficients of the original polynomial are in some subring, then the coefficients of the coefficients of the witness also do.

lemma *fund-sym-poly-wit-coeffs-aux*:
assumes *finite B vars p* $\subseteq B$ *symmetric-mpoly A p* $A \subseteq B$
shows *vars (coeff (fund-sym-poly-wit p) m)* $\subseteq B - A$
using *assms*
proof (*induction p rule: fund-sym-poly-wit.induct*)
case (*1 p*)
show *?case*
proof (*cases lead-monom p = 0 \vee vars p \cap A = {}*)
case *False*
have *vars (p - fund-sym-step-poly p)* $\subseteq B$
using *1.premis fund-sym-step-poly[of p]* **by** (*intro order.trans[OF vars-diff]*)
auto
with *1 False* **have** *vars (coeff (fund-sym-poly-wit (p - fund-sym-step-poly p)) m)* $\subseteq B - A$
by (*intro 1 symmetric-mpoly-diff fund-sym-step-poly*) *auto*
hence *vars (coeff (fund-sym-poly-wit (p - fund-sym-step-poly p) + monom (fund-sym-step-monom p) (fund-sym-step-coeff p)) m)* $\subseteq B - A$
unfolding *coeff-add coeff-monom* **using** *vars-fund-sym-step-coeff[of p] 1.premis*
by (*intro order.trans[OF vars-add] Un-least order.trans[OF vars-monom-subset]*)
(auto simp: when-def)
thus *?thesis* **using** *1.premis False* **unfolding** *fund-sym-poly-wit.simps[of p]* **by**
simp
qed (*insert 1.premis,*
auto simp: fund-sym-poly-wit.simps[of p] mpoly-coeff-Const lead-monom-eq-0-iff)
qed

lemma *fund-sym-poly-wit-coeffs*:
assumes *symmetric-mpoly A p*
shows *vars (coeff (fund-sym-poly-wit p) m)* \subseteq *vars p - A*
proof (*cases A \subseteq vars p*)
case *True*
with *fund-sym-poly-wit-coeffs-aux[of vars p p m]* *assms*
show *?thesis* **by** (*auto simp: vars-finite*)
next
case *False*
hence *vars p \cap A = {}*
using *symmetric-mpoly-imp-orthogonal-or-subset[OF assms]* **by** *auto*
thus *?thesis* **by** (*auto simp: fund-sym-poly-wit.simps[of p] mpoly-coeff-Const*)
qed

lemma *fund-sym-poly-wit-vars*: *vars (fund-sym-poly-wit p)* \subseteq $\{1..n\}$
proof (*cases symmetric-mpoly A p \wedge A \subseteq vars p*)
case *True*
define *B* **where** *B = vars p*

```

have finite B vars p ⊆ B symmetric-mpoly A p A ⊆ B
  using True unfolding B-def by (auto simp: vars-finite)
thus ?thesis
proof (induction p rule: fund-sym-poly-wit.induct)
  case (1 p)
  show ?case
  proof (cases lead-monom p = 0 ∨ vars p ∩ A = {})
    case False
    have vars (p - fund-sym-step-poly p) ⊆ B
      using 1.prem1 fund-sym-step-poly[of p] by (intro order.trans[OF vars-diff])
    auto
    hence vars (local.fund-sym-poly-wit (p - local.fund-sym-step-poly p)) ⊆ {1..n}
      using False 1.prem1
    by (intro 1 symmetric-mpoly-diff fund-sym-step-poly) (auto simp: lead-monom-eq-0-iff)
    hence vars (fund-sym-poly-wit (p - fund-sym-step-poly p) +
      monom (fund-sym-step-monom p) (local.fund-sym-step-coeff p)) ⊆ {1..n}
    by (intro order.trans[OF vars-add] Un-least order.trans[OF vars-monom-subset]
      keys-fund-sym-step-monom) auto
    thus ?thesis using 1.prem1 False unfolding fund-sym-poly-wit.simps[of p]
  by simp
  qed (insert 1.prem1,
    auto simp: fund-sym-poly-wit.simps[of p] mpoly-coeff-Const lead-monom-eq-0-iff)
  qed
next
  case False
  then consider ¬symmetric-mpoly A p | symmetric-mpoly A p vars p ∩ A = {}
    using symmetric-mpoly-imp-orthogonal-or-subset[of A p] by auto
  thus ?thesis
    by cases (auto simp: fund-sym-poly-wit.simps[of p])
  qed

lemma fund-sym-poly-wit-insertion-aux:
  assumes finite B vars p ⊆ B symmetric-mpoly A p A ⊆ B
  shows insertion (sym-mpoly A) (fund-sym-poly-wit p) = p
  using assms
proof (induction p rule: fund-sym-poly-wit.induct)
  case (1 p)
  from 1.prem1 have decr p
    using lookup-lead-monom-decreasing[of A p] by (auto simp: decr-def)
  show ?case
  proof (cases lead-monom p = 0 ∨ vars p ∩ A = {})
    case False
    have vars (p - fund-sym-step-poly p) ⊆ B
      using 1.prem1 fund-sym-step-poly[of p] by (intro order.trans[OF vars-diff])
    auto
    hence insertion (sym-mpoly A) (fund-sym-poly-wit (p - fund-sym-step-poly p))
    =
      p - fund-sym-step-poly p using 1 False
    by (intro 1 symmetric-mpoly-diff fund-sym-step-poly) auto
  
```

```

moreover have fund-sym-step-poly p =
  fund-sym-step-coeff p * ( $\prod x. \text{sym-mpoly } A \ x \ \hat{\ } \text{lookup}$ 
(fund-sym-step-monom p) x)
using 1.prem.s finite-subset[of A B] False <decr p> by (intro fund-sym-step-poly)
auto
ultimately show ?thesis
  unfolding fund-sym-poly-wit.simps[of p] by (auto simp: insertion-add)
qed (auto simp: fund-sym-poly-wit.simps[of p])
qed

```

```

lemma fund-sym-poly-wit-insertion:
  assumes symmetric-mpoly A p
  shows insertion (sym-mpoly A) (fund-sym-poly-wit p) = p
proof (cases A  $\subseteq$  vars p)
  case False
  hence vars p  $\cap$  A = {}
  using symmetric-mpoly-imp-orthogonal-or-subset[OF assms] by auto
  thus ?thesis
  by (auto simp: fund-sym-poly-wit.simps[of p])
next
  case True
  with fund-sym-poly-wit-insertion-aux[of vars p p] assms show ?thesis
  by (auto simp: vars-finite)
qed

```

```

lemma fund-sym-poly-wit-coeff:
  assumes  $\forall m. \text{coeff } p \ m \in C$  ring-closed C
  shows  $\forall m \ m'. \text{coeff } (\text{coeff } (\text{fund-sym-poly-wit } p) \ m) \ m' \in C$ 
  using assms(1)
proof (induction p rule: fund-sym-poly-wit.induct)
  case (1 p)
  interpret ring-closed C by fact
  show ?case
  proof (cases  $\neg$ symmetric-mpoly A p  $\vee$  lead-monom p = 0  $\vee$  vars p  $\cap$  A = {})
  case True
  thus ?thesis using 1.prem.s
  by (auto simp: fund-sym-poly-wit.simps[of p] mpoly-coeff-Const)
next
  case False
  have *:  $\forall m \ m'. \text{coeff } (\text{coeff } (\text{fund-sym-poly-wit } (p - \text{fund-sym-step-poly } p)) \ m) \ m' \in C$ 
  using False 1.prem.s assms coeff-fund-sym-step-poly [of p] by (intro 1) auto
  show ?thesis
  proof (intro allI, goal-cases)
  case (1 m m')
  thus ?case using * False coeff-fund-sym-step-coeff[of p m'] 1.prem.s
  by (auto simp: fund-sym-poly-wit.simps[of p] coeff-monom lead-coeff-def
when-def)
qed

```


qed
qed

2.13 Uniqueness

Next, we show that the polynomial representation of a symmetric polynomial in terms of the elementary symmetric polynomials not only exists, but is unique.

The key property here is that products of powers of elementary symmetric polynomials uniquely determine the exponent vectors, i. e. if e_1, \dots, e_n are the elementary symmetric polynomials, $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are vectors of natural numbers, then:

$$e_1^{a_1} \dots e_n^{a_n} = e_1^{b_1} \dots e_n^{b_n} \iff a = b$$

We show this now.

lemma *lead-monom-sym-mpoly-prod*:

assumes *finite A*

shows $\text{lead-monom } (\prod i = 1..n. \text{sym-mpoly } A \ i \ \hat{\ } h \ i :: 'a \ \text{mpoly}) =$
 $(\sum i = 1..n. \text{of-nat } (h \ i) * \text{lead-monom } (\text{sym-mpoly } A \ i :: 'a \ \text{mpoly}))$

proof –

have $(\prod i=1..n. \text{lead-coeff } (\text{sym-mpoly } A \ i \ \hat{\ } h \ i :: 'a \ \text{mpoly})) = 1$

using *assms unfolding n-def by (intro prod.neutral allI) (auto simp: lead-coeff-power)*

hence $\text{lead-monom } (\prod i=1..n. \text{sym-mpoly } A \ i \ \hat{\ } h \ i :: 'a \ \text{mpoly}) =$
 $(\sum i=1..n. \text{lead-monom } (\text{sym-mpoly } A \ i \ \hat{\ } h \ i :: 'a \ \text{mpoly}))$

by *(subst lead-monom-prod) auto*

also have $\dots = (\sum i=1..n. \text{of-nat } (h \ i) * \text{lead-monom } (\text{sym-mpoly } A \ i :: 'a \ \text{mpoly}))$

by *(intro sum.cong refl, subst lead-monom-power)*

(auto simp: lead-coeff-power assms n-def)

finally show *?thesis .*

qed

lemma *lead-monom-sym-mpoly-prod-notin*:

assumes *finite A k ∉ A*

shows $\text{lookup } (\text{lead-monom } (\prod i=1..n. \text{sym-mpoly } A \ i \ \hat{\ } h \ i :: 'a \ \text{mpoly})) \ k = 0$

proof –

have *xs: set xs = A distinct xs sorted xs and [simp]: length xs = n*

using *assms by (auto simp: xs-def n-def)*

have $\text{lead-monom } (\prod i = 1..n. \text{sym-mpoly } A \ i \ \hat{\ } h \ i :: 'a \ \text{mpoly}) =$

$(\sum i = 1..n. \text{of-nat } (h \ i) * \text{lead-monom } (\text{sym-mpoly } (\text{set } xs) \ i :: 'a \ \text{mpoly}))$

by *(subst lead-monom-sym-mpoly-prod) (use xs assms in auto)*

also have $\text{lookup } \dots \ k = 0$ **unfolding** *lookup-sum*

by *(intro sum.neutral ballI, subst lead-monom-sym-mpoly)*

(insert xs assms, auto simp: xs lead-monom-sym-mpoly lookup-monom-of-set set-conv-nth)

finally show *?thesis .*

qed

lemma *lead-monom-sym-mpoly-prod-in*:
assumes *finite A k < n*
shows $\text{lookup } (\text{lead-monom } (\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^h \ h \ i \ :: \ 'a \ \text{mpoly})) \ (xs \ ! \ k) =$
 $(\sum_{i=k+1..n} \ h \ i)$
proof –
have *xs: set xs = A distinct xs sorted xs and [simp]: length xs = n*
using *assms by (auto simp: xs-def n-def)*
have $\text{lead-monom } (\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^h \ h \ i \ :: \ 'a \ \text{mpoly}) =$
 $(\sum_{i=1..n} \ \text{of-nat } (h \ i) \ * \ \text{lead-monom } (\text{sym-mpoly } (\text{set } xs) \ i \ :: \ 'a \ \text{mpoly}))$
by (*subst lead-monom-sym-mpoly-prod (use xs assms in simp-all)*)
also have $\dots = (\sum_{i=1..n} \ \text{of-nat } (h \ i) \ * \ \text{monom-of-set } (\text{set } (\text{take } i \ xs)))$
using *xs by (intro sum.cong refl, subst lead-monom-sym-mpoly) auto*
also have $\text{lookup } \dots \ (xs \ ! \ k) = (\sum_{i \mid i \in \{1..n\} \wedge xs \ ! \ k \in \text{set } (\text{take } i \ xs)} \ h \ i)$
unfolding *lookup-sum lookup-monom-of-set by (intro sum.mono-neutral-cong-right)*
auto
also have $\{i. \ i \in \{1..n\} \wedge xs \ ! \ k \in \text{set } (\text{take } i \ xs)\} = \{k+1..n\}$
proof (*intro equalityI subsetI*)
fix *i assume i: i ∈ {k+1..n}*
hence *take i xs ! k = xs ! k k < n k < i using assms*
by *auto*
with *i show i ∈ {i. i ∈ {1..n} ∧ xs ! k ∈ set (take i xs)}*
by (*force simp: set-conv-nth*)
qed (*insert assms xs, auto simp: set-conv-nth Suc-le-eq nth-eq-iff-index-eq*)
finally show *?thesis .*
qed

lemma *lead-monom-sym-poly-powerprod-inj*:
assumes $\text{lead-monom } (\prod_{i} \text{sym-mpoly } A \ i \ ^h \ \text{lookup } m1 \ i \ :: \ 'a \ \text{mpoly}) =$
 $\text{lead-monom } (\prod_{i} \text{sym-mpoly } A \ i \ ^h \ \text{lookup } m2 \ i \ :: \ 'a \ \text{mpoly})$
assumes *finite A keys m1 ⊆ {1..n} keys m2 ⊆ {1..n}*
shows $m1 = m2$
proof (*rule poly-mapping-eqI*)
fix *k :: nat*
have *xs: set xs = A distinct xs sorted xs and [simp]: length xs = n*
using *assms by (auto simp: xs-def n-def)*

from *assms(3,4) have *: i ∈ {1..n} if lookup m1 i ≠ 0 ∨ lookup m2 i ≠ 0 for*
i
using *that by (auto simp: subset-iff in-keys-iff)*
have $** : (\prod_{i} \text{sym-mpoly } A \ i \ ^h \ \text{lookup } m \ i \ :: \ 'a \ \text{mpoly}) =$
 $(\prod_{i=1..n} \ \text{sym-mpoly } A \ i \ ^h \ \text{lookup } m \ i \ :: \ 'a \ \text{mpoly})$ **if** $m \in \{m1, m2\}$
for *m*
using *that * by (intro Prod-any.expand-superset subsetI *) (auto intro!: Nat.gr0I)*
have $*** : \text{lead-monom } (\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^h \ \text{lookup } m1 \ i \ :: \ 'a \ \text{mpoly}) =$
 $\text{lead-monom } (\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^h \ \text{lookup } m2 \ i \ :: \ 'a \ \text{mpoly})$

```

using assms by (simp add: **)

have sum-eq: sum (lookup m1) {Suc k..n} = sum (lookup m2) {Suc k..n} if k
< n for k
  using arg-cong[OF ***, of λm. lookup m (xs ! k)] ⟨finite A⟩ that
  by (subst (asm) (1 2) lead-monom-sym-mpoly-prod-in) auto

show lookup m1 k = lookup m2 k
proof (cases k ∈ {1..n})
  case False
  hence lookup m1 k = 0 lookup m2 k = 0 using assms by (auto simp: in-keys-iff)
  thus ?thesis by simp
next
  case True
  thus ?thesis
proof (induction n - k arbitrary: k rule: less-induct)
  case (less l)
  have sum (lookup m1) {Suc (l - 1)..n} = sum (lookup m2) {Suc (l - 1)..n}
  using less by (intro sum-eq) auto
  also have {Suc (l - 1)..n} = insert l {Suc l..n}
  using less by auto
  also have sum (lookup m1) ... = lookup m1 l + (∑ i=Suc l..n. lookup m1 i)
  by (subst sum.insert) auto
  also have (∑ i=Suc l..n. lookup m1 i) = (∑ i=Suc l..n. lookup m2 i)
  by (intro sum.cong less) auto
  also have sum (lookup m2) (insert l {Suc l..n}) = lookup m2 l + (∑ i=Suc
l..n. lookup m2 i)
  by (subst sum.insert) auto
  finally show lookup m1 l = lookup m2 l by simp
qed
qed
qed

```

We now show uniqueness by first showing that the zero polynomial has a unique representation. We fix some polynomial p with $p(e_1, \dots, e_n) = 0$ and then show, by contradiction, that $p = 0$.

We have

$$p(e_1, \dots, e_n) = \sum c_{a_1, \dots, a_n} e_1^{a_1} \dots e_n^{a_n}$$

and due to the injectivity of products of powers of elementary symmetric polynomials, the leading term of that sum is precisely the leading term of the summand with the biggest leading monomial, since summands cannot cancel each other.

However, we also know that $p(e_1, \dots, e_n) = 0$, so it follows that all summands must have leading term 0, and it is then easy to see that they must all be identically 0.

lemma *sym-mpoly-representation-unique-aux*:

fixes $p :: 'a$ *mpoly mpoly*

```

assumes finite A insertion (sym-mpoly A) p = 0
           $\bigwedge m. \text{vars} (\text{coeff } p \ m) \cap A = \{\} \ \text{vars } p \subseteq \{1..n\}$ 
shows  $p = 0$ 
proof (rule ccontr)
assume  $p \neq 0$ 
have xs: set xs = A distinct xs sorted xs and [simp]: length xs = n
      using assms by (auto simp: xs-def n-def)
define h where h = ( $\lambda m. \text{coeff } p \ m * (\prod i. \text{sym-mpoly } A \ i \wedge \text{lookup } m \ i)$ )
define M where M = {m. coeff p m ≠ 0}
define maxm where maxm = Max ((lead-monom o h) ` M)
have finite M
      by (auto intro!: finite-subset[OF - finite-coeff-support[of p]] simp: h-def M-def)
have keys-subset: keys m ⊆ {1..n} if coeff p m ≠ 0 for m
      using that assms coeff-notin-vars[of m p] by blast

have lead-coeff: lead-coeff (h m) = lead-coeff (coeff p m) (is ?th1)
      and lead-monom: lead-monom (h m) = lead-monom (coeff p m) +
           $\text{lead-monom} (\prod i. \text{sym-mpoly } A \ i \wedge \text{lookup } m \ i :: 'a \ \text{mpoly})$  (is
?th2)
      if [simp]: coeff p m ≠ 0 for m
proof -
      have  $(\prod i. \text{sym-mpoly } A \ i \wedge \text{lookup } m \ i :: 'a \ \text{mpoly}) =$ 
           $(\prod i \mid \text{lookup } m \ i \neq 0. \text{sym-mpoly } A \ i \wedge \text{lookup } m \ i :: 'a \ \text{mpoly})$ 
      by (intro Prod-any.expand-superset (auto intro!: Nat.gr0I))
      also have lead-coeff ... = 1
      using assms keys-subset[of m]
      by (intro lead-coeff-sym-mpoly-powerprod (auto simp: in-keys-iff subset-iff
n-def))
      finally have eq: lead-coeff ( $\prod i. \text{sym-mpoly } A \ i \wedge \text{lookup } m \ i :: 'a \ \text{mpoly}$ ) = 1 .
      thus ?th1 unfolding h-def using  $\langle \text{coeff } p \ m \neq 0 \rangle$  by (subst lead-coeff-mult)
auto
      show ?th2 unfolding h-def by (subst lead-monom-mult) (auto simp: eq)
qed

have insertion (sym-mpoly A) p = ( $\sum m \in M. h \ m$ )
      unfolding insertion-altdef h-def M-def by (intro Sum-any.expand-superset)
auto
also have lead-monom ... = maxm
      unfolding maxm-def
proof (rule lead-monom-sum)
      from p obtain m where coeff p m ≠ 0
      using mpoly-eqI[of p 0] by auto
      hence  $m \in M$ 
      using  $\langle \text{coeff } p \ m \neq 0 \rangle$  lead-coeff[of m] by (auto simp: M-def)
      thus  $M \neq \{\}$  by auto
next
      have restrict-lead-monom:
           $\text{restrictpm } A \ (\text{lead-monom} (h \ m)) =$ 
           $\text{lead-monom} (\prod i. \text{sym-mpoly } A \ i \wedge \text{lookup } m \ i :: 'a \ \text{mpoly})$ 

```

```

if [simp]: coeff p m ≠ 0 for m
proof –
  have restrictpm A (lead-monom (h m)) =
    restrictpm A (lead-monom (coeff p m)) +
    restrictpm A (lead-monom (∏ i. sym-mpoly A i ^ lookup m i :: 'a mpoly))
  by (auto simp: lead-monom restrictpm-add)
  also have restrictpm A (lead-monom (coeff p m)) = 0
  using assms by (intro restrictpm-orthogonal order.trans[OF keys-lead-monom-subset])
auto
  also have restrictpm A (lead-monom (∏ i. sym-mpoly A i ^ lookup m i :: 'a
mpoly)) =
    lead-monom (∏ i. sym-mpoly A i ^ lookup m i :: 'a mpoly)
  by (intro restrictpm-id order.trans[OF keys-lead-monom-subset]
order.trans[OF vars-Prod-any] UN-least order.trans[OF vars-power]
vars-sym-mpoly-subset)
  finally show ?thesis by simp
qed
show inj-on (lead-monom ∘ h) M
proof
  fix m1 m2 assume m12: m1 ∈ M m2 ∈ M (lead-monom ∘ h) m1 =
(lead-monom ∘ h) m2
  hence [simp]: coeff p m1 ≠ 0 coeff p m2 ≠ 0 by (auto simp: M-def h-def)
  have restrictpm A (lead-monom (h m1)) = restrictpm A (lead-monom (h
m2))
  using m12 by simp
  hence lead-monom (∏ i. sym-mpoly A i ^ lookup m1 i :: 'a mpoly) =
    lead-monom (∏ i. sym-mpoly A i ^ lookup m2 i :: 'a mpoly)
  by (simp add: restrict-lead-monom)
  thus m1 = m2
  by (rule lead-monom-sym-poly-powerprod-inj)
    (use ⟨finite A⟩ keys-subset[of m1] keys-subset[of m2] in auto)
qed
next
  fix m assume m ∈ M
  hence lead-coeff (h m) = lead-coeff (coeff p m)
  by (simp add: lead-coeff M-def)
  with ⟨m ∈ M⟩ show h m ≠ 0 by (auto simp: M-def)
qed fact+
finally have maxm = 0 by (simp add: assms)

have only-zero: m = 0 if m ∈ M for m
proof –
  from that have nz [simp]: coeff p m ≠ 0 by (auto simp: M-def h-def)
  from that have (lead-monom ∘ h) m ≤ maxm
  using ⟨finite M⟩ unfolding maxm-def by (intro Max-ge imageI finite-imageI)
  with ⟨maxm = 0⟩ have [simp]: lead-monom (h m) = 0 by simp
  have lookup-nzD: k ∈ {1..n} if lookup m k ≠ 0 for k
  using keys-subset[of m] that by (auto simp: in-keys-iff subset-iff)

```

have $\text{lead-monom } (\text{coeff } p \ m) + 0 \leq \text{lead-monom } (h \ m)$
unfolding $\text{lead-monom}[OF \ nz]$ **by** $(\text{intro } \text{add-left-mono}) \ \text{auto}$
also have $\dots = 0$ **by** simp
finally have $\text{lead-monom-0}: \text{lead-monom } (\text{coeff } p \ m) = 0$ **by** simp

have $\text{sum } (\text{lookup } m) \ \{1..n\} = 0$
proof $(\text{rule } \text{ccontr})$
assume $\text{sum } (\text{lookup } m) \ \{1..n\} \neq 0$
hence $\text{sum } (\text{lookup } m) \ \{1..n\} > 0$ **by** presburger
have $0 \neq \text{lead-coeff } (MPoly-Type.\text{coeff } p \ m)$
by auto
also have $\text{lead-coeff } (MPoly-Type.\text{coeff } p \ m) = \text{lead-coeff } (h \ m)$
by $(\text{simp } \text{add}: \text{lead-coeff})$
also have $\text{lead-coeff } (h \ m) = \text{coeff } (h \ m) \ 0$
by $(\text{simp } \text{add}: \text{lead-coeff-def})$
also have $\dots = \text{coeff } (\text{coeff } p \ m) \ 0 * \text{coeff } (\prod i. \text{sym-mpoly } A \ i \ ^{\text{lookup } m} \ i) \ 0$
by $(\text{simp } \text{add}: \text{h-def } \text{mpoly-coeff-times-0})$
also have $(\prod i. \text{sym-mpoly } A \ i \ ^{\text{lookup } m} \ i) = (\prod i=1..n. \text{sym-mpoly } A \ i \ ^{\text{lookup } m} \ i)$
by $(\text{intro } \text{Prod-any.expand-superset } \text{subsetI } \text{lookup-nzD}) \ (\text{auto } \text{intro!}: \text{Nat.grOI})$
also have $\text{coeff } \dots \ 0 = (\prod i=1..n. \ 0 \ ^{\text{lookup } m} \ i)$
unfolding $\text{mpoly-coeff-prod-0 } \ \text{mpoly-coeff-power-0}$
by $(\text{intro } \text{prod.cong}) \ (\text{auto } \text{simp}: \text{coeff-sym-mpoly-0})$
also have $\dots = 0 \ ^{(\sum i=1..n. \ \text{lookup } m \ i)}$
by $(\text{simp } \text{add}: \text{power-sum})$
also have $\dots = 0$
using $\text{zero-power}[OF \ \langle \text{sum } (\text{lookup } m) \ \{1..n\} > 0 \rangle]$ **by** simp
finally show False **by** auto
qed

hence $\text{lookup } m \ k = 0$ **for** k
using $\text{keys-subset}[of \ m]$ **by** $(\text{cases } k \in \{1..n\}) \ (\text{auto } \text{simp}: \text{in-keys-iff})$
thus $m = 0$ **by** $(\text{intro } \text{poly-mapping-eqI}) \ \text{auto}$
qed

have $0 = \text{insertion } (\text{sym-mpoly } A) \ p$
using assms **by** simp
also have $\text{insertion } (\text{sym-mpoly } A) \ p = (\sum m \in M. \ h \ m)$
by fact
also have $\dots = (\sum m \in \{0\}. \ h \ m)$
using only-zero **by** $(\text{intro } \text{sum.mono-neutral-left}) \ (\text{auto } \text{simp}: \text{h-def } M\text{-def})$
also have $\dots = \text{coeff } p \ 0$
by $(\text{simp } \text{add}: \text{h-def})$
finally have $0 \notin M$ **by** $(\text{auto } \text{simp}: M\text{-def})$
with only-zero **have** $M = \{\}$ **by** auto
hence $p = 0$ **by** $(\text{intro } \text{mpoly-eqI}) \ (\text{auto } \text{simp}: M\text{-def})$
with $\langle p \neq 0 \rangle$ **show** False **by** contradiction
qed

The general uniqueness theorem now follows easily. This essentially shows

that the substitution $Y_i \mapsto e_i(X_1, \dots, X_n)$ is an isomorphism between the ring $R[Y_1, \dots, Y_n]$ and the ring $R[X_1, \dots, X_n]^{S_n}$ of symmetric polynomials.

theorem *sym-mpoly-representation-unique*:

fixes $p :: 'a \text{ mpoly mpoly}$
assumes *finite A*
 $\text{insertion } (\text{sym-mpoly } A) \ p = \text{insertion } (\text{sym-mpoly } A) \ q$
 $\bigwedge m. \text{vars } (\text{coeff } p \ m) \cap A = \{\} \wedge m. \text{vars } (\text{coeff } q \ m) \cap A = \{\}$
 $\text{vars } p \subseteq \{1..n\} \ \text{vars } q \subseteq \{1..n\}$
shows $p = q$
proof –
have $p - q = 0$
proof (*rule sym-mpoly-representation-unique-aux*)
fix m **show** $\text{vars } (\text{coeff } (p - q) \ m) \cap A = \{\}$
using *vars-diff[of coeff p m coeff q m] assms(3,4)[of m] by auto*
qed (*insert assms vars-diff[of p q], auto simp: insertion-diff*)
thus *?thesis by simp*
qed

theorem *eq-fund-sym-poly-witI*:

fixes $p :: 'a \text{ mpoly}$ **and** $q :: 'a \text{ mpoly mpoly}$
assumes *finite A symmetric-mpoly A p*
 $\text{insertion } (\text{sym-mpoly } A) \ q = p$
 $\bigwedge m. \text{vars } (\text{coeff } q \ m) \cap A = \{\}$
 $\text{vars } q \subseteq \{1..n\}$
shows $q = \text{fund-sym-poly-wit } p$
using *fund-sym-poly-wit-insertion[of p] fund-sym-poly-wit-vars[of p]*
 $\text{fund-sym-poly-wit-coeffs[of p]}$
by (*intro sym-mpoly-representation-unique*)
(insert assms, auto simp: fund-sym-poly-wit-insertion)

2.14 A recursive characterisation of symmetry

In a similar spirit to the proof of the fundamental theorem, we obtain a nice recursive and executable characterisation of symmetry.

function (*domintros*) *check-symmetric-mpoly* **where**

check-symmetric-mpoly $p \longleftrightarrow$
 $(\text{vars } p \cap A = \{\}) \vee$
 $A \subseteq \text{vars } p \wedge \text{decr } p \wedge \text{check-symmetric-mpoly } (p - \text{fund-sym-step-poly } p)$
by *auto*

lemma *check-symmetric-mpoly-dom-aux*:

assumes *finite B vars p ⊆ B A ⊆ B*
shows *check-symmetric-mpoly-dom p*
using *assms(1–3)*
proof (*induction p rule: lead-monom-induct*)
case (*less p*)
have [*simp*]: *finite A by (rule finite-subset[of - B]) fact+*

```

show ?case
proof (cases lead-monom  $p = 0 \vee \neg \text{decr } p$ )
  case False
  hence [simp]:  $p \neq 0$  by auto
  have vars ( $p - \text{fund-sym-step-poly } p$ )  $\subseteq B$ 
    using fund-sym-step-poly[of  $p$ ] False less.premis less.hyps  $\langle A \subseteq B \rangle$ 
    by (intro order.trans[OF vars-diff]) auto
  hence check-symmetric-mpoly-dom ( $p - \text{local.fund-sym-step-poly } p$ )
    using False less.premis less.hyps
    by (intro less.IH fund-sym-step-poly symmetric-mpoly-diff
      lead-monom-fund-sym-step-poly-less) (auto simp: decr-def)
  thus ?thesis using check-symmetric-mpoly.domintros by blast
qed (auto intro: check-symmetric-mpoly.domintros simp: lead-monom-eq-0-iff)
qed

```

```

lemma check-symmetric-mpoly-dom [intro]: check-symmetric-mpoly-dom  $p$ 
proof -
  show ?thesis
  proof (cases  $A \subseteq \text{vars } p$ )
    assume  $A \subseteq \text{vars } p$ 
    thus ?thesis using check-symmetric-mpoly-dom-aux[of vars  $p$ ] by (auto simp:
vars-finite)
  qed (auto intro: check-symmetric-mpoly.domintros)
qed

```

```

termination check-symmetric-mpoly
  by (intro allI check-symmetric-mpoly-dom)

```

```

lemmas [simp del] = check-symmetric-mpoly.simps

```

```

lemma check-symmetric-mpoly-correct: check-symmetric-mpoly  $p \iff \text{symmetric-mpoly } A \ p$ 
proof (induction  $p$  rule: check-symmetric-mpoly.induct)
  case (1  $p$ )
  have symmetric-mpoly  $A$  ( $p - \text{fund-sym-step-poly } p$ )  $\iff \text{symmetric-mpoly } A \ p$ 
(is ?lhs = ?rhs)
  proof
    assume ?rhs
    thus ?lhs by (intro symmetric-mpoly-diff fund-sym-step-poly)
  next
    assume ?lhs
    hence symmetric-mpoly  $A$  ( $p - \text{fund-sym-step-poly } p + \text{fund-sym-step-poly } p$ )
    by (intro symmetric-mpoly-add fund-sym-step-poly)
    thus ?rhs by simp
  qed
moreover have decr  $p$  if symmetric-mpoly  $A \ p$ 
  using lookup-lead-monom-decreasing[of  $A \ p$ ] that by (auto simp: decr-def)
ultimately show check-symmetric-mpoly  $p \iff \text{symmetric-mpoly } A \ p$ 
  using 1 symmetric-mpoly-imp-orthogonal-or-subset[of  $A \ p$ ]

```


by (auto simp: Let-def check-symmetric-mpoly.simps[of p] intro: symmetric-mpoly-orthogonal)
qed

end

2.15 Symmetric functions of roots of a univariate polynomial

Consider a factored polynomial

$$p(X) = c_n X^n + c_{n-1} X^{n-1} + \dots + c_1 X + c_0 = (X - x_1) \dots (X - x_n) .$$

where c_n is a unit.

Then any symmetric polynomial expression $q(x_1, \dots, x_n)$ in the roots x_i can be written as a polynomial expression $q'(c_0, \dots, c_{n-1})$ in the c_i .

Moreover, if the coefficients of q and the inverse of c_n all lie in some subring, the coefficients of q' do as well.

context

fixes $C :: 'b :: \text{comm-ring-1 set}$

and $A :: \text{nat set}$

and $\text{root} :: \text{nat} \Rightarrow 'a :: \text{comm-ring-1}$

and $l :: 'a \Rightarrow 'b$

and $q :: 'b \text{ mpoly}$

and $n :: \text{nat}$

defines $n \equiv \text{card } A$

assumes C : ring-closed $C \forall m. \text{coeff } q \ m \in C$

assumes l : ring-homomorphism l

assumes finite: finite A

assumes sym: symmetric-mpoly $A \ q$ and vars: vars $q \subseteq A$

begin

interpretation ring-closed C by fact

interpretation ring-homomorphism l by fact

theorem symmetric-poly-of-roots-conv-poly-of-coeffs:

assumes c : $\text{cinv} * l \ c = 1 \ \text{cinv} \in C$

assumes $p = \text{Polynomial.smult } c \ (\prod i \in A. [:-\text{root } i, 1:])$

obtains q' where vars $q' \subseteq \{0..<n\}$

and $\bigwedge m. \text{coeff } q' \ m \in C$

and insertion $(l \circ \text{poly.coeff } p) \ q' = \text{insertion } (l \circ \text{root}) \ q$

proof –

define q' where $q' = \text{fund-sym-poly-wit } A \ q$

define q'' where $q'' =$

$\text{mapm-mpoly } (\lambda m \ x. (\prod i. (\text{cinv} * l \ (-1) \ ^i) \ ^{\text{lookup } m \ i}) * \text{insertion } (\lambda .$
 $0) \ x) \ q'$

define reindex where $\text{reindex} = (\lambda i. \text{if } i \leq n \text{ then } n - i \text{ else } i)$

have bij reindex

by (intro bij-betwI [of $\text{reindex} - - \text{reindex}$]) (auto simp: reindex-def)

have vars $q' \subseteq \{1..n\}$ unfolding q' -def n -def by (intro $\text{fund-sym-poly-wit-vars}$)

hence $\text{vars } q'' \subseteq \{1..n\}$
unfolding $q''\text{-def}$ **using** $\text{vars-mapm-mpoly-subset}$ **by** auto

have $\text{insertion } (l \circ \text{root}) (\text{insertion } (\text{sym-mpoly } A) q') =$
 $\text{insertion } (\lambda n. \text{insertion } (l \circ \text{root}) (\text{sym-mpoly } A n))$
 $(\text{map-mpoly } (\text{insertion } (l \circ \text{root})) q')$
by $(\text{rule insertion-insertion})$

also have $\text{insertion } (\text{sym-mpoly } A) q' = q$
unfolding $q'\text{-def}$ **by** $(\text{intro fund-sym-poly-wit-insertion sym})$

also have $\text{insertion } (\lambda i. \text{insertion } (l \circ \text{root}) (\text{sym-mpoly } A i))$
 $(\text{map-mpoly } (\text{insertion } (l \circ \text{root})) q') =$
 $\text{insertion } (\lambda i. \text{cinv} * l ((- 1) ^ i) * l (\text{poly.coeff } p (n - i)))$
 $(\text{map-mpoly } (\text{insertion } (l \circ \text{root})) q')$

proof $(\text{intro insertion-irrelevant-vars, goal-cases})$
case $(1 i)$
hence $i \in \text{vars } q'$ **using** $\text{vars-map-mpoly-subset}$ **by** auto
also have $\dots \subseteq \{1..n\}$ **unfolding** $q'\text{-def } n\text{-def}$
by $(\text{intro fund-sym-poly-wit-vars})$
finally have $i: i \in \{1..n\}$.
have $\text{insertion } (l \circ \text{root}) (\text{sym-mpoly } A i) =$
 $l (\sum Y \mid Y \subseteq A \wedge \text{card } Y = i. \text{prod root } Y)$
using $\langle \text{finite } A \rangle$ **by** $(\text{simp add: insertion-sym-mpoly})$
also have $\dots = \text{cinv} * l (c * (\sum Y \mid Y \subseteq A \wedge \text{card } Y = i. \text{prod root } Y))$
unfolding $\text{mult mult.assoc[symmetric]} \langle \text{cinv} * l c = 1 \rangle$ **by** simp
also have $c * (\sum Y \mid Y \subseteq A \wedge \text{card } Y = i. \text{prod root } Y) = ((-1) ^ i * \text{poly.coeff } p (n - i))$
using $\text{coeff-poly-from-roots[of } A \text{ } n - i \text{ root]} i \text{ assms finite}$
by $(\text{auto simp: } n\text{-def minus-one-power-iff})$
finally show $?case$ **by** (simp add: o-def)

qed
also have $\text{map-mpoly } (\text{insertion } (l \circ \text{root})) q' = \text{map-mpoly } (\text{insertion } (\lambda-. 0))$
 q'
using $\text{fund-sym-poly-wit-coeffs[OF sym]} \text{vars}$
by $(\text{intro map-mpoly-cong insertion-irrelevant-vars}) (\text{auto simp: } q'\text{-def})$

also have $\text{insertion } (\lambda i. \text{cinv} * l ((- 1) ^ i) * l (\text{poly.coeff } p (n - i))) \dots =$
 $\text{insertion } (\lambda i. l (\text{poly.coeff } p (n - i))) q''$
unfolding $\text{insertion-substitute-linear map-mpoly-conv-mapm-mpoly } q''\text{-def}$
by $(\text{subst mapm-mpoly-comp}) \text{auto}$

also have $\dots = \text{insertion } (l \circ \text{poly.coeff } p) (\text{mpoly-map-vars reindex } q'')$
using $\langle \text{bij reindex} \rangle$ **and** $\langle \text{vars } q'' \subseteq \{1..n\} \rangle$
by $(\text{subst insertion-mpoly-map-vars})$
 $(\text{auto simp: o-def reindex-def intro!: insertion-irrelevant-vars})$

finally have $\text{insertion } (l \circ \text{root}) q =$
 $\text{insertion } (l \circ \text{poly.coeff } p) (\text{mpoly-map-vars reindex } q'')$.

moreover have $\text{coeff } (\text{mpoly-map-vars reindex } q'') m \in C$ **for** m
unfolding $q''\text{-def } q'\text{-def}$ **using** $\langle \text{bij reindex} \rangle \text{fund-sym-poly-wit-coeff[of } q \text{ } C \text{ } A]$
 $C \langle \text{cinv} \in C \rangle$
by $(\text{auto simp: coeff-mpoly-map-vars})$

intro!: *mult-closed Prod-any-closed power-closed Sum-any-closed*)
moreover have *vars* (*mpoly-map-vars* *reindex* $q'' \subseteq \{0..<n\}$)
using $\langle \text{bij } \text{reindex} \rangle$ **and** $\langle \text{vars } q'' \subseteq \{1..n\} \rangle$
by (*subst vars-mpoly-map-vars*) (*auto simp: reindex-def subset-iff*)
ultimately show *?thesis using that*[*of mpoly-map-vars reindex q''*] **by** *auto*
qed

corollary *symmetric-poly-of-roots-conv-poly-of-coeffs-monic*:

assumes $p = (\prod_{i \in A}. [-\text{root } i, 1:])$
obtains q' **where** *vars* $q' \subseteq \{0..<n\}$
and $\bigwedge m. \text{coeff } q' m \in C$
and *insertion* ($l \circ \text{poly.coeff } p$) $q' = \text{insertion } (l \circ \text{root}) q$

proof –

obtain q' **where** *vars* $q' \subseteq \{0..<n\}$
and $\bigwedge m. \text{coeff } q' m \in C$
and *insertion* ($l \circ \text{poly.coeff } p$) $q' = \text{insertion } (l \circ \text{root}) q$
by (*rule symmetric-poly-of-roots-conv-poly-of-coeffs*[*of 1 1 p*])
(use assms in auto)
thus *?thesis by* (*intro that*[*of q'*]) *auto*
qed

As a corollary, we obtain the following: Let R, S be rings with $R \subseteq S$. Consider a polynomial $p \in R[X]$ whose leading coefficient c is a unit in R and that has a full set of roots $x_1, \dots, x_n \in S$, i. e. $p(X) = c(X - x_1) \dots (X - x_n)$. Let $q \in R[X_1, \dots, X_n]$ be some symmetric polynomial expression in the roots. Then $q(x_1, \dots, x_n) \in R$.

A typical use case is $R = \mathbb{Q}$ and $S = \mathbb{C}$, i. e. any symmetric polynomial expression with rational coefficients in the roots of a rational polynomial is again rational. Similarly, any symmetric polynomial expression with integer coefficients in the roots of a monic integer polynomial is again an integer.

This is remarkable, since the roots themselves are usually not rational (possibly not even real). This particular fact is a key ingredient used in the standard proof that π is transcendental.

corollary *symmetric-poly-of-roots-in-subring*:

assumes $\text{cinv} * l c = 1$ $\text{cinv} \in C$
assumes $p = \text{Polynomial.smult } c (\prod_{i \in A}. [-\text{root } i, 1:])$
assumes $\forall i. l (\text{poly.coeff } p i) \in C$
shows *insertion* ($\lambda x. l (\text{root } x)$) $q \in C$

proof –

obtain q'
where q' : *vars* $q' \subseteq \{0..<n\}$ $\bigwedge m. \text{coeff } q' m \in C$
insertion ($l \circ \text{poly.coeff } p$) $q' = \text{insertion } (l \circ \text{root}) q$
by (*rule symmetric-poly-of-roots-conv-poly-of-coeffs*[*of cinv c p*])
(use assms in simp-all)
have *insertion* ($l \circ \text{poly.coeff } p$) $q' \in C$ **using** C *assms unfolding insertion-altdef*
by (*intro Sum-any-closed mult-closed q' Prod-any-closed power-closed*) *auto*
also have *insertion* ($l \circ \text{poly.coeff } p$) $q' = \text{insertion } (l \circ \text{root}) q$ **by** *fact*

finally show *?thesis* **by** (*simp add: o-def*)
qed

corollary *symmetric-poly-of-roots-in-subring-monic*:

assumes $p = (\prod_{i \in A}. [-\text{root } i, 1:])$
assumes $\forall i. l (\text{poly.coeff } p \ i) \in C$
shows *insertion* $(\lambda x. l (\text{root } x)) \ q \in C$

proof –

interpret *ring-closed* C **by** *fact*

interpret *ring-homomorphism* l **by** *fact*

show *?thesis*

by (*rule symmetric-poly-of-roots-in-subring[of 1 1 p]*) (*use assms in auto*)

qed

end

end

3 Executable Operations for Symmetric Polynomials

theory *Symmetric-Polynomials-Code*

imports *Symmetric-Polynomials Polynomials.MPoly-Type-Class-FMap*

begin

Lastly, we shall provide some code equations to get executable code for operations related to symmetric polynomials, including, most notably, the fundamental theorem of symmetric polynomials and the recursive symmetry check.

lemma *Ball-subset-right*:

assumes $T \subseteq S \ \forall x \in S. P \ x$

shows $(\forall x \in S. P \ x) = (\forall x \in T. P \ x)$

using *assms* **by** *auto*

lemma *compute-less-pp[code]*:

$xs < (ys :: 'a :: \text{linorder} \Rightarrow_0 'b :: \{\text{zero}, \text{linorder}\}) \longleftrightarrow$

$(\exists i \in \text{keys } xs \cup \text{keys } ys. \text{lookup } xs \ i < \text{lookup } ys \ i \wedge$

$(\forall j \in \text{keys } xs \cup \text{keys } ys. j < i \longrightarrow \text{lookup } xs \ j = \text{lookup } ys \ j))$

proof *transfer*

fix $f \ g :: 'a \Rightarrow 'b$

let $?dom = \{i. f \ i \neq 0\} \cup \{i. g \ i \neq 0\}$

have *less-fun* $f \ g \longleftrightarrow (\exists k. f \ k < g \ k \wedge (\forall k' < k. f \ k' = g \ k'))$

unfolding *less-fun-def* ..

also have ... $\longleftrightarrow (\exists i. f \ i < g \ i \wedge (i \in ?dom \wedge (\forall j \in ?dom. j < i \longrightarrow f \ j = g \ j)))$

proof (*intro iff-exI conj-cong refl*)

fix k **assume** $f \ k < g \ k$

hence $k: k \in ?dom$ **by** *auto*
have $(\forall k' < k. f k' = g k') = (\forall k' \in \{..<k\}. f k' = g k')$
by *auto*
also have $\dots \longleftrightarrow (\forall j \in (\{k. f k \neq 0\} \cup \{k. g k \neq 0\}) \cap \{..<k\}. f j = g j)$
by (*intro Ball-subset-right*) *auto*
also have $\dots \longleftrightarrow (\forall j \in (\{k. f k \neq 0\} \cup \{k. g k \neq 0\}). j < k \longrightarrow f j = g j)$
by *auto*
finally show $(\forall k' < k. f k' = g k') \longleftrightarrow k \in ?dom \wedge (\forall j \in ?dom. j < k \longrightarrow f j = g j)$
using k **by** *simp*
qed
also have $\dots \longleftrightarrow (\exists i \in ?dom. f i < g i \wedge (\forall j \in ?dom. j < i \longrightarrow f j = g j))$
by (*simp add: Bex-def conj-ac*)
finally show $less\text{-}fun\ f\ g \longleftrightarrow (\exists i \in ?dom. f i < g i \wedge (\forall j \in ?dom. j < i \longrightarrow f j = g j))$.
qed

lemma *compute-le-pp*[code]:
 $xs \leq ys \longleftrightarrow xs = ys \vee xs < (ys :: - \Rightarrow_0 -)$
by (*auto simp: order.order-iff-strict*)

lemma *vars-code* [code]:
 $vars\ (MPoly\ p) = (\bigcup m \in keys\ p. keys\ m)$
unfolding *vars-def* **by** *transfer' simp*

lemma *mpoly-coeff-code* [code]: $coeff\ (MPoly\ p) = lookup\ p$
by *transfer' simp*

lemma *sym-mpoly-code* [code]:
 $sym\text{-}mpoly\ (set\ xs)\ k = (\sum X \in Set.filter\ (\lambda X. card\ X = k)\ (Pow\ (set\ xs)). monom\ (monom\text{-}of\text{-}set\ X)\ 1)$
by (*simp add: sym-mpoly-altdef Set.filter-def*)

lemma *monom-of-set-code* [code]:
 $monom\text{-}of\text{-}set\ (set\ xs) = Pm\text{-}fmap\ (fmap\text{-}of\text{-}list\ (map\ (\lambda x. (x, 1))\ xs))$
(is *?lhs = ?rhs***)**
proof (*intro poly-mapping-eqI*)
fix k
show $lookup\ ?lhs\ k = lookup\ ?rhs\ k$
by (*induction xs*) (*auto simp: lookup-monom-of-set fmllookup-default-def*)
qed

lemma *restrictpm-code* [code]:
 $restrictpm\ A\ (Pm\text{-}fmap\ m) = Pm\text{-}fmap\ (fmrestrict\text{-}set\ A\ m)$
by (*intro poly-mapping-eqI*) (*auto simp: lookup-restrictpm fmllookup-default-def*)

lemmas [code] = *check-symmetric-mpoly-correct* [symmetric]

notepad

```

begin
  define X Y Z :: int mpoly where X = Var 1 Y = Var 2 Z = Var 3
  define e1 e2 :: int mpoly mpoly where e1 = Var 1 e2 = Var 2
  have sym-mpoly {1, 2, 3} 2 = X * Y + X * Z + Y * Z
    unfolding X-Y-Z-def by eval
  have symmetric-mpoly {1, 2} (X ^ 3 + Y ^ 3)
    unfolding X-Y-Z-def by eval
  have fund-sym-poly-wit {1, 2} (X ^ 3 + Y ^ 3) = e1 ^ 3 - 3 * e1 * e2
    unfolding X-Y-Z-def e1-e2-def by eval
end

end

```

References

- [1] B. Blum-Smith and S. Coskey. The fundamental theorem on symmetric polynomials: History's first whiff of Galois theory. 48, 01 2013.