

Symmetric Polynomials

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Abstract

A symmetric polynomial is a polynomial in variables X_1, \dots, X_n that does not discriminate between its variables, i.e. it is invariant under any permutation of them. These polynomials are important in the study of the relationship between the coefficients of a univariate polynomial and its roots in its algebraic closure.

This article provides a definition of symmetric polynomials and the elementary symmetric polynomials e_1, \dots, e_n and proofs of their basic properties, including three notable ones:

- Vieta's formula, which gives an explicit expression for the k -th coefficient of a univariate monic polynomial in terms of its roots x_1, \dots, x_n , namely $c_k = (-1)^{n-k} e_{n-k}(x_1, \dots, x_n)$.
- Second, the Fundamental Theorem of Symmetric Polynomials, which states that any symmetric polynomial is itself a uniquely determined polynomial combination of the elementary symmetric polynomials.
- Third, as a corollary of the previous two, that given a polynomial over some ring R , any symmetric polynomial combination of its roots is also in R even when the roots are not.

Both the symmetry property itself and the witness for the Fundamental Theorem are executable.

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1 Vieta's Formulas

```
theory Vieta
imports
  HOL-Library.FuncSet
  HOL-Computational-Algebra.Computational-Algebra
begin
```

1.1 Auxiliary material

lemma *card-vimage-inter*:

assumes *inj*: *inj-on* *f* *A* **and** *subset*: $X \subseteq f^{-1} A$
shows $\text{card } (f^{-1} X \cap A) = \text{card } X$

proof –

have $\text{card } (f^{-1} X \cap A) = \text{card } (f^{-1} (f^{-1} X \cap A))$
by (*subst card-image*) (*auto intro!*: *inj-on-subset*[*OF inj*])
also have $f^{-1} (f^{-1} X \cap A) = X$
using *assms* **by** *auto*
finally show *?thesis* .

qed

lemma *bij-betw-image-fixed-card-subset*:

assumes *inj-on* *f* *A*
shows *bij-betw* $(\lambda X. f^{-1} X) \{X. X \subseteq A \wedge \text{card } X = k\} \{X. X \subseteq f^{-1} A \wedge \text{card } X = k\}$
using *assms inj-on-subset*[*OF assms*]
by (*intro bij-betwI*[*of - - -* $\lambda X. f^{-1} X \cap A$]) (*auto simp: card-image card-vimage-inter*)

lemma *image-image-fixed-card-subset*:

assumes *inj-on* *f* *A*
shows $(\lambda X. f^{-1} X)^{-1} \{X. X \subseteq A \wedge \text{card } X = k\} = \{X. X \subseteq f^{-1} A \wedge \text{card } X = k\}$
using *bij-betw-imp-surj-on*[*OF bij-betw-image-fixed-card-subset*[*OF assms, of k*]]
.

lemma *prod-uminus*: $(\prod_{x \in A}. -f\ x :: 'a :: \text{comm-ring-1}) = (-1)^{\wedge \text{card } A} *$
 $(\prod_{x \in A}. f\ x)$

by (*induction A rule: infinite-finite-induct*) (*auto simp: algebra-simps*)

theorem *prod-sum-PiE*:

fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c :: \text{comm-semiring-1}$
assumes *finite*: *finite* *A* **and** *finite*: $\bigwedge x. x \in A \implies \text{finite } (B\ x)$
shows $(\prod_{x \in A}. \sum_{y \in B\ x}. f\ x\ y) = (\sum_{g \in \text{PiE } A\ B}. \prod_{x \in A}. f\ x\ (g\ x))$
using *assms*

proof (*induction A rule: finite-induct*)

case *empty*
thus *?case* **by** *auto*

next

case (*insert* *x* *A*)
have $(\sum_{g \in \text{PiE } (\text{insert } x\ A)\ B}. \prod_{x \in \text{insert } x\ A}. f\ x\ (g\ x)) =$

$(\sum_{g \in Pi_E} (insert\ x\ A)\ B.\ f\ x\ (g\ x) * (\prod_{x' \in A} f\ x'\ (g\ x')))$
using *insert by simp*
also have $(\lambda g. \prod_{x' \in A} f\ x'\ (g\ x')) = (\lambda g. \prod_{x' \in A} f\ x'\ (if\ x' = x\ then\ undefined\ else\ g\ x'))$
using *insert by (intro ext prod.cong) auto*
also have $(\sum_{g \in Pi_E} (insert\ x\ A)\ B.\ f\ x\ (g\ x) * \dots\ g) =$
 $(\sum_{(y,g) \in B \times Pi_E\ A\ B} f\ x\ y * (\prod_{x' \in A} f\ x'\ (g\ x')))$
using *insert.premis insert.hyps*
by *(intro sum.reindex-bij-witness[of - $\lambda(y,g). g(x := y)$ $\lambda g. (g\ x, g(x := unde-
fined))$])*
(auto simp: PiE-def extensional-def)
also have $\dots = (\sum_{y \in B} x.\ \sum_{g \in Pi_E\ A\ B} f\ x\ y * (\prod_{x' \in A} f\ x'\ (g\ x')))$
by *(subst sum.cartesian-product) auto*
also have $\dots = (\sum_{y \in B} x.\ f\ x\ y) * (\sum_{g \in Pi_E\ A\ B} \prod_{x' \in A} f\ x'\ (g\ x'))$
using *insert by (subst sum.swap) (simp add: sum-distrib-left sum-distrib-right)*
also have $(\sum_{g \in Pi_E\ A\ B} \prod_{x' \in A} f\ x'\ (g\ x')) = (\prod_{x \in A} \sum_{y \in B} x.\ f\ x\ y)$
using *insert.premis by (intro insert.IH [symmetric]) auto*
also have $(\sum_{y \in B} x.\ f\ x\ y) * \dots = (\prod_{x \in insert\ x\ A} \sum_{y \in B} x.\ f\ x\ y)$
using *insert.hyps by simp*
finally show *?case ..*
qed

corollary *prod-add:*

fixes *f1 f2 :: 'a \Rightarrow 'c :: comm-semiring-1*
assumes *finite: finite A*
shows $(\prod_{x \in A} f1\ x + f2\ x) = (\sum_{X \in Pow\ A} (\prod_{x \in X} f1\ x) * (\prod_{x \in A-X} f2\ x))$
proof –
have $(\prod_{x \in A} f1\ x + f2\ x) = (\sum_{g \in A \rightarrow_E\ UNIV} \prod_{x \in A} if\ g\ x\ then\ f1\ x\ else\ f2\ x)$
using *prod-sum-PiE[of A $\lambda\cdot$. UNIV :: bool set $\lambda x\ y.$ if y then f1 x else f2 x]*
assms
by *(simp-all add: UNIV-bool add-ac)*
also have $\dots = (\sum_{X \in Pow\ A} \prod_{x \in A} if\ x \in X\ then\ f1\ x\ else\ f2\ x)$
by *(intro sum.reindex-bij-witness*
 $[of - \lambda X\ x. if\ x \in A\ then\ x \in X\ else\ undefined\ \lambda P. \{x \in A. P\ x\}])$ *auto*
also have $\dots = (\sum_{X \in Pow\ A} (\prod_{x \in X} f1\ x) * (\prod_{x \in A-X} f2\ x))$
proof *(intro sum.cong refl, goal-cases)*
case *(1 X)*
let *?f = $\lambda x.$ if x \in X then f1 x else f2 x*
have $prod\ f1\ X * prod\ f2\ (A - X) = prod\ ?f\ X * prod\ ?f\ (A - X)$
by *(intro arg-cong2[of - - - (*)] prod.cong) auto*
also have $\dots = prod\ ?f\ (X \cup (A - X))$
using *1 by (subst prod.union-disjoint) (auto intro: finite-subset[OF - finite])*
also have $X \cup (A - X) = A$ **using** *1 by auto*
finally show *?case ..*
qed
finally show *?thesis .*
qed

corollary *prod-diff1*:

fixes $f1\ f2 :: 'a \Rightarrow 'c :: \text{comm-ring-1}$
assumes *finite*: $\text{finite } A$
shows $(\prod_{x \in A}. f1\ x - f2\ x) = (\sum_{X \in \text{Pow } A}. (-1) \wedge \text{card } X * (\prod_{x \in X}. f2\ x) * (\prod_{x \in A-X}. f1\ x))$
proof –
have $(\prod_{x \in A}. f1\ x - f2\ x) = (\prod_{x \in A}. -f2\ x + f1\ x)$
by *simp*
also have $\dots = (\sum_{X \in \text{Pow } A}. (\prod_{x \in X}. -f2\ x) * \text{prod } f1\ (A - X))$
by *(rule prod-add) fact+*
also have $\dots = (\sum_{X \in \text{Pow } A}. (-1) \wedge \text{card } X * (\prod_{x \in X}. f2\ x) * \text{prod } f1\ (A - X))$
by *(simp add: prod-uminus)*
finally show *?thesis* .
qed

corollary *prod-diff2*:

fixes $f1\ f2 :: 'a \Rightarrow 'c :: \text{comm-ring-1}$
assumes *finite*: $\text{finite } A$
shows $(\prod_{x \in A}. f1\ x - f2\ x) = (\sum_{X \in \text{Pow } A}. (-1) \wedge (\text{card } A - \text{card } X) * (\prod_{x \in X}. f1\ x) * (\prod_{x \in A-X}. f2\ x))$
proof –
have $(\prod_{x \in A}. f1\ x - f2\ x) = (\prod_{x \in A}. f1\ x + (-f2\ x))$
by *simp*
also have $\dots = (\sum_{X \in \text{Pow } A}. (\prod_{x \in X}. f1\ x) * (\prod_{x \in A-X}. -f2\ x))$
by *(rule prod-add) fact+*
also have $\dots = (\sum_{X \in \text{Pow } A}. (-1) \wedge \text{card } (A - X) * (\prod_{x \in X}. f1\ x) * (\prod_{x \in A-X}. f2\ x))$
by *(simp add: prod-uminus mult-ac)*
also have $\dots = (\sum_{X \in \text{Pow } A}. (-1) \wedge (\text{card } A - \text{card } X) * (\prod_{x \in X}. f1\ x) * (\prod_{x \in A-X}. f2\ x))$
using *finite-subset[OF - assms]* **by** *(intro sum.cong refl, subst card-Diff-subset) auto*
finally show *?thesis* .
qed

1.2 Main proofs

Our goal is to determine the coefficients of some fully factored polynomial $p(X) = c(X - x_1) \dots (X - x_n)$ in terms of the x_i . It is clear that it is sufficient to consider monic polynomials (i.e. $c = 1$), since the general case follows easily from this one.

We start off by expanding the product over the linear factors:

lemma *poly-from-roots*:

fixes $f :: 'a \Rightarrow 'b :: \text{comm-ring-1}$ **assumes** *fin*: $\text{finite } A$
shows $(\prod_{x \in A}. [:-f\ x, 1:]) = (\sum_{X \in \text{Pow } A}. \text{monom } ((-1) \wedge \text{card } X * (\prod_{x \in X}. f\ x)) (\text{card } (A - X)))$

proof –
have $(\prod x \in A. [-f x, 1:]) = (\prod x \in A. [:0, 1:] - [:f x:])$
by *simp*
also have $\dots = (\sum X \in \text{Pow } A. (-1) \wedge \text{card } X * (\prod x \in X. [:f x:]) * \text{monom } 1$
 $(\text{card } (A - X)))$
using *fin* **by** (*subst prod-diff1*) (*auto simp: monom-altdef mult-ac*)
also have $\dots = (\sum X \in \text{Pow } A. \text{monom } ((-1) \wedge \text{card } X * (\prod x \in X. f x)) (\text{card}$
 $(A - X)))$
proof (*intro sum.cong refl, goal-cases*)
case $(1 X)$
have $(-1 :: 'b \text{ poly}) \wedge \text{card } X = [:(-1) \wedge \text{card } X:]$
by (*induction X rule: infinite-finite-induct*) (*auto simp: one-pCons alge-*
bra-simps)
moreover have $(\prod x \in X. [:f x:]) = [:\prod x \in X. f x:]$
by (*induction X rule: infinite-finite-induct*) *auto*
ultimately show *?case* **by** (*simp add: smult-monom*)
qed
finally show *?thesis* .
qed

Comparing coefficients yields Vieta's formula:

theorem *coeff-poly-from-roots*:
fixes $f :: 'a \Rightarrow 'b :: \text{comm-ring-1}$
assumes *fin: finite A and k: k ≤ card A*
shows $\text{coeff } (\prod x \in A. [-f x, 1:]) k =$
 $(-1) \wedge (\text{card } A - k) * (\sum X \mid X \subseteq A \wedge \text{card } X = \text{card } A - k. (\prod x \in X.$
 $f x))$
proof –
have $(\prod x \in A. [-f x, 1:]) = (\sum X \in \text{Pow } A. \text{monom } ((-1) \wedge \text{card } X * (\prod x \in X. f$
 $x)) (\text{card } (A - X)))$
by (*intro poly-from-roots fin*)
also have $\text{coeff } \dots k = (\sum X \mid X \subseteq A \wedge \text{card } X = \text{card } A - k. (-1) \wedge (\text{card } A$
 $- k) * (\prod x \in X. f x))$
unfolding *coeff-sum coeff-monom* **using** *finite-subset[OF - fin] k card-mono[OF*
fin]
by (*intro sum.mono-neutral-cong-right*) (*auto simp: card-Diff-subset*)
also have $\dots = (-1) \wedge (\text{card } A - k) * (\sum X \mid X \subseteq A \wedge \text{card } X = \text{card } A - k.$
 $(\prod x \in X. f x))$
by (*simp add: sum-distrib-left*)
finally show *?thesis* .
qed

If the roots are all distinct, we can get the following alternative representa-
tion:

corollary *coeff-poly-from-roots'*:
fixes $f :: 'a \Rightarrow 'b :: \text{comm-ring-1}$
assumes *fin: finite A and inj: inj-on f A and k: k ≤ card A*
shows $\text{coeff } (\prod x \in A. [-f x, 1:]) k =$
 $(-1) \wedge (\text{card } A - k) * (\sum X \mid X \subseteq f^{-1} A \wedge \text{card } X = \text{card } A - k. \prod X)$

proof –
have $\text{coeff} (\prod_{x \in A}. [-f\ x, 1])\ k =$
 $(-1)^{\wedge}(\text{card } A - k) * (\sum X \mid X \subseteq A \wedge \text{card } X = \text{card } A - k. (\prod_{x \in X}. f\ x))$
by (*intro coeff-poly-from-roots assms*)
also have $(\sum X \mid X \subseteq A \wedge \text{card } X = \text{card } A - k. (\prod_{x \in X}. f\ x)) =$
 $(\sum X \mid X \subseteq A \wedge \text{card } X = \text{card } A - k. \prod (f'X))$
by (*intro sum.cong refl, subst prod.reindex*) (*auto intro: inj-on-subset[OF inj]*)
also have $\dots = (\sum X \in (\lambda X. f'X) \cdot \{X. X \subseteq A \wedge \text{card } X = \text{card } A - k\}. \prod X)$
by (*subst sum.reindex*) (*auto intro!: inj-on-image inj-on-subset[OF inj]*)
also have $(\lambda X. f'X) \cdot \{X. X \subseteq A \wedge \text{card } X = \text{card } A - k\} = \{X. X \subseteq f' A$
 $\wedge \text{card } X = \text{card } A - k\}$
by (*intro image-image-fixed-card-subset inj*)
finally show *?thesis* .
qed
end

2 Symmetric Polynomials

theory *Symmetric-Polynomials*
imports
Vieta
Polynomials.More-MPoly-Type
HOL-Library.Permutations
begin

2.1 Auxiliary facts

An infinite set has infinitely many infinite subsets.

lemma *infinite-infinite-subsets*:

assumes *infinite A*
shows *infinite {X. X ⊆ A ∧ infinite X}*
proof –
have $\forall k. \exists X. X \subseteq A \wedge \text{infinite } X \wedge \text{card } (A - X) = k$ **for** $k :: \text{nat}$
proof
fix $k :: \text{nat}$ **obtain** Y **where** *finite Y card Y = k Y ⊆ A*
using *infinite-arbitrarily-large[of A k] assms* **by** *auto*
moreover from this have $A - (A - Y) = Y$ **by** *auto*
ultimately show $\exists X. X \subseteq A \wedge \text{infinite } X \wedge \text{card } (A - X) = k$
using *assms* **by** (*intro exI[of - A - Y]*) *auto*
qed
from *choice[OF this]* **obtain** f
where $f: \bigwedge k. f\ k \subseteq A \wedge \text{infinite } (f\ k) \wedge \text{card } (A - f\ k) = k$ **by** *blast*
have $k = l$ **if** $f\ k = f\ l$ **for** $k\ l$
proof (*rule ccontr*)
assume $k \neq l$
hence $\text{card } (A - f\ k) \neq \text{card } (A - f\ l)$

```

    using f[of k] f[of l] by auto
    with ⟨f k = f l⟩ show False by simp
qed
hence inj f by (auto intro: injI)
moreover have range f ⊆ {X. X ⊆ A ∧ infinite X}
  using f by auto
ultimately show ?thesis
  by (subst infinite-iff-countable-subset) auto
qed

```

An infinite set contains infinitely many finite subsets of any fixed nonzero cardinality.

```

lemma infinite-card-subsets:
  assumes infinite A k > 0
  shows infinite {X. X ⊆ A ∧ finite X ∧ card X = k}
proof -
  obtain B where B: B ⊆ A finite B card B = k - 1
    using infinite-arbitrarily-large[OF assms(1), of k - 1] by blast
  define f where f = (λx. insert x B)
  have f' (A - B) ⊆ {X. X ⊆ A ∧ finite X ∧ card X = k}
    using assms B by (auto simp: f-def)
  moreover have inj-on f (A - B)
    by (auto intro!: inj-onI simp: f-def)
  hence infinite (f' (A - B))
    using assms B by (subst finite-image-iff) auto
  ultimately show ?thesis
    by (rule infinite-super)
qed

```

```

lemma comp-bij-eq-iff:
  assumes bij f
  shows g ∘ f = h ∘ f ⟷ g = h
proof
  assume *: g ∘ f = h ∘ f
  show g = h
  proof
    fix x
    obtain y where [simp]: x = f y using bij-is-surj[OF assms] by auto
    have (g ∘ f) y = (h ∘ f) y by (simp only: *)
    thus g x = h x by simp
  qed
qed auto

```

```

lemma sum-list-replicate [simp]:
  sum-list (replicate n x) = of-nat n * (x :: 'a :: semiring-1)
  by (induction n) (auto simp: algebra-simps)

```

```

lemma ex-subset-of-card:
  assumes finite A card A ≥ k

```



```

shows  $\exists B. B \subseteq A \wedge \text{card } B = k$ 
using assms
proof (induction arbitrary: k rule: finite-induct)
  case empty
  thus ?case by auto
next
  case (insert x A k)
  show ?case
  proof (cases k = 0)
    case True
    thus ?thesis by (intro exI[of - {}]) auto
  next
  case False
  from insert have  $\exists B \subseteq A. \text{card } B = k - 1$  by (intro insert.IH) auto
  then obtain B where  $B: B \subseteq A \text{ card } B = k - 1$  by auto
  with insert have [simp]:  $x \notin B$  by auto
  have  $\text{insert } x B \subseteq \text{insert } x A$ 
    using B insert by auto
  moreover have  $\text{card } (\text{insert } x B) = k$ 
    using insert B finite-subset[of B A] False by (subst card.insert-remove) auto
  ultimately show ?thesis by blast
qed
qed

```

```

lemma length-sorted-list-of-set [simp]:  $\text{length } (\text{sorted-list-of-set } A) = \text{card } A$ 
using distinct-card[of sorted-list-of-set A] by (cases finite A) simp-all

```

```

lemma upt-add-eq-append':  $i \leq j \implies j \leq k \implies [i..<k] = [i..<j] @ [j..<k]$ 
using upt-add-eq-append[of i j k - j] by simp

```

2.2 Subrings and ring homomorphisms

```

locale ring-closed =
  fixes  $A :: 'a :: \text{comm-ring-1 set}$ 
  assumes zero-closed [simp]:  $0 \in A$ 
  assumes one-closed [simp]:  $1 \in A$ 
  assumes add-closed [simp]:  $x \in A \implies y \in A \implies (x + y) \in A$ 
  assumes mult-closed [simp]:  $x \in A \implies y \in A \implies (x * y) \in A$ 
  assumes uminus-closed [simp]:  $x \in A \implies -x \in A$ 
begin

```

```

lemma minus-closed [simp]:  $x \in A \implies y \in A \implies x - y \in A$ 
using add-closed[of x -y] uminus-closed[of y] by auto

```

```

lemma sum-closed [intro]:  $(\bigwedge x. x \in X \implies f x \in A) \implies \text{sum } f X \in A$ 
by (induction X rule: infinite-finite-induct) auto

```

```

lemma power-closed [intro]:  $x \in A \implies x \wedge^n \in A$ 
by (induction n) auto

```

lemma *Sum-any-closed* [intro]: $(\bigwedge x. f x \in A) \implies \text{Sum-any } f \in A$
unfolding *Sum-any.expand-set* **by** (rule *sum-closed*)

lemma *prod-closed* [intro]: $(\bigwedge x. x \in X \implies f x \in A) \implies \text{prod } f X \in A$
by (induction *X* rule: *infinite-finite-induct*) *auto*

lemma *Prod-any-closed* [intro]: $(\bigwedge x. f x \in A) \implies \text{Prod-any } f \in A$
unfolding *Prod-any.expand-set* **by** (rule *prod-closed*)

lemma *prod-fun-closed* [intro]: $(\bigwedge x. f x \in A) \implies (\bigwedge x. g x \in A) \implies \text{prod-fun } f g$
 $x \in A$
by (*auto simp: prod-fun-def when-def intro!*: *Sum-any-closed mult-closed*)

lemma *of-nat-closed* [simp, intro]: *of-nat* $n \in A$
by (induction *n*) *auto*

lemma *of-int-closed* [simp, intro]: *of-int* $n \in A$
by (induction *n*) *auto*

end

locale *ring-homomorphism* =
fixes $f :: 'a :: \text{comm-ring-1} \Rightarrow 'b :: \text{comm-ring-1}$
assumes *add[simp]*: $f (x + y) = f x + f y$
assumes *uminus[simp]*: $f (-x) = -f x$
assumes *mult[simp]*: $f (x * y) = f x * f y$
assumes *zero[simp]*: $f 0 = 0$
assumes *one [simp]*: $f 1 = 1$
begin

lemma *diff* [simp]: $f (x - y) = f x - f y$
using *add[of x -y]* **by** (*simp del: add*)

lemma *power* [simp]: $f (x \wedge n) = f x \wedge n$
by (induction *n*) *auto*

lemma *sum* [simp]: $f (\text{sum } g A) = (\sum x \in A. f (g x))$
by (induction *A* rule: *infinite-finite-induct*) *auto*

lemma *prod* [simp]: $f (\text{prod } g A) = (\prod x \in A. f (g x))$
by (induction *A* rule: *infinite-finite-induct*) *auto*

end

lemma *ring-homomorphism-id* [intro]: *ring-homomorphism* *id*
by *standard auto*

lemma *ring-homomorphism-id'* [intro]: *ring-homomorphism* $(\lambda x. x)$

by *standard auto*

lemma *ring-homomorphism-of-int* [intro]: *ring-homomorphism of-int*
by *standard auto*

2.3 Various facts about multivariate polynomials

lemma *poly-mapping-nat-ge-0* [simp]: $(m :: \text{nat} \Rightarrow_0 \text{nat}) \geq 0$

proof (*cases* $m = 0$)

case *False*

hence *Poly-Mapping.lookup* $m \neq \text{Poly-Mapping.lookup } 0$ by *transfer auto*

hence $\exists k. \text{Poly-Mapping.lookup } m \ k \neq 0$ by (*auto simp: fun-eq-iff*)

from *LeastI-ex*[OF *this*] *Least-le*[of $\lambda k. \text{Poly-Mapping.lookup } m \ k \neq 0$] **show**
?thesis

by (*force simp: less-eq-poly-mapping-def less-fun-def*)

qed *auto*

lemma *poly-mapping-nat-le-0* [simp]: $(m :: \text{nat} \Rightarrow_0 \text{nat}) \leq 0 \iff m = 0$

unfolding *less-eq-poly-mapping-def poly-mapping-eq-iff less-fun-def* by *auto*

lemma *of-nat-diff-poly-mapping-nat*:

assumes $m \geq n$

shows $\text{of-nat } (m - n) = (\text{of-nat } m - \text{of-nat } n :: 'a :: \text{monoid-add} \Rightarrow_0 \text{nat})$

by (*auto intro!: poly-mapping-eqI simp: lookup-of-nat lookup-minus when-def*)

lemma *mpoly-coeff-transfer* [transfer-rule]:

rel-fun cr-mpoly (=) *poly-mapping.lookup MPoly-Type.coeff*

unfolding *MPoly-Type.coeff-def* by *transfer-prover*

lemma *mapping-of-sum*: $(\sum x \in A. \text{mapping-of } (f \ x)) = \text{mapping-of } (\text{sum } f \ A)$

by (*induction A rule: infinite-finite-induct*) (*auto simp: plus-mpoly.rep-eq zero-mpoly.rep-eq*)

lemma *mapping-of-eq-0-iff* [simp]: $\text{mapping-of } p = 0 \iff p = 0$

by *transfer auto*

lemma *Sum-any-mapping-of*: $\text{Sum-any } (\lambda x. \text{mapping-of } (f \ x)) = \text{mapping-of } (\text{Sum-any } f)$

by (*simp add: Sum-any.expand-set mapping-of-sum*)

lemma *Sum-any-parametric-cr-mpoly* [transfer-rule]:

(*rel-fun* (*rel-fun* (=) *cr-mpoly*) *cr-mpoly*) *Sum-any Sum-any*

by (*auto simp: rel-fun-def cr-mpoly-def Sum-any-mapping-of*)

lemma *lookup-mult-of-nat* [simp]: $\text{lookup } (\text{of-nat } n * m) \ k = n * \text{lookup } m \ k$

proof –

have $\text{of-nat } n * m = (\sum i < n. m)$ by *simp*

also have $\text{lookup } \dots \ k = (\sum i < n. \text{lookup } m \ k)$

by (*simp only: lookup-sum*)

also have $\dots = n * \text{lookup } m \ k$

by *simp*
finally show *?thesis* .
qed

lemma *mpoly-eqI*:
assumes $\bigwedge mon. MPoly-Type.coeff\ p\ mon = MPoly-Type.coeff\ q\ mon$
shows $p = q$
using *assms* **by** (*transfer*, *transfer*) (*auto simp: fun-eq-iff*)

lemma *coeff-mpoly-times*:
 $MPoly-Type.coeff\ (p * q)\ mon = prod-fun\ (MPoly-Type.coeff\ p)\ (MPoly-Type.coeff\ q)\ mon$
by (*transfer'*, *transfer'*) *auto*

lemma (**in** *ring-closed*) *coeff-mult-closed* [*intro*]:
 $(\bigwedge x. coeff\ p\ x \in A) \implies (\bigwedge x. coeff\ q\ x \in A) \implies coeff\ (p * q)\ x \in A$
by (*auto simp: coeff-mpoly-times prod-fun-closed*)

lemma *coeff-notin-vars*:
assumes $\neg(keys\ m \subseteq vars\ p)$
shows $coeff\ p\ m = 0$
using *assms* **unfolding** *vars-def* **by** *transfer'* (*auto simp: in-keys-iff*)

lemma *finite-coeff-support* [*intro*]: $finite\ \{m. coeff\ p\ m \neq 0\}$
by *transfer simp*

lemma *insertion-altdef*:
 $insertion\ f\ p = Sum-any\ (\lambda m. coeff\ p\ m * Prod-any\ (\lambda i. f\ i \wedge lookup\ m\ i))$
by (*transfer'*, *transfer'*) (*simp add: insertion-fun-def*)

lemma *mpoly-coeff-uminus* [*simp*]: $coeff\ (-p)\ m = -coeff\ p\ m$
by *transfer auto*

lemma *Sum-any-uminus*: $Sum-any\ (\lambda x. -f\ x :: 'a :: ab-group-add) = -Sum-any\ f$
by (*simp add: Sum-any.expand-set sum-negf*)

lemma *insertion-uminus* [*simp*]: $insertion\ f\ (-p :: 'a :: comm-ring-1\ mpoly) = -insertion\ f\ p$
by (*simp add: insertion-altdef Sum-any-uminus*)

lemma *Sum-any-lookup*: $finite\ \{x. g\ x \neq 0\} \implies Sum-any\ (\lambda x. lookup\ (g\ x)\ y) = lookup\ (Sum-any\ g)\ y$
by (*auto simp: Sum-any.expand-set lookup-sum intro!: sum.mono-neutral-left*)

lemma *Sum-any-diff*:
assumes $finite\ \{x. f\ x \neq 0\}$
assumes $finite\ \{x. g\ x \neq 0\}$
shows $Sum-any\ (\lambda x. f\ x - g\ x :: 'a :: ab-group-add) = Sum-any\ f - Sum-any\ g$
proof –

have $\{x. f x - g x \neq 0\} \subseteq \{x. f x \neq 0\} \cup \{x. g x \neq 0\}$ **by** *auto*
moreover have *finite* $(\{x. f x \neq 0\} \cup \{x. g x \neq 0\})$
by (*subst finite-Un*) (*insert assms, auto*)
ultimately have *finite* $\{x. f x - g x \neq 0\}$
by (*rule finite-subset*)
with *assms show ?thesis*
by (*simp add: algebra-simps Sum-any.distrib [symmetric]*)
qed

lemma *insertion-diff*:

insertion f (p - q :: 'a :: comm-ring-1 mpoly) = insertion f p - insertion f q

proof (*transfer, transfer*)

fix $f :: \text{nat} \Rightarrow 'a$ **and** $p q :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow 'a$

assume *fin*: *finite* $\{x. p x \neq 0\}$ *finite* $\{x. q x \neq 0\}$

have *insertion-fun f* $(\lambda x. p x - q x) =$

$(\sum m. p m * (\prod v. f v \wedge \text{lookup } m v) - q m * (\prod v. f v \wedge \text{lookup } m v))$

by (*simp add: insertion-fun-def algebra-simps Sum-any-diff*)

also have $\dots = (\sum m. p m * (\prod v. f v \wedge \text{lookup } m v)) - (\sum m. q m * (\prod v. f v \wedge \text{lookup } m v))$

by (*subst Sum-any-diff*) (*auto intro: finite-subset[OF - fin(1)] finite-subset[OF - fin(2)]*)

also have $\dots = \text{insertion-fun } f p - \text{insertion-fun } f q$

by (*simp add: insertion-fun-def*)

finally show *insertion-fun f* $(\lambda x. p x - q x) = \dots$

qed

lemma *insertion-power*: *insertion f (p ^ n) = insertion f p ^ n*

by (*induction n*) (*simp-all add: insertion-mult*)

lemma *insertion-sum*: *insertion f (sum g A) = ($\sum x \in A. \text{insertion } f (g x)$)*

by (*induction A rule: infinite-finite-induct*) (*auto simp: insertion-add*)

lemma *insertion-prod*: *insertion f (prod g A) = ($\prod x \in A. \text{insertion } f (g x)$)*

by (*induction A rule: infinite-finite-induct*) (*auto simp: insertion-mult*)

lemma *coeff-Var*: *coeff (Var i) m = (1 when m = Poly-Mapping.single i 1)*

by *transfer'* (*auto simp: Var₀-def lookup-single when-def*)

lemma *vars-Var*: *vars (Var i :: 'a :: {one,zero} mpoly) = (if (0::'a) = 1 then {} else {i})*

unfolding *vars-def* **by** (*auto simp: Var.rep-eq Var₀-def*)

lemma *insertion-Var* [*simp*]: *insertion f (Var i) = f i*

proof –

have *insertion f (Var i)* $= (\sum m. (1 \text{ when } m = \text{Poly-Mapping.single } i \ 1) * (\prod i. f i \wedge \text{lookup } m i))$

by (*simp add: insertion-altdef coeff-Var*)

also have $\dots = (\prod j. f j \wedge \text{lookup } (\text{Poly-Mapping.single } i \ 1) \ j)$

by (*subst Sum-any.expand-superset[of {Poly-Mapping.single i 1}]*) (*auto simp:*

when-def
also have $\dots = f i$
by (*subst Prod-any.expand-superset*[of {i}]) (*auto simp: when-def lookup-single*)
finally show *?thesis* .
qed

lemma *insertion-Sum-any*:
assumes *finite* { $x. g x \neq 0$ }
shows *insertion* f (*Sum-any* g) = *Sum-any* ($\lambda x. \text{insertion } f (g x)$)
unfolding *Sum-any.expand-set insertion-sum*
by (*intro sum.mono-neutral-right*) (*auto intro!: finite-subset*[*OF - assms*])

lemma *keys-diff-subset*:
 $keys (f - g) \subseteq keys f \cup keys g$
by *transfer auto*

lemma *keys-empty-iff* [*simp*]: $keys p = \{\}$ $\longleftrightarrow p = 0$
by *transfer auto*

lemma *mpoly-coeff-0* [*simp*]: *MPoly-Type.coeff* 0 $m = 0$
by *transfer auto*

lemma *lookup-1*: *lookup* 1 $m = (\text{if } m = 0 \text{ then } 1 \text{ else } 0)$
by *transfer (simp add: when-def)*

lemma *mpoly-coeff-1*: *MPoly-Type.coeff* 1 $m = (\text{if } m = 0 \text{ then } 1 \text{ else } 0)$
by (*simp add: MPoly-Type.coeff-def one-mpoly.rep-eq lookup-1*)

lemma *lookup-Const₀*: *lookup* (*Const₀* c) $m = (\text{if } m = 0 \text{ then } c \text{ else } 0)$
unfolding *Const₀-def* **by** (*simp add: lookup-single when-def*)

lemma *mpoly-coeff-Const*: *MPoly-Type.coeff* (*Const* c) $m = (\text{if } m = 0 \text{ then } c \text{ else } 0)$
by (*simp add: MPoly-Type.coeff-def Const.rep-eq lookup-Const₀*)

lemma *coeff-smult* [*simp*]: *coeff* (*smult* $c p$) $m = (c :: 'a :: \text{mult-zero}) * \text{coeff } p m$
by *transfer (auto simp: map-lookup)*

lemma *in-keys-mapI*: $x \in keys m \implies f (\text{lookup } m x) \neq 0 \implies x \in keys (\text{Poly-Mapping.map } f m)$
by *transfer auto*

lemma *keys-uminus* [*simp*]: $keys (-m) = keys m$
by *transfer auto*

lemma *vars-uminus* [*simp*]: $vars (-p) = vars p$
unfolding *vars-def* **by** *transfer' auto*

lemma *vars-smult*: $vars (\text{smult } c p) \subseteq vars p$

unfolding *vars-def* **by** (*transfer'*, *transfer'*) *auto*

lemma *vars-0* [*simp*]: $\text{vars } 0 = \{\}$
unfolding *vars-def* **by** *transfer'* *simp*

lemma *vars-1* [*simp*]: $\text{vars } 1 = \{\}$
unfolding *vars-def* **by** *transfer'* *simp*

lemma *vars-sum*: $\text{vars } (\text{sum } f \ A) \subseteq (\bigcup x \in A. \text{vars } (f \ x))$
using *vars-add* **by** (*induction A rule: infinite-finite-induct*) *auto*

lemma *vars-prod*: $\text{vars } (\text{prod } f \ A) \subseteq (\bigcup x \in A. \text{vars } (f \ x))$
using *vars-mult* **by** (*induction A rule: infinite-finite-induct*) *auto*

lemma *vars-Sum-any*: $\text{vars } (\text{Sum-any } h) \subseteq (\bigcup i. \text{vars } (h \ i))$
unfolding *Sum-any.expand-set* **by** (*intro order.trans[OF vars-sum]*) *auto*

lemma *vars-Prod-any*: $\text{vars } (\text{Prod-any } h) \subseteq (\bigcup i. \text{vars } (h \ i))$
unfolding *Prod-any.expand-set* **by** (*intro order.trans[OF vars-prod]*) *auto*

lemma *vars-power*: $\text{vars } (p \ ^n) \subseteq \text{vars } p$
using *vars-mult* **by** (*induction n*) *auto*

lemma *vars-diff*: $\text{vars } (p1 - p2) \subseteq \text{vars } p1 \cup \text{vars } p2$
unfolding *vars-def*
proof *transfer'*
fix *p1 p2* :: (*nat* \Rightarrow_0 *nat*) \Rightarrow_0 'a
show $\bigcup (\text{keys } ' \ \text{keys } (p1 - p2)) \subseteq \bigcup (\text{keys } ' \ (\text{keys } p1)) \cup \bigcup (\text{keys } ' \ (\text{keys } p2))$
using *keys-diff-subset[of p1 p2]* **by** (*auto simp flip: not-in-keys-iff-lookup-eq-zero*)
qed

lemma *insertion-smult* [*simp*]: $\text{insertion } f \ (\text{smult } c \ p) = c * \text{insertion } f \ p$
unfolding *insertion-altdef*
by (*subst Sum-any-right-distrib*)
(auto intro: finite-subset[OF - finite-coeff-support[of p]] simp: mult.assoc)

lemma *coeff-add* [*simp*]: $\text{coeff } (p + q) \ m = \text{coeff } p \ m + \text{coeff } q \ m$
by *transfer'* (*simp add: lookup-add*)

lemma *coeff-diff* [*simp*]: $\text{coeff } (p - q) \ m = \text{coeff } p \ m - \text{coeff } q \ m$
by *transfer'* (*simp add: lookup-minus*)

lemma *insertion-monom* [*simp*]:
 $\text{insertion } f \ (\text{monom } m \ c) = c * \text{Prod-any } (\lambda x. f \ x \ ^{\text{lookup } m \ x})$
proof –
have $\text{insertion } f \ (\text{monom } m \ c) =$
 $(\sum m'. (c \ \text{when } m = m') * (\prod v. f \ v \ ^{\text{lookup } m' \ v}))$
by (*simp add: insertion-def insertion-aux-def insertion-fun-def lookup-single*)
also have $\dots = c * (\prod v. f \ v \ ^{\text{lookup } m \ v})$

by (*subst Sum-any.expand-superset*[of { m }] (*auto simp: when-def*))
finally show *?thesis* .
qed

lemma *insertion-aux-Const₀ [simp]*: *insertion-aux f (Const₀ c) = c*

proof –

have *insertion-aux f (Const₀ c) = (∑ m. (c when m = 0) * (∏ v. f v ^ lookup m v))*

by (*simp add: Const₀-def insertion-aux-def insertion-fun-def lookup-single*)

also have $\dots = (\sum m \in \{0\}. (c \text{ when } m = 0) * (\prod v. f v ^ \text{lookup } m v))$

by (*intro Sum-any.expand-superset*) (*auto simp: when-def*)

also have $\dots = c$ **by** *simp*

finally show *?thesis* .

qed

lemma *insertion-Const [simp]*: *insertion f (Const c) = c*

by (*simp add: insertion-def Const.rep-eq*)

lemma *coeffs-0 [simp]*: *coeffs 0 = {}*

by *transfer auto*

lemma *coeffs-1 [simp]*: *coeffs 1 = {1}*

by *transfer auto*

lemma *coeffs-Const*: *coeffs (Const c) = (if c = 0 then {} else {c})*

unfolding *Const-def Const₀-def* **by** *transfer' auto*

lemma *coeffs-subset*: *coeffs (Const c) ⊆ {c}*

by (*auto simp: coeffs-Const*)

lemma *keys-Const₀*: *keys (Const₀ c) = (if c = 0 then {} else {0})*

unfolding *Const₀-def* **by** *transfer' auto*

lemma *vars-Const [simp]*: *vars (Const c) = {}*

unfolding *vars-def* **by** *transfer' (auto simp: keys-Const₀)*

lemma *prod-fun-compose-bij*:

assumes *bij f and f*: $\bigwedge x y. f (x + y) = f x + f y$

shows *prod-fun m1 m2* $(f x) = \text{prod-fun } (m1 \circ f) (m2 \circ f) x$

proof –

have [*simp*]: $f x = f y \iff x = y$ **for** $x y$

using $\langle \text{bij } f \rangle$ **by** (*auto dest!: bij-is-inj inj-onD*)

have *prod-fun (m1 ∘ f) (m2 ∘ f) x =*

*Sum-any ((λ l. m1 l * Sum-any ((λ q. m2 q when f x = l + q) ∘ f)) ∘ f)*

by (*simp add: prod-fun-def f(1) [symmetric] o-def*)

also have $\dots = \text{Sum-any } ((\lambda l. m1 l * \text{Sum-any } ((\lambda q. m2 q \text{ when } f x = l + q))))$

by (*simp only: Sum-any.reindex-cong[OF assms(1) refl, symmetric]*)

also have $\dots = \text{prod-fun } m1 m2 (f x)$

by (*simp add: prod-fun-def*)

finally show *?thesis ..*
qed

lemma *add-nat-poly-mapping-zero-iff [simp]:*
 $(a + b :: 'a \Rightarrow_0 \text{nat}) = 0 \longleftrightarrow a = 0 \wedge b = 0$
by *transfer (auto simp: fun-eq-iff)*

lemma *prod-fun-nat-0:*
fixes $f g :: ('a \Rightarrow_0 \text{nat}) \Rightarrow 'b::\text{semiring-0}$
shows $\text{prod-fun } f g 0 = f 0 * g 0$

proof –

have $\text{prod-fun } f g 0 = (\sum l. f l * (\sum q. g q \text{ when } 0 = l + q))$

unfolding *prod-fun-def ..*

also have $(\lambda l. \sum q. g q \text{ when } 0 = l + q) = (\lambda l. \sum q \in \{0\}. g q \text{ when } 0 = l + q)$

by *(intro ext Sum-any.expand-superset) (auto simp: when-def)*

also have $(\sum l. f l * \dots l) = (\sum l \in \{0\}. f l * \dots l)$

by *(intro ext Sum-any.expand-superset) (auto simp: when-def)*

finally show *?thesis by simp*

qed

lemma *mpoly-coeff-times-0: coeff (p * q) 0 = coeff p 0 * coeff q 0*
by *(simp add: coeff-mpoly-times prod-fun-nat-0)*

lemma *mpoly-coeff-prod-0: coeff ($\prod x \in A. f x$) 0 = ($\prod x \in A. \text{coeff } (f x) 0$)*
by *(induction A rule: infinite-finite-induct) (auto simp: mpoly-coeff-times-0 mpoly-coeff-1)*

lemma *mpoly-coeff-power-0: coeff (p ^ n) 0 = coeff p 0 ^ n*
by *(induction n) (auto simp: mpoly-coeff-times-0 mpoly-coeff-1)*

lemma *prod-fun-max:*

fixes $f g :: 'a::\{\text{linorder, ordered-cancel-comm-monoid-add}\} \Rightarrow 'b::\text{semiring-0}$

assumes *zero:* $\bigwedge m. m > a \implies f m = 0 \wedge m > b \implies g m = 0$

assumes *fin:* $\text{finite } \{m. f m \neq 0\} \text{ finite } \{m. g m \neq 0\}$

shows $\text{prod-fun } f g (a + b) = f a * g b$

proof –

note $\text{fin}' = \text{finite-subset}[OF - \text{fin}(1)] \text{ finite-subset}[OF - \text{fin}(2)]$

have $\text{prod-fun } f g (a + b) = (\sum l. f l * (\sum q. g q \text{ when } a + b = l + q))$

by *(simp add: prod-fun-def Sum-any-right-distrib)*

also have $\dots = (\sum l. \sum q. f l * g q \text{ when } a + b = l + q)$

by *(subst Sum-any-right-distrib) (auto intro!: Sum-any.cong fin'(2) simp:*

when-def)

also {

fix $l q$ **assume** $lq: a + b = l + q$ $(a, b) \neq (l, q)$ **and** $nz: f l * g q \neq 0$

from nz **and** *zero* **have** $l \leq a & q \leq b$ **by** *(auto intro: leI)*

moreover from *this* **and** $lq(2)$ **have** $l < a \vee q < b$ **by** *auto*

ultimately have $l + q < a + b$

by *(auto intro: add-less-le-mono add-le-less-mono)*

with $lq(1)$ **have** *False* **by** *simp*

}

hence $(\sum l. \sum q. f l * g q \text{ when } a + b = l + q) = (\sum l. \sum q. f l * g q \text{ when } (a, b) = (l, q))$
by *(intro Sum-any.cong refl) (auto simp: when-def)*
also have $\dots = (\sum (l,q). f l * g q \text{ when } (a, b) = (l, q))$
by *(intro Sum-any.cartesian-product[of {(a, b)}]) auto*
also have $\dots = (\sum (l,q) \in \{(a,b)\}. f l * g q \text{ when } (a, b) = (l, q))$
by *(intro Sum-any.expand-superset) auto*
also have $\dots = f a * g b$ **by** *simp*
finally show *?thesis .*
qed

lemma *prod-fun-gt-max-eq-zero:*

fixes $f g :: 'a::\{\text{linorder, ordered-cancel-comm-monoid-add}\} \Rightarrow 'b::\text{semiring-0}$
assumes $m > a + b$
assumes $\text{zero: } \bigwedge m. m > a \implies f m = 0 \wedge m. m > b \implies g m = 0$
assumes $\text{fin: } \text{finite } \{m. f m \neq 0\} \text{ finite } \{m. g m \neq 0\}$
shows $\text{prod-fun } f g m = 0$

proof –

note $\text{fin}' = \text{finite-subset}[OF - \text{fin}(1)] \text{ finite-subset}[OF - \text{fin}(2)]$
have $\text{prod-fun } f g m = (\sum l. f l * (\sum q. g q \text{ when } m = l + q))$
by *(simp add: prod-fun-def Sum-any-right-distrib)*
also have $\dots = (\sum l. \sum q. f l * g q \text{ when } m = l + q)$
by *(subst Sum-any-right-distrib) (auto intro!: Sum-any.cong fin'(2) simp: when-def)*
also {
fix $l q$ **assume** $lq: m = l + q$ **and** $nz: f l * g q \neq 0$
from nz **and** zero **have** $l \leq a \wedge q \leq b$ **by** *(auto intro: leI)*
hence $l + q \leq a + b$ **by** *(intro add-mono)*
also have $\dots < m$ **by** *fact*
finally have $l + q < m$.
}
hence $(\sum l. \sum q. f l * g q \text{ when } m = l + q) = (\sum l. \sum q. f l * g q \text{ when } \text{False})$
by *(intro Sum-any.cong refl) (auto simp: when-def)*
also have $\dots = 0$ **by** *simp*
finally show *?thesis .*
qed

2.4 Restricting a monomial to a subset of variables

lift-definition *restrictpm* :: $'a \text{ set} \Rightarrow ('a \Rightarrow_0 'b :: \text{zero}) \Rightarrow ('a \Rightarrow_0 'b)$ **is**

$\lambda A f x. \text{if } x \in A \text{ then } f x \text{ else } 0$
by *(erule finite-subset[rotated]) auto*

lemma *lookup-restrictpm:* $\text{lookup } (\text{restrictpm } A m) x = (\text{if } x \in A \text{ then } \text{lookup } m x \text{ else } 0)$

by *transfer auto*

lemma *lookup-restrictpm-in* [*simp*]: $x \in A \implies \text{lookup } (\text{restrictpm } A m) x = \text{lookup } m x$

and *lookup-restrict-pm-not-in* [*simp*]: $x \notin A \implies \text{lookup } (\text{restrictpm } A \ m) \ x = 0$
by (*simp-all add: lookup-restrictpm*)

lemma *keys-restrictpm* [*simp*]: $\text{keys } (\text{restrictpm } A \ m) = \text{keys } m \cap A$
by *transfer auto*

lemma *restrictpm-add*: $\text{restrictpm } X \ (m1 + m2) = \text{restrictpm } X \ m1 + \text{restrictpm } X \ m2$
by *transfer auto*

lemma *restrictpm-id* [*simp*]: $\text{keys } m \subseteq X \implies \text{restrictpm } X \ m = m$
by *transfer (auto simp: fun-eq-iff)*

lemma *restrictpm-orthogonal* [*simp*]: $\text{keys } m \subseteq -X \implies \text{restrictpm } X \ m = 0$
by *transfer (auto simp: fun-eq-iff)*

lemma *restrictpm-add-disjoint*:
 $X \cap Y = \{\}$ $\implies \text{restrictpm } X \ m + \text{restrictpm } Y \ m = \text{restrictpm } (X \cup Y) \ m$
by *transfer (auto simp: fun-eq-iff)*

lemma *restrictpm-add-complements*:
 $\text{restrictpm } X \ m + \text{restrictpm } (-X) \ m = m \ \text{restrictpm } (-X) \ m + \text{restrictpm } X \ m$
 $m = m$
by (*subst restrictpm-add-disjoint; force*)+

2.5 Mapping over a polynomial

lift-definition *map-mpoly* :: $('a :: \text{zero} \Rightarrow 'b :: \text{zero}) \Rightarrow 'a \ \text{mpoly} \Rightarrow 'b \ \text{mpoly}$ **is**
 $\lambda(f :: 'a \Rightarrow 'b) \ (p :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a). \ \text{Poly-Mapping.map } f \ p .$

lift-definition *mapm-mpoly* :: $((\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow 'a :: \text{zero} \Rightarrow 'b :: \text{zero}) \Rightarrow 'a \ \text{mpoly} \Rightarrow 'b \ \text{mpoly}$ **is**
 $\lambda(f :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow 'a \Rightarrow 'b) \ (p :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a). \ \text{Poly-Mapping.mapp } f \ p .$

lemma *poly-mapping-map-conv-mapp*: $\text{Poly-Mapping.map } f = \text{Poly-Mapping.mapp } (\lambda-. \ f)$
by (*auto simp: Poly-Mapping.mapp-def Poly-Mapping.map-def map-fun-def o-def fun-eq-iff when-def in-keys-iff cong: if-cong*)

lemma *map-mpoly-conv-mapm-mpoly*: $\text{map-mpoly } f = \text{mapm-mpoly } (\lambda-. \ f)$
by *transfer' (auto simp: poly-mapping-map-conv-mapp)*

lemma *map-mpoly-comp*: $f \ 0 = 0 \implies \text{map-mpoly } f \ (\text{map-mpoly } g \ p) = \text{map-mpoly } (f \circ g) \ p$
by (*transfer', transfer'*) (*auto simp: when-def fun-eq-iff*)

lemma *mapp-mapp*:
 $(\bigwedge x. \ f \ x \ 0 = 0) \implies \text{Poly-Mapping.mapp } f \ (\text{Poly-Mapping.mapp } g \ m) =$

Poly-Mapping.mapp ($\lambda x y. f x (g x y)$) *m*
by *transfer'* (*auto simp: fun-eq-iff lookup-mapp in-keys-iff*)

lemma *mapm-mpoly-comp*:

$(\bigwedge x. f x 0 = 0) \implies \text{mapm-mpoly } f (\text{mapm-mpoly } g p) = \text{mapm-mpoly } (\lambda m c. f m (g m c)) p$
by *transfer'* (*simp add: mapp-mapp*)

lemma *coeff-map-mpoly*:

$\text{coeff } (\text{map-mpoly } f p) m = (\text{if } \text{coeff } p m = 0 \text{ then } 0 \text{ else } f (\text{coeff } p m))$
by (*transfer, transfer'*) *auto*

lemma *coeff-map-mpoly' [simp]*: $f 0 = 0 \implies \text{coeff } (\text{map-mpoly } f p) m = f (\text{coeff } p m)$

by (*subst coeff-map-mpoly*) *auto*

lemma *coeff-mapm-mpoly*: $\text{coeff } (\text{mapm-mpoly } f p) m = (\text{if } \text{coeff } p m = 0 \text{ then } 0 \text{ else } f m (\text{coeff } p m))$

by (*transfer, transfer'*) (*auto simp: in-keys-iff*)

lemma *coeff-mapm-mpoly' [simp]*: $(\bigwedge m. f m 0 = 0) \implies \text{coeff } (\text{mapm-mpoly } f p) m = f m (\text{coeff } p m)$

by (*subst coeff-mapm-mpoly*) *auto*

lemma *vars-map-mpoly-subset*: $\text{vars } (\text{map-mpoly } f p) \subseteq \text{vars } p$

unfolding *vars-def* **by** (*transfer', transfer'*) (*auto simp: map-mpoly.rep-eq*)

lemma *coeff-sum [simp]*: $\text{coeff } (\text{sum } f A) m = (\sum x \in A. \text{coeff } (f x) m)$

by (*induction A rule: infinite-finite-induct*) *auto*

lemma *coeff-Sum-any*: $\text{finite } \{x. f x \neq 0\} \implies \text{coeff } (\text{Sum-any } f) m = \text{Sum-any } (\lambda x. \text{coeff } (f x) m)$

by (*auto simp add: Sum-any.expand-set intro!: sum.mono-neutral-right*)

lemma *Sum-any-zeroI*: $(\bigwedge x. f x = 0) \implies \text{Sum-any } f = 0$

by (*auto simp: Sum-any.expand-set*)

lemma *insertion-Prod-any*:

$\text{finite } \{x. g x \neq 1\} \implies \text{insertion } f (\text{Prod-any } g) = \text{Prod-any } (\lambda x. \text{insertion } f (g x))$

by (*auto simp: Prod-any.expand-set insertion-prod intro!: prod.mono-neutral-right*)

lemma *insertion-insertion*:

$\text{insertion } g (\text{insertion } k p) =$

$\text{insertion } (\lambda x. \text{insertion } g (k x)) (\text{map-mpoly } (\text{insertion } g) p)$ (**is** *?lhs = ?rhs*)

proof –

have $\text{insertion } g (\text{insertion } k p) =$

$(\sum x. \text{insertion } g (\text{coeff } p x) * \text{insertion } g (\prod i. k i \wedge \text{lookup } x i))$

unfolding *insertion-altdef[of k p]*

```

    by (subst insertion-Sum-any)
      (auto intro: finite-subset[OF - finite-coeff-support[of p]] simp: insertion-mult)
  also have ... = (∑ x. insertion g (coeff p x) * (∏ i. insertion g (k i) ^ lookup
x i))
  proof (intro Sum-any.cong)
    fix x show insertion g (coeff p x) * insertion g (∏ i. k i ^ lookup x i) =
      insertion g (coeff p x) * (∏ i. insertion g (k i) ^ lookup x i)
    by (subst insertion-Prod-any)
      (auto simp: insertion-power intro!: finite-subset[OF - finite-lookup[of x]]
Nat.grOI)
  qed
  also have ... = insertion (λx. insertion g (k x)) (map-mpoly (insertion g) p)
    unfolding insertion-altdef[of - map-mpoly f p for f] by auto
  finally show ?thesis .
qed

```

```

lemma insertion-substitute-linear:
  insertion (λi. c i * f i) p =
    insertion f (mapm-mpoly (λm d. Prod-any (λi. c i ^ lookup m i) * d) p)
  unfolding insertion-altdef
proof (intro Sum-any.cong, goal-cases)
  case (1 m)
  have coeff (mapm-mpoly (λm. (*) (∏ i. c i ^ lookup m i)) p) m * (∏ i. f i ^
lookup m i) =
    MPoly-Type.coeff p m * ((∏ i. c i ^ lookup m i) * (∏ i. f i ^ lookup m i))
  by (simp add: mult-ac)
  also have (∏ i. c i ^ lookup m i) * (∏ i. f i ^ lookup m i) =
    (∏ i. (c i * f i) ^ lookup m i)
  by (subst Prod-any.distrib [symmetric])
    (auto simp: power-mult-distrib intro!: finite-subset[OF - finite-lookup[of m]]
Nat.grOI)
  finally show ?case by simp
qed

```

```

lemma vars-mapm-mpoly-subset: vars (mapm-mpoly f p) ⊆ vars p
  unfolding vars-def using keys-mapp-subset[of f] by (auto simp: mapm-mpoly.rep-eq)

```

```

lemma map-mpoly-cong:
  assumes ∧m. f (coeff p m) = g (coeff p m) p = q
  shows map-mpoly f p = map-mpoly g q
  using assms by (intro mpoly-eqI) (auto simp: coeff-map-mpoly)

```

2.6 The leading monomial and leading coefficient

The leading monomial of a multivariate polynomial is the one with the largest monomial w.r.t. the monomial ordering induced by the standard variable ordering. The leading coefficient is the coefficient of the leading monomial.

As a convention, the leading monomial of the zero polynomial is defined to be the same as that of any non-constant zero polynomial, i. e. the monomial $X_1^0 \dots X_n^0$.

lift-definition *lead-monom* :: 'a :: zero mpoly \Rightarrow (nat \Rightarrow_0 nat) is
 $\lambda f :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a. \text{Max} (\text{insert } 0 (\text{keys } f))$.

lemma *lead-monom-geI* [*intro*]:
assumes *coeff p m $\neq 0$*
shows $m \leq \text{lead-monom } p$
using *assms* **by** (*auto simp: lead-monom-def coeff-def in-keys-iff*)

lemma *coeff-gt-lead-monom-zero* [*simp*]:
assumes $m > \text{lead-monom } p$
shows $\text{coeff } p \ m = 0$
using *lead-monom-geI*[*of p m*] *assms* **by force**

lemma *lead-monom-nonzero-eq*:
assumes $p \neq 0$
shows $\text{lead-monom } p = \text{Max} (\text{keys} (\text{mapping-of } p))$
using *assms* **by transfer** (*simp add: max-def*)

lemma *lead-monom-0* [*simp*]: $\text{lead-monom } 0 = 0$
by (*simp add: lead-monom-def zero-mpoly.rep-eq*)

lemma *lead-monom-1* [*simp*]: $\text{lead-monom } 1 = 0$
by (*simp add: lead-monom-def one-mpoly.rep-eq*)

lemma *lead-monom-Const* [*simp*]: $\text{lead-monom} (\text{Const } c) = 0$
by (*simp add: lead-monom-def Const.rep-eq Const₀-def*)

lemma *lead-monom-uminus* [*simp*]: $\text{lead-monom} (-p) = \text{lead-monom } p$
by (*simp add: lead-monom-def uminus-mpoly.rep-eq*)

lemma *keys-mult-const* [*simp*]:
fixes $c :: 'a :: \{\text{semiring-0}, \text{semiring-no-zero-divisors}\}$
assumes $c \neq 0$
shows $\text{keys} (\text{Poly-Mapping.map } ((*) \ c) \ p) = \text{keys } p$
using *assms* **by transfer auto**

lemma *lead-monom-eq-0-iff*: $\text{lead-monom } p = 0 \iff \text{vars } p = \{\}$
unfolding *vars-def* **by transfer'** (*auto simp: Max-eq-iff*)

lemma *lead-monom-monom*: $\text{lead-monom} (\text{monom } m \ c) = (\text{if } c = 0 \text{ then } 0 \text{ else } m)$
by (*auto simp add: lead-monom-def monom.rep-eq Const₀-def max-def*)

lemma *lead-monom-monom'* [*simp*]: $c \neq 0 \implies \text{lead-monom} (\text{monom } m \ c) = m$
by (*simp add: lead-monom-monom*)

lemma *lead-monom-numeral* [*simp*]: $\text{lead-monom } (\text{numeral } n) = 0$
unfolding *monom-numeral*[*symmetric*] **by** (*subst lead-monom-monom*) *auto*

lemma *lead-monom-add*: $\text{lead-monom } (p + q) \leq \max (\text{lead-monom } p) (\text{lead-monom } q)$

proof *transfer*

fix $p\ q :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a$

show $\text{Max } (\text{insert } 0 (\text{keys } (p + q))) \leq \max (\text{Max } (\text{insert } 0 (\text{keys } p))) (\text{Max } (\text{insert } 0 (\text{keys } q)))$

proof (*rule Max.boundedI*)

fix m **assume** $m \in \text{insert } 0 (\text{keys } (p + q))$

thus $m \leq \max (\text{Max } (\text{insert } 0 (\text{keys } p))) (\text{Max } (\text{insert } 0 (\text{keys } q)))$

proof

assume $m \in \text{keys } (p + q)$

with *keys-add*[*of p q*] **have** $m \in \text{keys } p \vee m \in \text{keys } q$

by (*auto simp: in-keys-iff plus-poly-mapping.rep-eq*)

thus *?thesis* **by** (*auto simp: le-max-iff-disj*)

qed *auto*

qed *auto*

qed

lemma *lead-monom-diff*: $\text{lead-monom } (p - q) \leq \max (\text{lead-monom } p) (\text{lead-monom } q)$

proof *transfer*

fix $p\ q :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a$

show $\text{Max } (\text{insert } 0 (\text{keys } (p - q))) \leq \max (\text{Max } (\text{insert } 0 (\text{keys } p))) (\text{Max } (\text{insert } 0 (\text{keys } q)))$

proof (*rule Max.boundedI*)

fix m **assume** $m \in \text{insert } 0 (\text{keys } (p - q))$

thus $m \leq \max (\text{Max } (\text{insert } 0 (\text{keys } p))) (\text{Max } (\text{insert } 0 (\text{keys } q)))$

proof

assume $m \in \text{keys } (p - q)$

with *keys-diff-subset*[*of p q*] **have** $m \in \text{keys } p \vee m \in \text{keys } q$ **by** *auto*

thus *?thesis* **by** (*auto simp: le-max-iff-disj*)

qed *auto*

qed *auto*

qed

definition *lead-coeff* **where** $\text{lead-coeff } p = \text{coeff } p (\text{lead-monom } p)$

lemma *vars-empty-iff*: $\text{vars } p = \{\} \iff p = \text{Const } (\text{lead-coeff } p)$

proof

assume $\text{vars } p = \{\}$

hence [*simp*]: $\text{lead-monom } p = 0$

by (*simp add: lead-monom-eq-0-iff*)

have [*simp*]: $\text{mon} \neq 0 \iff (\text{mon} > (0 :: \text{nat} \Rightarrow_0 \text{nat}))$ **for** mon

by (*auto simp: order.strict-iff-order*)

thus $p = \text{Const } (\text{lead-coeff } p)$

by (*intro mpoly-eqI*) (*auto simp: mpoly-coeff-Const lead-coeff-def*)
next
assume $p = \text{Const } (\text{lead-coeff } p)$
also have $\text{vars } \dots = \{\}$ **by** *simp*
finally show $\text{vars } p = \{\}$.
qed

lemma *lead-coeff-0* [*simp*]: $\text{lead-coeff } 0 = 0$
by (*simp add: lead-coeff-def*)

lemma *lead-coeff-1* [*simp*]: $\text{lead-coeff } 1 = 1$
by (*simp add: lead-coeff-def mpoly-coeff-1*)

lemma *lead-coeff-Const* [*simp*]: $\text{lead-coeff } (\text{Const } c) = c$
by (*simp add: lead-coeff-def mpoly-coeff-Const*)

lemma *lead-coeff-monom* [*simp*]: $\text{lead-coeff } (\text{monom } p \ c) = c$
by (*simp add: lead-coeff-def coeff-monom when-def lead-monom-monom*)

lemma *lead-coeff-nonzero* [*simp*]: $p \neq 0 \implies \text{lead-coeff } p \neq 0$
unfolding *lead-coeff-def lead-monom-def*
by (*cases keys (mapping-of p) = \{\}*) (*auto simp: coeff-def max-def*)

lemma
fixes $c :: 'a :: \text{semiring-0}$
assumes $c * \text{lead-coeff } p \neq 0$
shows *lead-monom-smult* [*simp*]: $\text{lead-monom } (\text{smult } c \ p) = \text{lead-monom } p$
and *lead-coeff-smult* [*simp*]: $\text{lead-coeff } (\text{smult } c \ p) = c * \text{lead-coeff } p$
proof –
from *assms* **have** $*$: $\text{keys } (\text{mapping-of } p) \neq \{\}$
by *auto*
from *assms* **have** *coeff* (*MPoly-Type.smult* $c \ p$) ($\text{lead-monom } p$) $\neq 0$
by (*simp add: lead-coeff-def*)
hence *smult-nz*: *MPoly-Type.smult* $c \ p \neq 0$ **by** (*auto simp del: coeff-smult*)
with *assms* **have** $**$: $\text{keys } (\text{mapping-of } (\text{smult } c \ p)) \neq \{\}$
by *simp*

have $\text{Max } (\text{keys } (\text{mapping-of } (\text{smult } c \ p))) = \text{Max } (\text{keys } (\text{mapping-of } p))$

proof (*safe intro!*: *antisym Max.coboundedI*)

have *lookup* (*mapping-of* p) ($\text{Max } (\text{keys } (\text{mapping-of } p))$) = $\text{lead-coeff } p$

using $*$ **by** (*simp add: lead-coeff-def lead-monom-def max-def coeff-def*)

with *assms* **show** $\text{Max } (\text{keys } (\text{mapping-of } p)) \in \text{keys } (\text{mapping-of } (\text{smult } c \ p))$

using $*$ **by** (*auto simp: smult.rep-eq intro!: in-keys-mapI*)

from *smult-nz* **have** $\text{lead-coeff } (\text{smult } c \ p) \neq 0$

by (*intro lead-coeff-nonzero*) *auto*

hence *coeff* p ($\text{Max } (\text{keys } (\text{mapping-of } (\text{smult } c \ p)))) \neq 0$

using *assms* $*$ $**$ **by** (*auto simp: lead-coeff-def lead-monom-def max-def*)

thus $\text{Max } (\text{keys } (\text{mapping-of } (\text{smult } c \ p))) \in \text{keys } (\text{mapping-of } p)$

by (*auto simp: smult.rep-eq coeff-def in-keys-iff*)

qed *auto*
with * ** **show** $\text{lead-monom } (\text{smult } c \ p) = \text{lead-monom } p$
by (*simp add: lead-monom-def max-def*)
thus $\text{lead-coeff } (\text{smult } c \ p) = c * \text{lead-coeff } p$
by (*simp add: lead-coeff-def*)
qed

lemma *lead-coeff-mult-aux*:
 $\text{coeff } (p * q) (\text{lead-monom } p + \text{lead-monom } q) = \text{lead-coeff } p * \text{lead-coeff } q$
proof (*cases p = 0 \vee q = 0*)
case *False*
define *a b* **where** $a = \text{lead-monom } p$ **and** $b = \text{lead-monom } q$
have $\text{coeff } (p * q) (a + b) = \text{coeff } p \ a * \text{coeff } q \ b$
unfolding *coeff-mpoly-times*
by (*rule prod-fun-max*) (*insert False, auto simp: a-def b-def*)
thus *?thesis* **by** (*simp add: a-def b-def lead-coeff-def*)
qed *auto*

lemma *lead-monom-mult-le*: $\text{lead-monom } (p * q) \leq \text{lead-monom } p + \text{lead-monom } q$
proof (*cases p * q = 0*)
case *False*
show *?thesis*
proof (*intro leI notI*)
assume $\text{lead-monom } p + \text{lead-monom } q < \text{lead-monom } (p * q)$
hence $\text{lead-coeff } (p * q) = 0$
unfolding *lead-coeff-def coeff-mpoly-times* **by** (*rule prod-fun-gt-max-eq-zero*)
auto
with *False* **show** *False* **by** *simp*
qed
qed *auto*

lemma *lead-monom-mult*:
assumes $\text{lead-coeff } p * \text{lead-coeff } q \neq 0$
shows $\text{lead-monom } (p * q) = \text{lead-monom } p + \text{lead-monom } q$
by (*intro antisym lead-monom-mult-le lead-monom-geI*)
(insert assms, auto simp: lead-coeff-mult-aux)

lemma *lead-coeff-mult*:
assumes $\text{lead-coeff } p * \text{lead-coeff } q \neq 0$
shows $\text{lead-coeff } (p * q) = \text{lead-coeff } p * \text{lead-coeff } q$
using *assms* **by** (*simp add: lead-monom-mult lead-coeff-mult-aux lead-coeff-def*)

lemma *keys-lead-monom-subset*: $\text{keys } (\text{lead-monom } p) \subseteq \text{vars } p$
proof (*cases p = 0*)
case *False*
hence $\text{lead-coeff } p \neq 0$ **by** *simp*
hence $\text{coeff } p (\text{lead-monom } p) \neq 0$ **unfolding** *lead-coeff-def* .
thus *?thesis* **unfolding** *vars-def* **by** *transfer' (auto simp: max-def in-keys-iff)*

qed *auto*

lemma

assumes $(\prod i \in A. \text{lead-coeff } (f i)) \neq 0$
shows *lead-monom-prod*: $\text{lead-monom } (\prod i \in A. f i) = (\sum i \in A. \text{lead-monom } (f i))$ (**is** *?th1*)
and *lead-coeff-prod*: $\text{lead-coeff } (\prod i \in A. f i) = (\prod i \in A. \text{lead-coeff } (f i))$ (**is** *?th2*)

proof –

have *?th1* \wedge *?th2* **using** *assms*

proof (*induction A rule: infinite-finite-induct*)

case (*insert x A*)

from *insert* **have** *nz*: $\text{lead-coeff } (f x) \neq 0$ $(\prod i \in A. \text{lead-coeff } (f i)) \neq 0$ **by** *auto*

note *IH = insert.IH[OF this(2)]*

from *insert* **have** *nz'*: $\text{lead-coeff } (f x) * \text{lead-coeff } (\prod i \in A. f i) \neq 0$

by (*subst IH*) *auto*

from *insert.prem1 insert.hyps nz nz'* **show** *?case*

by (*auto simp: lead-monom-mult lead-coeff-mult IH*)

qed *auto*

thus *?th1 ?th2* **by** *blast+*

qed

lemma *lead-monom-sum-le*: $(\bigwedge x. x \in X \implies \text{lead-monom } (h x) \leq ub) \implies \text{lead-monom } (\text{sum } h X) \leq ub$

by (*induction X rule: infinite-finite-induct*) (*auto intro!: order.trans[OF lead-monom-add]*)

The leading monomial of a sum where the leading monomial the summands are distinct is simply the maximum of the leading monomials.

lemma *lead-monom-sum*:

assumes *inj-on* $(\text{lead-monom} \circ h)$ *X* **and** *finite X* **and** $X \neq \{\}$ **and** $\bigwedge x. x \in X \implies h x \neq 0$

defines $m \equiv \text{Max } ((\text{lead-monom} \circ h) \text{ ` } X)$

shows $\text{lead-monom } (\sum x \in X. h x) = m$

proof (*rule antisym*)

show $\text{lead-monom } (\text{sum } h X) \leq m$ **unfolding** *m-def* **using** *assms*

by (*intro lead-monom-sum-le Max-ge finite-imageI*) *auto*

next

from *assms* **have** $m \in (\text{lead-monom} \circ h) \text{ ` } X$

unfolding *m-def* **by** (*intro Max-in finite-imageI*) *auto*

then obtain *x* **where** $x \in X$ $m = \text{lead-monom } (h x)$ **by** *auto*

have $\text{coeff } (\sum x \in X. h x) m = (\sum x \in X. \text{coeff } (h x) m)$

by *simp*

also have $\dots = (\sum x \in \{x\}. \text{coeff } (h x) m)$

proof (*intro sum.mono-neutral-right ballI*)

fix *y* **assume** $y: y \in X - \{x\}$

hence $(\text{lead-monom} \circ h) y \leq m$

using *assms* **unfolding** *m-def* **by** (*intro Max-ge finite-imageI*) *auto*

moreover have $(\text{lead-monom} \circ h) y \neq (\text{lead-monom} \circ h) x$

using $\langle x \in X \rangle y$ *inj-onD*[*OF assms(1), of x y*] **by** *auto*

ultimately have $\text{lead-monom } (h\ y) < m$
using x **by** *auto*
thus $\text{coeff } (h\ y)\ m = 0$ **by** *simp*
qed (*insert x assms, auto*)
also have $\dots = \text{coeff } (h\ x)\ m$ **by** *simp*
also have $\dots = \text{lead-coeff } (h\ x)$ **using** x **by** (*simp add: lead-coeff-def*)
also have $\dots \neq 0$ **using** *assms x* **by** *auto*
finally show $\text{lead-monom } (\text{sum } h\ X) \geq m$ **by** (*intro lead-monom-geI*)
qed

lemma *lead-coeff-eq-0-iff* [*simp*]: $\text{lead-coeff } p = 0 \longleftrightarrow p = 0$
by (*cases p = 0*) *auto*

lemma
fixes $f :: - \Rightarrow 'a :: \text{semidom mpoly}$
assumes $\bigwedge i. i \in A \implies f\ i \neq 0$
shows *lead-monom-prod'* [*simp*]: $\text{lead-monom } (\prod i \in A. f\ i) = (\sum i \in A. \text{lead-monom } (f\ i))$ (*is ?th1*)
and *lead-coeff-prod'* [*simp*]: $\text{lead-coeff } (\prod i \in A. f\ i) = (\prod i \in A. \text{lead-coeff } (f\ i))$ (*is ?th2*)
proof –
from *assms* **have** $(\prod i \in A. \text{lead-coeff } (f\ i)) \neq 0$
by (*cases finite A*) *auto*
thus *?th1 ?th2* **by** (*simp-all add: lead-monom-prod lead-coeff-prod*)
qed

lemma
fixes $p :: 'a :: \text{comm-semiring-1 mpoly}$
assumes $\text{lead-coeff } p \wedge n \neq 0$
shows *lead-monom-power*: $\text{lead-monom } (p \wedge n) = \text{of-nat } n * \text{lead-monom } p$
and *lead-coeff-power*: $\text{lead-coeff } (p \wedge n) = \text{lead-coeff } p \wedge n$
using *assms lead-monom-prod*[*of* $\lambda-. p \{..<n\}$] *lead-coeff-prod*[*of* $\lambda-. p \{..<n\}$]
by *simp-all*

lemma
fixes $p :: 'a :: \text{semidom mpoly}$
assumes $p \neq 0$
shows *lead-monom-power'* [*simp*]: $\text{lead-monom } (p \wedge n) = \text{of-nat } n * \text{lead-monom } p$
and *lead-coeff-power'* [*simp*]: $\text{lead-coeff } (p \wedge n) = \text{lead-coeff } p \wedge n$
using *assms lead-monom-prod'*[*of* $\{..<n\} \lambda-. p$] *lead-coeff-prod'*[*of* $\{..<n\} \lambda-. p$]
by *simp-all*

2.7 Turning a set of variables into a monomial

Given a finite set $\{X_1, \dots, X_n\}$ of variables, the following is the monomial $X_1 \dots X_n$:

lift-definition *monom-of-set* :: $\text{nat set} \Rightarrow (\text{nat} \Rightarrow_0 \text{nat})$ **is**
 $\lambda X\ x. \text{if finite } X \wedge x \in X \text{ then } 1 \text{ else } 0$

by *auto*

lemma *lookup-monom-of-set*:

Poly-Mapping.lookup (monom-of-set X) i = (if finite X \wedge $i \in X$ then 1 else 0)

by *transfer auto*

lemma *lookup-monom-of-set-1* [*simp*]:

finite X $\implies i \in X \implies Poly-Mapping.lookup (monom-of-set X) i = 1$

and *lookup-monom-of-set-0* [*simp*]:

$i \notin X \implies Poly-Mapping.lookup (monom-of-set X) i = 0$

by (*simp-all add: lookup-monom-of-set*)

lemma *keys-monom-of-set*: *keys (monom-of-set X) = (if finite X then X else {})*

by *transfer auto*

lemma *keys-monom-of-set-finite* [*simp*]: *finite X $\implies keys (monom-of-set X) = X$*

by (*simp add: keys-monom-of-set*)

lemma *monom-of-set-eq-iff* [*simp*]: *finite X $\implies finite Y \implies monom-of-set X = monom-of-set Y \iff X = Y$*

by *transfer (auto simp: fun-eq-iff)*

lemma *monom-of-set-empty* [*simp*]: *monom-of-set {} = 0*

by *transfer auto*

lemma *monom-of-set-eq-zero-iff* [*simp*]: *monom-of-set X = 0 $\iff infinite X \vee X = \{\}$*

by *transfer (auto simp: fun-eq-iff)*

lemma *zero-eq-monom-of-set-iff* [*simp*]: *0 = monom-of-set X $\iff infinite X \vee X = \{\}$*

by *transfer (auto simp: fun-eq-iff)*

2.8 Permuting the variables of a polynomial

Next, we define the operation of permuting the variables of a monomial and polynomial.

lift-definition *permutep* :: *('a \Rightarrow 'a) \Rightarrow ('a \Rightarrow_0 'b) \Rightarrow ('a \Rightarrow_0 'b :: zero) is*

*$\lambda f p.$ if *bij* *f* then $p \circ f$ else p*

proof –

fix *f* :: 'a \Rightarrow 'a **and** *g* :: 'a \Rightarrow 'b

assume *: *finite {x. g x \neq 0}*

show *finite {x. (if *bij* *f* then $g \circ f$ else g) x \neq 0}*

proof (*cases bij f*)

case *True*

with * **have** *finite (f - ' {x. g x \neq 0})*

by (*intro finite-vimageI*) (*auto dest: bij-is-inj*)

with *True* **show** *?thesis* **by** *auto*

qed (*use * in auto*)

qed

lift-definition *mpoly-map-vars* :: (nat \Rightarrow nat) \Rightarrow 'a :: zero *mpoly* \Rightarrow 'a *mpoly* is
 $\lambda f p.$ *permutep* (*permutep* f) p .

lemma *keys-permutep*: *bij* f \Longrightarrow *keys* (*permutep* f m) = f - ' *keys* m
by *transfer auto*

lemma *permutep-id''* [*simp*]: *permutep* id = id
by *transfer' (auto simp: fun-eq-iff)*

lemma *permutep-id'''* [*simp*]: *permutep* ($\lambda x. x$) = id
by *transfer' (auto simp: fun-eq-iff)*

lemma *permutep-0* [*simp*]: *permutep* f 0 = 0
by *transfer auto*

lemma *permutep-single*:
bij f \Longrightarrow *permutep* f (*Poly-Mapping.single* a b) = *Poly-Mapping.single* (*inv-into*
UNIV f a) b
by *transfer (auto simp: fun-eq-iff when-def inv-f-f surj-f-inv-f bij-is-inj bij-is-surj)*

lemma *mpoly-map-vars-id* [*simp*]: *mpoly-map-vars* id = id
by *transfer auto*

lemma *mpoly-map-vars-id'* [*simp*]: *mpoly-map-vars* ($\lambda x. x$) = id
by *transfer auto*

lemma *lookup-permutep*:
Poly-Mapping.lookup (*permutep* f m) x = (if *bij* f then *Poly-Mapping.lookup* m (f
x) else *Poly-Mapping.lookup* m x)
by *transfer auto*

lemma *inj-permutep* [*intro*]: *inj* (*permutep* (f :: 'a \Rightarrow 'a) :: - \Rightarrow 'a \Rightarrow_0 'b :: zero)
unfolding *inj-def*

proof (*transfer, safe*)

fix f :: 'a \Rightarrow 'a **and** x y :: 'a \Rightarrow 'b

assume eq: (if *bij* f then x \circ f else x) = (if *bij* f then y \circ f else y)

show x = y

proof (*cases bij f*)

case True

show ?thesis

proof

fix t :: 'a

from (*bij* f) **obtain** s **where** t = f s

by (*auto dest!: bij-is-surj*)

with eq **and** True **show** x t = y t

by (*auto simp: fun-eq-iff*)

qed

qed (use eq in auto)
qed

lemma *surj-permutep* [intro]: *surj* (*permutep* ($f :: 'a \Rightarrow 'a$) :: $- \Rightarrow 'a \Rightarrow_0 'b :: \text{zero}$)
unfolding *surj-def*
proof (*transfer*, *safe*)
fix $f :: 'a \Rightarrow 'a$ **and** $y :: 'a \Rightarrow 'b$
assume $\text{fin}: \text{finite } \{t. y t \neq 0\}$
show $\exists x \in \{f. \text{finite } \{x. f x \neq 0\}\}. y = (\text{if } \text{bij } f \text{ then } x \circ f \text{ else } x)$
proof (*cases* *bij* f)
case *True*
with fin **have** *finite* (*the-inv* $f - ' \{t. y t \neq 0\}$)
by (*intro* *finite-vimageI*) (*auto* *simp: bij-is-inj bij-betw-the-inv-into*)
moreover **have** $y \circ \text{the-inv } f \circ f = y$
using *True* **by** (*simp* *add: fun-eq-iff the-inv-f-f bij-is-inj*)
ultimately **show** *?thesis* **by** (*intro* *beXI[of - y \circ the-inv f]*) (*auto* *simp: True*)
qed (use *fin* in *auto*)
qed

lemma *bij-permutep* [intro]: *bij* (*permutep* f)
using *inj-permutep[of f]* *surj-permutep[of f]* **by** (*simp* *add: bij-def*)

lemma *mpoly-map-vars-map-mpoly*:
 $\text{mpoly-map-vars } f (\text{map-mpoly } g p) = \text{map-mpoly } g (\text{mpoly-map-vars } f p)$
by (*transfer'*, *transfer'*) (*auto* *simp: fun-eq-iff*)

lemma *coeff-mpoly-map-vars*:
fixes $f :: \text{nat} \Rightarrow \text{nat}$ **and** $p :: 'a :: \text{zero } \text{mpoly}$
assumes *bij* f
shows $\text{MPoly-Type.coeff } (\text{mpoly-map-vars } f p) \text{ mon} =$
 $\text{MPoly-Type.coeff } p (\text{permutep } f \text{ mon})$
using *assms* **by** *transfer'* (*simp* *add: lookup-permutep bij-permutep*)

lemma *permutep-monom-of-set*:
assumes *bij* f
shows $\text{permutep } f (\text{monom-of-set } A) = \text{monom-of-set } (f - ' A)$
using *assms* **by** *transfer* (*auto* *simp: fun-eq-iff bij-is-inj finite-vimage-iff*)

lemma *permutep-comp*: $\text{bij } f \Longrightarrow \text{bij } g \Longrightarrow \text{permutep } (f \circ g) = \text{permutep } g \circ$
 $\text{permutep } f$
by *transfer'* (*auto* *simp: fun-eq-iff bij-comp*)

lemma *permutep-comp'*: $\text{bij } f \Longrightarrow \text{bij } g \Longrightarrow \text{permutep } (f \circ g) \text{ mon} = \text{permutep } g$
 $(\text{permutep } f \text{ mon})$
by *transfer* (*auto* *simp: fun-eq-iff bij-comp*)

lemma *mpoly-map-vars-comp*:
 $\text{bij } f \Longrightarrow \text{bij } g \Longrightarrow \text{mpoly-map-vars } f (\text{mpoly-map-vars } g p) = \text{mpoly-map-vars } (f$
 $\circ g) p$

by transfer (auto simp: bij-permutep permutep-comp)

lemma permutep-id [simp]: permutep id mon = mon
by transfer auto

lemma permutep-id' [simp]: permutep ($\lambda x. x$) mon = mon
by transfer auto

lemma inv-permutep [simp]:
fixes $f :: 'a \Rightarrow 'a$
assumes *bij f*
shows $\text{inv-into UNIV (permutep f) = permutep (inv-into UNIV f)}$
proof
fix $m :: 'a \Rightarrow_0 'b$
show $\text{inv-into UNIV (permutep f) m = permutep (inv-into UNIV f) m}$
using *permutep-comp'[of inv-into UNIV f f m] assms inj-iff[of f]*
by (intro inv-f-eq) (auto simp: bij-imp-bij-inv bij-is-inj)
qed

lemma mpoly-map-vars-monom:
 $\text{bij f} \implies \text{mpoly-map-vars f (monom m c) = monom (permutep (inv-into UNIV f) m) c}$
by transfer' (simp add: permutep-single bij-permutep)

lemma vars-mpoly-map-vars:
fixes $f :: \text{nat} \Rightarrow \text{nat}$ and $p :: 'a :: \text{zero mpoly}$
assumes *bij f*
shows $\text{vars (mpoly-map-vars f p) = f ` vars p}$
using *assms unfolding vars-def*
proof transfer'
fix $f :: \text{nat} \Rightarrow \text{nat}$ and $p :: (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a$
assume *f: bij f*
have *eq: f (inv-into UNIV f x) = x for x*
using *f by (subst surj-f-inv-f[of f]) (auto simp: bij-is-surj)*
show $\bigcup (\text{keys ` keys (permutep (permutep f) p)}) = f ` \bigcup (\text{keys ` keys p})$
proof safe
fix $m x$ assume *mx: m ∈ keys (permutep (permutep f) p) x ∈ keys m*
from *mx* have $\text{permutep f m} \in \text{keys p}$
by (auto simp: keys-permutep bij-permutep f)
with *mx* have $f (\text{inv-into UNIV f x}) \in f ` (\bigcup_{m \in \text{keys p}} \text{keys m})$
by (intro imageI) (auto intro!: bexI[of - permutep f m] simp: keys-permutep f
eq)
with *eq* show $x \in f ` (\bigcup_{m \in \text{keys p}} \text{keys m})$ by simp
next
fix $m x$ assume *mx: m ∈ keys p x ∈ keys m*
from *mx* have $\text{permutep id m} \in \text{keys p}$ by simp
also have $\text{id} = \text{inv-into UNIV f} \circ f$ using *f* by (intro ext) (auto simp: bij-is-inj
inv-f-f)
also have $\text{permutep} \dots m = \text{permutep f (permutep (inv-into UNIV f) m)}$

by (*simp add: permutep-comp f bij-imp-bij-inv*)
finally have **: *permutep f (permutep (inv-into UNIV f) m) ∈ keys p* .
moreover from *f mx have f x ∈ keys (permutep (inv-into UNIV f) m)*
 by (*auto simp: keys-permutep bij-imp-bij-inv inv-f-f bij-is-inj*)
ultimately show $f x \in \bigcup (\text{keys } \text{' } \text{keys (permutep (permutep f) p)})$ **using** *f*
 by (*auto simp: keys-permutep bij-permutep*)
qed
qed

lemma *permutep-eq-monom-of-set-iff [simp]*:
 assumes *bij f*
 shows $\text{permutep } f \text{ mon} = \text{monom-of-set } A \longleftrightarrow \text{mon} = \text{monom-of-set } (f \text{' } A)$
proof
 assume *eq: permutep f mon = monom-of-set A*
have *permutep (inv-into UNIV f) (permutep f mon) = monom-of-set (inv-into UNIV f - ' A)*
 using *assms by (simp add: eq bij-imp-bij-inv assms permutep-monom-of-set)*
also have *inv-into UNIV f - ' A = f ' A*
 using *assms by (force simp: bij-is-surj image-iff inv-f-f bij-is-inj surj-f-inv-f)*
also have *permutep (inv-into UNIV f) (permutep f mon) = permutep (f ∘ inv-into UNIV f) mon*
 using *assms by (simp add: permutep-comp bij-imp-bij-inv)*
also have *f ∘ inv-into UNIV f = id*
 by (*subst surj-iff [symmetric]*) (*insert assms, auto simp: bij-is-surj*)
finally show $\text{mon} = \text{monom-of-set } (f \text{' } A)$ **by** *simp*
qed (*insert assms, auto simp: permutep-monom-of-set inj-vimage-image-eq bij-is-inj*)

lemma *permutep-monom-of-set-permutes [simp]*:
 assumes $\pi \text{ permutes } A$
 shows $\text{permutep } \pi (\text{monom-of-set } A) = \text{monom-of-set } A$
 using *assms*
 by (*transfer (auto simp: if-splits fun-eq-iff permutes-in-image)*)

lemma *mpoly-map-vars-0 [simp]*: $\text{mpoly-map-vars } f \ 0 = 0$
 by (*transfer, transfer'*) (*simp add: o-def*)

lemma *permutep-eq-0-iff [simp]*: $\text{permutep } f \ m = 0 \longleftrightarrow m = 0$
proof *transfer*
fix $f :: 'a \Rightarrow 'a$ **and** $m :: 'a \Rightarrow 'b$ **assume** *finite {x. m x ≠ 0}*
show $((\text{if } \text{bij } f \text{ then } m \circ f \text{ else } m) = (\lambda k. 0)) = (m = (\lambda k. 0))$
proof (*cases bij f*)
 case *True*
hence $(\forall x. m (f x) = 0) \longleftrightarrow (\forall x. m x = 0)$
 using *bij-iff[of f]* **by** *metis*
with True show *?thesis by (auto simp: fun-eq-iff)*
qed (*auto simp: fun-eq-iff*)
qed

lemma *mpoly-map-vars-1 [simp]*: $\text{mpoly-map-vars } f \ 1 = 1$

by (*transfer*, *transfer'*) (*auto simp: o-def fun-eq-iff when-def*)

lemma *permutep-Const₀* [*simp*]: $(\bigwedge x. f x = 0 \longleftrightarrow x = 0) \implies \text{permutep } f (\text{Const}_0 c) = \text{Const}_0 c$

unfolding *Const₀-def* **by** *transfer'* (*auto simp: when-def fun-eq-iff*)

lemma *permutep-add* [*simp*]: $\text{permutep } f (m1 + m2) = \text{permutep } f m1 + \text{permutep } f m2$

unfolding *Const₀-def* **by** *transfer'* (*auto simp: when-def fun-eq-iff*)

lemma *permutep-diff* [*simp*]: $\text{permutep } f (m1 - m2) = \text{permutep } f m1 - \text{permutep } f m2$

unfolding *Const₀-def* **by** *transfer'* (*auto simp: when-def fun-eq-iff*)

lemma *permutep-uminus* [*simp*]: $\text{permutep } f (-m) = -\text{permutep } f m$

unfolding *Const₀-def* **by** *transfer'* (*auto simp: when-def fun-eq-iff*)

lemma *permutep-mult* [*simp*]:

$(\bigwedge x y. f (x + y) = f x + f y) \implies \text{permutep } f (m1 * m2) = \text{permutep } f m1 * \text{permutep } f m2$

unfolding *Const₀-def* **by** *transfer'* (*auto simp: when-def fun-eq-iff prod-fun-compose-bij*)

lemma *mpoly-map-vars-Const* [*simp*]: $\text{mpoly-map-vars } f (\text{Const } c) = \text{Const } c$

by *transfer* (*auto simp: o-def fun-eq-iff when-def*)

lemma *mpoly-map-vars-add* [*simp*]: $\text{mpoly-map-vars } f (p + q) = \text{mpoly-map-vars } f p + \text{mpoly-map-vars } f q$

by *transfer simp*

lemma *mpoly-map-vars-diff* [*simp*]: $\text{mpoly-map-vars } f (p - q) = \text{mpoly-map-vars } f p - \text{mpoly-map-vars } f q$

by *transfer simp*

lemma *mpoly-map-vars-uminus* [*simp*]: $\text{mpoly-map-vars } f (-p) = -\text{mpoly-map-vars } f p$

by *transfer simp*

lemma *permutep-smult*:

$\text{permutep } (\text{permutep } f) (\text{Poly-Mapping.map } ((*) c) p) = \text{Poly-Mapping.map } ((*) c) (\text{permutep } (\text{permutep } f) p)$

by *transfer'* (*auto split: if-splits simp: fun-eq-iff*)

lemma *mpoly-map-vars-smult* [*simp*]: $\text{mpoly-map-vars } f (\text{smult } c p) = \text{smult } c (\text{mpoly-map-vars } f p)$

by *transfer (simp add: permutep-smult)*

lemma *mpoly-map-vars-mult* [*simp*]: $\text{mpoly-map-vars } f (p * q) = \text{mpoly-map-vars } f p * \text{mpoly-map-vars } f q$

by *transfer simp*

lemma *mpoly-map-vars-sum* [*simp*]: $mpoly\text{-map}\text{-vars } f (sum\ g\ A) = (\sum_{x \in A}. mpoly\text{-map}\text{-vars } f (g\ x))$

by (*induction A rule: infinite-finite-induct*) *auto*

lemma *mpoly-map-vars-prod* [*simp*]: $mpoly\text{-map}\text{-vars } f (prod\ g\ A) = (\prod_{x \in A}. mpoly\text{-map}\text{-vars } f (g\ x))$

by (*induction A rule: infinite-finite-induct*) *auto*

lemma *mpoly-map-vars-eq-0-iff* [*simp*]: $mpoly\text{-map}\text{-vars } f\ p = 0 \longleftrightarrow p = 0$

by *transfer auto*

lemma *permutep-eq-iff* [*simp*]: $permutep\ f\ p = permutep\ f\ q \longleftrightarrow p = q$

by *transfer (auto simp: comp-bij-eq-iff)*

lemma *mpoly-map-vars-Sum-any* [*simp*]:

$mpoly\text{-map}\text{-vars } f (Sum\text{-any } g) = Sum\text{-any } (\lambda x. mpoly\text{-map}\text{-vars } f (g\ x))$

by (*simp add: Sum-any.expand-set*)

lemma *mpoly-map-vars-power* [*simp*]: $mpoly\text{-map}\text{-vars } f (p \hat{ } n) = mpoly\text{-map}\text{-vars } f\ p \hat{ } n$

by (*induction n*) *auto*

lemma *mpoly-map-vars-monom-single* [*simp*]:

assumes *bij f*

shows $mpoly\text{-map}\text{-vars } f (monom (Poly\text{-Mapping}.single\ i\ n)\ c) = monom (Poly\text{-Mapping}.single\ (f\ i)\ n)\ c$

using *assms* **by** (*simp add: mpoly-map-vars-monom permutep-single bij-imp-bij-inv inv-inv-eq*)

lemma *insertion-mpoly-map-vars*:

assumes *bij f*

shows $insertion\ g (mpoly\text{-map}\text{-vars } f\ p) = insertion\ (g \circ f)\ p$

proof –

have $insertion\ g (mpoly\text{-map}\text{-vars } f\ p) =$

$(\sum m. coeff\ p (permutep\ f\ m) * (\prod i. g\ i \hat{ } lookup\ m\ i))$

using *assms* **by** (*simp add: insertion-altdef coeff-mpoly-map-vars*)

also have $\dots = Sum\text{-any } (\lambda m. coeff\ p (permutep\ f\ m) *$

$Prod\text{-any } (\lambda i. g\ (f\ i) \hat{ } lookup\ m\ (f\ i)))$

by (*intro Sum-any.cong arg-cong[where ?f = λy. x * y for x]*

Prod-any.reindex-cong[OF assms]) (*auto simp: o-def*)

also have $\dots = Sum\text{-any } (\lambda m. coeff\ p\ m * (\prod i. g\ (f\ i) \hat{ } lookup\ m\ i))$

by (*intro Sum-any.reindex-cong [OF bij-permutep[of f], symmetric]*)

(*auto simp: o-def lookup-permutep assms*)

also have $\dots = insertion\ (g \circ f)\ p$

by (*simp add: insertion-altdef*)

finally show *?thesis* .

qed

```

lemma permutep-cong:
  assumes f permutes (-keys p) g permutes (-keys p) p = q
  shows permutep f p = permutep g q
proof (intro poly-mapping-eqI)
  fix k :: 'a
  show lookup (permutep f p) k = lookup (permutep g q) k
  proof (cases k ∈ keys p)
    case False
    with assms have f k ∉ keys p g k ∉ keys p
    using permutes-in-image[of - -keys p k] by auto
    thus ?thesis using assms by (auto simp: lookup-permutep permutes-bij in-keys-iff)
  qed (insert assms, auto simp: lookup-permutep permutes-bij permutes-not-in)
qed

```

```

lemma mpoly-map-vars-cong:
  assumes f permutes (-vars p) g permutes (-vars q) p = q
  shows mpoly-map-vars f p = mpoly-map-vars g (q :: 'a :: zero mpoly)
proof (intro mpoly-eqI)
  fix mon :: nat ⇒0 nat
  show coeff (mpoly-map-vars f p) mon = coeff (mpoly-map-vars g q) mon
  proof (cases keys mon ⊆ vars p)
    case True
    with assms have permutep f mon = permutep g mon
    by (intro permutep-cong assms(1,2)[THEN permutes-subset]) auto
    thus ?thesis using assms by (simp add: coeff-mpoly-map-vars permutes-bij)
  next
  case False
  hence ¬(keys mon ⊆ f ' vars q) ¬(keys mon ⊆ g ' vars q)
  using assms by (auto simp: subset-iff permutes-not-in)
  thus ?thesis using assms
  by (subst (1 2) coeff-notin-vars)
  (auto simp: coeff-notin-vars vars-mpoly-map-vars permutes-bij)
  qed
qed

```

2.9 Symmetric polynomials

A polynomial is symmetric on a set of variables if it is invariant under any permutation of that set.

definition *symmetric-mpoly* :: *nat set ⇒ 'a :: zero mpoly ⇒ bool* **where**
symmetric-mpoly A p = (∀ π. π permutes A ⟶ mpoly-map-vars π p = p)

lemma *symmetric-mpoly-empty* [*simp, intro*]: *symmetric-mpoly {} p*
by (*simp add: symmetric-mpoly-def*)

A polynomial is trivially symmetric on any set of variables that do not occur in it.

lemma *symmetric-mpoly-orthogonal*:

```

assumes vars p  $\cap$  A = {}
shows symmetric-mpoly A p
unfolding symmetric-mpoly-def
proof safe
  fix  $\pi$  assume  $\pi$ :  $\pi$  permutes A
  with assms have  $\pi$  x = x if x  $\in$  vars p for x
    using that permutes-not-in[of  $\pi$  A x] by auto
  from assms have mpoly-map-vars  $\pi$  p = mpoly-map-vars id p
    by (intro mpoly-map-vars-cong permutes-subset[OF  $\pi$ ] permutes-id) auto
  also have ... = p by simp
  finally show mpoly-map-vars  $\pi$  p = p .
qed

```

```

lemma symmetric-mpoly-monom [intro]:
  assumes keys m  $\cap$  A = {}
  shows symmetric-mpoly A (monom m c)
  using assms vars-monom-subset[of m c] by (intro symmetric-mpoly-orthogonal)
  auto

```

```

lemma symmetric-mpoly-subset:
  assumes symmetric-mpoly A p B  $\subseteq$  A
  shows symmetric-mpoly B p
  unfolding symmetric-mpoly-def
proof safe
  fix  $\pi$  assume  $\pi$  permutes B
  with assms have  $\pi$  permutes A using permutes-subset by blast
  with assms show mpoly-map-vars  $\pi$  p = p
    by (auto simp: symmetric-mpoly-def)
qed

```

If a polynomial is symmetric over some set of variables, that set must either be a subset of the variables occurring in the polynomial or disjoint from it.

```

lemma symmetric-mpoly-imp-orthogonal-or-subset:
  assumes symmetric-mpoly A p
  shows vars p  $\cap$  A = {}  $\vee$  A  $\subseteq$  vars p
proof (rule ccontr)
  assume  $\neg$ (vars p  $\cap$  A = {}  $\vee$  A  $\subseteq$  vars p)
  then obtain x y where xy: x  $\in$  vars p  $\cap$  A y  $\in$  A - vars p by auto
  define  $\pi$  where  $\pi$  = Fun.swap x y id
  from xy have  $\pi$ :  $\pi$  permutes A
    unfolding  $\pi$ -def by (intro permutes-swap-id) auto
  from xy have y  $\in$   $\pi$  ' vars p by (auto simp:  $\pi$ -def Fun.swap-def)
  also from  $\pi$  have  $\pi$  ' vars p = vars (mpoly-map-vars  $\pi$  p)
    by (auto simp: vars-mpoly-map-vars permutes-bij)
  also have mpoly-map-vars  $\pi$  p = p
    using assms  $\pi$  by (simp add: symmetric-mpoly-def)
  finally show False using xy by auto
qed

```

Symmetric polynomials are closed under ring operations.

lemma *symmetric-mpoly-add* [intro]:

symmetric-mpoly A p \implies *symmetric-mpoly A q* \implies *symmetric-mpoly A (p + q)*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-diff* [intro]:

symmetric-mpoly A p \implies *symmetric-mpoly A q* \implies *symmetric-mpoly A (p - q)*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-uminus* [intro]: *symmetric-mpoly A p* \implies *symmetric-mpoly A (-p)*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-uminus-iff* [simp]: *symmetric-mpoly A (-p)* \longleftrightarrow *symmetric-mpoly A p*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-smult* [intro]: *symmetric-mpoly A p* \implies *symmetric-mpoly A (smult c p)*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-mult* [intro]:

symmetric-mpoly A p \implies *symmetric-mpoly A q* \implies *symmetric-mpoly A (p * q)*

unfolding *symmetric-mpoly-def* **by** *simp*

lemma *symmetric-mpoly-0* [simp, intro]: *symmetric-mpoly A 0*

and *symmetric-mpoly-1* [simp, intro]: *symmetric-mpoly A 1*

and *symmetric-mpoly-Const* [simp, intro]: *symmetric-mpoly A (Const c)*

by (*simp-all add: symmetric-mpoly-def*)

lemma *symmetric-mpoly-power* [intro]:

symmetric-mpoly A p \implies *symmetric-mpoly A (p ^ n)*

by (*induction n*) (*auto intro!: symmetric-mpoly-mult*)

lemma *symmetric-mpoly-sum* [intro]:

($\bigwedge i. i \in B \implies$ *symmetric-mpoly A (f i)*) \implies *symmetric-mpoly A (sum f B)*

by (*induction B rule: infinite-finite-induct*) (*auto intro!: symmetric-mpoly-add*)

lemma *symmetric-mpoly-prod* [intro]:

($\bigwedge i. i \in B \implies$ *symmetric-mpoly A (f i)*) \implies *symmetric-mpoly A (prod f B)*

by (*induction B rule: infinite-finite-induct*) (*auto intro!: symmetric-mpoly-mult*)

An symmetric sum or product over polynomials yields a symmetric polynomial:

lemma *symmetric-mpoly-symmetric-sum*:

assumes *g permutes X*

assumes $\bigwedge x \pi. x \in X \implies \pi$ *permutes A* \implies *mpoly-map-vars* $\pi (f x) = f (g x)$

shows *symmetric-mpoly A* ($\sum_{x \in X}. f x$)

unfolding *symmetric-mpoly-def*

proof *safe*
fix π **assume** π : π *permutes* A
have $\text{mpoly-map-vars } \pi (\text{sum } f X) = (\sum x \in X. \text{mpoly-map-vars } \pi (f x))$
by *simp*
also have $\dots = (\sum x \in X. f (g x))$
by (*intro sum.cong assms* π *refl*)
also have $\dots = (\sum x \in g'X. f x)$
using *assms* **by** (*subst sum.reindex*) (*auto simp: permutes-inj-on*)
also have $g' X = X$
using *assms* **by** (*simp add: permutes-image*)
finally show $\text{mpoly-map-vars } \pi (\text{sum } f X) = \text{sum } f X$.
qed

lemma *symmetric-mpoly-symmetric-prod*:
assumes g *permutes* X
assumes $\bigwedge x \pi. x \in X \implies \pi$ *permutes* $A \implies \text{mpoly-map-vars } \pi (f x) = f (g x)$
shows *symmetric-mpoly* $A (\prod x \in X. f x)$
unfolding *symmetric-mpoly-def*

proof *safe*
fix π **assume** π : π *permutes* A
have $\text{mpoly-map-vars } \pi (\text{prod } f X) = (\prod x \in X. \text{mpoly-map-vars } \pi (f x))$
by *simp*
also have $\dots = (\prod x \in X. f (g x))$
by (*intro prod.cong assms* π *refl*)
also have $\dots = (\prod x \in g'X. f x)$
using *assms* **by** (*subst prod.reindex*) (*auto simp: permutes-inj-on*)
also have $g' X = X$
using *assms* **by** (*simp add: permutes-image*)
finally show $\text{mpoly-map-vars } \pi (\text{prod } f X) = \text{prod } f X$.
qed

If p is a polynomial that is symmetric on some subset of variables A , then for the leading monomial of p , the exponents of these variables are decreasing w. r. t. the variable ordering.

theorem *lookup-lead-monom-decreasing*:
assumes *symmetric-mpoly* $A p$
defines $m \equiv \text{lead-monom } p$
assumes $i \in A j \in A i \leq j$
shows $\text{lookup } m i \geq \text{lookup } m j$
proof (*cases* $p = 0$)
case [*simp*]: *False*
show *?thesis*
proof (*intro leI notI*)
assume *less*: $\text{lookup } m i < \text{lookup } m j$
define π **where** $\pi = \text{Fun.swap } i j \text{ id}$
from *assms* **have** π : π *permutes* A
unfolding π -*def* **by** (*intro permutes-swap-id*) *auto*
have [*simp*]: $\pi \circ \pi = \text{id } \pi i = j \pi j = i \bigwedge k. k \neq i \implies k \neq j \implies \pi k = k$
by (*auto simp:* π -*def Fun.swap-def fun-eq-iff*)

have $0 \neq \text{lead-coeff } p$ **by** *simp*
also have $\text{lead-coeff } p = \text{MPoly-Type.coeff } (\text{mpoly-map-vars } \pi \ p)$ (*permutep* π m)
using π **by** (*simp add: lead-coeff-def m-def coeff-mpoly-map-vars permutes-bij permutep-comp' [symmetric]*)
also have $\text{mpoly-map-vars } \pi \ p = p$
using π *assms* **by** (*simp add: symmetric-mpoly-def*)
finally have $\text{permutep } \pi \ m \leq m$ **by** (*auto simp: m-def*)

moreover have $\text{lookup } m \ i < \text{lookup } (\text{permutep } \pi \ m) \ i$
and $(\forall k < i. \text{lookup } m \ k = \text{lookup } (\text{permutep } \pi \ m) \ k)$
using *assms* π *less* **by** (*auto simp: lookup-permutep permutes-bij*)
hence $m < \text{permutep } \pi \ m$
by (*auto simp: less-poly-mapping-def less-fun-def*)
ultimately show *False* **by** *simp*
qed
qed (*auto simp: m-def*)

2.10 The elementary symmetric polynomials

The k -th elementary symmetric polynomial for a finite set of variables A , with k ranging between 1 and $|A|$, is the sum of the product of all subsets of A with cardinality k :

lift-definition $\text{sym-mpoly-aux} :: \text{nat set} \Rightarrow \text{nat} \Rightarrow (\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a :: \{\text{zero-neq-one}\}$
is

$\lambda X \ k \ \text{mon. if finite } X \wedge (\exists Y. Y \subseteq X \wedge \text{card } Y = k \wedge \text{mon} = \text{monom-of-set } Y)$
 $\text{then } 1 \ \text{else } 0$

proof –

fix $k :: \text{nat}$ **and** $X :: \text{nat set}$

show $\text{finite } \{x. (\text{if finite } X \wedge (\exists Y \subseteq X. \text{card } Y = k \wedge x = \text{monom-of-set } Y) \text{ then } 1 \ \text{else } 0) \neq$

$(0 :: 'a)\}$ (**is** *finite* $?A$)

proof (*cases finite* X)

case *True*

have $?A \subseteq \text{monom-of-set } \text{'Pow } X$ **by** *auto*

moreover from *True* **have** *finite* $(\text{monom-of-set } \text{'Pow } X)$ **by** *simp*

ultimately show *?thesis* **by** (*rule finite-subset*)

qed *auto*

qed

lemma *lookup-sym-mpoly-aux*:

$\text{Poly-Mapping.lookup } (\text{sym-mpoly-aux } X \ k) \ \text{mon} =$

$(\text{if finite } X \wedge (\exists Y. Y \subseteq X \wedge \text{card } Y = k \wedge \text{mon} = \text{monom-of-set } Y) \text{ then } 1$
 $\text{else } 0)$

by *transfer' simp*

lemma *lookup-sym-mpoly-aux-monom-of-set* [*simp*]:

assumes *finite* $X \ Y \subseteq X \ \text{card } Y = k$

shows $Poly\text{-Mapping.lookup } (sym\text{-mpoly-aux } X \ k) \ (monom\text{-of-set } Y) = 1$
using *assms* **by** *(auto simp: lookup-sym-mpoly-aux)*

lemma *keys-sym-mpoly-aux*: $m \in keys \ (sym\text{-mpoly-aux } A \ k) \implies keys \ m \subseteq A$
by *transfer' (auto split: if-splits simp: keys-monom-of-set)*

lift-definition *sym-mpoly* :: $nat \ set \Rightarrow nat \Rightarrow 'a :: \{zero\text{-neq-one}\} \ mpoly$ **is**
sym-mpoly-aux .

lemma *vars-sym-mpoly-subset*: $vars \ (sym\text{-mpoly } A \ k) \subseteq A$
using *keys-sym-mpoly-aux* **by** *(auto simp: vars-def sym-mpoly.rep-eq)*

lemma *coeff-sym-mpoly*:
 $MPoly\text{-Type.coeff } (sym\text{-mpoly } X \ k) \ mon =$
(if finite X \wedge ($\exists Y. Y \subseteq X \wedge card \ Y = k \wedge mon = monom\text{-of-set } Y$) then 1
else 0)
by *transfer' (simp add: lookup-sym-mpoly-aux)*

lemma *sym-mpoly-infinite*: $\neg finite \ A \implies sym\text{-mpoly } A \ k = 0$
by *(transfer, transfer) auto*

lemma *sym-mpoly-altdef*: $sym\text{-mpoly } A \ k = (\sum X \mid X \subseteq A \wedge card \ X = k. monom$
 $(monom\text{-of-set } X) \ 1)$

proof *(cases finite A)*

case *False*

hence *: $infinite \ \{X. X \subseteq A \wedge infinite \ X\}$
by *(rule infinite-infinite-subsets)*

have $infinite \ \{X. X \subseteq A \wedge card \ X = 0\}$
by *(rule infinite-super[OF - *]) auto*

moreover **have** **: $infinite \ \{X. X \subseteq A \wedge finite \ X \wedge card \ X = k\}$ **if** $k \neq 0$
using *that infinite-card-subsets[of A k] False* **by** *auto*

have $infinite \ \{X. X \subseteq A \wedge card \ X = k\}$ **if** $k \neq 0$
by *(rule infinite-super[OF - **[OF that]]) auto*

ultimately **show** *?thesis* **using** *False*
by *(cases k = 0) (simp-all add: sym-mpoly-infinite)*

next

case *True*

show *?thesis*

proof *(intro mpoly-eqI, goal-cases)*

case *(1 m)*

show *?case*

proof *(cases $\exists X. X \subseteq A \wedge card \ X = k \wedge m = monom\text{-of-set } X$)*

case *False*

thus *?thesis* **by** *(auto simp: coeff-sym-mpoly coeff-sum coeff-monom)*

next

case *True*

then **obtain** X **where** $X: X \subseteq A \ card \ X = k \ m = monom\text{-of-set } X$
by *blast*

have $coeff \ (\sum X \mid X \subseteq A \wedge card \ X = k.$


```

      monom (monom-of-set X) 1) m = (∑ X∈{X}. 1) unfolding coeff-sum
proof (intro sum.mono-neutral-cong-right ballI)
  fix Y assume Y: Y ∈ {X. X ⊆ A ∧ card X = k} - {X}
  hence X = Y if monom-of-set X = monom-of-set Y
    using that finite-subset[OF X(1)] finite-subset[of Y A] ⟨finite A⟩ by auto
  thus coeff (monom (monom-of-set Y) 1) m = 0
    using X Y by (auto simp: coeff-monom when-def )
  qed (insert X ⟨finite A⟩, auto simp: coeff-monom)
  thus ?thesis using ⟨finite A⟩ by (auto simp: coeff-sym-mpoly coeff-sum
coeff-monom)
qed
qed
qed

```

```

lemma coeff-sym-mpoly-monom-of-set [simp]:
  assumes finite X Y ⊆ X card Y = k
  shows MPoly-Type.coeff (sym-mpoly X k) (monom-of-set Y) = 1
  using assms by (auto simp: coeff-sym-mpoly)

```

```

lemma coeff-sym-mpoly-0: coeff (sym-mpoly X k) 0 = (if finite X ∧ k = 0 then 1
else 0)

```

```

proof -
  consider finite X k = 0 | finite X k ≠ 0 | infinite X by blast
  thus ?thesis
  proof cases
    assume finite X k = 0
    hence coeff (sym-mpoly X k) (monom-of-set {}) = 1
      by (subst coeff-sym-mpoly-monom-of-set) auto
    thus ?thesis unfolding monom-of-set-empty using ⟨finite X⟩ ⟨k = 0⟩ by simp
  next
    assume finite X k ≠ 0
    hence ¬(∃ Y. finite Y ∧ Y ⊆ X ∧ card Y = k ∧ monom-of-set Y = 0)
      by auto
    thus ?thesis using ⟨k ≠ 0⟩
      by (auto simp: coeff-sym-mpoly)
  next
    assume infinite X
    thus ?thesis by (simp add: coeff-sym-mpoly)
  qed
qed

```

```

lemma symmetric-sym-mpoly [intro]:
  assumes A ⊆ B
  shows symmetric-mpoly A (sym-mpoly B k :: 'a :: zero-neq-one mpoly)
  unfolding symmetric-mpoly-def
proof (safe intro!: mpoly-eqI)
  fix π and mon :: nat ⇒0 nat assume π: π permutes A
  from π have π!: π permutes B by (rule permutes-subset) fact
  from π have MPoly-Type.coeff (mpoly-map-vars π (sym-mpoly B k :: 'a mpoly))

```

$mon =$
 $MPoly-Type.coeff (sym-mpoly B k :: 'a mpoly) (permutep \pi mon)$
by (*simp add: coeff-mpoly-map-vars permutes-bij*)
also have $\dots = 1 \longleftrightarrow MPoly-Type.coeff (sym-mpoly B k :: 'a mpoly) mon = 1$
(is ?lhs = 1 \longleftrightarrow ?rhs = 1)
proof
assume $?rhs = 1$
then obtain Y **where** *finite B and* $Y: Y \subseteq B \text{ card } Y = k \text{ mon} = \text{monom-of-set } Y$
by (*auto simp: coeff-sym-mpoly split: if-splits*)
with π' **have** $\pi - ' Y \subseteq B \text{ card } (\pi - ' Y) = k \text{ permutep } \pi \text{ mon} = \text{monom-of-set } (\pi - ' Y)$
by (*auto simp: permutes-in-image card-vimage-inj permutep-monom-of-set permutes-bij permutes-inj permutes-surj*)
thus $?lhs = 1$ **using** $\langle \text{finite } B \rangle$ **by** (*auto simp: coeff-sym-mpoly*)
next
assume $?lhs = 1$
then obtain Y **where** *finite B and* $Y: Y \subseteq B \text{ card } Y = k \text{ permutep } \pi \text{ mon} = \text{monom-of-set } Y$
by (*auto simp: coeff-sym-mpoly split: if-splits*)
from $Y(1)$ **have** *inj-on* πY **using** *inj-on-subset*[of $\pi UNIV Y$] π'
by (*auto simp: permutes-inj*)
with $Y \pi'$ **have** $\pi - ' Y \subseteq B \text{ card } (\pi - ' Y) = k \text{ mon} = \text{monom-of-set } (\pi - ' Y)$
by (*auto simp: permutes-in-image card-image permutep-monom-of-set permutes-bij permutes-inj permutes-surj*)
thus $?rhs = 1$ **using** $\langle \text{finite } B \rangle$ **by** (*auto simp: coeff-sym-mpoly*)
qed
hence $?lhs = ?rhs$
by (*auto simp: coeff-sym-mpoly split: if-splits*)
finally show $MPoly-Type.coeff (mpoly-map-vars \pi (sym-mpoly B k :: 'a mpoly))$
 $mon =$
 $MPoly-Type.coeff (sym-mpoly B k :: 'a mpoly) mon .$
qed

lemma *insertion-sym-mpoly:*

assumes *finite X*
shows $\text{insertion } f (sym-mpoly X k) = (\sum Y \mid Y \subseteq X \wedge \text{card } Y = k. \text{prod } f Y)$
using *assms*
proof (*transfer, transfer*)
fix $f :: \text{nat} \Rightarrow 'a$ **and** $k :: \text{nat}$ **and** $X :: \text{nat set}$
assume $X: \text{finite } X$
have *insertion-fun* $f (\lambda mon.$
 $\text{if } \text{finite } X \wedge (\exists Y \subseteq X. \text{card } Y = k \wedge \text{mon} = \text{monom-of-set } Y) \text{ then } 1$
 $\text{else } 0) =$
 $(\sum m. (\prod v. f v \wedge \text{poly-mapping.lookup } m v) \text{ when } (\exists Y \subseteq X. \text{card } Y = k \wedge$
 $m = \text{monom-of-set } Y))$
by (*auto simp add: insertion-fun-def X when-def intro!: Sum-any.cong*)
also have $\dots = (\sum m \mid \exists Y \in \text{Pow } X. \text{card } Y = k \wedge m = \text{monom-of-set } Y. (\prod v.$
 $f v \wedge \text{poly-mapping.lookup } m v) \text{ when } (\exists Y \subseteq X. \text{card } Y = k \wedge m = \text{monom-of-set } Y))$

$Y)$
by (*rule Sum-any.expand-superset*) (*use X in auto*)
also have $\dots = (\sum m \mid \exists Y \in \text{Pow } X. \text{card } Y = k \wedge m = \text{monom-of-set } Y. (\prod v. f v \hat{\ } \text{poly-mapping.lookup } m v))$
by (*intro sum.cong*) (*auto simp: when-def*)
also have $\dots = (\sum Y \mid Y \subseteq X \wedge \text{card } Y = k. (\prod v. f v \hat{\ } \text{poly-mapping.lookup } (\text{monom-of-set } Y) v))$
by (*rule sum.reindex-bij-witness*[*of - monom-of-set keys*]) (*auto simp: finite-subset*[*OF - X*])
also have $\dots = (\sum Y \mid Y \subseteq X \wedge \text{card } Y = k. \prod v \in Y. f v)$
proof (*intro sum.cong when-cong refl, goal-cases*)
case ($1 Y$)
hence *finite Y* **by** (*auto dest: finite-subset*[*OF - X*])
with 1 **have** $(\prod v. f v \hat{\ } \text{poly-mapping.lookup } (\text{monom-of-set } Y) v) =$
 $(\prod v :: \text{nat. if } v \in Y \text{ then } f v \text{ else } 1)$
by (*intro Prod-any.cong*) (*auto simp: lookup-monom-of-set*)
also have $\dots = (\prod v \in Y. f v)$
by (*rule Prod-any.conditionalize* [*symmetric*]) *fact+*
finally show *?case* .
qed
finally show *insertion-fun f*
 $(\lambda \text{mon. if finite } X \wedge (\exists Y \subseteq X. \text{card } Y = k \wedge \text{mon} = \text{monom-of-set } Y) \text{ then } 1 \text{ else } 0) =$
 $(\sum Y \mid Y \subseteq X \wedge \text{card } Y = k. \text{prod } f Y) .$

qed

lemma *sym-mpoly-nz* [*simp*]:
assumes *finite A k ≤ card A*
shows *sym-mpoly A k ≠ (0 :: 'a :: zero-neq-one mpoly)*
proof –
from *assms* **obtain** *B* **where** $B \subseteq A$ *card B = k*
using *ex-subset-of-card* **by** *blast*
with *assms* **have** *coeff (sym-mpoly A k :: 'a mpoly) (monom-of-set B) = 1*
by (*intro coeff-sym-mpoly-monom-of-set*)
thus *?thesis* **by** *auto*

qed

lemma *coeff-sym-mpoly-0-or-1*: *coeff (sym-mpoly A k) m ∈ {0, 1}*
by (*transfer, transfer*) *auto*

lemma *lead-coeff-sym-mpoly* [*simp*]:
assumes *finite A k ≤ card A*
shows *lead-coeff (sym-mpoly A k) = 1*
proof –
from *assms* **have** *lead-coeff (sym-mpoly A k) ≠ 0* **by** *simp*
thus *?thesis* **using** *coeff-sym-mpoly-0-or-1*[*of A k lead-monom (sym-mpoly A k)*]
unfolding *lead-coeff-def* **by** *blast*
qed

lemma *lead-monom-sym-mpoly*:
assumes *sorted xs distinct xs k ≤ length xs*
shows *lead-monom (sym-mpoly (set xs) k :: 'a :: zero-neq-one mpoly) = monom-of-set (set (take k xs)) (is lead-monom ?p = -)*

proof –
let *?m = lead-monom ?p*
have *sym: symmetric-mpoly (set xs) (sym-mpoly (set xs) k)*
by (*intro symmetric-sym-mpoly auto*)
from *assms* **have** [*simp*]: *card (set xs) = length xs*
by (*subst distinct-card auto*)
from *assms* **have** *lead-coeff ?p = 1*
by (*subst lead-coeff-sym-mpoly auto*)
then obtain *X* **where** *X: X ⊆ set xs card X = k ?m = monom-of-set X*
unfolding *lead-coeff-def* **by** (*subst (asm) coeff-sym-mpoly (auto split: if-splits)*)
define *ys* **where** *ys = map (λx. if x ∈ X then 1 else 0 :: nat) xs*
have [*simp*]: *length ys = length xs* **by** (*simp add: ys-def*)

have *ys-altdef: ys = map (lookup ?m) xs*
unfolding *ys-def* **using** *X finite-subset[OF X(1)]*
by (*intro map-cong (auto simp: lookup-monom-of-set)*)

define *i* **where** *i = Min (insert (length xs) {i. i < length xs ∧ ys ! i = 0})*
have *i ≤ length xs* **by** (*auto simp: i-def*)
have *in-X: xs ! j ∈ X if j < i for j*
using *that* **unfolding** *i-def* **by** (*auto simp: ys-def*)
have *not-in-X: xs ! j ∉ X if i ≤ j j < length xs for j*
proof –
have *ne: {i. i < length xs ∧ ys ! i = 0} ≠ {}*
proof
assume [*simp*]: *{i. i < length xs ∧ ys ! i = 0} = {}*
from *that* **show** *False* **by** (*simp add: i-def*)
qed
hence *Min {i. i < length xs ∧ ys ! i = 0} ∈ {i. i < length xs ∧ ys ! i = 0}*
using *that* **by** (*intro Min-in auto*)
also have *Min {i. i < length xs ∧ ys ! i = 0} = i*
unfolding *i-def* **using** *ne* **by** (*subst Min-insert (auto simp: min-def)*)
finally have *i: ys ! i = 0 i < length xs* **by** *simp-all*

have *lookup ?m (xs ! j) ≤ lookup ?m (xs ! i)* **using** *that assms*
by (*intro lookup-lead-monom-decreasing[OF sym]*)
(auto intro!: sorted-nth-mono simp: set-conv-nth)
also have *... = 0* **using** *i* **by** (*simp add: ys-altdef*)
finally show *?thesis* **using** *that X finite-subset[OF X(1)]* **by** (*auto simp: lookup-monom-of-set*)
qed

from *X* **have** *k = card X*
by *simp*
also have *X = (λi. xs ! i) ` {i. i < length xs ∧ xs ! i ∈ X}*

```

    using X by (auto simp: set-conv-nth)
  also have card ... = (∑ i | i < length xs ∧ xs ! i ∈ X. 1)
    using assms by (subst card-image) (auto intro!: inj-on-nth)
  also have ... = (∑ i | i < length xs. if xs ! i ∈ X then 1 else 0)
    by (intro sum.mono-neutral-cong-left) auto
  also have ... = sum-list ys
    by (auto simp: sum-list-sum-nth ys-def intro!: sum.cong)
  also have ys = take i ys @ drop i ys by simp
  also have sum-list ... = sum-list (take i ys) + sum-list (drop i ys)
    by (subst sum-list-append) auto
  also have take i ys = replicate i 1 using ⟨i ≤ length xs⟩ in-X
    by (intro replicate-eqI) (auto simp: ys-def set-conv-nth)
  also have sum-list ... = i by simp
  also have drop i ys = replicate (length ys - i) 0 using ⟨i ≤ length xs⟩ not-in-X
    by (intro replicate-eqI) (auto simp: ys-def set-conv-nth)
  also have sum-list ... = 0 by simp
  finally have i = k by simp

  have X = set (filter (λx. x ∈ X) xs)
    using X by auto
  also have xs = take i xs @ drop i xs by simp
  also note filter-append
  also have filter (λx. x ∈ X) (take i xs) = take i xs
    using in-X by (intro filter-True) (auto simp: set-conv-nth)
  also have filter (λx. x ∈ X) (drop i xs) = []
    using not-in-X by (intro filter-False) (auto simp: set-conv-nth)
  finally have X = set (take i xs) by simp
  with ⟨i = k⟩ and X show ?thesis by simp
qed

```

2.11 Induction on the leading monomial

We show that the monomial ordering for a fixed set of variables is well-founded, so we can perform induction on the leading monomial of a polynomial.

definition *monom-less-on* **where**

$$\text{monom-less-on } A = \{(m1, m2). m1 < m2 \wedge \text{keys } m1 \subseteq A \wedge \text{keys } m2 \subseteq A\}$$

lemma *wf-monom-less-on*:

assumes *finite A*

shows *wf (monom-less-on A :: ((nat ⇒₀ 'b :: {zero, wellorder}) × -) set)*

proof (*rule wf-subset*)

define *n* **where** *n = Suc (Max (insert 0 A))*

have *less-n: k < n if k ∈ A for k*

using *that assms* **by** (*auto simp: n-def less-Suc-eq-le Max-ge-iff*)

define *f* :: *(nat ⇒₀ 'b) ⇒ 'b list* **where** *f = (λm. map (lookup m) [0..<n])*

show *wf (inv-image (lexn {(x,y). x < y} n) f)*

```

  by (intro wf-inv-image wf-learn wellorder-class.wf)
show monom-less-on A ⊆ inv-image (learn {(x, y). x < y} n) f
proof safe
  fix m1 m2 :: nat ⇒₀ 'b assume (m1, m2) ∈ monom-less-on A
  hence m12: m1 < m2 keys m1 ⊆ A keys m2 ⊆ A
  by (auto simp: monom-less-on-def)
  then obtain k where k: lookup m1 k < lookup m2 k ∀ i < k. lookup m1 i =
lookup m2 i
  by (auto simp: less-poly-mapping-def less-fun-def)
  have ¬(lookup m1 k = 0 ∧ lookup m2 k = 0)
  proof (intro notI)
    assume lookup m1 k = 0 ∧ lookup m2 k = 0
    hence [simp]: lookup m1 k = 0 lookup m2 k = 0 by blast+
    from k(1) show False by simp
  qed
  hence k ∈ A using m12 by (auto simp: in-keys-iff)
  hence k < n by (simp add: less-n)

define as where as = map (lookup m1) [0..

```

lemma *lead-monom-induct* [consumes 2, case-names less]:

```

  fixes p :: 'a :: zero mpoly
  assumes fin: finite A and vars: vars p ⊆ A
  assumes IH: ⋀ p'. vars p' ⊆ A ⇒ lead-monom p' < lead-monom p ⇒ P p' ⇒
    (⋀ p'. vars p' ⊆ A ⇒ lead-monom p' < lead-monom p ⇒ P p') ⇒

```

P p

shows *P p*

using *assms(2)*

proof (*induct m* ≡ *lead-monom p arbitrary: p rule: wf-induct-rule[OF wf-monom-less-on[OF fin]]*)

case (1 p)

show ?case

proof (*rule IH*)

fix p' :: 'a mpoly assume *: vars p' ⊆ A lead-monom p' < lead-monom p

```

  show P p'
  by (rule 1) (insert * 1.premis keys-lead-monom-subset, auto simp: monom-less-on-def)
qed (insert 1, auto)
qed

```

```

lemma lead-monom-induct' [case-names less]:
  fixes p :: 'a :: zero mpoly
  assumes IH:  $\bigwedge p. (\bigwedge p'. \text{vars } p' \subseteq \text{vars } p \implies \text{lead-monom } p' < \text{lead-monom } p \implies P p') \implies P p$ 
  shows P p
proof -
  have finite (vars p) vars p  $\subseteq$  vars p by (auto simp: vars-finite)
  thus ?thesis
    by (induction rule: lead-monom-induct) (use IH in blast)
qed

```

2.12 The fundamental theorem of symmetric polynomials

```

lemma lead-coeff-sym-mpoly-powerprod:
  assumes finite A  $\bigwedge x. x \in X \implies f x \in \{1..card A\}$ 
  shows lead-coeff ( $\prod_{x \in X}. \text{sym-mpoly } A (f (x::'a)) \wedge g x$ ) = 1
proof -
  have eq: lead-coeff (sym-mpoly A (f x)  $\wedge$  g x :: 'b mpoly) = 1 if x  $\in$  X for x
    using that assms by (subst lead-coeff-power) (auto simp: lead-coeff-sym-mpoly assms)
  hence ( $\prod_{x \in X}. \text{lead-coeff } (\text{sym-mpoly } A (f x) \wedge g x :: 'b \text{ mpoly})$ ) = ( $\prod_{x \in X}. 1$ )
    by (intro prod.cong eq refl)
  also have ... = 1 by simp
  finally have eq': ( $\prod_{x \in X}. \text{lead-coeff } (\text{sym-mpoly } A (f x) \wedge g x :: 'b \text{ mpoly})$ ) = 1 .
  show ?thesis by (subst lead-coeff-prod) (auto simp: eq eq')
qed

```

```

context
  fixes A :: nat set and xs n f and decr :: 'a :: comm-ring-1 mpoly  $\Rightarrow$  bool
  defines xs  $\equiv$  sorted-list-of-set A
  defines n  $\equiv$  card A
  defines f  $\equiv$  ( $\lambda i. \text{if } i < n \text{ then } xs ! i \text{ else } 0$ )
  defines decr  $\equiv$  ( $\lambda p. \forall i \in A. \forall j \in A. i \leq j \longrightarrow \text{lookup } (\text{lead-monom } p) i \geq \text{lookup } (\text{lead-monom } p) j$ )
begin

```

The computation of the witness for the fundamental theorem works like this: Given some polynomial p (that is assumed to be symmetric in the variables in A), we inspect its leading monomial, which is of the form $cX_1^{i_1} \dots X_n^{i_n}$ where the $A = \{X_1, \dots, X_n\}$, c contains only variables not in A , and the sequence i_j is decreasing. The latter holds because p is symmetric.

Now, we form the polynomial $q := ce_1^{i_1-i_2} e_2^{i_2-i_3} \dots e_n^{i_n}$, which has the same leading term as p . Then $p - q$ has a smaller leading monomial, so by induction, we can assume it to be of the required form and obtain a witness for

$p - q$.

Now, we only need to add $cY_1^{i_1-i_2} \dots Y_n^{i_n}$ to that witness and we obtain a witness for p .

definition *fund-sym-step-coeff* :: 'a mpoly \Rightarrow 'a mpoly **where**
fund-sym-step-coeff $p = \text{monom } (\text{restrictpm } (-A) (\text{lead-monom } p)) (\text{lead-coeff } p)$

definition *fund-sym-step-monom* :: 'a mpoly \Rightarrow (nat \Rightarrow_0 nat) **where**
fund-sym-step-monom $p =$ (
 let $g = (\lambda i. \text{if } i < n \text{ then lookup } (\text{lead-monom } p) (f \ i) \text{ else } 0)$
 in $(\sum_{i < n}. \text{Poly-Mapping.single } (\text{Suc } i) (g \ i - g (\text{Suc } i)))$)

definition *fund-sym-step-poly* :: 'a mpoly \Rightarrow 'a mpoly **where**
fund-sym-step-poly $p =$ (
 let $g = (\lambda i. \text{if } i < n \text{ then lookup } (\text{lead-monom } p) (f \ i) \text{ else } 0)$
 in $\text{fund-sym-step-coeff } p * (\prod_{i < n}. \text{sym-mpoly } A (\text{Suc } i) \wedge (g \ i - g (\text{Suc } i)))$)

The following function computes the witness, with the convention that it returns a constant polynomial if the input was not symmetric:

function (*domintros*) *fund-sym-poly-wit* :: 'a :: comm-ring-1 mpoly \Rightarrow 'a mpoly mpoly **where**
fund-sym-poly-wit $p =$
 (if $\neg \text{symmetric-mpoly } A \ p \vee \text{lead-monom } p = 0 \vee \text{vars } p \cap A = \{\}$ then *Const* p else
 $\text{fund-sym-poly-wit } (p - \text{fund-sym-step-poly } p) +$
 $\text{monom } (\text{fund-sym-step-monom } p) (\text{fund-sym-step-coeff } p)$)
by *auto*

lemma *coeff-fund-sym-step-coeff*: *coeff* (*fund-sym-step-coeff* p) $m \in \{\text{lead-coeff } p, 0\}$
by (*auto simp: fund-sym-step-coeff-def coeff-monom when-def*)

lemma *vars-fund-sym-step-coeff*: *vars* (*fund-sym-step-coeff* p) $\subseteq \text{vars } p - A$
unfolding *fund-sym-step-coeff-def* **using** *keys-lead-monom-subset[of p]*
by (*intro order.trans[OF vars-monom-subset]*) *auto*

lemma *keys-fund-sym-step-monom*: *keys* (*fund-sym-step-monom* p) $\subseteq \{1..n\}$
unfolding *fund-sym-step-monom-def* *Let-def*
by (*intro order.trans[OF keys-sum] UN-least, subst keys-single*) *auto*

lemma *coeff-fund-sym-step-poly*:
assumes $C: \forall m. \text{coeff } p \ m \in C$ **and** *ring-closed* C
shows $\text{coeff } (\text{fund-sym-step-poly } p) \ m \in C$
proof –
interpret *ring-closed* C **by** *fact*
have *: $\bigwedge m. \text{coeff } (p \wedge x) \ m \in C$ **if** $\bigwedge m. \text{coeff } p \ m \in C$ **for** $p \ x$
using *that by (induction x)*
 (*auto simp: coeff-mpoly-times mpoly-coeff-1 intro!: prod-fun-closed*)
have **: $\bigwedge m. \text{coeff } (\text{prod } f \ X) \ m \in C$ **if** $\bigwedge i \ m. i \in X \implies \text{coeff } (f \ i) \ m \in C$


```

for  $X$  and  $f :: nat \Rightarrow -$ 
using that by (induction X rule: infinite-finite-induct)
      (auto simp: coeff-mpoly-times mpoly-coeff-1 intro!: prod-fun-closed)
show ?thesis using C
unfolding fund-sym-step-poly-def Let-def fund-sym-step-coeff-def coeff-mpoly-times
by (intro prod-fun-closed)
      (auto simp: coeff-monom when-def lead-coeff-def coeff-sym-mpoly intro!: * **)
qed

```

We now show various relevant properties of the subtracted polynomial:

1. Its leading term is the same as that of the input polynomial.
2. It contains now new variables.
3. It is symmetric in the variables in A .

lemma *fund-sym-step-poly:*

```

shows finite A  $\implies p \neq 0 \implies \text{decr } p \implies \text{lead-monom (fund-sym-step-poly } p)$ 
= lead-monom p
and finite A  $\implies p \neq 0 \implies \text{decr } p \implies \text{lead-coeff (fund-sym-step-poly } p)$ 
= lead-coeff p
and finite A  $\implies p \neq 0 \implies \text{decr } p \implies \text{fund-sym-step-poly } p =$ 
fund-sym-step-coeff p * ( $\prod x. \text{sym-mpoly } A x \hat{\text{lookup (fund-sym-step-monom$ 
p) x})
and vars (fund-sym-step-poly p)  $\subseteq$  vars p  $\cup$  A
and symmetric-mpoly A (fund-sym-step-poly p)

```

proof –

```

define  $g$  where  $g = (\lambda i. \text{if } i < n \text{ then lookup (lead-monom } p) (f i) \text{ else } 0)$ 
define  $q$  where  $q = (\prod i < n. \text{sym-mpoly } A (\text{Suc } i) \hat{(g i - g (\text{Suc } i))} :: 'a \text{ mpoly})$ 
define  $c$  where  $c = \text{monom (restrictpm } (-A) (\text{lead-monom } p)) (\text{lead-coeff } p)$ 
have [simp]: fund-sym-step-poly p = c * q
by (simp add: fund-sym-step-poly-def fund-sym-step-coeff-def c-def q-def f-def g-def)

```

```

have vars (c * q)  $\subseteq$  vars p  $\cup$  A
using keys-lead-monom-subset[of p]
      vars-monom-subset[of restrictpm } (-A) (\text{lead-monom } p) \text{lead-coeff } p]
unfolding c-def q-def
by (intro order.trans[OF vars-mult] order.trans[OF vars-prod] order.trans[OF vars-power]
      Un-least UN-least order.trans[OF vars-sym-mpoly-subset]) auto
thus vars (fund-sym-step-poly p)  $\subseteq$  vars p  $\cup$  A
by simp
have symmetric-mpoly A (c * q) unfolding c-def q-def
by (intro symmetric-mpoly-mult symmetric-mpoly-monom symmetric-mpoly-prod
      symmetric-mpoly-power symmetric-sym-mpoly) auto
thus symmetric-mpoly A (fund-sym-step-poly p) by simp

```

```

assume finite: finite A and [simp]:  $p \neq 0$  and decr p
have set xs = A distinct xs and [simp]: length xs = n
  using finite by (auto simp: xs-def n-def)
  have [simp]: lead-coeff c = lead-coeff p lead-monom c = restrictpm (- A)
(lead-monom p)
  by (simp-all add: c-def lead-monom-monom)
hence f-range [simp]:  $f\ i \in A$  if  $i < n$  for  $i$ 
  using that (set xs = A) by (auto simp: f-def set-conv-nth)
have sorted xs by (simp add: xs-def)
hence f-mono:  $f\ i \leq f\ j$  if  $i \leq j$   $j < n$  for  $i\ j$  using that
  by (auto simp: f-def n-def intro: sorted-nth-mono)
hence g-mono:  $g\ i \geq g\ j$  if  $i \leq j$  for  $i\ j$ 
  unfolding g-def using that using (decr p) by (auto simp: decr-def)

have *: ( $\prod_{i < n}.$  lead-coeff (sym-mpoly A (Suc i)  $\wedge$  ( $g\ i - g\ (Suc\ i)$ ) :: 'a mpoly))
=
  ( $\prod_{i < \text{card}\ A}.$  1)
  using (finite A) by (intro prod.cong) (auto simp: n-def lead-coeff-power)
hence lead-coeff q = ( $\prod_{i < n}.$  lead-coeff (sym-mpoly A (Suc i)  $\wedge$  ( $g\ i - g\ (Suc\ i)$ ) :: 'a mpoly))
  by (simp add: lead-coeff-prod lead-coeff-power n-def q-def)
also have ... = ( $\prod_{i < n}.$  1)
  using (finite A) by (intro prod.cong) (auto simp: lead-coeff-power n-def)
finally have [simp]: lead-coeff q = 1 by simp

have lead-monom q = ( $\sum_{i < n}.$  lead-monom (sym-mpoly A (Suc i)  $\wedge$  ( $g\ i - g\ (Suc\ i)$ ) :: 'a mpoly))
  using * by (simp add: q-def lead-monom-prod lead-coeff-power n-def)
also have ... = ( $\sum_{i < n}.$  of-nat ( $g\ i - g\ (Suc\ i)$ ) * lead-monom (sym-mpoly A (Suc i) :: 'a mpoly))
  using (finite A) by (intro sum.cong) (auto simp: lead-monom-power n-def)
also have ... = ( $\sum_{i < n}.$  of-nat ( $g\ i - g\ (Suc\ i)$ ) * monom-of-set (set (take (Suc i) xs)))
  proof (intro sum.cong refl, goal-cases)
  case (1 i)
  have lead-monom (sym-mpoly A (Suc i) :: 'a mpoly) =
    lead-monom (sym-mpoly (set xs) (Suc i) :: 'a mpoly)
    by (simp add: (set xs = A))
  also from 1 have ... = monom-of-set (set (take (Suc i) xs))
    by (subst lead-monom-sym-mpoly) (auto simp: xs-def n-def)
  finally show ?case by simp
qed
finally have lead-monom-q:
  lead-monom q = ( $\sum_{i < n}.$  of-nat ( $g\ i - g\ (Suc\ i)$ ) * monom-of-set (set (take (Suc i) xs))) .

have lead-monom (c * q) = lead-monom c + lead-monom q
  by (simp add: lead-monom-mult)
also have ... = lead-monom p (is ?S = -)

```

```

proof (intro poly-mapping-eqI)
  fix i :: nat
  show lookup (lead-monom c + lead-monom q) i = lookup (lead-monom p) i
  proof (cases i ∈ A)
    case False
      hence lookup (lead-monom c + lead-monom q) i = lookup (lead-monom p) i
+
      (∑ j < n. (g j - g (Suc j)) * lookup (monom-of-set (set (take (Suc j)
xs))) i)
      (is - = - + ?S) by (simp add: lookup-add lead-monom-q lookup-sum)
      also from False have ?S = 0
      by (intro sum.neutral) (auto simp: lookup-monom-of-set ⟨set xs = A⟩ dest!:
in-set-takeD)
      finally show ?thesis by simp
  next
    case True
      with ⟨set xs = A⟩ obtain m where m: i = xs ! m m < n
      by (auto simp: set-conv-nth)
      have lookup (lead-monom c + lead-monom q) i =
      (∑ j < n. (g j - g (Suc j)) * lookup (monom-of-set (set (take (Suc j)
xs))) i)
      using True by (simp add: lookup-add lookup-sum lead-monom-q)
      also have ... = (∑ j | j < n ∧ i ∈ set (take (Suc j) xs). g j - g (Suc j))
      by (intro sum.mono-neutral-cong-right) auto
      also have {j. j < n ∧ i ∈ set (take (Suc j) xs)} = {m..<n}
      using m ⟨distinct xs⟩ by (force simp: set-conv-nth nth-eq-iff-index-eq)
      also have (∑ j ∈ ... g j - g (Suc j)) = (∑ j ∈ ... g j) - (∑ j ∈ ... g (Suc j))
      by (subst sum-subtractf-nat) (auto intro!: g-mono)
      also have (∑ j ∈ {m..<n}. g (Suc j)) = (∑ j ∈ {m<..n}. g j)
      by (intro sum.reindex-bij-witness[of - λj. j - 1 Suc]) auto
      also have ... = (∑ j ∈ {m<..<n}. g j)
      by (intro sum.mono-neutral-right) (auto simp: g-def)
      also have (∑ j ∈ {m..<n}. g j) - ... = (∑ j ∈ {m..<n} - {m<..<n}. g j)
      by (intro sum-diff-nat [symmetric]) auto
      also have {m..<n} - {m<..<n} = {m} using m by auto
      also have (∑ j ∈ ... g j) = lookup (lead-monom p) i
      using m by (auto simp: g-def not-less le-Suc-eq f-def)
      finally show ?thesis .
  qed
qed
finally show lead-monom (fund-sym-step-poly p) = lead-monom p by simp
show lead-coeff (fund-sym-step-poly p) = lead-coeff p
  by (simp add: lead-coeff-mult)

have *: lookup (fund-sym-step-monom p) k = (if k ∈ {1..n} then g (k - 1) - g
k else 0) for k
proof -
  have lookup (fund-sym-step-monom p) k =
  (∑ x ∈ (if k ∈ {1..n} then {k - 1} else {}). g (k - 1) - g k)

```

```

unfolding fund-sym-step-monom-def lookup-sum Let-def
by (intro sum.mono-neutral-cong-right)
      (auto simp: g-def lookup-single when-def split: if-splits)
thus ?thesis by simp
qed
hence ( $\prod x. \text{sym-mpoly } A \ x \ \hat{\ } \text{lookup } (\text{fund-sym-step-monom } p) \ x \ :: \ 'a \ \text{mpoly}$ ) =
      ( $\prod x \in \{1..n\}. \text{sym-mpoly } A \ x \ \hat{\ } \text{lookup } (\text{fund-sym-step-monom } p) \ x$ )
by (intro Prod-any.expand-superset) auto
also have ... = ( $\prod x < n. \text{sym-mpoly } A \ (\text{Suc } x) \ \hat{\ } \text{lookup } (\text{fund-sym-step-monom } p) \ (\text{Suc } x)$ )
by (intro prod.reindex-bij-witness[of - Suc  $\lambda i. i - 1$ ]) auto
also have ... = q
unfolding q-def by (intro prod.cong) (auto simp: *)
finally show fund-sym-step-poly p =
      fund-sym-step-coeff p * ( $\prod x. \text{sym-mpoly } A \ x \ \hat{\ } \text{lookup } (\text{fund-sym-step-monom } p) \ x$ )
by (simp add: c-def q-def f-def g-def fund-sym-step-monom-def fund-sym-step-coeff-def)
qed

```

If the input is well-formed, a single step of the procedure always decreases the leading monomial.

lemma *lead-monom-fund-sym-step-poly-less*:

```

assumes finite A and lead-monom p  $\neq$  0 and decr p
shows lead-monom (p - fund-sym-step-poly p) < lead-monom p
proof (cases p = fund-sym-step-poly p)
case True
thus ?thesis using assms by (auto simp: order.strict-iff-order)
next
case False
from assms have [simp]: p  $\neq$  0 by auto
let ?q = fund-sym-step-poly p and ?m = lead-monom p
have coeff (p - ?q) ?m = 0
using fund-sym-step-poly[of p] assms by (simp add: lead-coeff-def)
moreover have lead-coeff (p - ?q)  $\neq$  0 using False by auto
ultimately have lead-monom (p - ?q)  $\neq$  ?m
unfolding lead-coeff-def by auto
moreover have lead-monom (p - ?q)  $\leq$  ?m
using fund-sym-step-poly[of p] assms
by (intro order.trans[OF lead-monom-diff] max.boundedI) auto
ultimately show ?thesis by (auto simp: order.strict-iff-order)
qed

```

Finally, we prove that the witness is indeed well-defined for all inputs.

lemma *fund-sym-poly-wit-dom-aux*:

```

assumes finite B vars p  $\subseteq$  B A  $\subseteq$  B
shows fund-sym-poly-wit-dom p
using assms(1-3)
proof (induction p rule: lead-monom-induct)
case (less p)

```

```

have [simp]: finite A by (rule finite-subset[of - B]) fact+
show ?case
proof (cases lead-monom p = 0  $\vee$   $\neg$ symmetric-mpoly A p)
  case False
  hence [simp]: p  $\neq$  0 by auto
  note decr = lookup-lead-monom-decreasing[of A p]
  have vars (p - fund-sym-step-poly p)  $\subseteq$  B
    using fund-sym-step-poly[of p] decr False less.premis less.hyps  $\langle A \subseteq B \rangle$ 
    by (intro order.trans[OF vars-diff]) auto
  hence fund-sym-poly-wit-dom (p - local.fund-sym-step-poly p)
    using False less.premis less.hyps decr
    by (intro less.IH fund-sym-step-poly symmetric-mpoly-diff
      lead-monom-fund-sym-step-poly-less) (auto simp: decr-def)
  thus ?thesis using fund-sym-poly-wit.domintros by blast
qed (auto intro: fund-sym-poly-wit.domintros)
qed

```

```

lemma fund-sym-poly-wit-dom [intro]: fund-sym-poly-wit-dom p
proof -
  consider  $\neg$ symmetric-mpoly A p | vars p  $\cap$  A = {} | symmetric-mpoly A p A  $\subseteq$ 
  vars p
  using symmetric-mpoly-imp-orthogonal-or-subset[of A p] by blast
  thus ?thesis
proof cases
  assume symmetric-mpoly A p A  $\subseteq$  vars p
  thus ?thesis using fund-sym-poly-wit-dom-aux[of vars p p] by (auto simp:
  vars-finite)
qed (auto intro: fund-sym-poly-wit.domintros)
qed

```

```

termination fund-sym-poly-wit
by (intro allI fund-sym-poly-wit-dom)

```

Next, we prove that our witness indeed fulfils all the properties stated by the fundamental theorem:

1. If the original polynomial was in $R[X_1, \dots, X_n, \dots, X_m]$ where the X_1 to X_n are the symmetric variables, then the witness is a polynomial in $R[X_{n+1}, \dots, X_m][Y_1, \dots, Y_n]$. This means that its coefficients are polynomials in the variables of the original polynomial, minus the symmetric ones, and the (new and independent) variables of the witness polynomial range from 1 to n .
2. Substituting the i -th symmetric polynomial $e_i(X_1, \dots, X_n)$ for the Y_i variable for every i yields the original polynomial.
3. The coefficient ring R need not be the entire type; if the coefficients of the original polynomial are in some subring, then the coefficients of

the coefficients of the witness also do.

lemma *fund-sym-poly-wit-coeffs-aux*:
assumes *finite B vars p ⊆ B symmetric-mpoly A p A ⊆ B*
shows *vars (coeff (fund-sym-poly-wit p) m) ⊆ B - A*
using *assms*
proof (*induction p rule: fund-sym-poly-wit.induct*)
case (*1 p*)
show *?case*
proof (*cases lead-monom p = 0 ∨ vars p ∩ A = {}*)
case *False*
have *vars (p - fund-sym-step-poly p) ⊆ B*
using *1.premis fund-sym-step-poly[of p] by (intro order.trans[OF vars-diff])*
auto
with *1 False* **have** *vars (coeff (fund-sym-poly-wit (p - fund-sym-step-poly p)) m) ⊆ B - A*
by (*intro 1 symmetric-mpoly-diff fund-sym-step-poly*) *auto*
hence *vars (coeff (fund-sym-poly-wit (p - fund-sym-step-poly p) + monom (fund-sym-step-monom p) (fund-sym-step-coeff p)) m) ⊆ B - A*
unfolding *coeff-add coeff-monom* **using** *vars-fund-sym-step-coeff[of p] 1.premis*
by (*intro order.trans[OF vars-add] Un-least order.trans[OF vars-monom-subset]*)
(auto simp: when-def)
thus *?thesis* **using** *1.premis False* **unfolding** *fund-sym-poly-wit.simps[of p]* **by**
simp
qed (*insert 1.premis,*
auto simp: fund-sym-poly-wit.simps[of p] mpoly-coeff-Const lead-monom-eq-0-iff)
qed

lemma *fund-sym-poly-wit-coeffs*:
assumes *symmetric-mpoly A p*
shows *vars (coeff (fund-sym-poly-wit p) m) ⊆ vars p - A*
proof (*cases A ⊆ vars p*)
case *True*
with *fund-sym-poly-wit-coeffs-aux[of vars p p m] assms*
show *?thesis* **by** (*auto simp: vars-finite*)
next
case *False*
hence *vars p ∩ A = {}*
using *symmetric-mpoly-imp-orthogonal-or-subset[OF assms]* **by** *auto*
thus *?thesis* **by** (*auto simp: fund-sym-poly-wit.simps[of p] mpoly-coeff-Const*)
qed

lemma *fund-sym-poly-wit-vars: vars (fund-sym-poly-wit p) ⊆ {1..n}*
proof (*cases symmetric-mpoly A p ∧ A ⊆ vars p*)
case *True*
define *B* **where** *B = vars p*
have *finite B vars p ⊆ B symmetric-mpoly A p A ⊆ B*
using *True* **unfolding** *B-def* **by** (*auto simp: vars-finite*)

```

thus ?thesis
proof (induction p rule: fund-sym-poly-wit.induct)
  case (1 p)
  show ?case
  proof (cases lead-monom p = 0  $\vee$  vars p  $\cap$  A = {})
    case False
    have vars (p - fund-sym-step-poly p)  $\subseteq$  B
      using 1.prem1 fund-sym-step-poly[of p] by (intro order.trans[OF vars-diff])
  auto
  hence vars (local.fund-sym-poly-wit (p - local.fund-sym-step-poly p))  $\subseteq$  {1..n}
    using False 1.prem1
  by (intro 1.symmetric-mpoly-diff fund-sym-step-poly) (auto simp: lead-monom-eq-0-iff)
  hence vars (fund-sym-poly-wit (p - fund-sym-step-poly p) +
    monom (fund-sym-step-monom p) (local.fund-sym-step-coeff p))  $\subseteq$  {1..n}
  by (intro order.trans[OF vars-add] Un-least order.trans[OF vars-monom-subset]
    keys-fund-sym-step-monom) auto
  thus ?thesis using 1.prem1 False unfolding fund-sym-poly-wit.simps[of p]
by simp
  qed (insert 1.prem1,
    auto simp: fund-sym-poly-wit.simps[of p] mpoly-coeff-Const lead-monom-eq-0-iff)
  qed
next
  case False
  then consider  $\neg$ symmetric-mpoly A p | symmetric-mpoly A p vars p  $\cap$  A = {}
    using symmetric-mpoly-imp-orthogonal-or-subset[of A p] by auto
  thus ?thesis
    by cases (auto simp: fund-sym-poly-wit.simps[of p])
qed

lemma fund-sym-poly-wit-insertion-aux:
  assumes finite B vars p  $\subseteq$  B symmetric-mpoly A p A  $\subseteq$  B
  shows insertion (sym-mpoly A) (fund-sym-poly-wit p) = p
  using assms
proof (induction p rule: fund-sym-poly-wit.induct)
  case (1 p)
  from 1.prem1 have decr p
    using lookup-lead-monom-decreasing[of A p] by (auto simp: decr-def)
  show ?case
  proof (cases lead-monom p = 0  $\vee$  vars p  $\cap$  A = {})
    case False
    have vars (p - fund-sym-step-poly p)  $\subseteq$  B
      using 1.prem1 fund-sym-step-poly[of p] by (intro order.trans[OF vars-diff])
  auto
  hence insertion (sym-mpoly A) (fund-sym-poly-wit (p - fund-sym-step-poly p))
  =
    p - fund-sym-step-poly p using 1 False
  by (intro 1.symmetric-mpoly-diff fund-sym-step-poly) auto
  moreover have fund-sym-step-poly p =
    fund-sym-step-coeff p * ( $\prod$  x. sym-mpoly A x  $\wedge$  lookup

```

```

(fund-sym-step-monom p) x)
  using 1.prem finite-subset[of A B] False (decr p) by (intro fund-sym-step-poly)
auto
  ultimately show ?thesis
    unfolding fund-sym-poly-wit.simps[of p] by (auto simp: insertion-add)
  qed (auto simp: fund-sym-poly-wit.simps[of p])
qed

```

```

lemma fund-sym-poly-wit-insertion:
  assumes symmetric-mpoly A p
  shows insertion (sym-mpoly A) (fund-sym-poly-wit p) = p
proof (cases A  $\subseteq$  vars p)
  case False
  hence vars p  $\cap$  A = {}
  using symmetric-mpoly-imp-orthogonal-or-subset[OF assms] by auto
  thus ?thesis
    by (auto simp: fund-sym-poly-wit.simps[of p])
next
  case True
  with fund-sym-poly-wit-insertion-aux[of vars p p] assms show ?thesis
    by (auto simp: vars-finite)
qed

```

```

lemma fund-sym-poly-wit-coeff:
  assumes  $\forall m. \text{coeff } p \ m \in C$  ring-closed C
  shows  $\forall m \ m'. \text{coeff } (\text{coeff } (\text{fund-sym-poly-wit } p) \ m) \ m' \in C$ 
  using assms(1)
proof (induction p rule: fund-sym-poly-wit.induct)
  case (1 p)
  interpret ring-closed C by fact
  show ?case
  proof (cases  $\neg$ symmetric-mpoly A p  $\vee$  lead-monom p = 0  $\vee$  vars p  $\cap$  A = {})
    case True
    thus ?thesis using 1.prem
      by (auto simp: fund-sym-poly-wit.simps[of p] mpoly-coeff-Const)
  next
    case False
    have *:  $\forall m \ m'. \text{coeff } (\text{coeff } (\text{fund-sym-poly-wit } (p - \text{fund-sym-step-poly } p)) \ m) \ m' \in C$ 
    using False 1.prem assms coeff-fund-sym-step-poly [of p] by (intro 1) auto
    show ?thesis
    proof (intro allI, goal-cases)
      case (1 m m')
      thus ?case using * False coeff-fund-sym-step-coeff[of p m'] 1.prem
        by (auto simp: fund-sym-poly-wit.simps[of p] coeff-monom lead-coeff-def
when-def)
    qed
  qed
qed

```


2.13 Uniqueness

Next, we show that the polynomial representation of a symmetric polynomial in terms of the elementary symmetric polynomials not only exists, but is unique.

The key property here is that products of powers of elementary symmetric polynomials uniquely determine the exponent vectors, i. e. if e_1, \dots, e_n are the elementary symmetric polynomials, $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are vectors of natural numbers, then:

$$e_1^{a_1} \dots e_n^{a_n} = e_1^{b_1} \dots e_n^{b_n} \iff a = b$$

We show this now.

lemma *lead-monom-sym-mpoly-prod*:

assumes *finite A*

shows $\text{lead-monom } (\prod i = 1..n. \text{sym-mpoly } A \ i \ ^h \ i :: 'a \ \text{mpoly}) =$
 $(\sum i = 1..n. \text{of-nat } (h \ i) * \text{lead-monom } (\text{sym-mpoly } A \ i :: 'a \ \text{mpoly}))$

proof –

have $(\prod i = 1..n. \text{lead-coeff } (\text{sym-mpoly } A \ i \ ^h \ i :: 'a \ \text{mpoly})) = 1$

using *assms unfolding n-def by (intro prod.neutral allI) (auto simp: lead-coeff-power)*

hence $\text{lead-monom } (\prod i = 1..n. \text{sym-mpoly } A \ i \ ^h \ i :: 'a \ \text{mpoly}) =$
 $(\sum i = 1..n. \text{lead-monom } (\text{sym-mpoly } A \ i \ ^h \ i :: 'a \ \text{mpoly}))$

by *(subst lead-monom-prod) auto*

also have $\dots = (\sum i = 1..n. \text{of-nat } (h \ i) * \text{lead-monom } (\text{sym-mpoly } A \ i :: 'a \ \text{mpoly}))$

by *(intro sum.cong refl, subst lead-monom-power)*

(auto simp: lead-coeff-power assms n-def)

finally show *?thesis .*

qed

lemma *lead-monom-sym-mpoly-prod-notin*:

assumes *finite A k ∉ A*

shows $\text{lookup } (\text{lead-monom } (\prod i = 1..n. \text{sym-mpoly } A \ i \ ^h \ i :: 'a \ \text{mpoly})) \ k = 0$

proof –

have *xs: set xs = A distinct xs sorted xs and [simp]: length xs = n*

using *assms by (auto simp: xs-def n-def)*

have $\text{lead-monom } (\prod i = 1..n. \text{sym-mpoly } A \ i \ ^h \ i :: 'a \ \text{mpoly}) =$
 $(\sum i = 1..n. \text{of-nat } (h \ i) * \text{lead-monom } (\text{sym-mpoly } (\text{set } xs) \ i :: 'a \ \text{mpoly}))$

by *(subst lead-monom-sym-mpoly-prod) (use xs assms in auto)*

also have $\text{lookup } \dots \ k = 0$ **unfolding** *lookup-sum*

by *(intro sum.neutral ballI, subst lead-monom-sym-mpoly)*

(insert xs assms, auto simp: xs lead-monom-sym-mpoly lookup-monom-of-set set-conv-nth)

finally show *?thesis .*

qed

lemma *lead-monom-sym-mpoly-prod-in*:

assumes *finite A k < n*

shows $\text{lookup } (\text{lead-monom } (\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^\wedge \ h \ i :: 'a \ \text{mpoly})) \ (xs \ ! \ k) =$
 $(\sum_{i=k+1..n} \ h \ i)$

proof –

have xs : *set* $xs = A$ *distinct* xs *sorted* xs **and** $[simp]$: $\text{length } xs = n$
using $assms$ **by** $(\text{auto } simp: xs\text{-def } n\text{-def})$

have $\text{lead-monom } (\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^\wedge \ h \ i :: 'a \ \text{mpoly}) =$
 $(\sum_{i=1..n} \ \text{of-nat } (h \ i) * \text{lead-monom } (\text{sym-mpoly } (\text{set } xs) \ i :: 'a \ \text{mpoly}))$
by $(\text{subst } \text{lead-monom-sym-mpoly-prod})$ $(\text{use } xs \ \text{assms } \text{in } simp\text{-all})$

also have $\dots = (\sum_{i=1..n} \ \text{of-nat } (h \ i) * \text{monom-of-set } (\text{set } (\text{take } i \ xs)))$
using xs **by** $(\text{intro } sum.\text{cong } refl, \text{subst } \text{lead-monom-sym-mpoly})$ auto

also have $\text{lookup } \dots \ (xs \ ! \ k) = (\sum_{i \mid i \in \{1..n\} \wedge xs \ ! \ k \in \text{set } (\text{take } i \ xs)}. \ h \ i)$
unfolding $\text{lookup-sum } \text{lookup-monom-of-set}$ **by** $(\text{intro } sum.\text{mono-neutral-cong-right})$

auto

also have $\{i. \ i \in \{1..n\} \wedge xs \ ! \ k \in \text{set } (\text{take } i \ xs)\} = \{k+1..n\}$

proof $(\text{intro } equalityI \ \text{subsetI})$

fix i **assume** $i: \ i \in \{k+1..n\}$

hence $\text{take } i \ xs \ ! \ k = xs \ ! \ k \ k < n \ k < i$ **using** $assms$

by auto

with i **show** $i \in \{i. \ i \in \{1..n\} \wedge xs \ ! \ k \in \text{set } (\text{take } i \ xs)\}$

by $(\text{force } simp: \text{set-conv-nth})$

qed $(\text{insert } assms \ xs, \ \text{auto } simp: \text{set-conv-nth } Suc\text{-le-eq } nth\text{-eq-iff-index-eq})$

finally show $?thesis$.

qed

lemma *lead-monom-sym-poly-powerprod-inj*:

assumes $\text{lead-monom } (\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^\wedge \ \text{lookup } m1 \ i :: 'a \ \text{mpoly}) =$
 $\text{lead-monom } (\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^\wedge \ \text{lookup } m2 \ i :: 'a \ \text{mpoly})$

assumes $\text{finite } A$ $\text{keys } m1 \subseteq \{1..n\}$ $\text{keys } m2 \subseteq \{1..n\}$

shows $m1 = m2$

proof $(\text{rule } poly\text{-mapping-eqI})$

fix $k :: nat$

have xs : *set* $xs = A$ *distinct* xs *sorted* xs **and** $[simp]$: $\text{length } xs = n$
using $assms$ **by** $(\text{auto } simp: xs\text{-def } n\text{-def})$

from $assms(3,4)$ **have** $*$: $i \in \{1..n\}$ **if** $\text{lookup } m1 \ i \neq 0 \vee \text{lookup } m2 \ i \neq 0$ **for** i
using *that* **by** $(\text{auto } simp: \text{subset-iff } in\text{-keys-iff})$

have $**$: $(\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^\wedge \ \text{lookup } m \ i :: 'a \ \text{mpoly}) =$
 $(\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^\wedge \ \text{lookup } m \ i :: 'a \ \text{mpoly})$ **if** $m \in \{m1, m2\}$

for m

using *that* $*$ **by** $(\text{intro } Prod.\text{any.expand-superset } \text{subsetI } *)$ $(\text{auto } \text{intro}!: \text{Nat.gr0I})$

have $***$: $\text{lead-monom } (\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^\wedge \ \text{lookup } m1 \ i :: 'a \ \text{mpoly}) =$
 $\text{lead-monom } (\prod_{i=1..n} \text{sym-mpoly } A \ i \ ^\wedge \ \text{lookup } m2 \ i :: 'a \ \text{mpoly})$
using $assms$ **by** $(simp \ \text{add}: **)$

have $sum\text{-eq}$: $\text{sum } (\text{lookup } m1) \ \{Suc \ k..n\} = \text{sum } (\text{lookup } m2) \ \{Suc \ k..n\}$ **if** $k < n$ **for** k

using $arg\text{-cong}[OF \ ***, \ \text{of } \lambda m. \ \text{lookup } m \ (xs \ ! \ k)]$ $(\text{finite } A)$ *that*

```

by (subst (asm) (1 2) lead-monom-sym-mpoly-prod-in) auto

show lookup m1 k = lookup m2 k
proof (cases k ∈ {1..n})
  case False
  hence lookup m1 k = 0 lookup m2 k = 0 using assms by (auto simp: in-keys-iff)
  thus ?thesis by simp
next
  case True
  thus ?thesis
proof (induction n - k arbitrary: k rule: less-induct)
  case (less l)
  have sum (lookup m1) {Suc (l - 1)..n} = sum (lookup m2) {Suc (l - 1)..n}
  using less by (intro sum-eq) auto
  also have {Suc (l - 1)..n} = insert l {Suc l..n}
  using less by auto
  also have sum (lookup m1) ... = lookup m1 l + (∑ i=Suc l..n. lookup m1 i)
  by (subst sum.insert) auto
  also have (∑ i=Suc l..n. lookup m1 i) = (∑ i=Suc l..n. lookup m2 i)
  by (intro sum.cong less) auto
  also have sum (lookup m2) (insert l {Suc l..n}) = lookup m2 l + (∑ i=Suc
l..n. lookup m2 i)
  by (subst sum.insert) auto
  finally show lookup m1 l = lookup m2 l by simp
qed
qed
qed

```

We now show uniqueness by first showing that the zero polynomial has a unique representation. We fix some polynomial p with $p(e_1, \dots, e_n) = 0$ and then show, by contradiction, that $p = 0$.

We have

$$p(e_1, \dots, e_n) = \sum c_{a_1, \dots, a_n} e_1^{a_1} \dots e_n^{a_n}$$

and due to the injectivity of products of powers of elementary symmetric polynomials, the leading term of that sum is precisely the leading term of the summand with the biggest leading monomial, since summands cannot cancel each other.

However, we also know that $p(e_1, \dots, e_n) = 0$, so it follows that all summands must have leading term 0, and it is then easy to see that they must all be identically 0.

lemma *sym-mpoly-representation-unique-aux:*

```

fixes p :: 'a mpoly mpoly
assumes finite A insertion (sym-mpoly A) p = 0
      ∧ m. vars (coeff p m) ∩ A = {} vars p ⊆ {1..n}
shows p = 0
proof (rule ccontr)
  assume p: p ≠ 0

```

```

have xs: set xs = A distinct xs sorted xs and [simp]: length xs = n
  using assms by (auto simp: xs-def n-def)
define h where h = (λm. coeff p m * (∏ i. sym-mpoly A i ^ lookup m i))
define M where M = {m. coeff p m ≠ 0}
define maxm where maxm = Max ((lead-monom ∘ h) ` M)
have finite M
  by (auto intro!: finite-subset[OF - finite-coeff-support[of p]] simp: h-def M-def)
have keys-subset: keys m ⊆ {1..n} if coeff p m ≠ 0 for m
  using that assms coeff-notin-vars[of m p] by blast

have lead-coeff: lead-coeff (h m) = lead-coeff (coeff p m) (is ?th1)
and lead-monom: lead-monom (h m) = lead-monom (coeff p m) +
  lead-monom (∏ i. sym-mpoly A i ^ lookup m i :: 'a mpoly) (is
?th2)
if [simp]: coeff p m ≠ 0 for m
proof -
  have  $(\prod i. \text{sym-mpoly } A \ i \ ^{\text{lookup } m \ i} :: 'a \ \text{mpoly}) =$ 
 $(\prod i \mid \text{lookup } m \ i \neq 0. \text{sym-mpoly } A \ i \ ^{\text{lookup } m \ i} :: 'a \ \text{mpoly})$ 
  by (intro Prod-any.expand-superset) (auto intro!: Nat.gr0I)
  also have lead-coeff ... = 1
  using assms keys-subset[of m]
  by (intro lead-coeff-sym-mpoly-powerprod) (auto simp: in-keys-iff subset-iff
n-def)
  finally have eq: lead-coeff (∏ i. sym-mpoly A i ^ lookup m i :: 'a mpoly) = 1 .
  thus ?th1 unfolding h-def using <coeff p m ≠ 0> by (subst lead-coeff-mult)
auto
  show ?th2 unfolding h-def by (subst lead-monom-mult) (auto simp: eq)
qed

have insertion (sym-mpoly A) p = (∑ m∈M. h m)
  unfolding insertion-altdef h-def M-def by (intro Sum-any.expand-superset)
auto
also have lead-monom ... = maxm
  unfolding maxm-def
proof (rule lead-monom-sum)
  from p obtain m where coeff p m ≠ 0
  using mpoly-eqI[of p 0] by auto
  hence m ∈ M
  using <coeff p m ≠ 0> lead-coeff[of m] by (auto simp: M-def)
  thus M ≠ {} by auto
next
have restrict-lead-monom:
  restrictpm A (lead-monom (h m)) =
  lead-monom (∏ i. sym-mpoly A i ^ lookup m i :: 'a mpoly)
  if [simp]: coeff p m ≠ 0 for m
proof -
  have restrictpm A (lead-monom (h m)) =
  restrictpm A (lead-monom (coeff p m)) +
  restrictpm A (lead-monom (∏ i. sym-mpoly A i ^ lookup m i :: 'a mpoly))

```

```

    by (auto simp: lead-monom restrictpm-add)
    also have restrictpm A (lead-monom (coeff p m)) = 0
    using assms by (intro restrictpm-orthogonal order.trans[OF keys-lead-monom-subset])
  auto
    also have restrictpm A (lead-monom ( $\prod i$ . sym-mpoly A i  $\wedge$  lookup m i :: 'a mpoly)) =
      lead-monom ( $\prod i$ . sym-mpoly A i  $\wedge$  lookup m i :: 'a mpoly)
    by (intro restrictpm-id order.trans[OF keys-lead-monom-subset]
        order.trans[OF vars-Prod-any] UN-least order.trans[OF vars-power]
        vars-sym-mpoly-subset)
    finally show ?thesis by simp
  qed
  show inj-on (lead-monom  $\circ$  h) M
  proof
    fix m1 m2 assume m12: m1  $\in$  M m2  $\in$  M (lead-monom  $\circ$  h) m1 =
      (lead-monom  $\circ$  h) m2
    hence [simp]: coeff p m1  $\neq$  0 coeff p m2  $\neq$  0 by (auto simp: M-def h-def)
    have restrictpm A (lead-monom (h m1)) = restrictpm A (lead-monom (h m2))
    using m12 by simp
    hence lead-monom ( $\prod i$ . sym-mpoly A i  $\wedge$  lookup m1 i :: 'a mpoly) =
      lead-monom ( $\prod i$ . sym-mpoly A i  $\wedge$  lookup m2 i :: 'a mpoly)
    by (simp add: restrict-lead-monom)
    thus m1 = m2
    by (rule lead-monom-sym-poly-powerprod-inj)
      (use ⟨finite A⟩ keys-subset[of m1] keys-subset[of m2] in auto)
  qed
  next
  fix m assume m  $\in$  M
  hence lead-coeff (h m) = lead-coeff (coeff p m)
  by (simp add: lead-coeff M-def)
  with ⟨m  $\in$  M⟩ show h m  $\neq$  0 by (auto simp: M-def)
  qed fact+
  finally have maxm = 0 by (simp add: assms)

  have only-zero: m = 0 if m  $\in$  M for m
  proof -
    from that have nz [simp]: coeff p m  $\neq$  0 by (auto simp: M-def h-def)
    from that have (lead-monom  $\circ$  h) m  $\leq$  maxm
    using ⟨finite M⟩ unfolding maxm-def by (intro Max-ge imageI finite-imageI)
    with ⟨maxm = 0⟩ have [simp]: lead-monom (h m) = 0 by simp
    have lookup-nzD: k  $\in$  {1..n} if lookup m k  $\neq$  0 for k
    using keys-subset[of m] that by (auto simp: in-keys-iff subset-iff)

    have lead-monom (coeff p m) + 0  $\leq$  lead-monom (h m)
    unfolding lead-monom[OF nz] by (intro add-left-mono) auto
    also have ... = 0 by simp
    finally have lead-monom-0: lead-monom (coeff p m) = 0 by simp

    have sum (lookup m) {1..n} = 0

```

proof (*rule ccontr*)
assume $\text{sum } (\text{lookup } m) \{1..n\} \neq 0$
hence $\text{sum } (\text{lookup } m) \{1..n\} > 0$ **by** *presburger*
have $0 \neq \text{lead-coeff } (\text{MPoly-Type.coeff } p \ m)$
by *auto*
also have $\text{lead-coeff } (\text{MPoly-Type.coeff } p \ m) = \text{lead-coeff } (h \ m)$
by (*simp add: lead-coeff*)
also have $\text{lead-coeff } (h \ m) = \text{coeff } (h \ m) \ 0$
by (*simp add: lead-coeff-def*)
also have $\dots = \text{coeff } (\text{coeff } p \ m) \ 0 * \text{coeff } (\prod i. \text{sym-mpoly } A \ i \ ^{\text{lookup } m} \ i) \ 0$
by (*simp add: h-def mpoly-coeff-times-0*)
also have $(\prod i. \text{sym-mpoly } A \ i \ ^{\text{lookup } m} \ i) = (\prod i=1..n. \text{sym-mpoly } A \ i \ ^{\text{lookup } m} \ i)$
by (*intro Prod-any.expand-superset subsetI lookup-nzD*) (*auto intro!: Nat.gr0I*)
also have $\text{coeff } \dots \ 0 = (\prod i=1..n. \ 0 \ ^{\text{lookup } m} \ i)$
unfolding *mpoly-coeff-prod-0 mpoly-coeff-power-0*
by (*intro prod.cong*) (*auto simp: coeff-sym-mpoly-0*)
also have $\dots = 0 \ ^{(\sum i=1..n. \ \text{lookup } m \ i)}$
by (*simp add: power-sum*)
also have $\dots = 0$
using *zero-power[OF (sum (lookup m) {1..n} > 0)]* **by** *simp*
finally show *False* **by** *auto*
qed
hence $\text{lookup } m \ k = 0$ **for** k
using *keys-subset[of m]* **by** (*cases k ∈ {1..n}*) (*auto simp: in-keys-iff*)
thus $m = 0$ **by** (*intro poly-mapping-eqI*) *auto*
qed

have $0 = \text{insertion } (\text{sym-mpoly } A) \ p$
using *assms* **by** *simp*
also have $\text{insertion } (\text{sym-mpoly } A) \ p = (\sum m \in M. h \ m)$
by *fact*
also have $\dots = (\sum m \in \{0\}. h \ m)$
using *only-zero* **by** (*intro sum.mono-neutral-left*) (*auto simp: h-def M-def*)
also have $\dots = \text{coeff } p \ 0$
by (*simp add: h-def*)
finally have $0 \notin M$ **by** (*auto simp: M-def*)
with *only-zero* **have** $M = \{\}$ **by** *auto*
hence $p = 0$ **by** (*intro mpoly-eqI*) (*auto simp: M-def*)
with $\langle p \neq 0 \rangle$ **show** *False* **by** *contradiction*
qed

The general uniqueness theorem now follows easily. This essentially shows that the substitution $Y_i \mapsto e_i(X_1, \dots, X_n)$ is an isomorphism between the ring $R[Y_1, \dots, Y_n]$ and the ring $R[X_1, \dots, X_n]^{S_n}$ of symmetric polynomials.

theorem *sym-mpoly-representation-unique*:
fixes $p :: 'a \ \text{mpoly} \ \text{mpoly}$
assumes *finite A*

```

insertion (sym-mpoly A) p = insertion (sym-mpoly A) q
 $\bigwedge m. \text{vars} (\text{coeff } p \ m) \cap A = \{\}$   $\bigwedge m. \text{vars} (\text{coeff } q \ m) \cap A = \{\}$ 
vars p  $\subseteq$  {1..n} vars q  $\subseteq$  {1..n}
shows p = q
proof -
have p - q = 0
proof (rule sym-mpoly-representation-unique-aux)
fix m show vars (coeff (p - q) m)  $\cap$  A =  $\{\}$ 
using vars-diff[of coeff p m coeff q m] assms(3,4)[of m] by auto
qed (insert assms vars-diff[of p q], auto simp: insertion-diff)
thus ?thesis by simp
qed

```

```

theorem eq-fund-sym-poly-witI:
fixes p :: 'a mpoly and q :: 'a mpoly mpoly
assumes finite A symmetric-mpoly A p
insertion (sym-mpoly A) q = p
 $\bigwedge m. \text{vars} (\text{coeff } q \ m) \cap A = \{\}$ 
vars q  $\subseteq$  {1..n}
shows q = fund-sym-poly-wit p
using fund-sym-poly-wit-insertion[of p] fund-sym-poly-wit-vars[of p]
fund-sym-poly-wit-coeffs[of p]
by (intro sym-mpoly-representation-unique)
(insert assms, auto simp: fund-sym-poly-wit-insertion)

```

2.14 A recursive characterisation of symmetry

In a similar spirit to the proof of the fundamental theorem, we obtain a nice recursive and executable characterisation of symmetry.

```

function (domintros) check-symmetric-mpoly where
check-symmetric-mpoly p  $\longleftrightarrow$ 
(vars p  $\cap$  A =  $\{\}$ )  $\vee$ 
A  $\subseteq$  vars p  $\wedge$  decr p  $\wedge$  check-symmetric-mpoly (p - fund-sym-step-poly p)
by auto

```

```

lemma check-symmetric-mpoly-dom-aux:
assumes finite B vars p  $\subseteq$  B A  $\subseteq$  B
shows check-symmetric-mpoly-dom p
using assms(1-3)
proof (induction p rule: lead-monom-induct)
case (less p)
have [simp]: finite A by (rule finite-subset[of - B]) fact+
show ?case
proof (cases lead-monom p = 0  $\vee$   $\neg$  decr p)
case False
hence [simp]: p  $\neq$  0 by auto
have vars (p - fund-sym-step-poly p)  $\subseteq$  B
using fund-sym-step-poly[of p] False less.premis less.hyps (A  $\subseteq$  B)

```

```

    by (intro order.trans[OF vars-diff]) auto
  hence check-symmetric-mpoly-dom (p - local.fund-sym-step-poly p)
    using False less.premis less.hyps
  by (intro less.IH fund-sym-step-poly symmetric-mpoly-diff
      lead-monom-fund-sym-step-poly-less) (auto simp: decr-def)
  thus ?thesis using check-symmetric-mpoly.domintros by blast
qed (auto intro: check-symmetric-mpoly.domintros simp: lead-monom-eq-0-iff)
qed

```

lemma *check-symmetric-mpoly-dom* [intro]: *check-symmetric-mpoly-dom* p

proof –

show ?thesis

proof (cases $A \subseteq \text{vars } p$)

assume $A \subseteq \text{vars } p$

thus ?thesis using *check-symmetric-mpoly-dom-aux*[of vars p p] by (auto simp: vars-finite)

qed (auto intro: *check-symmetric-mpoly.domintros*)

qed

termination *check-symmetric-mpoly*

 by (intro allI *check-symmetric-mpoly-dom*)

lemmas [*simp del*] = *check-symmetric-mpoly.simps*

lemma *check-symmetric-mpoly-correct*: *check-symmetric-mpoly* p \longleftrightarrow *symmetric-mpoly* A p

proof (*induction* p rule: *check-symmetric-mpoly.induct*)

case (1 p)

have *symmetric-mpoly* A (p - *fund-sym-step-poly* p) \longleftrightarrow *symmetric-mpoly* A p
 (**is** ?lhs = ?rhs)

proof

assume ?rhs

thus ?lhs by (intro *symmetric-mpoly-diff fund-sym-step-poly*)

next

assume ?lhs

hence *symmetric-mpoly* A (p - *fund-sym-step-poly* p + *fund-sym-step-poly* p)

 by (intro *symmetric-mpoly-add fund-sym-step-poly*)

thus ?rhs by *simp*

qed

moreover have *decr* p **if** *symmetric-mpoly* A p

 using *lookup-lead-monom-decreasing*[of A p] **that** by (auto simp: *decr-def*)

ultimately show *check-symmetric-mpoly* p \longleftrightarrow *symmetric-mpoly* A p

 using 1 *symmetric-mpoly-imp-orthogonal-or-subset*[of A p]

 by (auto simp: *Let-def check-symmetric-mpoly.simps*[of p] intro: *symmetric-mpoly-orthogonal*)

qed

end

2.15 Symmetric functions of roots of a univariate polynomial

Consider a factored polynomial

$$p(X) = c_n X^n + c_{n-1} X^{n-1} + \dots + c_1 X + c_0 = (X - x_1) \dots (X - x_n) .$$

where c_n is a unit.

Then any symmetric polynomial expression $q(x_1, \dots, x_n)$ in the roots x_i can be written as a polynomial expression $q'(c_0, \dots, c_{n-1})$ in the c_i .

Moreover, if the coefficients of q and the inverse of c_n all lie in some subring, the coefficients of q' do as well.

context

```

fixes C :: 'b :: comm-ring-1 set
  and A :: nat set
  and root :: nat  $\Rightarrow$  'a :: comm-ring-1
  and l :: 'a  $\Rightarrow$  'b
  and q :: 'b mpoly
  and n :: nat
defines n  $\equiv$  card A
assumes C: ring-closed C  $\forall$  m. coeff q m  $\in$  C
assumes l: ring-homomorphism l
assumes finite: finite A
assumes sym: symmetric-mpoly A q and vars: vars q  $\subseteq$  A

```

begin

interpretation ring-closed C **by** fact

interpretation ring-homomorphism l **by** fact

theorem symmetric-poly-of-roots-conv-poly-of-coeffs:

```

assumes c: cinv * l c = 1 cinv  $\in$  C
assumes p = Polynomial.smult c ( $\prod$  i $\in$ A. [:-root i, 1:])
obtains q' where vars q'  $\subseteq$  {0.. $n$ }
  and  $\bigwedge$  m. coeff q' m  $\in$  C
  and insertion (l  $\circ$  poly.coeff p) q' = insertion (l  $\circ$  root) q

```

proof –

define q' **where** q' = fund-sym-poly-wit A q

define q'' **where** q'' =

mapm-mpoly (λ m x. (\prod i. (cinv * l (- 1) \wedge i) \wedge lookup m i) * insertion (λ . 0) x) q'

define reindex **where** reindex = (λ i. if i \leq n then n - i else i)

have bij reindex

by (intro bij-betwI[of reindex - - reindex]) (auto simp: reindex-def)

have vars q' \subseteq {1..n} **unfolding** q'-def n-def **by** (intro fund-sym-poly-wit-vars)

hence vars q'' \subseteq {1..n}

unfolding q''-def **using** vars-mapm-mpoly-subset **by** auto

have insertion (l \circ root) (insertion (sym-mpoly A) q') =
insertion (λ n. insertion (l \circ root) (sym-mpoly A n))

$(\text{map-mpoly } (\text{insertion } (l \circ \text{root})) q')$
by $(\text{rule insertion-insertion})$
also have $\text{insertion } (\text{sym-mpoly } A) q' = q$
unfolding q' -def **by** $(\text{intro fund-sym-poly-wit-insertion sym})$
also have $\text{insertion } (\lambda i. \text{insertion } (l \circ \text{root}) (\text{sym-mpoly } A i))$
 $(\text{map-mpoly } (\text{insertion } (l \circ \text{root})) q') =$
 $\text{insertion } (\lambda i. \text{cinv} * l ((- 1) ^ i) * l (\text{poly.coeff } p (n - i)))$
 $(\text{map-mpoly } (\text{insertion } (l \circ \text{root})) q')$
proof $(\text{intro insertion-irrelevant-vars, goal-cases})$
case $(1 i)$
hence $i \in \text{vars } q'$ **using** $\text{vars-map-mpoly-subset}$ **by** auto
also have $\dots \subseteq \{1..n\}$ **unfolding** q' -def n -def
by $(\text{intro fund-sym-poly-wit-vars})$
finally have $i: i \in \{1..n\}$.
have $\text{insertion } (l \circ \text{root}) (\text{sym-mpoly } A i) =$
 $l (\sum Y \mid Y \subseteq A \wedge \text{card } Y = i. \text{prod root } Y)$
using $\langle \text{finite } A \rangle$ **by** $(\text{simp add: insertion-sym-mpoly})$
also have $\dots = \text{cinv} * l (c * (\sum Y \mid Y \subseteq A \wedge \text{card } Y = i. \text{prod root } Y))$
unfolding $\text{mult mult.assoc[symmetric]} \langle \text{cinv} * l c = 1 \rangle$ **by** simp
also have $c * (\sum Y \mid Y \subseteq A \wedge \text{card } Y = i. \text{prod root } Y) = ((-1) ^ i *$
 $\text{poly.coeff } p (n - i))$
using $\text{coeff-poly-from-roots[of } A \ n - i \ \text{root}] i \ \text{assms finite}$
by $(\text{auto simp: n-def minus-one-power-iff})$
finally show $?case$ **by** (simp add: o-def)
qed
also have $\text{map-mpoly } (\text{insertion } (l \circ \text{root})) q' = \text{map-mpoly } (\text{insertion } (\lambda-. 0))$
 q'
using $\text{fund-sym-poly-wit-coeffs[OF sym]} \ \text{vars}$
by $(\text{intro map-mpoly-cong insertion-irrelevant-vars})$ $(\text{auto simp: } q'$ -def)
also have $\text{insertion } (\lambda i. \text{cinv} * l ((- 1) ^ i) * l (\text{poly.coeff } p (n - i))) \dots =$
 $\text{insertion } (\lambda i. l (\text{poly.coeff } p (n - i))) q''$
unfolding $\text{insertion-substitute-linear map-mpoly-conv-mapm-mpoly } q''$ -def
by $(\text{subst mapm-mpoly-comp})$ auto
also have $\dots = \text{insertion } (l \circ \text{poly.coeff } p) (\text{mpoly-map-vars reindex } q'')$
using $\langle \text{bij reindex} \rangle$ **and** $\langle \text{vars } q'' \subseteq \{1..n\} \rangle$
by $(\text{subst insertion-mpoly-map-vars})$
 $(\text{auto simp: o-def reindex-def intro!: insertion-irrelevant-vars})$
finally have $\text{insertion } (l \circ \text{root}) q =$
 $\text{insertion } (l \circ \text{poly.coeff } p) (\text{mpoly-map-vars reindex } q'')$.

moreover have $\text{coeff } (\text{mpoly-map-vars reindex } q'') m \in C$ **for** m
unfolding q'' -def q' -def **using** $\langle \text{bij reindex} \rangle$ $\text{fund-sym-poly-wit-coeff[of } q \ C \ A]$
 $C \ \langle \text{cinv} \in C \rangle$
by $(\text{auto simp: coeff-mpoly-map-vars})$
 $\text{intro!: mult-closed Prod-any-closed power-closed Sum-any-closed}$
moreover have $\text{vars } (\text{mpoly-map-vars reindex } q'') \subseteq \{0..<n\}$
using $\langle \text{bij reindex} \rangle$ **and** $\langle \text{vars } q'' \subseteq \{1..n\} \rangle$
by $(\text{subst vars-mpoly-map-vars})$ $(\text{auto simp: reindex-def subset-iff})$ +
ultimately show $?thesis$ **using** $\text{that[of mpoly-map-vars reindex } q'']$ **by** auto

qed

corollary *symmetric-poly-of-roots-conv-poly-of-coeffs-monic:*

assumes $p = (\prod_{i \in A}. [-\text{root } i, 1:])$

obtains q' **where** $\text{vars } q' \subseteq \{0..<n\}$

and $\bigwedge m. \text{coeff } q' m \in C$

and $\text{insertion } (l \circ \text{poly.coeff } p) q' = \text{insertion } (l \circ \text{root}) q$

proof –

obtain q' **where** $\text{vars } q' \subseteq \{0..<n\}$

and $\bigwedge m. \text{coeff } q' m \in C$

and $\text{insertion } (l \circ \text{poly.coeff } p) q' = \text{insertion } (l \circ \text{root}) q$

by (*rule symmetric-poly-of-roots-conv-poly-of-coeffs[of 1 1 p]*)

(*use assms in auto*)

thus *?thesis* **by** (*intro that[of q'] auto*)

qed

As a corollary, we obtain the following: Let R, S be rings with $R \subseteq S$. Consider a polynomial $p \in R[X]$ whose leading coefficient c is a unit in R and that has a full set of roots $x_1, \dots, x_n \in S$, i. e. $p(X) = c(X - x_1) \dots (X - x_n)$. Let $q \in R[X_1, \dots, X_n]$ be some symmetric polynomial expression in the roots. Then $q(x_1, \dots, x_n) \in R$.

A typical use case is $R = \mathbb{Q}$ and $S = \mathbb{C}$, i. e. any symmetric polynomial expression with rational coefficients in the roots of a rational polynomial is again rational. Similarly, any symmetric polynomial expression with integer coefficients in the roots of a monic integer polynomial is again an integer.

This is remarkable, since the roots themselves are usually not rational (possibly not even real). This particular fact is a key ingredient used in the standard proof that π is transcendental.

corollary *symmetric-poly-of-roots-in-subring:*

assumes $\text{cinv} * l c = 1$ $\text{cinv} \in C$

assumes $p = \text{Polynomial.smult } c (\prod_{i \in A}. [-\text{root } i, 1:])$

assumes $\forall i. l (\text{poly.coeff } p i) \in C$

shows $\text{insertion } (\lambda x. l (\text{root } x)) q \in C$

proof –

obtain q'

where q' : $\text{vars } q' \subseteq \{0..<n\} \bigwedge m. \text{coeff } q' m \in C$

$\text{insertion } (l \circ \text{poly.coeff } p) q' = \text{insertion } (l \circ \text{root}) q$

by (*rule symmetric-poly-of-roots-conv-poly-of-coeffs[of cinv c p]*)

(*use assms in simp-all*)

have $\text{insertion } (l \circ \text{poly.coeff } p) q' \in C$ **using** C **assms** **unfolding** *insertion-altdef*

by (*intro Sum-any-closed mult-closed q' Prod-any-closed power-closed*) *auto*

also have $\text{insertion } (l \circ \text{poly.coeff } p) q' = \text{insertion } (l \circ \text{root}) q$ **by fact**

finally show *?thesis* **by** (*simp add: o-def*)

qed

corollary *symmetric-poly-of-roots-in-subring-monic:*

assumes $p = (\prod_{i \in A}. [-\text{root } i, 1:])$

```

assumes  $\forall i. l \text{ (poly.coeff } p \ i) \in C$ 
shows insertion  $(\lambda x. l \text{ (root } x)) \ q \in C$ 
proof –
  interpret ring-closed  $C$  by fact
  interpret ring-homomorphism  $l$  by fact
  show ?thesis
    by (rule symmetric-poly-of-roots-in-subring[of 1 1 p]) (use assms in auto)
qed

end

end

```

3 Executable Operations for Symmetric Polynomials

```

theory Symmetric-Polynomials-Code
imports Symmetric-Polynomials Polynomials.MPoly-Type-Class-FMap
begin

```

Lastly, we shall provide some code equations to get executable code for operations related to symmetric polynomials, including, most notably, the fundamental theorem of symmetric polynomials and the recursive symmetry check.

```

lemma Ball-subset-right:
assumes  $T \subseteq S \ \forall x \in S. P \ x$ 
shows  $(\forall x \in S. P \ x) = (\forall x \in T. P \ x)$ 
using assms by auto

```

```

lemma compute-less-pp[code]:
   $xs < (ys :: 'a :: \text{linorder} \Rightarrow_0 'b :: \{\text{zero, linorder}\}) \iff$ 
     $(\exists i \in \text{keys } xs \cup \text{keys } ys. \text{lookup } xs \ i < \text{lookup } ys \ i \wedge$ 
     $(\forall j \in \text{keys } xs \cup \text{keys } ys. j < i \implies \text{lookup } xs \ j = \text{lookup } ys \ j))$ 
proof transfer
  fix  $f \ g :: 'a \Rightarrow 'b$ 
  let  $?dom = \{i. f \ i \neq 0\} \cup \{i. g \ i \neq 0\}$ 
  have less-fun  $f \ g \iff (\exists k. f \ k < g \ k \wedge (\forall k' < k. f \ k' = g \ k'))$ 
    unfolding less-fun-def ..
  also have  $\dots \iff (\exists i. f \ i < g \ i \wedge (i \in ?dom \wedge (\forall j \in ?dom. j < i \implies f \ j = g \ j)))$ 
proof (intro iff-exI conj-cong refl)
  fix  $k$  assume  $f \ k < g \ k$ 
  hence  $k: k \in ?dom$  by auto
  have  $(\forall k' < k. f \ k' = g \ k') = (\forall k' \in \{..<k\}. f \ k' = g \ k')$ 
    by auto
  also have  $\dots \iff (\forall j \in (\{k. f \ k \neq 0\} \cup \{k. g \ k \neq 0\}) \cap \{..<k\}. f \ j = g \ j)$ 
    by (intro Ball-subset-right) auto

```

also have ... $\longleftrightarrow (\forall j \in (\{k. f k \neq 0\} \cup \{k. g k \neq 0\}). j < k \longrightarrow f j = g j)$
by *auto*
finally show $(\forall k' < k. f k' = g k') \longleftrightarrow k \in ?dom \wedge (\forall j \in ?dom. j < k \longrightarrow f j = g j)$
using *k by simp*
qed
also have ... $\longleftrightarrow (\exists i \in ?dom. f i < g i \wedge (\forall j \in ?dom. j < i \longrightarrow f j = g j))$
by (*simp add: Bex-def conj-ac*)
finally show *less-fun f g* $\longleftrightarrow (\exists i \in ?dom. f i < g i \wedge (\forall j \in ?dom. j < i \longrightarrow f j = g j))$.
qed

lemma *compute-le-pp*[code]:
 $xs \leq ys \longleftrightarrow xs = ys \vee xs < (ys :: - \Rightarrow_0 -)$
by (*auto simp: order.order-iff-strict*)

lemma *vars-code* [code]:
 $vars (MPoly p) = (\bigcup m \in keys p. keys m)$
unfolding *vars-def* **by** *transfer' simp*

lemma *mpoly-coeff-code* [code]: $coeff (MPoly p) = lookup p$
by *transfer' simp*

lemma *sym-mpoly-code* [code]:
 $sym_mpoly (set xs) k = (\sum X \in Set.filter (\lambda X. card X = k) (Pow (set xs)). monom (monom-of-set X) 1)$
by (*simp add: sym-mpoly-altdef Set.filter-def*)

lemma *monom-of-set-code* [code]:
 $monom-of-set (set xs) = Pm-fmap (fmap-of-list (map (\lambda x. (x, 1)) xs))$
(is ?lhs = ?rhs)

proof (*intro poly-mapping-eqI*)
fix *k*
show $lookup ?lhs k = lookup ?rhs k$
by (*induction xs*) (*auto simp: lookup-monom-of-set fmlookup-default-def*)
qed

lemma *restrictpm-code* [code]:
 $restrictpm A (Pm-fmap m) = Pm-fmap (fmrestrict-set A m)$
by (*intro poly-mapping-eqI*) (*auto simp: lookup-restrictpm fmlookup-default-def*)

lemmas [code] = *check-symmetric-mpoly-correct* [symmetric]

notepad

begin

define *X Y Z* :: *int mpoly* **where** *X = Var 1 Y = Var 2 Z = Var 3*
define *e1 e2* :: *int mpoly mpoly* **where** *e1 = Var 1 e2 = Var 2*
have *sym-mpoly* {1, 2, 3} $2 = X * Y + X * Z + Y * Z$
unfolding *X-Y-Z-def* **by** *eval*

```

have symmetric-mpoly {1, 2} (X ^ 3 + Y ^ 3)
  unfolding X-Y-Z-def by eval
have fund-sym-poly-wit {1, 2} (X ^ 3 + Y ^ 3) = e1 ^ 3 - 3 * e1 * e2
  unfolding X-Y-Z-def e1-e2-def by eval
end

end

```

References

- [1] B. Blum-Smith and S. Coskey. The fundamental theorem on symmetric polynomials: History's first whiff of Galois theory. 48, 01 2013.