The Sunflower Lemma of Erdős and Rado

René Thiemann

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Abstract

We formally define sunflowers and provide a formalization of the sunflower lemma of Erdős and Rado: whenever a set of size-k-sets has a larger cardinality than $(r-1)^k \cdot k!$, then it contains a sunflower of cardinality r.

1 Sunflowers

Sunflowers are sets of sets, such that whenever an element is contained in at least two of the sets, then it is contained in all of the sets.

theory *Sunflower*
  imports *Main*
    HOL−Library.FuncSet
begin

definition *sunflower* :: *'a set set* ⇒ *bool* **where**
  $sunflower\ S = (\forall\ x.\ (\exists\ A\ B.\ A \in S \wedge B \in S \wedge A \neq B \wedge$
    $x \in A \wedge x \in B)$
    $\longrightarrow (\forall\ A.\ A \in S \longrightarrow x \in A))$

lemma *sunflower-subset*: $F \subseteq G \Longrightarrow sunflower\ G \Longrightarrow sunflower\ F$
  ⟨*proof*⟩

lemma *pairwise-disjnt-imp-sunflower*:
  $pairwise\ disjnt\ F \Longrightarrow sunflower\ F$
  ⟨*proof*⟩

lemma *card2-sunflower*: **assumes** *finite S* **and** $card\ S \leq 2$
  shows *sunflower S*
⟨*proof*⟩

lemma *empty-sunflower*: *sunflower* {}
  ⟨*proof*⟩

lemma *singleton-sunflower*: *sunflower* {*A*}
  ⟨*proof*⟩

lemma *doubleton-sunflower*: *sunflower* $\{A,B\}$
⟨*proof*⟩

lemma *sunflower-imp-union-intersect-unique*:
assumes *sunflower S*
and $x \in (\bigcup S) - (\bigcap S)$
shows $\exists! A. A \in S \wedge x \in A$
⟨*proof*⟩

lemma *union-intersect-unique-imp-sunflower*:
assumes $\bigwedge x. x \in (\bigcup S) - (\bigcap S) \Longrightarrow \exists_{\leq 1} A. A \in S \wedge x \in A$
shows *sunflower S*
⟨*proof*⟩

lemma *sunflower-iff-union-intersect-unique*:
sunflower $S \longleftrightarrow (\forall x \in \bigcup S - \bigcap S. \exists! A. A \in S \wedge x \in A)$
(**is** $?l = ?r$)
⟨*proof*⟩

lemma *sunflower-iff-intersect-Uniq*:
sunflower $S \longleftrightarrow (\forall x. x \in \bigcap S \vee (\exists_{\leq 1} A. A \in S \wedge x \in A))$
(**is** $?l = ?r$)
⟨*proof*⟩

If there exists sunflowers whenever all elements are sets of the same cardinality r, then there also exists sunflowers whenever all elements are sets with cardinality at most r.

lemma *sunflower-card-subset-lift*: **fixes** F :: *'a set set*
assumes *sunflower*: $\bigwedge G$ :: *('a + nat) set set*.
$(\forall A \in G. \textit{finite } A \wedge \textit{card } A = k) \Longrightarrow \textit{card } G > c$
$\Longrightarrow \exists S. S \subseteq G \wedge \textit{sunflower } S \wedge \textit{card } S = r$
and *kF*: $\forall A \in F. \textit{finite } A \wedge \textit{card } A \leq k$
and *cardF*: $\textit{card } F > c$
shows $\exists S. S \subseteq F \wedge \textit{sunflower } S \wedge \textit{card } S = r$
⟨*proof*⟩

We provide another sunflower lifting lemma that ensures non-empty cores. Here, all elements must be taken from a finite set, and the bound is multiplied the cardinality.

lemma *sunflower-card-core-lift*:
assumes *finE*: *finite* $(E$ :: *'a set*$)$
and *sunflower*: $\bigwedge G$ :: *'a set set*.
$(\forall A \in G. \textit{finite } A \wedge \textit{card } A \leq k) \Longrightarrow \textit{card } G > c$
$\Longrightarrow \exists S. S \subseteq G \wedge \textit{sunflower } S \wedge \textit{card } S = r$
and *F*: $\forall A \in F. A \subseteq E \wedge s \leq \textit{card } A \wedge \textit{card } A \leq k$
and *cardF*: $\textit{card } F > (\textit{card } E \textit{ choose } s) * c$
and *s*: $s \neq 0$
and *r*: $r \neq 0$

shows $\exists\ S.\ S \subseteq F \wedge \mathit{sunflower}\ S \wedge \mathit{card}\ S = r \wedge \mathit{card}\ (\bigcap\ S) \geq s$
$\langle \mathit{proof} \rangle$

lemma *sunflower-nonempty-core-lift*:
assumes *finE*: $\mathit{finite}\ (E :: {}'a\ \mathit{set})$
and *sunflower*: $\bigwedge\ G :: {}'a\ \mathit{set}\ \mathit{set}.$
$(\forall\ A \in G.\ \mathit{finite}\ A \wedge \mathit{card}\ A \leq k) \Longrightarrow \mathit{card}\ G > c$
$\Longrightarrow \exists\ S.\ S \subseteq G \wedge \mathit{sunflower}\ S \wedge \mathit{card}\ S = r$
and *F*: $\forall\ A \in F.\ A \subseteq E \wedge \mathit{card}\ A \leq k$
and *empty*: $\{\} \notin F$
and *cardF*: $\mathit{card}\ F > \mathit{card}\ E * c$
shows $\exists\ S.\ S \subseteq F \wedge \mathit{sunflower}\ S \wedge \mathit{card}\ S = r \wedge (\bigcap\ S) \neq \{\}$
$\langle \mathit{proof} \rangle$

end

2 The Sunflower Lemma

We formalize the proof of the sunflower lemma of Erdős and Rado [2], as it is presented in the textbook [3, Chapter 6]. We further integrate Exercise 6.2 from the textbook, which provides a lower bound on the existence of sunflowers.

theory *Erdos-Rado-Sunflower*
imports
Sunflower
begin

When removing an element from all subsets, then one can afterwards add these elements to a sunflower and get a new sunflower.

lemma *sunflower-remove-element-lift*:
assumes *S*: $S \subseteq \{\ A - \{a\} \mid A\ .\ A \in F \wedge a \in A\}$
and *sf*: *sunflower S*
shows $\exists\ Sa.\ \mathit{sunflower}\ Sa \wedge Sa \subseteq F \wedge \mathit{card}\ Sa = \mathit{card}\ S \wedge Sa = \mathit{insert}\ a\ `\ S$
$\langle \mathit{proof} \rangle$

The sunflower-lemma of Erdős and Rado: if a set has a certain size and all elements have the same cardinality, then a sunflower exists.

lemma *Erdos-Rado-sunflower-same-card*:
assumes $\forall\ A \in F.\ \mathit{finite}\ A \wedge \mathit{card}\ A = k$
and $\mathit{card}\ F > (r - 1)\hat{}k * \mathit{fact}\ k$
shows $\exists\ S.\ S \subseteq F \wedge \mathit{sunflower}\ S \wedge \mathit{card}\ S = r \wedge \{\} \notin S$
$\langle \mathit{proof} \rangle$

Using *sunflower-card-subset-lift* we can easily replace the condition that the cardinality is exactly k by the requirement that the cardinality is at most k. However, then $\{\} \notin S$ cannot be ensured. Consider $r = 1 \wedge 0 < k \wedge F = \{\{\}\}$.

lemma *Erdos-Rado-sunflower*:
  assumes $\forall\ A \in F.\ \mathit{finite}\ A \wedge \mathit{card}\ A \leq k$
    and $\mathit{card}\ F > (r - 1)\hat{}k * \mathit{fact}\ k$
  shows $\exists\ S.\ S \subseteq F \wedge \mathit{sunflower}\ S \wedge \mathit{card}\ S = r$
  $\langle \mathit{proof} \rangle$

We further provide a lower bound on the existence of sunflowers, i.e., Exercise 6.2 of the textbook [3]. To be more precise, we prove that there is a set of sets of cardinality $(r - 1)^k$, where each element is a set of cardinality k, such that there is no subset which is a sunflower with cardinality of at least r.

lemma *sunflower-lower-bound*:
  assumes inf: $\mathit{infinite}\ (\mathit{UNIV} :: {'a}\ \mathit{set})$
    and r: $r \neq 0$
    and rk: $r = 1 \Longrightarrow k \neq 0$
  shows $\exists\ F.$
    $\mathit{card}\ F = (r - 1)\hat{}k \wedge \mathit{finite}\ F \wedge$
    $(\forall\ A \in F.\ \mathit{finite}\ (A :: {'a}\ \mathit{set}) \wedge \mathit{card}\ A = k) \wedge$
    $(\nexists\ S.\ S \subseteq F \wedge \mathit{sunflower}\ S \wedge \mathit{card}\ S \geq r)$
$\langle \mathit{proof} \rangle$

The difference between the lower and the upper bound on the existence of sunflowers as they have been formalized is *fact k*. There is more recent work with tighter bounds [1], but we only integrate the initial result of Erdős and Rado in this theory.

We further provide the Erdős Rado lemma lifted to obtain non-empty cores or cores of arbitrary cardinality.

lemma *Erdos-Rado-sunflower-card-core*:
  assumes *finite E*
    and $\forall\ A \in F.\ A \subseteq E \wedge s \leq \mathit{card}\ A \wedge \mathit{card}\ A \leq k$
    and $\mathit{card}\ F > (\mathit{card}\ E\ \mathit{choose}\ s) * (r - 1)\hat{}k * \mathit{fact}\ k$
    and $s \neq 0$
    and $r \neq 0$
  shows $\exists\ S.\ S \subseteq F \wedge \mathit{sunflower}\ S \wedge \mathit{card}\ S = r \wedge \mathit{card}\ (\bigcap\ S) \geq s$
  $\langle \mathit{proof} \rangle$

lemma *Erdos-Rado-sunflower-nonempty-core*:
  assumes *finite E*
    and $\forall\ A \in F.\ A \subseteq E \wedge \mathit{card}\ A \leq k$
    and $\{\} \notin F$
    and $\mathit{card}\ F > \mathit{card}\ E * (r - 1)\hat{}k * \mathit{fact}\ k$
  shows $\exists\ S.\ S \subseteq F \wedge \mathit{sunflower}\ S \wedge \mathit{card}\ S = r \wedge \bigcap\ S \neq \{\}$
  $\langle \mathit{proof} \rangle$

end

References

[1] Ryan Alweiss, Shachar Lovett, Kewen Wu, and Jiapeng Zhang. Improved bounds for the sunflower lemma. In *Proccedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2020*, pages 624–630. ACM, 2020. doi:10.1145/3357713.3384234.

[2] Paul Erdős and Richard Rado. Intersection theorems for systems of sets. *Journal of the London Mathematical Society*, 35:85–90, 1960. doi:10.1112/jlms/s1-35.1.85.

[3] Stasys Jukna. *Extremal Combinatorics*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2011. doi:10.1007/978-3-642-17364-6_6.