

A Hierarchy of Algebras for Boolean Subsets

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Abstract

We present a collection of axiom systems for the construction of Boolean subalgebras of larger overall algebras. The subalgebras are defined as the range of a complement-like operation on a semilattice. This technique has been used, for example, with the antidomain operation, dynamic negation and Stone algebras. We present a common ground for these constructions based on a new equational axiomatisation of Boolean algebras.

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1 Overview

A Boolean algebra often arises as a subalgebra of some overall algebra. To avoid introducing a separate type for the subalgebra, the overall algebra can be enriched with a special operation leading into the intended subalgebra and axioms to guarantee that the range of this operation has a Boolean structure. Examples for this are the antidomain operation in idempotent (left) semirings [6, 7, 8], dynamic negation [17], the operation yielding tests in [13, 16], and the pseudocomplement operation in Stone algebras [9, 12, 14]. The present development looks at a common ground pattern.

In Sections 2 and 3 we relate various axiomatisations of Boolean algebras from the literature and present a new equational one tailored to our needs. Section 4 adapts this for the construction of Boolean subalgebras of larger overall algebras. In Section 5 we add successively stronger assumptions to the overall algebra. Sections 6, 7 and 8 show how Stone algebras, domain semirings and antidomain semirings fit into this hierarchy.

This Isabelle/HOL theory formally verifies results in [15]. See that paper for further details and related work. Some proofs in this theory have been translated from proofs found by Prover9 [21] using a program we wrote.

```
theory Subset-Boolean-Algebras
```

```
imports Stone-Algebras.P-Algebras
```

```
begin
```

2 Boolean Algebras

We show that Isabelle/HOL's *boolean-algebra* class is equivalent to Huntington's axioms [18]. See [24] for related results.

2.1 Huntington's Axioms

Definition 1

```
class huntington = sup + uminus +
  assumes associative:  $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ 
  assumes commutative:  $x \sqcup y = y \sqcup x$ 
  assumes huntington:  $x = -(-x \sqcup y) \sqcup -(-x \sqcup -y)$ 
begin

lemma top-unique:
   $x \sqcup -x = y \sqcup -y$ 
proof -
  have  $x \sqcup -x = y \sqcup -(-y \sqcup -x) \sqcup -(-y \sqcup -x)$ 
    by (smt associative commutative huntington)
  thus ?thesis
    by (metis associative huntington)
qed

end
```

2.2 Equivalence to boolean-algebra Class

Definition 2

```
class extended = sup + inf + minus + uminus + bot + top + ord +
  assumes top-def:  $top = (THE x . \forall y . x = y \sqcup -y)$ 
  assumes bot-def:  $bot = -(THE x . \forall y . x = y \sqcup -y)$ 
  assumes inf-def:  $x \sqcap y = -(-x \sqcup -y)$ 
  assumes minus-def:  $x - y = -(-x \sqcup y)$ 
  assumes less-eq-def:  $x \leq y \iff x \sqcup y = y$ 
  assumes less-def:  $x < y \iff x \sqcup y = y \wedge \neg (y \sqcup x = x)$ 

class huntington-extended = huntington + extended
begin

lemma top-char:
   $top = x \sqcup -x$ 
  using top-def top-unique by auto

lemma bot-char:
   $bot = -top$ 
  by (simp add: bot-def top-def)

subclass boolean-algebra
proof
  show 1:  $\bigwedge x y. (x < y) = (x \leq y \wedge \neg y \leq x)$ 
    by (simp add: less-def less-eq-def)
  show 2:  $\bigwedge x. x \leq x$ 
  proof -
    fix x
```

```

have x ⊔ top = top ⊔ --x
  by (metis (full-types) associative top-char)
thus x ≤ x
  by (metis (no-types) associative huntington less-eq-def top-char)
qed
show 3:  $\bigwedge x y z. x \leq y \implies y \leq z \implies x \leq z$ 
  by (metis associative less-eq-def)
show 4:  $\bigwedge x y. x \leq y \implies y \leq x \implies x = y$ 
  by (simp add: commutative less-eq-def)
show 5:  $\bigwedge x y. x \sqcap y \leq x$ 
  using 2 by (metis associative huntington inf-def less-eq-def)
show 6:  $\bigwedge x y. x \sqcap y \leq y$ 
  using 5 commutative inf-def by fastforce
show 8:  $\bigwedge x y. x \leq x \sqcup y$ 
  using 2 associative less-eq-def by auto
show 9:  $\bigwedge y x. y \leq x \sqcup y$ 
  using 8 commutative by fastforce
show 10:  $\bigwedge y x z. y \leq x \implies z \leq x \implies y \sqcup z \leq x$ 
  by (metis associative less-eq-def)
show 11:  $\bigwedge x. \text{bot} \leq x$ 
  using 8 by (metis bot-char huntington top-char)
show 12:  $\bigwedge x. x \leq \text{top}$ 
  using 6 11 by (metis huntington bot-def inf-def less-eq-def top-def)
show 13:  $\bigwedge x y z. x \sqcup y \sqcap z = (x \sqcup y) \sqcap (x \sqcup z)$ 
proof -
  have 2:  $\bigwedge x y z. x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ 
    by (simp add: associative)
  have 3:  $\bigwedge x y z. (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$ 
    using 2 by metis
  have 4:  $\bigwedge x y. x \sqcup y = y \sqcup x$ 
    by (simp add: commutative)
  have 5:  $\bigwedge x y. x = -(-x \sqcup y) \sqcup -(-x \sqcup -y)$ 
    by (simp add: huntington)
  have 6:  $\bigwedge x y. -(-x \sqcup y) \sqcup -(-x \sqcup -y) = x$ 
    using 5 by metis
  have 7:  $\bigwedge x y. x \sqcap y = -(-x \sqcup -y)$ 
    by (simp add: inf-def)
  have 10:  $\bigwedge x y z. x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$ 
    using 3 4 by metis
  have 11:  $\bigwedge x y z. -(-x \sqcup y) \sqcup -(-x \sqcup -y) \sqcup z = x \sqcup z$ 
    using 3 6 by metis
  have 12:  $\bigwedge x y. -(x \sqcup -y) \sqcup -(-y \sqcup -x) = y$ 
    using 4 6 by metis
  have 13:  $\bigwedge x y. -(-x \sqcup y) \sqcup -(-y \sqcup -x) = x$ 
    using 4 6 by metis
  have 14:  $\bigwedge x y. -x \sqcup -(-(-x \sqcup y) \sqcup -(-x \sqcup -y)) = -x \sqcup y$ 
    using 6 by metis
  have 18:  $\bigwedge x y z. -(x \sqcup -y) \sqcup -(-y \sqcup -x) \sqcup z = y \sqcup z$ 
    using 3 12 by metis

```

have 20: $\bigwedge x y . - (- x \sqcup - y) \sqcup - (y \sqcup - x) = x$
using 4 12 by metis
have 21: $\bigwedge x y . - (x \sqcup - y) \sqcup - (- x \sqcup - y) = y$
using 4 12 by metis
have 22: $\bigwedge x y . - x \sqcup - (- (y \sqcup - x) \sqcup - - (- x \sqcup - y)) = y \sqcup - x$
using 6 12 by metis
have 23: $\bigwedge x y . - x \sqcup - (- x \sqcup (- y \sqcup - (y \sqcup - x))) = y \sqcup - x$
using 3 4 6 12 by metis
have 24: $\bigwedge x y . - x \sqcup - (- (- x \sqcup - y) \sqcup - - (- x \sqcup y)) = - x \sqcup - y$
using 6 12 by metis
have 28: $\bigwedge x y . - (- x \sqcup - y) \sqcup - (- y \sqcup x) = y$
using 4 13 by metis
have 30: $\bigwedge x y . - x \sqcup - (- y \sqcup (- x \sqcup - (- x \sqcup y))) = - x \sqcup y$
using 3 4 6 13 by metis
have 32: $\bigwedge x y z . - (- x \sqcup y) \sqcup (z \sqcup - (- y \sqcup - x)) = z \sqcup x$
using 10 13 by metis
have 37: $\bigwedge x y z . - (- x \sqcup - y) \sqcup (- (y \sqcup - x) \sqcup z) = x \sqcup z$
using 3 20 by metis
have 39: $\bigwedge x y z . - (- x \sqcup - y) \sqcup (z \sqcup - (y \sqcup - x)) = z \sqcup x$
using 10 20 by metis
have 40: $\bigwedge x y z . - (x \sqcup - y) \sqcup (- (- x \sqcup - y) \sqcup z) = y \sqcup z$
using 3 21 by metis
have 43: $\bigwedge x y . - x \sqcup - (- y \sqcup (- x \sqcup - (y \sqcup - x))) = y \sqcup - x$
using 3 4 6 21 by metis
have 47: $\bigwedge x y z . - (x \sqcup y) \sqcup - (- (- x \sqcup z) \sqcup - (- (- x \sqcup - z) \sqcup y)) =$
 $- x \sqcup z$
using 6 11 by metis
have 55: $\bigwedge x y . x \sqcup - (- y \sqcup - - x) = y \sqcup - (- x \sqcup y)$
using 4 11 12 by metis
have 58: $\bigwedge x y . x \sqcup - (- - y \sqcup - x) = x \sqcup - (- x \sqcup y)$
using 4 11 13 by metis
have 63: $\bigwedge x y . x \sqcup - (- - x \sqcup - y) = y \sqcup - (- x \sqcup y)$
using 4 11 21 by metis
have 71: $\bigwedge x y . x \sqcup - (- y \sqcup x) = y \sqcup - (- x \sqcup y)$
using 4 11 28 by metis
have 75: $\bigwedge x y . x \sqcup - (- y \sqcup x) = y \sqcup - (y \sqcup - x)$
using 4 71 by metis
have 78: $\bigwedge x y . - x \sqcup (y \sqcup - (- x \sqcup (y \sqcup - - (- x \sqcup - y)))) = - x \sqcup -$
 $(- x \sqcup - y)$
using 3 4 6 71 by metis
have 86: $\bigwedge x y . - (- x \sqcup - (- y \sqcup x)) \sqcup - (y \sqcup - (- x \sqcup y)) = - y \sqcup x$
using 4 20 71 by metis
have 172: $\bigwedge x y . - x \sqcup - (- x \sqcup - y) = y \sqcup - (- - x \sqcup y)$
using 14 75 by metis
have 201: $\bigwedge x y . x \sqcup - (- y \sqcup - - x) = y \sqcup - (y \sqcup - x)$
using 4 55 by metis
have 236: $\bigwedge x y . x \sqcup - (- - y \sqcup - x) = x \sqcup - (y \sqcup - x)$
using 4 58 by metis
have 266: $\bigwedge x y . - x \sqcup - (- (- x \sqcup - (y \sqcup - - x)) \sqcup - - (- x \sqcup - - (-$

$- x \sqcup y))) = - x \sqcup - (- - x \sqcup y)$
using 14 58 236 by metis
have 678: $\bigwedge x y z . - (- x \sqcup - (- y \sqcup x)) \sqcup (- (y \sqcup - (- x \sqcup y)) \sqcup z) = - y \sqcup (x \sqcup z)$
using 3 4 37 71 by smt
have 745: $\bigwedge x y z . - (- x \sqcup - (- y \sqcup x)) \sqcup (z \sqcup - (y \sqcup - (- x \sqcup y))) = z \sqcup (- y \sqcup x)$
using 4 39 71 by metis
have 800: $\bigwedge x y . - - x \sqcup (- y \sqcup (- (y \sqcup - - x) \sqcup - (- x \sqcup (- - x \sqcup (- y \sqcup - (y \sqcup - - x)))))) = x \sqcup - (y \sqcup - - x)$
using 3 23 63 by metis
have 944: $\bigwedge x y . x \sqcup - (x \sqcup - - (- (- x \sqcup - y) \sqcup - - (- x \sqcup y))) = - (- x \sqcup - y) \sqcup - (- (- x \sqcup - y) \sqcup - - (- x \sqcup y))$
using 4 24 71 by metis
have 948: $\bigwedge x y . - x \sqcup - (- (y \sqcup - (y \sqcup - - x)) \sqcup - - (- x \sqcup (- y \sqcup - x))) = - x \sqcup - (- y \sqcup - x)$
using 24 75 by metis
have 950: $\bigwedge x y . - x \sqcup - (- (y \sqcup - (- - x \sqcup y)) \sqcup - - (- x \sqcup (- x \sqcup - y))) = - x \sqcup - (- x \sqcup - y)$
using 24 75 by metis
have 961: $\bigwedge x y . - x \sqcup - (- (y \sqcup - (- - x \sqcup y)) \sqcup - - (- x \sqcup (- - - x \sqcup - y))) = y \sqcup - (- - x \sqcup y)$
using 24 63 by metis
have 966: $\bigwedge x y . - x \sqcup - (- (y \sqcup - (y \sqcup - - x)) \sqcup - - (- x \sqcup (- y \sqcup - - - x))) = y \sqcup - (y \sqcup - - x)$
using 24 201 by metis
have 969: $\bigwedge x y . - x \sqcup - (- (- x \sqcup - (y \sqcup - - x)) \sqcup - - (- x \sqcup (- - y \sqcup - - - x))) = - x \sqcup - (y \sqcup - - x)$
using 24 236 by metis
have 1096: $\bigwedge x y z . - x \sqcup (- (- x \sqcup - y) \sqcup z) = y \sqcup (- (- - x \sqcup y) \sqcup z)$
using 3 172 by metis
have 1098: $\bigwedge x y z . - x \sqcup (y \sqcup - (- x \sqcup - z)) = y \sqcup (z \sqcup - (- - x \sqcup z))$
using 10 172 by metis
have 1105: $\bigwedge x y . x \sqcup - x = y \sqcup - y$
using 4 10 12 32 172 by metis
have 1109: $\bigwedge x y z . x \sqcup (- x \sqcup y) = z \sqcup (- z \sqcup y)$
using 3 1105 by metis
have 1110: $\bigwedge x y z . x \sqcup - x = y \sqcup (z \sqcup - (y \sqcup z))$
using 3 1105 by metis
have 1114: $\bigwedge x y . - (- x \sqcup - - x) = - (y \sqcup - y)$
using 7 1105 by metis
have 1115: $\bigwedge x y z . x \sqcup (y \sqcup - y) = z \sqcup (x \sqcup - z)$
using 10 1105 by metis
have 1117: $\bigwedge x y . - (x \sqcup - - x) \sqcup - (y \sqcup - y) = - x$
using 4 13 1105 by metis
have 1121: $\bigwedge x y . - (x \sqcup - x) \sqcup - (y \sqcup - - y) = - y$
using 4 28 1105 by metis
have 1122: $\bigwedge x . - - x = x$
using 4 28 1105 1117 by metis

have 1134: $\bigwedge x y z . - (x \sqcup - y) \sqcup (z \sqcup - z) = y \sqcup (- y \sqcup - x)$
using 18 1105 1122 by metis
have 1140: $\bigwedge x . - x \sqcup - (x \sqcup (x \sqcup - x)) = - x \sqcup - x$
using 4 22 1105 1122 1134 by metis
have 1143: $\bigwedge x y . x \sqcup (- x \sqcup y) = y \sqcup (x \sqcup - y)$
using 37 1105 1122 1134 by metis
have 1155: $\bigwedge x y . - (x \sqcup - x) \sqcup - (y \sqcup y) = - y$
using 1121 1122 by metis
have 1156: $\bigwedge x y . - (x \sqcup x) \sqcup - (y \sqcup - y) = - x$
using 1117 1122 by metis
have 1157: $\bigwedge x y . - (x \sqcup - x) = - (y \sqcup - y)$
using 4 1114 1122 by metis
have 1167: $\bigwedge x y z . - x \sqcup (y \sqcup - (- x \sqcup - z)) = y \sqcup (z \sqcup - (x \sqcup z))$
using 1098 1122 by metis
have 1169: $\bigwedge x y z . - x \sqcup (- (- x \sqcup - y) \sqcup z) = y \sqcup (- (x \sqcup y) \sqcup z)$
using 1096 1122 by metis
have 1227: $\bigwedge x y . - x \sqcup - (- x \sqcup (y \sqcup (x \sqcup - (- x \sqcup - (y \sqcup x)))))) = - x$
 $\sqcup - (y \sqcup x)$
using 3 4 969 1122 by smt
have 1230: $\bigwedge x y . - x \sqcup - (- x \sqcup (- y \sqcup (- x \sqcup - (y \sqcup - (y \sqcup x)))))) = y$
 $\sqcup - (y \sqcup x)$
using 3 4 966 1122 by smt
have 1234: $\bigwedge x y . - x \sqcup - (- x \sqcup (- x \sqcup (- y \sqcup - (y \sqcup - (x \sqcup y)))))) = y$
 $\sqcup - (x \sqcup y)$
using 3 4 961 1122 by metis
have 1239: $\bigwedge x y . - x \sqcup - (- x \sqcup - y) = y \sqcup - (x \sqcup y)$
using 3 4 950 1122 1234 by metis
have 1240: $\bigwedge x y . - x \sqcup - (- y \sqcup - x) = y \sqcup - (y \sqcup x)$
using 3 4 948 1122 1230 by metis
have 1244: $\bigwedge x y . x \sqcup - (x \sqcup (y \sqcup (y \sqcup - (x \sqcup y)))) = - (- x \sqcup - y) \sqcup -$
 $(y \sqcup (y \sqcup - (x \sqcup y)))$
using 3 4 944 1122 1167 by metis
have 1275: $\bigwedge x y . x \sqcup (- y \sqcup (- (y \sqcup x) \sqcup - (x \sqcup (- x \sqcup (- y \sqcup - (y \sqcup$
 $x)))))) = x \sqcup - (y \sqcup x)$
using 10 800 1122 by metis
have 1346: $\bigwedge x y . - x \sqcup - (x \sqcup (y \sqcup (y \sqcup (x \sqcup - (x \sqcup (y \sqcup x)))))) = - x \sqcup$
 $- (x \sqcup y)$
using 3 4 10 266 1122 1167 by smt
have 1377: $\bigwedge x y . - x \sqcup (y \sqcup - (- x \sqcup (y \sqcup (- x \sqcup - y)))) = y \sqcup - (x \sqcup y)$
using 78 1122 1239 by metis
have 1394: $\bigwedge x y . - (- x \sqcup - y) \sqcup - (y \sqcup (y \sqcup (- x \sqcup - (x \sqcup y)))) = x$
using 3 4 10 20 30 1122 1239 by smt
have 1427: $\bigwedge x y . - (- x \sqcup - y) \sqcup - (y \sqcup - (x \sqcup (x \sqcup - (x \sqcup y)))) = x \sqcup$
 $(x \sqcup - (x \sqcup y))$
using 3 4 30 40 1240 by smt
have 1436: $\bigwedge x . - x \sqcup - (x \sqcup (x \sqcup (- x \sqcup - x))) = - x \sqcup (- x \sqcup - (x \sqcup$
 $- x))$
using 3 4 30 1140 1239 by smt
have 1437: $\bigwedge x y . - (x \sqcup y) \sqcup - (x \sqcup - y) = - x$

using 6 1122 by metis
have 1438: $\bigwedge x y . - (x \sqcup y) \sqcup - (y \sqcup - x) = - y$
using 12 1122 by metis
have 1439: $\bigwedge x y . - (x \sqcup y) \sqcup - (- y \sqcup x) = - x$
using 13 1122 by metis
have 1440: $\bigwedge x y . - (x \sqcup - y) \sqcup - (y \sqcup x) = - x$
using 20 1122 by metis
have 1441: $\bigwedge x y . - (x \sqcup y) \sqcup - (- x \sqcup y) = - y$
using 21 1122 by metis
have 1568: $\bigwedge x y . x \sqcup (- y \sqcup - x) = y \sqcup (- y \sqcup x)$
using 10 1122 1143 by metis
have 1598: $\bigwedge x . - x \sqcup - (x \sqcup (x \sqcup (x \sqcup - x))) = - x \sqcup (- x \sqcup - (x \sqcup - x))$
using 4 1436 1568 by metis
have 1599: $\bigwedge x y . - x \sqcup (y \sqcup - (x \sqcup (- x \sqcup (- x \sqcup y)))) = y \sqcup - (x \sqcup y)$
using 10 1377 1568 by smt
have 1617: $\bigwedge x . x \sqcup (- x \sqcup (- x \sqcup - (x \sqcup - x))) = x \sqcup - x$
using 3 4 10 71 1122 1155 1568 1598 by metis
have 1632: $\bigwedge x y z . - (x \sqcup - x) \sqcup - (- y \sqcup (- (z \sqcup - z) \sqcup - (y \sqcup - (x \sqcup - x)))) = y \sqcup - (x \sqcup - x)$
using 43 1157 by metis
have 1633: $\bigwedge x y z . - (x \sqcup - x) \sqcup - (- y \sqcup (- (x \sqcup - x) \sqcup - (y \sqcup - (z \sqcup - z)))) = y \sqcup - (x \sqcup - x)$
using 43 1157 by metis
have 1636: $\bigwedge x y . x \sqcup - (y \sqcup (- y \sqcup - (x \sqcup x))) = x \sqcup x$
using 43 1109 1122 by metis
have 1645: $\bigwedge x y . x \sqcup - x = y \sqcup (y \sqcup - y)$
using 3 1110 1156 by metis
have 1648: $\bigwedge x y z . - (x \sqcup (y \sqcup (- y \sqcup - x))) \sqcup - (z \sqcup - z) = - (y \sqcup - y)$
using 3 1115 1156 by metis
have 1657: $\bigwedge x y z . x \sqcup - x = y \sqcup (z \sqcup - z)$
using 1105 1645 by metis
have 1664: $\bigwedge x y z . x \sqcup - x = y \sqcup (z \sqcup - y)$
using 1115 1645 by metis
have 1672: $\bigwedge x y z . x \sqcup - x = y \sqcup (- y \sqcup z)$
using 3 4 1657 by metis
have 1697: $\bigwedge x y z . - x \sqcup (y \sqcup x) = z \sqcup - z$
using 1122 1664 by metis
have 1733: $\bigwedge x y z . - (x \sqcup y) \sqcup - (- (z \sqcup - z) \sqcup - (- (- x \sqcup - x) \sqcup y)) = x \sqcup - x$
using 4 47 1105 1122 by metis
have 1791: $\bigwedge x y z . x \sqcup - (y \sqcup (- y \sqcup z)) = x \sqcup - (x \sqcup - x)$
using 4 71 1122 1672 by metis
have 1818: $\bigwedge x y z . x \sqcup - (- y \sqcup (z \sqcup y)) = x \sqcup - (x \sqcup - x)$
using 4 71 1122 1697 by metis
have 1861: $\bigwedge x y z . - (x \sqcup - x) \sqcup - (y \sqcup - (z \sqcup - z)) = - y$
using 1437 1657 by metis
have 1867: $\bigwedge x y z . - (x \sqcup - x) \sqcup - (- y \sqcup - (z \sqcup y)) = y$
using 1122 1437 1697 by metis

have 1868: $\bigwedge x y . x \sqcup - (y \sqcup - y) = x$
using 1122 1155 1633 1861 by metis
have 1869: $\bigwedge x y z . - (x \sqcup - x) \sqcup - (- y \sqcup (- (z \sqcup - z) \sqcup - y)) = y$
using 1632 1868 by metis
have 1870: $\bigwedge x y . - (x \sqcup - x) \sqcup - y = - y$
using 1861 1868 by metis
have 1872: $\bigwedge x y z . x \sqcup - (- y \sqcup (z \sqcup y)) = x$
using 1818 1868 by metis
have 1875: $\bigwedge x y z . x \sqcup - (y \sqcup (- y \sqcup z)) = x$
using 1791 1868 by metis
have 1883: $\bigwedge x y . - (x \sqcup (y \sqcup (- y \sqcup - x))) = - (y \sqcup - y)$
using 1648 1868 by metis
have 1885: $\bigwedge x . x \sqcup (x \sqcup - x) = x \sqcup - x$
using 4 1568 1617 1868 by metis
have 1886: $\bigwedge x . - x \sqcup - x = - x$
using 1598 1868 1885 by metis
have 1890: $\bigwedge x . - (x \sqcup x) = - x$
using 1156 1868 by metis
have 1892: $\bigwedge x y . - (x \sqcup - x) \sqcup y = y$
using 1122 1869 1870 1886 by metis
have 1893: $\bigwedge x y . - (- x \sqcup - (y \sqcup x)) = x$
using 1867 1892 by metis
have 1902: $\bigwedge x y . x \sqcup (y \sqcup - (x \sqcup y)) = x \sqcup - x$
using 3 4 1122 1733 1886 1892 by metis
have 1908: $\bigwedge x . x \sqcup x = x$
using 1636 1875 1890 by metis
have 1910: $\bigwedge x y . x \sqcup - (y \sqcup x) = - y \sqcup x$
using 1599 1875 by metis
have 1921: $\bigwedge x y . x \sqcup (- y \sqcup - (y \sqcup x)) = - y \sqcup x$
using 1275 1875 1910 by metis
have 1951: $\bigwedge x y . - x \sqcup - (y \sqcup x) = - x$
using 1227 1872 1893 1908 by metis
have 1954: $\bigwedge x y z . x \sqcup (y \sqcup - (x \sqcup z)) = y \sqcup (- z \sqcup x)$
using 745 1122 1910 1951 by metis
have 1956: $\bigwedge x y z . x \sqcup (- (x \sqcup y) \sqcup z) = - y \sqcup (x \sqcup z)$
using 678 1122 1910 1951 by metis
have 1959: $\bigwedge x y . x \sqcup - (x \sqcup y) = - y \sqcup x$
using 86 1122 1910 1951 by metis
have 1972: $\bigwedge x y . x \sqcup (- x \sqcup y) = x \sqcup - x$
using 1902 1910 by metis
have 2000: $\bigwedge x y . - (- x \sqcup - y) \sqcup - (y \sqcup (- x \sqcup y)) = x \sqcup - (y \sqcup (- x \sqcup$
 $y))$
using 4 1244 1910 1959 by metis
have 2054: $\bigwedge x y . x \sqcup - (y \sqcup (- x \sqcup y)) = x$
using 1394 1921 2000 by metis
have 2057: $\bigwedge x y . - (x \sqcup (y \sqcup - y)) = - (y \sqcup - y)$
using 1883 1972 by metis
have 2061: $\bigwedge x y . x \sqcup (- y \sqcup x) = x \sqcup - y$
using 4 1122 1427 1910 1959 2054 by metis

```

have 2090:  $\bigwedge x y z . x \sqcup (- (y \sqcup x) \sqcup z) = x \sqcup (- y \sqcup z)$ 
  using 1122 1169 1956 by metis
have 2100:  $\bigwedge x y . - x \sqcup - (x \sqcup y) = - x$ 
  using 4 1346 1868 1885 1910 1959 1972 2057 by metis
have 2144:  $\bigwedge x y . x \sqcup - (y \sqcup - x) = x$ 
  using 1122 1440 2000 2061 by metis
have 2199:  $\bigwedge x y . x \sqcup (x \sqcup y) = x \sqcup y$ 
  using 3 1908 by metis
have 2208:  $\bigwedge x y z . x \sqcup (- (y \sqcup - x) \sqcup z) = x \sqcup z$ 
  using 3 2144 by metis
have 2349:  $\bigwedge x y z . - (x \sqcup y) \sqcup - (x \sqcup (y \sqcup z)) = - (x \sqcup y)$ 
  using 3 2100 by metis
have 2432:  $\bigwedge x y z . - (x \sqcup (y \sqcup z)) \sqcup - (y \sqcup (z \sqcup - x)) = - (y \sqcup z)$ 
  using 3 1438 by metis
have 2530:  $\bigwedge x y z . - (- (x \sqcup y) \sqcup z) = - (y \sqcup (- x \sqcup z)) \sqcup - (- y \sqcup z)$ 
  using 4 1122 1439 2090 2208 by smt
have 3364:  $\bigwedge x y z . - (- x \sqcup y) \sqcup (z \sqcup - (x \sqcup y)) = z \sqcup - y$ 
  using 3 4 1122 1441 1910 1954 2199 by metis
have 5763:  $\bigwedge x y z . - (x \sqcup y) \sqcup - (- x \sqcup (y \sqcup z)) = - (x \sqcup y) \sqcup - (y \sqcup z)$ 
  using 4 2349 3364 by metis
have 6113:  $\bigwedge x y z . - (x \sqcup (y \sqcup z)) \sqcup - (z \sqcup - x) = - (y \sqcup z) \sqcup - (z \sqcup -$ 
x)
  using 4 2432 3364 5763 by metis
show  $\bigwedge x y z . x \sqcup y \sqcap z = (x \sqcup y) \sqcap (x \sqcup z)$ 
proof -
  fix x y z
  have  $- (y \sqcap z \sqcup x) = - (- (- y \sqcup z) \sqcup - (- y \sqcup - z) \sqcup x) \sqcup - (x \sqcup - -$ 
z)
    using 1437 2530 6113 by (smt commutative inf-def)
  thus  $x \sqcup y \sqcap z = (x \sqcup y) \sqcap (x \sqcup z)$ 
  using 12 1122 by (metis commutative inf-def)
qed
qed
show 14:  $\bigwedge x . x \sqcap - x = \text{bot}$ 
proof -
  fix x
  have  $(\text{bot} \sqcup x) \sqcap (\text{bot} \sqcup -x) = \text{bot}$ 
  using huntington bot-def inf-def by auto
  thus  $x \sqcap -x = \text{bot}$ 
  using 11 less-eq-def by force
qed
show 15:  $\bigwedge x . x \sqcup - x = \text{top}$ 
  using 5 14 by (metis (no-types, lifting) huntington bot-def less-eq-def top-def)
show 16:  $\bigwedge x y . x - y = x \sqcap - y$ 
  using 15 by (metis commutative huntington inf-def minus-def)
show 7:  $\bigwedge x y z . x \leq y \implies x \leq z \implies x \leq y \sqcap z$ 
  by (simp add: 13 less-eq-def)
qed

```

```

end

context boolean-algebra
begin

sublocale ba-he: huntington-extended
proof
  show  $\bigwedge x y z. x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$ 
    by (simp add: sup-assoc)
  show  $\bigwedge x y. x \sqcup y = y \sqcup x$ 
    by (simp add: sup-commute)
  show  $\bigwedge x y. x = -(-x \sqcup y) \sqcup -(-x \sqcup -y)$ 
    by simp
  show  $top = (THE x. \forall y. x = y \sqcup -y)$ 
    by auto
  show  $bot = - (THE x. \forall y. x = y \sqcup -y)$ 
    by auto
  show  $\bigwedge x y. x \sqcap y = -(-x \sqcup -y)$ 
    by simp
  show  $\bigwedge x y. x - y = -(-x \sqcup y)$ 
    by (simp add: diff-eq)
  show  $\bigwedge x y. (x \leq y) = (x \sqcup y = y)$ 
    by (simp add: le-iff-sup)
  show  $\bigwedge x y. (x < y) = (x \sqcup y = y \wedge y \sqcup x \neq x)$ 
    using sup.strict-order-iff sup-commute by auto
qed

end

```

2.3 Stone Algebras

We relate Stone algebras to Boolean algebras.

```

class stone-algebra-extended = stone-algebra + minus +
  assumes stone-minus-def[simp]:  $x - y = x \sqcap -y$ 

```

```

class regular-stone-algebra = stone-algebra-extended +
  assumes double-complement[simp]:  $--x = x$ 

```

```
begin
```

```
subclass boolean-algebra
```

```
proof
```

```
  show  $\bigwedge x. x \sqcap -x = bot$ 
```

```
    by simp
```

```
  show  $\bigwedge x. x \sqcup -x = top$ 
```

```
    using regular-dense-top by fastforce
```

```
  show  $\bigwedge x y. x - y = x \sqcap -y$ 
```

```
    by simp
```

```
qed
```

```

end

context boolean-algebra
begin

sublocale ba-rsa: regular-stone-algebra
proof
  show  $\bigwedge x y. x - y = x \sqcap - y$ 
    by (simp add: diff-eq)
  show  $\bigwedge x. - - x = x$ 
    by simp
qed

end

```

3 Alternative Axiomatisations of Boolean Algebras

We consider four axiomatisations of Boolean algebras based only on join and complement. The first three are from the literature and the fourth, a version using equational axioms, is new. The motivation for Byrne's and the new axiomatisation is that the axioms are easier to understand than Huntington's third axiom. We also include Meredith's axiomatisation.

3.1 Lee Byrne's Formulation A

The following axiomatisation is from [2, Formulation A]; see also [10].

Theorem 3

```

class boolean-algebra-1 = sup + uminus +
  assumes ba1-associative:  $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ 
  assumes ba1-commutative:  $x \sqcup y = y \sqcup x$ 
  assumes ba1-complement:  $x \sqcup -y = z \sqcup -z \longleftrightarrow x \sqcup y = x$ 
begin

subclass huntington
proof
  show 1:  $\bigwedge x y z. x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$ 
    by (simp add: ba1-associative)
  show  $\bigwedge x y. x \sqcup y = y \sqcup x$ 
    by (simp add: ba1-commutative)
  show  $\bigwedge x y. x = -(-x \sqcup y) \sqcup -(-x \sqcup -y)$ 
  proof -
    have 2:  $\forall x y. y \sqcup (y \sqcup x) = y \sqcup x$ 
      using 1 by (metis ba1-complement)
    hence  $\forall x. - - x = x$ 
      by (smt ba1-associative ba1-commutative ba1-complement)
  qed
end

```

```

    hence  $\forall x y. y \sqcup -(y \sqcup -x) = y \sqcup x$ 
      by (smt ba1-associative ba1-commutative ba1-complement)
    thus  $\bigwedge x y. x = -(-x \sqcup y) \sqcup -(-x \sqcup -y)$ 
      using 2 by (smt ba1-commutative ba1-complement)
  qed
end

```

```

context huntington
begin

```

```

sublocale h-ba1: boolean-algebra-1
proof

```

```

  show  $\bigwedge x y z. x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$ 
    by (simp add: associative)
  show  $\bigwedge x y. x \sqcup y = y \sqcup x$ 
    by (simp add: commutative)
  show  $\bigwedge x y z. (x \sqcup -y = z \sqcup -z) = (x \sqcup y = x)$ 
  proof
    fix x y z
    have 1:  $\bigwedge x y z. -(-x \sqcup y) \sqcup (-(-x \sqcup -y) \sqcup z) = x \sqcup z$ 
      using associative huntington by force
    have 2:  $\bigwedge x y. -(x \sqcup -y) \sqcup -(-y \sqcup -x) = y$ 
      by (metis commutative huntington)
    show  $x \sqcup -y = z \sqcup -z \implies x \sqcup y = x$ 
      by (metis 1 2 associative commutative top-unique)
    show  $x \sqcup y = x \implies x \sqcup -y = z \sqcup -z$ 
      by (metis associative huntington commutative top-unique)
  qed
end

```

```

end

```

3.2 Lee Byrne's Formulation B

The following axiomatisation is from [2, Formulation B].

Theorem 4

```

class boolean-algebra-2 = sup + uminus +
  assumes ba2-associative-commutative:  $(x \sqcup y) \sqcup z = (y \sqcup z) \sqcup x$ 
  assumes ba2-complement:  $x \sqcup -y = z \sqcup -z \longleftrightarrow x \sqcup y = x$ 
begin

```

```

subclass boolean-algebra-1

```

```

proof
  show  $\bigwedge x y z. x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$ 
    by (smt ba2-associative-commutative ba2-complement)
  show  $\bigwedge x y. x \sqcup y = y \sqcup x$ 

```

```

    by (metis ba2-associative-commutative ba2-complement)
  show  $\bigwedge x y z. (x \sqcup - y = z \sqcup - z) = (x \sqcup y = x)$ 
    by (simp add: ba2-complement)
qed

```

```
end
```

```

context boolean-algebra-1
begin

```

```

sublocale ba1-ba2: boolean-algebra-2
proof

```

```

  show  $\bigwedge x y z. x \sqcup y \sqcup z = y \sqcup z \sqcup x$ 
    using ba1-associative commutative by force
  show  $\bigwedge x y z. (x \sqcup - y = z \sqcup - z) = (x \sqcup y = x)$ 
    by (simp add: ba1-complement)
qed

```

```
end
```

3.3 Meredith's Equational Axioms

The following axiomatisation is from [22, page 221 (1) {A,N}].

```

class boolean-algebra-mp = sup + uminus +
  assumes ba-mp-1:  $\neg(-x \sqcup y) \sqcup x = x$ 
  assumes ba-mp-2:  $\neg(-x \sqcup y) \sqcup (z \sqcup y) = y \sqcup (z \sqcup x)$ 
begin

```

```

subclass huntington

```

```

proof
  show  $\bigwedge x y z. x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$ 
    by (metis ba-mp-1 ba-mp-2)
  show  $\bigwedge x y. x \sqcup y = y \sqcup x$ 
    by (metis ba-mp-1 ba-mp-2)
  show  $\bigwedge x y. x = \neg(-x \sqcup y) \sqcup \neg(-x \sqcup -y)$ 
    by (metis ba-mp-1 ba-mp-2)
qed

```

```
end
```

```

context huntington
begin

```

```

sublocale mp-h: boolean-algebra-mp

```

```

proof
  show 1:  $\bigwedge x y. \neg(-x \sqcup y) \sqcup x = x$ 
    by (metis h-ba1.ba1-associative h-ba1.ba1-complement huntington)
  show  $\bigwedge x y z. \neg(-x \sqcup y) \sqcup (z \sqcup y) = y \sqcup (z \sqcup x)$ 
  proof -

```

```

fix x y z
have y =  $\neg(\neg x \sqcup \neg y) \sqcup y$ 
  using 1 h-ba1.ba1-commutative by auto
thus  $\neg(\neg x \sqcup y) \sqcup (z \sqcup y) = y \sqcup (z \sqcup x)$ 
  by (metis h-ba1.ba1-associative h-ba1.ba1-commutative huntington)
qed
qed

end

```

3.4 An Equational Axiomatisation based on Semilattices

The following version is an equational axiomatisation based on semilattices. We add the double complement rule and that *top* is unique. The final axiom *ba3-export* encodes the logical statement $P \vee Q = P \vee (\neg P \wedge Q)$. Its dual appears in [1].

Theorem 5

```

class boolean-algebra-3 = sup + uminus +
  assumes ba3-associative:  $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ 
  assumes ba3-commutative:  $x \sqcup y = y \sqcup x$ 
  assumes ba3-idempotent[simp]:  $x \sqcup x = x$ 
  assumes ba3-double-complement[simp]:  $\neg\neg x = x$ 
  assumes ba3-top-unique:  $x \sqcup \neg x = y \sqcup \neg y$ 
  assumes ba3-export:  $x \sqcup \neg(x \sqcup y) = x \sqcup \neg y$ 
begin

subclass huntington
proof
  show  $\bigwedge x y z. x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$ 
    by (simp add: ba3-associative)
  show  $\bigwedge x y. x \sqcup y = y \sqcup x$ 
    by (simp add: ba3-commutative)
  show  $\bigwedge x y. x = \neg(\neg x \sqcup y) \sqcup \neg(\neg x \sqcup \neg y)$ 
    by (metis ba3-commutative ba3-double-complement ba3-export ba3-idempotent
ba3-top-unique)
qed

end

context huntington
begin

sublocale h-ba3: boolean-algebra-3
proof
  show  $\bigwedge x y z. x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$ 
    by (simp add: h-ba1.ba1-associative)
  show  $\bigwedge x y. x \sqcup y = y \sqcup x$ 
    by (simp add: h-ba1.ba1-commutative)

```

```

show 3:  $\bigwedge x. x \sqcup x = x$ 
  using h-ba1.ba1-complement by blast
show 4:  $\bigwedge x. \neg \neg x = x$ 
  by (metis h-ba1.ba1-commutative huntington top-unique)
show  $\bigwedge x y. x \sqcup \neg x = y \sqcup \neg y$ 
  by (simp add: top-unique)
show  $\bigwedge x y. x \sqcup \neg (x \sqcup y) = x \sqcup \neg y$ 
  using 3 4 by (smt h-ba1.ba1-ba2.ba2-associative-commutative
h-ba1.ba1-complement)
qed

end

```

4 Subset Boolean Algebras

We apply Huntington's axioms to the range of a unary operation, which serves as complement on the range. This gives a Boolean algebra structure on the range without imposing any further constraints on the set. The obtained structure is used as a reference in the subsequent development and to inherit the results proved here. This is taken from [13, 16] and follows the development of Boolean algebras in [20].

Definition 6

```

class subset-boolean-algebra = sup + uminus +
  assumes sub-associative:  $\neg x \sqcup (\neg y \sqcup \neg z) = (\neg x \sqcup \neg y) \sqcup \neg z$ 
  assumes sub-commutative:  $\neg x \sqcup \neg y = \neg y \sqcup \neg x$ 
  assumes sub-complement:  $\neg x = \neg(\neg \neg x \sqcup \neg y) \sqcup \neg(\neg \neg x \sqcup \neg \neg y)$ 
  assumes sub-sup-closed:  $\neg x \sqcup \neg y = \neg \neg(\neg x \sqcup \neg y)$ 
begin

```

uniqueness of *top*, resulting in the lemma *top-def* to replace the assumption *sub-top-def*

```

lemma top-unique:
   $\neg x \sqcup \neg \neg x = \neg y \sqcup \neg \neg y$ 
  by (metis sub-associative sub-commutative sub-complement)

```

consequences for join and complement

```

lemma double-negation[simp]:
   $\neg \neg \neg x = \neg x$ 
  by (metis sub-complement sub-sup-closed)

```

```

lemma complement-1:
   $\neg \neg x = \neg(\neg x \sqcup \neg y) \sqcup \neg(\neg x \sqcup \neg \neg y)$ 
  by (metis double-negation sub-complement)

```

```

lemma sup-right-zero-var:
   $\neg x \sqcup (\neg y \sqcup \neg \neg y) = \neg z \sqcup \neg \neg z$ 
  by (smt complement-1 sub-associative sub-sup-closed top-unique)

```


lemma *sup-right-unit-idempotent*:
 $-x \sqcup -x = -x \sqcup -(-y \sqcup --y)$
by (*metis complement-1 double-negation sub-sup-closed sup-right-zero-var*)

lemma *sup-idempotent[simp]*:
 $-x \sqcup -x = -x$
by (*smt complement-1 double-negation sub-associative sup-right-unit-idempotent*)

lemma *complement-2*:
 $-x = -(-(-x \sqcup -y) \sqcup -(-x \sqcup --y))$
using *complement-1* **by** *auto*

lemma *sup-eq-cases*:
 $-x \sqcup -y = -x \sqcup -z \implies --x \sqcup -y = --x \sqcup -z \implies -y = -z$
by (*metis complement-2 sub-commutative*)

lemma *sup-eq-cases-2*:
 $-y \sqcup -x = -z \sqcup -x \implies -y \sqcup --x = -z \sqcup --x \implies -y = -z$
using *sub-commutative sup-eq-cases* **by** *auto*

end

Definition 7

class *subset-extended* = *sup* + *inf* + *minus* + *uminus* + *bot* + *top* + *ord* +
assumes *sub-top-def*: $top = (THE\ x . \forall y . x = -y \sqcup --y)$
assumes *sub-bot-def*: $bot = -(THE\ x . \forall y . x = -y \sqcup --y)$
assumes *sub-inf-def*: $-x \sqcap -y = -(-x \sqcup --y)$
assumes *sub-minus-def*: $-x - -y = -(-x \sqcup -y)$
assumes *sub-less-eq-def*: $-x \leq -y \iff -x \sqcup -y = -y$
assumes *sub-less-def*: $-x < -y \iff -x \sqcup -y = -y \wedge \neg (-y \sqcup -x = -x)$

class *subset-boolean-algebra-extended* = *subset-boolean-algebra* + *subset-extended*
begin

lemma *top-def*:
 $top = -x \sqcup --x$
using *sub-top-def top-unique* **by** *blast*

consequences for meet

lemma *inf-closed*:
 $-x \sqcap -y = --(-x \sqcap -y)$
by (*simp add: sub-inf-def*)

lemma *inf-associative*:
 $-x \sqcap (-y \sqcap -z) = (-x \sqcap -y) \sqcap -z$
using *sub-associative sub-inf-def sub-sup-closed* **by** *auto*

lemma *inf-commutative*:
 $-x \sqcap -y = -y \sqcap -x$
by (*simp add: sub-commutative sub-inf-def*)

lemma *inf-idempotent[simp]*:
 $-x \sqcap -x = -x$
by (*simp add: sub-inf-def*)

lemma *inf-absorb[simp]*:
 $(-x \sqcup -y) \sqcap -x = -x$
by (*metis complement-1 sup-idempotent sub-inf-def sub-associative sub-sup-closed*)

lemma *sup-absorb[simp]*:
 $-x \sqcup (-x \sqcap -y) = -x$
by (*metis sub-associative sub-complement sub-inf-def sup-idempotent*)

lemma *inf-demorgan*:
 $-(-x \sqcap -y) = --x \sqcup --y$
using *sub-inf-def sub-sup-closed by auto*

lemma *sub-sup-demorgan*:
 $-(-x \sqcup -y) = --x \sqcap --y$
by (*simp add: sub-inf-def*)

lemma *sup-cases*:
 $-x = (-x \sqcap -y) \sqcup (-x \sqcap --y)$
by (*metis inf-closed inf-demorgan sub-complement*)

lemma *inf-cases*:
 $-x = (-x \sqcup -y) \sqcap (-x \sqcup --y)$
by (*metis complement-2 sub-sup-closed sub-sup-demorgan*)

lemma *inf-complement-intro*:
 $(-x \sqcup -y) \sqcap --x = -y \sqcap --x$
proof –
have $(-x \sqcup -y) \sqcap --x = (-x \sqcup -y) \sqcap (--x \sqcup -y) \sqcap --x$
by (*metis inf-absorb inf-associative sub-sup-closed*)
also have $\dots = -y \sqcap --x$
by (*metis inf-cases sub-commutative*)
finally show *?thesis*
qed

lemma *sup-complement-intro*:
 $-x \sqcup -y = -x \sqcup (--x \sqcap -y)$
by (*metis inf-absorb inf-commutative inf-complement-intro sub-sup-closed sup-cases*)

lemma *inf-left-dist-sup*:

$$-x \sqcap (-y \sqcup -z) = (-x \sqcap -y) \sqcup (-x \sqcap -z)$$

proof –

have $-x \sqcap (-y \sqcup -z) = (-x \sqcap (-y \sqcup -z) \sqcap -y) \sqcup (-x \sqcap (-y \sqcup -z) \sqcap --y)$

by (*metis sub-inf-def sub-sup-closed sup-cases*)

also have $\dots = (-x \sqcap -y) \sqcup (-x \sqcap -z \sqcap --y)$

by (*metis inf-absorb inf-associative inf-complement-intro sub-sup-closed*)

also have $\dots = (-x \sqcap -y) \sqcup ((-x \sqcap -y \sqcap -z) \sqcup (-x \sqcap -z \sqcap --y))$

using *sub-associative sub-inf-def sup-absorb* **by** *auto*

also have $\dots = (-x \sqcap -y) \sqcup ((-x \sqcap -z \sqcap -y) \sqcup (-x \sqcap -z \sqcap --y))$

by (*metis inf-associative inf-commutative*)

also have $\dots = (-x \sqcap -y) \sqcup (-x \sqcap -z)$

by (*metis sub-inf-def sup-cases*)

finally show *?thesis*

•
qed

lemma *sup-left-dist-inf*:

$$-x \sqcup (-y \sqcap -z) = (-x \sqcup -y) \sqcap (-x \sqcup -z)$$

proof –

have $-x \sqcup (-y \sqcap -z) = -(--x \sqcap (--y \sqcup --z))$

by (*metis sub-inf-def sub-sup-closed sub-sup-demorgan*)

also have $\dots = (-x \sqcup -y) \sqcap (-x \sqcup -z)$

by (*metis inf-left-dist-sup sub-sup-closed sub-sup-demorgan*)

finally show *?thesis*

•
qed

lemma *sup-right-dist-inf*:

$$(-y \sqcap -z) \sqcup -x = (-y \sqcup -x) \sqcap (-z \sqcup -x)$$

using *sub-commutative sub-inf-def sup-left-dist-inf* **by** *auto*

lemma *inf-right-dist-sup*:

$$(-y \sqcup -z) \sqcap -x = (-y \sqcap -x) \sqcup (-z \sqcap -x)$$

by (*metis inf-commutative inf-left-dist-sup sub-sup-closed*)

lemma *case-duality*:

$$(--x \sqcap -y) \sqcup (-x \sqcap -z) = (-x \sqcup -y) \sqcap (--x \sqcup -z)$$

proof –

have *1*: $--x \sqcap --y \sqcap ----x = --x \sqcap -y$

using *inf-commutative inf-complement-intro sub-sup-closed sub-sup-demorgan*

by *auto*

have *2*: $-(----x \sqcup -(--x \sqcup -z)) = ----x \sqcap ----z$

by (*metis (no-types) double-negation sup-complement-intro sub-sup-demorgan*)

have *3*: $-(--x \sqcap --y) \sqcap -x = -x$

using *inf-commutative inf-left-dist-sup sub-sup-closed sub-sup-demorgan* **by**

auto

hence $-(--x \sqcap --y) = -x \sqcup -y$

using *sub-sup-closed sub-sup-demorgan* **by** *auto*

thus *?thesis*
by (*metis double-negation 1 2 3 inf-associative inf-left-dist-sup sup-complement-intro*)
qed

lemma case-duality-2:

$(-x \sqcap -y) \sqcup (---x \sqcap -z) = (-x \sqcup -z) \sqcap (---x \sqcup -y)$
using *case-duality sub-commutative sub-inf-def by auto*

lemma complement-cases:

$((-v \sqcap -w) \sqcup (---v \sqcap -x)) \sqcap -((v \sqcap -y) \sqcup (---v \sqcap -z)) = (-v \sqcap -w \sqcap ---y) \sqcup (---v \sqcap -x \sqcap ---z)$

proof –

have 1: $(---v \sqcup -w) = ---(-v \sqcup -w) \wedge (-v \sqcup -x) = ---(-v \sqcup -x) \wedge (---v \sqcup ---y) = ---(-v \sqcup ---y) \wedge (-v \sqcup ---z) = ---(-v \sqcup ---z)$
using *sub-inf-def sub-sup-closed by auto*
have 2: $(-v \sqcup (-x \sqcap ---z)) = ---(-v \sqcup (-x \sqcap ---z))$
using *sub-inf-def sub-sup-closed by auto*
have $((-v \sqcap -w) \sqcup (---v \sqcap -x)) \sqcap -((v \sqcap -y) \sqcup (---v \sqcap -z)) = ((-v \sqcap -w) \sqcup (---v \sqcap -x)) \sqcap (-(v \sqcap -y) \sqcap -(---v \sqcap -z))$
using *sub-inf-def by auto*
also have $... = ((-v \sqcap -w) \sqcup (---v \sqcap -x)) \sqcap ((---v \sqcup ---y) \sqcap (-v \sqcup ---z))$
using *inf-demorgan by auto*
also have $... = (---v \sqcup -w) \sqcap (-v \sqcup -x) \sqcap ((---v \sqcup ---y) \sqcap (-v \sqcup ---z))$
by (*metis case-duality double-negation*)
also have $... = (---v \sqcup -w) \sqcap ((-v \sqcup -x) \sqcap ((---v \sqcup ---y) \sqcap (-v \sqcup ---z)))$
by (*metis 1 inf-associative sub-inf-def*)
also have $... = (---v \sqcup -w) \sqcap ((-v \sqcup -x) \sqcap (---v \sqcup ---y) \sqcap (-v \sqcup ---z))$
by (*metis 1 inf-associative*)
also have $... = (---v \sqcup -w) \sqcap ((---v \sqcup ---y) \sqcap (-v \sqcup -x) \sqcap (-v \sqcup ---z))$
by (*metis 1 inf-commutative*)
also have $... = (---v \sqcup -w) \sqcap ((---v \sqcup ---y) \sqcap ((-v \sqcup -x) \sqcap (-v \sqcup ---z)))$
by (*metis 1 inf-associative*)
also have $... = (---v \sqcup -w) \sqcap ((---v \sqcup ---y) \sqcap (-v \sqcup (-x \sqcap ---z)))$
by (*simp add: sup-left-dist-inf*)
also have $... = (---v \sqcup -w) \sqcap (---v \sqcup ---y) \sqcap (-v \sqcup (-x \sqcap ---z))$
using *1 2 by (metis inf-associative)*
also have $... = (---v \sqcup (-w \sqcap ---y)) \sqcap (-v \sqcup (-x \sqcap ---z))$
by (*simp add: sup-left-dist-inf*)
also have $... = (-v \sqcap (-w \sqcap ---y)) \sqcup (---v \sqcap (-x \sqcap ---z))$
by (*metis case-duality complement-1 complement-2 sub-inf-def*)
also have $... = (-v \sqcap -w \sqcap ---y) \sqcup (---v \sqcap -x \sqcap ---z)$
by (*simp add: inf-associative*)
finally show *?thesis*

qed

lemma inf-cases-2: $---x = -(-x \sqcap -y) \sqcap -(-x \sqcap ---y)$
using *sub-inf-def sup-cases by auto*

consequences for *top* and *bot*

lemma *sup-complement*[*simp*]:

$$-x \sqcup --x = top$$

using *top-def* **by** *auto*

lemma *inf-complement*[*simp*]:

$$-x \sqcap --x = bot$$

by (*metis sub-bot-def sub-inf-def sub-top-def top-def*)

lemma *complement-bot*[*simp*]:

$$-bot = top$$

using *inf-complement inf-demorgan sup-complement* **by** *fastforce*

lemma *complement-top*[*simp*]:

$$-top = bot$$

using *sub-bot-def sub-top-def* **by** *blast*

lemma *sup-right-zero*[*simp*]:

$$-x \sqcup top = top$$

using *sup-right-zero-var* **by** *auto*

lemma *sup-left-zero*[*simp*]:

$$top \sqcup -x = top$$

by (*metis complement-bot sub-commutative sup-right-zero*)

lemma *inf-right-unit*[*simp*]:

$$-x \sqcap bot = bot$$

by (*metis complement-bot complement-top double-negation sub-sup-demorgan sup-right-zero*)

lemma *inf-left-unit*[*simp*]:

$$bot \sqcap -x = bot$$

by (*metis complement-top inf-commutative inf-right-unit*)

lemma *sup-right-unit*[*simp*]:

$$-x \sqcup bot = -x$$

using *sup-right-unit-idempotent* **by** *auto*

lemma *sup-left-unit*[*simp*]:

$$bot \sqcup -x = -x$$

by (*metis complement-top sub-commutative sup-right-unit*)

lemma *inf-right-zero*[*simp*]:

$$-x \sqcap top = -x$$

by (*metis inf-left-dist-sup sup-cases top-def*)

lemma *sub-inf-left-zero*[*simp*]:

$$top \sqcap -x = -x$$

using *inf-absorb top-def* **by** *fastforce*

lemma *bot-double-complement*[simp]:

$--bot = bot$

by *simp*

lemma *top-double-complement*[simp]:

$--top = top$

by *simp*

consequences for the order

lemma *reflexive*:

$-x \leq -x$

by (*simp add: sub-less-eq-def*)

lemma *transitive*:

$-x \leq -y \implies -y \leq -z \implies -x \leq -z$

by (*metis sub-associative sub-less-eq-def*)

lemma *antisymmetric*:

$-x \leq -y \implies -y \leq -x \implies -x = -y$

by (*simp add: sub-commutative sub-less-eq-def*)

lemma *sub-bot-least*:

$bot \leq -x$

using *sup-left-unit complement-top sub-less-eq-def* **by** *blast*

lemma *top-greatest*:

$-x \leq top$

using *complement-bot sub-less-eq-def sup-right-zero* **by** *blast*

lemma *upper-bound-left*:

$-x \leq -x \sqcup -y$

by (*metis sub-associative sub-less-eq-def sub-sup-closed sup-idempotent*)

lemma *upper-bound-right*:

$-y \leq -x \sqcup -y$

using *sub-commutative upper-bound-left* **by** *fastforce*

lemma *sub-sup-left-isotone*:

assumes $-x \leq -y$

shows $-x \sqcup -z \leq -y \sqcup -z$

proof $-$

have $-x \sqcup -y = -y$

by (*meson assms sub-less-eq-def*)

thus *?thesis*

by (*metis (full-types) sub-associative sub-commutative sub-sup-closed upper-bound-left*)

qed

lemma *sub-sup-right-isotone*:

$$-x \leq -y \implies -z \sqcup -x \leq -z \sqcup -y$$

by (*simp add: sub-commutative sub-sup-left-isotone*)

lemma *sup-isotone*:

assumes $-p \leq -q$

and $-r \leq -s$

shows $-p \sqcup -r \leq -q \sqcup -s$

proof –

have $\bigwedge x y. \neg -x \leq -y \sqcup -r \vee -x \leq -y \sqcup -s$

by (*metis (full-types) assms(2) sub-sup-closed sub-sup-right-isotone transitive*)

thus *?thesis*

by (*metis (no-types) assms(1) sub-sup-closed sub-sup-left-isotone*)

qed

lemma *sub-complement-antitone*:

$$-x \leq -y \implies --y \leq --x$$

by (*metis inf-absorb inf-demorgan sub-less-eq-def*)

lemma *less-eq-inf*:

$$-x \leq -y \iff -x \sqcap -y = -x$$

by (*metis inf-absorb inf-commutative sub-less-eq-def upper-bound-right sup-absorb*)

lemma *inf-complement-left-antitone*:

$$-x \leq -y \implies -(-y \sqcap -z) \leq -(x \sqcap -z)$$

by (*simp add: sub-complement-antitone inf-demorgan sub-sup-left-isotone*)

lemma *sub-inf-left-isotone*:

$$-x \leq -y \implies -x \sqcap -z \leq -y \sqcap -z$$

using *sub-complement-antitone inf-closed inf-complement-left-antitone* **by** *fastforce*

lemma *sub-inf-right-isotone*:

$$-x \leq -y \implies -z \sqcap -x \leq -z \sqcap -y$$

by (*simp add: inf-commutative sub-inf-left-isotone*)

lemma *inf-isotone*:

assumes $-p \leq -q$

and $-r \leq -s$

shows $-p \sqcap -r \leq -q \sqcap -s$

proof –

have $\forall w x y z. (-w \leq -x \sqcap -y \vee \neg -w \leq -x \sqcap -z) \vee \neg -z \leq -y$

by (*metis (no-types) inf-closed sub-inf-right-isotone transitive*)

thus *?thesis*

by (*metis (no-types) assms inf-closed sub-inf-left-isotone*)

qed

lemma *least-upper-bound*:

$-x \leq -z \wedge -y \leq -z \iff -x \sqcup -y \leq -z$
by (*metis sub-sup-closed transitive upper-bound-right sup-idempotent sup-isotone upper-bound-left*)

lemma *lower-bound-left*:

$-x \sqcap -y \leq -x$
by (*metis sub-inf-def upper-bound-right sup-absorb*)

lemma *lower-bound-right*:

$-x \sqcap -y \leq -y$
using *inf-commutative lower-bound-left* **by** *fastforce*

lemma *greatest-lower-bound*:

$-x \leq -y \wedge -x \leq -z \iff -x \leq -y \sqcap -z$
by (*metis inf-closed sub-inf-left-isotone less-eq-inf transitive lower-bound-left lower-bound-right*)

lemma *less-eq-sup-top*:

$-x \leq -y \iff \neg\neg x \sqcup -y = \text{top}$
by (*metis complement-1 inf-commutative inf-complement-intro sub-inf-left-zero less-eq-inf sub-complement sup-complement-intro top-def*)

lemma *less-eq-inf-bot*:

$-x \leq -y \iff -x \sqcap \neg\neg y = \text{bot}$
by (*metis complement-bot complement-top double-negation inf-demorgan less-eq-sup-top sub-inf-def*)

lemma *shunting*:

$-x \sqcap -y \leq -z \iff -y \leq \neg\neg x \sqcup -z$
proof (*cases* $\neg\neg x \sqcup (-z \sqcup \neg\neg y) = \text{top}$)
case *True*
have $\forall v w. -v \leq -w \vee -w \sqcup \neg\neg v \neq \text{top}$
using *less-eq-sup-top sub-commutative* **by** *blast*
thus *?thesis*
by (*metis True sub-associative sub-commutative sub-inf-def sub-sup-closed*)
next
case *False*
hence $\neg\neg x \sqcup (-z \sqcup \neg\neg y) \neq \text{top} \wedge \neg\neg y \leq -z \sqcup \neg\neg x$
by (*metis (no-types) less-eq-sup-top sub-associative sub-commutative sub-sup-closed*)
thus *?thesis*
using *less-eq-sup-top sub-associative sub-commutative sub-inf-def sub-sup-closed* **by** *auto*
qed

lemma *shunting-right*:

$-x \sqcap -y \leq -z \iff -x \leq -z \sqcup \neg\neg y$
by (*metis inf-commutative sub-commutative shunting*)

lemma *sup-less-eq-cases*:

assumes $-z \leq -x \sqcup -y$

and $-z \leq --x \sqcup -y$

shows $-z \leq -y$

proof –

have $-z \leq (-x \sqcup -y) \sqcap (--x \sqcup -y)$

by (*metis assms greatest-lower-bound sub-sup-closed*)

also have $\dots = -y$

by (*metis inf-cases sub-commutative*)

finally show *?thesis*

qed

lemma *sup-less-eq-cases-2*:

$-x \sqcup -y \leq -x \sqcup -z \implies --x \sqcup -y \leq --x \sqcup -z \implies -y \leq -z$

by (*metis least-upper-bound sup-less-eq-cases sub-sup-closed*)

lemma *sup-less-eq-cases-3*:

$-y \sqcup -x \leq -z \sqcup -x \implies -y \sqcup --x \leq -z \sqcup --x \implies -y \leq -z$

by (*simp add: sup-less-eq-cases-2 sub-commutative*)

lemma *inf-less-eq-cases*:

$-x \sqcap -y \leq -z \implies --x \sqcap -y \leq -z \implies -y \leq -z$

by (*simp add: shunting sup-less-eq-cases*)

lemma *inf-less-eq-cases-2*:

$-x \sqcap -y \leq -x \sqcap -z \implies --x \sqcap -y \leq --x \sqcap -z \implies -y \leq -z$

by (*metis greatest-lower-bound inf-closed inf-less-eq-cases*)

lemma *inf-less-eq-cases-3*:

$-y \sqcap -x \leq -z \sqcap -x \implies -y \sqcap --x \leq -z \sqcap --x \implies -y \leq -z$

by (*simp add: inf-commutative inf-less-eq-cases-2*)

lemma *inf-eq-cases*:

$-x \sqcap -y = -x \sqcap -z \implies --x \sqcap -y = --x \sqcap -z \implies -y = -z$

by (*metis inf-commutative sup-cases*)

lemma *inf-eq-cases-2*:

$-y \sqcap -x = -z \sqcap -x \implies -y \sqcap --x = -z \sqcap --x \implies -y = -z$

using *inf-commutative inf-eq-cases* **by** *auto*

lemma *wnf-lemma-1*:

$((-x \sqcup -y) \sqcap (--x \sqcup -z)) \sqcup -x = -x \sqcup -y$

proof –

have $\forall u v w. (-u \sqcap (-v \sqcup --w)) \sqcup -w = -u \sqcup -w$

by (*metis inf-right-zero sub-associative sub-sup-closed sup-complement sup-idempotent sup-right-dist-inf*)

thus *?thesis*

by (*metis (no-types) sub-associative sub-commutative sub-sup-closed*)

sup-idempotent)
qed

lemma *wnf-lemma-2*:
 $((-x \sqcup -y) \sqcap (-z \sqcup --y)) \sqcup -y = -x \sqcup -y$
using *sub-commutative wnf-lemma-1* **by** *fastforce*

lemma *wnf-lemma-3*:
 $((-x \sqcup -z) \sqcap (--x \sqcup -y)) \sqcup --x = --x \sqcup -y$
by (*metis case-duality case-duality-2 double-negation sub-commutative wnf-lemma-2*)

lemma *wnf-lemma-4*:
 $((-z \sqcup -y) \sqcap (-x \sqcup --y)) \sqcup --y = -x \sqcup --y$
using *sub-commutative wnf-lemma-3* **by** *auto*

end

class *subset-boolean-algebra'* = *sup* + *uminus* +
assumes *sub-associative'*: $-x \sqcup (-y \sqcup -z) = (-x \sqcup -y) \sqcup -z$
assumes *sub-commutative'*: $-x \sqcup -y = -y \sqcup -x$
assumes *sub-complement'*: $-x = -(-x \sqcup -y) \sqcup -(-x \sqcup -y)$
assumes *sub-sup-closed'*: $\exists z. -x \sqcup -y = -z$
begin

subclass *subset-boolean-algebra*

proof

show $\bigwedge x y z. -x \sqcup (-y \sqcup -z) = -x \sqcup -y \sqcup -z$

by (*simp add: sub-associative'*)

show $\bigwedge x y. -x \sqcup -y = -y \sqcup -x$

by (*simp add: sub-commutative'*)

show $\bigwedge x y. -x = -(-x \sqcup -y) \sqcup -(-x \sqcup -y)$

by (*simp add: sub-complement'*)

show $\bigwedge x y. -x \sqcup -y = --(-x \sqcup -y)$

proof -

fix *x y*

have $\forall x y. -y \sqcup (-(-y \sqcup -x) \sqcup -(-x \sqcup -y)) = -y \sqcup --x$

by (*metis (no-types) sub-associative' sub-commutative' sub-complement'*)

hence $\forall x. ---x = -x$

by (*metis (no-types) sub-commutative' sub-complement'*)

thus $-x \sqcup -y = --(-x \sqcup -y)$

by (*metis sub-sup-closed'*)

qed

qed

end

We introduce a type for the range of complement and show that it is an instance of *boolean-algebra*.

typedef (overloaded) 'a boolean-subset = { x::'a::uminus . $\exists y . x = -y$ }
by *auto*

lemma *simp-boolean-subset*[*simp*]:
 $\exists y . \text{Rep-boolean-subset } x = -y$
using *Rep-boolean-subset* **by** *simp*

setup-lifting *type-definition-boolean-subset*

Theorem 8.1

instantiation *boolean-subset* :: (subset-boolean-algebra) *huntington*
begin

lift-definition *sup-boolean-subset* :: 'a boolean-subset \Rightarrow 'a boolean-subset \Rightarrow 'a
boolean-subset **is** *sup*
using *sub-sup-closed* **by** *auto*

lift-definition *uminus-boolean-subset* :: 'a boolean-subset \Rightarrow 'a boolean-subset **is**
uminus
by *auto*

instance

proof

show $\bigwedge x y z :: 'a \text{ boolean-subset. } x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
apply *transfer*

using *sub-associative* **by** *blast*

show $\bigwedge x y :: 'a \text{ boolean-subset. } x \sqcup y = y \sqcup x$
apply *transfer*

using *sub-commutative* **by** *blast*

show $\bigwedge x y :: 'a \text{ boolean-subset. } x = -(-x \sqcup y) \sqcup -(-x \sqcup -y)$
apply *transfer*

using *sub-complement* **by** *blast*

qed

end

Theorem 8.2

instantiation *boolean-subset* :: (subset-boolean-algebra-extended)
huntington-extended
begin

lift-definition *inf-boolean-subset* :: 'a boolean-subset \Rightarrow 'a boolean-subset \Rightarrow 'a
boolean-subset **is** *inf*
using *inf-closed* **by** *auto*

lift-definition *minus-boolean-subset* :: 'a boolean-subset \Rightarrow 'a boolean-subset \Rightarrow 'a
boolean-subset **is** *minus*
using *sub-minus-def* **by** *auto*

lift-definition *bot-boolean-subset* :: 'a boolean-subset is bot
by (*metis complement-top*)

lift-definition *top-boolean-subset* :: 'a boolean-subset is top
by (*metis complement-bot*)

lift-definition *less-eq-boolean-subset* :: 'a boolean-subset \Rightarrow 'a boolean-subset \Rightarrow
bool is *less-eq* .

lift-definition *less-boolean-subset* :: 'a boolean-subset \Rightarrow 'a boolean-subset \Rightarrow *bool*
is *less* .

instance

proof

show $1: top = (THE\ x.\ \forall\ y::'a\ \text{boolean-subset}.\ x = y \sqcup -\ y)$

proof (*rule the-equality[symmetric]*)

show $\forall\ y::'a\ \text{boolean-subset}.\ top = y \sqcup -\ y$

apply *transfer*

by *auto*

show $\bigwedge\ x::'a\ \text{boolean-subset}.\ \forall\ y.\ x = y \sqcup -\ y \implies x = top$

apply *transfer*

by *force*

qed

have $(bot::'a\ \text{boolean-subset}) = -\ top$

apply *transfer*

by *simp*

thus $bot = -\ (THE\ x.\ \forall\ y::'a\ \text{boolean-subset}.\ x = y \sqcup -\ y)$

using 1 **by** *simp*

show $\bigwedge\ x\ y::'a\ \text{boolean-subset}.\ x \sqcap y = -\ (-\ x \sqcup -\ y)$

apply *transfer*

using *sub-inf-def* **by** *blast*

show $\bigwedge\ x\ y::'a\ \text{boolean-subset}.\ x -\ y = -\ (-\ x \sqcup y)$

apply *transfer*

using *sub-minus-def* **by** *blast*

show $\bigwedge\ x\ y::'a\ \text{boolean-subset}.\ (x \leq y) = (x \sqcup y = y)$

apply *transfer*

using *sub-less-eq-def* **by** *blast*

show $\bigwedge\ x\ y::'a\ \text{boolean-subset}.\ (x < y) = (x \sqcup y = y \wedge y \sqcup x \neq x)$

apply *transfer*

using *sub-less-def* **by** *blast*

qed

end

5 Subset Boolean algebras with Additional Structure

We now discuss axioms that make the range of a unary operation a Boolean algebra, but add further properties that are common to the intended models. In the intended models, the unary operation can be a complement, a pseudocomplement or the antidomain operation. For simplicity, we mostly call the unary operation ‘complement’.

We first look at structures based only on join and complement, and then add axioms for the remaining operations of Boolean algebras. In the intended models, the operation that is meet on the range of the complement can be a meet in the whole algebra or composition.

5.1 Axioms Derived from the New Axiomatisation

The axioms of the first algebra are based on *boolean-algebra-3*.

Definition 9

```

class subset-boolean-algebra-1 = sup + uminus +
  assumes sba1-associative:  $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ 
  assumes sba1-commutative:  $x \sqcup y = y \sqcup x$ 
  assumes sba1-idempotent[simp]:  $x \sqcup x = x$ 
  assumes sba1-double-complement[simp]:  $---x = -x$ 
  assumes sba1-bot-unique:  $-(x \sqcup -x) = -(y \sqcup -y)$ 
  assumes sba1-export:  $-x \sqcup -( -x \sqcup y) = -x \sqcup -y$ 
begin

```

Theorem 11.1

```

subclass subset-boolean-algebra

```

```

proof

```

```

  show  $\bigwedge x y z. -x \sqcup (-y \sqcup -z) = -x \sqcup -y \sqcup -z$ 

```

```

    by (simp add: sba1-associative)

```

```

  show  $\bigwedge x y. -x \sqcup -y = -y \sqcup -x$ 

```

```

    by (simp add: sba1-commutative)

```

```

  show  $\bigwedge x y. -x = -(- -x \sqcup -y) \sqcup -(- -x \sqcup - -y)$ 

```

```

    by (smt sba1-bot-unique sba1-commutative sba1-double-complement sba1-export
sba1-idempotent)

```

```

  thus  $\bigwedge x y. -x \sqcup -y = - -(-x \sqcup -y)$ 

```

```

    by (metis sba1-double-complement sba1-export)

```

```

qed

```

```

definition sba1-bot  $\equiv$  THE  $x . \forall z . x = -(z \sqcup -z)$ 

```

```

lemma sba1-bot:

```

```

  sba1-bot =  $-(z \sqcup -z)$ 

```

```

  using sba1-bot-def sba1-bot-unique by auto

```

```

end

```

Boolean algebra operations based on join and complement

Definition 10

```

class subset-extended-1 = sup + inf + minus + uminus + bot + top + ord +
  assumes ba-bot: bot = (THE x .  $\forall z . x = -(z \sqcup -z)$ )
  assumes ba-top: top = -(THE x .  $\forall z . x = -(z \sqcup -z)$ )
  assumes ba-inf:  $-x \sqcap -y = -(-x \sqcup -y)$ 
  assumes ba-minus:  $-x - -y = -(-x \sqcup -y)$ 
  assumes ba-less-eq:  $x \leq y \longleftrightarrow x \sqcup y = y$ 
  assumes ba-less:  $x < y \longleftrightarrow x \sqcup y = y \wedge \neg (y \sqcup x = x)$ 

```

```

class subset-extended-2 = subset-extended-1 +
  assumes ba-bot-unique:  $-(x \sqcup -x) = -(y \sqcup -y)$ 
begin

```

```

lemma ba-bot-def:
  bot =  $-(z \sqcup -z)$ 
  using ba-bot ba-bot-unique by auto

```

```

lemma ba-top-def:
  top =  $--(z \sqcup -z)$ 
  using ba-bot-def ba-top by simp

```

end

Subset forms Boolean Algebra, extended by Boolean algebra operations

```

class subset-boolean-algebra-1-extended = subset-boolean-algebra-1 +
  subset-extended-1
begin

```

```

subclass subset-extended-2
proof
  show  $\bigwedge x y . -(x \sqcup -x) = -(y \sqcup -y)$ 
    by (simp add: sba1-bot-unique)
qed

```

```

subclass semilattice-sup
proof
  show  $\bigwedge x y . (x < y) = (x \leq y \wedge \neg y \leq x)$ 
    by (simp add: ba-less ba-less-eq)
  show  $\bigwedge x . x \leq x$ 
    by (simp add: ba-less-eq)
  show  $\bigwedge x y z . x \leq y \implies y \leq z \implies x \leq z$ 
    by (metis sba1-associative ba-less-eq)
  show  $\bigwedge x y . x \leq y \implies y \leq x \implies x = y$ 
    by (simp add: sba1-commutative ba-less-eq)
  show  $\bigwedge x y . x \leq x \sqcup y$ 
    by (simp add: sba1-associative ba-less-eq)
  thus  $\bigwedge y x . y \leq x \sqcup y$ 
    by (simp add: sba1-commutative)
  show  $\bigwedge y x z . y \leq x \implies z \leq x \implies y \sqcup z \leq x$ 
    by (metis sba1-associative ba-less-eq)

```

qed

Theorem 11.2

subclass *subset-boolean-algebra-extended*

proof

show $top = (THE\ x.\ \forall\ y.\ x = -\ y \sqcup -\ -\ y)$

by (*smt ba-bot ba-bot-def ba-top sub-sup-closed the-equality*)

thus $bot = -\ (THE\ x.\ \forall\ y.\ x = -\ y \sqcup -\ -\ y)$

using *ba-bot-def ba-top-def* by *force*

show $\bigwedge x\ y.\ -\ x \sqcap -\ y = -\ (-\ -\ x \sqcup -\ -\ y)$

by (*simp add: ba-inf*)

show $\bigwedge x\ y.\ -\ x \sqcup -\ -\ y = -\ (-\ -\ x \sqcup -\ y)$

by (*simp add: ba-minus*)

show $\bigwedge x\ y.\ (-\ x \leq -\ y) = (-\ x \sqcup -\ y = -\ y)$

using *le-iff-sup* by *auto*

show $\bigwedge x\ y.\ (-\ x < -\ y) = (-\ x \sqcup -\ y = -\ y \wedge -\ y \sqcup -\ x \neq -\ x)$

by (*simp add: ba-less*)

qed

end

5.2 Stronger Assumptions based on Join and Complement

We add further axioms covering properties common to the antidomain and (pseudo)complement instances.

Definition 12

```
class subset-boolean-algebra-2 = sup + uminus +
  assumes sba2-associative:  $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ 
  assumes sba2-commutative:  $x \sqcup y = y \sqcup x$ 
  assumes sba2-idempotent[simp]:  $x \sqcup x = x$ 
  assumes sba2-bot-unit:  $x \sqcup -(y \sqcup -y) = x$ 
  assumes sba2-sub-sup-demorgan:  $-(x \sqcup y) = -(-x \sqcup -y)$ 
  assumes sba2-export:  $-x \sqcup -(x \sqcup y) = -x \sqcup -y$ 
begin
```

Theorem 13.1

subclass *subset-boolean-algebra-1*

proof

show $\bigwedge x\ y\ z.\ x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$

by (*simp add: sba2-associative*)

show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$

by (*simp add: sba2-commutative*)

show $\bigwedge x.\ x \sqcup x = x$

by *simp*

show $\bigwedge x.\ -\ -\ -\ x = -\ x$

by (*metis sba2-idempotent sba2-sub-sup-demorgan*)

show $\bigwedge x\ y.\ -\ (x \sqcup -\ x) = -\ (y \sqcup -\ y)$

by (*metis sba2-bot-unit sba2-commutative*)

show $\bigwedge x y. \neg x \sqcup \neg (\neg x \sqcup y) = \neg x \sqcup \neg y$
by (*simp add: sba2-export*)
qed

Theorem 13.2

lemma *double-complement-dist-sup*:
 $\neg\neg(x \sqcup y) = \neg\neg x \sqcup \neg\neg y$
by (*metis sba2-commutative sba2-export sba2-idempotent sba2-sub-sup-demorgan*)

lemma *maddux-3-3[simp]*:
 $\neg(x \sqcup y) \sqcup \neg(x \sqcup \neg y) = \neg x$
by (*metis double-complement-dist-sup sba1-double-complement sba2-commutative sub-complement*)

lemma *huntington-3-pp[simp]*:
 $\neg(\neg x \sqcup \neg y) \sqcup \neg(\neg x \sqcup y) = \neg\neg x$
using *sba2-commutative maddux-3-3* **by** *fastforce*

end

class *subset-boolean-algebra-2-extended* = *subset-boolean-algebra-2* +
subset-extended-1
begin

subclass *subset-boolean-algebra-1-extended* ..

subclass *bounded-semilattice-sup-bot*
proof
show $\bigwedge x. \text{bot} \leq x$
using *sba2-bot-unit ba-bot-def sup-right-divisibility* **by** *auto*
qed

Theorem 13.3

lemma *complement-antitone*:
 $x \leq y \implies \neg y \leq \neg x$
by (*metis le-iff-sup maddux-3-3 sba2-export sup-monoid.add-commute*)

lemma *double-complement-isotone*:
 $x \leq y \implies \neg\neg x \leq \neg\neg y$
by (*simp add: complement-antitone*)

lemma *sup-demorgan*:
 $\neg(x \sqcup y) = \neg x \sqcap \neg y$
using *sba2-sub-sup-demorgan ba-inf* **by** *auto*

end

5.3 Axioms for Meet

We add further axioms of *inf* covering properties common to the antidomain and pseudocomplement instances. We omit the left distributivity rule and the right zero rule as they do not hold in some models. In particular, the operation *inf* does not have to be commutative.

Definition 14

```

class subset-boolean-algebra-3-extended = subset-boolean-algebra-2-extended +
  assumes sba3-inf-associative:  $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$ 
  assumes sba3-inf-right-dist-sup:  $(x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$ 
  assumes sba3-inf-complement-bot:  $\neg x \sqcap x = \text{bot}$ 
  assumes sba3-inf-left-unit[simp]:  $\text{top} \sqcap x = x$ 
  assumes sba3-complement-inf-double-complement:  $\neg(x \sqcap \neg y) = \neg(x \sqcap y)$ 
begin

```

Theorem 15

lemma *inf-left-zero*:

$$\text{bot} \sqcap x = \text{bot}$$

by (*metis inf-right-unit sba3-inf-associative sba3-inf-complement-bot*)

lemma *inf-double-complement-export*:

$$\neg(\neg x \sqcap y) = \neg x \sqcap \neg y$$

by (*metis inf-closed sba3-complement-inf-double-complement*)

lemma *inf-left-isotone*:

$$x \leq y \implies x \sqcap z \leq y \sqcap z$$

using *sba3-inf-right-dist-sup sup-right-divisibility* **by** *auto*

lemma *inf-complement-export*:

$$\neg(\neg x \sqcap y) = \neg x \sqcap \neg y$$

by (*metis inf-double-complement-export sba1-double-complement*)

lemma *double-complement-above*:

$$\neg\neg x \sqcap x = x$$

by (*metis sup-monoid.add-0-right complement-bot inf-demorgan sba1-double-complement sba3-inf-complement-bot sba3-inf-right-dist-sup sba3-inf-left-unit*)

lemma $x \leq y \implies z \sqcap x \leq z \sqcap y$ **nitpick** [*expect=genuine*] **oops**

lemma $x \sqcap \text{top} = x$ **nitpick** [*expect=genuine*] **oops**

lemma $x \sqcap y = y \sqcap x$ **nitpick** [*expect=genuine*] **oops**

end

5.4 Stronger Assumptions for Meet

The following axioms also hold in both models, but follow from the axioms of *subset-boolean-algebra-5-operations*.

Definition 16

```

class subset-boolean-algebra-4-extended = subset-boolean-algebra-3-extended +
  assumes sba4-inf-right-unit[simp]:  $x \sqcap top = x$ 
  assumes inf-right-isotone:  $x \leq y \implies z \sqcap x \leq z \sqcap y$ 
begin

lemma  $x \sqcup top = top$  nitpick [expect=genuine] oops
lemma  $x \sqcap bot = bot$  nitpick [expect=genuine] oops
lemma  $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$  nitpick [expect=genuine] oops
lemma  $(x \sqcap y = bot) = (x \leq -y)$  nitpick [expect=genuine] oops

end

```

6 Boolean Algebras in Stone Algebras

We specialise *inf* to meet and complement to pseudocomplement. This puts Stone algebras into the picture; for these it is well known that regular elements form a Boolean subalgebra [12].

Definition 17

```

class subset-boolean-algebra-5-extended = subset-boolean-algebra-3-extended +
  assumes sba5-inf-commutative:  $x \sqcap y = y \sqcap x$ 
  assumes sba5-inf-absorb:  $x \sqcap (x \sqcup y) = x$ 
begin

subclass distrib-lattice-bot
proof
  show  $\bigwedge x y. x \sqcap y \leq x$ 
    by (metis sba5-inf-commutative sba3-inf-right-dist-sup sba5-inf-absorb
sup-right-divisibility)
  show  $\bigwedge x y. x \sqcap y \leq y$ 
    by (metis inf-left-isotone sba5-inf-absorb sba5-inf-commutative sup-ge2)
  show  $\bigwedge x y z. x \leq y \implies x \leq z \implies x \leq y \sqcap z$ 
    by (metis inf-left-isotone sba5-inf-absorb sup.orderE sup-monoid.add-commute)
  show  $\bigwedge x y z. x \sqcup y \sqcap z = (x \sqcup y) \sqcap (x \sqcup z)$ 
    by (metis sba3-inf-right-dist-sup sba5-inf-absorb sba5-inf-commutative
sup-assoc)
qed

lemma inf-demorgan-2:
   $-(x \sqcap y) = -x \sqcup -y$ 
  using sba3-complement-inf-double-complement sba5-inf-commutative
sub-sup-closed sub-sup-demorgan by auto

lemma inf-export:
   $x \sqcap -(x \sqcap y) = x \sqcap -y$ 
  using inf-demorgan-2 sba3-inf-complement-bot sba3-inf-right-dist-sup
sba5-inf-commutative by auto

```

lemma *complement-inf[simp]*:
 $x \sqcap -x = \text{bot}$
using *sba3-inf-complement-bot sba5-inf-commutative* **by** *auto*

Theorem 18.2

subclass *stone-algebra*
proof
show $\bigwedge x. x \leq \text{top}$
by (*simp add: inf.absorb-iff2*)
show $\bigwedge x y. (x \sqcap y = \text{bot}) = (x \leq - y)$
by (*metis (full-types) complement-bot complement-inf inf.cobounded1 inf.order-iff inf-export sba3-complement-inf-double-complement sba3-inf-left-unit*)
show $\bigwedge x. - x \sqcup - - x = \text{top}$
by *simp*
qed

Theorem 18.1

subclass *subset-boolean-algebra-4-extended*
proof
show $\bigwedge x. x \sqcap \text{top} = x$
by *simp*
show $\bigwedge x y z. x \leq y \implies z \sqcap x \leq z \sqcap y$
using *inf.sup-right-isotone* **by** *blast*
qed
end

context *stone-algebra-extended*
begin

Theorem 18.3

subclass *subset-boolean-algebra-5-extended*
proof
show $\bigwedge x y z. x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
using *sup-assoc* **by** *auto*
show $\bigwedge x y. x \sqcup y = y \sqcup x$
by (*simp add: sup-commute*)
show $\bigwedge x. x \sqcup x = x$
by *simp*
show $\bigwedge x y. x \sqcup - (y \sqcup - y) = x$
by *simp*
show $\bigwedge x y. - (x \sqcup y) = - (- - x \sqcup - - y)$
by *auto*
show $\bigwedge x y. - x \sqcup - (- x \sqcup y) = - x \sqcup - y$
by (*metis maddux-3-21-pp p-dist-sup regular-closed-p*)
show $\text{bot} = (\text{THE } x. \forall z. x = - (z \sqcup - z))$
by *simp*
thus $\text{top} = - (\text{THE } x. \forall z. x = - (z \sqcup - z))$
using *p-bot* **by** *blast*

```

show  $\bigwedge x y. \neg x \sqcap \neg y = \neg (\neg \neg x \sqcup \neg \neg y)$ 
  by simp
show  $\bigwedge x y. \neg x \neg \neg y = \neg (\neg \neg x \sqcup \neg y)$ 
  by auto
show  $\bigwedge x y. (x \leq y) = (x \sqcup y = y)$ 
  by (simp add: le-iff-sup)
thus  $\bigwedge x y. (x < y) = (x \sqcup y = y \wedge y \sqcup x \neq x)$ 
  by (simp add: less-le-not-le)
show  $\bigwedge x y z. x \sqcap (y \sqcap z) = x \sqcap y \sqcap z$ 
  by (simp add: inf.sup-monoid.add-assoc)
show  $\bigwedge x y z. (x \sqcup y) \sqcap z = x \sqcap z \sqcup y \sqcap z$ 
  by (simp add: inf-sup-distrib2)
show  $\bigwedge x. \neg x \sqcap x = \text{bot}$ 
  by simp
show  $\bigwedge x. \text{top} \sqcap x = x$ 
  by simp
show  $\bigwedge x y. \neg (x \sqcap \neg \neg y) = \neg (x \sqcap y)$ 
  by simp
show  $\bigwedge x y. x \sqcap y = y \sqcap x$ 
  by (simp add: inf-commute)
show  $\bigwedge x y. x \sqcap (x \sqcup y) = x$ 
  by simp
qed
end

```

7 Domain Semirings

The following development of tests in IL-semirings, prepre-domain semirings, pre-domain semirings and domain semirings is mostly based on [23]; see also [4]. See [5] for domain axioms in idempotent semirings. See [3, 19] for domain axioms in semigroups and monoids. Some variants have been implemented in [11].

7.1 Idempotent Left Semirings

Definition 19

```

class il-semiring = sup + inf + bot + top + ord +
  assumes il-associative:  $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ 
  assumes il-commutative:  $x \sqcup y = y \sqcup x$ 
  assumes il-idempotent[simp]:  $x \sqcup x = x$ 
  assumes il-bot-unit:  $x \sqcup \text{bot} = x$ 
  assumes il-inf-associative:  $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$ 
  assumes il-inf-right-dist-sup:  $(x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$ 
  assumes il-inf-left-unit[simp]:  $\text{top} \sqcap x = x$ 
  assumes il-inf-right-unit[simp]:  $x \sqcap \text{top} = x$ 
  assumes il-sub-inf-left-zero[simp]:  $\text{bot} \sqcap x = \text{bot}$ 

```

assumes *il-sub-inf-right-isotone*: $x \leq y \implies z \sqcap x \leq z \sqcap y$
assumes *il-less-eq*: $x \leq y \iff x \sqcup y = y$
assumes *il-less-def*: $x < y \iff x \leq y \wedge \neg(y \leq x)$
begin

lemma *il-unit-bot*: $\text{bot} \sqcup x = x$
using *il-bot-unit il-commutative* **by** *fastforce*

subclass *order*

proof

show $\bigwedge x y. (x < y) = (x \leq y \wedge \neg y \leq x)$
by (*simp add: il-less-def*)
show $\bigwedge x. x \leq x$
by (*simp add: il-less-eq*)
show $\bigwedge x y z. x \leq y \implies y \leq z \implies x \leq z$
by (*metis il-associative il-less-eq*)
show $\bigwedge x y. x \leq y \implies y \leq x \implies x = y$
by (*simp add: il-commutative il-less-eq*)
qed

lemma *il-sub-inf-right-isotone-var*:
 $(x \sqcap y) \sqcup (x \sqcap z) \leq x \sqcap (y \sqcup z)$
by (*smt il-associative il-commutative il-idempotent il-less-eq*
il-sub-inf-right-isotone)

lemma *il-sub-inf-left-isotone*:
 $x \leq y \implies x \sqcap z \leq y \sqcap z$
by (*metis il-inf-right-dist-sup il-less-eq*)

lemma *il-sub-inf-left-isotone-var*:
 $(y \sqcap x) \sqcup (z \sqcap x) \leq (y \sqcup z) \sqcap x$
by (*simp add: il-inf-right-dist-sup*)

lemma *sup-left-isotone*:
 $x \leq y \implies x \sqcup z \leq y \sqcup z$
by (*smt il-associative il-commutative il-idempotent il-less-eq*)

lemma *sup-right-isotone*:
 $x \leq y \implies z \sqcup x \leq z \sqcup y$
by (*simp add: il-commutative sup-left-isotone*)

lemma *bot-least*:
 $\text{bot} \leq x$
by (*simp add: il-less-eq il-unit-bot*)

lemma *less-eq-bot*:
 $x \leq \text{bot} \iff x = \text{bot}$
by (*simp add: il-bot-unit il-less-eq*)

abbreviation *are-complementary* :: 'a ⇒ 'a ⇒ bool
where *are-complementary* x y ≡ x ⊔ y = top ∧ x ⊓ y = bot ∧ y ⊓ x = bot

abbreviation *test* :: 'a ⇒ bool
where *test* x ≡ ∃ y . *are-complementary* x y

definition *tests* :: 'a set
where *tests* = { x . *test* x }

lemma *bot-test*:
test bot
by (*simp* *add: il-unit-bot*)

lemma *top-test*:
test top
by (*simp* *add: il-bot-unit*)

lemma *test-sub-identity*:
test x ⇒ x ≤ top
using *il-associative il-less-eq* **by** *auto*

lemma *neg-unique*:
are-complementary x y ⇒ *are-complementary* x z ⇒ y = z
by (*metis* *order.antisym il-inf-left-unit il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone-var*)

definition *neg* :: 'a ⇒ 'a (!)
where !x ≡ *THE* y . *are-complementary* x y

lemma *neg-char*:
assumes *test* x
shows *are-complementary* x (!x)
proof (*unfold neg-def*)
from *assms* **obtain** y **where** 1: *are-complementary* x y
by *auto*
show *are-complementary* x (*THE* y . *are-complementary* x y)
proof (*rule theI*)
show *are-complementary* x y
using 1 **by** *simp*
show $\bigwedge z. \textit{are-complementary } x z \implies z = y$
using 1 *neg-unique* **by** *blast*
qed
qed

lemma *are-complementary-symmetric*:
are-complementary x y ⇔ *are-complementary* y x
using *il-commutative* **by** *auto*

lemma *neg-test*:

$test\ x \implies test\ (!x)$
using *are-complementary-symmetric neg-char* **by** *blast*

lemma *are-complementary-test*:
 $test\ x \implies are-complementary\ x\ y \implies test\ y$
using *il-commutative* **by** *auto*

lemma *neg-involutive*:
 $test\ x \implies !(!x) = x$
using *are-complementary-symmetric neg-char neg-unique* **by** *blast*

lemma *test-inf-left-below*:
 $test\ x \implies x \sqcap y \leq y$
by (*metis il-associative il-idempotent il-inf-left-unit il-inf-right-dist-sup il-less-eq*)

lemma *test-inf-right-below*:
 $test\ x \implies y \sqcap x \leq y$
by (*metis il-inf-right-unit il-sub-inf-right-isotone test-sub-identity*)

lemma *neg-bot*:
 $!bot = top$
using *il-unit-bot neg-char* **by** *fastforce*

lemma *neg-top*:
 $!top = bot$
using *bot-test neg-bot neg-involutive* **by** *fastforce*

lemma *test-inf-idempotent*:
 $test\ x \implies x \sqcap x = x$
by (*metis il-bot-unit il-inf-left-unit il-inf-right-dist-sup*)

lemma *test-inf-semicommutative*:
assumes *test x*
and *test y*
shows $x \sqcap y \leq y \sqcap x$
proof –
have $x \sqcap y = (y \sqcap x \sqcap y) \sqcup (!y \sqcap x \sqcap y)$
by (*metis assms(2) il-inf-left-unit il-inf-right-dist-sup neg-char*)
also have $\dots \leq (y \sqcap x \sqcap y) \sqcup (!y \sqcap y)$
proof –
obtain z **where** *are-complementary y z*
using *assms(2)* **by** *blast*
hence $y \sqcap (x \sqcap y) \sqcup !y \sqcap (x \sqcap y) \leq y \sqcap (x \sqcap y)$
by (*metis assms(1) calculation il-sub-inf-left-isotone il-bot-unit il-idempotent il-inf-associative il-less-eq neg-char test-inf-right-below*)
thus *?thesis*
by (*simp add: il-associative il-inf-associative il-less-eq*)
qed
also have $\dots \leq (y \sqcap x) \sqcup (!y \sqcap y)$

by (*metis* *assms*(2) *il-bot-unit* *il-inf-right-unit* *il-sub-inf-right-isotone* *neg-char* *test-sub-identity*)
also have ... = $y \sqcap x$
by (*simp* *add*: *assms*(2) *il-bot-unit* *neg-char*)
finally show ?thesis

qed

lemma *test-inf-commutative*:
 $test\ x \implies test\ y \implies x \sqcap y = y \sqcap x$
by (*simp* *add*: *order.antisym* *test-inf-semicommutative*)

lemma *test-inf-bot*:
 $test\ x \implies x \sqcap bot = bot$
using *il-inf-associative* *test-inf-idempotent* **by** *fastforce*

lemma *test-absorb-1*:
 $test\ x \implies test\ y \implies x \sqcup (x \sqcap y) = x$
using *il-commutative* *il-less-eq* *test-inf-right-below* **by** *auto*

lemma *test-absorb-2*:
 $test\ x \implies test\ y \implies x \sqcup (y \sqcap x) = x$
by (*metis* *test-absorb-1* *test-inf-commutative*)

lemma *test-absorb-3*:
 $test\ x \implies test\ y \implies x \sqcap (x \sqcup y) = x$
apply (*rule* *order.antisym*)
apply (*metis* *il-associative* *il-inf-right-unit* *il-less-eq* *il-sub-inf-right-isotone* *test-sub-identity*)
by (*metis* *il-sub-inf-right-isotone-var* *test-absorb-1* *test-inf-idempotent*)

lemma *test-absorb-4*:
 $test\ x \implies test\ y \implies (x \sqcup y) \sqcap x = x$
by (*smt* *il-inf-right-dist-sup* *test-inf-idempotent* *il-commutative* *il-less-eq* *test-inf-left-below*)

lemma *test-import-1*:
assumes *test* *x*
and *test* *y*
shows $x \sqcup (!x \sqcap y) = x \sqcup y$
proof –
have $x \sqcup (!x \sqcap y) = x \sqcup ((y \sqcup !y) \sqcap x) \sqcup (!x \sqcap y)$
by (*simp* *add*: *assms*(2) *neg-char*)
also have ... = $x \sqcup (!y \sqcap x) \sqcup (x \sqcap y) \sqcup (!x \sqcap y)$
by (*smt* *assms* *il-associative* *il-commutative* *il-inf-right-dist-sup* *test-inf-commutative*)
also have ... = $x \sqcup ((x \sqcup !x) \sqcap y)$
by (*smt* *calculation* *il-associative* *il-commutative* *il-idempotent* *il-inf-right-dist-sup*)

also have $\dots = x \sqcup y$
by (*simp add: assms(1) neg-char*)
finally show *?thesis*
 \cdot
qed

lemma *test-import-2*:
assumes *test x*
and *test y*
shows $x \sqcup (y \sqcap !x) = x \sqcup y$
proof –
obtain *z* **where** *1: are-complementary y z*
using *assms(2)* **by** *moura*
obtain *w* **where** *2: are-complementary x w*
using *assms(1)* **by** *auto*
hence $x \sqcap !x = \text{bot}$
using *neg-char* **by** *blast*
hence $!x \sqcap y = y \sqcap !x$
using *1 2* **by** (*metis il-commutative neg-char test-inf-commutative*)
thus *?thesis*
using *1 2* **by** (*metis test-import-1*)
qed

lemma *test-import-3*:
assumes *test x*
shows $(!x \sqcup y) \sqcap x = y \sqcap x$
by (*simp add: assms(1) il-inf-right-dist-sup il-unit-bot neg-char*)

lemma *test-import-4*:
assumes *test x*
and *test y*
shows $(!x \sqcup y) \sqcap x = x \sqcap y$
by (*metis assms test-import-3 test-inf-commutative*)

lemma *test-inf*:
 $\text{test } x \implies \text{test } y \implies \text{test } z \implies z \leq x \sqcap y \iff z \leq x \wedge z \leq y$
apply (*rule iffI*)
using *dual-order.trans test-inf-left-below test-inf-right-below* **apply** *blast*
by (*smt il-less-eq il-sub-inf-right-isotone test-absorb-4*)

lemma *test-shunting*:
assumes *test x*
and *test y*
shows $x \sqcap y \leq z \iff x \leq !y \sqcup z$
proof
assume *1: x \sqcap y \leq z*
have $x = (!y \sqcap x) \sqcup (y \sqcap x)$
by (*metis assms(2) il-commutative il-inf-left-unit il-inf-right-dist-sup neg-char*)
also have $\dots \leq !y \sqcup (y \sqcap x)$

by (*simp add: assms(1) sup-left-isotone test-inf-right-below*)
 also have $\dots \leq !y \sqcup z$
 using 1 by (*simp add: assms sup-right-isotone test-inf-commutative*)
 finally show $x \leq !y \sqcup z$

.
 next
 assume $x \leq !y \sqcup z$
 hence $x \sqcap y \leq (!y \sqcup z) \sqcap y$
 using *il-sub-inf-left-isotone* by *blast*
 also have $\dots = z \sqcap y$
 by (*simp add: assms(2) test-import-3*)
 also have $\dots \leq z$
 by (*simp add: assms(2) test-inf-right-below*)
 finally show $x \sqcap y \leq z$

.
 qed

lemma *test-shunting-bot*:
 assumes *test x*
 and *test y*
 shows $x \leq y \iff x \sqcap !y \leq \text{bot}$
 by (*simp add: assms il-bot-unit neg-involutive neg-test test-shunting*)

lemma *test-shunting-bot-eq*:
 assumes *test x*
 and *test y*
 shows $x \leq y \iff x \sqcap !y = \text{bot}$
 by (*simp add: assms test-shunting-bot less-eq-bot*)

lemma *neg-antitone*:
 assumes *test x*
 and *test y*
 and $x \leq y$
 shows $!y \leq !x$

proof –
 have 1: $x \sqcap !y = \text{bot}$
 using *assms test-shunting-bot-eq* by *blast*
 have 2: $x \sqcup !x = \text{top}$
 by (*simp add: assms(1) neg-char*)
 have *are-complementary y (!y)*
 by (*simp add: assms(2) neg-char*)
 thus ?thesis
 using 1 2 by (*metis il-unit-bot il-commutative il-inf-left-unit*
il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone test-sub-identity)

qed

lemma *test-sup-neg-1*:
 assumes *test x*
 and *test y*

shows $(x \sqcup y) \sqcup (!x \sqcap !y) = \text{top}$
proof –
have $x \sqcup !x = \text{top}$
by (*simp add: assms(1) neg-char*)
hence $x \sqcup (y \sqcup !x) = \text{top}$
by (*metis assms(2) il-associative il-commutative il-idempotent*)
hence $x \sqcup (y \sqcup !x \sqcap !y) = \text{top}$
by (*simp add: assms neg-test test-import-2*)
thus *?thesis*
by (*simp add: il-associative*)
qed

lemma *test-sup-neg-2*:
assumes *test x*
and *test y*
shows $(x \sqcup y) \sqcap (!x \sqcap !y) = \text{bot}$
proof –
have *1: are-complementary y (!y)*
by (*simp add: assms(2) neg-char*)
obtain *z where 2: are-complementary x z*
using *assms(1) by auto*
hence $!x = z$
using *neg-char neg-unique by blast*
thus *?thesis*
using *1 2 by (metis are-complementary-symmetric il-inf-associative neg-involutive test-import-3 test-inf-bot test-inf-commutative)*
qed

lemma *de-morgan-1*:
assumes *test x*
and *test y*
and *test (x \sqcap y)*
shows $!(x \sqcap y) = !x \sqcup !y$
proof (*rule order.antisym*)
have *1: test (!(x \sqcap y))*
by (*simp add: assms neg-test*)
have $x \leq (x \sqcap y) \sqcup !y$
by (*metis (full-types) assms il-commutative neg-char test-shunting test-shunting-bot-eq*)
hence $x \sqcap !(x \sqcap y) \leq !y$
using *1 by (simp add: assms(1,3) neg-involutive test-shunting)*
hence $!(x \sqcap y) \sqcap x \leq !y$
using *1 by (metis assms(1) test-inf-commutative)*
thus $!(x \sqcap y) \leq !x \sqcup !y$
using *1 assms(1) test-shunting by blast*
have *2: !x \leq !(x \sqcap y)*
by (*simp add: assms neg-antitone test-inf-right-below*)
have $!y \leq !(x \sqcap y)$
by (*simp add: assms neg-antitone test-inf-left-below*)

thus $!x \sqcup !y \leq !(x \sqcap y)$
using 2 **by** (*metis il-associative il-less-eq*)
qed

lemma *de-morgan-2*:

assumes *test x*
and *test y*
and *test (x \sqcup y)*
shows $!(x \sqcup y) = !x \sqcap !y$
proof (*rule order.antisym*)
have 1: $!(x \sqcup y) \leq !x$
by (*metis assms il-inf-left-unit il-sub-inf-left-isotone neg-antitone test-absorb-3 test-sub-identity*)
have $!(x \sqcup y) \leq !y$
by (*metis assms il-commutative il-inf-left-unit il-sub-inf-left-isotone neg-antitone test-absorb-3 test-sub-identity*)
thus $!(x \sqcup y) \leq !x \sqcap !y$
using 1 **by** (*simp add: assms neg-test test-inf*)
have $top \leq x \sqcup y \sqcup !(x \sqcup y)$
by (*simp add: assms(3) neg-char*)
hence $top \sqcap !x \leq y \sqcup !(x \sqcup y)$
by (*smt assms(1) assms(3) il-commutative il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone il-unit-bot neg-char test-sub-identity*)
thus $!x \sqcap !y \leq !(x \sqcup y)$
by (*simp add: assms(1) assms(2) neg-involutive neg-test test-shunting*)
qed

lemma *test-inf-closed-sup-complement*:

assumes *test x*
and *test y*
and $\forall u v . test\ u \wedge test\ v \longrightarrow test\ (u \sqcap v)$
shows $!x \sqcap !y \sqcap (x \sqcup y) = bot$
proof –
have 1: $!(!x \sqcap !y) = x \sqcup y$
by (*simp add: assms de-morgan-1 neg-involutive neg-test*)
have *test (!(!x \sqcap !y))*
by (*metis assms neg-test*)
thus *?thesis*
using 1 **by** (*metis assms(1,2) de-morgan-2 neg-char*)
qed

lemma *test-sup-complement-sup-closed*:

assumes *test x*
and *test y*
and $\forall u v . test\ u \wedge test\ v \longrightarrow !u \sqcap !v \sqcap (u \sqcup v) = bot$
shows *test (x \sqcup y)*
by (*meson assms test-sup-neg-1 test-sup-neg-2*)

lemma *test-inf-closed-sup-closed*:

```

assumes test x
  and test y
  and  $\forall u v . test\ u \wedge test\ v \longrightarrow test\ (u \sqcap v)$ 
  shows test (x  $\sqcup$  y)
using assms test-inf-closed-sup-complement test-sup-complement-sup-closed by
simp
end

```

7.2 Prepredomain Semirings

```

class dom =
  fixes d :: 'a  $\Rightarrow$  'a

class ppd-semiring = il-semiring + dom +
  assumes d-closed: test (d x)
  assumes d1: x  $\leq$  d x  $\sqcap$  x
begin

lemma d-sub-identity:
  d x  $\leq$  top
  using d-closed test-sub-identity by blast

lemma d1-eq:
  x = d x  $\sqcap$  x
proof -
  have x = (d x  $\sqcup$  top)  $\sqcap$  x
  using d-sub-identity il-less-eq by auto
  thus ?thesis
  using d1 il-commutative il-inf-right-dist-sup il-less-eq by force
qed

lemma d-increasing-sub-identity:
  x  $\leq$  top  $\Longrightarrow$  x  $\leq$  d x
  by (metis d1-eq il-inf-right-unit il-sub-inf-right-isotone)

lemma d-top:
  d top = top
  by (simp add: d-increasing-sub-identity d-sub-identity dual-order.antisym)

lemma d-bot-only:
  d x = bot  $\Longrightarrow$  x = bot
  by (metis d1-eq il-sub-inf-left-zero)

lemma d-strict: d bot  $\leq$  bot nitpick [expect=genuine] oops
lemma d-isotone-var: d x  $\leq$  d (x  $\sqcup$  y) nitpick [expect=genuine] oops
lemma d-fully-strict: d x = bot  $\longleftrightarrow$  x = bot nitpick [expect=genuine] oops
lemma test-d-fixpoint: test x  $\Longrightarrow$  d x = x nitpick [expect=genuine] oops

```

end

7.3 Predomain Semirings

```
class pd-semiring = ppd-semiring +
  assumes d2: test p  $\implies$  d (p  $\sqcap$  x)  $\leq$  p
begin
```

```
lemma d-strict:
  d bot  $\leq$  bot
  using bot-test d2 by fastforce
```

```
lemma d-strict-eq:
  d bot = bot
  using d-strict il-bot-unit il-less-eq by auto
```

```
lemma test-d-fixpoint:
  test x  $\implies$  d x = x
  by (metis order.antisym d1-eq d2 test-inf-idempotent test-inf-right-below)
```

```
lemma d-surjective:
  test x  $\implies$   $\exists$  y . d y = x
  using test-d-fixpoint by blast
```

```
lemma test-d-fixpoint-iff:
  test x  $\longleftrightarrow$  d x = x
  by (metis d-closed test-d-fixpoint)
```

```
lemma d-surjective-iff:
  test x  $\longleftrightarrow$  ( $\exists$  y . d y = x)
  using d-surjective d-closed by blast
```

```
lemma tests-d-range:
  tests = range d
  using tests-def image-def d-surjective-iff by auto
```

```
lemma lp:
  assumes test y
  shows d x  $\leq$  y  $\longleftrightarrow$  x  $\leq$  y  $\sqcap$  x
  by (metis assms d1-eq d2 order.eq-iff il-sub-inf-left-isotone test-inf-left-below)
```

```
lemma gla:
  assumes test y
  shows y  $\leq$  !(d x)  $\longleftrightarrow$  y  $\sqcap$  x  $\leq$  bot
proof -
  obtain ad where 1:  $\forall$  x. are-complementary (d x) (ad x)
  using d-closed by moura
  hence 2:  $\forall$  x y. d (d y  $\sqcap$  x)  $\leq$  d y
  using d2 by blast
```

have 3: $\forall x. ad\ x \sqcap x = bot$
using 1 **by** (*metis d1-eq il-inf-associative il-sub-inf-left-zero*)
have 4: $\forall x\ y. d\ y \sqcap x \sqcup ad\ y \sqcap x = top \sqcap x$
using 1 **by** (*metis il-inf-right-dist-sup*)
have 5: $\forall x\ y\ z. z \sqcap y \leq x \sqcap y \vee (z \sqcup x) \sqcap y \neq x \sqcap y$
by (*simp add: il-inf-right-dist-sup il-less-eq*)
have 6: $\forall x. !(d\ x) = ad\ x$
using 1 *neg-char neg-unique* **by** *blast*
have 7: $\forall x. top \sqcap x = x$
by *auto*
hence $\forall x. y \sqcap x \sqcup !y \sqcap x = x$
by (*metis assms il-inf-right-dist-sup neg-char*)
thus *?thesis*
using 1 2 3 4 5 6 7 **by** (*metis assms d1-eq il-commutative il-less-eq test-d-fixpoint*)
qed

lemma *gla-var*:
 $test\ y \implies y \sqcap d\ x \leq bot \iff y \sqcap x \leq bot$
using *gla d-closed il-bot-unit test-shunting* **by** *auto*

lemma *llp-var*:
assumes *test y*
shows $y \leq !(d\ x) \iff x \leq !y \sqcap x$
apply (*rule iffI*)
apply (*metis (no-types, opaque-lifting) assms gla Least-equality il-inf-left-unit il-inf-right-dist-sup il-less-eq il-unit-bot order.refl neg-char*)
by (*metis assms gla gla-var llp il-commutative il-sub-inf-right-isotone neg-char*)

lemma *d-idempotent*:
 $d\ (d\ x) = d\ x$
using *d-closed test-d-fixpoint-iff* **by** *auto*

lemma *d-neg*:
 $test\ x \implies d\ (!x) = !x$
using *il-commutative neg-char test-d-fixpoint-iff* **by** *fastforce*

lemma *d-fully-strict*:
 $d\ x = bot \iff x = bot$
using *d-strict-eq d-bot-only* **by** *blast*

lemma *d-ad-comp*:
 $!(d\ x) \sqcap x = bot$
proof –
have $\forall x. !(d\ x) \sqcap d\ x = bot$
by (*simp add: d-closed neg-char*)
thus *?thesis*
by (*metis d1-eq il-inf-associative il-sub-inf-left-zero*)
qed

lemma *d-isotone*:
assumes $x \leq y$
shows $d\ x \leq d\ y$
proof –
obtain *ad* **where** $1: \forall x. \text{are-complementary } (d\ x) (ad\ x)$
using *d-closed* **by** *moura*
hence $ad\ y \sqcap x \leq bot$
by (*metis* *assms* *d1-eq* *il-inf-associative* *il-sub-inf-left-zero*
il-sub-inf-right-isotone)
thus *?thesis*
using 1 **by** (*metis* *d2* *il-bot-unit* *il-inf-left-unit* *il-inf-right-dist-sup* *il-less-eq*)
qed

lemma *d-isotone-var*:
 $d\ x \leq d\ (x \sqcup y)$
using *d-isotone* *il-associative* *il-less-eq* **by** *auto*

lemma *d3-conv*:
 $d\ (x \sqcap y) \leq d\ (x \sqcap d\ y)$
by (*metis* (*mono-tags*, *opaque-lifting*) *d1-eq* *d2* *d-closed* *il-inf-associative*)

lemma *d-test-inf-idempotent*:
 $d\ x \sqcap d\ x = d\ x$
by (*metis* *d-idempotent* *d1-eq*)

lemma *d-test-inf-closed*:
assumes *test* x
and *test* y
shows $d\ (x \sqcap y) = x \sqcap y$
proof (*rule* *order.antisym*)
have $d\ (x \sqcap y) = d\ (x \sqcap y) \sqcap d\ (x \sqcap y)$
by (*simp* *add*: *d-test-inf-idempotent*)
also have $\dots \leq x \sqcap d\ (x \sqcap y)$
by (*simp* *add*: *assms*(1) *d2* *il-sub-inf-left-isotone*)
also have $\dots \leq x \sqcap y$
by (*metis* *assms* *d-isotone* *il-sub-inf-right-isotone* *test-inf-left-below*
test-d-fixpoint)
finally show $d\ (x \sqcap y) \leq x \sqcap y$
 \cdot
show $x \sqcap y \leq d\ (x \sqcap y)$
using *assms* *d-increasing-sub-identity* *dual-order.trans* *test-inf-left-below*
test-sub-identity **by** *blast*
qed

lemma *test-inf-closed*:
 $test\ x \implies test\ y \implies test\ (x \sqcap y)$
using *d-test-inf-closed* *test-d-fixpoint-iff* **by** *simp*

lemma *test-sup-closed*:

test x \implies *test y* \implies *test (x \sqcup y)*

using *test-inf-closed test-inf-closed-sup-closed* **by** *simp*

lemma *d-export*:

assumes *test x*

shows $d (x \sqcap y) = x \sqcap d y$

proof (*rule order.antisym*)

have 1: $d (x \sqcap y) \leq x$

by (*simp add: assms d2*)

have $d (x \sqcap y) \leq d y$

by (*metis assms d-isotone-var il-inf-left-unit il-inf-right-dist-sup*)

thus $d (x \sqcap y) \leq x \sqcap d y$

using 1 **by** (*metis assms d-idempotent llp dual-order.trans*

il-sub-inf-right-isotone)

have $y = (!x \sqcap y) \sqcup (x \sqcap y)$

by (*metis assms il-commutative il-inf-left-unit il-inf-right-dist-sup neg-char*)

also have $\dots = (!x \sqcap y) \sqcup (d (x \sqcap y) \sqcap x \sqcap y)$

by (*metis d1-eq il-inf-associative*)

also have $\dots = (!x \sqcap y) \sqcup (d (x \sqcap y) \sqcap y)$

using 1 **by** (*smt calculation d1-eq il-associative il-commutative*

il-inf-associative il-inf-right-dist-sup il-less-eq il-sub-inf-right-isotone-var)

also have $\dots = (!x \sqcup d (x \sqcap y)) \sqcap y$

by (*simp add: il-inf-right-dist-sup*)

finally have $y \leq (!x \sqcup d (x \sqcap y)) \sqcap y$

by *simp*

hence $d y \leq !x \sqcup d (x \sqcap y)$

using *assms llp test-sup-closed neg-test d-closed* **by** *simp*

hence $d y \sqcap x \leq d (x \sqcap y)$

by (*simp add: assms d-closed test-shunting*)

thus $x \sqcap d y \leq d (x \sqcap y)$

by (*metis assms d-closed test-inf-commutative*)

qed

lemma *test-inf-left-dist-sup*:

assumes *test x*

and *test y*

and *test z*

shows $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$

proof –

have $x \sqcap (y \sqcup z) = (y \sqcup z) \sqcap x$

using *assms test-sup-closed test-inf-commutative* **by** *smt*

also have $\dots = (y \sqcap x) \sqcup (z \sqcap x)$

using *il-inf-right-dist-sup* **by** *simp*

also have $\dots = (x \sqcap y) \sqcup (x \sqcap z)$

using *assms test-sup-closed test-inf-commutative* **by** *smt*

finally show *?thesis*

qed

lemma $!x \sqcup !y = !(!(x \sqcup !y))$ **nitpick** [*expect=genuine*] **oops**

lemma $d x = !(!x)$ **nitpick** [*expect=genuine*] **oops**

sublocale *subset-boolean-algebra* **where** *uminus* = $\lambda x . !(d x)$

proof

show $\bigwedge x y z. !(d x) \sqcup (!(d y) \sqcup !(d z)) = !(d x) \sqcup !(d y) \sqcup !(d z)$

using *il-associative* **by** *blast*

show $\bigwedge x y. !(d x) \sqcup !(d y) = !(d y) \sqcup !(d x)$

by (*simp add: il-commutative*)

show $\bigwedge x y. !(d x) \sqcup !(d y) = !(d (!(d (!(d x) \sqcup !(d y)))))$

proof –

fix $x y$

have $\text{test } (!(d x)) \wedge \text{test } (!(d y))$

by (*simp add: d-closed neg-test*)

hence $\text{test } (!(d x) \sqcup !(d y))$

by (*simp add: test-sup-closed*)

thus $!(d x) \sqcup !(d y) = !(d (!(d (!(d x) \sqcup !(d y)))))$

by (*simp add: d-neg neg-involutive test-d-fixpoint*)

qed

show $\bigwedge x y. !(d x) = !(d (!(d (!(d x))) \sqcup !(d y))) \sqcup !(d (!(d (!(d x))) \sqcup !(d (!(d y))))))$

proof –

fix $x y$

have $!(d (!(d (!(d x))) \sqcup !(d y))) \sqcup !(d (!(d (!(d x))) \sqcup !(d (!(d y)))))) = !(d x \sqcup !(d y)) \sqcup !(d x \sqcup d y)$

using *d-closed neg-test test-sup-closed neg-involutive test-d-fixpoint* **by** *auto*

also have $\dots = (!(d x) \sqcap d y) \sqcup (!(d x) \sqcap !(d y))$

using *d-closed neg-test test-sup-closed neg-involutive de-morgan-2* **by** *auto*

also have $\dots = !(d x) \sqcap (d y \sqcup !(d y))$

using *d-closed neg-test test-inf-left-dist-sup* **by** *auto*

also have $\dots = !(d x) \sqcap \text{top}$

by (*simp add: neg-char d-closed*)

finally show $!(d x) = !(d (!(d (!(d x))) \sqcup !(d y))) \sqcup !(d (!(d (!(d x))) \sqcup !(d (!(d y))))))$

by *simp*

qed

qed

lemma *d-dist-sup*:

$d (x \sqcup y) = d x \sqcup d y$

proof (*rule order.antisym*)

have $x \leq d x \sqcap x$

by (*simp add: d1*)

also have $\dots \leq (d x \sqcup d y) \sqcap (x \sqcup y)$

using *il-associative il-inf-right-dist-sup il-less-eq il-sub-inf-right-isotone* **by**

auto

finally have $1: x \leq (d x \sqcup d y) \sqcap (x \sqcup y)$

.

```

have  $y \leq d\ y \sqcap y$ 
  by (simp add: d1)
also have  $\dots \leq (d\ y \sqcup d\ x) \sqcap (y \sqcup x)$ 
  using il-associative il-idempotent il-inf-right-dist-sup il-less-eq
il-sub-inf-right-isotone by simp
finally have  $y \leq (d\ x \sqcup d\ y) \sqcap (x \sqcup y)$ 
  using il-commutative by auto
hence  $x \sqcup y \leq (d\ x \sqcup d\ y) \sqcap (x \sqcup y)$ 
  using 1 by (metis il-associative il-less-eq)
thus  $d\ (x \sqcup y) \leq d\ x \sqcup d\ y$ 
  using llp test-sup-closed neg-test d-closed by simp
show  $d\ x \sqcup d\ y \leq d\ (x \sqcup y)$ 
  using d-isotone-var il-associative il-commutative il-less-eq by fastforce
qed

```

end

```

class pd-semiring-extended = pd-semiring + uminus +
  assumes uminus-def:  $-x = !(d\ x)$ 
begin

```

```

subclass subset-boolean-algebra
  by (metis subset-boolean-algebra-axioms uminus-def ext)

```

end

7.4 Domain Semirings

```

class d-semiring = pd-semiring +
  assumes d3:  $d\ (x \sqcap d\ y) \leq d\ (x \sqcap y)$ 
begin

```

```

lemma d3-eq:  $d\ (x \sqcap d\ y) = d\ (x \sqcap y)$ 
  by (simp add: order.antisym d3 d3-conv)

```

end

Axioms (d1), (d2) and (d3) are independent in IL-semirings.

```

context il-semiring
begin

```

```

context
  fixes  $d :: 'a \Rightarrow 'a$ 
  assumes d-closed: test ( $d\ x$ )
begin

```

```

context
  assumes d1:  $x \leq d\ x \sqcap x$ 
  assumes d2: test  $p \implies d\ (p \sqcap x) \leq p$ 
begin

```

```

lemma d3:  $d (x \sqcap d y) \leq d (x \sqcap y)$  nitpick [expect=genuine] oops

end

context
  assumes d1:  $x \leq d x \sqcap x$ 
  assumes d3:  $d (x \sqcap d y) \leq d (x \sqcap y)$ 
begin

lemma d2:  $test\ p \implies d (p \sqcap x) \leq p$  nitpick [expect=genuine] oops

end

context
  assumes d2:  $test\ p \implies d (p \sqcap x) \leq p$ 
  assumes d3:  $d (x \sqcap d y) \leq d (x \sqcap y)$ 
begin

lemma d1:  $x \leq d x \sqcap x$  nitpick [expect=genuine] oops

end

end

end

class d-semiring-var = ppd-semiring +
  assumes d3-var:  $d (x \sqcap d y) \leq d (x \sqcap y)$ 
  assumes d-strict-eq-var:  $d\ bot = bot$ 
begin

lemma d2-var:
  assumes test p
  shows  $d (p \sqcap x) \leq p$ 
proof –
  have  $!p \sqcap p \sqcap x = bot$ 
  by (simp add: asms neg-char)
  hence  $d (!p \sqcap p \sqcap x) = bot$ 
  by (simp add: d-strict-eq-var)
  hence  $d (!p \sqcap d (p \sqcap x)) = bot$ 
  by (metis d3-var il-inf-associative less-eq-bot)
  hence  $!p \sqcap d (p \sqcap x) = bot$ 
  using d-bot-only by blast
  thus ?thesis
  by (metis (no-types, opaque-lifting) asms d-sub-identity il-bot-unit
il-inf-left-unit il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone neg-char)
qed

```

```

subclass d-semiring
proof
  show  $\bigwedge p x. \text{test } p \implies d (p \sqcap x) \leq p$ 
    by (simp add: d2-var)
  show  $\bigwedge x y. d (x \sqcap d y) \leq d (x \sqcap y)$ 
    by (simp add: d3-var)
qed

end

```

8 Antidomain Semirings

We now develop prepreantidomain semirings, preantidomain semirings and antidomain semirings. See [6, 7, 8] for related work on internal axioms for antidomain.

8.1 Prepreantidomain Semirings

Definition 20

```

class ppa-semiring = il-semiring + uminus +
  assumes a-inf-complement-bot:  $-x \sqcap x = \text{bot}$ 
  assumes a-stone[simp]:  $-x \sqcup --x = \text{top}$ 
begin

```

Theorem 21

```

lemma l1:
   $-\text{top} = \text{bot}$ 
  by (metis a-inf-complement-bot il-inf-right-unit)

```

```

lemma l2:
   $-\text{bot} = \text{top}$ 
  by (metis l1 a-stone il-unit-bot)

```

```

lemma l3:
   $-x \leq -y \implies -x \sqcap y = \text{bot}$ 
  by (metis a-inf-complement-bot il-bot-unit il-inf-right-dist-sup il-less-eq)

```

```

lemma l5:
   $--x \leq --y \implies -y \leq -x$ 
  by (metis (mono-tags, opaque-lifting) l3 a-stone bot-least il-bot-unit
il-inf-left-unit il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone
sup-right-isotone)

```

```

lemma l4:
   $---x = -x$ 
  by (metis l5 a-inf-complement-bot a-stone order.antisym bot-least il-inf-left-unit
il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone il-unit-bot)

```

lemma l6:

$$-x \sqcap --x = \text{bot}$$

by (*metis l3 l5 a-inf-complement-bot a-stone il-inf-left-unit il-inf-right-dist-sup il-inf-right-unit il-less-eq il-sub-inf-right-isotone il-unit-bot*)

lemma l7:

$$-x \sqcap -y = -y \sqcap -x$$

using l6 *a-inf-complement-bot a-stone test-inf-commutative* **by** *blast*

lemma l8:

$$x \leq --x \sqcap x$$

by (*metis a-inf-complement-bot a-stone il-idempotent il-inf-left-unit il-inf-right-dist-sup il-less-eq il-unit-bot*)

sublocale *ppa-ppd*: *ppd-semiring* **where** $d = \lambda x . --x$

proof

show $\bigwedge x. \text{test } (- - x)$

using l4 l6 **by** *force*

show $\bigwedge x. x \leq - - x \sqcap x$

by (*simp add: l8*)

qed

end

8.2 Preantidomain Semirings

Definition 22

class *pa-semiring* = *ppa-semiring* +

assumes *pad2*: $--x \leq -(-x \sqcap y)$

begin

Theorem 23

lemma l10:

$$-x \sqcap y = \text{bot} \implies -x \leq -y$$

by (*metis a-stone il-inf-left-unit il-inf-right-dist-sup il-unit-bot l4 pad2*)

lemma l10-iff:

$$-x \sqcap y = \text{bot} \iff -x \leq -y$$

using l10 l3 **by** *blast*

lemma l13:

$$--(- -x \sqcap y) \leq --x$$

by (*metis l4 l5 pad2*)

lemma l14:

$$-(x \sqcap --y) \leq -(x \sqcap y)$$

by (*metis il-inf-associative l4 pad2 ppa-ppd.d1-eq*)

lemma l9:

$x \leq y \implies -y \leq -x$

by (*metis l10 a-inf-complement-bot il-commutative il-less-eq il-sub-inf-right-isotone il-unit-bot*)

lemma l11:

$-x \sqcup -y = -(-x \sqcap -y)$

proof –

have 1: $\bigwedge x y . x \leq y \longleftrightarrow x \sqcup y = y$

by (*simp add: il-less-eq*)

have 4: $\bigwedge x y . \neg(x \leq y) \vee x \sqcup y = y$

using 1 by *metis*

have 5: $\bigwedge x y z . (x \sqcap y) \sqcup (x \sqcap z) \leq x \sqcap (y \sqcup z)$

by (*simp add: il-sub-inf-right-isotone-var*)

have 6: $\bigwedge x y . - -x \leq -(-x \sqcap y)$

by (*simp add: pad2*)

have 7: $\bigwedge x y z . x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$

by (*simp add: il-associative*)

have 8: $\bigwedge x y z . (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$

using 7 by *metis*

have 9: $\bigwedge x y . x \sqcup y = y \sqcup x$

by (*simp add: il-commutative*)

have 10: $\bigwedge x . x \sqcup \text{bot} = x$

by (*simp add: il-bot-unit*)

have 11: $\bigwedge x . x \sqcup x = x$

by *simp*

have 12: $\bigwedge x y z . x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$

by (*simp add: il-inf-associative*)

have 13: $\bigwedge x y z . (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$

using 12 by *metis*

have 14: $\bigwedge x . \text{top} \sqcap x = x$

by *simp*

have 15: $\bigwedge x . x \sqcap \text{top} = x$

by *simp*

have 16: $\bigwedge x y z . (x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$

by (*simp add: il-inf-right-dist-sup*)

have 17: $\bigwedge x y z . (x \sqcap y) \sqcup (z \sqcap y) = (x \sqcup z) \sqcap y$

using 16 by *metis*

have 18: $\bigwedge x . \text{bot} \sqcap x = \text{bot}$

by *simp*

have 19: $\bigwedge x . -x \sqcup - -x = \text{top}$

by *simp*

have 20: $\bigwedge x . -x \sqcap x = \text{bot}$

by (*simp add: a-inf-complement-bot*)

have 23: $\bigwedge x y z . ((x \sqcap y) \sqcup (x \sqcap z)) \sqcup (x \sqcap (y \sqcup z)) = x \sqcap (y \sqcup z)$

using 4 5 by *metis*

have 24: $\bigwedge x y z . (x \sqcap (y \sqcup z)) \sqcup ((x \sqcap y) \sqcup (x \sqcap z)) = x \sqcap (y \sqcup z)$

using 9 23 by metis
have 25: $\bigwedge x y . \neg \neg x \sqcup \neg (\neg x \sqcap y) = \neg (\neg x \sqcap y)$
using 4 6 by metis
have 26: $\bigwedge x y z . x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
using 8 9 by metis
have 27: $\bigwedge x y z . (x \sqcap y) \sqcup ((x \sqcap z) \sqcup (x \sqcap (y \sqcup z))) = x \sqcap (y \sqcup z)$
using 9 24 26 by metis
have 30: $\bigwedge x . \text{bot} \sqcup x = x$
using 9 10 by metis
have 31: $\bigwedge x y . x \sqcup (x \sqcup y) = x \sqcup y$
using 8 11 by metis
have 34: $\bigwedge u x y z . ((x \sqcup y) \sqcap z) \sqcup u = (x \sqcap z) \sqcup ((y \sqcap z) \sqcup u)$
using 8 17 by metis
have 35: $\bigwedge u x y z . (x \sqcap (y \sqcap z)) \sqcup (u \sqcap z) = ((x \sqcap y) \sqcup u) \sqcap z$
using 13 17 by metis
have 36: $\bigwedge u x y z . (x \sqcap y) \sqcup (z \sqcap (u \sqcap y)) = (x \sqcup (z \sqcap u)) \sqcap y$
using 13 17 by metis
have 39: $\bigwedge x y . \neg x \sqcup (\neg \neg x \sqcup y) = \text{top} \sqcup y$
using 8 19 by metis
have 41: $\bigwedge x y . \neg x \sqcap (x \sqcap y) = \text{bot}$
using 13 18 20 by metis
have 42: $\neg \text{top} = \text{bot}$
using 15 20 by metis
have 43: $\bigwedge x y . (\neg x \sqcup y) \sqcap x = y \sqcap x$
using 17 20 30 by metis
have 44: $\bigwedge x y . (x \sqcup \neg y) \sqcap y = x \sqcap y$
using 9 17 20 30 by metis
have 46: $\bigwedge x . \neg \text{bot} \sqcup \neg \neg x = \neg \text{bot}$
using 9 20 25 by metis
have 50: $\neg \text{bot} = \text{top}$
using 19 30 42 by metis
have 51: $\bigwedge x . \text{top} \sqcup \neg \neg x = \text{top}$
using 46 50 by metis
have 63: $\bigwedge x y . x \sqcup ((x \sqcap \neg y) \sqcup (x \sqcap \neg \neg y)) = x$
using 9 15 19 26 27 by metis
have 66: $\bigwedge x y . (\neg (x \sqcup y) \sqcap x) \sqcup (\neg (x \sqcup y) \sqcap y) = \text{bot}$
using 9 20 27 30 by metis
have 67: $\bigwedge x y z . (x \sqcap \neg \neg y) \sqcup (x \sqcap \neg (\neg y \sqcap z)) = x \sqcap \neg (\neg y \sqcap z)$
using 11 25 27 by metis
have 70: $\bigwedge x y . x \sqcup (x \sqcap \neg \neg y) = x$
using 9 15 27 31 51 by metis
have 82: $\bigwedge x . \text{top} \sqcup \neg x = \text{top}$
using 9 19 31 by metis
have 89: $\bigwedge x y . x \sqcup (\neg y \sqcap x) = x$
using 14 17 82 by metis
have 102: $\bigwedge x y z . x \sqcup (y \sqcup (x \sqcap \neg \neg z)) = y \sqcup x$
using 26 70 by metis
have 104: $\bigwedge x y . x \sqcup (x \sqcap \neg y) = x$
using 9 63 102 by metis

have 112: $\bigwedge x y z . (-x \sqcap y) \sqcup ((- - x \sqcap y) \sqcup z) = y \sqcup z$
using 14 19 34 by metis
have 117: $\bigwedge x y z . x \sqcup ((x \sqcap - y) \sqcup z) = x \sqcup z$
using 8 104 by metis
have 120: $\bigwedge x y z . x \sqcup (y \sqcup (x \sqcap - z)) = y \sqcup x$
using 26 104 by metis
have 124: $\bigwedge x . - - x \sqcap x = x$
using 14 19 43 by metis
have 128: $\bigwedge x y . - - x \sqcap (x \sqcap y) = x \sqcap y$
using 13 124 by metis
have 131: $\bigwedge x . - x \sqcup - - - x = - x$
using 9 25 124 by metis
have 133: $\bigwedge x . - - - x = - x$
using 9 104 124 131 by metis
have 135: $\bigwedge x y . - x \sqcup - (- - x \sqcap y) = - (- - x \sqcap y)$
using 25 133 by metis
have 137: $\bigwedge x y . (- x \sqcup y) \sqcap - - x = y \sqcap - - x$
using 43 133 by metis
have 145: $\bigwedge x y z . ((- (x \sqcap y) \sqcap x) \sqcup z) \sqcap y = z \sqcap y$
using 20 30 35 by metis
have 183: $\bigwedge x y z . (x \sqcup (- - (y \sqcap z) \sqcap y)) \sqcap z = (x \sqcup y) \sqcap z$
using 17 36 124 by metis
have 289: $\bigwedge x y . - x \sqcup - (- x \sqcap y) = top$
using 25 39 82 by metis
have 316: $\bigwedge x y . - (- x \sqcap y) \sqcap x = x$
using 14 43 289 by metis
have 317: $\bigwedge x y z . - (- x \sqcap y) \sqcap (x \sqcap z) = x \sqcap z$
using 13 316 by metis
have 320: $\bigwedge x y . - x \sqcup - - (- x \sqcap y) = - x$
using 9 25 316 by metis
have 321: $\bigwedge x y . - - (- x \sqcap y) \sqcap x = bot$
using 41 316 by metis
have 374: $\bigwedge x y . - x \sqcup - (x \sqcap y) = - (x \sqcap y)$
using 25 128 133 by metis
have 388: $\bigwedge x y . - (x \sqcap y) \sqcap - x = - x$
using 128 316 by metis
have 389: $\bigwedge x y . - - (x \sqcap y) \sqcap - x = bot$
using 128 321 by metis
have 405: $\bigwedge x y z . - (x \sqcap y) \sqcap (- x \sqcap z) = - x \sqcap z$
using 13 388 by metis
have 406: $\bigwedge x y z . - (x \sqcap (y \sqcap z)) \sqcap - (x \sqcap y) = - (x \sqcap y)$
using 13 388 by metis
have 420: $\bigwedge x y . - x \sqcap - - (- x \sqcap y) = - - (- x \sqcap y)$
using 316 388 by metis
have 422: $\bigwedge x y z . - - (x \sqcap y) \sqcap (- x \sqcap z) = bot$
using 13 18 389 by metis
have 758: $\bigwedge x y z . x \sqcup (x \sqcap (- y \sqcap - z)) = x$
using 13 104 117 by metis
have 1092: $\bigwedge x y . - (x \sqcup y) \sqcap x = bot$

using 9 30 31 66 **by** *metis*
have 1130: $\bigwedge x y z . (\neg (x \sqcup y) \sqcup z) \sqcap x = z \sqcap x$
using 17 30 1092 **by** *metis*
have 1156: $\bigwedge x y . \neg \neg x \sqcap \neg (\neg x \sqcap y) = \neg \neg x$
using 67 104 124 133 **by** *metis*
have 2098: $\bigwedge x y . \neg \neg (x \sqcup y) \sqcap x = x$
using 14 19 1130 **by** *metis*
have 2125: $\bigwedge x y . \neg \neg (x \sqcup y) \sqcap y = y$
using 9 2098 **by** *metis*
have 2138: $\bigwedge x y . \neg x \sqcup \neg \neg (x \sqcup y) = \text{top}$
using 9 289 2098 **by** *metis*
have 2139: $\bigwedge x y . \neg x \sqcap \neg (x \sqcup y) = \neg (x \sqcup y)$
using 316 2098 **by** *metis*
have 2192: $\bigwedge x y . \neg \neg x \sqcap (\neg y \sqcap x) = \neg y \sqcap x$
using 89 2125 **by** *metis*
have 2202: $\bigwedge x y . \neg x \sqcup \neg \neg (y \sqcup x) = \text{top}$
using 9 289 2125 **by** *metis*
have 2344: $\bigwedge x y . \neg (\neg x \sqcap y) \sqcup \neg \neg y = \text{top}$
using 89 2202 **by** *metis*
have 2547: $\bigwedge x y z . \neg x \sqcup ((\neg \neg x \sqcap \neg y) \sqcup z) = \neg x \sqcup (\neg y \sqcup z)$
using 112 117 **by** *metis*
have 3023: $\bigwedge x y . \neg x \sqcup \neg (\neg y \sqcap \neg x) = \text{top}$
using 9 133 2344 **by** *metis*
have 3134: $\bigwedge x y . \neg (\neg x \sqcap \neg y) \sqcap y = y$
using 14 43 3023 **by** *metis*
have 3135: $\bigwedge x y . \neg x \sqcap (\neg y \sqcap \neg x) = \neg y \sqcap \neg x$
using 14 44 3023 **by** *metis*
have 3962: $\bigwedge x y . \neg \neg (x \sqcup y) \sqcap \neg \neg x = \neg \neg x$
using 14 137 2138 **by** *metis*
have 5496: $\bigwedge x y z . \neg \neg (x \sqcap y) \sqcap \neg (x \sqcup z) = \text{bot}$
using 422 2139 **by** *metis*
have 9414: $\bigwedge x y . \neg \neg (\neg x \sqcap y) \sqcap y = \neg x \sqcap y$
using 9 104 183 320 **by** *metis*
have 9520: $\bigwedge x y z . \neg \neg (\neg x \sqcap y) \sqcap \neg \neg (x \sqcap z) = \text{bot}$
using 374 5496 **by** *metis*
have 11070: $\bigwedge x y z . \neg (\neg \neg x \sqcap y) \sqcup (\neg x \sqcap \neg z) = \neg (\neg \neg x \sqcap y)$
using 317 758 **by** *metis*
have 12371: $\bigwedge x y . \neg x \sqcap \neg (\neg \neg x \sqcap y) = \neg x$
using 133 1156 **by** *metis*
have 12377: $\bigwedge x y . \neg x \sqcap \neg (x \sqcap y) = \neg x$
using 128 133 1156 **by** *metis*
have 12384: $\bigwedge x y . \neg (x \sqcup y) \sqcap \neg y = \neg (x \sqcup y)$
using 133 1156 2125 **by** *metis*
have 12394: $\bigwedge x y . \neg \neg (\neg x \sqcap \neg y) = \neg x \sqcap \neg y$
using 1156 3134 9414 **by** *metis*
have 12640: $\bigwedge x y . \neg x \sqcap \neg (\neg y \sqcap x) = \neg x$
using 89 12384 **by** *metis*
have 24648: $\bigwedge x y . (\neg x \sqcap \neg y) \sqcup \neg (\neg x \sqcap \neg y) = \text{top}$
using 19 12394 **by** *metis*

have 28270: $\bigwedge x y z . - - (x \sqcap y) \sqcup - (- x \sqcap z) = - (- x \sqcap z)$
using 374 405 **by** *metis*
have 28339: $\bigwedge x y . - (- - (x \sqcap y) \sqcap x) = - (x \sqcap y)$
using 124 406 12371 **by** *metis*
have 28423: $\bigwedge x y . - (- x \sqcap - y) = - (- y \sqcap - x)$
using 13 3135 12394 28339 **by** *metis*
have 28487: $\bigwedge x y . - x \sqcap - y = - y \sqcap - x$
using 2098 3962 12394 28423 **by** *metis*
have 52423: $\bigwedge x y . - (- x \sqcap - (- x \sqcap y)) \sqcap y = y$
using 14 145 24648 28487 **by** *metis*
have 52522: $\bigwedge x y . - x \sqcap - (- x \sqcap y) = - x \sqcap - y$
using 13 12377 12394 12640 28487 52423 **by** *metis*
have 61103: $\bigwedge x y z . - (- - x \sqcap y) \sqcup z = - x \sqcup (- y \sqcup z)$
using 112 2547 12371 52522 **by** *metis*
have 61158: $\bigwedge x y . - - (- x \sqcap y) = - x \sqcap - - y$
using 420 52522 **by** *metis*
have 61231: $\bigwedge x y z . - x \sqcap (- - y \sqcap - (x \sqcap z)) = - x \sqcap - - y$
using 13 15 50 133 9520 52522 61158 **by** *metis*
have 61313: $\bigwedge x y . - x \sqcup - y = - (- - y \sqcap x)$
using 120 11070 61103 **by** *metis*
have 61393: $\bigwedge x y . - (- x \sqcap - - y) = - (- x \sqcap y)$
using 13 28270 61158 61231 61313 **by** *metis*
have 61422: $\bigwedge x y . - (- - x \sqcap y) = - (- - y \sqcap x)$
using 13 135 2192 61158 61313 **by** *metis*
show *?thesis*
using 61313 61393 61422 **by** *metis*
qed

lemma *l12*:

$$- x \sqcap - y = - (x \sqcup y)$$

proof -

have 1: $\bigwedge x y . x \leq y \longleftrightarrow x \sqcup y = y$
by (*simp add: il-less-eq*)
have 4: $\bigwedge x y . \neg(x \leq y) \vee x \sqcup y = y$
using 1 **by** *metis*
have 5: $\bigwedge x y z . (x \sqcap y) \sqcup (x \sqcap z) \leq x \sqcap (y \sqcup z)$
by (*simp add: il-sub-inf-right-isotone-var*)
have 6: $\bigwedge x y . - - x \leq - (- x \sqcap y)$
by (*simp add: pad2*)
have 7: $\bigwedge x y z . x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
by (*simp add: il-associative*)
have 8: $\bigwedge x y z . (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
using 7 **by** *metis*
have 9: $\bigwedge x y . x \sqcup y = y \sqcup x$
by (*simp add: il-commutative*)
have 10: $\bigwedge x . x \sqcup \text{bot} = x$
by (*simp add: il-bot-unit*)
have 11: $\bigwedge x . x \sqcup x = x$
by *simp*

have 12: $\bigwedge x y z . x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
 by (*simp add: il-inf-associative*)
have 13: $\bigwedge x y z . (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
 using 12 by *metis*
have 14: $\bigwedge x . top \sqcap x = x$
 by *simp*
have 15: $\bigwedge x . x \sqcap top = x$
 by *simp*
have 16: $\bigwedge x y z . (x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$
 by (*simp add: il-inf-right-dist-sup*)
have 17: $\bigwedge x y z . (x \sqcap y) \sqcup (z \sqcap y) = (x \sqcup z) \sqcap y$
 using 16 by *metis*
have 18: $\bigwedge x . bot \sqcap x = bot$
 by *simp*
have 19: $\bigwedge x . - x \sqcup - - x = top$
 by *simp*
have 20: $\bigwedge x . - x \sqcap x = bot$
 by (*simp add: a-inf-complement-bot*)
have 22: $\bigwedge x y z . ((x \sqcap y) \sqcup (x \sqcap z)) \sqcup (x \sqcap (y \sqcup z)) = x \sqcap (y \sqcup z)$
 using 4 5 by *metis*
have 23: $\bigwedge x y z . (x \sqcap (y \sqcup z)) \sqcup ((x \sqcap y) \sqcup (x \sqcap z)) = x \sqcap (y \sqcup z)$
 using 9 22 by *metis*
have 24: $\bigwedge x y . - - x \sqcup - (- x \sqcap y) = - (- x \sqcap y)$
 using 4 6 by *metis*
have 25: $\bigwedge x y z . x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
 using 8 9 by *metis*
have 26: $\bigwedge x y z . (x \sqcap y) \sqcup ((x \sqcap z) \sqcup (x \sqcap (y \sqcup z))) = x \sqcap (y \sqcup z)$
 using 9 23 25 by *metis*
have 29: $\bigwedge x . bot \sqcup x = x$
 using 9 10 by *metis*
have 30: $\bigwedge x y . x \sqcup (x \sqcup y) = x \sqcup y$
 using 8 11 by *metis*
have 32: $\bigwedge x y . x \sqcup (y \sqcup x) = y \sqcup x$
 using 8 9 11 by *metis*
have 33: $\bigwedge u x y z . ((x \sqcup y) \sqcap z) \sqcup u = (x \sqcap z) \sqcup ((y \sqcap z) \sqcup u)$
 using 8 17 by *metis*
have 34: $\bigwedge u x y z . (x \sqcap (y \sqcap z)) \sqcup (u \sqcap z) = ((x \sqcap y) \sqcup u) \sqcap z$
 using 13 17 by *metis*
have 35: $\bigwedge u x y z . (x \sqcap y) \sqcup (z \sqcap (u \sqcap y)) = (x \sqcup (z \sqcap u)) \sqcap y$
 using 13 17 by *metis*
have 36: $\bigwedge x y . (top \sqcup x) \sqcap y = y \sqcup (x \sqcap y)$
 using 14 17 by *metis*
have 37: $\bigwedge x y . (x \sqcup top) \sqcap y = y \sqcup (x \sqcap y)$
 using 9 14 17 by *metis*
have 38: $\bigwedge x y . - x \sqcup (- - x \sqcup y) = top \sqcup y$
 using 8 19 by *metis*
have 40: $\bigwedge x y . - x \sqcap (x \sqcap y) = bot$
 using 13 18 20 by *metis*
have 41: $- top = bot$

using 15 20 **by** *metis*
have 42: $\bigwedge x y . (- x \sqcup y) \sqcap x = y \sqcap x$
using 17 20 29 **by** *metis*
have 43: $\bigwedge x y . (x \sqcup - y) \sqcap y = x \sqcap y$
using 9 17 20 29 **by** *metis*
have 45: $\bigwedge x . - \text{bot} \sqcup - - x = - \text{bot}$
using 9 20 24 **by** *metis*
have 46: $\bigwedge u x y z . (x \sqcap y) \sqcup (z \sqcup (u \sqcap y)) = z \sqcup ((x \sqcup u) \sqcap y)$
using 17 25 **by** *metis*
have 47: $\bigwedge x y . - x \sqcup (y \sqcup - - x) = y \sqcup \text{top}$
using 19 25 **by** *metis*
have 49: $- \text{bot} = \text{top}$
using 19 29 41 **by** *metis*
have 50: $\bigwedge x . \text{top} \sqcup - - x = \text{top}$
using 45 49 **by** *metis*
have 54: $\bigwedge u x y z . (x \sqcap y) \sqcup ((x \sqcap z) \sqcup ((x \sqcap (y \sqcup z)) \sqcup u)) = (x \sqcap (y \sqcup z))$
 $\sqcup u$
using 8 26 **by** *metis*
have 58: $\bigwedge u x y z . (x \sqcap (y \sqcap z)) \sqcup ((x \sqcap (y \sqcap u)) \sqcup (x \sqcap (y \sqcap (z \sqcup u)))) = x$
 $\sqcap (y \sqcap (z \sqcup u))$
using 13 26 **by** *metis*
have 60: $\bigwedge x y . x \sqcup ((x \sqcap y) \sqcup (x \sqcap (y \sqcup \text{top}))) = x \sqcap (y \sqcup \text{top})$
using 15 25 26 **by** *metis*
have 62: $\bigwedge x y . x \sqcup ((x \sqcap - y) \sqcup (x \sqcap - - y)) = x$
using 9 15 19 25 26 **by** *metis*
have 65: $\bigwedge x y . (- (x \sqcup y) \sqcap x) \sqcup (- (x \sqcup y) \sqcap y) = \text{bot}$
using 9 20 26 29 **by** *metis*
have 66: $\bigwedge x y z . (x \sqcap - - y) \sqcup (x \sqcap - (- y \sqcap z)) = x \sqcap - (- y \sqcap z)$
using 11 24 26 **by** *metis*
have 69: $\bigwedge x y . x \sqcup (x \sqcap - - y) = x$
using 9 15 26 30 50 **by** *metis*
have 81: $\bigwedge x . \text{top} \sqcup - x = \text{top}$
using 9 19 30 **by** *metis*
have 82: $\bigwedge x y z . (x \sqcap y) \sqcup (x \sqcap (y \sqcup z)) = x \sqcap (y \sqcup z)$
using 11 26 30 **by** *metis*
have 83: $\bigwedge x y . x \sqcup (x \sqcap (y \sqcup \text{top})) = x \sqcap (y \sqcup \text{top})$
using 60 82 **by** *metis*
have 88: $\bigwedge x y . x \sqcup (- y \sqcap x) = x$
using 14 17 81 **by** *metis*
have 89: $\bigwedge x y . \text{top} \sqcup (x \sqcup - y) = x \sqcup \text{top}$
using 25 81 **by** *metis*
have 91: $\bigwedge x y z . x \sqcup (y \sqcup (z \sqcup x)) = y \sqcup (z \sqcup x)$
using 8 32 **by** *metis*
have 94: $\bigwedge x y z . x \sqcup (y \sqcup (- z \sqcap x)) = y \sqcup x$
using 25 88 **by** *metis*
have 101: $\bigwedge x y z . x \sqcup (y \sqcup (x \sqcap - - z)) = y \sqcup x$
using 25 69 **by** *metis*
have 102: $\bigwedge x . x \sqcup (x \sqcap \text{bot}) = x$
using 41 49 69 **by** *metis*

have 103: $\bigwedge x y . x \sqcup (x \sqcap - y) = x$
using 9 62 101 by metis
have 109: $\bigwedge x y . x \sqcup (y \sqcup (x \sqcap \text{bot})) = y \sqcup x$
using 25 102 by metis
have 111: $\bigwedge x y z . (- x \sqcap y) \sqcup ((- - x \sqcap y) \sqcup z) = y \sqcup z$
using 14 19 33 by metis
have 116: $\bigwedge x y z . x \sqcup ((x \sqcap - y) \sqcup z) = x \sqcup z$
using 8 103 by metis
have 119: $\bigwedge x y z . x \sqcup (y \sqcup (x \sqcap - z)) = y \sqcup x$
using 25 103 by metis
have 123: $\bigwedge x . - - x \sqcap x = x$
using 14 19 42 by metis
have 127: $\bigwedge x y . - - x \sqcap (x \sqcap y) = x \sqcap y$
using 13 123 by metis
have 130: $\bigwedge x . - x \sqcup - - - x = - x$
using 9 24 123 by metis
have 132: $\bigwedge x . - - - x = - x$
using 9 103 123 130 by metis
have 134: $\bigwedge x y . - x \sqcup - (- - x \sqcap y) = - (- - x \sqcap y)$
using 24 132 by metis
have 136: $\bigwedge x y . (- x \sqcup y) \sqcap - - x = y \sqcap - - x$
using 42 132 by metis
have 138: $\bigwedge x . - x \sqcap - x = - x$
using 123 132 by metis
have 144: $\bigwedge x y z . ((- (x \sqcap y) \sqcap x) \sqcup z) \sqcap y = z \sqcap y$
using 20 29 34 by metis
have 157: $\bigwedge x y . (- x \sqcup y) \sqcap - x = (\text{top} \sqcup y) \sqcap - x$
using 17 36 138 by metis
have 182: $\bigwedge x y z . (x \sqcup (- - (y \sqcap z) \sqcap y)) \sqcap z = (x \sqcup y) \sqcap z$
using 17 35 123 by metis
have 288: $\bigwedge x y . - x \sqcup - (- x \sqcap y) = \text{top}$
using 24 38 81 by metis
have 315: $\bigwedge x y . - (- x \sqcap y) \sqcap x = x$
using 14 42 288 by metis
have 316: $\bigwedge x y z . - (- x \sqcap y) \sqcap (x \sqcap z) = x \sqcap z$
using 13 315 by metis
have 319: $\bigwedge x y . - x \sqcup - - (- x \sqcap y) = - x$
using 9 24 315 by metis
have 320: $\bigwedge x y . - - (- x \sqcap y) \sqcap x = \text{bot}$
using 40 315 by metis
have 373: $\bigwedge x y . - x \sqcup - (x \sqcap y) = - (x \sqcap y)$
using 24 127 132 by metis
have 387: $\bigwedge x y . - (x \sqcap y) \sqcap - x = - x$
using 127 315 by metis
have 388: $\bigwedge x y . - - (x \sqcap y) \sqcap - x = \text{bot}$
using 127 320 by metis
have 404: $\bigwedge x y z . - (x \sqcap y) \sqcap (- x \sqcap z) = - x \sqcap z$
using 13 387 by metis
have 405: $\bigwedge x y z . - (x \sqcap (y \sqcap z)) \sqcap - (x \sqcap y) = - (x \sqcap y)$

using 13 387 by metis
have 419: $\bigwedge x y . - x \sqcap - - (- x \sqcap y) = - - (- x \sqcap y)$
using 315 387 by metis
have 420: $\bigwedge x y . - - x \sqcap - - (x \sqcap y) = - - (x \sqcap y)$
using 387 by metis
have 421: $\bigwedge x y z . - - (x \sqcap y) \sqcap (- x \sqcap z) = bot$
using 13 18 388 by metis
have 536: $\bigwedge x y . (x \sqcup - - y) \sqcap y = (x \sqcup top) \sqcap y$
using 42 47 by metis
have 662: $\bigwedge u x y z . (x \sqcap y) \sqcup ((x \sqcap (z \sqcup y)) \sqcup u) = (x \sqcap (z \sqcup y)) \sqcup u$
using 9 32 54 by metis
have 705: $\bigwedge u x y z . (x \sqcap (y \sqcup z)) \sqcup ((x \sqcap (y \sqcup (z \sqcap bot))) \sqcup u) = (x \sqcap (y \sqcup z)) \sqcup u$
using 25 54 109 662 by metis
have 755: $\bigwedge x y z . (x \sqcap - y) \sqcup (z \sqcup x) = z \sqcup x$
using 32 91 116 by metis
have 757: $\bigwedge x y z . x \sqcup (x \sqcap (- y \sqcap - z)) = x$
using 13 103 116 by metis
have 930: $\bigwedge x y z . (- (x \sqcap (y \sqcup z)) \sqcap (x \sqcap y)) \sqcup (- (x \sqcap (y \sqcup z)) \sqcap (x \sqcap z)) = bot$
using 9 20 29 58 by metis
have 1091: $\bigwedge x y . - (x \sqcup y) \sqcap x = bot$
using 9 29 30 65 by metis
have 1092: $\bigwedge x y . - (x \sqcup y) \sqcap y = bot$
using 29 30 65 1091 by metis
have 1113: $\bigwedge u x y z . - (x \sqcup ((y \sqcup z) \sqcap u)) \sqcap (x \sqcup (z \sqcap u)) = bot$
using 29 46 65 1091 by metis
have 1117: $\bigwedge x y z . - (x \sqcup y) \sqcap (x \sqcup (- z \sqcap y)) = bot$
using 29 65 94 1092 by metis
have 1128: $\bigwedge x y z . - (x \sqcup (y \sqcup z)) \sqcap (x \sqcup y) = bot$
using 8 1091 by metis
have 1129: $\bigwedge x y z . (- (x \sqcup y) \sqcup z) \sqcap x = z \sqcap x$
using 17 29 1091 by metis
have 1155: $\bigwedge x y . - - x \sqcap - (- x \sqcap y) = - - x$
using 66 103 123 132 by metis
have 1578: $\bigwedge x y z . - (x \sqcap (y \sqcup z)) \sqcap (x \sqcap y) = bot$
using 82 1091 by metis
have 1594: $\bigwedge x y z . - (x \sqcap (y \sqcup z)) \sqcap (x \sqcap z) = bot$
using 29 930 1578 by metis
have 2094: $\bigwedge x y z . - (x \sqcup (y \sqcap (z \sqcup top))) \sqcap (x \sqcup y) = bot$
using 83 1128 by metis
have 2097: $\bigwedge x y . - - (x \sqcup y) \sqcap x = x$
using 14 19 1129 by metis
have 2124: $\bigwedge x y . - - (x \sqcup y) \sqcap y = y$
using 9 2097 by metis
have 2135: $\bigwedge x y . - - ((top \sqcup x) \sqcap y) \sqcap y = y$
using 36 2097 by metis
have 2136: $\bigwedge x y . - - ((x \sqcup top) \sqcap y) \sqcap y = y$
using 37 2097 by metis

have 2137: $\bigwedge x y . - x \sqcup - - (x \sqcup y) = top$
using 9 288 2097 by metis
have 2138: $\bigwedge x y . - x \sqcap - (x \sqcup y) = - (x \sqcup y)$
using 315 2097 by metis
have 2151: $\bigwedge x y . - x \sqcup - (x \sqcup y) = - x$
using 9 132 373 2097 by metis
have 2191: $\bigwedge x y . - - x \sqcap (- y \sqcap x) = - y \sqcap x$
using 88 2124 by metis
have 2201: $\bigwedge x y . - x \sqcup - - (y \sqcup x) = top$
using 9 288 2124 by metis
have 2202: $\bigwedge x y . - x \sqcap - (y \sqcup x) = - (y \sqcup x)$
using 315 2124 by metis
have 2320: $\bigwedge x y . - (x \sqcap (y \sqcup top)) = - x$
using 83 373 2151 by metis
have 2343: $\bigwedge x y . - (- x \sqcap y) \sqcup - - y = top$
using 88 2201 by metis
have 2546: $\bigwedge x y z . - x \sqcup ((- - x \sqcap - y) \sqcup z) = - x \sqcup (- y \sqcup z)$
using 111 116 by metis
have 2706: $\bigwedge x y z . - x \sqcup (y \sqcup - - ((top \sqcup z) \sqcap - x)) = y \sqcup - - ((top \sqcup z)$
 $\sqcap - x)$
using 755 2135 by metis
have 2810: $\bigwedge x y . - x \sqcap - ((y \sqcup top) \sqcap x) = - ((y \sqcup top) \sqcap x)$
using 315 2136 by metis
have 3022: $\bigwedge x y . - x \sqcup - (- y \sqcap - x) = top$
using 9 132 2343 by metis
have 3133: $\bigwedge x y . - (- x \sqcap - y) \sqcap y = y$
using 14 42 3022 by metis
have 3134: $\bigwedge x y . - x \sqcap (- y \sqcap - x) = - y \sqcap - x$
using 14 43 3022 by metis
have 3961: $\bigwedge x y . - - (x \sqcup y) \sqcap - - x = - - x$
using 14 136 2137 by metis
have 4644: $\bigwedge x y z . - (x \sqcap - y) \sqcap (x \sqcap - (y \sqcup z)) = bot$
using 1594 2151 by metis
have 5495: $\bigwedge x y z . - - (x \sqcap y) \sqcap - (x \sqcup z) = bot$
using 421 2138 by metis
have 9413: $\bigwedge x y . - - (- x \sqcap y) \sqcap y = - x \sqcap y$
using 9 103 182 319 by metis
have 9519: $\bigwedge x y z . - - (- x \sqcap y) \sqcap - - (x \sqcap z) = bot$
using 373 5495 by metis
have 11069: $\bigwedge x y z . - (- - x \sqcap y) \sqcup (- x \sqcap - z) = - (- - x \sqcap y)$
using 316 757 by metis
have 12370: $\bigwedge x y . - x \sqcap - (- - x \sqcap y) = - x$
using 132 1155 by metis
have 12376: $\bigwedge x y . - x \sqcap - (x \sqcap y) = - x$
using 127 132 1155 by metis
have 12383: $\bigwedge x y . - (x \sqcup y) \sqcap - y = - (x \sqcup y)$
using 132 1155 2124 by metis
have 12393: $\bigwedge x y . - - (- x \sqcap - y) = - x \sqcap - y$
using 1155 3133 9413 by metis

have 12407: $\bigwedge x y . \neg \neg x \sqcap \neg \neg (x \sqcup y) = \neg \neg x$
using 1155 2138 **by** *metis*
have 12639: $\bigwedge x y . \neg x \sqcap \neg (\neg y \sqcap x) = \neg x$
using 88 12383 **by** *metis*
have 24647: $\bigwedge x y . (\neg x \sqcap \neg y) \sqcup \neg (\neg x \sqcap \neg y) = \text{top}$
using 19 12393 **by** *metis*
have 28269: $\bigwedge x y z . \neg \neg (x \sqcap y) \sqcup \neg (\neg x \sqcap z) = \neg (\neg x \sqcap z)$
using 373 404 **by** *metis*
have 28338: $\bigwedge x y . \neg (\neg \neg (x \sqcap y) \sqcap x) = \neg (x \sqcap y)$
using 123 405 12370 **by** *metis*
have 28422: $\bigwedge x y . \neg (\neg x \sqcap \neg y) = \neg (\neg y \sqcap \neg x)$
using 13 3134 12393 28338 **by** *metis*
have 28485: $\bigwedge x y . \neg x \sqcap \neg y = \neg y \sqcap \neg x$
using 2097 3961 12393 28422 **by** *metis*
have 30411: $\bigwedge x y . \neg x \sqcap (x \sqcup (x \sqcap y)) = \text{bot}$
using 9 82 2094 2320 **by** *metis*
have 30469: $\bigwedge x . \neg x \sqcap (x \sqcup \neg x) = \text{bot}$
using 9 123 132 30411 **by** *metis*
have 37513: $\bigwedge x y . \neg (\neg x \sqcap \neg y) \sqcap \neg (y \sqcup x) = \text{bot}$
using 2202 4644 **by** *metis*
have 52421: $\bigwedge x y . \neg (\neg x \sqcap \neg (\neg x \sqcap y)) \sqcap y = y$
using 14 144 24647 28485 **by** *metis*
have 52520: $\bigwedge x y . \neg x \sqcap \neg (\neg x \sqcap y) = \neg x \sqcap \neg y$
using 13 12376 12393 12639 28485 52421 **by** *metis*
have 52533: $\bigwedge x y z . \neg \neg (x \sqcup (y \sqcap (z \sqcup \text{top}))) \sqcap (x \sqcup y) = x \sqcup y$
using 15 49 2094 52421 **by** *metis*
have 61101: $\bigwedge x y z . \neg (\neg \neg x \sqcap y) \sqcup z = \neg x \sqcup (\neg y \sqcup z)$
using 111 2546 12370 52520 **by** *metis*
have 61156: $\bigwedge x y . \neg \neg (\neg x \sqcap y) = \neg x \sqcap \neg \neg y$
using 419 52520 **by** *metis*
have 61162: $\bigwedge x y . \neg (x \sqcup (x \sqcap y)) = \neg x$
using 15 49 2138 30411 52520 **by** *metis*
have 61163: $\bigwedge x . \neg (x \sqcup \neg \neg x) = \neg x$
using 15 49 2138 30469 52520 **by** *metis*
have 61229: $\bigwedge x y z . \neg x \sqcap (\neg \neg y \sqcap \neg (x \sqcap z)) = \neg x \sqcap \neg \neg y$
using 13 15 49 132 9519 52520 61156 **by** *metis*
have 61311: $\bigwedge x y . \neg x \sqcup \neg y = \neg (\neg \neg y \sqcap x)$
using 119 11069 61101 **by** *metis*
have 61391: $\bigwedge x y . \neg (\neg x \sqcap \neg \neg y) = \neg (\neg x \sqcap y)$
using 13 28269 61156 61229 61311 **by** *metis*
have 61420: $\bigwedge x y . \neg (\neg \neg x \sqcap y) = \neg (\neg \neg y \sqcap x)$
using 13 134 2191 61156 61311 **by** *metis*
have 61454: $\bigwedge x y . \neg (x \sqcup \neg (\neg y \sqcap \neg x)) = \neg y \sqcap \neg x$
using 9 132 3133 61156 61162 **by** *metis*
have 61648: $\bigwedge x y . \neg x \sqcap (x \sqcup (\neg y \sqcap \neg \neg x)) = \text{bot}$
using 1117 61163 **by** *metis*
have 62434: $\bigwedge x y . \neg (\neg \neg x \sqcap y) \sqcap x = \neg y \sqcap x$
using 43 61311 **by** *metis*
have 63947: $\bigwedge x y . \neg (\neg x \sqcap y) \sqcap \neg (\neg y \sqcup x) = \text{bot}$

using 37513 61391 **by** *metis*
have 64227: $\bigwedge x y . - (x \sqcup (- y \sqcap - x)) = - x$
using 15 49 2138 52520 61648 **by** *metis*
have 64239: $\bigwedge x y . - (x \sqcup (- - x \sqcup y)) = - (x \sqcup y)$
using 9 25 12407 64227 **by** *metis*
have 64241: $\bigwedge x y . - (x \sqcup (- - x \sqcap - y)) = - x$
using 28485 64227 **by** *metis*
have 64260: $\bigwedge x y . - (x \sqcup - - (x \sqcap y)) = - x$
using 420 64241 **by** *metis*
have 64271: $\bigwedge x y . - (- x \sqcup (y \sqcup - - (y \sqcap x))) = - (- x \sqcup y)$
using 9 25 42 64260 **by** *metis*
have 64281: $\bigwedge x y . - (- x \sqcup y) = - (y \sqcup - - ((top \sqcup y) \sqcap - x))$
using 9 25 157 2706 64260 **by** *metis*
have 64282: $\bigwedge x y . - (x \sqcup - - ((x \sqcup top) \sqcap y)) = - (x \sqcup - - y)$
using 9 25 132 536 2810 28485 61311 64260 **by** *metis*
have 65110: $\bigwedge x y . - ((- x \sqcap y) \sqcup (- y \sqcup x)) = bot$
using 9 14 49 37513 63947 **by** *metis*
have 65231: $\bigwedge x y . - (x \sqcup ((- x \sqcap y) \sqcup - y)) = bot$
using 9 25 65110 **by** *metis*
have 65585: $\bigwedge x y . - (x \sqcup - y) = - - y \sqcap - x$
using 61311 61454 64239 **by** *metis*
have 65615: $\bigwedge x y . - x \sqcap - ((x \sqcup top) \sqcap y) = - y \sqcap - x$
using 132 28485 64282 65585 **by** *metis*
have 65616: $\bigwedge x y . - (- x \sqcup y) = - y \sqcap - ((top \sqcup y) \sqcap - x)$
using 132 28485 64281 65585 **by** *metis*
have 65791: $\bigwedge x y . - x \sqcap - ((top \sqcup x) \sqcap - y) = - - y \sqcap - x$
using 89 132 12376 28485 64271 65585 65615 65616 **by** *metis*
have 65933: $\bigwedge x y . - (- x \sqcup y) = - - x \sqcap - y$
using 65616 65791 **by** *metis*
have 66082: $\bigwedge x y z . - (x \sqcup (y \sqcup - z)) = - - z \sqcap - (x \sqcup y)$
using 8 65585 **by** *metis*
have 66204: $\bigwedge x y . - - x \sqcap - (y \sqcup (- y \sqcap x)) = bot$
using 65231 66082 **by** *metis*
have 66281: $\bigwedge x y z . - (x \sqcup (- y \sqcup z)) = - - y \sqcap - (x \sqcup z)$
using 25 65933 **by** *metis*
have 67527: $\bigwedge x y . - - (x \sqcup (- x \sqcap y)) \sqcap y = y$
using 14 49 62434 66204 **by** *metis*
have 67762: $\bigwedge x y . - (- - x \sqcap (y \sqcup (- y \sqcap x))) = - x$
using 61420 67527 **by** *metis*
have 68018: $\bigwedge x y z . - (x \sqcup y) \sqcap (x \sqcup (y \sqcap (z \sqcup top))) = bot$
using 8 83 1113 2320 **by** *metis*
have 71989: $\bigwedge x y z . - (x \sqcup (y \sqcap (z \sqcup top))) = - (x \sqcup y)$
using 9 29 52533 67762 68018 **by** *metis*
have 71997: $\bigwedge x y z . - ((x \sqcap (y \sqcup top)) \sqcup z) = - (x \sqcup z)$
using 17 2320 71989 **by** *metis*
have 72090: $\bigwedge x y z . - (x \sqcup ((x \sqcap y) \sqcup z)) = - (x \sqcup z)$
using 10 14 705 71997 **by** *metis*
have 72139: $\bigwedge x y . - (x \sqcup y) = - x \sqcap - y$
using 25 123 132 2138 65933 66281 72090 **by** *metis*

show ?thesis
 using 72139 by metis
 qed

lemma l15:

$-(x \sqcup y) = -x \sqcup -y$
 by (simp add: l11 l12 l4)

lemma l13-var:

$-(-x \sqcap y) = -x \sqcap -y$

proof -

have 1: $\bigwedge x y . x \leq y \longleftrightarrow x \sqcup y = y$
 by (simp add: il-less-eq)

have 4: $\bigwedge x y . \neg(x \leq y) \vee x \sqcup y = y$
 using 1 by metis

have 5: $\bigwedge x y z . (x \sqcap y) \sqcup (x \sqcap z) \leq x \sqcap (y \sqcup z)$
 by (simp add: il-sub-inf-right-isotone-var)

have 6: $\bigwedge x y . -x \leq -(x \sqcap y)$
 by (simp add: pad2)

have 7: $\bigwedge x y z . x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
 by (simp add: il-associative)

have 8: $\bigwedge x y z . (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
 using 7 by metis

have 9: $\bigwedge x y . x \sqcup y = y \sqcup x$
 by (simp add: il-commutative)

have 10: $\bigwedge x . x \sqcup \text{bot} = x$
 by (simp add: il-bot-unit)

have 11: $\bigwedge x . x \sqcup x = x$
 by simp

have 12: $\bigwedge x y z . x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
 by (simp add: il-inf-associative)

have 13: $\bigwedge x y z . (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
 using 12 by metis

have 14: $\bigwedge x . \text{top} \sqcap x = x$
 by simp

have 15: $\bigwedge x . x \sqcap \text{top} = x$
 by simp

have 16: $\bigwedge x y z . (x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$
 by (simp add: il-inf-right-dist-sup)

have 17: $\bigwedge x y z . (x \sqcap y) \sqcup (z \sqcap y) = (x \sqcup z) \sqcap y$
 using 16 by metis

have 19: $\bigwedge x . -x \sqcup -x = \text{top}$
 by simp

have 20: $\bigwedge x . -x \sqcap x = \text{bot}$
 by (simp add: a-inf-complement-bot)

have 22: $\bigwedge x y z . ((x \sqcap y) \sqcup (x \sqcap z)) \sqcup (x \sqcap (y \sqcup z)) = x \sqcap (y \sqcup z)$
 using 4 5 by metis

have 23: $\bigwedge x y z . (x \sqcap (y \sqcup z)) \sqcup ((x \sqcap y) \sqcup (x \sqcap z)) = x \sqcap (y \sqcup z)$
 using 9 22 by metis

have 24: $\bigwedge x y . \neg \neg x \sqcup \neg (x \sqcap y) = \neg (x \sqcap y)$
using 4 6 by metis
have 25: $\bigwedge x y z . x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
using 8 9 by metis
have 26: $\bigwedge x y z . (x \sqcap y) \sqcup ((x \sqcap z) \sqcup (x \sqcap (y \sqcup z))) = x \sqcap (y \sqcup z)$
using 9 23 25 by metis
have 29: $\bigwedge x . \text{bot} \sqcup x = x$
using 9 10 by metis
have 30: $\bigwedge x y . x \sqcup (x \sqcup y) = x \sqcup y$
using 8 11 by metis
have 34: $\bigwedge u x y z . (x \sqcap (y \sqcap z)) \sqcup (u \sqcap z) = ((x \sqcap y) \sqcup u) \sqcap z$
using 13 17 by metis
have 35: $\bigwedge u x y z . (x \sqcap y) \sqcup (z \sqcap (u \sqcap y)) = (x \sqcup (z \sqcap u)) \sqcap y$
using 13 17 by metis
have 38: $\bigwedge x y . \neg x \sqcup (\neg \neg x \sqcup y) = \text{top} \sqcup y$
using 8 19 by metis
have 41: $\neg \text{top} = \text{bot}$
using 15 20 by metis
have 42: $\bigwedge x y . (\neg x \sqcup y) \sqcap x = y \sqcap x$
using 17 20 29 by metis
have 43: $\bigwedge x y . (x \sqcup \neg y) \sqcap y = x \sqcap y$
using 9 17 20 29 by metis
have 45: $\bigwedge x . \neg \text{bot} \sqcup \neg \neg x = \neg \text{bot}$
using 9 20 24 by metis
have 49: $\neg \text{bot} = \text{top}$
using 19 29 41 by metis
have 50: $\bigwedge x . \text{top} \sqcup \neg \neg x = \text{top}$
using 45 49 by metis
have 62: $\bigwedge x y . x \sqcup ((x \sqcap \neg y) \sqcup (x \sqcap \neg \neg y)) = x$
using 9 15 19 25 26 by metis
have 65: $\bigwedge x y . (\neg (x \sqcup y) \sqcap x) \sqcup (\neg (x \sqcup y) \sqcap y) = \text{bot}$
using 9 20 26 29 by metis
have 66: $\bigwedge x y z . (x \sqcap \neg \neg y) \sqcup (x \sqcap \neg (\neg y \sqcap z)) = x \sqcap \neg (\neg y \sqcap z)$
using 11 24 26 by metis
have 69: $\bigwedge x y . x \sqcup (x \sqcap \neg \neg y) = x$
using 9 15 26 30 50 by metis
have 81: $\bigwedge x . \text{top} \sqcup \neg x = \text{top}$
using 9 19 30 by metis
have 88: $\bigwedge x y . x \sqcup (\neg y \sqcap x) = x$
using 14 17 81 by metis
have 101: $\bigwedge x y z . x \sqcup (y \sqcup (x \sqcap \neg \neg z)) = y \sqcup x$
using 25 69 by metis
have 103: $\bigwedge x y . x \sqcup (x \sqcap \neg y) = x$
using 9 62 101 by metis
have 123: $\bigwedge x . \neg \neg x \sqcap x = x$
using 14 19 42 by metis
have 127: $\bigwedge x y . \neg \neg x \sqcap (x \sqcap y) = x \sqcap y$
using 13 123 by metis
have 130: $\bigwedge x . \neg x \sqcup \neg \neg \neg x = \neg x$

using 9 24 123 by metis
 have 132: $\bigwedge x . \neg \neg \neg x = \neg x$
 using 9 103 123 130 by metis
 have 136: $\bigwedge x y . (\neg x \sqcup y) \sqcap \neg \neg x = y \sqcap \neg \neg x$
 using 42 132 by metis
 have 144: $\bigwedge x y z . ((\neg (x \sqcap y) \sqcap x) \sqcup z) \sqcap y = z \sqcap y$
 using 20 29 34 by metis
 have 182: $\bigwedge x y z . (x \sqcup (\neg \neg (y \sqcap z) \sqcap y)) \sqcap z = (x \sqcup y) \sqcap z$
 using 17 35 123 by metis
 have 288: $\bigwedge x y . \neg x \sqcup \neg (\neg x \sqcap y) = \text{top}$
 using 24 38 81 by metis
 have 315: $\bigwedge x y . \neg (\neg x \sqcap y) \sqcap x = x$
 using 14 42 288 by metis
 have 319: $\bigwedge x y . \neg x \sqcup \neg \neg (\neg x \sqcap y) = \neg x$
 using 9 24 315 by metis
 have 387: $\bigwedge x y . \neg (x \sqcap y) \sqcap \neg x = \neg x$
 using 127 315 by metis
 have 405: $\bigwedge x y z . \neg (x \sqcap (y \sqcap z)) \sqcap \neg (x \sqcap y) = \neg (x \sqcap y)$
 using 13 387 by metis
 have 419: $\bigwedge x y . \neg x \sqcap \neg \neg (\neg x \sqcap y) = \neg \neg (\neg x \sqcap y)$
 using 315 387 by metis
 have 1091: $\bigwedge x y . \neg (x \sqcup y) \sqcap x = \text{bot}$
 using 9 29 30 65 by metis
 have 1129: $\bigwedge x y z . (\neg (x \sqcup y) \sqcup z) \sqcap x = z \sqcap x$
 using 17 29 1091 by metis
 have 1155: $\bigwedge x y . \neg \neg x \sqcap \neg (\neg x \sqcap y) = \neg \neg x$
 using 66 103 123 132 by metis
 have 2097: $\bigwedge x y . \neg \neg (x \sqcup y) \sqcap x = x$
 using 14 19 1129 by metis
 have 2124: $\bigwedge x y . \neg \neg (x \sqcup y) \sqcap y = y$
 using 9 2097 by metis
 have 2137: $\bigwedge x y . \neg x \sqcup \neg \neg (x \sqcup y) = \text{top}$
 using 9 288 2097 by metis
 have 2201: $\bigwedge x y . \neg x \sqcup \neg \neg (y \sqcup x) = \text{top}$
 using 9 288 2124 by metis
 have 2343: $\bigwedge x y . \neg (\neg x \sqcap y) \sqcup \neg \neg y = \text{top}$
 using 88 2201 by metis
 have 3022: $\bigwedge x y . \neg x \sqcup \neg (\neg y \sqcap \neg x) = \text{top}$
 using 9 132 2343 by metis
 have 3133: $\bigwedge x y . \neg (\neg x \sqcap \neg y) \sqcap y = y$
 using 14 42 3022 by metis
 have 3134: $\bigwedge x y . \neg x \sqcap (\neg y \sqcap \neg x) = \neg y \sqcap \neg x$
 using 14 43 3022 by metis
 have 3961: $\bigwedge x y . \neg \neg (x \sqcup y) \sqcap \neg \neg x = \neg \neg x$
 using 14 136 2137 by metis
 have 9413: $\bigwedge x y . \neg \neg (\neg x \sqcap y) \sqcap y = \neg x \sqcap y$
 using 9 103 182 319 by metis
 have 12370: $\bigwedge x y . \neg x \sqcap \neg (\neg \neg x \sqcap y) = \neg x$
 using 132 1155 by metis

have 12376: $\bigwedge x y . - x \sqcap - (x \sqcap y) = - x$
using 127 132 1155 **by** *metis*
have 12383: $\bigwedge x y . - (x \sqcup y) \sqcap - y = - (x \sqcup y)$
using 132 1155 2124 **by** *metis*
have 12393: $\bigwedge x y . - - (- x \sqcap - y) = - x \sqcap - y$
using 1155 3133 9413 **by** *metis*
have 12639: $\bigwedge x y . - x \sqcap - (- y \sqcap x) = - x$
using 88 12383 **by** *metis*
have 24647: $\bigwedge x y . (- x \sqcap - y) \sqcup - (- x \sqcap - y) = \text{top}$
using 19 12393 **by** *metis*
have 28338: $\bigwedge x y . - (- - (x \sqcap y) \sqcap x) = - (x \sqcap y)$
using 123 405 12370 **by** *metis*
have 28422: $\bigwedge x y . - (- x \sqcap - y) = - (- y \sqcap - x)$
using 13 3134 12393 28338 **by** *metis*
have 28485: $\bigwedge x y . - x \sqcap - y = - y \sqcap - x$
using 2097 3961 12393 28422 **by** *metis*
have 52421: $\bigwedge x y . - (- x \sqcap - (- x \sqcap y)) \sqcap y = y$
using 14 144 24647 28485 **by** *metis*
have 52520: $\bigwedge x y . - x \sqcap - (- x \sqcap y) = - x \sqcap - y$
using 13 12376 12393 12639 28485 52421 **by** *metis*
have 61156: $\bigwedge x y . - - (- x \sqcap y) = - x \sqcap - - y$
using 419 52520 **by** *metis*
show *?thesis*
using 61156 **by** *metis*
qed

Theorem 25.1

subclass *subset-boolean-algebra-2*

proof

show $\bigwedge x y z . x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
by (*simp add: il-associative*)
show $\bigwedge x y . x \sqcup y = y \sqcup x$
by (*simp add: il-commutative*)
show $\bigwedge x . x \sqcup x = x$
by *simp*
show $\bigwedge x y . x \sqcup - (y \sqcup - y) = x$
using *il-bot-unit l12 l6* **by** *auto*
show $\bigwedge x y . - (x \sqcup y) = - (- - x \sqcup - - y)$
by (*metis l15 l4*)
show $\bigwedge x y . - x \sqcup - (- x \sqcup y) = - x \sqcup - y$
by (*smt l11 l15 il-inf-right-dist-sup il-unit-bot l6 l7*)
qed

lemma *aa-test*:

$p = --p \implies \text{test } p$
by (*metis ppa-ppd.d-closed*)

lemma *test-aa-increasing*:

$\text{test } p \implies p \leq --p$

```

    by (simp add: ppa-ppd.d-increasing-sub-identity test-sub-identity)

lemma test p  $\implies$   $-- (p \sqcap x) \leq p$  nitpick [expect=genuine] oops
lemma test p  $\implies$   $--p \leq p$  nitpick [expect=genuine] oops

end

class pa-algebra = pa-semiring + minus +
  assumes pa-minus-def:  $-x - -y = -(--x \sqcup -y)$ 
begin

subclass subset-boolean-algebra-2-extended
proof
  show bot = (THE x.  $\forall z. x = -(z \sqcup -z)$ )
    using l12 l6 by auto
  thus top = -(THE x.  $\forall z. x = -(z \sqcup -z)$ )
    using l2 by blast
  show  $\bigwedge x y. -x \sqcap -y = -(--x \sqcup --y)$ 
    by (metis l12 l4)
  show  $\bigwedge x y. -x - -y = -(--x \sqcup -y)$ 
    by (simp add: pa-minus-def)
  show  $\bigwedge x y. (x \leq y) = (x \sqcup y = y)$ 
    by (simp add: il-less-eq)
  show  $\bigwedge x y. (x < y) = (x \sqcup y = y \wedge y \sqcup x \neq x)$ 
    by (simp add: il-less-eq less-le-not-le)
qed

lemma  $\bigwedge x y. -(x \sqcap --y) = -(x \sqcap y)$  nitpick [expect=genuine] oops

end

```

8.3 Antidomain Semirings

Definition 24

```

class a-semiring = ppa-semiring +
  assumes ad3:  $-(x \sqcap y) \leq -(x \sqcap --y)$ 
begin

lemma l16:
   $--x \leq -( - x \sqcap y)$ 
proof -
  have 1:  $\bigwedge x y. x \leq y \iff x \sqcup y = y$ 
    by (simp add: il-less-eq)
  have 3:  $\bigwedge x y z. x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ 
    by (simp add: il-associative)
  have 4:  $\bigwedge x y z. (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$ 
    using 3 by metis
  have 5:  $\bigwedge x y. x \sqcup y = y \sqcup x$ 
    by (simp add: il-commutative)

```

have 6: $\bigwedge x . x \sqcup \text{bot} = x$
by (*simp add: il-bot-unit*)
have 7: $\bigwedge x . x \sqcup x = x$
by *simp*
have 8: $\bigwedge x y . \neg(x \leq y) \vee x \sqcup y = y$
using 1 by *metis*
have 9: $\bigwedge x y . x \leq y \vee x \sqcup y \neq y$
using 1 by *metis*
have 10: $\bigwedge x y z . x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
by (*simp add: il-inf-associative*)
have 11: $\bigwedge x y z . (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
using 10 by *metis*
have 12: $\bigwedge x . \text{top} \sqcap x = x$
by *simp*
have 13: $\bigwedge x . x \sqcap \text{top} = x$
by *simp*
have 14: $\bigwedge x y z . (x \sqcap y) \sqcup (x \sqcap z) \leq x \sqcap (y \sqcup z)$
by (*simp add: il-sub-inf-right-isotone-var*)
have 15: $\bigwedge x y z . (x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$
by (*simp add: il-inf-right-dist-sup*)
have 16: $\bigwedge x y z . (x \sqcap y) \sqcup (z \sqcap y) = (x \sqcup z) \sqcap y$
using 15 by *metis*
have 17: $\bigwedge x . \text{bot} \sqcap x = \text{bot}$
by *simp*
have 18: $\bigwedge x . - x \sqcup - - x = \text{top}$
by *simp*
have 19: $\bigwedge x . - x \sqcap x = \text{bot}$
by (*simp add: a-inf-complement-bot*)
have 20: $\bigwedge x y . - (x \sqcap y) \leq - (x \sqcap - - y)$
by (*simp add: ad3*)
have 22: $\bigwedge x y z . x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
using 4 5 by *metis*
have 25: $\bigwedge x . \text{bot} \sqcup x = x$
using 5 6 by *metis*
have 26: $\bigwedge x y . x \sqcup (x \sqcup y) = x \sqcup y$
using 4 7 by *metis*
have 33: $\bigwedge x y z . (x \sqcap y) \sqcup ((x \sqcap z) \sqcup (x \sqcap (y \sqcup z))) = x \sqcap (y \sqcup z)$
using 5 8 14 22 by *metis*
have 47: $\bigwedge x y . - x \sqcup (- - x \sqcup y) = \text{top} \sqcup y$
using 4 18 by *metis*
have 48: $\bigwedge x y . - - x \sqcup (y \sqcup - x) = y \sqcup \text{top}$
using 4 5 18 by *metis*
have 51: $\bigwedge x y . - x \sqcap (x \sqcap y) = \text{bot}$
using 11 17 19 by *metis*
have 52: $-\text{top} = \text{bot}$
using 13 19 by *metis*
have 56: $\bigwedge x y . (- x \sqcup y) \sqcap x = y \sqcap x$
using 16 19 25 by *metis*
have 57: $\bigwedge x y . (x \sqcup - y) \sqcap y = x \sqcap y$

using 5 16 19 25 **by** *metis*
have 58: $\bigwedge x y . \neg (x \sqcap y) \sqcup \neg (x \sqcap \neg \neg y) = \neg (x \sqcap \neg \neg y)$
using 8 20 **by** *metis*
have 60: $\bigwedge x . \neg x \leq \neg \neg \neg x$
using 12 20 **by** *metis*
have 69: $\neg \text{bot} = \text{top}$
using 18 25 52 **by** *metis*
have 74: $\bigwedge x y . x \leq x \sqcup y$
using 9 26 **by** *metis*
have 78: $\bigwedge x . \text{top} \sqcup \neg x = \text{top}$
using 5 18 26 **by** *metis*
have 80: $\bigwedge x y . x \leq y \sqcup x$
using 5 74 **by** *metis*
have 86: $\bigwedge x y z . x \sqcup y \leq x \sqcup (z \sqcup y)$
using 22 80 **by** *metis*
have 95: $\bigwedge x . \neg x \sqcup \neg \neg \neg x = \neg \neg \neg x$
using 8 60 **by** *metis*
have 143: $\bigwedge x y . x \sqcup (x \sqcap \neg y) = x$
using 5 13 26 33 78 **by** *metis*
have 370: $\bigwedge x y z . x \sqcup (y \sqcap \neg z) \leq x \sqcup y$
using 86 143 **by** *metis*
have 907: $\bigwedge x . \neg x \sqcap \neg x = \neg x$
using 12 18 57 **by** *metis*
have 928: $\bigwedge x y . \neg x \sqcap (\neg x \sqcap y) = \neg x \sqcap y$
using 11 907 **by** *metis*
have 966: $\bigwedge x y . \neg (\neg x \sqcap \neg \neg (x \sqcap y)) = \text{top}$
using 51 58 69 78 **by** *metis*
have 1535: $\bigwedge x . \neg x \sqcup \neg \neg \neg \neg x = \text{top}$
using 47 78 95 **by** *metis*
have 1630: $\bigwedge x y z . (x \sqcup y) \sqcap \neg z \leq (x \sqcap \neg z) \sqcup y$
using 16 370 **by** *metis*
have 2422: $\bigwedge x . \neg x \sqcap \neg \neg \neg x = \neg \neg \neg x$
using 12 57 1535 **by** *metis*
have 6567: $\bigwedge x y . \neg x \sqcap \neg \neg (x \sqcap y) = \text{bot}$
using 12 19 966 **by** *metis*
have 18123: $\bigwedge x . \neg \neg \neg x = \neg x$
using 95 143 2422 **by** *metis*
have 26264: $\bigwedge x y . \neg x \leq (\neg y \sqcap \neg x) \sqcup \neg \neg y$
using 12 18 1630 **by** *metis*
have 26279: $\bigwedge x y . \neg \neg (x \sqcap y) \leq \neg \neg x$
using 25 6567 26264 **by** *metis*
have 26307: $\bigwedge x y . \neg \neg (\neg x \sqcap y) \leq \neg x$
using 928 18123 26279 **by** *metis*
have 26339: $\bigwedge x y . \neg x \sqcup \neg \neg (\neg x \sqcap y) = \neg x$
using 5 8 26307 **by** *metis*
have 26564: $\bigwedge x y . \neg x \sqcup \neg (\neg x \sqcap y) = \text{top}$
using 5 48 78 18123 26339 **by** *metis*
have 26682: $\bigwedge x y . \neg (\neg x \sqcap y) \sqcap x = x$
using 12 56 26564 **by** *metis*

```

have 26864:  $\bigwedge x y. \neg \neg x \leq \neg (\neg x \sqcap y)$ 
  using 18123 26279 26682 by metis
show ?thesis
  using 26864 by metis
qed

```

Theorem 25.2

```

subclass pa-semiring
proof
  show  $\bigwedge x y. \neg \neg x \leq \neg (\neg x \sqcap y)$ 
    by (rule l16)
qed

```

```

lemma l17:
   $\neg(x \sqcap y) = \neg(x \sqcap \neg\neg y)$ 
  by (simp add: ad3 order.antisym l14)

```

```

lemma a-complement-inf-double-complement:
   $\neg(x \sqcap \neg\neg y) = \neg(x \sqcap y)$ 
  using l17 by auto

```

```

sublocale a-d: d-semiring-var where d =  $\lambda x. \neg\neg x$ 
proof
  show  $\bigwedge x y. \neg \neg (x \sqcap \neg\neg y) \leq \neg \neg (x \sqcap y)$ 
    using l17 by auto
  show  $\neg \neg \text{bot} = \text{bot}$ 
    by (simp add: l1 l2)
qed

```

```

lemma test p  $\implies \neg \neg (p \sqcap x) \leq p$ 
  by (fact a-d.d2)

```

end

```

class a-algebra = a-semiring + minus +
  assumes a-minus-def:  $\neg x \neg y = \neg(\neg\neg x \sqcup \neg y)$ 
begin

```

```

subclass pa-algebra
proof
  show  $\bigwedge x y. \neg x \neg y = \neg(\neg\neg x \sqcup \neg y)$ 
    by (simp add: a-minus-def)
qed

```

Theorem 25.4

```

subclass subset-boolean-algebra-4-extended
proof
  show  $\bigwedge x y z. x \sqcap (y \sqcap z) = x \sqcap y \sqcap z$ 
    by (simp add: il-inf-associative)

```

```

show  $\bigwedge x y z. (x \sqcup y) \sqcap z = x \sqcap z \sqcup y \sqcap z$ 
  by (simp add: il-inf-right-dist-sup)
show  $\bigwedge x. \neg x \sqcap x = \text{bot}$ 
  by (simp add: a-inf-complement-bot)
show  $\bigwedge x. \text{top} \sqcap x = x$ 
  by simp
show  $\bigwedge x y. \neg (x \sqcap \neg y) = \neg (x \sqcap y)$ 
  using l17 by auto
show  $\bigwedge x. x \sqcap \text{top} = x$ 
  by simp
show  $\bigwedge x y z. x \leq y \implies z \sqcap x \leq z \sqcap y$ 
  by (simp add: il-sub-inf-right-isotone)
qed

```

end

```

context subset-boolean-algebra-4-extended
begin

```

```

subclass il-semiring

```

```

proof

```

```

show  $\bigwedge x y z. x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$ 
  by (simp add: sup-assoc)
show  $\bigwedge x y. x \sqcup y = y \sqcup x$ 
  by (simp add: sup-commute)
show  $\bigwedge x. x \sqcup x = x$ 
  by simp
show  $\bigwedge x. x \sqcup \text{bot} = x$ 
  by simp
show  $\bigwedge x y z. x \sqcap (y \sqcap z) = x \sqcap y \sqcap z$ 
  by (simp add: sba3-inf-associative)
show  $\bigwedge x y z. (x \sqcup y) \sqcap z = x \sqcap z \sqcup y \sqcap z$ 
  by (simp add: sba3-inf-right-dist-sup)
show  $\bigwedge x. \text{top} \sqcap x = x$ 
  by simp
show  $\bigwedge x. x \sqcap \text{top} = x$ 
  by simp
show  $\bigwedge x. \text{bot} \sqcap x = \text{bot}$ 
  by (simp add: inf-left-zero)
show  $\bigwedge x y z. x \leq y \implies z \sqcap x \leq z \sqcap y$ 
  by (simp add: inf-right-isotone)
show  $\bigwedge x y. (x \leq y) = (x \sqcup y = y)$ 
  by (simp add: le-iff-sup)
show  $\bigwedge x y. (x < y) = (x \leq y \wedge \neg y \leq x)$ 
  by (simp add: less-le-not-le)

```

```

qed

```

```

subclass a-semiring

```

```

proof

```

```

show  $\bigwedge x. \neg x \sqcap x = \text{bot}$ 
  by (simp add: sba3-inf-complement-bot)
show  $\bigwedge x. \neg x \sqcup \neg x = \text{top}$ 
  by simp
show  $\bigwedge x y. \neg (x \sqcap y) \leq \neg (x \sqcap \neg y)$ 
  by (simp add: sba3-complement-inf-double-complement)
qed

```

sublocale *sba4-a: a-algebra*

proof

```

show  $\bigwedge x y. \neg x \neg y = \neg (\neg x \sqcup \neg y)$ 
  by (simp add: sub-minus-def)
qed

```

end

context *stone-algebra*

begin

Theorem 25.3

subclass *il-semiring*

proof

```

show  $\bigwedge x y z. x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$ 
  by (simp add: sup-assoc)
show  $\bigwedge x y. x \sqcup y = y \sqcup x$ 
  by (simp add: sup-commute)
show  $\bigwedge x. x \sqcup x = x$ 
  by simp
show  $\bigwedge x. x \sqcup \text{bot} = x$ 
  by simp
show  $\bigwedge x y z. x \sqcap (y \sqcap z) = x \sqcap y \sqcap z$ 
  by (simp add: inf.sup-monoid.add-assoc)
show  $\bigwedge x y z. (x \sqcup y) \sqcap z = x \sqcap z \sqcup y \sqcap z$ 
  by (simp add: inf-sup-distrib2)
show  $\bigwedge x. \text{top} \sqcap x = x$ 
  by simp
show  $\bigwedge x. x \sqcap \text{top} = x$ 
  by simp
show  $\bigwedge x. \text{bot} \sqcap x = \text{bot}$ 
  by simp
show  $\bigwedge x y z. x \leq y \implies z \sqcap x \leq z \sqcap y$ 
  using inf.sup-right-isotone by blast
show  $\bigwedge x y. (x \leq y) = (x \sqcup y = y)$ 
  by (simp add: le-iff-sup)
show  $\bigwedge x y. (x < y) = (x \leq y \wedge \neg y \leq x)$ 
  by (simp add: less-le-not-le)
qed

```

subclass *a-semiring*

```

proof
  show  $\bigwedge x. \neg x \sqcap x = \text{bot}$ 
    by simp
  show  $\bigwedge x. \neg x \sqcup \neg \neg x = \text{top}$ 
    by simp
  show  $\bigwedge x y. \neg (x \sqcap y) \leq \neg (x \sqcap \neg \neg y)$ 
    by simp
qed

end

end

```

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