The Sturm–Tarski Theorem

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Abstract

We have formalised the Sturm–Tarski theorem (also referred as the Tarski theorem): Given polynomials $p, q \in \mathbb{R}[x]$, the Sturm–Tarski theorem computes the sum of the signs of q over the roots of p by calculating some remainder sequences. Note, the better-known Sturm theorem is an instance of the Sturm–Tarski theorem when $q = 1$. The proof follows the classic book by Basu et al. [1] and Cyril Cohen's work in Coq [2]. With the Sturm–Tarski theorem proved, it is possible to further build a quantifier elimination procedure for real numbers as Cohen did in Coq. Another application of the Sturm–Tarski theorem is to build sign determination procedures for polynomials at real algebraic points, as described in our formalisation of real algebraic numbers [3].

1 Misc polynomial lemmas for the Sturm–Tarski theorem

theory *PolyMisc* **imports**
HOL−Computational-Algebra.Polynomial-Factorial
begin

lemma *coprime-poly-0*:
poly p x ≠ *0* ∨ *poly q x* ≠ *0* **if** *coprime p q*
for *x* :: *'a* :: *field*
⟨*proof*⟩

lemma *smult-cancel*:
fixes *p*::*'a*::*idom poly*
assumes *c*≠*0* **and** *smult*: *smult c p* = *smult c q*
shows *p*=*q*
⟨*proof*⟩

lemma *dvd-monic*:
fixes *p q*:: *'a* :: *idom poly*
assumes *monic*:*lead-coeff p*=*1* **and** *p dvd* (*smult c q*) **and** *c*≠*0*
shows *p dvd q* ⟨*proof*⟩

lemma *poly-power-n-eq*:
fixes $x::'a :: idom$
assumes $n\neq 0$
shows $poly\ ([:-a,1:]\hat{}n)\ x=0 \longleftrightarrow (x=a)$ ⟨*proof*⟩

lemma *poly-power-n-odd*:
fixes $x\ a::real$
assumes $odd\ n$
shows $poly\ ([:-a,1:]\hat{}n)\ x>0 \longleftrightarrow (x>a)$ ⟨*proof*⟩

lemma *gcd-coprime-poly*:
fixes $p\ q::'a::\{factorial\text{-}ring\text{-}gcd,semiring\text{-}gcd\text{-}mult\text{-}normalize\}\ poly$
assumes nz: $p \neq 0 \vee q \neq 0$ **and** p': $p = p' * gcd\ p\ q$ **and**
q': $q = q' * gcd\ p\ q$
shows $coprime\ p'\ q'$
⟨*proof*⟩

lemma *poly-mod*:
$poly\ (p\ mod\ q)\ x = poly\ p\ x$ **if** $poly\ q\ x = 0$
⟨*proof*⟩

lemma *pseudo-divmod-0*[*simp*]: $pseudo\text{-}divmod\ f\ 0 = (0,f)$
⟨*proof*⟩

lemma *map-poly-eq-iff*:
assumes $f\ 0=0\ inj\ f$
shows $map\text{-}poly\ f\ x = map\text{-}poly\ f\ y \longleftrightarrow x=y$
⟨*proof*⟩

lemma *pseudo-mod-0*[*simp*]:
shows $pseudo\text{-}mod\ p\ 0= p\ pseudo\text{-}mod\ 0\ q = 0$
⟨*proof*⟩

lemma *pseudo-mod-mod*:
assumes $g\neq 0$
shows $smult\ (lead\text{-}coeff\ g\ \hat{}\ (Suc\ (degree\ f) - degree\ g))\ (f\ mod\ g) = pseudo\text{-}mod\ f\ g$
⟨*proof*⟩

lemma *poly-pseudo-mod*:
assumes $poly\ q\ x=0\ q\neq 0$
shows $poly\ (pseudo\text{-}mod\ p\ q)\ x = (lead\text{-}coeff\ q\ \hat{}\ (Suc\ (degree\ p) - degree\ q)) * poly\ p\ x$
⟨*proof*⟩

lemma *degree-less-timesD*:
fixes $q::'a::idom\ poly$
assumes $q*g=r$ **and** $deg: r=0 \vee degree\ g > degree\ r$ **and** $g\neq 0$
shows $q=0 \wedge r=0$

⟨*proof*⟩

end

2 Sturm–Tarski Theorem

theory *Sturm-Tarski*
  imports *Complex-Main PolyMisc HOL−Computational-Algebra.Field-as-Ring*
begin

2.1 Misc

lemma *eventually-at-right*:
  fixes *x*::*'a*::{*archimedean-field*,*linorder-topology*}
  shows *eventually P* (*at-right x*) ⟷ (∃ *b*>*x*. ∀ *y*>*x*. *y* < *b* ⟶ *P y*)
⟨*proof*⟩

lemma *eventually-at-left*:
  fixes *x*::*'a*::{*archimedean-field*,*linorder-topology*}
  shows *eventually P* (*at-left x*) ⟷ (∃ *b*<*x*. ∀ *y*>*b*. *y* < *x* ⟶ *P y*)
⟨*proof*⟩

lemma *eventually-neg*:
  assumes *F*≠*bot* **and** *eve*:*eventually* (λ*x*. *P x*) *F*
  shows ¬ *eventually* (λ*x*. ¬ *P x*) *F*
⟨*proof*⟩

lemma *poly-tendsto*[*simp*]:
    (*poly p* ⟶ *poly p x*) (*at* (*x*::*real*))
    (*poly p* ⟶ *poly p x*) (*at-left* (*x*::*real*))
    (*poly p* ⟶ *poly p x*) (*at-right* (*x*::*real*))
  ⟨*proof*⟩

lemma *not-eq-pos-or-neg-iff-1*:
  fixes *p*::*real poly*
  shows (∀ *z*. *lb*<*z*∧*z*≤*ub*⟶*poly p z*≠*0*) ⟷
    (∀ *z*. *lb*<*z*∧*z*≤*ub*⟶*poly p z*>*0*)∨(∀ *z*. *lb*<*z*∧*z*≤*ub*⟶*poly p z*<*0*) (**is** *?Q* ⟷ *?P*)
⟨*proof*⟩

lemma *not-eq-pos-or-neg-iff-2*:
  fixes *p*::*real poly*
  shows (∀ *z*. *lb*≤*z*∧*z*<*ub*⟶*poly p z*≠*0*)
   ⟷(∀ *z*. *lb*≤*z*∧*z*<*ub*⟶*poly p z*>*0*)∨(∀ *z*. *lb*≤*z*∧*z*<*ub*⟶*poly p z*<*0*) (**is** *?Q*⟷*?P*)
⟨*proof*⟩

lemma *next-non-root-interval*:
  fixes *p*::*real poly*
  assumes *p*≠*0*

obtains *ub* **where** *ub*>*lb* **and** (∀ *z*. *lb*<*z*∧*z*≤*ub*⟶*poly p z*≠*0*)
⟨*proof*⟩

lemma *last-non-root-interval*:
fixes *p*::*real poly*
assumes *p*≠*0*
obtains *lb* **where** *lb*<*ub* **and** (∀ *z*. *lb*≤*z*∧*z*<*ub*⟶*poly p z*≠*0*)
⟨*proof*⟩

2.2 Sign

definition *sign*:: *'a*::{*zero*,*linorder*} ⇒ *int* **where**
sign x≡(*if x*>*0 then 1 else if x*=*0 then 0 else* −*1*)

lemma *sign-simps*[*simp*]:
x>*0* ⟹ *sign x*=*1*
x=*0* ⟹ *sign x*=*0*
x<*0* ⟹ *sign x*=−*1*
⟨*proof*⟩

lemma *sign-cases* [*case-names neg zero pos*]:
(*sign x* = −*1* ⟹ *P*) ⟹ (*sign x* = *0* ⟹ *P*) ⟹ (*sign x* =*1* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma *sign-times*:
fixes *x*::*'a*::*linordered-ring-strict*
shows *sign* (*x*∗*y*) = *sign x* ∗ *sign y*
⟨*proof*⟩

lemma *sign-power*:
fixes *x*::*'a*::*linordered-idom*
shows *sign* (*x*^*n*) = (*if n*=*0 then 1 else if even n then* |*sign x*| *else sign x*)
⟨*proof*⟩

lemma *sgn-sign-eq*:*sgn* = *sign*
⟨*proof*⟩

lemma *sign-sgn*[*simp*]: *sign* (*sgn x*) = *sign* (*x*::*'b*::*linordered-idom*)
⟨*proof*⟩

lemma *sign-uminus*[*simp*]:*sign* (− *x*) = − *sign* (*x*::*'b*::*linordered-idom*)
⟨*proof*⟩

2.3 Bound of polynomials

definition *sgn-pos-inf* :: (*'a* ::*linordered-idom*) *poly* ⇒ *'a* **where**
sgn-pos-inf p ≡ *sgn* (*lead-coeff p*)
definition *sgn-neg-inf* :: (*'a* ::*linordered-idom*) *poly* ⇒ *'a* **where**

sgn-neg-inf p ≡ if even (degree p) then sgn (lead-coeff p) else −sgn (lead-coeff p)

lemma *sgn-inf-sym*:
fixes *p*::*real poly*
shows *sgn-pos-inf (pcompose p [:0,−1:]) = sgn-neg-inf p* (**is** *?L=?R*)
⟨*proof*⟩

lemma *poly-pinfty-gt-lc*:
fixes *p*:: *real poly*
assumes *lead-coeff p > 0*
shows ∃ *n*. ∀ *x ≥ n*. *poly p x ≥ lead-coeff p* ⟨*proof*⟩

lemma *poly-sgn-eventually-at-top*:
fixes *p*::*real poly*
shows *eventually* (λ*x*. *sgn* (*poly p x*) = *sgn-pos-inf p*) *at-top*
⟨*proof*⟩

lemma *poly-sgn-eventually-at-bot*:
fixes *p*::*real poly*
shows *eventually* (λ*x*. *sgn* (*poly p x*) = *sgn-neg-inf p*) *at-bot*
⟨*proof*⟩

lemma *root-ub*:
fixes *p*:: *real poly*
assumes *p≠0*
obtains *ub* **where** ∀ *x*. *poly p x=0* ⟶ *x<ub*
and ∀ *x≥ub*. *sgn* (*poly p x*) = *sgn-pos-inf p*
⟨*proof*⟩

lemma *root-lb*:
fixes *p*:: *real poly*
assumes *p≠0*
obtains *lb* **where** ∀ *x*. *poly p x=0* ⟶ *x>lb*
and ∀ *x≤lb*. *sgn* (*poly p x*) = *sgn-neg-inf p*
⟨*proof*⟩

2.4 Variation and cross

definition *variation* :: *real* ⇒ *real* ⇒ *int* **where**
variation x y=(*if x∗y≥0 then 0 else if x<y then 1 else −1*)

definition *cross* :: *real poly* ⇒ *real* ⇒ *real* ⇒ *int* **where**
cross p a b=variation (*poly p a*) (*poly p b*)

lemma *variation-0*[*simp*]: *variation 0 y=0 variation x 0=0*
⟨*proof*⟩

lemma *variation-comm*: *variation x y= − variation y x* ⟨*proof*⟩

lemma *cross-0*[*simp*]: *cross 0 a b=0* ⟨*proof*⟩

lemma *variation-cases*:
⟦*x>0*;*y>0*⟧⟹*variation x y = 0*
⟦*x>0*;*y<0*⟧⟹*variation x y = −1*
⟦*x<0*;*y>0*⟧⟹*variation x y = 1*
⟦*x<0*;*y<0*⟧⟹*variation x y = 0*
⟨*proof*⟩

lemma *variation-congr*:
assumes *sgn x=sgn x′ sgn y=sgn y′*
shows *variation x y=variation x′ y′* ⟨*proof*⟩

lemma *variation-mult-pos*:
assumes *c>0*
shows *variation* (*c*∗*x*) *y* =*variation x y* **and** *variation x* (*c*∗*y*) =*variation x y*
⟨*proof*⟩

lemma *variation-mult-neg-1*:
assumes *c<0*
shows *variation* (*c*∗*x*) *y* =*variation x y* + (*if y=0 then 0 else sign x*)
⟨*proof*⟩

lemma *variation-mult-neg-2*:
assumes *c<0*
shows *variation x* (*c*∗*y*) = *variation x y* + (*if x=0 then 0 else* − *sign y*)
⟨*proof*⟩

lemma *cross-no-root*:
assumes *a<b* **and** *no-root*:∀ *x*. *a<x*∧*x<b* ⟶ *poly p x*≠*0*
shows *cross p a b=0*
⟨*proof*⟩

2.5 Tarski query

definition *taq* :: *′a*::*linordered-idom set* ⇒ *′a poly* ⇒ *int* **where**
taq s q ≡ ∑ *x*∈*s*. *sign* (*poly q x*)

2.6 Sign at the right

definition *sign-r-pos* :: *real poly* ⇒ *real* ⇒ *bool*
where
sign-r-pos p x≡ (*eventually* (λ*x*. *poly p x>0*) (*at-right x*))

lemma *sign-r-pos-rec*:
fixes *p*:: *real poly*
assumes *p*≠*0*
shows *sign-r-pos p x*= (*if poly p x=0 then sign-r-pos* (*pderiv p*) *x else poly p x>0*
)
⟨*proof*⟩

lemma *sign-r-pos-0*[*simp*]:¬ *sign-r-pos 0* (*x*::*real*)
⟨*proof*⟩

lemma *sign-r-pos-minus*:
fixes *p*:: *real poly*
assumes *p≠0*
shows *sign-r-pos p x* = (¬ *sign-r-pos* (−*p*) *x*)
⟨*proof*⟩

lemma *sign-r-pos-smult*:
fixes *p* :: *real poly*
assumes *c≠0 p≠0*
shows *sign-r-pos* (*smult c p*) *x*= (*if c>0 then sign-r-pos p x else* ¬ *sign-r-pos p x*)
(**is** *?L=?R*)
⟨*proof*⟩

lemma *sign-r-pos-mult*:
fixes *p q* :: *real poly*
assumes *p≠0 q≠0*
shows *sign-r-pos* (*p*∗*q*) *x*= (*sign-r-pos p x* ⟷ *sign-r-pos q x*)
⟨*proof*⟩

lemma *sign-r-pos-add*:
fixes *p q* :: *real poly*
assumes *poly p x=0 poly q x≠0*
shows *sign-r-pos* (*p*+*q*) *x*=*sign-r-pos q x*
⟨*proof*⟩

lemma *sign-r-pos-mod*:
fixes *p q* :: *real poly*
assumes *poly p x=0 poly q x≠0*
shows *sign-r-pos* (*q mod p*) *x*=*sign-r-pos q x*
⟨*proof*⟩

lemma *sign-r-pos-pderiv*:
fixes *p*:: *real poly*
assumes *poly p x=0 p≠0*
shows *sign-r-pos* (*pderiv p* ∗ *p*) *x*
⟨*proof*⟩

lemma *sign-r-pos-power*:
fixes *p*:: *real poly* **and** *a*::*real*
shows *sign-r-pos* ([:−*a*,*1*:]^*n*) *a*
⟨*proof*⟩

2.7 Jump

definition *jump-poly* :: *real poly* ⇒ *real poly* ⇒*real* ⇒ *int*
where
jump-poly q p x≡ (*if p*≠*0* ∧ *q*≠*0* ∧ *odd*((*order x p*) − (*order x q*)) *then*
if sign-r-pos (*q*∗*p*) *x then 1 else* −*1*
else 0)

lemma *jump-poly-not-root*:*poly p x*≠*0*⟹ *jump-poly q p x*=*0*
⟨*proof*⟩

lemma *jump-poly0*[*simp*]:
jump-poly 0 p x = *0*
jump-poly q 0 x = *0*
⟨*proof*⟩

lemma *jump-poly-smult-1*:
fixes *p q*::*real poly* **and** *c*::*real*
shows *jump-poly* (*smult c q*) *p x*= *sign c* ∗ *jump-poly q p x* (**is** *?L*=*?R*)
⟨*proof*⟩

lemma *jump-poly-mult*:
fixes *p q p'*::*real poly*
assumes *p'*≠*0*
shows *jump-poly* (*p'*∗*q*) (*p'*∗*p*) *x*= *jump-poly q p x*
⟨*proof*⟩

lemma *jump-poly-1-mult*:
fixes *p1 p2*::*real poly*
assumes *poly p1 x*≠*0* ∨ *poly p2 x*≠*0*
shows *jump-poly 1* (*p1*∗*p2*) *x*= *sign* (*poly p2 x*) ∗ *jump-poly 1 p1 x*
+ *sign* (*poly p1 x*) ∗ *jump-poly 1 p2 x* (**is** *?L*=*?R*)
⟨*proof*⟩

lemma *jump-poly-mod*:
fixes *p q*::*real poly*
shows *jump-poly q p x*= *jump-poly* (*q mod p*) *p x*
⟨*proof*⟩

lemma *jump-poly-coprime*:
fixes *p q*:: *real poly*
assumes *poly p x*=*0 coprime p q*
shows *jump-poly q p x*= *jump-poly 1* (*q*∗*p*) *x*
⟨*proof*⟩

lemma *jump-poly-sgn*:
fixes *p q*:: *real poly*
assumes *p*≠*0 poly p x*=*0*
shows *jump-poly* (*pderiv p* ∗ *q*) *p x* = *sign* (*poly q x*)
⟨*proof*⟩

2.8 Cauchy index

definition *cindex-poly*:: *real* ⇒ *real* ⇒ *real poly* ⇒ *real poly* ⇒ *int*
where
cindex-poly a b q p≡ (∑ *x*∈{*x*. *poly p x*=*0* ∧ *a*< *x*∧ *x*< *b*}. *jump-poly q p x*)

lemma *cindex-poly-0*[*simp*]: *cindex-poly a b 0 p* = *0 cindex-poly a b q 0* = *0*
⟨*proof*⟩

lemma *cindex-poly-cross*:
fixes *p*::*real poly* **and** *a b*::*real*
assumes *a*<*b poly p a*≠*0 poly p b*≠*0*
shows *cindex-poly a b 1 p* = *cross p a b*
⟨*proof*⟩

lemma *cindex-poly-mult*:
fixes *p q p*′::*real poly*
assumes *p*′≠ *0*
shows *cindex-poly a b* (*p*′ ∗ *q*) (*p*′ ∗ *p*) = *cindex-poly a b q p*
⟨*proof*⟩

lemma *cindex-poly-smult-1*:
fixes *p q*::*real poly* **and** *c*::*real*
shows *cindex-poly a b* (*smult c q*) *p* = (*sign c*) ∗ *cindex-poly a b q p*
⟨*proof*⟩

lemma *cindex-poly-mod*:
fixes *p q*::*real poly*
shows *cindex-poly a b q p* = *cindex-poly a b* (*q mod p*) *p*
⟨*proof*⟩

lemma *cindex-poly-inverse-add*:
fixes *p q*::*real poly*
assumes *coprime p q*
shows *cindex-poly a b q p* + *cindex-poly a b p q*=*cindex-poly a b 1* (*q*∗*p*)
(**is** *?L*=*?R*)
⟨*proof*⟩

lemma *cindex-poly-inverse-add-cross*:
fixes *p q*::*real poly*
assumes *a* < *b poly* (*p* ∗ *q*) *a* ≠*0 poly* (*p* ∗ *q*) *b* ≠*0*
shows *cindex-poly a b q p* + *cindex-poly a b p q* = *cross* (*p* ∗ *q*) *a b* (**is** *?L*=*?R*)
⟨*proof*⟩

lemma *cindex-poly-rec*:
fixes *p q*::*real poly*
assumes *a* < *b poly* (*p* ∗ *q*) *a* ≠*0 poly* (*p* ∗ *q*) *b* ≠*0*
shows *cindex-poly a b q p* = *cross* (*p* ∗ *q*) *a b* + *cindex-poly a b* (− (*p mod q*)) *q* (**is** *?L*=*?R*)
⟨*proof*⟩

lemma *cindex-poly-congr*:
fixes p q:: *real poly*
assumes $a<a'$ $a'<b'$ $b'<b$
assumes $\forall x.\ ((a<x \wedge x \leq a') \vee (b' \leq x \wedge x<b)) \longrightarrow poly\ p\ x \neq 0$
shows *cindex-poly a b q p*=*cindex-poly a' b' q p*
⟨*proof*⟩

lemma *greaterThanLessThan-unfold*:$\{a<..<b\} = \{x.\ a<x \wedge x<b\}$
⟨*proof*⟩

lemma *cindex-poly-taq*:
fixes p q::*real poly*
shows *taq* $\{x.\ poly\ p\ x = 0 \wedge a < x \wedge x < b\}$ *q*=*cindex-poly a b* (*pderiv p* $*$ *q*) *p*
⟨*proof*⟩

2.9 Signed remainder sequence

function *smods*:: *real poly* ⇒ *real poly* ⇒ (*real poly*) *list* **where**
smods p q= (*if p*=*0 then* [] *else Cons p* (*smods q* (−(*p mod q*))))
⟨*proof*⟩
termination
⟨*proof*⟩

lemma *smods-nil-eq*:*smods p q* = [] ⟷ (*p*=*0*) ⟨*proof*⟩
lemma *smods-singleton*:[*x*] = *smods p q* ⟹ (*p*≠*0* ∧ *q*=*0* ∧ *x*=*p*)
⟨*proof*⟩

lemma *smods-0*[*simp*]:
smods 0 q = []
smods p 0 = (*if p*=*0 then* [] *else* [*p*])
⟨*proof*⟩

lemma *no-0-in-smods*: *0*∉*set* (*smods p q*)
⟨*proof*⟩

fun *changes*:: ('*a* ::*linordered-idom*) *list* ⇒ *int* **where**
changes [] = *0*|
changes [-] = *0* |
changes (*x1*#*x2*#*xs*) = (*if x1*∗*x2*<*0 then* 1+*changes* (*x2*#*xs*)
else if x2=*0 then changes* (*x1*#*xs*)
else changes (*x2*#*xs*))

lemma *changes-map-sgn-eq*:
changes xs = *changes* (*map sgn xs*)
⟨*proof*⟩

lemma *changes-map-sign-eq*:
changes xs = *changes* (*map sign xs*)

⟨*proof*⟩

lemma *changes-map-sign-of-int-eq*:
changes xs = changes (map ((of-int::-⇒'c::{ring-1,linordered-idom}) o sign) xs)
⟨*proof*⟩

definition *changes-poly-at*::('*a* ::*linordered-idom*) *poly list* ⇒ '*a* ⇒ *int* **where**
changes-poly-at ps a= changes (map (λp. poly p a) ps)

definition *changes-poly-pos-inf*:: ('*a* ::*linordered-idom*) *poly list* ⇒ *int* **where**
changes-poly-pos-inf ps = changes (map sgn-pos-inf ps)

definition *changes-poly-neg-inf*:: ('*a* ::*linordered-idom*) *poly list* ⇒ *int* **where**
changes-poly-neg-inf ps = changes (map sgn-neg-inf ps)

lemma *changes-poly-at-0*[*simp*]:
changes-poly-at [] a =0
changes-poly-at [p] a=0
⟨*proof*⟩

definition *changes-itv-smods*:: *real* ⇒ *real* ⇒*real poly* ⇒ *real poly* ⇒ *int* **where**
changes-itv-smods a b p q= (let ps= smods p q in changes-poly-at ps a − changes-poly-at ps b)

definition *changes-gt-smods*:: *real* ⇒*real poly* ⇒ *real poly* ⇒ *int* **where**
changes-gt-smods a p q= (let ps= smods p q in changes-poly-at ps a − changes-poly-pos-inf ps)

definition *changes-le-smods*:: *real* ⇒*real poly* ⇒ *real poly* ⇒ *int* **where**
changes-le-smods b p q= (let ps= smods p q in changes-poly-neg-inf ps − changes-poly-at ps b)

definition *changes-R-smods*:: *real poly* ⇒ *real poly* ⇒ *int* **where**
changes-R-smods p q= (let ps= smods p q in changes-poly-neg-inf ps − changes-poly-pos-inf ps)

lemma *changes-R-smods-0*[*simp*]:
changes-R-smods 0 q = 0
changes-R-smods p 0 = 0
⟨*proof*⟩

lemma *changes-itv-smods-0*[*simp*]:
changes-itv-smods a b 0 q = 0
changes-itv-smods a b p 0 = 0
⟨*proof*⟩

lemma *changes-itv-smods-rec*:
assumes *a<b poly (p*q) a≠0 poly (p*q) b≠0*
shows *changes-itv-smods a b p q = cross (p*q) a b + changes-itv-smods a b q*

$(-(p\ mod\ q))$
⟨*proof*⟩

lemma *changes-smods-congr*:
fixes $p\ q$:: *real poly*
assumes $a{\neq}a'$ *poly* $p\ a{\neq}0$
assumes $\forall\, p{\in}set\ (smods\ p\ q).\ \forall\, x.\ ((a{<}x{\wedge}x{\leq}a') \vee (a'{\leq}x \wedge x{<}a)) \longrightarrow poly\ p\ x \neq 0$
shows *changes-poly-at* $(smods\ p\ q)\ a$ = *changes-poly-at* $(smods\ p\ q)\ a'$
⟨*proof*⟩

lemma *changes-itv-smods-congr*:
fixes $p\ q$:: *real poly*
assumes $a{<}a'\ a'{<}b'\ b'{<}b$ *poly* $p\ a{\neq}0$ *poly* $p\ b{\neq}0$
assumes *no-root*:$\forall\, p{\in}set\ (smods\ p\ q).\ \forall\, x.\ ((a{<}x{\wedge}x{\leq}a') \vee (b'{\leq}x \wedge x{<}b)) \longrightarrow poly\ p\ x \neq 0$
shows *changes-itv-smods* $a\ b\ p\ q$=*changes-itv-smods* $a'\ b'\ p\ q$
⟨*proof*⟩

lemma *cindex-poly-changes-itv-mods*:
assumes $a{<}b$ *poly* $p\ a{\neq}0$ *poly* $p\ b{\neq}0$
shows *cindex-poly* $a\ b\ q\ p$ = *changes-itv-smods* $a\ b\ p\ q$ ⟨*proof*⟩

lemma *root-list-ub*:
fixes *ps*:: (*real poly*) *list* **and** *a*::*real*
assumes $0{\notin}set\ ps$
obtains *ub* **where** $\forall\, p{\in}set\ ps.\ \forall\, x.\ poly\ p\ x{=}0 \longrightarrow x{<}ub$
and $\forall\, x{\geq}ub.\ \forall\, p{\in}set\ ps.\ sgn\ (poly\ p\ x)$ = *sgn-pos-inf* p **and** $ub{>}a$
⟨*proof*⟩

lemma *root-list-lb*:
fixes *ps*:: (*real poly*) *list* **and** *b*::*real*
assumes $0{\notin}set\ ps$
obtains *lb* **where** $\forall\, p{\in}set\ ps.\ \forall\, x.\ poly\ p\ x{=}0 \longrightarrow x{>}lb$
and $\forall\, x{\leq}lb.\ \forall\, p{\in}set\ ps.\ sgn\ (poly\ p\ x)$ = *sgn-neg-inf* p **and** $lb{<}b$
⟨*proof*⟩

theorem *sturm-tarski-interval*:
assumes $a{<}b$ *poly* $p\ a{\neq}0$ *poly* $p\ b{\neq}0$
shows *taq* $\{x.\ poly\ p\ x{=}0 \wedge a{<}x \wedge x{<}b\}\ q$ = *changes-itv-smods* $a\ b\ p$ $(pderiv\ p * q)$
⟨*proof*⟩

theorem *sturm-tarski-above*:
assumes *poly* $p\ a{\neq}0$
shows *taq* $\{x.\ poly\ p\ x{=}0 \wedge a{<}x\}\ q$ = *changes-gt-smods* $a\ p$ $(pderiv\ p * q)$
⟨*proof*⟩

theorem *sturm-tarski-below*:

assumes *poly p b≠0*
shows *taq {x. poly p x=0 ∧ x<b} q = changes-le-smods b p (pderiv p * q)*
⟨*proof*⟩

theorem *sturm-tarski-R*:
shows *taq {x. poly p x=0} q = changes-R-smods p (pderiv p * q)*
⟨*proof*⟩

theorem *sturm-interval*:
assumes *a < b poly p a ≠ 0 poly p b ≠ 0*
shows *card {x. poly p x = 0 ∧ a < x ∧ x < b} = changes-itv-smods a b p (pderiv p)*
⟨*proof*⟩

theorem *sturm-above*:
assumes *poly p a ≠ 0*
shows *card {x. poly p x = 0 ∧ a < x} = changes-gt-smods a p (pderiv p)*
⟨*proof*⟩

theorem *sturm-below*:
assumes *poly p b ≠ 0*
shows *card {x. poly p x = 0 ∧ x < b} = changes-le-smods b p (pderiv p)*
⟨*proof*⟩

theorem *sturm-R*:
shows *card {x. poly p x=0} = changes-R-smods p (pderiv p)*
⟨*proof*⟩

end

3 An implementation for calculating pseudo remainder sequences

theory *Pseudo-Remainder-Sequence*
imports *Sturm-Tarski*
HOL−Computational-Algebra.Computational-Algebra

Polynomial-Interpolation.Ring-Hom-Poly
begin

3.1 Misc

function *spmods* :: *'a::idom poly ⇒ 'a poly ⇒ ('a poly) list* **where**
spmods p q= (if p=0 then [] else
let
m=(if even(degree p+1−degree q) then −1 else −lead-coeff q)
in

Cons p (spmods q (smult m (pseudo-mod p q))))
⟨*proof*⟩
termination
⟨*proof*⟩

declare *spmods.simps*[*simp del*]

lemma *spmods-0*[*simp*]:
spmods 0 q = []
spmods p 0 = (if p=0 then [] else [p])
⟨*proof*⟩

lemma *spmods-nil-eq*:*spmods p q = [] ⟷ (p=0)*
⟨*proof*⟩

lemma *changes-poly-at-alternative*:
changes-poly-at ps a= changes (map (λp. sign(poly p a)) ps)
changes-poly-at ps a= changes (map (λp. sgn(poly p a)) ps)
⟨*proof*⟩

lemma *smods-smult-length*:
assumes *a≠0 b≠0*
shows *length (smods p q) = length (smods (smult a p) (smult b q))* ⟨*proof*⟩

lemma *smods-smult-nth*[*rule-format*]:
fixes *p q*::*real poly*
assumes *a≠0 b≠0*
defines *xs≡smods p q* **and** *ys≡smods (smult a p) (smult b q)*
shows *∀ n<length xs. ys!n = (if even n then smult a (xs!n) else smult b (xs!n))*
⟨*proof*⟩

lemma *smods-smult-sgn-map-eq*:
fixes *x*::*real*
assumes *m>0*
defines *f≡λp. sgn(poly p x)*
shows *map f (smods p (smult m q)) = map f (smods p q)*
map sgn-pos-inf (smods p (smult m q)) = map sgn-pos-inf (smods p q)
map sgn-neg-inf (smods p (smult m q)) = map sgn-neg-inf (smods p q)
⟨*proof*⟩

lemma *changes-poly-at-smods-smult*:
assumes *m>0*
shows *changes-poly-at (smods p (smult m q)) x =changes-poly-at (smods p q) x*
⟨*proof*⟩

lemma *spmods-smods-sgn-map-eq*:
fixes *p q*::*real poly* **and** *x*::*real*
defines *f≡λp. sgn (poly p x)*
shows *map f (smods p q) = map f (spmods p q)*

map sgn-pos-inf (smods p q) = map sgn-pos-inf (spmods p q)
map sgn-neg-inf (smods p q) = map sgn-neg-inf (spmods p q)
⟨*proof*⟩

3.2 Converting *rat poly* to *int poly* by clearing the denominators

definition *int-of-rat*::*rat* ⇒ *int* **where**
int-of-rat = inv of-int

lemma *of-rat-inj*[*simp*]: *inj of-rat*
⟨*proof*⟩

lemma (**in** *ring-char-0*) *of-int-inj*[*simp*]: *inj of-int*
⟨*proof*⟩

lemma *int-of-rat-id*: *int-of-rat o of-int = id*
⟨*proof*⟩

lemma *int-of-rat-0*[*simp*]:*int-of-rat 0 = 0*
⟨*proof*⟩

lemma *int-of-rat-inv*:*r*∈ℤ ⟹ *of-int* (*int-of-rat r*) = *r*
⟨*proof*⟩

lemma *int-of-rat-0-iff*:*x*∈ℤ ⟹ *int-of-rat x = 0* ⟷ *x = 0*
⟨*proof*⟩

lemma [*code*]:*int-of-rat r* = (*let* (*a*,*b*) = *quotient-of r in*
if b=1 then a else Code.abort (*STR ''Failed to convert rat to int''*)
(λ-. *int-of-rat r*))
⟨*proof*⟩

definition *de-lcm*::*rat poly* ⇒ *int* **where**
de-lcm p = *Lcm*(*set*(*map* (λ*x*. *snd* (*quotient-of x*)) (*coeffs p*)))

lemma *de-lcm-pCons*:*de-lcm* (*pCons a p*) = *lcm* (*snd* (*quotient-of a*)) (*de-lcm p*)
⟨*proof*⟩

lemma *de-lcm-0*[*simp*]:*de-lcm 0 = 1* ⟨*proof*⟩

lemma *de-lcm-pos*[*simp*]:*de-lcm p* > *0*
⟨*proof*⟩

lemma *de-lcm-ints*:
fixes *x*::*rat*
shows *x*∈*set* (*coeffs p*) ⟹ *rat-of-int* (*de-lcm p*) ∗ *x* ∈ ℤ
⟨*proof*⟩

definition *clear-de*::*rat poly* ⇒ *int poly* **where**
clear-de p = (*SOME q*. (*map-poly of-int q*) = *smult* (*of-int* (*de-lcm p*)) *p*)

lemma *clear-de*:*of-int-poly*(*clear-de p*) = *smult* (*of-int* (*de-lcm p*)) *p*
⟨*proof*⟩

lemma *clear-de-0*[*simp*]:*clear-de 0* = *0*
⟨*proof*⟩

lemma [*code abstract*]: *coeffs* (*clear-de p*) =
(*let lcm* = *de-lcm p in map* (λ*x*. *int-of-rat* (*of-int lcm* ∗ *x*)) (*coeffs p*))
⟨*proof*⟩

3.3 Sign variations for pseudo-remainder sequences

locale *order-hom* =
fixes *hom* :: *'a* :: *ord* ⇒ *'b* :: *ord*
assumes *hom-less*: *x* < *y* ⟷ *hom x* < *hom y*
and *hom-less-eq*: *x* ≤ *y* ⟷ *hom x* ≤ *hom y*

locale *linordered-idom-hom* = *order-hom hom* + *inj-idom-hom hom*
for *hom* :: *'a* :: *linordered-idom* ⇒ *'b* :: *linordered-idom*
begin

lemma *sgn-sign*:*sgn* (*hom x*) = *of-int* (*sign x*)
⟨*proof*⟩

end

locale *hom-pseudo-smods*= *comm-semiring-hom hom*
+ *r1*:*linordered-idom-hom* R_1 + *r2*:*linordered-idom-hom* R_2
for *hom*::*'a*::*linordered-idom* ⇒ *'b*::{*comm-semiring-1*,*linordered-idom*}
and R_1::*'a* ⇒ *real*
and R_2::*'b* ⇒ *real* +
assumes *R-hom*:R_1 *x* = R_2 (*hom x*)
begin

lemma *map-poly-R-hom-commute*:
poly (*map-poly* R_1 *p*) (R_2 *x*) = R_2 (*poly* (*map-poly hom p*) *x*)
⟨*proof*⟩

definition *changes-hpoly-at*::*'a poly list* ⇒ *'b* ⇒ *int* **where**
changes-hpoly-at ps a= *changes* (*map* (λ*p*. *eval-poly hom p a*) *ps*)

lemma *changes-hpoly-at-Nil*[*simp*]: *changes-hpoly-at* [] *a* = *0*
⟨*proof*⟩

definition *changes-itv-spmods*:: *'b ⇒ 'b ⇒ 'a poly ⇒ 'a poly ⇒ int* **where**
changes-itv-spmods a b p q= (*let ps* = *spmods p q in*
changes-hpoly-at ps a − *changes-hpoly-at ps b*)

definition *changes-gt-spmods*:: *'b ⇒ 'a poly ⇒ 'a poly ⇒ int* **where**
changes-gt-spmods a p q= (*let ps* = *spmods p q in*
changes-hpoly-at ps a − *changes-poly-pos-inf ps*)

definition *changes-le-spmods*:: *'b ⇒ 'a poly ⇒ 'a poly ⇒ int* **where**
changes-le-spmods b p q= (*let ps*= *spmods p q in*
changes-poly-neg-inf ps − *changes-hpoly-at ps b*)

definition *changes-R-spmods*:: *'a poly ⇒ 'a poly ⇒ int* **where**
changes-R-spmods p q= (*let ps*= *spmods p q in changes-poly-neg-inf ps*
− *changes-poly-pos-inf ps*)

lemma *changes-spmods-smods*:
shows *changes-itv-spmods a b p q*
= *changes-itv-smods* (R_2 *a*) (R_2 *b*) (*map-poly* R_1 *p*) (*map-poly* R_1 *q*)
and *changes-R-spmods p q* = *changes-R-smods* (*map-poly* R_1 *p*) (*map-poly* R_1 *q*)
and *changes-gt-spmods a p q* = *changes-gt-smods* (R_2 *a*) (*map-poly* R_1 *p*) (*map-poly* R_1 *q*)
and *changes-le-spmods b p q* = *changes-le-smods* (R_2 *b*) (*map-poly* R_1 *p*) (*map-poly* R_1 *q*)
⟨*proof*⟩

end

end

4 TaQ for polynomials with rational coefficients

theory *Tarski-Query-Impl* **imports**
Pseudo-Remainder-Sequence Sturm-Tarski
begin

global-interpretation *rat-int*:*hom-pseudo-smods rat-of-int real-of-int real-of-rat*
defines
ri-changes-itv-spmods = *rat-int.changes-itv-spmods* **and**
ri-changes-gt-spmods = *rat-int.changes-gt-spmods* **and**
ri-changes-le-spmods = *rat-int.changes-le-spmods* **and**
ri-changes-R-spmods = *rat-int.changes-R-spmods*
⟨*proof*⟩

definition *TaQ-R-rats*::*rat poly ⇒ rat poly ⇒ int* **where**
TaQ-R-rats p q = *taq* {*x*. *poly* (*map-poly real-of-rat p*) *x* = (*0*::*real*)}
(*map-poly real-of-rat q*)

definition *TaQ-itv-rats*::*rat ⇒ rat ⇒ rat poly ⇒ rat poly ⇒ int* **where**

```
  TaQ-itv-rats a b p q = taq {x. poly (map-poly real-of-rat p) x = (0::real)
      ∧ of-rat a < x ∧ x < of-rat b} (map-poly real-of-rat q)

definition TaQ-gt-rats:: rat ⇒ rat poly ⇒ rat poly ⇒ int where
  TaQ-gt-rats a p q = taq {x. poly (map-poly real-of-rat p) x = (0::real)
      ∧ of-rat a < x } (map-poly real-of-rat q)

definition TaQ-le-rats::rat ⇒ rat poly ⇒ rat poly ⇒ int where
  TaQ-le-rats b p q = taq {x. poly (map-poly real-of-rat p) x = (0::real)
      ∧ x < of-rat b} (map-poly real-of-rat q)

lemma taq-smult-pos:
  assumes a>0
  shows taq s (smult a p) = taq s p
  ⟨proof⟩

lemma taq-proots-R-code[code]:
  TaQ-R-rats p q = (let
    ip = clear-de p;
    iq = clear-de q
    in ri-changes-R-spmods ip (pderiv ip * iq))
⟨proof⟩

lemma taq-proots-itv-code[code]:
  TaQ-itv-rats a b p q = (if a≥b then
      0
    else if poly p a ≠ 0 ∧ poly p b ≠0 then
      (let
      ip = clear-de p;
      iq = clear-de q
      in ri-changes-itv-spmods a b ip (pderiv ip * iq))
    else
       Code.abort (STR ''Roots at border yet to be supported'')
                          (λ-. TaQ-itv-rats a b p q)
    )
⟨proof⟩

lemma taq-proots-gt-code[code]:
  TaQ-gt-rats a p q = (
    if poly p a ≠ 0 then
      (let
      ip = clear-de p;
      iq = clear-de q
      in ri-changes-gt-spmods a ip (pderiv ip * iq))
    else
       Code.abort (STR ''Roots at border yet to be supported'')
                          (λ-. TaQ-gt-rats a p q)
    )
⟨proof⟩
```

```
lemma taq-proots-le-code[code]:
  TaQ-le-rats b p q = (
    if poly p b ≠0 then
      (let
      ip = clear-de p;
      iq = clear-de q
      in ri-changes-le-spmods b ip (pderiv ip ∗ iq))
    else
       Code.abort (STR ''Roots at border yet to be supported'')
                          (λ-. TaQ-le-rats b p q)
    )
⟨proof⟩

end
```

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