Stirling's formula

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Abstract

This work contains a proof of Stirling's formula both for the factorial $n! \sim \sqrt{2\pi n}(n/e)^n$ on natural numbers and the real Gamma function $\Gamma(x) \sim \sqrt{2\pi/x}(x/e)^x$. The proof is based on work by Graham Jameson [3].

This is then extended to the full asymptotic expansion

$$\log \Gamma(z) = \left(z - \tfrac{1}{2}\right) \log z - z + \tfrac{1}{2} \log(2\pi) + \sum_{k=1}^{n-1} \frac{B_{k+1}}{k(k+1)} z^{-k} - \frac{1}{n} \int_0^\infty B_n([t])(t+z)^{-n}\, \mathrm{d}t$$

uniformly for all complex $z \neq 0$ in the cone $\arg(z) \leq \alpha$ for any $\alpha \in (0, \pi)$, with which the above asymptotic relation for Γ is also extended to complex arguments.

Contents

1 Stirling's Formula

theory *Stirling-Formula*
imports
HOL−Analysis.Analysis
Landau-Symbols.Landau-More
HOL−Real-Asymp.Real-Asymp
begin

context
begin

First, we define the S^*_n from Jameson's article:

qualified definition S' :: *nat* ⇒ *real* ⇒ *real* **where**
  $S'\ n\ x = 1/(2{*}x) + (\sum r{=}1..{<}n.\ 1/(\mathit{of\text{-}nat}\ r{+}x)) + 1\ /(2{*}(n + x))$

Next, the trapezium (also called T in Jameson's article):

qualified definition T :: *real* ⇒ *real* **where**
  $T\ x = 1/(2{*}x) + 1/(2{*}(x{+}1))$

Now we define The difference $\Delta(x)$:

qualified definition D :: *real* ⇒ *real* **where**
  $D\ x = T\ x - \mathit{ln}\ (x + 1) + \mathit{ln}\ x$

qualified lemma *S′-telescope-trapezium*:
  assumes $n > 0$
  shows $S'\ n\ x = (\sum r{<}n.\ T\ (\mathit{of\text{-}nat}\ r + x))$
⟨*proof*⟩ **lemma** *stirling-trapezium*:
  assumes *x*: $(x{::}\mathit{real}) > 0$
  shows $D\ x \in \{0\ ..\ 1/(12{*}x\hat{\ }2) - 1/(12 * (x{+}1)\hat{\ }2)\}$
⟨*proof*⟩

The following functions correspond to the $p_n(x)$ from the article. The special case $n = 0$ would not, strictly speaking, be necessary, but it allows some theorems to work even without the precondition $n \neq 0$:

qualified definition p :: *nat* ⇒ *real* ⇒ *real* **where**
  $p\ n\ x = (\mathit{if}\ n = 0\ \mathit{then}\ 1/x\ \mathit{else}\ (\sum r{<}n.\ D\ (\mathit{real}\ r + x)))$

We can write the Digamma function in terms of S':

qualified lemma *S′-LIMSEQ-Digamma*:
  assumes *x*: $x \neq 0$
  shows $(\lambda n.\ \mathit{ln}\ (\mathit{real}\ n) - S'\ n\ x - 1/(2{*}x)) \longrightarrow \mathit{Digamma}\ x$
⟨*proof*⟩

Moreover, we can give an expansion of S' with the p as variation terms.

qualified lemma *S′-approx*:
  $S'\ n\ x = \mathit{ln}\ (\mathit{real}\ n + x) - \mathit{ln}\ x + p\ n\ x$
⟨*proof*⟩

We define the limit of the p (simply called $p(x)$ in Jameson's article):

qualified definition P :: *real* ⇒ *real* **where**
  $P\ x = (\sum n.\ D\ (\mathit{real}\ n + x))$

qualified lemma *D-summable*:
  assumes *x*: $x > 0$

shows *summable* ($\lambda n.$ *D* (*real* $n + x$))
⟨*proof*⟩ **lemma** *p-LIMSEQ*:
assumes x: $x > 0$
shows ($\lambda n.$ $p\ n\ x$) $\longrightarrow$ $P\ x$
⟨*proof*⟩

This gives us an expansion of the Digamma function:

lemma *Digamma-approx*:
assumes x: (x :: *real*) > 0
shows *Digamma* $x = \ln x - 1 / (2 * x) - P\ x$
⟨*proof*⟩

Next, we derive some bounds on P:

qualified lemma *p-ge-0*: $x > 0 \Longrightarrow p\ n\ x \geq 0$
⟨*proof*⟩ **lemma** *P-ge-0*: $x > 0 \Longrightarrow P\ x \geq 0$
⟨*proof*⟩ **lemma** *p-upper-bound*:
assumes $x > 0$ $n > 0$
shows $p\ n\ x \leq 1/(12*x\hat{}2)$
⟨*proof*⟩ **lemma** *P-upper-bound*:
assumes $x > 0$
shows $P\ x \leq 1/(12*x\hat{}2)$
⟨*proof*⟩

We can now use this approximation of the Digamma function to approximate *ln-Gamma* (since the former is the derivative of the latter). We therefore define the function g from Jameson's article, which measures the difference between *ln-Gamma* and its approximation:

qualified definition g :: *real* $\Rightarrow$ *real* **where**
$g\ x$ = *ln-Gamma* $x - (x - 1/2) * \ln x + x$

qualified lemma *DERIV-g*: $x > 0 \Longrightarrow$ (g *has-field-derivative* $-P\ x$) (*at* x)
⟨*proof*⟩ **lemma** *isCont-P*:
assumes $x > 0$
shows *isCont* $P\ x$
⟨*proof*⟩ **lemma** *P-continuous-on* [*THEN continuous-on-subset*]: *continuous-on* {*0*<..} P
⟨*proof*⟩ **lemma** *P-integrable*:
assumes a: $a > 0$
shows P *integrable-on* {a..}
⟨*proof*⟩ **definition** c :: *real* **where** c = *integral* {*1*..} ($\lambda x.\ -P\ x$) + $g\ 1$

We can now give bounds on g:

qualified lemma *g-bounds*:
assumes x: $x \geq 1$
shows $g\ x \in \{c..c + 1/(12*x)\}$
⟨*proof*⟩

Finally, we have bounds on *ln-Gamma*, *Gamma*, and *fact*.

qualified lemma *ln-Gamma-bounds-aux*:
  $x \geq 1 \Longrightarrow$ *ln-Gamma* $x \geq c + (x - 1/2) * \ln x - x$
  $x \geq 1 \Longrightarrow$ *ln-Gamma* $x \leq c + (x - 1/2) * \ln x - x + 1/(12*x)$
  ⟨*proof*⟩ **lemma** *Gamma-bounds-aux*:
  assumes *x*: $x \geq 1$
  shows *Gamma* $x \geq \exp c * x$ *powr* $(x - 1/2) / \exp x$
      Gamma $x \leq \exp c * x$ *powr* $(x - 1/2) / \exp x * \exp (1/(12*x))$
⟨*proof*⟩ **lemma** *Gamma-asymp-equiv-aux*:
  Gamma ∼[*at-top*] ($\lambda x.$ *exp c* $*$ *x powr* $(x - 1/2)$ / *exp x*)
⟨*proof*⟩
  include *asymp-equiv-syntax*
  ⟨*proof*⟩ **lemma** *exp-1-powr-real* [*simp*]: *exp* (*1*::*real*) *powr x* = *exp x*
  ⟨*proof*⟩ **lemma** *fact-asymp-equiv-aux*:
  fact ∼[*at-top*] ($\lambda x.$ *exp c* $*$ *sqrt* (*real x*) $*$ (*real x* / *exp 1*) *powr real x*)
⟨*proof*⟩
  include *asymp-equiv-syntax*
  ⟨*proof*⟩

We cal also bound *Digamma* above and below.

lemma *Digamma-plus-1-gt-ln*:
  assumes *x*: $x > (0 :: real)$
  shows *Digamma* $(x + 1) > \ln x$
⟨*proof*⟩

lemma *Digamma-less-ln*:
  assumes *x*: $x > (0 :: real)$
  shows *Digamma* $x < \ln x$
⟨*proof*⟩

We still need to determine the constant term c, which we do using Wallis' product formula:

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{\pi}{2}$$

qualified lemma *powr-mult-2*: (*x*::*real*) > *0* $\Longrightarrow$ *x powr* (*y* $*$ *2*) = (*x*^*2*) *powr y*
  ⟨*proof*⟩ **lemma** *exp-mult-2*: *exp* (*y* $*$ *2* :: *real*) = *exp y* $*$ *exp y*
  ⟨*proof*⟩ **lemma** *exp-c*: *exp c* = *sqrt* (*2*$*$*pi*)
⟨*proof*⟩
  include *asymp-equiv-syntax*
  ⟨*proof*⟩ **lemma** *c*: *c* = *ln* (*2*$*$*pi*) / *2*
⟨*proof*⟩

This gives us the final bounds:

theorem *Gamma-bounds*:
  assumes $x \geq 1$
  shows *Gamma* $x \geq$ *sqrt* (*2*$*$*pi*/*x*) $*$ (*x* / *exp 1*) *powr x* (**is** *?th1*)
      Gamma $x \leq$ *sqrt* (*2*$*$*pi*/*x*) $*$ (*x* / *exp 1*) *powr x* $*$ *exp* (*1* / (*12* $*$ *x*)) (**is** *?th2*)

⟨*proof*⟩

theorem *ln-Gamma-bounds*:
assumes $x \geq 1$
shows $ln\text{-}Gamma\ x \geq ln\ (2{*}pi/x)\ /\ 2 + x * ln\ x - x$ (**is** *?th1*)
$ln\text{-}Gamma\ x \leq ln\ (2{*}pi/x)\ /\ 2 + x * ln\ x - x + 1/(12{*}x)$ (**is** *?th2*)
⟨*proof*⟩

theorem *fact-bounds*:
assumes $n > 0$
shows $(fact\ n :: real) \geq sqrt\ (2{*}pi{*}n) * (n\ /\ exp\ 1)\ \hat{}\ n$ (**is** *?th1*)
$(fact\ n :: real) \leq sqrt\ (2{*}pi{*}n) * (n\ /\ exp\ 1)\ \hat{}\ n * exp\ (1\ /\ (12 * n))$ (**is** *?th2*)
⟨*proof*⟩

theorem *ln-fact-bounds*:
assumes $n > 0$
shows $ln\ (fact\ n :: real) \geq ln\ (2{*}pi{*}n)/2 + n * ln\ n - n$ (**is** *?th1*)
$ln\ (fact\ n :: real) \leq ln\ (2{*}pi{*}n)/2 + n * ln\ n - n + 1/(12{*}real\ n)$ (**is** *?th2*)
⟨*proof*⟩

theorem *Gamma-asymp-equiv*:
$Gamma \sim[at\text{-}top]\ (\lambda x.\ sqrt\ (2{*}pi/x) * (x\ /\ exp\ 1)\ powr\ x :: real)$
⟨*proof*⟩

theorem *fact-asymp-equiv*:
$fact \sim[at\text{-}top]\ (\lambda n.\ sqrt\ (2{*}pi{*}n) * (n\ /\ exp\ 1)\ \hat{}\ n :: real)$
⟨*proof*⟩

corollary *stirling-tendsto-sqrt-pi*:
$(\lambda n.\ fact\ n\ /\ (sqrt\ (2 * n) * (n\ /\ exp\ 1)\ \hat{}\ n)) \longrightarrow sqrt\ pi$
⟨*proof*⟩

end

end

2 Complete asymptotics of the logarithmic Gamma function

theory *Gamma-Asymptotics*
imports
HOL−Complex-Analysis.Complex-Analysis
Bernoulli.Bernoulli-FPS
Bernoulli.Periodic-Bernpoly
Stirling-Formula
begin

2.1 Auxiliary Facts

lemma *stirling-limit-aux1*:
$((\lambda y.\ Ln\ (1 + z * of\text{-}real\ y)\ /\ of\text{-}real\ y) \longrightarrow z)\ (at\text{-}right\ 0)$ **for** $z :: complex$
⟨*proof*⟩

lemma *stirling-limit-aux2*:
$((\lambda y.\ y * Ln\ (1 + z\ /\ of\text{-}real\ y)) \longrightarrow z)\ at\text{-}top$ **for** $z :: complex$
⟨*proof*⟩

lemma *Union-atLeastAtMost*:
assumes $N > 0$
shows $(\bigcup n\in\{0..<N\}.\ \{real\ n..real\ (n + 1)\}) = \{0..real\ N\}$
⟨*proof*⟩

2.2 Cones in the complex plane

definition $complex\text{-}cone :: real \Rightarrow real \Rightarrow complex\ set$ **where**
$complex\text{-}cone\ a\ b = \{z.\ \exists y\in\{a..b\}.\ z = rcis\ (norm\ z)\ y\}$

abbreviation $complex\text{-}cone' :: real \Rightarrow complex\ set$ **where**
$complex\text{-}cone'\ a \equiv complex\text{-}cone\ (-a)\ a$

lemma *zero-in-complex-cone* [*simp*, *intro*]: $a \le b \Longrightarrow 0 \in complex\text{-}cone\ a\ b$
⟨*proof*⟩

lemma *complex-coneE*:
assumes $z \in complex\text{-}cone\ a\ b$
obtains $r\ \alpha$ **where** $r \ge 0\ \ \alpha \in \{a..b\}\ \ z = rcis\ r\ \alpha$
⟨*proof*⟩

lemma *arg-cis* [*simp*]:
assumes $x \in \{-pi<..pi\}$
shows $Arg\ (cis\ x) = x$
⟨*proof*⟩

lemma *arg-mult-of-real-left* [*simp*]:
assumes $r > 0$
shows $Arg\ (of\text{-}real\ r * z) = Arg\ z$
⟨*proof*⟩

lemma *arg-mult-of-real-right* [*simp*]:
assumes $r > 0$
shows $Arg\ (z * of\text{-}real\ r) = Arg\ z$
⟨*proof*⟩

lemma *arg-rcis* [*simp*]:
assumes $x \in \{-pi<..pi\}\ \ r > 0$
shows $Arg\ (rcis\ r\ x) = x$
⟨*proof*⟩

lemma *rcis-in-complex-cone* [*intro*]:
  assumes $\alpha \in \{a..b\}$ $r \geq 0$
  shows *rcis r* $\alpha \in$ *complex-cone a b*
  ⟨*proof*⟩

lemma *arg-imp-in-complex-cone*:
  assumes *Arg z* $\in \{a..b\}$
  shows $z \in$ *complex-cone a b*
⟨*proof*⟩

lemma *complex-cone-altdef*:
  assumes $-pi < a$ $a \leq b$ $b \leq pi$
  shows *complex-cone a b* = *insert 0* $\{z.$ *Arg z* $\in \{a..b\}\}$
⟨*proof*⟩

lemma *nonneg-of-real-in-complex-cone* [*simp*, *intro*]:
  assumes $x \geq 0$ $a \leq 0$ $0 \leq b$
  shows *of-real x* $\in$ *complex-cone a b*
⟨*proof*⟩

lemma *one-in-complex-cone* [*simp*, *intro*]: $a \leq 0 \Longrightarrow 0 \leq b \Longrightarrow 1 \in$ *complex-cone a b*
  ⟨*proof*⟩

lemma *of-nat-in-complex-cone* [*simp*, *intro*]: $a \leq 0 \Longrightarrow 0 \leq b \Longrightarrow$ *of-nat n* $\in$ *complex-cone a b*
  ⟨*proof*⟩

2.3 Another integral representation of the Beta function

lemma *complex-cone-inter-nonpos-Reals*:
  assumes $-pi < a$ $a \leq b$ $b < pi$
  shows *complex-cone a b* $\cap \mathbb{R}_{\leq 0} = \{0\}$
⟨*proof*⟩

theorem
  assumes *a*: $a > 0$ **and** *b*: $b > (0 :: real)$
  shows *has-integral-Beta-real'*:
    (($\lambda u.$ *u powr* $(b - 1)$ / $(1 + u)$ *powr* $(a + b)$) *has-integral Beta a b*) $\{0<..\}$
  and *Beta-conv-nn-integral*:
    Beta a b = ($\int^+ u.$ *ennreal* (*indicator* $\{0<..\}$ $u * u$ *powr* $(b - 1)$ / $(1 + u)$ *powr* $(a + b)$) ∂*lborel*)
⟨*proof*⟩

lemma *has-integral-Beta2*:
  fixes *a* :: *real*
  assumes $a < -1/2$
  shows (($\lambda x.$ $(1 + x$ ^ $2)$ *powr a*) *has-integral Beta* $(- a - 1 / 2)$ $(1 / 2)$ /

2) {*0*<..}
⟨*proof*⟩

lemma *has-integral-Beta3*:
fixes *a b* :: *real*
assumes *a* < −*1*/*2* **and** *b* > *0*
shows ((λ*x*. (*b* + *x* ^ *2*) *powr a*) *has-integral*
Beta (−*a* − *1*/*2*) (*1*/*2*) / *2* ∗ *b powr* (*a* + *1*/*2*)) {*0*<..}
⟨*proof*⟩

2.4 Asymptotics of the real $\log\Gamma$ function and its derivatives

This is the error term that occurs in the expansion of *ln-Gamma*. It can be shown to be of order $O(s^{-n})$.

definition *stirling-integral* :: *nat* ⇒ *'a* :: {*real-normed-div-algebra*, *banach*} ⇒ *'a*
where
stirling-integral n s =
lim (λ*N*. *integral* {*0*..*N*} (λ*x*. *of-real* (*pbernpoly n x*) / (*of-real x* + *s*) ^ *n*))

context
fixes *s* :: *complex* **assumes** *s*: $s \notin \mathbb{R}_{\leq 0}$
fixes *approx* :: *nat* ⇒ *complex*
defines *approx* ≡ (λ*N*.
(∑*n* = *1*..<*N*. *s* / *of-nat n* − *ln* (*1* + *s* / *of-nat n*)) − (*euler-mascheroni* ∗ *s* + *ln s*) − — ⟶ *ln-Gamma s*
(*ln-Gamma* (*of-nat N*) − *ln* (*2* ∗ *pi* / *of-nat N*) / *2* − *of-nat N* ∗ *ln* (*of-nat N*) + *of-nat N*) − — ⟶ *0*
s ∗ (*harm* (*N* − *1*) − *ln* (*of-nat* (*N* − *1*)) − *euler-mascheroni*) + — ⟶ *0*
s ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat* (*N* − *1*))) − — ⟶ *0*
(*1*/*2*) ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat N*)) + — ⟶ *0*
of-nat N ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat N*)) − — ⟶ *s*
(*s* − *1*/*2*) ∗ *ln s* − *ln* (*2* ∗ *pi*) / *2*)
begin

qualified lemma
assumes *N*: *N* > *0*
shows *integrable-pbernpoly-1*:
(λ*x*. *of-real* (−*pbernpoly 1 x*) / (*of-real x* + *s*)) *integrable-on* {*0*..*real N*}
and *integral-pbernpoly-1-aux*:
integral {*0*..*real N*} (λ*x*. −*of-real* (*pbernpoly 1 x*) / (*of-real x* + *s*)) = *approx N*
and *has-integral-pbernpoly-1*:
((λ*x*. *pbernpoly 1 x* /(*x* + *s*)) *has-integral*
(∑*m*<*N*. (*of-nat m* + *1* / *2* + *s*) ∗ (*ln* (*of-nat m* + *s*) −
ln (*of-nat m* + *1* + *s*)) + *1*)) {*0*..*real N*}
⟨*proof*⟩

lemma *integrable-ln-Gamma-aux*:
shows (λ*x*. *of-real* (*pbernpoly n x*) / (*of-real x* + *s*) ^ *n*) *integrable-on* {*0*..*real*

N}
⟨*proof*⟩

This following proof is based on "Rudiments of the theory of the gamma function" by Bruce Berndt [1].

lemma *tendsto-of-real-0-I*:
(*f* ⟶ *0*) *G* ⟹ ((λ*x*. (*of-real* (*f x*))) ⟶ (*0* ::′*a*::*real-normed-div-algebra*)) *G*
⟨*proof*⟩ **lemma** *integral-pbernpoly-1*:
(λ*N*. *integral* {*0*..*real N*} (λ*x*. *pbernpoly 1 x* / (*x* + *s*)))
⟶ −*ln-Gamma s* − *s* + (*s* − *1* / *2*) ∗ *ln s* + *ln* (*2* ∗ *pi*) / *2*
⟨*proof*⟩ **lemma** *pbernpoly-integral-conv-pbernpoly-integral-Suc*:
assumes *n* ≥ *1*
shows *integral* {*0*..*real N*} (λ*x*. *pbernpoly n x* / (*x* + *s*) ^ *n*) =
of-real (*pbernpoly* (*Suc n*) (*real N*)) / (*of-nat* (*Suc n*) ∗ (*s* + *of-nat N*) ^ *n*) −
of-real (*bernoulli* (*Suc n*)) / (*of-nat* (*Suc n*) ∗ *s* ^ *n*) + *of-nat n* / *of-nat* (*Suc n*) ∗
integral {*0*..*real N*} (λ*x*. *of-real* (*pbernpoly* (*Suc n*) *x*) / (*of-real x* + *s*) ^ *Suc n*)
⟨*proof*⟩

lemma *pbernpoly-over-power-tendsto-0*:
assumes *n* > *0*
shows (λ*x*. *of-real* (*pbernpoly* (*Suc n*) (*real x*)) / (*of-nat* (*Suc n*) ∗ (*s* + *of-nat x*) ^ *n*)) ⟶ *0*
⟨*proof*⟩

lemma *convergent-stirling-integral*:
assumes *n* > *0*
shows *convergent* (λ*N*. *integral* {*0*..*real N*}
(λ*x*. *of-real* (*pbernpoly n x*) / (*of-real x* + *s*) ^ *n*)) (**is** *convergent* (*?f n*))
⟨*proof*⟩

lemma *stirling-integral-conv-stirling-integral-Suc*:
assumes *n* > *0*
shows *stirling-integral n s* =
of-nat n / *of-nat* (*Suc n*) ∗ *stirling-integral* (*Suc n*) *s* −
of-real (*bernoulli* (*Suc n*)) / (*of-nat* (*Suc n*) ∗ *s* ^ *n*)
⟨*proof*⟩

lemma *stirling-integral-1-unfold*:
assumes *m* > *0*
shows *stirling-integral 1 s* = *stirling-integral m s* / *of-nat m* −
(∑ *k*=*1*..<*m*. *of-real* (*bernoulli* (*Suc k*)) / (*of-nat k* ∗ *of-nat* (*Suc k*) ∗ *s* ^ *k*))
⟨*proof*⟩

lemma *ln-Gamma-stirling-complex*:

assumes *m > 0*
shows *ln-Gamma s = (s − 1 / 2) ∗ ln s − s + ln (2 ∗ pi) / 2 +*
(∑ k=1..<m. of-real (bernoulli (Suc k)) / (of-nat k ∗ of-nat (Suc k) ∗ s ^ k)) −
stirling-integral m s / of-nat m
⟨*proof*⟩

lemma *LIMSEQ-stirling-integral*:
n > 0 =⇒ (λx. integral {0..real x} (λx. of-real (pbernpoly n x) / (of-real x + s) ^ n))
−−−−→ stirling-integral n s ⟨*proof*⟩

end

lemmas *has-integral-of-real = has-integral-linear[OF - bounded-linear-of-real, unfolded o-def]*
lemmas *integral-of-real = integral-linear[OF - bounded-linear-of-real, unfolded o-def]*

lemma *integrable-ln-Gamma-aux-real*:
assumes *0 < s*
shows *(λx. pbernpoly n x / (x + s) ^ n) integrable-on {0..real N}*
⟨*proof*⟩

lemma
assumes *x > 0 n > 0*
shows *stirling-integral-complex-of-real*:
stirling-integral n (complex-of-real x) = of-real (stirling-integral n x)
and *LIMSEQ-stirling-integral-real*:
(λN. integral {0..real N} (λt. pbernpoly n t / (t + x) ^ n))
−−−−→ stirling-integral n x
and *stirling-integral-real-convergent*:
convergent (λN. integral {0..real N} (λt. pbernpoly n t / (t + x) ^ n))
⟨*proof*⟩

lemma *ln-Gamma-stirling-real*:
assumes *x > (0 :: real) m > (0::nat)*
shows *ln-Gamma x = (x − 1 / 2) ∗ ln x − x + ln (2 ∗ pi) / 2 +*
(∑ k = 1..<m. bernoulli (Suc k) / (of-nat k ∗ of-nat (Suc k) ∗ x ^ k)) −
stirling-integral m x / of-nat m
⟨*proof*⟩

lemma *stirling-integral-bound-aux*:
assumes *n*: *n > (1::nat)*
obtains *c* **where** *⋀s. Re s > 0 =⇒ norm (stirling-integral n s) ≤ c / Re s ^ (n − 1)*
⟨*proof*⟩

lemma *stirling-integral-bound-aux-integral1*:
fixes $a\ b\ c :: real$ **and** $n :: nat$
assumes $a \geq 0\ b > 0\ c \geq 0\ n > 1\ l < a - b\ r > a + b$
shows $((\lambda x.\ c\ /\ max\ b\ |x - a|\ \hat{}\ n)$ *has-integral*
$2*c*(n\ /\ (n - 1))/b\hat{}(n-1) - c/(n-1) * (1/(a-l)\hat{}(n-1) + 1/(r-a)\hat{}(n-1)))$
$\{l..r\}$
⟨*proof*⟩

lemma *stirling-integral-bound-aux-integral2*:
fixes $a\ b\ c :: real$ **and** $n :: nat$
assumes $a \geq 0\ b > 0\ c \geq 0\ n > 1$
obtains I **where** $((\lambda x.\ c\ /\ max\ b\ |x - a|\ \hat{}\ n)$ *has-integral* $I)\ \{l..r\}$
$I \leq 2 * c * (n\ /\ (n - 1))\ /\ b\ \hat{}(n-1)$
⟨*proof*⟩

lemma *stirling-integral-bound-aux'*:
assumes n: $n > (1::nat)$ **and** α: $\alpha \in \{0<..<pi\}$
obtains c **where** $\bigwedge s::complex.\ s \in complex\text{-}cone'\ \alpha - \{0\} \Longrightarrow$
$norm\ (stirling\text{-}integral\ n\ s) \leq c\ /\ norm\ s\ \hat{}\ (n - 1)$
⟨*proof*⟩

lemma *stirling-integral-bound*:
assumes $n > 0$
obtains c **where**
$\bigwedge s.\ Re\ s > 0 \Longrightarrow norm\ (stirling\text{-}integral\ n\ s) \leq c\ /\ Re\ s\ \hat{}\ n$
⟨*proof*⟩

lemma *stirling-integral-bound'*:
assumes $n > 0$ **and** $\alpha \in \{0<..<pi\}$
obtains c **where**
$\bigwedge s::complex.\ s \in complex\text{-}cone'\ \alpha - \{0\} \Longrightarrow norm\ (stirling\text{-}integral\ n\ s) \leq c\ /$
$norm\ s\ \hat{}\ n$
⟨*proof*⟩

lemma *stirling-integral-holomorphic* [*holomorphic-intros*]:
assumes m: $m > 0$ **and** $A \cap \mathbb{R}_{\leq 0} = \{\}$
shows *stirling-integral m holomorphic-on A*
⟨*proof*⟩

lemma *stirling-integral-continuous-on-complex* [*continuous-intros*]:
assumes m: $m > 0$ **and** $A \cap \mathbb{R}_{\leq 0} = \{\}$
shows *continuous-on A* (*stirling-integral m* :: - ⇒ *complex*)
⟨*proof*⟩

lemma *has-field-derivative-stirling-integral-complex*:
fixes $x :: complex$
assumes $x \notin \mathbb{R}_{\leq 0}\ n > 0$
shows (*stirling-integral n has-field-derivative deriv* (*stirling-integral n*) *x*) (*at*

x)
⟨*proof*⟩

lemma
assumes *n*: $n > 0$ **and** $x > 0$
shows *deriv-stirling-integral-complex-of-real*:
(*deriv* ^^ *j*) (*stirling-integral n*) (*complex-of-real x*) =
complex-of-real ((*deriv* ^^ *j*) (*stirling-integral n*) *x*) (**is** *?lhs x* = *?rhs x*)
and *differentiable-stirling-integral-real*:
(*deriv* ^^ *j*) (*stirling-integral n*) *field-differentiable at x* (**is** *?thesis2*)
⟨*proof*⟩

Unfortunately, asymptotic power series cannot, in general, be differentiated. However, since *ln-Gamma* is holomorphic on the entire positive real half-space, we can differentiate its asymptotic expansion after all.

To do this, we use an ad-hoc version of the more general approach outlined in Erdelyi's "Asymptotic Expansions" for holomorphic functions: We bound the value of the j-th derivative of the remainder term at some value x by applying Cauchy's integral formula along a circle centred at x with radius $\frac{1}{2}x$.

lemma *deriv-stirling-integral-real-bound*:
assumes *m*: $m > 0$
shows (*deriv* ^^ *j*) (*stirling-integral m*) ∈ *O*(λ*x*::*real*. *1* / *x* ^ (*m* + *j*))
⟨*proof*⟩

definition *stirling-sum* **where**
stirling-sum j m x =
(−*1*) ^ *j* ∗ (∑ *k* = *1*..<*m*. (*of-real* (*bernoulli* (*Suc k*)) ∗ *pochhammer* (*of-nat* *k*) *j* / (*of-nat k* ∗
of-nat (*Suc k*))) ∗ *inverse x* ^ (*k* + *j*))

definition *stirling-sum′* **where**
stirling-sum′ j m x =
(−*1*) ^ (*Suc j*) ∗ (∑ *k*≤*m*. (*of-real* (*bernoulli′ k*) ∗
pochhammer (*of-nat* (*Suc k*)) (*j* − *1*) ∗ *inverse x* ^ (*k* + *j*)))

lemma *stirling-sum-complex-of-real*:
stirling-sum j m (*complex-of-real x*) = *complex-of-real* (*stirling-sum j m x*)
⟨*proof*⟩

lemma *stirling-sum′-complex-of-real*:
stirling-sum′ j m (*complex-of-real x*) = *complex-of-real* (*stirling-sum′ j m x*)
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum-complex* [*derivative-intros*]:
Re x > *0* ⟹ (*stirling-sum j m has-field-derivative stirling-sum* (*Suc j*) *m x*) (*at*

x)
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum-real* [*derivative-intros*]:
x > (*0*::*real*) ⟹ (*stirling-sum j m has-field-derivative stirling-sum* (*Suc j*) *m x*) (*at x*)
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum′-complex* [*derivative-intros*]:
assumes *j* > *0 Re x* > *0*
shows (*stirling-sum′ j m has-field-derivative stirling-sum′* (*Suc j*) *m x*) (*at x*)
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum′-real* [*derivative-intros*]:
assumes *j* > *0 x* > (*0*::*real*)
shows (*stirling-sum′ j m has-field-derivative stirling-sum′* (*Suc j*) *m x*) (*at x*)
⟨*proof*⟩

lemma *higher-deriv-stirling-sum-complex*:
Re x > *0* ⟹ (*deriv* ^^ *i*) (*stirling-sum j m*) *x* = *stirling-sum* (*i* + *j*) *m x*
⟨*proof*⟩

definition *Polygamma-approx* :: *nat* ⇒ *nat* ⇒ *′a* ⇒ *′a* :: {*real-normed-field*, *ln*}
where
Polygamma-approx j m =
(*deriv* ^^ *j*) (λ*x*. (*x* − *1* / *2*) ∗ *ln x* − *x* + *of-real* (*ln* (*2* ∗ *pi*)) / *2* + *stirling-sum 0 m x*)

lemma *Polygamma-approx-Suc*: *Polygamma-approx* (*Suc j*) *m* = *deriv* (*Polygamma-approx j m*)
⟨*proof*⟩

lemma *Polygamma-approx-0*:
Polygamma-approx 0 m x = (*x* − *1*/*2*) ∗ *ln x* − *x* + *of-real* (*ln* (*2*∗*pi*)) / *2* + *stirling-sum 0 m x*
⟨*proof*⟩

lemma *Polygamma-approx-1-complex*:
Re x > *0* ⟹
Polygamma-approx (*Suc 0*) *m x* = *ln x* − *1* / (*2*∗*x*) + *stirling-sum* (*Suc 0*) *m x*
⟨*proof*⟩

lemma *Polygamma-approx-1-real*:
x > (*0* :: *real*) ⟹
Polygamma-approx (*Suc 0*) *m x* = *ln x* − *1* / (*2*∗*x*) + *stirling-sum* (*Suc 0*) *m x*
⟨*proof*⟩

lemma *stirling-sum-2-conv-stirling-sum′-1*:
fixes x :: ′*a* :: {*real-div-algebra*, *field-char-0*}
assumes $m > 0$ $x \neq 0$
shows *stirling-sum′ 1 m x* = 1 / x + 1 / (2 ∗ x ^ 2) + *stirling-sum 2 m x*
⟨*proof*⟩

lemma *Polygamma-approx-2-real*:
assumes $x > (0::real)$ $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc* 0)) *m x* = *stirling-sum′ 1 m x*
⟨*proof*⟩

lemma *Polygamma-approx-2-complex*:
assumes *Re* $x > 0$ $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc* 0)) *m x* = *stirling-sum′ 1 m x*
⟨*proof*⟩

lemma *Polygamma-approx-ge-2-real*:
assumes $x > (0::real)$ $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc j*)) *m x* = *stirling-sum′* (*Suc j*) *m x*
⟨*proof*⟩

lemma *Polygamma-approx-ge-2-complex*:
assumes *Re* $x > 0$ $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc j*)) *m x* = *stirling-sum′* (*Suc j*) *m x*
⟨*proof*⟩

lemma *Polygamma-approx-complex-of-real*:
assumes $x > 0$ $m > 0$
shows *Polygamma-approx j m* (*complex-of-real x*) = *of-real* (*Polygamma-approx j m x*)
⟨*proof*⟩

lemma *higher-deriv-Polygamma-approx* [*simp*]:
(*deriv* ^^ *j*) (*Polygamma-approx i m*) = *Polygamma-approx* (*j* + *i*) *m*
⟨*proof*⟩

lemma *stirling-sum-holomorphic* [*holomorphic-intros*]:
0 ∉ *A* ⟹ *stirling-sum j m holomorphic-on A*
⟨*proof*⟩

lemma *Polygamma-approx-holomorphic* [*holomorphic-intros*]:
Polygamma-approx j m holomorphic-on {*s*. *Re s* > 0}
⟨*proof*⟩

lemma *higher-deriv-lnGamma-stirling*:
assumes *m*: $m > 0$
shows (λ*x*::*real*. (*deriv* ^^ *j*) *ln-Gamma x* − *Polygamma-approx j m x*) ∈ *O*(λ*x*. 1 / x ^ (m + j))

⟨*proof*⟩

lemma *Polygamma-approx-1-real′*:
assumes *x*: (*x*::*real*) > *0* **and** *m*: *m* > *0*
shows *Polygamma-approx 1 m x* = *ln x* − (∑ *k* = *Suc 0*..*m*. *bernoulli′ k* ∗ *inverse x* ^ *k* / *real k*)
⟨*proof*⟩

theorem
assumes *m*: *m* > *0*
shows *ln-Gamma-real-asymptotics*:
(λ*x*. *ln-Gamma x* − ((*x* − *1* / *2*) ∗ *ln x* − *x* + *ln* (*2* ∗ *pi*) / *2* +
(∑ *k* = *1*..<*m*. *bernoulli* (*Suc k*) / (*real k* ∗ *real* (*Suc k*)) / *x*^*k*)))
∈ *O*(λ*x*. *1* / *x* ^ *m*) (**is** *?th1*)
and *Digamma-real-asymptotics*:
(λ*x*. *Digamma x* − (*ln x* − (∑ *k*=*1*..*m*. *bernoulli′ k* / *real k* / *x* ^ *k*)))
∈ *O*(λ*x*. *1* / (*x* ^ *Suc m*)) (**is** *?th2*)
and *Polygamma-real-asymptotics*: *j* > *0* ⟹
(λ*x*. *Polygamma j x* − (− *1*) ^ *Suc j* ∗ (∑ *k*≤*m*. *bernoulli′ k* ∗
pochhammer (*real* (*Suc k*)) (*j* − *1*) / *x* ^ (*k* + *j*)))
∈ *O*(λ*x*. *1* / *x* ^ (*m*+*j*+*1*)) (**is** - ⟹ *?th3*)
⟨*proof*⟩

2.5 Asymptotics of the complex Gamma function

The m-th order remainder of Stirling's formula for $\log\Gamma$ is $O(s^{-m})$ uniformly over any complex cone $\mathrm{Arg}(z) \le \alpha$, $z \ne 0$ for any angle $\alpha \in (0, \pi)$. This means that there is bounded by cz^{-m} for some constant c for all z in this cone.

context
fixes *F* **and** α
assumes α: α ∈ {*0*<..<*pi*}
defines *F* ≡ *principal* (*complex-cone′* α − {*0*})
begin

lemma *stirling-integral-bigo*:
fixes *m* :: *nat*
assumes *m*: *m* > *0*
shows *stirling-integral m* ∈ *O*[*F*](λ*s*. *1* / *s* ^ *m*)
⟨*proof*⟩

end

The following is a more explicit statement of this:

theorem *ln-Gamma-complex-asymptotics-explicit*:
fixes *m* :: *nat* **and** α :: *real*
assumes *m* > *0* **and** α ∈ {*0*<..<*pi*}
obtains *C* :: *real* **and** *R* :: *complex* ⇒ *complex*

where $\forall$ *s::complex. s* $\notin \mathbb{R}_{\leq 0} \longrightarrow$
ln-Gamma s $=$ *(s* $-$ *1/2)* $*$ *ln s* $-$ *s* $+$ *ln (2* $*$ *pi) / 2* $+$
$(\sum$ *k=1..<m. bernoulli (k+1) / (k* $*$ *(k+1)* $*$ *s ^ k))* $-$ *R s*
and $\forall$ *s. s* $\neq$ *0* $\wedge$ *|Arg s|* $\leq \alpha \longrightarrow$ *norm (R s)* $\leq$ *C / norm s ^ m*
⟨*proof*⟩

Lastly, we can also derive the asymptotics of Γ itself:

$$\Gamma(z) \sim \sqrt{2\pi/z}\left(\frac{z}{e}\right)^z$$

uniformly for $|z| \to \infty$ within the cone $\mathrm{Arg}(z) \leq \alpha$ for $\alpha \in (0, \pi)$:

context
fixes *F* **and** α
assumes α: $\alpha \in$ *{0<..<pi}*
defines *F* $\equiv$ *inf at-infinity (principal (complex-cone′* α*))*
begin

lemma *Gamma-complex-asymp-equiv*:
Gamma ∼[F] (λs. sqrt (2 $*$ *pi)* $*$ *(s / exp 1) powr s / s powr (1 / 2))*
⟨*proof*⟩

end

end

References

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