Stirling's formula

Manuel Eberl

December 14, 2021

Abstract

This work contains a proof of Stirling's formula both for the factorial $n! \sim \sqrt{2\pi n}(n/e)^n$ on natural numbers and the real Gamma function $\Gamma(x) \sim \sqrt{2\pi/x}(x/e)^x$. The proof is based on work by Graham Jameson [3].

This is then extended to the full asymptotic expansion

$$\log\Gamma(z) = \left(z - \tfrac{1}{2}\right)\log z - z + \tfrac{1}{2}\log(2\pi) + \sum_{k=1}^{n-1} \frac{B_{k+1}}{k(k+1)} z^{-k} - \frac{1}{n}\int_0^\infty B_n([t])(t+z)^{-n}\,\mathrm{d}t$$

uniformly for all complex $z \neq 0$ in the cone $\arg(z) \leq \alpha$ for any $\alpha \in (0, \pi)$, with which the above asymptotic relation for Γ is also extended to complex arguments.

Contents

1 Stirling's Formula

theory *Stirling-Formula*
imports
HOL−Analysis.Analysis
Landau-Symbols.Landau-More
begin

context
begin

First, we define the S_n^* from Jameson's article:

private definition S' :: *nat* $\Rightarrow$ *real* $\Rightarrow$ *real* **where**
$S'\ n\ x = 1/(2*x) + (\sum r{=}1..{<}n.\ 1/(\textit{of-nat}\ r{+}x)) + 1\ /(2*(n + x))$

Next, the trapezium (also called T in Jameson's article):

private definition T :: *real* $\Rightarrow$ *real* **where**
$T\ x = 1/(2*x) + 1/(2*(x{+}1))$

Now we define The difference $\Delta(x)$:

private definition D :: *real* $\Rightarrow$ *real* **where**
$D\ x = T\ x - \textit{ln}\ (x + 1) + \textit{ln}\ x$

private lemma S'*-telescope-trapezium*:
assumes $n > 0$
shows $S'\ n\ x = (\sum r{<}n.\ T\ (\textit{of-nat}\ r + x))$
⟨*proof*⟩ **lemma** *stirling-trapezium*:
assumes x: $(x{::}\textit{real}) > 0$
shows $D\ x \in \{0\ ..\ 1/(12{*}x\hat{\ }2) - 1/(12 * (x{+}1)\hat{\ }2)\}$
⟨*proof*⟩

The following functions correspond to the $p_n(x)$ from the article. The special case $n = 0$ would not, strictly speaking, be necessary, but it allows some theorems to work even without the precondition $n \neq 0$:

private definition p :: *nat* $\Rightarrow$ *real* $\Rightarrow$ *real* **where**
$p\ n\ x = (\textit{if}\ n = 0\ \textit{then}\ 1/x\ \textit{else}\ (\sum r{<}n.\ D\ (\textit{real}\ r + x)))$

We can write the Digamma function in terms of S':

private lemma S'*-LIMSEQ-Digamma*:
assumes x: $x \neq 0$
shows $(\lambda n.\ \textit{ln}\ (\textit{real}\ n) - S'\ n\ x - 1/(2*x)) \longrightarrow \textit{Digamma}\ x$
⟨*proof*⟩

Moreover, we can give an expansion of S' with the p as variation terms.

private lemma S'*-approx*:
$S'\ n\ x = \textit{ln}\ (\textit{real}\ n + x) - \textit{ln}\ x + p\ n\ x$
⟨*proof*⟩

We define the limit of the p (simply called $p(x)$ in Jameson's article):

private definition P :: *real* $\Rightarrow$ *real* **where**
$P\ x = (\sum n.\ D\ (\textit{real}\ n + x))$

private lemma *D-summable*:
assumes x: $x > 0$
shows *summable* $(\lambda n.\ D\ (\textit{real}\ n + x))$

$\langle proof \rangle$ **lemma** *p-LIMSEQ*:
assumes x: $x > 0$
shows $(\lambda n.\ p\ n\ x) \longrightarrow P\ x$
$\langle proof \rangle$

This gives us an expansion of the Digamma function:

lemma *Digamma-approx*:
assumes x: $(x :: real) > 0$
shows $Digamma\ x = ln\ x - 1\ /\ (2 * x) - P\ x$
$\langle proof \rangle$

Next, we derive some bounds on P:

private lemma *p-ge-0*: $x > 0 \Longrightarrow p\ n\ x \geq 0$
$\langle proof \rangle$ **lemma** *P-ge-0*: $x > 0 \Longrightarrow P\ x \geq 0$
$\langle proof \rangle$ **lemma** *p-upper-bound*:
assumes $x > 0\ n > 0$
shows $p\ n\ x \leq 1/(12{*}x\hat{}2)$
$\langle proof \rangle$ **lemma** *P-upper-bound*:
assumes $x > 0$
shows $P\ x \leq 1/(12{*}x\hat{}2)$
$\langle proof \rangle$

We can now use this approximation of the Digamma function to approximate *ln-Gamma* (since the former is the derivative of the latter). We therefore define the function g from Jameson's article, which measures the difference between *ln-Gamma* and its approximation:

private definition $g :: real \Rightarrow real$ **where**
$g\ x = ln\text{-}Gamma\ x - (x - 1/2) * ln\ x + x$

private lemma *DERIV-g*: $x > 0 \Longrightarrow (g\ has\text{-}field\text{-}derivative\ -P\ x)\ (at\ x)$
$\langle proof \rangle$ **lemma** *isCont-P*:
assumes $x > 0$
shows $isCont\ P\ x$
$\langle proof \rangle$ **lemma** *P-continuous-on* [*THEN continuous-on-subset*]: *continuous-on* $\{0{<}..\}$ P
$\langle proof \rangle$ **lemma** *P-integrable*:
assumes a: $a > 0$
shows $P\ integrable\text{-}on\ \{a..\}$
$\langle proof \rangle$ **definition** $c :: real$ **where** $c = integral\ \{1..\}\ (\lambda x.\ -P\ x) + g\ 1$

We can now give bounds on g:

private lemma *g-bounds*:
assumes x: $x \geq 1$
shows $g\ x \in \{c..c + 1/(12{*}x)\}$
$\langle proof \rangle$

Finally, we have bounds on *ln-Gamma*, *Gamma*, and *fact*.

private lemma *ln-Gamma-bounds-aux*:

$x \geq 1 \Longrightarrow$ *ln-Gamma* $x \geq c + (x - 1/2) * \ln x - x$
$x \geq 1 \Longrightarrow$ *ln-Gamma* $x \leq c + (x - 1/2) * \ln x - x + 1/(12{*}x)$
⟨*proof*⟩ **lemma** *Gamma-bounds-aux*:
assumes *x*: $x \geq 1$
shows *Gamma* $x \geq$ *exp* $c * x$ *powr* $(x - 1/2)$ / *exp* x
Gamma $x \leq$ *exp* $c * x$ *powr* $(x - 1/2)$ / *exp* $x *$ *exp* $(1/(12{*}x))$
⟨*proof*⟩ **lemma** *Gamma-asymp-equiv-aux*:
Gamma ∼[*at-top*] ($\lambda x.$ *exp* $c * x$ *powr* $(x - 1/2)$ / *exp* x)
⟨*proof*⟩
include *asymp-equiv-notation*
⟨*proof*⟩ **lemma** *exp-1-powr-real* [*simp*]: *exp* (*1*::*real*) *powr* x = *exp* x
⟨*proof*⟩ **lemma** *fact-asymp-equiv-aux*:
fact ∼[*at-top*] ($\lambda x.$ *exp* $c *$ *sqrt* (*real* x) $*$ (*real* x / *exp* *1*) *powr* *real* x)
⟨*proof*⟩
include *asymp-equiv-notation*
⟨*proof*⟩

We still need to determine the constant term c, which we do using Wallis' product formula:

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{\pi}{2}$$

private lemma *powr-mult-2*: (*x*::*real*) > *0* $\Longrightarrow$ x *powr* $(y * 2)$ = (x^*2*) *powr* y
⟨*proof*⟩ **lemma** *exp-mult-2*: *exp* ($y * 2$ :: *real*) = *exp* $y *$ *exp* y
⟨*proof*⟩ **lemma** *exp-c*: *exp* c = *sqrt* (*2*∗*pi*)
⟨*proof*⟩
include *asymp-equiv-notation*
⟨*proof*⟩ **lemma** *c*: c = *ln* (*2*∗*pi*) / *2*
⟨*proof*⟩

This gives us the final bounds:

theorem *Gamma-bounds*:
assumes $x \geq 1$
shows *Gamma* $x \geq$ *sqrt* (*2*∗*pi*/x) ∗ (x / *exp* *1*) *powr* x (**is** *?th1*)
Gamma $x \leq$ *sqrt* (*2*∗*pi*/x) ∗ (x / *exp* *1*) *powr* x ∗ *exp* (*1* / (*12* ∗ x)) (**is** *?th2*)
⟨*proof*⟩

theorem *ln-Gamma-bounds*:
assumes $x \geq 1$
shows *ln-Gamma* $x \geq$ *ln* (*2*∗*pi*/x) / *2* + x ∗ *ln* $x - x$ (**is** *?th1*)
ln-Gamma $x \leq$ *ln* (*2*∗*pi*/x) / *2* + x ∗ *ln* $x - x$ + *1*/(*12*∗x) (**is** *?th2*)
⟨*proof*⟩

theorem *fact-bounds*:
assumes $n > 0$
shows (*fact* n :: *real*) $\geq$ *sqrt* (*2*∗*pi*∗n) ∗ (n / *exp* *1*) ^ n (**is** *?th1*)
(*fact* n :: *real*) $\leq$ *sqrt* (*2*∗*pi*∗n) ∗ (n / *exp* *1*) ^ n ∗ *exp* (*1* / (*12* ∗ n)) (**is** *?th2*)

⟨*proof*⟩

theorem *ln-fact-bounds*:
assumes $n > 0$
shows $ln\ (fact\ n :: real) \geq ln\ (2*pi*n)/2 + n * ln\ n - n$ (**is** *?th1*)
$ln\ (fact\ n :: real) \leq ln\ (2*pi*n)/2 + n * ln\ n - n + 1/(12*real\ n)$ (**is** *?th2*)
⟨*proof*⟩

theorem *Gamma-asymp-equiv*:
$Gamma \sim[at\text{-}top]\ (\lambda x.\ sqrt\ (2*pi/x) * (x\ /\ exp\ 1)\ powr\ x :: real)$
⟨*proof*⟩

theorem *fact-asymp-equiv*:
$fact \sim[at\text{-}top]\ (\lambda n.\ sqrt\ (2*pi*n) * (n\ /\ exp\ 1)\ \widehat{}\ n :: real)$
⟨*proof*⟩

end

end

2 Complete asymptotics of the logarithmic Gamma function

theory *Gamma-Asymptotics*
imports
HOL−Complex-Analysis.Complex-Analysis
HOL−Real-Asymp.Real-Asymp
Bernoulli.Bernoulli-FPS
Bernoulli.Periodic-Bernpoly
Stirling-Formula
begin

2.1 Auxiliary Facts

lemma *stirling-limit-aux1*:
$((\lambda y.\ Ln\ (1 + z * of\text{-}real\ y)\ /\ of\text{-}real\ y) \longrightarrow z)\ (at\text{-}right\ 0)$ **for** $z :: complex$
⟨*proof*⟩

lemma *stirling-limit-aux2*:
$((\lambda y.\ y * Ln\ (1 + z\ /\ of\text{-}real\ y)) \longrightarrow z)\ at\text{-}top$ **for** $z :: complex$
⟨*proof*⟩

lemma *Union-atLeastAtMost*:
assumes $N > 0$
shows $(\bigcup n \in \{0..<N\}.\ \{real\ n..real\ (n + 1)\}) = \{0..real\ N\}$
⟨*proof*⟩

2.2 Cones in the complex plane

definition *complex-cone* :: *real* ⇒ *real* ⇒ *complex set* **where**
  complex-cone a b = {*z*. ∃ *y*∈{*a*..*b*}. *z* = *rcis* (*norm z*) *y*}

abbreviation *complex-cone′* :: *real* ⇒ *complex set* **where**
  complex-cone′ a ≡ *complex-cone* (−*a*) *a*

lemma *zero-in-complex-cone* [*simp*, *intro*]: *a* ≤ *b* ⟹ *0* ∈ *complex-cone a b*
  ⟨*proof*⟩

lemma *complex-coneE*:
  assumes *z* ∈ *complex-cone a b*
  obtains *r α* **where** *r* ≥ *0* *α* ∈ {*a*..*b*} *z* = *rcis r α*
⟨*proof*⟩

lemma *arg-cis* [*simp*]:
  assumes *x* ∈ {−*pi*<..*pi*}
  shows *Arg* (*cis x*) = *x*
  ⟨*proof*⟩

lemma *arg-mult-of-real-left* [*simp*]:
  assumes *r* > *0*
  shows *Arg* (*of-real r* ∗ *z*) = *Arg z*
⟨*proof*⟩

lemma *arg-mult-of-real-right* [*simp*]:
  assumes *r* > *0*
  shows *Arg* (*z* ∗ *of-real r*) = *Arg z*
  ⟨*proof*⟩

lemma *arg-rcis* [*simp*]:
  assumes *x* ∈ {−*pi*<..*pi*} *r* > *0*
  shows *Arg* (*rcis r x*) = *x*
  ⟨*proof*⟩

lemma *rcis-in-complex-cone* [*intro*]:
  assumes *α* ∈ {*a*..*b*} *r* ≥ *0*
  shows *rcis r α* ∈ *complex-cone a b*
  ⟨*proof*⟩

lemma *arg-imp-in-complex-cone*:
  assumes *Arg z* ∈ {*a*..*b*}
  shows *z* ∈ *complex-cone a b*
⟨*proof*⟩

lemma *complex-cone-altdef*:
  assumes −*pi* < *a* *a* ≤ *b* *b* ≤ *pi*
  shows *complex-cone a b* = *insert 0* {*z*. *Arg z* ∈ {*a*..*b*}}
⟨*proof*⟩

lemma *nonneg-of-real-in-complex-cone* [*simp*, *intro*]:
  assumes $x \geq 0$ $a \leq 0$ $0 \leq b$
  shows *of-real* $x \in$ *complex-cone* a b
⟨*proof*⟩

lemma *one-in-complex-cone* [*simp*, *intro*]: $a \leq 0 \Longrightarrow 0 \leq b \Longrightarrow 1 \in$ *complex-cone* a b
  ⟨*proof*⟩

lemma *of-nat-in-complex-cone* [*simp*, *intro*]: $a \leq 0 \Longrightarrow 0 \leq b \Longrightarrow$ *of-nat* $n \in$ *complex-cone* a b
  ⟨*proof*⟩

2.3 Another integral representation of the Beta function

lemma *complex-cone-inter-nonpos-Reals*:
  assumes $-pi < a$ $a \leq b$ $b < pi$
  shows *complex-cone* a $b \cap \mathbb{R}_{\leq 0} = \{0\}$
⟨*proof*⟩

theorem
  assumes a: $a > 0$ **and** b: $b > (0 :: real)$
  shows *has-integral-Beta-real'*:
    $((\lambda u.\ u\ powr\ (b - 1) / (1 + u)\ powr\ (a + b))$ *has-integral Beta* a $b)$ $\{0<..\}$
  and *Beta-conv-nn-integral*:
    Beta a $b = (\int^{+} u.\ ennreal\ (indicator\ \{0<..\}\ u * u\ powr\ (b - 1) / (1 + u)\ powr\ (a + b))\ \partial lborel)$
⟨*proof*⟩

lemma *has-integral-Beta2*:
  fixes $a :: real$
  assumes $a < -1/2$
  shows $((\lambda x.\ (1 + x \hat{}\ 2)\ powr\ a)$ *has-integral Beta* $(- a - 1 / 2)\ (1 / 2) / 2)$ $\{0<..\}$
⟨*proof*⟩

lemma *has-integral-Beta3*:
  fixes a $b :: real$
  assumes $a < -1/2$ **and** $b > 0$
  shows $((\lambda x.\ (b + x \hat{}\ 2)\ powr\ a)$ *has-integral*
    Beta $(-a - 1/2)\ (1/2) / 2 * b\ powr\ (a + 1/2))$ $\{0<..\}$
⟨*proof*⟩

2.4 Asymptotics of the real $\log \Gamma$ function and its derivatives

This is the error term that occurs in the expansion of *ln-Gamma*. It can be shown to be of order $O(s^{-n})$.

definition *stirling-integral* :: *nat* $\Rightarrow$ *'a* :: {*real-normed-div-algebra*, *banach*} $\Rightarrow$ *'a*

where
stirling-integral n s =
lim (λ*N. integral* {*0..N*} (λ*x. of-real* (*pbernpoly n x*) / (*of-real x* + *s*) ^ *n*))

context
fixes *s* :: *complex* **assumes** *s*: $s \notin \mathbb{R}_{\leq 0}$
fixes *approx* :: *nat* ⇒ *complex*
defines *approx* ≡ (λ*N*.
(∑ *n* = *1*..<*N*. *s* / *of-nat n* − *ln* (*1* + *s* / *of-nat n*)) − (*euler-mascheroni* ∗ *s* + *ln s*) − — ⟶ *ln-Gamma s*
(*ln-Gamma* (*of-nat N*) − *ln* (*2* ∗ *pi* / *of-nat N*) / *2* − *of-nat N* ∗ *ln* (*of-nat N*) + *of-nat N*) − — ⟶ *0*
s ∗ (*harm* (*N* − *1*) − *ln* (*of-nat* (*N* − *1*)) − *euler-mascheroni*) + — ⟶ *0*
s ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat* (*N* − *1*))) − — ⟶ *0*
(*1*/*2*) ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat N*)) + — ⟶ *0*
of-nat N ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat N*)) − — ⟶ *s*
(*s* − *1*/*2*) ∗ *ln s* − *ln* (*2* ∗ *pi*) / *2*)
begin

qualified lemma
assumes *N*: *N* > *0*
shows *integrable-pbernpoly-1*:
(λ*x. of-real* (−*pbernpoly 1 x*) / (*of-real x* + *s*)) *integrable-on* {*0..real N*}
and *integral-pbernpoly-1-aux*:
integral {*0..real N*} (λ*x.* −*of-real* (*pbernpoly 1 x*) / (*of-real x* + *s*)) = *approx N*
and *has-integral-pbernpoly-1*:
((λ*x. pbernpoly 1 x* /(*x* + *s*)) *has-integral*
(∑ *m*<*N*. (*of-nat m* + *1* / *2* + *s*) ∗ (*ln* (*of-nat m* + *s*) −
ln (*of-nat m* + *1* + *s*)) + *1*)) {*0..real N*}
⟨*proof*⟩

lemma *integrable-ln-Gamma-aux*:
shows (λ*x. of-real* (*pbernpoly n x*) / (*of-real x* + *s*) ^ *n*) *integrable-on* {*0..real N*}
⟨*proof*⟩

This following proof is based on "Rudiments of the theory of the gamma function" by Bruce Berndt [1].

lemma *tendsto-of-real-0-I*:
(*f* ⟶ *0*) *G* ⟹ ((λ*x.* (*of-real* (*f x*))) ⟶ (*0* ::′*a*::*real-normed-div-algebra*)) *G*
⟨*proof*⟩ **lemma** *integral-pbernpoly-1*:
(λ*N. integral* {*0..real N*} (λ*x. pbernpoly 1 x* / (*x* + *s*)))
⟶ −*ln-Gamma s* − *s* + (*s* − *1* / *2*) ∗ *ln s* + *ln* (*2* ∗ *pi*) / *2*
⟨*proof*⟩ **lemma** *pbernpoly-integral-conv-pbernpoly-integral-Suc*:
assumes *n* ≥ *1*
shows *integral* {*0..real N*} (λ*x. pbernpoly n x* / (*x* + *s*) ^ *n*) =
of-real (*pbernpoly* (*Suc n*) (*real N*)) / (*of-nat* (*Suc n*) ∗ (*s* + *of-nat N*)

$\hat{}\ n) -$
$\quad$ *of-real* (*bernoulli* (*Suc n*)) / (*of-nat* (*Suc n*) $*$ *s* $\hat{}$ *n*) + *of-nat n* / *of-nat* (*Suc n*) $*$
$\quad\quad$ *integral* {*0..real N*} (λx. *of-real* (*pbernpoly* (*Suc n*) *x*) / (*of-real x* + *s*) $\hat{}$ *Suc n*)
⟨*proof*⟩

lemma *pbernpoly-over-power-tendsto-0*:
 assumes $n > 0$
 shows (λx. *of-real* (*pbernpoly* (*Suc n*) (*real x*)) / (*of-nat* (*Suc n*) $*$ (*s* + *of-nat x*) $\hat{}$ *n*)) $\longrightarrow$ *0*
⟨*proof*⟩

lemma *convergent-stirling-integral*:
 assumes $n > 0$
 shows *convergent* (λN. *integral* {*0..real N*}
 (λx. *of-real* (*pbernpoly n x*) / (*of-real x* + *s*) $\hat{}$ *n*)) (**is** *convergent* (*?f n*))
⟨*proof*⟩

lemma *stirling-integral-conv-stirling-integral-Suc*:
 assumes $n > 0$
 shows *stirling-integral n s* =
 of-nat n / *of-nat* (*Suc n*) $*$ *stirling-integral* (*Suc n*) *s* −
 of-real (*bernoulli* (*Suc n*)) / (*of-nat* (*Suc n*) $*$ *s* $\hat{}$ *n*)
⟨*proof*⟩

lemma *stirling-integral-1-unfold*:
 assumes $m > 0$
 shows *stirling-integral 1 s* = *stirling-integral m s* / *of-nat m* −
 ($\sum$ *k=1..<m*. *of-real* (*bernoulli* (*Suc k*)) / (*of-nat k* $*$ *of-nat* (*Suc k*) $*$ *s* $\hat{}$ *k*))
⟨*proof*⟩

lemma *ln-Gamma-stirling-complex*:
 assumes $m > 0$
 shows *ln-Gamma s* = (*s* − *1* / *2*) $*$ *ln s* − *s* + *ln* (*2* $*$ *pi*) / *2* +
 ($\sum$ *k=1..<m*. *of-real* (*bernoulli* (*Suc k*)) / (*of-nat k* $*$ *of-nat* (*Suc k*) $*$ *s* $\hat{}$ *k*)) −
 stirling-integral m s / *of-nat m*
⟨*proof*⟩

lemma *LIMSEQ-stirling-integral*:
 $n > 0 \Longrightarrow$ (λx. *integral* {*0..real x*} (λx. *of-real* (*pbernpoly n x*) / (*of-real x* + *s*) $\hat{}$ *n*))
 $\longrightarrow$ *stirling-integral n s* ⟨*proof*⟩

end

lemmas *has-integral-of-real* = *has-integral-linear*[*OF - bounded-linear-of-real*, *un-*

folded o-def]
lemmas *integral-of-real* = *integral-linear*[*OF - bounded-linear-of-real*, *unfolded o-def*]

lemma *integrable-ln-Gamma-aux-real*:
  assumes $0 < s$
  shows $(\lambda x.\ pbernpoly\ n\ x\ /\ (x + s)\ \hat{}\ n)$ *integrable-on* $\{0..real\ N\}$
⟨*proof*⟩

lemma
  assumes $x > 0$ $n > 0$
  shows *stirling-integral-complex-of-real*:
    stirling-integral n (*complex-of-real* x) = *of-real* (*stirling-integral* n x)
  and *LIMSEQ-stirling-integral-real*:
    $(\lambda N.\ integral\ \{0..real\ N\}\ (\lambda t.\ pbernpoly\ n\ t\ /\ (t + x)\ \hat{}\ n))$
    $\longrightarrow$ *stirling-integral* n x
  and *stirling-integral-real-convergent*:
    convergent $(\lambda N.\ integral\ \{0..real\ N\}\ (\lambda t.\ pbernpoly\ n\ t\ /\ (t + x)\ \hat{}\ n))$
⟨*proof*⟩

lemma *ln-Gamma-stirling-real*:
  assumes $x > (0 :: real)$ $m > (0::nat)$
  shows $ln\text{-}Gamma\ x = (x - 1\ /\ 2) * ln\ x - x + ln\ (2 * pi)\ /\ 2 +$
    $(\sum k = 1..<m.\ bernoulli\ (Suc\ k)\ /\ (of\text{-}nat\ k * of\text{-}nat\ (Suc\ k) * x\ \hat{}\ k))$
$-$
    stirling-integral m x / *of-nat* m
⟨*proof*⟩

lemma *stirling-integral-bound-aux*:
  assumes n: $n > (1::nat)$
  obtains c **where** $\bigwedge s.\ Re\ s > 0 \Longrightarrow norm\ (stirling\text{-}integral\ n\ s) \leq\ c\ /\ Re\ s\ \hat{}$
$(n - 1)$
⟨*proof*⟩

lemma *stirling-integral-bound-aux-integral1*:
  fixes a b c :: *real* **and** n :: *nat*
  assumes $a \geq 0$ $b > 0$ $c \geq 0$ $n > 1$ $l < a - b$ $r > a + b$
  shows $((\lambda x.\ c\ /\ max\ b\ |x - a|\ \hat{}\ n)$ *has-integral*
    $2*c*(n\ /\ (n - 1))/b\hat{}(n-1) - c/(n-1) * (1/(a-l)\hat{}(n-1) + 1/(r-a)\hat{}(n-1)))$
$\{l..r\}$
⟨*proof*⟩

lemma *stirling-integral-bound-aux-integral2*:
  fixes a b c :: *real* **and** n :: *nat*
  assumes $a \geq 0$ $b > 0$ $c \geq 0$ $n > 1$
  obtains I **where** $((\lambda x.\ c\ /\ max\ b\ |x - a|\ \hat{}\ n)$ *has-integral* $I)$ $\{l..r\}$
    $I \leq 2 * c * (n\ /\ (n - 1))\ /\ b\ \hat{}(n-1)$
⟨*proof*⟩

lemma *stirling-integral-bound-aux'*:
assumes *n*: $n > (1::nat)$ **and** α: $\alpha \in \{0<..<pi\}$
obtains *c* **where** $\bigwedge s::complex.\ s \in complex\text{-}cone'\ \alpha - \{0\} \Longrightarrow$
$norm\ (stirling\text{-}integral\ n\ s) \leq c\ /\ norm\ s\ \hat{}\ (n - 1)$
⟨*proof*⟩

lemma *stirling-integral-bound*:
assumes $n > 0$
obtains *c* **where**
$\bigwedge s.\ Re\ s > 0 \Longrightarrow norm\ (stirling\text{-}integral\ n\ s) \leq c\ /\ Re\ s\ \hat{}\ n$
⟨*proof*⟩

lemma *stirling-integral-bound'*:
assumes $n > 0$ **and** $\alpha \in \{0<..<pi\}$
obtains *c* **where**
$\bigwedge s::complex.\ s \in complex\text{-}cone'\ \alpha - \{0\} \Longrightarrow norm\ (stirling\text{-}integral\ n\ s) \leq c\ /$ $norm\ s\ \hat{}\ n$
⟨*proof*⟩

lemma *stirling-integral-holomorphic* [*holomorphic-intros*]:
assumes *m*: $m > 0$ **and** $A \cap \mathbb{R}_{\leq 0} = \{\}$
shows *stirling-integral m holomorphic-on A*
⟨*proof*⟩

lemma *stirling-integral-continuous-on-complex* [*continuous-intros*]:
assumes *m*: $m > 0$ **and** $A \cap \mathbb{R}_{\leq 0} = \{\}$
shows *continuous-on A* (*stirling-integral m* :: - ⇒ *complex*)
⟨*proof*⟩

lemma *has-field-derivative-stirling-integral-complex*:
fixes *x* :: *complex*
assumes $x \notin \mathbb{R}_{\leq 0}$ $n > 0$
shows (*stirling-integral n has-field-derivative deriv* (*stirling-integral n*) *x*) (*at x*)
⟨*proof*⟩

lemma
assumes *n*: $n > 0$ **and** $x > 0$
shows *deriv-stirling-integral-complex-of-real*:
(*deriv* ^^ *j*) (*stirling-integral n*) (*complex-of-real x*) =
complex-of-real ((*deriv* ^^ *j*) (*stirling-integral n*) *x*) (**is** *?lhs x* = *?rhs x*)
and *differentiable-stirling-integral-real*:
(*deriv* ^^ *j*) (*stirling-integral n*) *field-differentiable at x* (**is** *?thesis2*)
⟨*proof*⟩

Unfortunately, asymptotic power series cannot, in general, be differentiated.

However, since *ln-Gamma* is holomorphic on the entire positive real half-space, we can differentiate its asymptotic expansion after all.

To do this, we use an ad-hoc version of the more general approach outlined in Erdelyi's "Asymptotic Expansions" for holomorphic functions: We bound the value of the j-th derivative of the remainder term at some value x by applying Cauchy's integral formula along a circle centred at x with radius $\frac{1}{2}x$.

lemma *deriv-stirling-integral-real-bound*:
assumes *m*: $m > 0$
shows $(deriv \ \widehat{}\widehat{} \ j)\ (stirling\text{-}integral\ m) \in O(\lambda x{::}real.\ 1\ /\ x\ \widehat{}\ (m + j))$
⟨*proof*⟩

definition *stirling-sum* **where**
$stirling\text{-}sum\ j\ m\ x =$
$(-1)\ \widehat{}\ j * (\sum k = 1..{<}m.\ (of\text{-}real\ (bernoulli\ (Suc\ k)) * pochhammer\ (of\text{-}nat\ k)\ j\ /\ (of\text{-}nat\ k *$
$of\text{-}nat\ (Suc\ k))) * inverse\ x\ \widehat{}\ (k + j))$

definition *stirling-sum′* **where**
$stirling\text{-}sum'\ j\ m\ x =$
$(-1)\ \widehat{}\ (Suc\ j) * (\sum k{\le}m.\ (of\text{-}real\ (bernoulli'\ k) *$
$pochhammer\ (of\text{-}nat\ (Suc\ k))\ (j - 1) * inverse\ x\ \widehat{}\ (k + j)))$

lemma *stirling-sum-complex-of-real*:
$stirling\text{-}sum\ j\ m\ (complex\text{-}of\text{-}real\ x) = complex\text{-}of\text{-}real\ (stirling\text{-}sum\ j\ m\ x)$
⟨*proof*⟩

lemma *stirling-sum′-complex-of-real*:
$stirling\text{-}sum'\ j\ m\ (complex\text{-}of\text{-}real\ x) = complex\text{-}of\text{-}real\ (stirling\text{-}sum'\ j\ m\ x)$
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum-complex* [*derivative-intros*]:
$Re\ x > 0 \Longrightarrow (stirling\text{-}sum\ j\ m\ has\text{-}field\text{-}derivative\ stirling\text{-}sum\ (Suc\ j)\ m\ x)\ (at\ x)$
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum-real* [*derivative-intros*]:
$x > (0{::}real) \Longrightarrow (stirling\text{-}sum\ j\ m\ has\text{-}field\text{-}derivative\ stirling\text{-}sum\ (Suc\ j)\ m\ x)\ (at\ x)$
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum′-complex* [*derivative-intros*]:
assumes $j > 0\ Re\ x > 0$
shows $(stirling\text{-}sum'\ j\ m\ has\text{-}field\text{-}derivative\ stirling\text{-}sum'\ (Suc\ j)\ m\ x)\ (at\ x)$
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum′-real* [*derivative-intros*]:
assumes $j > 0\ x > (0{::}real)$

shows $(stirling\text{-}sum'\ j\ m\ has\text{-}field\text{-}derivative\ stirling\text{-}sum'\ (Suc\ j)\ m\ x)\ (at\ x)$
$\langle proof \rangle$

lemma *higher-deriv-stirling-sum-complex*:
$Re\ x > 0 \Longrightarrow (deriv\ \widehat{\ }\widehat{\ }\ i)\ (stirling\text{-}sum\ j\ m)\ x = stirling\text{-}sum\ (i + j)\ m\ x$
$\langle proof \rangle$

definition $Polygamma\text{-}approx :: nat \Rightarrow nat \Rightarrow 'a \Rightarrow 'a :: \{real\text{-}normed\text{-}field,\ ln\}$
where
$Polygamma\text{-}approx\ j\ m =$
$(deriv\ \widehat{\ }\widehat{\ }\ j)\ (\lambda x.\ (x - 1\ /\ 2) * ln\ x - x + of\text{-}real\ (ln\ (2 * pi))\ /\ 2 + stirling\text{-}sum\ 0\ m\ x)$

lemma *Polygamma-approx-Suc*: $Polygamma\text{-}approx\ (Suc\ j)\ m = deriv\ (Polygamma\text{-}approx\ j\ m)$
$\langle proof \rangle$

lemma *Polygamma-approx-0*:
$Polygamma\text{-}approx\ 0\ m\ x = (x - 1/2) * ln\ x - x + of\text{-}real\ (ln\ (2*pi))\ /\ 2 + stirling\text{-}sum\ 0\ m\ x$
$\langle proof \rangle$

lemma *Polygamma-approx-1-complex*:
$Re\ x > 0 \Longrightarrow$
$Polygamma\text{-}approx\ (Suc\ 0)\ m\ x = ln\ x - 1\ /\ (2*x) + stirling\text{-}sum\ (Suc\ 0)\ m\ x$
$\langle proof \rangle$

lemma *Polygamma-approx-1-real*:
$x > (0 :: real) \Longrightarrow$
$Polygamma\text{-}approx\ (Suc\ 0)\ m\ x = ln\ x - 1\ /\ (2*x) + stirling\text{-}sum\ (Suc\ 0)\ m\ x$
$\langle proof \rangle$

lemma *stirling-sum-2-conv-stirling-sum'-1*:
fixes $x :: 'a :: \{real\text{-}div\text{-}algebra,\ field\text{-}char\text{-}0\}$
assumes $m > 0\ x \neq 0$
shows $stirling\text{-}sum'\ 1\ m\ x = 1\ /\ x + 1\ /\ (2 * x\widehat{\ }2) + stirling\text{-}sum\ 2\ m\ x$
$\langle proof \rangle$

lemma *Polygamma-approx-2-real*:
assumes $x > (0::real)\ m > 0$
shows $Polygamma\text{-}approx\ (Suc\ (Suc\ 0))\ m\ x = stirling\text{-}sum'\ 1\ m\ x$
$\langle proof \rangle$

lemma *Polygamma-approx-2-complex*:
assumes $Re\ x > 0\ m > 0$
shows $Polygamma\text{-}approx\ (Suc\ (Suc\ 0))\ m\ x = stirling\text{-}sum'\ 1\ m\ x$

⟨*proof*⟩

lemma *Polygamma-approx-ge-2-real*:
assumes $x > (0::real)$ $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc* *j*)) *m* *x* = *stirling-sum'* (*Suc* *j*) *m* *x*
⟨*proof*⟩

lemma *Polygamma-approx-ge-2-complex*:
assumes *Re* $x > 0$ $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc* *j*)) *m* *x* = *stirling-sum'* (*Suc* *j*) *m* *x*
⟨*proof*⟩

lemma *Polygamma-approx-complex-of-real*:
assumes $x > 0$ $m > 0$
shows *Polygamma-approx* *j* *m* (*complex-of-real* *x*) = *of-real* (*Polygamma-approx* *j* *m* *x*)
⟨*proof*⟩

lemma *higher-deriv-Polygamma-approx* [*simp*]:
(*deriv* ^^ *j*) (*Polygamma-approx* *i* *m*) = *Polygamma-approx* (*j* + *i*) *m*
⟨*proof*⟩

lemma *stirling-sum-holomorphic* [*holomorphic-intros*]:
$0 \notin A \Longrightarrow$ *stirling-sum* *j* *m* *holomorphic-on* *A*
⟨*proof*⟩

lemma *Polygamma-approx-holomorphic* [*holomorphic-intros*]:
Polygamma-approx *j* *m* *holomorphic-on* $\{s.\ Re\ s > 0\}$
⟨*proof*⟩

lemma *higher-deriv-lnGamma-stirling*:
assumes *m*: $m > 0$
shows (λx::*real*. (*deriv* ^^ *j*) *ln-Gamma* *x* − *Polygamma-approx* *j* *m* *x*) ∈ $O(\lambda x.\ 1\ /\ x\ \hat{}\ (m + j))$
⟨*proof*⟩

lemma *Polygamma-approx-1-real'*:
assumes *x*: $(x::real) > 0$ **and** *m*: $m > 0$
shows *Polygamma-approx* *1* *m* *x* = *ln* *x* − ($\sum k = Suc\ 0..m.$ *bernoulli'* *k* ∗ *inverse* *x* ^ *k* / *real* *k*)
⟨*proof*⟩

theorem
assumes *m*: $m > 0$
shows *ln-Gamma-real-asymptotics*:
(λx. *ln-Gamma* *x* − ((*x* − *1* / *2*) ∗ *ln* *x* − *x* + *ln* (*2* ∗ *pi*) / *2* +
($\sum k = 1..<m.$ *bernoulli* (*Suc* *k*) / (*real* *k* ∗ *real* (*Suc* *k*)) / *x*^*k*)))
∈ $O(\lambda x.\ 1\ /\ x\ \hat{}\ m)$ (**is** *?th1*)
and *Digamma-real-asymptotics*:

$(\lambda x.\ \mathit{Digamma}\ x - (\mathit{ln}\ x - (\sum k{=}1..m.\ \mathit{bernoulli}'\ k\ /\ \mathit{real}\ k\ /\ x\ \hat{}\ k)))$
$\in O(\lambda x.\ 1\ /\ (x\ \hat{}\ \mathit{Suc}\ m))$ (**is** *?th2*)
and *Polygamma-real-asymptotics*: $j > 0 \Longrightarrow$
$(\lambda x.\ \mathit{Polygamma}\ j\ x - (-\ 1)\ \hat{}\ \mathit{Suc}\ j * (\sum k{\le}m.\ \mathit{bernoulli}'\ k *$
$\mathit{pochhammer}\ (\mathit{real}\ (\mathit{Suc}\ k))\ (j - 1)\ /\ x\ \hat{}\ (k + j)))$
$\in O(\lambda x.\ 1\ /\ x\ \hat{}\ (m{+}j{+}1))$ (**is** - $\Longrightarrow$ *?th3*)
⟨*proof*⟩

2.5 Asymptotics of the complex Gamma function

The m-th order remainder of Stirling's formula for $\log \Gamma$ is $O(s^{-m})$ uniformly over any complex cone $\mathrm{Arg}(z) \le \alpha$, $z \ne 0$ for any angle $\alpha \in (0, \pi)$. This means that there is bounded by cz^{-m} for some constant c for all z in this cone.

context
fixes F **and** α
assumes α: $\alpha \in \{0{<}..{<}pi\}$
defines $F \equiv \mathit{principal}\ (\mathit{complex\text{-}cone}'\ \alpha - \{0\})$
begin

lemma *stirling-integral-bigo*:
fixes $m :: \mathit{nat}$
assumes m: $m > 0$
shows $\mathit{stirling\text{-}integral}\ m \in O[F](\lambda s.\ 1\ /\ s\ \hat{}\ m)$
⟨*proof*⟩

end

The following is a more explicit statement of this:

theorem *ln-Gamma-complex-asymptotics-explicit*:
fixes $m :: \mathit{nat}$ **and** $\alpha :: \mathit{real}$
assumes $m > 0$ **and** $\alpha \in \{0{<}..{<}pi\}$
obtains $C :: \mathit{real}$ **and** $R :: \mathit{complex} \Rightarrow \mathit{complex}$
where $\forall s{::}\mathit{complex}.\ s \notin \mathbb{R}_{\le 0} \longrightarrow$
$\mathit{ln\text{-}Gamma}\ s = (s - 1/2) * \mathit{ln}\ s - s + \mathit{ln}\ (2 * \mathit{pi})\ /\ 2 +$
$(\sum k{=}1..{<}m.\ \mathit{bernoulli}\ (k{+}1)\ /\ (k * (k{+}1) * s\ \hat{}\ k)) - R\ s$
and $\forall s.\ s \ne 0 \wedge |\mathit{Arg}\ s| \le \alpha \longrightarrow \mathit{norm}\ (R\ s) \le C\ /\ \mathit{norm}\ s\ \hat{}\ m$
⟨*proof*⟩

Lastly, we can also derive the asymptotics of Γ itself:

$$\Gamma(z) \sim \sqrt{2\pi/z}\left(\frac{z}{e}\right)^z$$

uniformly for $|z| \to \infty$ within the cone $\mathrm{Arg}(z) \le \alpha$ for $\alpha \in (0, \pi)$:

context
fixes F **and** α
assumes α: $\alpha \in \{0{<}..{<}pi\}$

defines $F \equiv$ *inf at-infinity* (*principal* (*complex-cone'* α))
begin

lemma *Gamma-complex-asymp-equiv*:
Gamma $\sim$[F] (λ*s. sqrt* (*2* $*$ *pi*) $*$ (*s* / *exp 1*) *powr s* / *s powr* (*1* / *2*))
⟨*proof*⟩

end

end

References

[1] B. Berndt. *Rudiments of the Theory of the Gamma Function*. University of Chicago, 1976.

[2] A. Erdélyi. *Asymptotic Expansions*. Dover Publications, 1956.

[3] G. J. O. Jameson. A simple proof of Stirling's formula for the Gamma function. *The Mathematical Gazette*, 99:68–74, 3 2015.