Stirling's formula

Manuel Eberl

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Abstract

This work contains a proof of Stirling's formula both for the factorial $n! \sim \sqrt{2\pi n}(n/e)^n$ on natural numbers and the real Gamma function $\Gamma(x) \sim \sqrt{2\pi/x}(x/e)^x$. The proof is based on work by Graham Jameson [3].

This is then used to derive the complete asymptotic expansion of the logarithmic Gamma function and its derivatives (the Polygamma functions) in terms of Bernoulli numbers.

Contents

1 Stirling's Formula

theory *Stirling-Formula*
imports
HOL−Analysis.Analysis
Landau-Symbols.Landau-Symbols
begin

context
begin

First, we define the S_n^* from Jameson's article:

private definition S' :: *nat* ⇒ *real* ⇒ *real* **where**
$S'\ n\ x = 1/(2{*}x) + (\sum r{=}1..{<}n.\ 1/(\textit{of-nat}\ r{+}x)) + 1\ /(2{*}(n + x))$

Next, the trapezium (also called T in Jameson's article):

private definition T :: *real* ⇒ *real* **where**
$T\ x = 1/(2{*}x) + 1/(2{*}(x{+}1))$

Now we define The difference $\Delta(x)$:

private definition D :: *real* ⇒ *real* **where**
  $D\ x = T\ x - ln\ (x + 1) + ln\ x$

private lemma *S′-telescope-trapezium*:
  assumes $n > 0$
  shows $S'\ n\ x = (\sum r<n.\ T\ (of\text{-}nat\ r + x))$
⟨*proof*⟩ **lemma** *stirling-trapezium*:
  assumes x: $(x::real) > 0$
  shows $D\ x \in \{0\ ..\ 1/(12*x\hat{\ }2) - 1/(12 * (x+1)\hat{\ }2)\}$
⟨*proof*⟩

The following functions correspond to the $p_n(x)$ from the article. The special case $n = 0$ would not, strictly speaking, be necessary, but it allows some theorems to work even without the precondition $n \neq 0$:

private definition p :: *nat* ⇒ *real* ⇒ *real* **where**
  $p\ n\ x = (if\ n = 0\ then\ 1/x\ else\ (\sum r<n.\ D\ (real\ r + x)))$

We can write the Digamma function in terms of S':

private lemma *S′-LIMSEQ-Digamma*:
  assumes x: $x \neq 0$
  shows $(\lambda n.\ ln\ (real\ n) - S'\ n\ x - 1/(2*x)) \longrightarrow Digamma\ x$
⟨*proof*⟩

Moreover, we can give an expansion of S' with the p as variation terms.

private lemma *S′-approx*:
  $S'\ n\ x = ln\ (real\ n + x) - ln\ x + p\ n\ x$
⟨*proof*⟩

We define the limit of the p (simply called $p(x)$ in Jameson's article):

private definition P :: *real* ⇒ *real* **where**
  $P\ x = (\sum n.\ D\ (real\ n + x))$

private lemma *D-summable*:
  assumes x: $x > 0$
  shows $summable\ (\lambda n.\ D\ (real\ n + x))$
⟨*proof*⟩ **lemma** *p-LIMSEQ*:
  assumes x: $x > 0$
  shows $(\lambda n.\ p\ n\ x) \longrightarrow P\ x$
⟨*proof*⟩

This gives us an expansion of the Digamma function:

lemma *Digamma-approx*:
  assumes x: $(x :: real) > 0$
  shows $Digamma\ x = ln\ x - 1\ /\ (2 * x) - P\ x$
⟨*proof*⟩

Next, we derive some bounds on P:

private lemma *p-ge-0*: $x > 0 \Longrightarrow p\ n\ x \geq 0$
⟨*proof*⟩ **lemma** *P-ge-0*: $x > 0 \Longrightarrow P\ x \geq 0$
⟨*proof*⟩ **lemma** *p-upper-bound*:
assumes $x > 0$ $n > 0$
shows $p\ n\ x \leq 1/(12{*}x\hat{\ }2)$
⟨*proof*⟩ **lemma** *P-upper-bound*:
assumes $x > 0$
shows $P\ x \leq 1/(12{*}x\hat{\ }2)$
⟨*proof*⟩

We can now use this approximation of the Digamma function to approximate *ln-Gamma* (since the former is the derivative of the latter). We therefore define the function g from Jameson's article, which measures the difference between *ln-Gamma* and its approximation:

private definition g :: *real* ⇒ *real* **where**
$g\ x = \textit{ln-Gamma}\ x - (x - 1/2) * \ln x + x$

private lemma *DERIV-g*: $x > 0 \Longrightarrow (g\ \textit{has-field-derivative}\ -P\ x)\ (at\ x)$
⟨*proof*⟩ **lemma** *isCont-P*:
assumes $x > 0$
shows *isCont P x*
⟨*proof*⟩ **lemma** *P-continuous-on* [*THEN continuous-on-subset*]: *continuous-on* {0<..} *P*
⟨*proof*⟩ **lemma** *P-integrable*:
assumes *a*: $a > 0$
shows *P integrable-on* {*a*..}
⟨*proof*⟩ **definition** c :: *real* **where** $c = \textit{integral}\ \{1..\}\ (\lambda x.\ -P\ x) + g\ 1$

We can now give bounds on g:

private lemma *g-bounds*:
assumes *x*: $x \geq 1$
shows $g\ x \in \{c..c + 1/(12{*}x)\}$
⟨*proof*⟩

Finally, we have bounds on *ln-Gamma*, *Gamma*, and *fact*.

private lemma *ln-Gamma-bounds-aux*:
$x \geq 1 \Longrightarrow \textit{ln-Gamma}\ x \geq c + (x - 1/2) * \ln x - x$
$x \geq 1 \Longrightarrow \textit{ln-Gamma}\ x \leq c + (x - 1/2) * \ln x - x + 1/(12{*}x)$
⟨*proof*⟩ **lemma** *Gamma-bounds-aux*:
assumes *x*: $x \geq 1$
shows $\textit{Gamma}\ x \geq \exp c * x\ \textit{powr}\ (x - 1/2) / \exp x$
$\textit{Gamma}\ x \leq \exp c * x\ \textit{powr}\ (x - 1/2) / \exp x * \exp (1/(12{*}x))$
⟨*proof*⟩ **lemma** *Gamma-asymp-equiv-aux*:
$\textit{Gamma} \sim (\lambda x.\ \exp c * x\ \textit{powr}\ (x - 1/2) / \exp x)$
⟨*proof*⟩ **lemma** *exp-1-powr-real* [*simp*]: $\exp (1{::}\textit{real})\ \textit{powr}\ x = \exp x$
⟨*proof*⟩ **lemma** *fact-asymp-equiv-aux*:
$\textit{fact} \sim (\lambda x.\ \exp c * \textit{sqrt}\ (\textit{real}\ x) * (\textit{real}\ x / \exp 1)\ \textit{powr}\ \textit{real}\ x)$
⟨*proof*⟩

We still need to determine the constant term c, which we do using Wallis' product formula:

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \frac{\pi}{2}$$

private lemma *powr-mult-2*: *(x::real) > 0* $\Longrightarrow$ *x powr (y * 2) = (x^2) powr y*
⟨*proof*⟩ **lemma** *exp-mult-2*: *exp (y * 2 :: real) = exp y * exp y*
⟨*proof*⟩ **lemma** *exp-c*: *exp c = sqrt (2*pi)*
⟨*proof*⟩ **lemma** *c*: *c = ln (2*pi) / 2*
⟨*proof*⟩

This gives us the final bounds:

theorem *Gamma-bounds*:
assumes *x* ≥ *1*
shows *Gamma x* ≥ *sqrt (2*pi/x) * (x / exp 1) powr x* (**is** *?th1*)
Gamma x ≤ *sqrt (2*pi/x) * (x / exp 1) powr x * exp (1 / (12 * x))* (**is** *?th2*)
⟨*proof*⟩

theorem *ln-Gamma-bounds*:
assumes *x* ≥ *1*
shows *ln-Gamma x* ≥ *ln (2*pi/x) / 2 + x * ln x − x* (**is** *?th1*)
ln-Gamma x ≤ *ln (2*pi/x) / 2 + x * ln x − x + 1/(12*x)* (**is** *?th2*)
⟨*proof*⟩

theorem *fact-bounds*:
assumes *n > 0*
shows *(fact n :: real)* ≥ *sqrt (2*pi*n) * (n / exp 1) ^ n* (**is** *?th1*)
(fact n :: real) ≤ *sqrt (2*pi*n) * (n / exp 1) ^ n * exp (1 / (12 * n))* (**is** *?th2*)
⟨*proof*⟩

theorem *ln-fact-bounds*:
assumes *n > 0*
shows *ln (fact n :: real)* ≥ *ln (2*pi*n)/2 + n * ln n − n* (**is** *?th1*)
ln (fact n :: real) ≤ *ln (2*pi*n)/2 + n * ln n − n + 1/(12*real n)* (**is** *?th2*)
⟨*proof*⟩

theorem *Gamma-asymp-equiv*:
Gamma ∼ *(λx. sqrt (2*pi/x) * (x / exp 1) powr x :: real)*
⟨*proof*⟩

theorem *fact-asymp-equiv*:
fact ∼ *(λn. sqrt (2*pi*n) * (n / exp 1) ^ n :: real)*
⟨*proof*⟩

end

end

2 Complete asymptotics of the logarithmic Gamma function

theory *Ln-Gamma-Asymptotics*
imports
HOL−Analysis.Analysis
Bernoulli.Bernoulli-FPS
Bernoulli.Periodic-Bernpoly
Stirling-Formula
begin

2.1 Auxiliary Facts

lemma *filterlim-at-infinity-conv-norm-at-top*:
$filterlim\ f\ at\text{-}infinity\ G \longleftrightarrow filterlim\ (\lambda x.\ norm\ (f\ x))\ at\text{-}top\ G$
⟨*proof*⟩

corollary *Ln-times-of-nat*:
$[\![r > 0;\ z \neq 0]\!] \Longrightarrow Ln(of\text{-}nat\ r * z :: complex) = ln\ (of\text{-}nat\ r) + Ln(z)$
⟨*proof*⟩

lemma *tendsto-of-real-0-I*:
$(f \longrightarrow 0)\ G \Longrightarrow ((\lambda x.\ (of\text{-}real\ (f\ x))) \longrightarrow (0 ::'a::real\text{-}normed\text{-}div\text{-}algebra))\ G$
⟨*proof*⟩

lemma *negligible-atLeastAtMostI*: $b \le a \Longrightarrow negligible\ \{a..(b::real)\}$
⟨*proof*⟩

lemma *integrable-on-negligible*:
$negligible\ A \Longrightarrow (f :: 'n :: euclidean\text{-}space \Rightarrow 'a :: banach)\ integrable\text{-}on\ A$
⟨*proof*⟩

lemma *vector-derivative-cong-eq*:
assumes $eventually\ (\lambda x.\ x \in A \longrightarrow f\ x = g\ x)\ (nhds\ x)\ x = y\ A = B\ x \in A$
shows $vector\text{-}derivative\ f\ (at\ x\ within\ A) = vector\text{-}derivative\ g\ (at\ y\ within\ B)$
⟨*proof*⟩

lemma *differentiable-of-real* [*simp*]: $of\text{-}real\ differentiable\ at\ x\ within\ A$
⟨*proof*⟩

lemma *higher-deriv-cong-ev*:
assumes $eventually\ (\lambda x.\ f\ x = g\ x)\ (nhds\ x)\ x = y$
shows $(deriv\ \hat{}\hat{}\ n)\ f\ x = (deriv\ \hat{}\hat{}\ n)\ g\ y$
⟨*proof*⟩

lemma *deriv-of-real* [*simp*]:
at x within A $\neq$ *bot* $\Longrightarrow$ *vector-derivative of-real* (*at x within A*) = *1*
⟨*proof*⟩

lemma *deriv-Re* [*simp*]: *deriv Re* = (λ-. *1*)
⟨*proof*⟩

lemma *vector-derivative-of-real-left*:
assumes *f differentiable at x*
shows *vector-derivative* (λx. *of-real* (*f x*)) (*at x*) = *of-real* (*deriv f x*)
⟨*proof*⟩

lemma *vector-derivative-of-real-right*:
assumes *f field-differentiable at* (*of-real x*)
shows *vector-derivative* (λx. *f* (*of-real x*)) (*at x*) = *deriv f* (*of-real x*)
⟨*proof*⟩

lemma *Ln-holomorphic* [*holomorphic-intros*]:
assumes $A \cap \mathbb{R}_{\leq 0} = \{\}$
shows *Ln holomorphic-on* (*A* :: *complex set*)
⟨*proof*⟩

lemma *ln-Gamma-holomorphic* [*holomorphic-intros*]:
assumes $A \cap \mathbb{R}_{\leq 0} = \{\}$
shows *ln-Gamma holomorphic-on* (*A* :: *complex set*)
⟨*proof*⟩

lemma *higher-deriv-Polygamma*:
assumes $z \notin \mathbb{Z}_{\leq 0}$
shows (*deriv* ^^ *n*) (*Polygamma m*) *z* =
Polygamma (*m* + *n*) (*z* :: *'a* :: {*real-normed-field*,*euclidean-space*})
⟨*proof*⟩

lemma *higher-deriv-cmult*:
assumes *f holomorphic-on A x* $\in$ *A open A*
shows (*deriv* ^^ *j*) (λx. *c* ∗ *f x*) *x* = *c* ∗ (*deriv* ^^ *j*) *f x*
⟨*proof*⟩

lemma *higher-deriv-ln-Gamma-complex*:
assumes (*x*::*complex*) $\notin \mathbb{R}_{\leq 0}$
shows (*deriv* ^^ *j*) *ln-Gamma x* = (*if j* = *0 then ln-Gamma x else Polygamma* (*j* − *1*) *x*)
⟨*proof*⟩

lemma *higher-deriv-ln-Gamma-real*:
assumes (*x*::*real*) > *0*
shows (*deriv* ^^ *j*) *ln-Gamma x* = (*if j* = *0 then ln-Gamma x else Polygamma* (*j* − *1*) *x*)

⟨proof⟩

lemma *higher-deriv-ln-Gamma-complex-of-real*:
assumes $(x :: real) > 0$
shows (*deriv* ^^ *j*) *ln-Gamma* (*complex-of-real x*) = *of-real* ((*deriv* ^^ *j*) *ln-Gamma x*)
⟨proof⟩

lemma *stirling-limit-aux1*:
$((\lambda y.\ Ln\ (1 + z * of\text{-}real\ y)\ /\ of\text{-}real\ y) \longrightarrow z)\ (at\text{-}right\ 0)$ **for** $z :: complex$
⟨proof⟩

lemma *stirling-limit-aux2*:
$((\lambda y.\ y * Ln\ (1 + z\ /\ of\text{-}real\ y)) \longrightarrow z)\ at\text{-}top$ **for** $z :: complex$
⟨proof⟩

lemma *Union-atLeastAtMost*:
assumes $N > 0$
shows $(\bigcup n\in\{0..<N\}.\ \{real\ n..real\ (n + 1)\}) = \{0..real\ N\}$
⟨proof⟩

2.2 Asymptotics of *ln-Gamma*

This is the error term that occurs in the expansion of *ln-Gamma*. It can be shown to be of order $O(s^{-n})$.

definition *stirling-integral* :: *nat* ⇒ *'a* :: {*real-normed-div-algebra*, *banach*} ⇒ *'a*
where
stirling-integral n s =
lim (λN. *integral* {0..N} (λx. *of-real* (*pbernpoly n x*) / (*of-real x* + *s*) ^ *n*))

context
fixes *s* :: *complex* **assumes** *s*: *Re s* > *0*
fixes *approx* :: *nat* ⇒ *complex*
defines *approx* ≡ (λN.
($\sum$ *n* = *1..<N*. *s* / *of-nat n* − *ln* (*1* + *s* / *of-nat n*)) − (*euler-mascheroni* ∗ *s* + *ln s*) − (∗ ⟶ *ln-Gamma s* ∗)
(*ln-Gamma* (*of-nat N*) − *ln* (*2* ∗ *pi* / *of-nat N*) / *2* − *of-nat N* ∗ *ln* (*of-nat N*) + *of-nat N*) − (∗ ⟶ *0* ∗)
s ∗ (*harm* (*N* − *1*) − *ln* (*of-nat* (*N* − *1*)) − *euler-mascheroni*) + (∗ ⟶ *0* ∗)
s ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat* (*N* − *1*))) − (∗ ⟶ *0* ∗)
(*1/2*) ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat N*)) + (∗ ⟶ *0* ∗)
of-nat N ∗ (*ln* (*of-nat N* + *s*) − *ln* (*of-nat N*)) − (∗ ⟶ *s* ∗)
(*s* − *1/2*) ∗ *ln s* − *ln* (*2* ∗ *pi*) / *2*)
begin

qualified lemma
assumes *N*: *N* > *0*
shows *integrable-pbernpoly-1*:

$(\lambda x.\ \textit{of-real}\ (-\textit{pbernpoly 1 x})\ /\ (\textit{of-real x} + s))$ *integrable-on* $\{0..\textit{real } N\}$
and *integral-pbernpoly-1-aux*:
$\textit{integral}\ \{0..\textit{real } N\}\ (\lambda x.\ -\textit{of-real}\ (\textit{pbernpoly 1 x})\ /\ (\textit{of-real x} + s)) =$ *approx N*
and *has-integral-pbernpoly-1*:
$((\lambda x.\ \textit{pbernpoly 1 x}\ /(x + s))$ *has-integral*
$(\sum m<N.\ (\textit{of-nat m} + 1\ /\ 2 + s) * (\textit{ln}\ (\textit{of-nat m} + s) -$
$\textit{ln}\ (\textit{of-nat m} + 1 + s)) + 1))\ \{0..\textit{real } N\}$
⟨*proof*⟩

lemma *integrable-ln-Gamma-aux*:
shows $(\lambda x.\ \textit{of-real}\ (\textit{pbernpoly n x})\ /\ (\textit{of-real x} + s)\ \hat{}\ n)$ *integrable-on* $\{0..\textit{real } N\}$
⟨*proof*⟩

This following proof is based on "Rudiments of the theory of the gamma function" by Bruce Berndt [1].

qualified lemma *integral-pbernpoly-1*:
$(\lambda N.\ \textit{integral}\ \{0..\textit{real } N\}\ (\lambda x.\ \textit{pbernpoly 1 x}\ /\ (x + s)))$
$\longrightarrow -\textit{ln-Gamma s} - s + (s - 1\ /\ 2) * \textit{ln s} + \textit{ln}\ (2 * \textit{pi})\ /\ 2$
⟨*proof*⟩ **lemma** *pbernpoly-integral-conv-pbernpoly-integral-Suc*:
assumes $n \geq 1$
shows $\textit{integral}\ \{0..\textit{real } N\}\ (\lambda x.\ \textit{pbernpoly n x}\ /\ (x + s)\ \hat{}\ n) =$
$\textit{of-real}\ (\textit{pbernpoly}\ (\textit{Suc n})\ (\textit{real N}))\ /\ (\textit{of-nat}\ (\textit{Suc n}) * (s + \textit{of-nat N})\ \hat{}\ n) -$
$\textit{of-real}\ (\textit{bernoulli}\ (\textit{Suc n}))\ /\ (\textit{of-nat}\ (\textit{Suc n}) * s\ \hat{}\ n) + \textit{of-nat n}\ /\ \textit{of-nat}\ (\textit{Suc n}) *$
$\textit{integral}\ \{0..\textit{real } N\}\ (\lambda x.\ \textit{of-real}\ (\textit{pbernpoly}\ (\textit{Suc n})\ x)\ /\ (\textit{of-real x} + s)\ \hat{}\ \textit{Suc n})$
⟨*proof*⟩

lemma *pbernpoly-over-power-tendsto-0*:
assumes $n > 0$
shows $(\lambda x.\ \textit{of-real}\ (\textit{pbernpoly}\ (\textit{Suc n})\ (\textit{real x}))\ /\ (\textit{of-nat}\ (\textit{Suc n}) * (s + \textit{of-nat x})\ \hat{}\ n)) \longrightarrow 0$
⟨*proof*⟩

lemma *convergent-stirling-integral*:
assumes $n > 0$
shows *convergent* $(\lambda N.\ \textit{integral}\ \{0..\textit{real } N\}$
$(\lambda x.\ \textit{of-real}\ (\textit{pbernpoly n x})\ /\ (\textit{of-real x} + s)\ \hat{}\ n))$ (**is** *convergent* (*?f n*))
⟨*proof*⟩

lemma *stirling-integral-conv-stirling-integral-Suc*:
assumes $n > 0$
shows *stirling-integral n s* =
$\textit{of-nat n}\ /\ \textit{of-nat}\ (\textit{Suc n}) * \textit{stirling-integral}\ (\textit{Suc n})\ s -$
$\textit{of-real}\ (\textit{bernoulli}\ (\textit{Suc n}))\ /\ (\textit{of-nat}\ (\textit{Suc n}) * s\ \hat{}\ n)$
⟨*proof*⟩

lemma *stirling-integral-1-unfold*:
assumes $m > 0$
shows *stirling-integral 1 s = stirling-integral m s / of-nat m* $-$
($\sum$ *k=1..<m. of-real (bernoulli (Suc k)) / (of-nat k * of-nat (Suc k) *
s ˆ k))*
⟨*proof*⟩

lemma *ln-Gamma-stirling-complex*:
assumes $m > 0$
shows *ln-Gamma s = (s − 1 / 2) * ln s − s + ln (2 * pi) / 2 +*
($\sum$ *k=1..<m. of-real (bernoulli (Suc k)) / (of-nat k * of-nat (Suc k) *
s ˆ k))* $-$
stirling-integral m s / of-nat m
⟨*proof*⟩

lemma *LIMSEQ-stirling-integral*:
n > 0 $\Longrightarrow$ *(λx. integral {0..real x} (λx. of-real (pbernpoly n x) / (of-real x + s)
ˆ n))*
$\longrightarrow$ *stirling-integral n s* ⟨*proof*⟩

end

lemmas *has-integral-of-real = has-integral-linear[OF - bounded-linear-of-real, unfolded o-def]*
lemmas *integral-of-real = integral-linear[OF - bounded-linear-of-real, unfolded o-def]*

lemma *integrable-ln-Gamma-aux-real*:
assumes $0 < s$
shows *(λx. pbernpoly n x / (x + s) ˆ n) integrable-on {0..real N}*
⟨*proof*⟩

lemma
assumes *x > 0 n > 0*
shows *stirling-integral-complex-of-real*:
stirling-integral n (complex-of-real x) = of-real (stirling-integral n x)
and *LIMSEQ-stirling-integral-real*:
(λN. integral {0..real N} (λt. pbernpoly n t / (t + x) ˆ n))
$\longrightarrow$ *stirling-integral n x*
and *stirling-integral-real-convergent*:
convergent (λN. integral {0..real N} (λt. pbernpoly n t / (t + x) ˆ n))
⟨*proof*⟩

lemma *ln-Gamma-stirling-real*:
assumes *x > (0 :: real) m > (0::nat)*
shows *ln-Gamma x = (x − 1 / 2) * ln x − x + ln (2 * pi) / 2 +*
($\sum$ *k = 1..<m. bernoulli (Suc k) / (of-nat k * of-nat (Suc k) * x ˆ k))*
$-$
stirling-integral m x / of-nat m

⟨proof⟩

context
begin

private lemma *stirling-integral-bound-aux*:
assumes *n*: $n > (1::nat)$
obtains *c* **where** $\bigwedge s.\ Re\ s > 0 \Longrightarrow norm\ (stirling\text{-}integral\ n\ s) \leq\ c\ /\ Re\ s\ \hat{}\ (n - 1)$
⟨proof⟩

lemma *stirling-integral-bound*:
assumes $n > 0$
obtains *c* **where**
$\bigwedge s.\ Re\ s > 0 \Longrightarrow norm\ (stirling\text{-}integral\ n\ s) \leq c\ /\ Re\ s\ \hat{}\ n$
⟨proof⟩

end

lemma *stirling-integral-holomorphic* [*holomorphic-intros*]:
assumes *m*: $m > 0$ **and** $\forall s\in A.\ Re\ s > 0$
shows *stirling-integral m holomorphic-on A*
⟨proof⟩

lemma *stirling-integral-continuous-on* [*continuous-intros*]:
assumes *m*: $m > 0$ **and** $\forall s\in A.\ Re\ s > 0$
shows *continuous-on A* (*stirling-integral m*)
⟨proof⟩

lemma *has-field-derivative-stirling-integral*:
assumes $Re\ x > 0\ n > 0$
shows (*stirling-integral n has-field-derivative deriv* (*stirling-integral n*) *x*) (*at x*)
⟨proof⟩

lemma
assumes *n*: $n > 0$ **and** $x > 0$
shows *deriv-stirling-integral-complex-of-real*:
(*deriv* ^^ *j*) (*stirling-integral n*) (*complex-of-real x*) =
complex-of-real ((*deriv* ^^ *j*) (*stirling-integral n*) *x*) (**is** *?lhs x* = *?rhs x*)
and *differentiable-stirling-integral-real*:
(*deriv* ^^ *j*) (*stirling-integral n*) *field-differentiable at x* (**is** *?thesis2*)
⟨proof⟩

Unfortunately, asymptotic power series cannot, in general, be differentiated.

However, since *ln-Gamma* is holomorphic on the entire positive real half-space, we can differentiate its asymptotic expansion after all.

To do this, we use an ad-hoc version of the more general approach outlined in Erdelyi's "Asymptotic Expansions" for holomorphic functions: We bound the value of the j-th derivative of the remainder term at some value x by applying Cauchy's integral formula along a circle centred at x with radius $\frac{1}{2}x$.

lemma *deriv-stirling-integral-real-bound*:
  assumes *m*: *m > 0*
  shows (*deriv ^^ j*) (*stirling-integral m*) ∈ *O*(λ*x*::*real*. *1* / *x* ^ (*m* + *j*))
⟨*proof*⟩

definition *stirling-sum* **where**
  stirling-sum j m x =
    (−*1*) ^ *j* ∗ (∑ *k* = *1*..<*m*. (*of-real* (*bernoulli* (*Suc k*)) ∗ *pochhammer* (*of-nat k*) *j* / (*of-nat k* ∗
      of-nat (*Suc k*))) ∗ *inverse x* ^ (*k* + *j*))

definition *stirling-sum′* **where**
  stirling-sum′ j m x =
    (−*1*) ^ (*Suc j*) ∗ (∑ *k*≤*m*. (*of-real* (*bernoulli′ k*) ∗
      pochhammer (*of-nat* (*Suc k*)) (*j* − *1*) ∗ *inverse x* ^ (*k* + *j*)))

lemma *stirling-sum-complex-of-real*:
  stirling-sum j m (*complex-of-real x*) = *complex-of-real* (*stirling-sum j m x*)
  ⟨*proof*⟩

lemma *stirling-sum′-complex-of-real*:
  stirling-sum′ j m (*complex-of-real x*) = *complex-of-real* (*stirling-sum′ j m x*)
  ⟨*proof*⟩

lemma *has-field-derivative-stirling-sum-complex* [*derivative-intros*]:
  Re x > *0* ⟹ (*stirling-sum j m has-field-derivative stirling-sum* (*Suc j*) *m x*) (*at x*)
    ⟨*proof*⟩

lemma *has-field-derivative-stirling-sum-real* [*derivative-intros*]:
  x > (*0*::*real*) ⟹ (*stirling-sum j m has-field-derivative stirling-sum* (*Suc j*) *m x*) (*at x*)
    ⟨*proof*⟩

lemma *has-field-derivative-stirling-sum′-complex* [*derivative-intros*]:
  assumes *j* > *0 Re x* > *0*
  shows (*stirling-sum′ j m has-field-derivative stirling-sum′* (*Suc j*) *m x*) (*at x*)
⟨*proof*⟩

lemma *has-field-derivative-stirling-sum′-real* [*derivative-intros*]:
  assumes *j* > *0 x* > (*0*::*real*)

shows $(stirling\text{-}sum'\ j\ m\ has\text{-}field\text{-}derivative\ stirling\text{-}sum'\ (Suc\ j)\ m\ x)\ (at\ x)$
$\langle proof \rangle$

lemma *higher-deriv-stirling-sum-complex*:
$Re\ x > 0 \Longrightarrow (deriv\ \hat{}\hat{}\ i)\ (stirling\text{-}sum\ j\ m)\ x = stirling\text{-}sum\ (i + j)\ m\ x$
$\langle proof \rangle$

definition $Polygamma\text{-}approx :: nat \Rightarrow nat \Rightarrow 'a \Rightarrow 'a :: \{real\text{-}normed\text{-}field,\ ln\}$
where
$Polygamma\text{-}approx\ j\ m =$
$(deriv\ \hat{}\hat{}\ j)\ (\lambda x.\ (x - 1\ /\ 2) * ln\ x - x + of\text{-}real\ (ln\ (2 * pi))\ /\ 2 + stirling\text{-}sum\ 0\ m\ x)$

lemma *Polygamma-approx-Suc*: $Polygamma\text{-}approx\ (Suc\ j)\ m = deriv\ (Polygamma\text{-}approx\ j\ m)$
$\langle proof \rangle$

lemma *Polygamma-approx-0*:
$Polygamma\text{-}approx\ 0\ m\ x = (x - 1/2) * ln\ x - x + of\text{-}real\ (ln\ (2*pi))\ /\ 2 + stirling\text{-}sum\ 0\ m\ x$
$\langle proof \rangle$

lemma *Polygamma-approx-1-complex*:
$Re\ x > 0 \Longrightarrow$
$Polygamma\text{-}approx\ (Suc\ 0)\ m\ x = ln\ x - 1\ /\ (2*x) + stirling\text{-}sum\ (Suc\ 0)\ m\ x$
$\langle proof \rangle$

lemma *Polygamma-approx-1-real*:
$x > (0 :: real) \Longrightarrow$
$Polygamma\text{-}approx\ (Suc\ 0)\ m\ x = ln\ x - 1\ /\ (2*x) + stirling\text{-}sum\ (Suc\ 0)\ m\ x$
$\langle proof \rangle$

lemma *stirling-sum-2-conv-stirling-sum'-1*:
fixes $x :: 'a :: \{real\text{-}div\text{-}algebra,\ field\text{-}char\text{-}0\}$
assumes $m > 0\ x \neq 0$
shows $stirling\text{-}sum'\ 1\ m\ x = 1\ /\ x + 1\ /\ (2 * x\hat{}2) + stirling\text{-}sum\ 2\ m\ x$
$\langle proof \rangle$

lemma *Polygamma-approx-2-real*:
assumes $x > (0::real)\ m > 0$
shows $Polygamma\text{-}approx\ (Suc\ (Suc\ 0))\ m\ x = stirling\text{-}sum'\ 1\ m\ x$
$\langle proof \rangle$

lemma *Polygamma-approx-2-complex*:
assumes $Re\ x > 0\ m > 0$
shows $Polygamma\text{-}approx\ (Suc\ (Suc\ 0))\ m\ x = stirling\text{-}sum'\ 1\ m\ x$

$\langle proof \rangle$

lemma *Polygamma-approx-ge-2-real*:
assumes $x > (0::real)$ $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc* *j*)) *m* *x* = *stirling-sum'* (*Suc* *j*) *m* *x*
$\langle proof \rangle$

lemma *Polygamma-approx-ge-2-complex*:
assumes *Re* $x > 0$ $m > 0$
shows *Polygamma-approx* (*Suc* (*Suc* *j*)) *m* *x* = *stirling-sum'* (*Suc* *j*) *m* *x*
$\langle proof \rangle$

lemma *Polygamma-approx-complex-of-real*:
assumes $x > 0$ $m > 0$
shows *Polygamma-approx j m* (*complex-of-real x*) = *of-real* (*Polygamma-approx j m x*)
$\langle proof \rangle$

lemma *higher-deriv-Polygamma-approx* [*simp*]:
(*deriv* ^^ *j*) (*Polygamma-approx i m*) = *Polygamma-approx* (*j* + *i*) *m*
$\langle proof \rangle$

lemma *stirling-sum-holomorphic* [*holomorphic-intros*]:
$0 \notin A \Longrightarrow$ *stirling-sum j m holomorphic-on A*
$\langle proof \rangle$

lemma *Polygamma-approx-holomorphic* [*holomorphic-intros*]:
Polygamma-approx j m holomorphic-on {*s*. *Re* $s > 0$}
$\langle proof \rangle$

lemma *higher-deriv-lnGamma-stirling*:
assumes *m*: $m > 0$
shows (λx::*real*. (*deriv* ^^ *j*) *ln-Gamma x* − *Polygamma-approx j m x*) ∈ *O*(λx. 1 / *x* ^ (*m* + *j*))
$\langle proof \rangle$

lemma *Polygamma-approx-1-real'*:
assumes *x*: (*x*::*real*) > 0 **and** *m*: $m > 0$
shows *Polygamma-approx* 1 *m x* = *ln x* − ($\sum$*k* = *Suc* 0..*m*. *bernoulli'* *k* ∗ *inverse x* ^ *k* / *real k*)
$\langle proof \rangle$

theorem
assumes *m*: $m > 0$
shows *ln-Gamma-real-asymptotics*:
(λx. *ln-Gamma x* − ((*x* − 1 / 2) ∗ *ln x* − *x* + *ln* (2 ∗ *pi*) / 2 +
($\sum$*k* = 1..<*m*. *bernoulli* (*Suc k*) / (*real k* ∗ *real* (*Suc k*)) / *x*^*k*)))
∈ *O*(λx. 1 / *x* ^ *m*) (**is** *?th1*)
and *Digamma-real-asymptotics*:

$(\lambda x.\ \mathit{Digamma}\ x - (\ln x - (\sum k{=}1..m.\ \mathit{bernoulli}'\ k\ /\ \mathit{real}\ k\ /\ x\ \hat{}\ k)))$
$\in O(\lambda x.\ 1\ /\ (x\ \hat{}\ \mathit{Suc}\ m))$ (**is** *?th2*)
and *Polygamma-real-asymptotics*: $j > 0 \Longrightarrow$
$(\lambda x.\ \mathit{Polygamma}\ j\ x - (-\ 1)\ \hat{}\ \mathit{Suc}\ j * (\sum k{\le}m.\ \mathit{bernoulli}'\ k *$
$\mathit{pochhammer}\ (\mathit{real}\ (\mathit{Suc}\ k))\ (j - 1)\ /\ x\ \hat{}\ (k + j)))$
$\in O(\lambda x.\ 1\ /\ x\ \hat{}\ (m{+}j{+}1))$ (**is** - $\Longrightarrow$ *?th3*)
⟨*proof*⟩

end

References

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