Stirling's formula

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Abstract

This work contains a proof of Stirling's formula both for the factorial $n! \sim \sqrt{2\pi n}(n/e)^n$ on natural numbers and the real Gamma function $\Gamma(x) \sim \sqrt{2\pi/x}(x/e)^x$. The proof is based on work by Graham Jameson [1].

Contents

theory *Stirling-Formula*
imports
  ~~/src/HOL/Analysis
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begin

context
begin

First, we define the S_n^* from Jameson's article:

private definition S' :: *nat* ⇒ *real* ⇒ *real* **where**
  $S'\ n\ x = 1/(2{*}x) + (\sum r{=}1..{<}n.\ 1/(\textit{of-nat}\ r{+}x)) + 1\ /(2{*}(n + x))$

Next, the trapezium (also called T in Jameson's article):

private definition T :: *real* ⇒ *real* **where**
  $T\ x = 1/(2{*}x) + 1/(2{*}(x{+}1))$

Now we define The difference $\Delta(x)$:

private definition D :: *real* ⇒ *real* **where**
  $D\ x = T\ x - \ln\ (x + 1) + \ln\ x$

private lemma *S′-telescope-trapezium*:
  assumes $n > 0$
  shows $S'\ n\ x = (\sum r{<}n.\ T\ (\textit{of-nat}\ r + x))$
⟨*proof*⟩ **lemma** *stirling-trapezium*:
  assumes x: $(x{::}real) > 0$
  shows $D\ x \in \{0\ ..\ 1/(12{*}x\hat{}2) - 1/(12 * (x{+}1)\hat{}2)\}$

⟨*proof*⟩

The following functions correspond to the $p_n(x)$ from the article. The special case $n = 0$ would not, strictly speaking, be necessary, but it allows some theorems to work even without the precondition $n \neq 0$:

private definition $p :: nat \Rightarrow real \Rightarrow real$ **where**
$p\ n\ x = (if\ n = 0\ then\ 1/x\ else\ (\sum r{<}n.\ D\ (real\ r + x)))$

We can write the Digamma function in terms of S':

private lemma *S′-LIMSEQ-Digamma*:
assumes x: $x \neq 0$
shows $(\lambda n.\ ln\ (real\ n) - S'\ n\ x - 1/(2{*}x)) \longrightarrow Digamma\ x$
⟨*proof*⟩

Moreover, we can give an expansion of S' with the p as variation terms.

private lemma *S′-approx*:
$S'\ n\ x = ln\ (real\ n + x) - ln\ x + p\ n\ x$
⟨*proof*⟩

We define the limit of the p (simply called $p(x)$ in Jameson's article):

private definition $P :: real \Rightarrow real$ **where**
$P\ x = (\sum n.\ D\ (real\ n + x))$

private lemma *D-summable*:
assumes x: $x > 0$
shows $summable\ (\lambda n.\ D\ (real\ n + x))$
⟨*proof*⟩ **lemma** *p-LIMSEQ*:
assumes x: $x > 0$
shows $(\lambda n.\ p\ n\ x) \longrightarrow P\ x$
⟨*proof*⟩

This gives us an expansion of the Digamma function:

lemma *Digamma-approx*:
assumes x: $(x :: real) > 0$
shows $Digamma\ x = ln\ x - 1\ /\ (2 * x) - P\ x$
⟨*proof*⟩

Next, we derive some bounds on P:

private lemma *p-ge-0*: $x > 0 \Longrightarrow p\ n\ x \geq 0$
⟨*proof*⟩ **lemma** *P-ge-0*: $x > 0 \Longrightarrow P\ x \geq 0$
⟨*proof*⟩ **lemma** *p-upper-bound*:
assumes $x > 0\ n > 0$
shows $p\ n\ x \leq 1/(12{*}x\hat{\ }2)$
⟨*proof*⟩ **lemma** *P-upper-bound*:
assumes $x > 0$
shows $P\ x \leq 1/(12{*}x\hat{\ }2)$
⟨*proof*⟩

We can now use this approximation of the Digamma function to approximate *ln-Gamma* (since the former is the derivative of the latter). We therefore define the function g from Jameson's article, which measures the difference between *ln-Gamma* and its approximation:

private definition *g* :: *real* ⇒ *real* **where**
  g x = *ln-Gamma x* − (*x* − *1/2*) ∗ *ln x* + *x*

private lemma *DERIV-g*: *x* > *0* ⟹ (*g has-field-derivative* −*P x*) (*at x*)
  ⟨*proof*⟩ **lemma** *isCont-P*:
  assumes *x* > *0*
  shows *isCont P x*
⟨*proof*⟩ **lemma** *P-continuous-on* [*THEN continuous-on-subset*]: *continuous-on* {*0*<..} *P*
  ⟨*proof*⟩ **lemma** *P-integrable*:
  assumes *a*: *a* > *0*
  shows *P integrable-on* {*a*..}
⟨*proof*⟩ **definition** *c* :: *real* **where** *c* = *integral* {*1*..} (λ*x*. −*P x*) + *g 1*

We can now give bounds on g:

private lemma *g-bounds*:
  assumes *x*: *x* ≥ *1*
  shows *g x* ∈ {*c*..*c* + *1*/(*12*∗*x*)}
⟨*proof*⟩

Finally, we have bounds on *ln-Gamma*, *Gamma*, and *fact*.

private lemma *ln-Gamma-bounds-aux*:
  x ≥ *1* ⟹ *ln-Gamma x* ≥ *c* + (*x* − *1/2*) ∗ *ln x* − *x*
  x ≥ *1* ⟹ *ln-Gamma x* ≤ *c* + (*x* − *1/2*) ∗ *ln x* − *x* + *1*/(*12*∗*x*)
  ⟨*proof*⟩ **lemma** *Gamma-bounds-aux*:
  assumes *x*: *x* ≥ *1*
  shows *Gamma x* ≥ *exp c* ∗ *x powr* (*x* − *1/2*) / *exp x*
    Gamma x ≤ *exp c* ∗ *x powr* (*x* − *1/2*) / *exp x* ∗ *exp* (*1*/(*12*∗*x*))
⟨*proof*⟩ **lemma** *Gamma-asymp-equiv-aux*:
  Gamma ∼ (λ*x*. *exp c* ∗ *x powr* (*x* − *1/2*) / *exp x*)
⟨*proof*⟩ **lemma** *exp-1-powr-real* [*simp*]: *exp* (*1*::*real*) *powr x* = *exp x*
  ⟨*proof*⟩ **lemma** *fact-asymp-equiv-aux*:
  fact ∼ (λ*x*. *exp c* ∗ *sqrt* (*real x*) ∗ (*real x* / *exp 1*) *powr real x*)
⟨*proof*⟩

We still need to determine the constant term c, which we do using Wallis' product formula:

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \frac{\pi}{2}$$

private lemma *powr-mult-2*: (*x*::*real*) > *0* ⟹ *x powr* (*y* ∗ *2*) = (*x*ˆ*2*) *powr y*
  ⟨*proof*⟩ **lemma** *exp-mult-2*: *exp* (*y* ∗ *2* :: *real*) = *exp y* ∗ *exp y*
  ⟨*proof*⟩ **lemma** *exp-c*: *exp c* = *sqrt* (*2*∗*pi*)

⟨*proof*⟩ **lemma** *c*: *c* = *ln* (*2*∗*pi*) / *2*
⟨*proof*⟩

This gives us the final bounds:

theorem *Gamma-bounds*:
assumes *x* ≥ *1*
shows *Gamma x* ≥ *sqrt* (*2*∗*pi*/*x*) ∗ (*x* / *exp 1*) *powr x* (**is** *?th1*)
Gamma x ≤ *sqrt* (*2*∗*pi*/*x*) ∗ (*x* / *exp 1*) *powr x* ∗ *exp* (*1* / (*12* ∗ *x*)) (**is** *?th2*)
⟨*proof*⟩

theorem *ln-Gamma-bounds*:
assumes *x* ≥ *1*
shows *ln-Gamma x* ≥ *ln* (*2*∗*pi*/*x*) / *2* + *x* ∗ *ln x* − *x* (**is** *?th1*)
ln-Gamma x ≤ *ln* (*2*∗*pi*/*x*) / *2* + *x* ∗ *ln x* − *x* + *1*/(*12*∗*x*) (**is** *?th2*)
⟨*proof*⟩

theorem *fact-bounds*:
assumes *n* > *0*
shows (*fact n* :: *real*) ≥ *sqrt* (*2*∗*pi*∗*n*) ∗ (*n* / *exp 1*) ^ *n* (**is** *?th1*)
(*fact n* :: *real*) ≤ *sqrt* (*2*∗*pi*∗*n*) ∗ (*n* / *exp 1*) ^ *n* ∗ *exp* (*1* / (*12* ∗ *n*)) (**is** *?th2*)
⟨*proof*⟩

theorem *ln-fact-bounds*:
assumes *n* > *0*
shows *ln* (*fact n* :: *real*) ≥ *ln* (*2*∗*pi*∗*n*)/*2* + *n* ∗ *ln n* − *n* (**is** *?th1*)
ln (*fact n* :: *real*) ≤ *ln* (*2*∗*pi*∗*n*)/*2* + *n* ∗ *ln n* − *n* + *1*/(*12*∗*real n*) (**is** *?th2*)
⟨*proof*⟩

theorem *Gamma-asymp-equiv*:
Gamma ∼ (λ*x*. *sqrt* (*2*∗*pi*/*x*) ∗ (*x* / *exp 1*) *powr x* :: *real*)
⟨*proof*⟩

theorem *fact-asymp-equiv*:
fact ∼ (λ*n*. *sqrt* (*2*∗*pi*∗*n*) ∗ (*n* / *exp 1*) ^ *n* :: *real*)
⟨*proof*⟩

end

end

References

[1] G. J. O. Jameson. A simple proof of Stirling's formula for the Gamma function. *The Mathematical Gazette*, 99:68–74, 3 2015.