The Stern-Brocot Tree

Peter Gammie Andreas Lochbihler

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Abstract

The Stern-Brocot tree contains all rational numbers exactly once and in their lowest terms. We formalise the Stern-Brocot tree as a coinductive tree using recursive and iterative specifications, which we have proven equivalent, and show that it indeed contains all the numbers as stated. Following Hinze, we prove that the Stern-Brocot tree can be linearised looplessly into Stern's diatonic sequence (also known as Dijkstra's fusc function) and that it is a permutation of the Bird tree.

The reasoning stays at an abstract level by appealing to the uniqueness of solutions of guarded recursive equations and lifting algebraic laws point-wise to trees and streams using applicative functors.

Contents

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1 A codatatype of infinite binary trees

theory *Cotree* **imports**
  Main
  Applicative-Lifting.Applicative
  HOL−Library.BNF-Corec
  HOL−Library.Adhoc-Overloading
begin

context notes [[*bnf-internals*]]
begin
  codatatype *'a tree* = *Node* (*root*: *'a*) (*left*: *'a tree*) (*right*: *'a tree*)
end

lemma *rel-treeD*:
  assumes *rel-tree A x y*
  shows *rel-tree-rootD*: *A* (*root x*) (*root y*)
  and *rel-tree-leftD*: *rel-tree A* (*left x*) (*left y*)
  and *rel-tree-rightD*: *rel-tree A* (*right x*) (*right y*)
using *assms*
by(*cases x y rule*: *tree.exhaust*[*case-product tree.exhaust*], *simp-all*)+

lemmas [*simp*] = *tree.map-sel tree.map-comp*

lemma *set-tree-induct*[*consumes 1*, *case-names root left right*]:
  assumes *x*: $x \in$ *set-tree t*
  and *root*: $\bigwedge t.$ *P* (*root t*) *t*
  and *left*: $\bigwedge x\ t.$ ⟦ $x \in$ *set-tree* (*left t*); *P x* (*left t*) ⟧ $\Longrightarrow$ *P x t*
  and *right*: $\bigwedge x\ t.$ ⟦ $x \in$ *set-tree* (*right t*); *P x* (*right t*) ⟧ $\Longrightarrow$ *P x t*
  shows *P x t*
using *x*
proof(*rule tree.set-induct*)
  fix *l x r*
  from *root*[*of Node x l r*] **show** *P x* (*Node x l r*) **by** *simp*
qed(*auto intro*: *left right*)

lemma *corec-tree-cong*:
  assumes $\bigwedge x.$ *stopL* $x \Longrightarrow$ *STOPL* $x =$ *STOPL'* x
  and $\bigwedge x.$ ~ *stopL* $x \Longrightarrow$ *LEFT* $x =$ *LEFT'* x
  and $\bigwedge x.$ *stopR* $x \Longrightarrow$ *STOPR* $x =$ *STOPR'* x
  and $\bigwedge x. \neg$ *stopR* $x \Longrightarrow$ *RIGHT* $x =$ *RIGHT'* x
  shows *corec-tree ROOT stopL STOPL LEFT stopR STOPR RIGHT* =
    corec-tree ROOT stopL STOPL' LEFT' stopR STOPR' RIGHT'
  (**is** *?lhs* = *?rhs*)
proof
  fix *x*
  show *?lhs x* = *?rhs x*
    by(*coinduction arbitrary*: *x rule*: *tree.coinduct-strong*)(*auto simp add*: *assms*)
qed

context
  fixes *g1* :: *'a* ⇒ *'b*
  and *g22* :: *'a* ⇒ *'a*
  and *g32* :: *'a* ⇒ *'a*
begin

corec *unfold-tree* :: *'a* ⇒ *'b tree*
where *unfold-tree a* = *Node* (*g1 a*) (*unfold-tree* (*g22 a*)) (*unfold-tree* (*g32 a*))

lemma *unfold-tree-simps* [*simp*]:
  root (*unfold-tree a*) = *g1 a*
  left (*unfold-tree a*) = *unfold-tree* (*g22 a*)
  right (*unfold-tree a*) = *unfold-tree* (*g32 a*)
by(*subst unfold-tree.code*; *simp*; *fail*)+

end

lemma *unfold-tree-unique*:
  assumes $\bigwedge s.$ *root* (*f s*) = *ROOT s*
  and $\bigwedge s.$ *left* (*f s*) = *f* (*LEFT s*)
  and $\bigwedge s.$ *right* (*f s*) = *f* (*RIGHT s*)
  shows *f s* = *unfold-tree ROOT LEFT RIGHT s*
by(*rule unfold-tree.unique*[*THEN fun-cong*])(*auto simp add*: *fun-eq-iff assms intro*: *tree.expand*)

1.1 Applicative functor for *'a tree*

context fixes *x* :: *'a* **begin**
corec *pure-tree* :: *'a tree*
where *pure-tree* = *Node x pure-tree pure-tree*
end

lemmas *pure-tree-unfold* = *pure-tree.code*

lemma *pure-tree-simps* [*simp*]:

root (*pure-tree* *x*) = *x*
left (*pure-tree* *x*) = *pure-tree* *x*
right (*pure-tree* *x*) = *pure-tree* *x*
by(*subst pure-tree-unfold*; *simp*; *fail*)+

adhoc-overloading *pure pure-tree*

lemma *pure-tree-parametric* [*transfer-rule*]: (*rel-fun A* (*rel-tree A*)) *pure pure*
by(*rule rel-funI*)(*coinduction*, *auto*)

lemma *map-pure-tree* [*simp*]: *map-tree f* (*pure x*) = *pure* (*f x*)
by(*coinduction arbitrary*: *x*) *auto*

lemmas *pure-tree-unique* = *pure-tree.unique*

primcorec (*transfer*) *ap-tree* :: ($'a \Rightarrow 'b$) *tree* $\Rightarrow$ $'a$ *tree* $\Rightarrow$ $'b$ *tree*
where
root (*ap-tree f x*) = *root f* (*root x*)
| *left* (*ap-tree f x*) = *ap-tree* (*left f*) (*left x*)
| *right* (*ap-tree f x*) = *ap-tree* (*right f*) (*right x*)

adhoc-overloading *Applicative.ap ap-tree*

unbundle *applicative-syntax*

lemma *ap-tree-pure-Node* [*simp*]:
pure f $\diamond$ *Node x l r* = *Node* (*f x*) (*pure f* $\diamond$ *l*) (*pure f* $\diamond$ *r*)
by(*rule tree.expand*) *auto*

lemma *ap-tree-Node-Node* [*simp*]:
Node f fl fr $\diamond$ *Node x l r* = *Node* (*f x*) (*fl* $\diamond$ *l*) (*fr* $\diamond$ *r*)
by(*rule tree.expand*) *auto*

Applicative functor laws

lemma *map-tree-ap-tree-pure-tree*:
pure f $\diamond$ *u* = *map-tree f u*
by(*coinduction arbitrary*: *u*) *auto*

lemma *ap-tree-identity*: *pure id* $\diamond$ *t* = *t*
by(*simp add*: *map-tree-ap-tree-pure-tree tree.map-id*)

lemma *ap-tree-composition*:
pure ($\circ$) $\diamond$ *r1* $\diamond$ *r2* $\diamond$ *r3* = *r1* $\diamond$ (*r2* $\diamond$ *r3*)
by(*coinduction arbitrary*: *r1 r2 r3*) *auto*

lemma *ap-tree-homomorphism*:
pure f $\diamond$ *pure x* = *pure* (*f x*)
by(*simp add*: *map-tree-ap-tree-pure-tree*)

lemma *ap-tree-interchange*:
  $t \diamond \mathit{pure}\ x = \mathit{pure}\ (\lambda f.\ f\ x) \diamond t$
by(*coinduction arbitrary*: *t*) *auto*

lemma *ap-tree-K-tree*: $\mathit{pure}\ (\lambda x\ y.\ x) \diamond u \diamond v = u$
by(*coinduction arbitrary*: *u v*)(*auto*)

lemma *ap-tree-C-tree*: $\mathit{pure}\ (\lambda f\ x\ y.\ f\ y\ x) \diamond u \diamond v \diamond w = u \diamond w \diamond v$
by(*coinduction arbitrary*: *u v w*)(*auto*)

lemma *ap-tree-W-tree*: $\mathit{pure}\ (\lambda f\ x.\ f\ x\ x) \diamond f \diamond x = f \diamond x \diamond x$
by(*coinduction arbitrary*: *f x*)(*auto*)

applicative *tree* (*K*, *W*) **for**
  pure: *pure-tree*
  ap: *ap-tree*
  rel: *rel-tree*
  set: *set-tree*
proof −
  fix $R :: {'b} \Rightarrow {'c} \Rightarrow \mathit{bool}$ **and** $f :: ({'a} \Rightarrow {'b})\ \mathit{tree}$ **and** *g x*
  assume [*transfer-rule*]: *rel-tree* (*rel-fun* (*eq-on* (*set-tree x*)) *R*) *f g*
  have [*transfer-rule*]: *rel-tree* (*eq-on* (*set-tree x*)) *x x* **by**(*rule tree.rel-refl-strong*) *simp*
  show *rel-tree R* $(f \diamond x)$ $(g \diamond x)$ **by** *transfer-prover*
qed(*rule ap-tree-homomorphism ap-tree-composition*[*unfolded o-def*[*abs-def*]] *ap-tree-K-tree ap-tree-W-tree ap-tree-interchange pure-tree-parametric*)+

declare *map-tree-ap-tree-pure-tree*[*symmetric*, *applicative-unfold*]

lemma *ap-tree-strong-extensional*:
  $(\bigwedge x.\ f \diamond \mathit{pure}\ x = g \diamond \mathit{pure}\ x) \Longrightarrow f = g$
proof(*coinduction arbitrary*: *f g*)
  case [*rule-format*]: (*Eq-tree f g*)
  have *root f* = *root g*
  proof
    fix *x*
    show *root f x* = *root g x*
      using *Eq-tree*[*of x*] **by**(*subst* (*asm*) (*1 2*) *ap-tree.ctr*) *simp*
  qed
  moreover {
    fix *x*
    have *left f* $\diamond$ *pure x* = *left g* $\diamond$ *pure x*
      using *Eq-tree*[*of x*] **by**(*subst* (*asm*) (*1 2*) *ap-tree.ctr*) *simp*
  } **moreover** {
    fix *x*
    have *right f* $\diamond$ *pure x* = *right g* $\diamond$ *pure x*
      using *Eq-tree*[*of x*] **by**(*subst* (*asm*) (*1 2*) *ap-tree.ctr*) *simp*
  } **ultimately show** *?case* **by** *simp*
qed

lemma *ap-tree-extensional*:
 $(\bigwedge x.\ f \diamond x = g \diamond x) \Longrightarrow f = g$
by(*rule ap-tree-strong-extensional*) *simp*

1.2 Standard tree combinators

1.2.1 Recurse combinator

This will be the main combinator to define trees recursively

Uniqueness for this gives us the unique fixed-point theorem for guarded recursive definitions.

lemma *map-unfold-tree* [*simp*]: **fixes** $l\ r\ x$
 defines $\mathit{unf} \equiv \mathit{unfold\text{-}tree}\ (\lambda f.\ f\ x)\ (\lambda f.\ f \circ l)\ (\lambda f.\ f \circ r)$
 shows $\mathit{map\text{-}tree}\ G\ (\mathit{unf}\ F) = \mathit{unf}\ (G \circ F)$
by(*coinduction arbitrary*: $F\ G$)(*auto 4 3 simp add*: *unf-def o-assoc*)

friend-of-corec *map-tree* :: $('a \Rightarrow 'a) \Rightarrow 'a\ \mathit{tree} \Rightarrow 'a\ \mathit{tree}$ **where**
 $\mathit{map\text{-}tree}\ f\ t = \mathit{Node}\ (f\ (\mathit{root}\ t))\ (\mathit{map\text{-}tree}\ f\ (\mathit{left}\ t))\ (\mathit{map\text{-}tree}\ f\ (\mathit{right}\ t))$
subgoal by (*rule tree.expand*; *simp*)
subgoal by (*fold relator-eq*; *transfer-prover*)
done

context fixes $l :: 'a \Rightarrow 'a$ **and** $r :: 'a \Rightarrow 'a$ **and** $x :: 'a$ **begin**
corec *tree-recurse* :: $'a\ \mathit{tree}$
where $\mathit{tree\text{-}recurse} = \mathit{Node}\ x\ (\mathit{map\text{-}tree}\ l\ \mathit{tree\text{-}recurse})\ (\mathit{map\text{-}tree}\ r\ \mathit{tree\text{-}recurse})$
end

lemma *tree-recurse-simps* [*simp*]:
 $\mathit{root}\ (\mathit{tree\text{-}recurse}\ l\ r\ x) = x$
 $\mathit{left}\ (\mathit{tree\text{-}recurse}\ l\ r\ x) = \mathit{map\text{-}tree}\ l\ (\mathit{tree\text{-}recurse}\ l\ r\ x)$
 $\mathit{right}\ (\mathit{tree\text{-}recurse}\ l\ r\ x) = \mathit{map\text{-}tree}\ r\ (\mathit{tree\text{-}recurse}\ l\ r\ x)$
by(*subst tree-recurse.code*; *simp*; *fail*)+

lemma *tree-recurse-unfold*:
 $\mathit{tree\text{-}recurse}\ l\ r\ x = \mathit{Node}\ x\ (\mathit{map\text{-}tree}\ l\ (\mathit{tree\text{-}recurse}\ l\ r\ x))\ (\mathit{map\text{-}tree}\ r\ (\mathit{tree\text{-}recurse}\ l\ r\ x))$
by(*fact tree-recurse.code*)

lemma *tree-recurse-fusion*:
 assumes $h \circ l = l' \circ h$ **and** $h \circ r = r' \circ h$
 shows $\mathit{map\text{-}tree}\ h\ (\mathit{tree\text{-}recurse}\ l\ r\ x) = \mathit{tree\text{-}recurse}\ l'\ r'\ (h\ x)$
by(*rule tree-recurse.unique*)(*simp add*: *tree.expand assms*)

1.2.2 Tree iteration

context fixes $l :: 'a \Rightarrow 'a$ **and** $r :: 'a \Rightarrow 'a$ **begin**
primcorec *tree-iterate* :: $'a \Rightarrow 'a\ \mathit{tree}$
where $\mathit{tree\text{-}iterate}\ s = \mathit{Node}\ s\ (\mathit{tree\text{-}iterate}\ (l\ s))\ (\mathit{tree\text{-}iterate}\ (r\ s))$

end

lemma *unfold-tree-tree-iterate*:
  unfold-tree out l r = map-tree out $\circ$ *tree-iterate l r*
by(*rule ext*)(*rule unfold-tree-unique*[*symmetric*]; *simp*)

lemma *tree-iterate-fusion*:
  assumes $h \circ l = l' \circ h$
  assumes $h \circ r = r' \circ h$
  shows *map-tree h* (*tree-iterate l r x*) = *tree-iterate* l' r' (*h x*)
apply(*coinduction arbitrary*: *x*)
using *assms* **by**(*auto simp add*: *fun-eq-iff*)

1.2.3 Tree traversal

datatype *dir* = *L* | *R*
type-synonym *path* = *dir list*

definition *traverse-tree* :: *path* $\Rightarrow$ *'a tree* $\Rightarrow$ *'a tree*
where *traverse-tree path* $\equiv$ *foldr* (λ*d f*. *f* $\circ$ *case-dir left right d*) *path id*

lemma *traverse-tree-simps*[*simp*]:
  traverse-tree [] = *id*
  traverse-tree (*d* # *path*) = *traverse-tree path* $\circ$ (*case d of L* $\Rightarrow$ *left* | *R* $\Rightarrow$ *right*)
by (*simp-all add*: *traverse-tree-def*)

lemma *traverse-tree-map-tree* [*simp*]:
  traverse-tree path (*map-tree f t*) = *map-tree f* (*traverse-tree path t*)
by (*induct path arbitrary*: *t*) (*simp-all split*: *dir.splits*)

lemma *traverse-tree-append* [*simp*]:
  traverse-tree (*path* @ *ext*) *t* = *traverse-tree ext* (*traverse-tree path t*)
by (*induct path arbitrary*: *t*) *simp-all*

traverse-tree is an applicative-functor homomorphism.

lemma *traverse-tree-pure-tree* [*simp*]:
  traverse-tree path (*pure x*) = *pure x*
by (*induct path arbitrary*: *x*) (*simp-all split*: *dir.splits*)

lemma *traverse-tree-ap* [*simp*]:
  traverse-tree path (*f* $\diamond$ *x*) = *traverse-tree path f* $\diamond$ *traverse-tree path x*
by (*induct path arbitrary*: *f x*) (*simp-all split*: *dir.splits*)

context fixes *l r* :: *'a* $\Rightarrow$ *'a* **begin**

primrec *traverse-dir* :: *dir* $\Rightarrow$ *'a* $\Rightarrow$ *'a*
where
  traverse-dir L = *l*
| *traverse-dir R* = *r*

abbreviation *traverse-path* :: *path* ⇒ *'a* ⇒ *'a*
where *traverse-path* ≡ *fold traverse-dir*

end

lemma *traverse-tree-tree-iterate*:
  traverse-tree path (*tree-iterate l r s*) =
   tree-iterate l r (*traverse-path l r path s*)
by (*induct path arbitrary*: *s*) (*simp-all split*: *dir.splits*)

? shows that if the tree construction function is suitably monoidal then recursion and iteration define the same tree.

lemma *tree-recurse-iterate*:
  assumes *monoid*:
    $\bigwedge x\ y\ z.\ f\ (f\ x\ y)\ z = f\ x\ (f\ y\ z)$
    $\bigwedge x.\ f\ x\ \varepsilon = x$
    $\bigwedge x.\ f\ \varepsilon\ x = x$
  shows *tree-recurse* (*f l*) (*f r*) ε = *tree-iterate* ($\lambda x.\ f\ x\ l$) ($\lambda x.\ f\ x\ r$) ε
apply(*rule tree-recurse.unique*[*symmetric*])
apply(*rule tree.expand*)
apply(*simp add*: *tree-iterate-fusion*[**where** $r'{=}\lambda x.\ f\ x\ r$ **and** $l'{=}\lambda x.\ f\ x\ l$] *fun-eq-iff monoid*)
done

1.2.4 Mirroring

primcorec *mirror* :: *'a tree* ⇒ *'a tree*
where
  root (*mirror t*) = *root t*
| *left* (*mirror t*) = *mirror* (*right t*)
| *right* (*mirror t*) = *mirror* (*left t*)

lemma *mirror-unfold*: *mirror* (*Node x l r*) = *Node x* (*mirror r*) (*mirror l*)
by(*rule tree.expand*) *simp*

lemma *mirror-pure*: *mirror* (*pure x*) = *pure x*
by(*coinduction rule*: *tree.coinduct*) *simp*

lemma *mirror-ap-tree*: *mirror* ($f \diamond x$) = *mirror f* $\diamond$ *mirror x*
by(*coinduction arbitrary*: *f x*) *auto*

end

1.3 Pointwise arithmetic on infinite binary trees

theory *Cotree-Algebra*
imports *Cotree*
begin

1.3.1 Constants and operators

instantiation *tree* :: (*zero*) *zero* **begin**
definition [*applicative-unfold*]: *0* = *pure-tree 0*
instance ..
end

instantiation *tree* :: (*one*) *one* **begin**
definition [*applicative-unfold*]: *1* = *pure-tree 1*
instance ..
end

instantiation *tree* :: (*plus*) *plus* **begin**
definition [*applicative-unfold*]: *plus x y* = *pure* (+) ⋄ *x* ⋄ (*y* :: *'a tree*)
instance ..
end

lemma *plus-tree-simps* [*simp*]:
  root (t + t') = *root* t + *root* t'
  left (t + t') = *left* t + *left* t'
  right (t + t') = *right* t + *right* t'
by(*simp-all add*: *plus-tree-def*)

friend-of-corec *plus* **where** t + t' = *Node* (*root* t + *root* t') (*left* t + *left* t') (*right* t + *right* t')
subgoal by(*rule tree.expand*; *simp*)
subgoal by *transfer-prover*
done

instantiation *tree* :: (*minus*) *minus* **begin**
definition [*applicative-unfold*]: *minus x y* = *pure* (−) ⋄ *x* ⋄ (*y* :: *'a tree*)
instance ..
end

lemma *minus-tree-simps* [*simp*]:
  root (t − t') = *root* t − *root* t'
  left (t − t') = *left* t − *left* t'
  right (t − t') = *right* t − *right* t'
by(*simp-all add*: *minus-tree-def*)

instantiation *tree* :: (*uminus*) *uminus* **begin**
definition [*applicative-unfold tree*]: *uminus* = ((⋄) (*pure uminus*) :: *'a tree* ⇒ *'a tree*)
instance ..
end

instantiation *tree* :: (*times*) *times* **begin**
definition [*applicative-unfold*]: *times x y* = *pure* (∗) ⋄ *x* ⋄ (*y* :: *'a tree*)
instance ..
end

lemma *times-tree-simps* [*simp*]:
  root $(t * t') =$ *root* $t *$ *root* t'
  left $(t * t') =$ *left* $t *$ *left* t'
  right $(t * t') =$ *right* $t *$ *right* t'
by(*simp-all add*: *times-tree-def*)

instance *tree* :: (*Rings.dvd*) *Rings.dvd* **..**

instantiation *tree* :: (*modulo*) *modulo* **begin**
definition [*applicative-unfold*]: x *div* $y =$ *pure-tree* (*div*) $\diamond\ x \diamond (y$ :: *'a tree*)
definition [*applicative-unfold*]: x *mod* $y =$ *pure-tree* (*mod*) $\diamond\ x \diamond (y$ :: *'a tree*)
instance ..
end

lemma *mod-tree-simps* [*simp*]:
  root (t *mod* t') = *root* t *mod* *root* t'
  left (t *mod* t') = *left* t *mod* *left* t'
  right (t *mod* t') = *right* t *mod* *right* t'
by(*simp-all add*: *modulo-tree-def*)

1.3.2 Algebraic instances

instance *tree* :: (*semigroup-add*) *semigroup-add*
using *add.assoc* **by** *intro-classes applicative-lifting*

instance *tree* :: (*ab-semigroup-add*) *ab-semigroup-add*
using *add.commute* **by** *intro-classes applicative-lifting*

instance *tree* :: (*semigroup-mult*) *semigroup-mult*
using *mult.assoc* **by** *intro-classes applicative-lifting*

instance *tree* :: (*ab-semigroup-mult*) *ab-semigroup-mult*
using *mult.commute* **by** *intro-classes applicative-lifting*

instance *tree* :: (*monoid-add*) *monoid-add*
by *intro-classes* (*applicative-lifting*, *simp*)+

instance *tree* :: (*comm-monoid-add*) *comm-monoid-add*
by *intro-classes* (*applicative-lifting*, *simp*)

instance *tree* :: (*comm-monoid-diff*) *comm-monoid-diff*
by *intro-classes* (*applicative-lifting*, *simp add*: *diff-diff-add*)+

instance *tree* :: (*monoid-mult*) *monoid-mult*
by *intro-classes* (*applicative-lifting*, *simp*)+

instance *tree* :: (*comm-monoid-mult*) *comm-monoid-mult*
by *intro-classes* (*applicative-lifting*, *simp*)

instance *tree* :: (*cancel-semigroup-add*) *cancel-semigroup-add*
proof
  fix *a b c* :: *'a tree*
  assume $a + b = a + c$
  thus $b = c$
  proof (*coinduction arbitrary*: *a b c*)
    case (*Eq-tree a b c*)
    hence *root* $(a + b)$ = *root* $(a + c)$
          left $(a + b)$ = *left* $(a + c)$
          right $(a + b)$ = *right* $(a + c)$
      by *simp-all*
    thus *?case* **by** (*auto*)
  qed
next
  fix *a b c* :: *'a tree*
  assume $b + a = c + a$
  thus $b = c$
  proof (*coinduction arbitrary*: *a b c*)
    case (*Eq-tree a b c*)
    hence *root* $(b + a)$ = *root* $(c + a)$
          left $(b + a)$ = *left* $(c + a)$
          right $(b + a)$ = *right* $(c + a)$
      by *simp-all*
    thus *?case* **by** (*auto*)
  qed
qed

instance *tree* :: (*cancel-ab-semigroup-add*) *cancel-ab-semigroup-add*
by *intro-classes* (*applicative-lifting*, *simp add*: *diff-diff-eq*)+

instance *tree* :: (*cancel-comm-monoid-add*) *cancel-comm-monoid-add* **..**

instance *tree* :: (*group-add*) *group-add*
by *intro-classes* (*applicative-lifting*, *simp*)+

instance *tree* :: (*ab-group-add*) *ab-group-add*
by *intro-classes* (*applicative-lifting*, *simp*)+

instance *tree* :: (*semiring*) *semiring*
by *intro-classes* (*applicative-lifting*, *simp add*: *ring-distribs*)+

instance *tree* :: (*mult-zero*) *mult-zero*
by *intro-classes* (*applicative-lifting*, *simp*)+

instance *tree* :: (*semiring-0*) *semiring-0* **..**

instance *tree* :: (*semiring-0-cancel*) *semiring-0-cancel* **..**

instance *tree* :: (*comm-semiring*) *comm-semiring*
by *intro-classes*(*rule distrib-right*)

instance *tree* :: (*comm-semiring-0*) *comm-semiring-0* **..**

instance *tree* :: (*comm-semiring-0-cancel*) *comm-semiring-0-cancel* **..**

lemma *pure-tree-inject*[*simp*]: *pure-tree* x = *pure-tree* y $\longleftrightarrow$ $x = y$
proof
  assume *pure-tree* x = *pure-tree* y
  hence *root* (*pure-tree* x) = *root* (*pure-tree* y) **by** *simp*
  thus $x = y$ **by** *simp*
qed *simp*

instance *tree* :: (*zero-neq-one*) *zero-neq-one*
by *intro-classes* (*applicative-unfold tree*)

instance *tree* :: (*semiring-1*) *semiring-1* **..**

instance *tree* :: (*comm-semiring-1*) *comm-semiring-1* **..**

instance *tree* :: (*semiring-1-cancel*) *semiring-1-cancel* **..**

instance *tree* :: (*comm-semiring-1-cancel*) *comm-semiring-1-cancel*
by(*intro-classes*; *applicative-lifting*, *rule right-diff-distrib'*)

instance *tree* :: (*ring*) *ring* **..**

instance *tree* :: (*comm-ring*) *comm-ring* **..**

instance *tree* :: (*ring-1*) *ring-1* **..**

instance *tree* :: (*comm-ring-1*) *comm-ring-1* **..**

instance *tree* :: (*numeral*) *numeral* **..**

instance *tree* :: (*neg-numeral*) *neg-numeral* **..**

instance *tree* :: (*semiring-numeral*) *semiring-numeral* **..**

lemma *of-nat-tree*: *of-nat* n = *pure-tree* (*of-nat* n)
proof (*induction* n)
  case *0* **show** *?case* **by** (*simp add*: *zero-tree-def*)
next
  case (*Suc* n)
  have *1* + *pure* (*of-nat* n) = *pure* (*1* + *of-nat* n) **by** *applicative-nf rule*
  with *Suc.IH* **show** *?case* **by** *simp*
qed

instance *tree* :: (*semiring-char-0*) *semiring-char-0*
by *intro-classes* (*simp add*: *inj-on-def of-nat-tree*)

lemma *numeral-tree-simps* [*simp*]:
  root (*numeral n*) = *numeral n*
  left (*numeral n*) = *numeral n*
  right (*numeral n*) = *numeral n*
by(*induct n*)(*auto simp add*: *numeral.simps plus-tree-def one-tree-def*)

lemma *numeral-tree-conv-pure* [*applicative-unfold*]: *numeral n* = *pure* (*numeral n*)
by(*rule pure-tree-unique*)(*rule tree.expand*; *simp*)

instance *tree* :: (*ring-char-0*) *ring-char-0* **..**

end

2 The Stern-Brocot Tree

theory *Stern-Brocot-Tree*
imports
  HOL.Rat
  HOL−Library.Sublist
  Cotree-Algebra
  Applicative-Lifting.Stream-Algebra
begin

The Stern-Brocot tree is discussed at length by Graham et al. (1994, §4.5). In essence the tree enumerates the rational numbers in their lowest terms by constructing the *mediant* of two bounding fractions.

type-synonym *fraction* = *nat* × *nat*

definition *mediant* :: *fraction* × *fraction* ⇒ *fraction*
where *mediant* ≡ λ((*a*, *c*), (*b*, *d*)). (*a* + *b*, *c* + *d*)

definition *stern-brocot* :: *fraction tree*
where
  stern-brocot = *unfold-tree*
    (λ(*lb*, *ub*). *mediant* (*lb*, *ub*))
    (λ(*lb*, *ub*). (*lb*, *mediant* (*lb*, *ub*)))
    (λ(*lb*, *ub*). (*mediant* (*lb*, *ub*), *ub*))
    ((*0*, *1*), (*1*, *0*))

This process is visualised in Figure 2. Intuitively each node is labelled with the mediant of it's rightmost and leftmost ancestors.

Our ultimate goal is to show that the Stern-Brocot tree contains all rationals (in lowest terms), and that each occurs exactly once in the tree. A proof is sketched in Graham et al. (1994, §4.5).

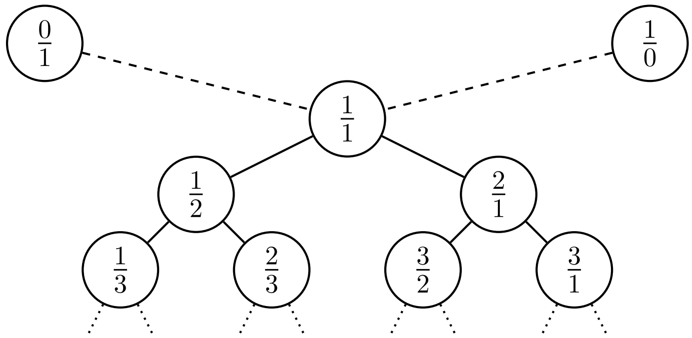

Figure 1: Constructing the Stern-Brocot tree iteratively.

2.1 Specification via a recursion equation

Hinze (2009) derives the following recurrence relation for the Stern-Brocot tree. We will show in §2.5 that his derivation is sound with respect to the standard iterative definition of the tree shown above.

abbreviation *succ* :: *fraction* ⇒ *fraction*
where *succ* ≡ λ(*m*, *n*). (*m* + *n*, *n*)

abbreviation *recip* :: *fraction* ⇒ *fraction*
where *recip* ≡ λ(*m*, *n*). (*n*, *m*)

corec *stern-brocot-recurse* :: *fraction tree*
where
  stern-brocot-recurse =
   Node (*1*, *1*)
     (*map-tree recip* (*map-tree succ* (*map-tree recip stern-brocot-recurse*)))
     (*map-tree succ stern-brocot-recurse*)

Actually, we would like to write the specification below, but (⋄) cannot be registered as friendly due to varying type parameters

lemma *stern-brocot-unfold*:
  stern-brocot-recurse =
   Node (*1*, *1*)
     (*pure recip* ⋄ (*pure succ* ⋄ (*pure recip* ⋄ *stern-brocot-recurse*)))
     (*pure succ* ⋄ *stern-brocot-recurse*)
by(*fact stern-brocot-recurse.code*[*unfolded map-tree-ap-tree-pure-tree*[*symmetric*]])

lemma *stern-brocot-simps* [*simp*]:
  root stern-brocot-recurse = (*1*, *1*)
  left stern-brocot-recurse = *pure recip* ⋄ (*pure succ* ⋄ (*pure recip* ⋄ *stern-brocot-recurse*))
  right stern-brocot-recurse = *pure succ* ⋄ *stern-brocot-recurse*
by (*subst stern-brocot-unfold*, *simp*)+

lemma *stern-brocot-conv*:

stern-brocot-recurse = *tree-recurse* (*recip* ∘ *succ* ∘ *recip*) *succ* (*1*, *1*)
apply(*rule tree-recurse.unique*)
apply(*subst stern-brocot-unfold*)
apply(*simp add*: *o-assoc*)
apply(*rule conjI*; *applicative-nf*; *simp*)
done

2.2 Basic properties

The recursive definition is useful for showing some basic properties of the tree, such as that the pairs of numbers at each node are coprime, and have non-zero denominators. Both are simple inductions on the path.

lemma *stern-brocot-denominator-non-zero*:
case root (*traverse-tree path stern-brocot-recurse*) *of* (*m*, *n*) ⇒ *m* > *0* ∧ *n* > *0*
by(*induct path*)(*auto split*: *dir.splits*)

lemma *stern-brocot-coprime*:
case root (*traverse-tree path stern-brocot-recurse*) *of* (*m*, *n*) ⇒ *coprime m n*
by (*induct path*) (*auto split*: *dir.splits simp add*: *coprime-iff-gcd-eq-1*, *metis gcd.commute gcd-add1*)

2.3 All the rationals

For every pair of positive naturals, we can construct a path into the Stern-Brocot tree such that the naturals at the end of the path define the same rational as the pair we started with. Intuitively, the choices made by Euclid's algorithm define this path.

function *mk-path* :: *nat* ⇒ *nat* ⇒ *path* **where**
m = *n* ⟹ *mk-path* (*Suc m*) (*Suc n*) = []
| *m* < *n* ⟹ *mk-path* (*Suc m*) (*Suc n*) = *L* # *mk-path* (*Suc m*) (*n* − *m*)
| *m* > *n* ⟹ *mk-path* (*Suc m*) (*Suc n*) = *R* # *mk-path* (*m* − *n*) (*Suc n*)
| *mk-path 0* - = *undefined*
| *mk-path* - *0* = *undefined*
by *atomize-elim*(*auto*, *arith*)
termination *mk-path* **by** *lexicographic-order*

lemmas *mk-path-induct*[*case-names equal less greater*] = *mk-path.induct*

abbreviation *rat-of* :: *fraction* ⇒ *rat*
where *rat-of* ≡ λ(*x*, *y*). *Fract* (*int x*) (*int y*)

theorem *stern-brocot-rationals*:
⟦ *m* > *0*; *n* > *0* ⟧ ⟹
root (*traverse-tree* (*mk-path m n*) (*pure rat-of* ⋄ *stern-brocot-recurse*)) = *Fract* (*int m*) (*int n*)
proof(*induction m n rule*: *mk-path-induct*)
case (*less m n*)

with *stern-brocot-denominator-non-zero*[**where** *path*=*mk-path* (*Suc m*) ($n - m$)]
show *?case*
by (*simp add*: *eq-rat field-simps of-nat-diff split*: *prod.split-asm*)
next
case (*greater m n*)
with *stern-brocot-denominator-non-zero*[**where** *path*=*mk-path* ($m - n$) (*Suc n*)]
show *?case*
by (*simp add*: *eq-rat field-simps of-nat-diff split*: *prod.split-asm*)
qed (*simp-all add*: *eq-rat*)

2.4 No repetitions

We establish that the Stern-Brocot tree does not contain repetitions, i.e., that each rational number appears at most once in it. Note that this property is stronger than merely requiring that pairs of naturals not be repeated, though it is implied by that property and *stern-brocot-coprime*.

Intuitively, the tree enjoys the *binary search tree* ordering property when we map our pairs of naturals into rationals. This suffices to show that each rational appears at most once in the tree. To establish this seems to require more structure than is present in the recursion equations, and so we follow Backhouse and Ferreira (2008) and Hinze (2009) by introducing another definition of the tree, which summarises the path to each node using a matrix.

We then derive an iterative version and use invariant reasoning on that. We begin by defining some matrix machinery. This is all elementary and primitive (we do not need much algebra).

type-synonym *matrix* = *fraction* × *fraction*
type-synonym *vector* = *fraction*

definition *times-matrix* :: *matrix* ⇒ *matrix* ⇒ *matrix* (**infixl** ⊗ *70*)
where *times-matrix* = $(\lambda((a,\ c),\ (b,\ d))\ ((a',\ c'),\ (b',\ d')).$
$((a * a' + b * c',\ c * a' + d * c'),$
$(a * b' + b * d',\ c * b' + d * d')))$

definition *times-vector* :: *matrix* ⇒ *vector* ⇒ *vector* (**infixr** ⊙ *70*)
where *times-vector* = $(\lambda((a,\ c),\ (b,\ d))\ (a',\ c').\ (a * a' + b * c',\ c * a' + d * c'))$

context begin

private definition *F* :: *matrix* **where** $F = ((0,\ 1),\ (1,\ 0))$
private definition *I* :: *matrix* **where** $I = ((1,\ 0),\ (0,\ 1))$
private definition *LL* :: *matrix* **where** $LL = ((1,\ 1),\ (0,\ 1))$
private definition *UR* :: *matrix* **where** $UR = ((1,\ 0),\ (1,\ 1))$

definition *Det* :: *matrix* ⇒ *nat* **where** $Det \equiv \lambda((a,\ c),\ (b,\ d)).\ a * d - b * c$

lemma *Dets* [*iff*]:

Det I = *1*
Det LL = *1*
Det UR = *1*
unfolding *Det-def I-def LL-def UR-def* **by** *simp-all*

lemma *LL-UR-Det*:
Det m = *1* $\Longrightarrow$ *Det* (*m* ⊗ *LL*) = *1*
Det m = *1* $\Longrightarrow$ *Det* (*LL* ⊗ *m*) = *1*
Det m = *1* $\Longrightarrow$ *Det* (*m* ⊗ *UR*) = *1*
Det m = *1* $\Longrightarrow$ *Det* (*UR* ⊗ *m*) = *1*
by (*cases m*, *simp add*: *Det-def LL-def UR-def times-matrix-def split-def field-simps*)+

lemma *mediant-I-F* [*simp*]:
mediant F = (*1*, *1*)
mediant I = (*1*, *1*)
by (*simp-all add*: *F-def I-def mediant-def*)

lemma *times-matrix-I* [*simp*]:
I ⊗ *x* = *x*
x ⊗ *I* = *x*
by (*simp-all add*: *times-matrix-def I-def split-def*)

lemma *times-matrix-assoc* [*simp*]:
(*x* ⊗ *y*) ⊗ *z* = *x* ⊗ (*y* ⊗ *z*)
by (*simp add*: *times-matrix-def field-simps split-def*)

lemma *LL-UR-pos*:
0 < *snd* (*mediant m*) $\Longrightarrow$ *0* < *snd* (*mediant* (*m* ⊗ *LL*))
0 < *snd* (*mediant m*) $\Longrightarrow$ *0* < *snd* (*mediant* (*m* ⊗ *UR*))
by (*cases m*) (*simp-all add*: *LL-def UR-def times-matrix-def split-def field-simps mediant-def*)

lemma *recip-succ-recip*: *recip* ∘ *succ* ∘ *recip* = (λ(*x*, *y*). (*x*, *x* + *y*))
by (*clarsimp simp*: *fun-eq-iff*)

Backhouse and Ferreira work with the identity matrix *I* at the root. This has the advantage that all relevant matrices have determinants of *1*.

definition *stern-brocot-iterate-aux* :: *matrix* ⇒ *matrix tree*
where *stern-brocot-iterate-aux* ≡ *tree-iterate* (λ*s*. *s* ⊗ *LL*) (λ*s*. *s* ⊗ *UR*)

definition *stern-brocot-iterate* :: *fraction tree*
where *stern-brocot-iterate* ≡ *map-tree mediant* (*stern-brocot-iterate-aux I*)

lemma *stern-brocot-recurse-iterate*: *stern-brocot-recurse* = *stern-brocot-iterate* (**is** *?lhs* = *?rhs*)
proof −
have *?rhs* = *map-tree mediant* (*tree-recurse* ((⊗) *LL*) ((⊗) *UR*) *I*)
using *tree-recurse-iterate*[**where** *f*=(⊗) **and** *l*=*LL* **and** *r*=*UR* **and** *ε*=*I*]
by (*simp add*: *stern-brocot-iterate-def stern-brocot-iterate-aux-def*)

also have . . . = *tree-recurse* ((⊙) *LL*) ((⊙) *UR*) (*1*, *1*)
unfolding *mediant-I-F*(*2*)[*symmetric*]
by (*rule tree-recurse-fusion*)(*simp-all add*: *fun-eq-iff mediant-def times-matrix-def times-vector-def LL-def UR-def*)[*2*]
also have . . . = *?lhs*
by (*simp add*: *stern-brocot-conv recip-succ-recip times-vector-def LL-def UR-def*)
finally show *?thesis* **by** *simp*
qed

The following are the key ordering properties derived by Backhouse and Ferreira (2008). They hinge on the matrices containing only natural numbers.

lemma *tree-ordering-left*:
assumes *DX*: *Det X* = *1*
assumes *DY*: *Det Y* = *1*
assumes *MX*: *0* < *snd* (*mediant X*)
shows *rat-of* (*mediant* (*X* ⊗ *LL* ⊗ *Y*)) < *rat-of* (*mediant X*)
proof −
from *DX DY* **have** *F*: *0* < *snd* (*mediant* (*X* ⊗ *LL* ⊗ *Y*))
by (*auto simp*: *Det-def times-matrix-def LL-def split-def mediant-def*)
obtain *x11 x12 x21 x22* **where** *X*: *X* = ((*x11*, *x12*), (*x21*, *x22*)) **by**(*cases X*) *auto*
obtain *y11 y12 y21 y22* **where** *Y*: *Y* = ((*y11*, *y12*), (*y21*, *y22*)) **by**(*cases Y*) *auto*
from *DX DY* **have** ∗: (*x12* ∗ *x21*) ∗ (*y12* + *y22*) < (*x11* ∗ *x22*) ∗ (*y12* + *y22*)
by(*simp add*: *X Y Det-def*)(*cases y12*, *simp-all add*: *field-simps*)
from *DX DY MX F* **show** *?thesis*
apply (*simp add*: *split-def X Y of-nat-mult* [*symmetric*] *del*: *of-nat-mult*)
apply (*clarsimp simp*: *Det-def times-matrix-def LL-def UR-def mediant-def split-def*)
using ∗ **by** (*simp add*: *field-simps*)
qed

lemma *tree-ordering-right*:
assumes *DX*: *Det X* = *1*
assumes *DY*: *Det Y* = *1*
assumes *MX*: *0* < *snd* (*mediant X*)
shows *rat-of* (*mediant X*) < *rat-of* (*mediant* (*X* ⊗ *UR* ⊗ *Y*))
proof −
from *DX DY* **have** *F*: *0* < *snd* (*mediant* (*X* ⊗ *UR* ⊗ *Y*))
by (*auto simp*: *Det-def times-matrix-def UR-def split-def mediant-def*)
obtain *x11 x12 x21 x22* **where** *X*: *X* = ((*x11*, *x12*), (*x21*, *x22*)) **by**(*cases X*) *auto*
obtain *y11 y12 y21 y22* **where** *Y*: *Y* = ((*y11*, *y12*), (*y21*, *y22*)) **by**(*cases Y*) *auto*
show *?thesis* **using** *DX DY MX F*
apply (*simp add*: *X Y split-def of-nat-mult* [*symmetric*] *del*: *of-nat-mult*)
apply (*simp add*: *Det-def times-matrix-def LL-def UR-def mediant-def split-def algebra-simps*)
apply (*simp add*: *add-mult-distrib2*[*symmetric*] *mult.assoc*[*symmetric*])

apply (*cases y21*; *simp*)
done
qed

lemma *stern-brocot-iterate-aux-Det*:
assumes *Det m* = *1 0* < *snd* (*mediant m*)
shows *Det* (*root* (*traverse-tree path* (*stern-brocot-iterate-aux m*))) = *1*
and *0* < *snd* (*mediant* (*root* (*traverse-tree path* (*stern-brocot-iterate-aux m*))))
using *assms*
by (*induct path arbitrary*: *m*)
(*simp-all add*: *stern-brocot-iterate-aux-def LL-UR-Det LL-UR-pos split*: *dir.splits*)

lemma *stern-brocot-iterate-aux-decompose*:
$\exists$ *m''*. *m* $\otimes$ *m''* = *root* (*traverse-tree path* (*stern-brocot-iterate-aux m*)) $\wedge$ *Det m''* = *1*
proof(*induction path arbitrary*: *m*)
case *Nil* **show** *?case*
by (*auto simp add*: *stern-brocot-iterate-aux-def intro*: *exI*[**where** *x*=*I*] *simp del*: *split-paired-Ex*)
next
case (*Cons d ds m*)
from *Cons.IH*[**where** *m*=*m* $\otimes$ *UR*] *Cons.IH*[**where** *m*=*m* $\otimes$ *LL*] **show** *?case*
by(*simp add*: *stern-brocot-iterate-aux-def split*: *dir.splits del*: *split-paired-Ex*)(*fastforce simp*: *LL-UR-Det*)
qed

lemma *stern-brocot-fractions-not-repeated-strict-prefix*:
assumes *root* (*traverse-tree path stern-brocot-iterate*) = *root* (*traverse-tree path' stern-brocot-iterate*)
assumes *pp'*: *strict-prefix path path'*
shows *False*
proof −
from *pp'* **obtain** *d ds* **where** *pp'*: *path'* = *path* @ [*d*] @ *ds* **by** (*auto elim*!: *strict-prefixE'*)
define *m* **where** *m* = *root* (*traverse-tree path* (*stern-brocot-iterate-aux I*))
then have *Dm*: *Det m* = *1* **and** *Pm*: *0* < *snd* (*mediant m*)
using *stern-brocot-iterate-aux-Det*[**where** *path*=*path* **and** *m*=*I*] **by** *simp-all*
define *m'* **where** *m'* = *root* (*traverse-tree path'* (*stern-brocot-iterate-aux I*))
then have *Dm'*: *Det m'* = *1*
using *stern-brocot-iterate-aux-Det*[**where** *path*=*path'* **and** *m*=*I*] **by** *simp*
let *?M* = *case d of L* $\Rightarrow$ *m* $\otimes$ *LL* | *R* $\Rightarrow$ *m* $\otimes$ *UR*
from *pp'* **have** *root* (*traverse-tree ds* (*stern-brocot-iterate-aux ?M*)) = *m'*
by(*simp add*: *m-def m'-def stern-brocot-iterate-aux-def traverse-tree-tree-iterate split*: *dir.splits*)
then obtain *m''* **where** *mm'm''*: *?M* $\otimes$ *m''*= *m'* **and** *Dm''*: *Det m''* = *1*
using *stern-brocot-iterate-aux-decompose*[**where** *path*=*ds* **and** *m*=*?M*] **by** *clarsimp*
hence *case d of L* $\Rightarrow$ *rat-of* (*mediant m'*) < *rat-of* (*mediant m*) | *R* $\Rightarrow$ *rat-of* (*mediant m*) < *rat-of* (*mediant m'*)

using *tree-ordering-left*[*OF Dm Dm'' Pm*] *tree-ordering-right*[*OF Dm Dm'' Pm*]
by (*simp split*: *dir.splits*)
with *assms* **show** *False*
by (*simp add*: *stern-brocot-iterate-def m-def m'-def split*: *dir.splits*)
qed

lemma *stern-brocot-fractions-not-repeated-parallel*:
assumes *root* (*traverse-tree path stern-brocot-iterate*) = *root* (*traverse-tree path' stern-brocot-iterate*)
assumes *p*: *path* = *pref* @ *d* # *ds*
assumes *p'*: *path'* = *pref* @ *d'* # *ds'*
assumes *dd'*: $d \neq d'$
shows *False*
proof –
define *m* **where** *m* = *root* (*traverse-tree pref* (*stern-brocot-iterate-aux I*))
then have *Dm*: *Det m* = *1* **and** *Pm*: *0* < *snd* (*mediant m*)
using *stern-brocot-iterate-aux-Det*[**where** *path*=*pref* **and** *m*=*I*] **by** *simp-all*
define *pm* **where** *pm* = *root* (*traverse-tree path* (*stern-brocot-iterate-aux I*))
then have *Dpm*: *Det pm* = *1*
using *stern-brocot-iterate-aux-Det*[**where** *path*=*path* **and** *m*=*I*] **by** *simp*
let *?M* = *case d of L* ⇒ *m* ⊗ *LL* | *R* ⇒ *m* ⊗ *UR*
from *p*
have *root* (*traverse-tree ds* (*stern-brocot-iterate-aux ?M*)) = *pm*
by(*simp add*: *stern-brocot-iterate-aux-def m-def pm-def traverse-tree-tree-iterate split*: *dir.splits*)
then obtain *pm'*
where *pm'*: *?M* ⊗ *pm'*= *pm* **and** *Dpm'*: *Det pm'* = *1*
using *stern-brocot-iterate-aux-decompose*[**where** *path*=*ds* **and** *m*=*?M*] **by** *clarsimp*
hence *case d of L* ⇒ *rat-of* (*mediant pm*) < *rat-of* (*mediant m*) | *R* ⇒ *rat-of* (*mediant m*) < *rat-of* (*mediant pm*)
using *tree-ordering-left*[*OF Dm Dpm' Pm*, *unfolded pm'*]
tree-ordering-right[*OF Dm Dpm' Pm*, *unfolded pm'*]
by (*simp split*: *dir.splits*)
moreover
define *p'm* **where** *p'm* = *root* (*traverse-tree path'* (*stern-brocot-iterate-aux I*))
then have *Dp'm*: *Det p'm* = *1*
using *stern-brocot-iterate-aux-Det*[**where** *path*=*path'* **and** *m*=*I*] **by** *simp*
let *?M'* = *case d' of L* ⇒ *m* ⊗ *LL* | *R* ⇒ *m* ⊗ *UR*
from *p'*
have *root* (*traverse-tree ds'* (*stern-brocot-iterate-aux ?M'*)) = *p'm*
by(*simp add*: *stern-brocot-iterate-aux-def m-def p'm-def traverse-tree-tree-iterate split*: *dir.splits*)
then obtain *p'm'*
where *p'm'*: *?M'* ⊗ *p'm'* = *p'm* **and** *Dp'm'*: *Det p'm'* = *1*
using *stern-brocot-iterate-aux-decompose*[**where** *path*=*ds'* **and** *m*=*?M'*] **by** *clarsimp*
hence *case d' of L* ⇒ *rat-of* (*mediant p'm*) < *rat-of* (*mediant m*) | *R* ⇒ *rat-of* (*mediant m*) < *rat-of* (*mediant p'm*)

using *tree-ordering-left*[*OF Dm Dp'm' Pm*, *unfolded pm'*]
tree-ordering-right[*OF Dm Dp'm' Pm*, *unfolded pm'*]
by (*simp split*: *dir.splits*)
ultimately show *False* **using** *pm' p'm' assms*
by(*simp add*: *m-def pm-def p'm-def stern-brocot-iterate-def split*: *dir.splits*)
qed

lemma *lists-not-eq*:
assumes *xs* ≠ *ys*
obtains
(*c1*) *strict-prefix xs ys*
| (*c2*) *strict-prefix ys xs*
| (*c3*) *ps x y xs' ys'*
where *xs* = *ps* @ *x* # *xs'* **and** *ys* = *ps* @ *y* # *ys'* **and** *x* ≠ *y*
using *assms*
by (*cases xs ys rule*: *prefix-cases*)
(*blast dest*: *parallel-decomp prefix-order.neq-le-trans*)+

lemma *stern-brocot-fractions-not-repeated*:
assumes *root* (*traverse-tree path stern-brocot-iterate*) = *root* (*traverse-tree path' stern-brocot-iterate*)
shows *path* = *path'*
proof(*rule ccontr*)
assume *path* ≠ *path'*
then show *False* **using** *assms*
by (*cases path path' rule*: *lists-not-eq*)
(*blast intro*: *stern-brocot-fractions-not-repeated-strict-prefix sym stern-brocot-fractions-not-repeated-parallel*)+
qed

The function *Fract* is injective under certain conditions.

lemma *rat-inv-eq*:
assumes *Fract a b* = *Fract c d*
assumes *b* > *0*
assumes *d* > *0*
assumes *coprime a b*
assumes *coprime c d*
shows *a* = *c* ∧ *b* = *d*
proof −
from ‹*b* > *0*› ‹*d* > *0*› ‹*Fract a b* = *Fract c d*›
have ∗: *a* ∗ *d* = *c* ∗ *b* **by** (*simp add*: *eq-rat*)
from *arg-cong*[**where** *f*=*sgn*, *OF this*] ‹*b* > *0*› ‹*d* > *0*›
have *sgn a* = *sgn c* **by** (*simp add*: *sgn-mult*)
with ∗ **show** *?thesis*
using ‹*b* > *0*› ‹*d* > *0*› *coprime-crossproduct-int*[*OF* ‹*coprime a b*› ‹*coprime c d*›]
by (*simp add*: *abs-sgn*)
qed

theorem *stern-brocot-rationals-not-repeated*:
assumes *root* (*traverse-tree path* (*pure rat-of* ⋄ *stern-brocot-recurse*))
= *root* (*traverse-tree path'* (*pure rat-of* ⋄ *stern-brocot-recurse*))
shows *path* = *path'*
using *assms*
using *stern-brocot-coprime*[**where** *path*=*path*]
stern-brocot-coprime[**where** *path*=*path'*]
stern-brocot-denominator-non-zero[**where** *path*=*path*]
stern-brocot-denominator-non-zero[**where** *path*=*path'*]
by(*auto simp*: *gcd-int-def dest*!: *rat-inv-eq intro*: *stern-brocot-fractions-not-repeated simp add*: *stern-brocot-recurse-iterate*[*symmetric*] *split*: *prod.splits*)

2.5 Equivalence of recursive and iterative version

Hinze shows that it does not matter whether we use I or F at the root provided we swap the left and right matrices too.

definition *stern-brocot-Hinze-iterate* :: *fraction tree*
where *stern-brocot-Hinze-iterate* = *map-tree mediant* (*tree-iterate* ($\lambda s.\ s \otimes UR$) ($\lambda s.\ s \otimes LL$) F)

lemma *mediant-times-F*: *mediant* ∘ ($\lambda s.\ s \otimes F$) = *mediant*
by(*simp add*: *times-matrix-def F-def mediant-def split-def o-def add.commute*)

lemma *stern-brocot-iterate*: *stern-brocot* = *stern-brocot-iterate*
proof −
have *stern-brocot* = *stern-brocot-Hinze-iterate*
unfolding *stern-brocot-def stern-brocot-Hinze-iterate-def*
by(*subst unfold-tree-tree-iterate*)(*simp add*: *F-def times-matrix-def mediant-def UR-def LL-def split-def*)
also have ... = *map-tree mediant* (*map-tree* ($\lambda s.\ s \otimes F$) (*tree-iterate* ($\lambda s.\ s \otimes LL$) ($\lambda s.\ s \otimes UR$) I))
unfolding *stern-brocot-Hinze-iterate-def*
by(*subst tree-iterate-fusion*[**where** l'=$\lambda s.\ s \otimes UR$ **and** r'=$\lambda s.\ s \otimes LL$])
(*simp-all add*: *fun-eq-iff times-matrix-def UR-def LL-def F-def I-def*)
also have ... = *stern-brocot-iterate*
by(*simp only*: *tree.map-comp mediant-times-F stern-brocot-iterate-def stern-brocot-iterate-aux-def*)
finally show *?thesis* .
qed

theorem *stern-brocot-mediant-recurse*: *stern-brocot* = *stern-brocot-recurse*
by(*simp add*: *stern-brocot-recurse-iterate stern-brocot-iterate*)

end

no-notation *times-matrix* (**infixl** ⊗ *70*)
and *times-vector* (**infixl** ⊙ *70*)

3 Linearising the Stern-Brocot Tree

3.1 Turning a tree into a stream

corec *tree-chop* :: $'a$ *tree* $\Rightarrow$ $'a$ *tree*
where *tree-chop* t = *Node* (*root* (*left* t)) (*right* t) (*tree-chop* (*left* t))

lemma *tree-chop-sel* [*simp*]:
  root (*tree-chop* t) = *root* (*left* t)
  left (*tree-chop* t) = *right* t
  right (*tree-chop* t) = *tree-chop* (*left* t)
by(*subst tree-chop.code*; *simp*; *fail*)+

tree-chop is a idiom homomorphism

lemma *tree-chop-pure-tree* [*simp*]:
  tree-chop (*pure* x) = *pure* x
by(*coinduction rule*: *tree.coinduct-strong*) *auto*

lemma *tree-chop-ap-tree* [*simp*]:
  tree-chop ($f \diamond x$) = *tree-chop* f $\diamond$ *tree-chop* x
by(*coinduction arbitrary*: f x *rule*: *tree.coinduct-strong*) *auto*

lemma *tree-chop-plus*: *tree-chop* ($t + t'$) = *tree-chop* t + *tree-chop* t'
by(*simp add*: *plus-tree-def*)

corec *stream* :: $'a$ *tree* $\Rightarrow$ $'a$ *stream*
where *stream* t = *root* t ## *stream* (*tree-chop* t)

lemma *stream-sel* [*simp*]:
  shd (*stream* t) = *root* t
  stl (*stream* t) = *stream* (*tree-chop* t)
by(*subst stream.code*; *simp*; *fail*)+

stream is an idiom homomorphism.

lemma *stream-pure* [*simp*]: *stream* (*pure* x) = *pure* x
by *coinduction auto*

lemma *stream-ap* [*simp*]: *stream* ($f \diamond x$) = *stream* f $\diamond$ *stream* x
by(*coinduction arbitrary*: f x) *auto*

lemma *stream-plus* [*simp*]: *stream* ($t + t'$) = *stream* t + *stream* t'
by(*simp add*: *plus-stream-def plus-tree-def*)

lemma *stream-minus* [*simp*]: *stream* ($t - t'$) = *stream* t − *stream* t'
by(*simp add*: *minus-stream-def minus-tree-def*)

lemma *stream-times* [*simp*]: *stream* ($t * t'$) = *stream* t $*$ *stream* t'
by(*simp add*: *times-stream-def times-tree-def*)

lemma *stream-mod* [*simp*]: *stream* (t *mod* t') = *stream t mod stream* t'
by(*simp add*: *modulo-stream-def modulo-tree-def*)

lemma *stream-1* [*simp*]: *stream 1* = *1*
by(*simp add*: *one-tree-def one-stream-def*)

lemma *stream-numeral* [*simp*]: *stream* (*numeral n*) = *numeral n*
by(*induct n*)(*simp-all only*: *numeral.simps stream-plus stream-1*)

3.2 Split the Stern-Brocot tree into numerators and denumerators

corec *num-den* :: *bool* ⇒ *nat tree*
where
  num-den x =
   Node 1
    (*if x then num-den True else num-den True* + *num-den False*)
    (*if x then num-den True* + *num-den False else num-den False*)

abbreviation *num* **where** *num* ≡ *num-den True*
abbreviation *den* **where** *den* ≡ *num-den False*

lemma *num-unfold*: *num* = *Node 1 num* (*num* + *den*)
by(*subst num-den.code*; *simp*)

lemma *den-unfold*: *den* = *Node 1* (*num* + *den*) *den*
by(*subst num-den.code*; *simp*)

lemma *num-simps* [*simp*]:
  root num = *1*
  left num = *num*
  right num = *num* + *den*
by(*subst num-unfold*, *simp*)+

lemma *den-simps* [*simp*]:
  root den = *1*
  left den = *num* + *den*
  right den = *den*
by (*subst den-unfold*, *simp*)+

lemma *stern-brocot-num-den*:
  pure-tree Pair ◇ *num* ◇ *den* = *stern-brocot-recurse*
apply(*rule stern-brocot-recurse.unique*)
apply(*subst den-unfold*)
apply(*subst num-unfold*)
apply(*simp*; *intro conjI*)
apply(*applicative-lifting*; *simp*)+
done

lemma *den-eq-chop-num*: *den* = *tree-chop num*
by(*coinduction rule*: *tree.coinduct-strong*) *simp*

lemma *num-conv*: *num* = *pure fst* ◇ *stern-brocot-recurse*
unfolding *stern-brocot-num-den*[*symmetric*]
apply(*simp add*: *map-tree-ap-tree-pure-tree stern-brocot-num-den*[*symmetric*])
apply(*applicative-lifting*; *simp*)
done

lemma *den-conv*: *den* = *pure snd* ◇ *stern-brocot-recurse*
unfolding *stern-brocot-num-den*[*symmetric*]
apply(*simp add*: *map-tree-ap-tree-pure-tree stern-brocot-num-den*[*symmetric*])
apply(*applicative-lifting*; *simp*)
done

corec *num-mod-den* :: *nat tree*
where *num-mod-den* = *Node 0 num num-mod-den*

lemma *num-mod-den-simps* [*simp*]:
  root num-mod-den = *0*
  left num-mod-den = *num*
  right num-mod-den = *num-mod-den*
by(*subst num-mod-den.code*; *simp*; *fail*)+

The arithmetic transformations need the precondition that *den* contains only positive numbers, no *0*. Hinze (2009, p502) gets a bit sloppy here; it is not straightforward to adapt his lifting framework Hinze (2010) to conditional equations.

lemma *mod-tree-lemma1*:
  fixes *x* :: *nat tree*
  assumes ∀ *i*∈*set-tree y*. *0* < *i*
  shows *x mod* (*x* + *y*) = *x*
proof −
  have *rel-tree* (=) (*x mod* (*x* + *y*)) *x* **by** *applicative-lifting*(*simp add*: *assms*)
  thus *?thesis* **by**(*unfold tree.rel-eq*)
qed

lemma *mod-tree-lemma2*:
  fixes *x y* :: *'a* :: *unique-euclidean-semiring tree*
  shows (*x* + *y*) *mod y* = *x mod y*
by *applicative-lifting simp*

lemma *set-tree-pathD*: *x* ∈ *set-tree t* ⟹ ∃ *p*. *x* = *root* (*traverse-tree p t*)
by(*induct rule*: *set-tree-induct*)(*auto intro*: *exI*[**where** *x*=[]] *exI*[**where** *x*=*L* # *p* **for** *p*] *exI*[**where** *x*=*R* # *p* **for** *p*])

lemma *den-gt-0*: *0* < *x* **if** *x* ∈ *set-tree den*
proof −
 from *that* **obtain** *p* **where** *x* = *root* (*traverse-tree p den*) **by**(*blast dest*: *set-tree-pathD*)

with *stern-brocot-denominator-non-zero*[*of p*] **show** $0 < x$ **by**(*simp add*: *den-conv split-beta*)
qed

lemma *num-mod-den*: *num mod den = num-mod-den*
by(*rule num-mod-den.unique*)(*rule tree.expand*, *simp add*: *mod-tree-lemma2 mod-tree-lemma1 den-gt-0*)

lemma *tree-chop-den*: *tree-chop den* $=$ *num* $+$ *den* $- 2 *$ (*num mod den*)
proof $-$
have *le*: $0 < y \Longrightarrow 2 * (x \ mod \ y) \leq x + y$ **for** $x\ y$ :: *nat*
by (*simp add*: *mult-2 add-mono*)

We switch to *int* such that all cancellation laws are available.

define *den′* **where** *den′* = *pure int* ⋄ *den*
define *num′* **where** *num′* = *pure int* ⋄ *num*
define *num-mod-den′* **where** *num-mod-den′* = *pure int* ⋄ *num-mod-den*

have [*simp*]: *root num′* = *1* *left num′* = *num′* **unfolding** *den′-def num′-def* **by** *simp-all*
have [*simp*]: *right num′* = *num′* + *den′* **unfolding** *den′-def num′-def ap-tree.sel pure-tree-simps num-simps*
by *applicative-lifting simp*

have *num-mod-den′-simps* [*simp*]: *root num-mod-den′* = *0* *left num-mod-den′* = *num′* *right num-mod-den′* = *num-mod-den′*
by(*simp-all add*: *num-mod-den′-def num′-def*)
have *den′-eq-chop-num′*: *den′* = *tree-chop num′* **by**(*simp add*: *den′-def num′-def den-eq-chop-num*)
have *num-mod-den′2-unique*: $\bigwedge x.\ x =$ *Node* 0 ($2 *$ *num′*) $x \Longrightarrow x = 2 *$ *num-mod-den′*
by(*corec-unique*)(*rule tree.expand*; *simp*)
have *num′-plus-den′-minus-chop-den′*: *num′* + *den′* − *tree-chop den′* = $2 *$ *num-mod-den′*
by(*rule num-mod-den′2-unique*)(*rule tree.expand*, *simp add*: *tree-chop-plus den′-eq-chop-num′*)

have *tree-chop den* = *pure nat* ⋄ (*tree-chop den′*)
unfolding *den-conv tree-chop-ap-tree tree-chop-pure-tree den′-def* **by** *applicative-nf simp*
also have *tree-chop den′* = *num′* + *den′* − *tree-chop den′* + *tree-chop den′* − $2 *$ *num-mod-den′*
by(*subst num′-plus-den′-minus-chop-den′*) *simp*
also have . . . = *num′* + *den′* − $2 *$ (*num′ mod den′*)
unfolding *num-mod-den′-def num′-def den′-def num-mod-den*[*symmetric*]
by *applicative-lifting*(*simp add*: *zmod-int*)
also have [*unfolded tree.rel-eq*]: *rel-tree* (=) . . . (*pure int* ⋄ (*num* + *den* − $2 *$ (*num mod den*)))
unfolding *num′-def den′-def* **by**(*applicative-lifting*)(*simp add*: *of-nat-diff zmod-int le den-gt-0*)

also have *pure nat* $\diamond$ (*pure int* $\diamond$ (*num* + *den* − *2* ∗ (*num mod den*))) = *num* + *den* − *2* ∗ (*num mod den*) **by**(*applicative-nf*) *simp*
finally show *?thesis* **.**
qed

3.3 Loopless linearisation of the Stern-Brocot tree.

This is a loopless linearisation of the Stern-Brocot tree that gives Stern's diatomic sequence, which is also known as Dijkstra's fusc function Dijkstra (1982a,b). Loopless à la Bird (2006) means that the first element of the stream can be computed in linear time and every further element in constant time.

friend-of-corec *smap* :: (*'a* ⇒ *'a*) ⇒ *'a stream* ⇒ *'a stream*
where *smap f xs* = *SCons* (*f* (*shd xs*)) (*smap f* (*stl xs*))
subgoal by(*rule stream.expand*) *simp*
subgoal by(*fold relator-eq*)(*transfer-prover*)
done

definition *step* :: *nat* × *nat* ⇒ *nat* × *nat*
where *step* = (λ(*n*, *d*). (*d*, *n* + *d* − *2* ∗ (*n mod d*)))

corec *stern-brocot-loopless* :: *fraction stream*
where *stern-brocot-loopless* = (*1*, *1*) ## *smap step stern-brocot-loopless*

lemmas *stern-brocot-loopless-rec* = *stern-brocot-loopless.code*

friend-of-corec *plus* **where** *s* + *s'* = (*shd s* + *shd s'*) ## (*stl s* + *stl s'*)
subgoal by (*rule stream.expand*; *simp add*: *plus-stream-shd plus-stream-stl*)
subgoal by *transfer-prover*
done

friend-of-corec *minus* **where** *t* − *t'* = (*shd t* − *shd t'*) ## (*stl t* − *stl t'*)
subgoal by (*rule stream.expand*; *simp add*: *minus-stream-def*)
subgoal by *transfer-prover*
done

friend-of-corec *times* **where** *t* ∗ *t'* = (*shd t* ∗ *shd t'*) ## (*stl t* ∗ *stl t'*)
subgoal by (*rule stream.expand*; *simp add*: *times-stream-def*)
subgoal by *transfer-prover*
done

friend-of-corec *modulo* **where** *t mod t'* = (*shd t mod shd t'*) ## (*stl t mod stl t'*)
subgoal by (*rule stream.expand*; *simp add*: *modulo-stream-def*)
subgoal by *transfer-prover*
done

corec *fusc'* :: *nat stream*
where *fusc'* = *1* ## (((*1* ## *fusc'*) + *fusc'*) − *2* ∗ ((*1* ## *fusc'*) *mod fusc'*))

definition *fusc* **where** *fusc* = *1* ## *fusc′*

lemma *fusc-unfold*: *fusc* = *1* ## *fusc′* **by**(*fact fusc-def*)

lemma *fusc′-unfold*: *fusc′* = *1* ## (*fusc* + *fusc′* − *2* ∗ (*fusc mod fusc′*))
by(*subst fusc′.code*)(*simp add*: *fusc-def*)

lemma *fusc-simps* [*simp*]:
  shd fusc = *1*
  stl fusc = *fusc′*
by(*simp-all add*: *fusc-unfold*)

lemma *fusc′-simps* [*simp*]:
  shd fusc′ = *1*
  stl fusc′ = *fusc* + *fusc′* − *2* ∗ (*fusc mod fusc′*)
by(*subst fusc′-unfold*, *simp*)+

3.4 Equivalence with Dijkstra's fusc function

lemma *stern-brocot-loopless-siterate*: *stern-brocot-loopless* = *siterate step* (*1*, *1*)
by(*rule stern-brocot-loopless.unique*[*symmetric*])(*rule stream.expand*; *simp add*: *smap-siterate*[*symmetric*])

lemma *fusc-fusc′-iterate*: *pure Pair* ⋄ *fusc* ⋄ *fusc′* = *stern-brocot-loopless*
apply(*rule stern-brocot-loopless.unique*)
apply(*rule stream.expand*; *simp add*: *step-def*)
apply(*applicative-lifting*; *simp*)
done

theorem *stern-brocot-loopless*:
  stream stern-brocot-recurse = *stern-brocot-loopless* (**is** *?lhs* = *?rhs*)
proof(*rule stern-brocot-loopless.unique*)
 have *eq*: *?lhs* = *stream* (*pure-tree Pair* ⋄ *num* ⋄ *den*) **by** (*simp only*: *stern-brocot-num-den*)
  have *num*: *stream num* = *1* ## *stream den*
    by (*rule stream.expand*) (*simp add*: *den-eq-chop-num*)
  have *den*: *stream den* = *1* ## (*stream num* + *stream den* − *2* ∗ (*stream num*
mod stream den))
    by (*rule stream.expand*)(*simp add*: *tree-chop-den*)
  show *?lhs* = (*1*, *1*) ## *smap step ?lhs* **unfolding** *eq*
    by(*rule stream.expand*)(*simp add*: *den-eq-chop-num*[*symmetric*] *tree-chop-den*;
applicative-lifting; *simp add*: *step-def*)
qed

end

4 The Bird tree

We define the Bird tree following Hinze (2009) and prove that it is a permutation of the Stern-Brocot tree. As a corollary, we derive that the Bird tree also contains all rational numbers in lowest terms exactly once.

theory *Bird-Tree* **imports** *Stern-Brocot-Tree* **begin**

corec *bird* :: *fraction tree*
where
bird = *Node* (*1*, *1*) (*map-tree recip* (*map-tree succ bird*)) (*map-tree succ* (*map-tree recip bird*))

lemma *bird-unfold*:
bird = *Node* (*1*, *1*) (*pure recip* ◇ (*pure succ* ◇ *bird*)) (*pure succ* ◇ (*pure recip* ◇ *bird*))
using *bird.code* **unfolding** *map-tree-ap-tree-pure-tree*[*symmetric*] **.**

lemma *bird-simps* [*simp*]:
root bird = (*1*, *1*)
left bird = *pure recip* ◇ (*pure succ* ◇ *bird*)
right bird = *pure succ* ◇ (*pure recip* ◇ *bird*)
by(*subst bird-unfold*, *simp*)+

lemma *mirror-bird*: *mirror bird* = *pure recip* ◇ *bird* (**is** *?lhs* = *?rhs*)
proof(*rule sym*)
let *?F* = λ*t*. *Node* (*1*, *1*) (*map-tree succ* (*map-tree recip t*)) (*map-tree recip* (*map-tree succ t*))
have ∗: *mirror bird* = *?F* (*mirror bird*)
by(*rule tree.expand*; *simp add*: *mirror-ap-tree mirror-pure map-tree-ap-tree-pure-tree*[*symmetric*])
show *t* = *mirror bird* **when** *t* = *?F t* **for** *t* **using** *that* **by** *corec-unique* (*fact* ∗)
show *pure recip* ◇ *bird* = *?F* (*pure recip* ◇ *bird*)
by(*rule tree.expand*; *simp add*: *map-tree-ap-tree-pure-tree*; *applicative-lifting*; *simp add*: *split-beta*)
qed

primcorec *even-odd-mirror* :: *bool* ⇒ *'a tree* ⇒ *'a tree*
where
⋀*even*. *root* (*even-odd-mirror even t*) = *root t*
| ⋀*even*. *left* (*even-odd-mirror even t*) = *even-odd-mirror* (¬ *even*) (*if even then right t else left t*)
| ⋀*even*. *right* (*even-odd-mirror even t*) = *even-odd-mirror* (¬ *even*) (*if even then left t else right t*)

definition *even-mirror* :: *'a tree* ⇒ *'a tree*
where *even-mirror* = *even-odd-mirror True*

definition *odd-mirror* :: *'a tree* ⇒ *'a tree*
where *odd-mirror* = *even-odd-mirror False*

lemma *even-mirror-simps* [*simp*]:
  root (*even-mirror t*) = *root t*
  left (*even-mirror t*) = *odd-mirror* (*right t*)
  right (*even-mirror t*) = *odd-mirror* (*left t*)
  and *odd-mirror-simps* [*simp*]:
  root (*odd-mirror t*) = *root t*
  left (*odd-mirror t*) = *even-mirror* (*left t*)
  right (*odd-mirror t*) = *even-mirror* (*right t*)
by(*simp-all add*: *even-mirror-def odd-mirror-def*)

lemma *even-odd-mirror-pure* [*simp*]: **fixes** *even* **shows**
  even-odd-mirror even (*pure-tree x*) = *pure-tree x*
by(*coinduction arbitrary*: *even*) *auto*

lemma *even-odd-mirror-ap-tree* [*simp*]: **fixes** *even* **shows**
  even-odd-mirror even ($f \diamond x$) = *even-odd-mirror even f* $\diamond$ *even-odd-mirror even x*
by(*coinduction arbitrary*: *even f x*) *auto*

lemma [*simp*]:
  shows *even-mirror-pure*: *even-mirror* (*pure-tree x*) = *pure-tree x*
  and *odd-mirror-pure*: *odd-mirror* (*pure-tree x*) = *pure-tree x*
by(*simp-all add*: *even-mirror-def odd-mirror-def*)

lemma [*simp*]:
  shows *even-mirror-ap-tree*: *even-mirror* ($f \diamond x$) = *even-mirror f* $\diamond$ *even-mirror x*
  and *odd-mirror-ap-tree*: *odd-mirror* ($f \diamond x$) = *odd-mirror f* $\diamond$ *odd-mirror x*
by(*simp-all add*: *even-mirror-def odd-mirror-def*)

fun *even-mirror-path* :: *path* $\Rightarrow$ *path*
  and *odd-mirror-path* :: *path* $\Rightarrow$ *path*
where
  even-mirror-path [] = []
| *even-mirror-path* (*d* # *ds*) = (*case d of L* $\Rightarrow$ *R* | *R* $\Rightarrow$ *L*) # *odd-mirror-path ds*
| *odd-mirror-path* [] = []
| *odd-mirror-path* (*d* # *ds*) = *d* # *even-mirror-path ds*

lemma *even-mirror-traverse-tree* [*simp*]:
  root (*traverse-tree path* (*even-mirror t*)) = *root* (*traverse-tree* (*even-mirror-path path*) *t*)
  and *odd-mirror-traverse-tree* [*simp*]:
  root (*traverse-tree path* (*odd-mirror t*)) = *root* (*traverse-tree* (*odd-mirror-path path*) *t*)
by (*induct path arbitrary*: *t*) (*simp-all split*: *dir.splits*)

lemma *even-odd-mirror-path-involution* [*simp*]:
  even-mirror-path (*even-mirror-path path*) = *path*
  odd-mirror-path (*odd-mirror-path path*) = *path*
by (*induct path*) (*simp-all split*: *dir.splits*)

lemma *even-odd-mirror-path-injective* [*simp*]:
even-mirror-path path = *even-mirror-path path*′ ⟷ *path* = *path*′
odd-mirror-path path = *odd-mirror-path path*′ ⟷ *path* = *path*′
by (*induct path arbitrary*: *path*′) (*case-tac path*′, *simp-all split*: *dir.splits*)+

lemma *odd-mirror-bird-stern-brocot*:
odd-mirror bird = *stern-brocot-recurse*
proof −
let *?rsrs* = *map-tree* (*recip* ∘ *succ* ∘ *recip* ∘ *succ*)
let *?rssr* = *map-tree* (*recip* ∘ *succ* ∘ *succ* ∘ *recip*)
let *?srrs* = *map-tree* (*succ* ∘ *recip* ∘ *recip* ∘ *succ*)
let *?srsr* = *map-tree* (*succ* ∘ *recip* ∘ *succ* ∘ *recip*)
let *?R* = λ*t*. *Node* (*1*, *1*) (*Node* (*1*, *2*) (*?rssr t*) (*?rsrs t*)) (*Node* (*2*, *1*) (*?srsr t*) (*?srrs t*))

have ∗: *stern-brocot-recurse* = *?R stern-brocot-recurse*
by(*rule tree.expand*; *simp*; *intro conjI*; *rule tree.expand*; *simp*; *intro conjI*) — Expand the tree twice
(*applicative-lifting*, *simp add*: *split-beta*)+
show *f* = *stern-brocot-recurse* **when** *f* = *?R f* **for** *f* **using** *that* ∗ **by** *corec-unique*
show *odd-mirror bird* = *?R* (*odd-mirror bird*)
by(*rule tree.expand*; *simp*; *intro conjI*; *rule tree.expand*; *simp*; *intro conjI*) — Expand the tree twice
(*applicative-lifting*; *simp*)+
qed

theorem *bird-rationals*:
assumes *m* > *0* *n* > *0*
shows *root* (*traverse-tree* (*odd-mirror-path* (*mk-path m n*)) (*pure rat-of* ⋄ *bird*)) = *Fract* (*int m*) (*int n*)
using *stern-brocot-rationals*[*OF assms*]
by (*simp add*: *odd-mirror-bird-stern-brocot*[*symmetric*])

theorem *bird-rationals-not-repeated*:
root (*traverse-tree path* (*pure rat-of* ⋄ *bird*)) = *root* (*traverse-tree path*′ (*pure rat-of* ⋄ *bird*))
⟹ *path* = *path*′
using *stern-brocot-rationals-not-repeated*[**where** *path*=*odd-mirror-path path* **and** *path*′=*odd-mirror-path path*′]
by (*simp add*: *odd-mirror-bird-stern-brocot*[*symmetric*])

end

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