Standard Borel Spaces

Michikazu Hirata

October 12, 2023

Abstract

This entry includes a formalization of standard Borel spaces and (a variant of) the Borel isomorphism theorem. A separable complete metrizable topological space is called a polish space and a measurable space generated from a polish space is called a standard Borel space. We formalize the notion of standard Borel spaces by establishing set-based metric spaces, and then prove (a variant of) the Borel isomorphism theorem. The theorem states that a standard Borel spaces is either a countable discrete space or isomorphic to $\mathbb{R}$.

Contents

We refer to the HOL-Analysis library, the textbooks by Matsuzaka [2] and Srivastava [3], and the lecture note by Biskup [1].

1 Lemmas

theory *Lemmas-StandardBorel*
imports *HOL−Probability.Probability*
begin

1.1 Lemmas for Abstract Topology

1.1.1 Generated By

lemma *topology-generated-by-sub*:
assumes $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow (openin\ X\ U)$
and $openin\ (topology\text{-}generated\text{-}by\ \mathcal{U})\ U$
shows $openin\ X\ U$
⟨*proof*⟩

lemma *topology-generated-by-open*:
$S = topology\text{-}generated\text{-}by\ \{U \mid U\ .\ openin\ S\ U\}$
⟨*proof*⟩

lemma *topology-generated-by-eq*:
assumes $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow (openin\ (topology\text{-}generated\text{-}by\ \mathcal{O})\ U)$
and $\bigwedge U.\ U \in \mathcal{O} \Longrightarrow (openin\ (topology\text{-}generated\text{-}by\ \mathcal{U})\ U)$
shows $topology\text{-}generated\text{-}by\ \mathcal{O} = topology\text{-}generated\text{-}by\ \mathcal{U}$
⟨*proof*⟩

lemma *topology-generated-by-homeomorphic-spaces*:

assumes *homeomorphic-map* X Y f X = *topology-generated-by* $\mathcal{O}$
shows Y = *topology-generated-by* $((`) \ f \ ` \ \mathcal{O})$
⟨*proof*⟩

lemma *open-map-generated-topo*:
assumes $\bigwedge u.\ u \in U \Longrightarrow$ *openin* S $(f \ ` \ u)$ *inj-on* f (*topspace* (*topology-generated-by* U))
shows *open-map* (*topology-generated-by* U) S f
⟨*proof*⟩

lemma *subtopology-generated-by*:
subtopology (*topology-generated-by* $\mathcal{O}$) T = *topology-generated-by* $\{T \cap U \mid U.\ U \in \mathcal{O}\}$
⟨*proof*⟩

lemma *prod-topology-generated-by*:
topology-generated-by $\{ U \times V \mid U\ V.\ U \in \mathcal{O} \land V \in \mathcal{U}\}$ = *prod-topology* (*topology-generated-by* $\mathcal{O}$) (*topology-generated-by* $\mathcal{U}$)
⟨*proof*⟩

lemma *prod-topology-generated-by-open*:
prod-topology S S' = *topology-generated-by* $\{U \times V \mid U\ V.$ *openin* S $U \land$ *openin* S' $V\}$
⟨*proof*⟩

lemma *product-topology-cong*:
assumes $\bigwedge i.\ i \in I \Longrightarrow S\ i = K\ i$
shows *product-topology* S I = *product-topology* K I
⟨*proof*⟩

lemma *topology-generated-by-without-empty*:
topology-generated-by $\mathcal{O}$ = *topology-generated-by* $\{ U \in \mathcal{O}.\ U \neq \{\}\}$
⟨*proof*⟩

lemma *topology-from-bij*:
assumes *bij-betw* f A (*topspace* S)
shows *homeomorphic-map* (*pullback-topology* A f S) S f *topspace* (*pullback-topology* A f S) = A
⟨*proof*⟩

lemma *openin-pullback-topology'*:
assumes *bij-betw* f A (*topspace* S)
shows *openin* (*pullback-topology* A f S) $u \longleftrightarrow$ (*openin* S $(f \ ` \ u)$) $\land u \subseteq A$
⟨*proof*⟩

1.1.2 Isolated Point

definition *isolated-points-of* :: *'a topology* ⇒ *'a set* ⇒ *'a set* (**infixr** *isolated'-points'-of* *80*) **where**

X *isolated-points-of* $A \equiv \{x{\in}\mathit{topspace}\ X \cap A.\ x \notin X\ \mathit{derived\text{-}set\text{-}of}\ A\}$

lemma *isolated-points-of-eq*:
$X\ \mathit{isolated\text{-}points\text{-}of}\ A = \{x{\in}\mathit{topspace}\ X \cap A.\ \exists\, U.\ x \in U \wedge \mathit{openin}\ X\ U \wedge U \cap (A - \{x\}) = \{\}\}$
⟨*proof*⟩

lemma *in-isolated-points-of*:
$x \in X\ \mathit{isolated\text{-}points\text{-}of}\ A \longleftrightarrow x \in \mathit{topspace}\ X \wedge x \in A \wedge (\exists\, U.\ x \in U \wedge \mathit{openin}\ X\ U \wedge U \cap (A - \{x\}) = \{\})$
⟨*proof*⟩

lemma *derived-set-of-eq*:
$x \in X\ \mathit{derived\text{-}set\text{-}of}\ A \longleftrightarrow x \in X\ \mathit{closure\text{-}of}\ (A - \{x\})$
⟨*proof*⟩

1.1.3 Perfect Set

definition *perfect-set* :: $'a\ \mathit{topology} \Rightarrow {'a}\ \mathit{set} \Rightarrow \mathit{bool}$ **where**
$\mathit{perfect\text{-}set}\ X\ A \longleftrightarrow \mathit{closedin}\ X\ A \wedge X\ \mathit{isolated\text{-}points\text{-}of}\ A = \{\}$

abbreviation $\mathit{perfect\text{-}space}\ X \equiv \mathit{perfect\text{-}set}\ X\ (\mathit{topspace}\ X)$

lemma *perfect-setI*:
assumes $\mathit{closedin}\ X\ A$
and $\bigwedge x\ T.\ [\![x \in A;\ x \in T;\ \mathit{openin}\ X\ T]\!] \Longrightarrow \exists\, y{\neq}x.\ y \in T \wedge y \in A$
shows $\mathit{perfect\text{-}set}\ X\ A$
⟨*proof*⟩

lemma *perfect-spaceI*:
assumes $\bigwedge x\ T.\ [\![x \in T;\ \mathit{openin}\ X\ T]\!] \Longrightarrow \exists\, y{\neq}x.\ y \in T$
shows $\mathit{perfect\text{-}space}\ X$
⟨*proof*⟩

lemma *perfect-setD*:
assumes $\mathit{perfect\text{-}set}\ X\ A$
shows $\mathit{closedin}\ X\ A$ $A \subseteq \mathit{topspace}\ X$ $\bigwedge x\ T.\ [\![x \in A;\ x \in T;\ \mathit{openin}\ X\ T]\!] \Longrightarrow \exists\, y{\neq}x.\ y \in T \wedge y \in A$
⟨*proof*⟩

lemma *perfect-space-perfect*:
$\mathit{perfect\text{-}set}\ \mathit{euclidean}\ (\mathit{UNIV} :: {'a} :: \mathit{perfect\text{-}space}\ \mathit{set})$
⟨*proof*⟩

lemma *perfect-set-subtopology*:
assumes $\mathit{perfect\text{-}set}\ X\ A$
shows $\mathit{perfect\text{-}space}\ (\mathit{subtopology}\ X\ A)$
⟨*proof*⟩

1.1.4 Bases and Sub-Bases in Abstract Topology

definition *subbase-of* :: $['a\ topology, 'a\ set\ set] \Rightarrow bool$ **where**
subbase-of $S\ \mathcal{O} \longleftrightarrow S = \textit{topology-generated-by}\ \mathcal{O}$

definition *base-of* :: $['a\ topology, 'a\ set\ set] \Rightarrow bool$ **where**
base-of $S\ \mathcal{O} \longleftrightarrow (\forall\, U.\ \textit{openin}\ S\ U \longleftrightarrow (\exists \mathcal{U}.\ U = \bigcup \mathcal{U} \wedge \mathcal{U} \subseteq \mathcal{O}))$

definition *second-countable* :: $'a\ topology \Rightarrow bool$ **where**
second-countable $S \longleftrightarrow (\exists\, \mathcal{O}.\ \textit{countable}\ \mathcal{O} \wedge \textit{base-of}\ S\ \mathcal{O})$

definition *zero-dimensional* :: $'a\ topology \Rightarrow bool$ **where**
zero-dimensional $S \longleftrightarrow (\exists\, \mathcal{O}.\ \textit{base-of}\ S\ \mathcal{O} \wedge (\forall\, u{\in}\mathcal{O}.\ \textit{openin}\ S\ u \wedge \textit{closedin}\ S\ u))$

lemma *openin-base*:
  assumes *base-of* $S\ \mathcal{O}\quad U = \bigcup \mathcal{U}$ **and** $\mathcal{U} \subseteq \mathcal{O}$
  shows *openin* $S\ U$
  ⟨*proof*⟩

lemma *base-is-subbase*:
  assumes *base-of* $S\ \mathcal{O}$
  shows *subbase-of* $S\ \mathcal{O}$
  ⟨*proof*⟩

lemma *subbase-of-subset*:
  assumes *subbase-of* $S\ \mathcal{O}$ **and** $U \in \mathcal{O}$
  shows $U \subseteq \textit{topspace}\ S$
  ⟨*proof*⟩

lemma *subbase-of-openin*:
  assumes *subbase-of* $S\ \mathcal{O}$ **and** $U \in \mathcal{O}$
  shows *openin* $S\ U$
  ⟨*proof*⟩

lemma *base-of-subset*:
  assumes *base-of* $S\ \mathcal{O}$ **and** $U \in \mathcal{O}$
  shows $U \subseteq \textit{topspace}\ S$
  ⟨*proof*⟩

lemma *base-of-openin*:
  assumes *base-of* $S\ \mathcal{O}$ **and** $U \in \mathcal{O}$
  shows *openin* $S\ U$
  ⟨*proof*⟩

lemma *base-of-def2*:
  assumes $\bigwedge U.\ U \in \mathcal{O} \Longrightarrow \textit{openin}\ S\ U$
  shows *base-of* $S\ \mathcal{O} \longleftrightarrow (\forall\, U.\ \textit{openin}\ S\ U \longrightarrow (\forall\, x{\in}U.\ \exists\, W{\in}\mathcal{O}.\ x \in W \wedge W \subseteq U))$
⟨*proof*⟩

lemma *base-of-def2'*:
base-of S $\mathcal{O}$ $\longleftrightarrow$ $(\forall\, b{\in}\mathcal{O}.\ openin\ S\ b) \wedge (\forall\, x.\ openin\ S\ x \longrightarrow (\exists\, B'{\subseteq}\mathcal{O}.\ \bigcup\ B' = x))$
$\langle proof \rangle$

corollary *base-of-in-subset*:
assumes *base-of* S $\mathcal{O}$ *openin* S u $x \in u$
shows $\exists\, v{\in}\mathcal{O}.\ x \in v \wedge v \subseteq u$
$\langle proof \rangle$

lemma *base-of-without-empty*:
assumes *base-of* S $\mathcal{O}$
shows *base-of* S $\{U \in \mathcal{O}.\ U \neq \{\}\}$
$\langle proof \rangle$

lemma *second-countable-ex-without-empty*:
assumes *second-countable* S
shows $\exists\, \mathcal{O}.\ countable\ \mathcal{O} \wedge base\text{-}of\ S\ \mathcal{O} \wedge (\forall\, U{\in}\mathcal{O}.\ U \neq \{\})$
$\langle proof \rangle$

lemma *subtopology-subbase-of*:
assumes *subbase-of* S $\mathcal{O}$
shows *subbase-of* (*subtopology* S T) $\{T \cap U \mid U.\ U \in \mathcal{O}\}$
$\langle proof \rangle$

lemma *subtopology-base-of*:
assumes *base-of* S $\mathcal{O}$
shows *base-of* (*subtopology* S T) $\{T \cap U \mid U.\ U \in \mathcal{O}\}$
$\langle proof \rangle$

lemma *second-countable-subtopology*:
assumes *second-countable* S
shows *second-countable* (*subtopology* S T)
$\langle proof \rangle$

lemma *Lindelof-of*:
assumes *second-countable* S $\bigwedge u.\ u \in U \Longrightarrow openin\ S\ u$ $\bigcup\ U = topspace\ S$
shows $\exists\, U'.\ countable\ U' \wedge U' \subseteq U \wedge \bigcup\ U' = topspace\ S$
$\langle proof \rangle$

lemma *open-map-with-base*:
assumes *base-of* S $\mathcal{O}$ $\bigwedge A.\ A \in \mathcal{O} \Longrightarrow openin\ S'\ (f\ `\ A)$
shows *open-map* S S' f
$\langle proof \rangle$

Construct a base from a subbase.

definition *finite-intersections* :: *'a set set* $\Rightarrow$ *'a set set* **where**
finite-intersections $\mathcal{O} \equiv \{\ \bigcap \mathcal{O}' \mid \mathcal{O}'.\ \mathcal{O}' \neq \{\} \wedge finite\ \mathcal{O}' \wedge \mathcal{O}' \subseteq \mathcal{O}\}$

lemma *finite-intersections-inI*:
assumes $U = \bigcap \mathcal{O}'$ $\mathcal{O}' \neq \{\}$ *finite* $\mathcal{O}'$ **and** $\mathcal{O}' \subseteq \mathcal{O}$
shows $U \in$ *finite-intersections* $\mathcal{O}$
⟨*proof*⟩

lemma *finite-intersections-Uin*:
assumes $U \in \mathcal{O}$
shows $U \in$ *finite-intersections* $\mathcal{O}$
⟨*proof*⟩

lemma *finite-intersections-int*:
assumes $U \in$ *finite-intersections* $\mathcal{O}$ **and** $V \in$ *finite-intersections* $\mathcal{O}$
shows $U \cap V \in$ *finite-intersections* $\mathcal{O}$
⟨*proof*⟩

lemma *finite-intersections-countable*:
assumes *countable* $\mathcal{O}$
shows *countable* (*finite-intersections* $\mathcal{O}$)
⟨*proof*⟩

lemma *finite-intersections-openin*:
assumes $U \in$ *finite-intersections* $\mathcal{O}$
shows *openin* (*topology-generated-by* $\mathcal{O}$) U
⟨*proof*⟩

lemma *topology-generated-by-finite-intersections*:
topology-generated-by $\mathcal{O}$ = *topology-generated-by* (*finite-intersections* $\mathcal{O}$)
⟨*proof*⟩

lemma *topology-generated-by-is-union-of-finite-intersections*:
openin (*topology-generated-by* $\mathcal{O}$) $U \longleftrightarrow (\exists \mathcal{U}.\ U = \bigcup \mathcal{U} \wedge \mathcal{U} \subseteq$ *finite-intersections* $\mathcal{O})$
⟨*proof*⟩

lemma *base-from-subbase*:
assumes *subbase-of* S $\mathcal{O}$
shows *base-of* S (*finite-intersections* $\mathcal{O}$)
⟨*proof*⟩

lemma *countable-base-from-countable-subbase*:
assumes *countable* $\mathcal{O}$ **and** *subbase-of* S $\mathcal{O}$
shows *second-countable* S
⟨*proof*⟩

lemma *prod-topology-second-countable*:
assumes *second-countable* S **and** *second-countable* S'
shows *second-countable* (*prod-topology* S S')
⟨*proof*⟩

Abstract version of the theorem $\exists K.$ *topological-basis* K $\wedge$ *countable* K $\wedge$

$(\forall\, k{\in}K.\ \exists X.\ k = Pi_E\ \ UNIV\ X \wedge (\forall\, i.\ open\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq UNIV\})$.

lemma *product-topology-countable-base-of*:
assumes *countable I* **and** $\bigwedge i.\ i \in I \Longrightarrow$ *second-countable* $(S\ i)$
shows $\exists \mathcal{O}'.$ *countable* $\mathcal{O}' \wedge$ *base-of* (*product-topology* $S\ I$) $\mathcal{O}' \wedge$
$(\forall\, k \in \mathcal{O}'.\ \exists X.\ k = (\Pi_E\ i{\in}I.\ X\ i) \wedge (\forall\, i.\ openin\ (S\ i)\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq topspace\ (S\ i)\} \wedge \{i.\ X\ i \neq topspace\ (S\ i)\} \subseteq I)$
⟨*proof*⟩

lemma *product-topology-second-countable*:
assumes *countable I* **and** $\bigwedge i.\ i \in I \Longrightarrow$ *second-countable* $(S\ i)$
shows *second-countable* (*product-topology* $S\ I$)
⟨*proof*⟩

lemma *Cantor-Bendixon*:
assumes *second-countable X*
shows $\exists\, U\ P.\ countable\ U \wedge openin\ X\ U \wedge perfect\text{-}set\ X\ P \wedge U \cup P = topspace\ X \wedge U \cap P = \{\} \wedge (\forall\, a{\neq}\{\}.\ openin\ (subtopology\ X\ P)\ a \longrightarrow uncountable\ a)$
⟨*proof*⟩

1.1.5 Dense and Separable in Abstract Topology

definition *dense-of* :: [*'a topology*, *'a set*] ⇒ *bool* **where**
dense-of $S\ U \longleftrightarrow (U \subseteq topspace\ S \wedge (\forall\, V.\ openin\ S\ V \longrightarrow V \neq \{\} \longrightarrow U \cap V \neq \{\}))$

lemma *dense-of-def2*:
dense-of $S\ U \longleftrightarrow (U \subseteq topspace\ S \wedge (S\ closure\text{-}of\ U) = topspace\ S)$
⟨*proof*⟩

lemma *dense-of-subset*:
assumes *dense-of S U*
shows $U \subseteq topspace\ S$
⟨*proof*⟩

lemma *dense-of-nonempty*:
assumes $topspace\ S \neq \{\}$ *dense-of S U*
shows $U \neq \{\}$
⟨*proof*⟩

definition *separable* :: *'a topology* ⇒ *bool* **where**
separable $S \longleftrightarrow (\exists\, U.\ countable\ U \wedge dense\text{-}of\ S\ U)$

lemma *dense-ofI*:
assumes $U \subseteq topspace\ S$
and $\bigwedge V.\ openin\ S\ V \Longrightarrow V \neq \{\} \Longrightarrow U \cap V \neq \{\}$
shows *dense-of S U*
⟨*proof*⟩

lemma *separable-if-second-countable*:
  assumes *second-countable* S
  shows *separable* S
⟨*proof*⟩

lemma *dense-of-prod*:
  assumes *dense-of* S U **and** *dense-of* S' U'
  shows *dense-of* (*prod-topology* S S') $(U \times U')$
⟨*proof*⟩

lemma *separable-prod*:
  assumes *separable* S **and** *separable* S'
  shows *separable* (*prod-topology* S S')
⟨*proof*⟩

lemma *dense-of-product*:
  assumes $\bigwedge i.\ i \in I \Longrightarrow$ *dense-of* $(T\ i)$ $(U\ i)$
  shows *dense-of* (*product-topology* T I) $(\Pi_E\ i{\in}I.\ U\ i)$
⟨*proof*⟩

lemma *separable-countable-product*:
  assumes *countable* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *separable* $(T\ i)$
  shows *separable* (*product-topology* T I)
⟨*proof*⟩

lemma *separable-finite-product*:
  assumes *finite* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *separable* $(T\ i)$
  shows *separable* (*product-topology* T I)
  ⟨*proof*⟩

lemma *homeomorphic-separable*:
  assumes *separable* X X *homeomorphic-space* Y
  shows *separable* Y
⟨*proof*⟩

1.1.6 G_δ Set in Abstract Topology

definition *g-delta-of* :: ['*a topology*, '*a set*] ⇒ *bool* **where**
g-delta-of S A $\longleftrightarrow$ $(\exists \mathcal{U}.\ \mathcal{U} \neq \{\} \wedge$ *countable* $\mathcal{U} \wedge (\forall b{\in}\mathcal{U}.$ *openin* $S\ b) \wedge A = \bigcap \mathcal{U})$

lemma *g-delta-ofI*:
  assumes $U \neq \{\}$ *countable* U $\bigwedge b.\ b \in U \Longrightarrow$ *openin* S b $A = \bigcap\ U$
  shows *g-delta-of* S A
  ⟨*proof*⟩

lemma *g-delta-ofD*:
  assumes *g-delta-of* S A
  shows $\exists \mathcal{U}.\ \mathcal{U} \neq \{\} \wedge$ *countable* $\mathcal{U} \wedge (\forall b{\in}\mathcal{U}.$ *openin* $S\ b) \wedge A = \bigcap \mathcal{U}$

⟨*proof*⟩

lemma *g-delta-ofD′*:
assumes *g-delta-of S A*
shows $\exists\, U.\ (\forall\, n{::}nat.\ openin\ S\ (U\ n)) \wedge A = \bigcap\ (range\ U)$
⟨*proof*⟩

lemma *g-delta-of-subset*:
assumes *g-delta-of S A*
shows $A \subseteq topspace\ S$
⟨*proof*⟩

lemma *g-delta-of-open-set*[*simp*]:
assumes *openin S A*
shows *g-delta-of S A*
⟨*proof*⟩

lemma *g-delta-of-empty*[*simp*]: *g-delta-of S* {}
⟨*proof*⟩

lemma *g-delta-of-topspace*[*simp*]: *g-delta-of S* (*topspace S*)
⟨*proof*⟩

lemma *g-delta-of-inter*:
assumes *g-delta-of S A* **and** *g-delta-of S B*
shows *g-delta-of S* $(A \cap B)$
⟨*proof*⟩

lemma *g-delta-of-Int*:
assumes $\bigwedge a.\ a \in \mathcal{U} \Longrightarrow$ *g-delta-of X a countable* $\mathcal{U}$ $\mathcal{U} \neq \{\}$
shows *g-delta-of X* $(\bigcap\ \mathcal{U})$
⟨*proof*⟩

lemma *g-delta-of-continuous-map*:
assumes *continuous-map X Y f g-delta-of Y a*
shows *g-delta-of X* $(f\ -`\ a \cap topspace\ X)$
⟨*proof*⟩

lemma *g-delta-of-inj-open-map*:
assumes *open-map X Y f inj-on f* (*topspace X*) *g-delta-of X a*
shows *g-delta-of Y* $(f\ `\ a)$
⟨*proof*⟩

lemma *g-delta-of-homeo-morphic*:
assumes *g-delta-of X a homeomorphic-map X Y f*
shows *g-delta-of Y* $(f\ `\ a)$
⟨*proof*⟩

lemma *g-delta-of-prod*:

assumes *g-delta-of X A g-delta-of Y B*
shows *g-delta-of* (*prod-topology X Y*) ($A \times B$)
⟨*proof*⟩

lemma *g-delta-of-prod1*:
assumes *g-delta-of X A*
shows *g-delta-of* (*prod-topology X Y*) ($A \times$ *topspace Y*)
⟨*proof*⟩

lemma *g-delta-of-prod2*:
assumes *g-delta-of Y B*
shows *g-delta-of* (*prod-topology X Y*) (*topspace X* $\times B$)
⟨*proof*⟩

lemma *g-delta-of-subtopology*:
assumes *g-delta-of X A* $A \subseteq S$
shows *g-delta-of* (*subtopology X S*) *A*
⟨*proof*⟩

lemma *g-delta-of-subtopology-inverse*:
assumes *g-delta-of* (*subtopology X S*) *A g-delta-of X S*
shows *g-delta-of X A*
⟨*proof*⟩

lemma *continuous-map-imp-closed-graph'*:
assumes *continuous-map X Y f Hausdorff-space Y*
shows *closedin* (*prod-topology Y X*) (($\lambda x.$ (*f x*,*x*)) ‘ *topspace X*)
⟨*proof*⟩

1.1.7 Upper-Semicontinuous

definition *upper-semicontinuous-map* :: [*'a topology*, *'a* $\Rightarrow$ *'b* :: *linorder-topology*] $\Rightarrow$ *bool* **where**
upper-semicontinuous-map X f $\longleftrightarrow$ ($\forall a.$ *openin X* {*x*∈*topspace X*. *f x* < *a*})

lemma *continuous-upper-semicontinuous*:
assumes *continuous-map X* (*euclidean* :: (*'b* :: *linorder-topology*) *topology*) *f*
shows *upper-semicontinuous-map X f*
⟨*proof*⟩

lemma *upper-semicontinuous-map-iff-closed*:
upper-semicontinuous-map X f $\longleftrightarrow$ ($\forall a.$ *closedin X* {*x*∈*topspace X*. *f x* ≥ *a*})
⟨*proof*⟩

lemma *upper-semicontinuous-map-real-iff*:
fixes *f* :: *'a* $\Rightarrow$ *real*
shows *upper-semicontinuous-map X f* $\longleftrightarrow$ *upper-semicontinuous-map X* ($\lambda x.$ *ereal* (*f x*))
⟨*proof*⟩

1.2 Lemmas for Limits

lemma *qlim-eq-lim-mono-at-bot*:
  fixes $g :: rat \Rightarrow {'a} :: linorder\text{-}topology$
  assumes $mono\ f\ (g \longrightarrow a)\ at\text{-}bot\ \bigwedge r{::}rat.\ f\ (real\text{-}of\text{-}rat\ r) = g\ r$
  shows $(f \longrightarrow a)\ at\text{-}bot$
⟨*proof*⟩

lemma *qlim-eq-lim-mono-at-top*:
  fixes $g :: rat \Rightarrow {'a} :: linorder\text{-}topology$
  assumes $mono\ f\ (g \longrightarrow a)\ at\text{-}top\ \bigwedge r{::}rat.\ f\ (real\text{-}of\text{-}rat\ r) = g\ r$
  shows $(f \longrightarrow a)\ at\text{-}top$
⟨*proof*⟩

lemma *tendsto-enn2real*:
  assumes $k < top$ **and** $(f \longrightarrow k)\ F$
  shows $((\lambda n.\ enn2real\ (f\ n)) \longrightarrow enn2real\ k)\ F$
⟨*proof*⟩

lemma *LIMSEQ-inverse-not0*:
  fixes $xn :: nat \Rightarrow real$
  assumes $\bigwedge n.\ xn\ n \neq 0\ xn \longrightarrow x\ (\lambda n.\ 1\ /\ (xn\ n)) \longrightarrow b$
  shows $x \neq 0$
⟨*proof*⟩

lemma *obtain-subsequence*:
  fixes $xn :: nat \Rightarrow \text{-}$
  assumes $infinite\ \{n.\ P\ n\ (xn\ n)\}$
  obtains $a :: nat \Rightarrow nat$ **where** $strict\text{-}mono\ a\ \bigwedge n.\ P\ (a\ n)\ (xn\ (a\ n))$
⟨*proof*⟩

1.3 Lemmas for Measure Theory

lemma *measurable-preserve-sigma-sets*:
  assumes $sets\ M = sigma\text{-}sets\ \Omega\ S\ S \subseteq Pow\ \Omega$
    $\bigwedge a.\ a \in S \Longrightarrow f\ `\ a \in sets\ N\ inj\text{-}on\ f\ (space\ M)\ f\ `\ space\ M \in sets\ N$
   and $b \in sets\ M$
  shows $f\ `\ b \in sets\ N$
⟨*proof*⟩

lemma *integral-measurable-subprob-algebra2*:
  fixes $f :: \text{-} \Rightarrow \text{-} \Rightarrow \text{-}{::}\{banach,second\text{-}countable\text{-}topology\}$
  assumes $[measurable]$: $(\lambda(x,\ y).\ f\ x\ y) \in borel\text{-}measurable\ (M \bigotimes_M N)\ L \in measurable\ M\ (subprob\text{-}algebra\ N)$
  shows $(\lambda x.\ integral^L\ (L\ x)\ (f\ x)) \in borel\text{-}measurable\ M$
⟨*proof*⟩

inductive-set $sigma\text{-}sets\text{-}cinter :: {'a}\ set \Rightarrow {'a}\ set\ set \Rightarrow {'a}\ set\ set$
  for $sp :: {'a}\ set$ **and** $A :: {'a}\ set\ set$
  where

Basic-c[*intro*, *simp*]: *a* ∈ *A* ⟹ *a* ∈ *sigma-sets-cinter sp A*
| *Top-c*[*simp*]: *sp* ∈ *sigma-sets-cinter sp A*
| *Inter-c*: (⋀*i*::*nat*. *a i* ∈ *sigma-sets-cinter sp A*) ⟹ (⋂ *i*. *a i*) ∈ *sigma-sets-cinter sp A*
| *Union-c*: (⋀*i*::*nat*. *a i* ∈ *sigma-sets-cinter sp A*) ⟹ (⋃ *i*. *a i*) ∈ *sigma-sets-cinter sp A*

inductive-set *sigma-sets-cinter-dunion* :: *'a set* ⇒ *'a set set* ⇒ *'a set set*
for *sp* :: *'a set* **and** *A* :: *'a set set*
where
Basic-cd[*intro*, *simp*]: *a* ∈ *A* ⟹ *a* ∈ *sigma-sets-cinter-dunion sp A*
| *Top-cd*[*simp*]: *sp* ∈ *sigma-sets-cinter-dunion sp A*
| *Inter-cd*: (⋀*i*::*nat*. *a i* ∈ *sigma-sets-cinter-dunion sp A*) ⟹ (⋂ *i*. *a i*) ∈ *sigma-sets-cinter-dunion sp A*
| *Union-cd*: (⋀*i*::*nat*. *a i* ∈ *sigma-sets-cinter-dunion sp A*) ⟹ *disjoint-family a* ⟹ (⋃ *i*. *a i*) ∈ *sigma-sets-cinter-dunion sp A*

lemma *sigma-sets-cinter-dunion-subset*: *sigma-sets-cinter-dunion sp A* ⊆ *sigma-sets-cinter sp A*
⟨*proof*⟩

lemma *sigma-sets-cinter-into-sp*:
assumes *A* ⊆ *Pow sp x* ∈ *sigma-sets-cinter sp A*
shows *x* ⊆ *sp*
⟨*proof*⟩

lemma *sigma-sets-cinter-dunion-into-sp*:
assumes *A* ⊆ *Pow sp x* ∈ *sigma-sets-cinter-dunion sp A*
shows *x* ⊆ *sp*
⟨*proof*⟩

lemma *sigma-sets-cinter-int*:
assumes *a* ∈ *sigma-sets-cinter sp A b* ∈ *sigma-sets-cinter sp A*
shows *a* ∩ *b* ∈ *sigma-sets-cinter sp A*
⟨*proof*⟩

lemma *sigma-sets-cinter-dunion-int*:
assumes *a* ∈ *sigma-sets-cinter-dunion sp A b* ∈ *sigma-sets-cinter-dunion sp A*
shows *a* ∩ *b* ∈ *sigma-sets-cinter-dunion sp A*
⟨*proof*⟩

lemma *sigma-sets-cinter-un*:
assumes *a* ∈ *sigma-sets-cinter sp A b* ∈ *sigma-sets-cinter sp A*
shows *a* ∪ *b* ∈ *sigma-sets-cinter sp A*
⟨*proof*⟩

Measurable isomorphisms.

definition *measurable-isomorphic-map*::[*'a measure*, *'b measure*, *'a* ⇒ *'b*] ⇒ *bool*
where

measurable-isomorphic-map M N f $\longleftrightarrow$ *bij-betw* f (*space* M) (*space* N) $\wedge$ $f \in M$ $\rightarrow_M$ N $\wedge$ *the-inv-into* (*space* M) $f \in N \rightarrow_M M$

lemma *measurable-isomorphic-map-sets-cong*:
  assumes *sets* M = *sets* M' *sets* N = *sets* N'
  shows *measurable-isomorphic-map* M N f $\longleftrightarrow$ *measurable-isomorphic-map* M' N' f
  ⟨*proof*⟩

lemma *measurable-isomorphic-map-surj*:
  assumes *measurable-isomorphic-map* M N f
  shows f ‘ *space* M = *space* N
  ⟨*proof*⟩

lemma *measurable-isomorphic-mapI*:
  assumes *bij-betw* f (*space* M) (*space* N) $f \in M \rightarrow_M N$ *the-inv-into* (*space* M) f $\in N \rightarrow_M M$
  shows *measurable-isomorphic-map* M N f
  ⟨*proof*⟩

lemma *measurable-isomorphic-map-byWitness*:
  assumes $f \in M \rightarrow_M N$ $g \in N \rightarrow_M M$ $\bigwedge x.\ x \in$ *space* $M \Longrightarrow g\ (f\ x) = x$ $\bigwedge x.\ x$ $\in$ *space* $N \Longrightarrow f\ (g\ x) = x$
  shows *measurable-isomorphic-map* M N f
⟨*proof*⟩

lemma *measurable-isomorphic-map-restrict-space*:
  assumes $f \in M \rightarrow_M N$ $\bigwedge A.\ A \in$ *sets* $M \Longrightarrow f$ ‘ $A \in$ *sets* N *inj-on* f (*space* M)
  shows *measurable-isomorphic-map* M (*restrict-space* N (f ‘ *space* M)) f
⟨*proof*⟩

lemma *measurable-isomorphic-mapD'*:
  assumes *measurable-isomorphic-map* M N f
  shows $\bigwedge A.\ A \in$ *sets* $M \Longrightarrow f$ ‘ $A \in$ *sets* N $f \in M \rightarrow_M N$
    $\exists g.$ *bij-betw* g (*space* N) (*space* M) $\wedge$ $g \in N \rightarrow_M M$ $\wedge$ ($\forall x \in$ *space* $M.\ g\ (f\ x) = x$) $\wedge$ ($\forall x\in$ *space* $N.\ f\ (g\ x) = x$) $\wedge$ ($\forall A\in$*sets* $N.\ g$ ‘ $A \in$ *sets* M)
⟨*proof*⟩

lemma *measurable-isomorphic-map-inv*:
  assumes *measurable-isomorphic-map* M N f
  shows *measurable-isomorphic-map* N M (*the-inv-into* (*space* M) f)
  ⟨*proof*⟩

lemma *measurable-isomorphic-map-comp*:
  assumes *measurable-isomorphic-map* M N f **and** *measurable-isomorphic-map* N L g
  shows *measurable-isomorphic-map* M L ($g \circ f$)
⟨*proof*⟩

definition *measurable-isomorphic*::['a measure, 'b measure] $\Rightarrow$ *bool* (**infixr** *measurable'-isomorphic 50*) **where**
M measurable-isomorphic N $\longleftrightarrow$ ($\exists f$. *measurable-isomorphic-map M N f*)

lemma *measurable-isomorphic-sets-cong*:
assumes *sets M = sets M' sets N = sets N'*
shows *M measurable-isomorphic N* $\longleftrightarrow$ *M' measurable-isomorphic N'*
⟨*proof*⟩

lemma *measurable-isomorphicD*:
assumes *M measurable-isomorphic N*
shows $\exists f\ g.\ f \in M \to_M N \wedge g \in N \to_M M \wedge (\forall x \in space\ M.\ g\ (f\ x) = x) \wedge (\forall y \in space\ N.\ f\ (g\ y) = y) \wedge (\forall A \in sets\ M.\ f\ `\ A \in sets\ N) \wedge (\forall A \in sets\ N.\ g\ `\ A \in sets\ M)$
⟨*proof*⟩

lemma *measurable-isomorphic-byWitness*:
assumes $f \in M \to_M N$ $\bigwedge x.\ x \in space\ M \Longrightarrow g\ (f\ x) = x$
and $g \in N \to_M M$ $\bigwedge y.\ y \in space\ N \Longrightarrow f\ (g\ y) = y$
shows *M measurable-isomorphic N*
⟨*proof*⟩

lemma *measurable-isomorphic-refl*:
M measurable-isomorphic M
⟨*proof*⟩

lemma *measurable-isomorphic-sym*:
assumes *M measurable-isomorphic N*
shows *N measurable-isomorphic M*
⟨*proof*⟩

lemma *measurable-isomorphic-trans*:
assumes *M measurable-isomorphic N* **and** *N measurable-isomorphic L*
shows *M measurable-isomorphic L*
⟨*proof*⟩

lemma *measurable-isomorphic-empty*:
assumes *space M* = {} *space N* = {}
shows *M measurable-isomorphic N*
⟨*proof*⟩

lemma *measurable-isomorphic-empty1*:
assumes *space M* = {} *M measurable-isomorphic N*
shows *space N* = {}
⟨*proof*⟩

lemma *measurable-ismorphic-empty2*:
assumes *space N* = {} *M measurable-isomorphic N*

shows *space* $M = \{\}$
⟨*proof*⟩

lemma *measurable-lift-product*:
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i \in (M\ i) \rightarrow_M (N\ i)$
shows $(\lambda x\ i.\ \mathit{if}\ i \in I\ \mathit{then}\ f\ i\ (x\ i)\ \mathit{else}\ \mathit{undefined}) \in (\Pi_M\ i{\in}I.\ M\ i) \rightarrow_M (\Pi_M\ i{\in}I.\ N\ i)$
⟨*proof*⟩

lemma *measurable-isomorphic-map-lift-product*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *measurable-isomorphic-map* $(M\ i)\ (N\ i)\ (h\ i)$
shows *measurable-isomorphic-map* $(\Pi_M\ i{\in}I.\ M\ i)\ (\Pi_M\ i{\in}I.\ N\ i)\ (\lambda x\ i.\ \mathit{if}\ i \in I\ \mathit{then}\ h\ i\ (x\ i)\ \mathit{else}\ \mathit{undefined})$
⟨*proof*⟩

lemma *measurable-isomorphic-lift-product*:
assumes $\bigwedge i.\ i \in I \Longrightarrow (M\ i)$ *measurable-isomorphic* $(N\ i)$
shows $(\Pi_M\ i{\in}I.\ M\ i)$ *measurable-isomorphic* $(\Pi_M\ i{\in}I.\ N\ i)$
⟨*proof*⟩

https://math24.net/cantor-schroder-bernstein-theorem.html

lemma *Schroeder-Bernstein-measurable′*:
assumes f ‘ (*space* M) $\in$ *sets* N g ‘ (*space* N) $\in$ *sets* M
and *measurable-isomorphic-map* M (*restrict-space* N (f ‘ (*space* M))) f **and** *measurable-isomorphic-map* N (*restrict-space* M (g ‘ (*space* N))) g
shows $\exists h.$ *measurable-isomorphic-map* M N h
⟨*proof*⟩

lemma *Schroeder-Bernstein-measurable*:
assumes $f \in M \rightarrow_M N$ $\bigwedge A.\ A \in$ *sets* $M \Longrightarrow f$ ‘ $A \in$ *sets* N *inj-on* f (*space* M)
and $g \in N \rightarrow_M M$ $\bigwedge A.\ A \in$ *sets* $N \Longrightarrow g$ ‘ $A \in$ *sets* M *inj-on* g (*space* N)
shows $\exists h.$ *measurable-isomorphic-map* M N h
⟨*proof*⟩

lemma *measurable-isomorphic-from-embeddings*:
assumes M *measurable-isomorphic* (*restrict-space* N B) N *measurable-isomorphic* (*restrict-space* M A)
and $A \in$ *sets* M $B \in$ *sets* N
shows M *measurable-isomorphic* N
⟨*proof*⟩

lemma *measurable-isomorphic-antisym*:
assumes B *measurable-isomorphic* (*restrict-space* C c) A *measurable-isomorphic* (*restrict-space* B b)
and $c \in$ *sets* C $b \in$ *sets* B C *measurable-isomorphic* A
shows C *measurable-isomorphic* B
⟨*proof*⟩

lemma *countable-infinite-isomorphisc-to-nat-index*:

assumes *countable* I **and** *infinite* I
shows $(\Pi_M\ x{\in}I.\ M)$ *measurable-isomorphic* $(\Pi_M\ (x{::}nat){\in}UNIV.\ M)$
$\langle proof \rangle$

lemma *PiM-PiM-isomorphic-to-PiM*:
$(\Pi_M\ i{\in}I.\ \Pi_M\ j{\in}J.\ M\ i\ j)$ *measurable-isomorphic* $(\Pi_M\ (i,j){\in}I{\times}J.\ M\ i\ j)$
$\langle proof \rangle$

lemma *measurable-isomorphic-map-sigma-sets*:
assumes *sets* M = *sigma-sets* (*space* M) U *measurable-isomorphic-map* M N f
shows *sets* N = *sigma-sets* (*space* N) ((') f ' U)
$\langle proof \rangle$

end

2 Set-Based Metric Spaces

theory *Set-Based-Metric-Space*
imports *Lemmas-StandardBorel*
begin

2.1 Set-Based Metric Spaces

locale *metric-set* =
fixes $S :: 'a\ set$
and $dist :: 'a \Rightarrow 'a \Rightarrow real$
assumes *dist-geq0*: $\bigwedge x\ y.\ dist\ x\ y \geq 0$
and *dist-notin*: $\bigwedge x\ y.\ x \notin S \Longrightarrow dist\ x\ y = 0$
and *dist-0*: $\bigwedge x\ y.\ x \in S \Longrightarrow y \in S \Longrightarrow (x = y) = (dist\ x\ y = 0)$
and *dist-sym*: $\bigwedge x\ y.\ dist\ x\ y = dist\ y\ x$
and *dist-tr*: $\bigwedge x\ y\ z.\ x \in S \Longrightarrow y \in S \Longrightarrow z \in S \Longrightarrow dist\ x\ z \leq dist\ x\ y + dist\ y\ z$

lemma *metric-class-metric-set*[*simp*]: *metric-set UNIV dist*
$\langle proof \rangle$

context *metric-set*
begin

abbreviation *dist-typeclass* ≡ *Real-Vector-Spaces.dist*

lemma *dist-notin'*:
assumes $y \notin S$
shows $dist\ x\ y = 0$
$\langle proof \rangle$

lemma *dist-ge0*:
assumes $x \in S$ $y \in S$
shows $x \neq y \longleftrightarrow dist\ x\ y > 0$

$\langle proof \rangle$

lemma *dist-0'*[*simp*]: $dist\ x\ x = 0$
$\langle proof \rangle$

lemma *dist-tr-abs*:
assumes $x \in S$ $y \in S$ $z \in S$
shows $|dist\ x\ y - dist\ y\ z| \leq dist\ x\ z$
$\langle proof \rangle$

Ball

definition *open-ball* :: $'a \Rightarrow real \Rightarrow 'a\ set$ **where**
$open\text{-}ball\ a\ r \equiv if\ a \in S\ then\ \{x \in S.\ dist\ a\ x < r\}\ else\ \{\}$

lemma *open-ball-subset-ofS*: $open\text{-}ball\ a\ \varepsilon \subseteq S$
$\langle proof \rangle$

lemma *open-ballD*:
assumes $x \in open\text{-}ball\ a\ \varepsilon$
shows $dist\ a\ x < \varepsilon$
$\langle proof \rangle$

lemma *open-ballD'*:
assumes $x \in open\text{-}ball\ a\ \varepsilon$
shows $x \in S$ $a \in S$ $\varepsilon > 0$
$\langle proof \rangle$

lemma *open-ball-inverse*:
$x \in open\text{-}ball\ y\ \varepsilon \longleftrightarrow y \in open\text{-}ball\ x\ \varepsilon$
$\langle proof \rangle$

lemma *open-ball-ina*[*simp*]:
assumes $a \in S$ **and** $\varepsilon > 0$
shows $a \in open\text{-}ball\ a\ \varepsilon$
$\langle proof \rangle$

lemma *open-ball-nin-le*:
assumes $a \in S$ $\varepsilon > 0$ $b \in S$ $b \notin open\text{-}ball\ a\ \varepsilon$
shows $\varepsilon \leq dist\ a\ b$
$\langle proof \rangle$

lemma *open-ball-le*:
assumes $r \leq l$
shows $open\text{-}ball\ a\ r \subseteq open\text{-}ball\ a\ l$
$\langle proof \rangle$

lemma *open-ball-le-0*:
assumes $\varepsilon \leq 0$
shows $open\text{-}ball\ a\ \varepsilon = \{\}$

$\langle proof \rangle$

lemma *open-ball-nin*:
assumes $a \notin S$
shows *open-ball* $a\ \varepsilon = \{\}$
$\langle proof \rangle$

definition *closed-ball* :: $'a \Rightarrow real \Rightarrow 'a\ set$ **where**
closed-ball $a\ r \equiv if\ a \in S\ then\ \{x \in S.\ dist\ a\ x \leq r\}\ else\ \{\}$

lemma *closed-ball-subset-ofS*:
closed-ball $a\ \varepsilon \subseteq S$
$\langle proof \rangle$

lemma *closed-ballD*:
assumes $x \in$ *closed-ball* $a\ \varepsilon$
shows $dist\ a\ x \leq \varepsilon$
$\langle proof \rangle$

lemma *closed-ballD'*:
assumes $x \in$ *closed-ball* $a\ \varepsilon$
shows $x \in S\ a \in S\ \varepsilon \geq 0$
$\langle proof \rangle$

lemma *closed-ball-ina*[*simp*]:
assumes $a \in S$ **and** $\varepsilon \geq 0$
shows $a \in$ *closed-ball* $a\ \varepsilon$
$\langle proof \rangle$

lemma *closed-ball-le*:
assumes $r \leq l$
shows *closed-ball* $a\ r \subseteq$ *closed-ball* $a\ l$
$\langle proof \rangle$

lemma *closed-ball-le-0*:
assumes $\varepsilon < 0$
shows *closed-ball* $a\ \varepsilon = \{\}$
$\langle proof \rangle$

lemma *closed-ball-0*:
assumes $a \in S$
shows *closed-ball* $a\ 0 = \{a\}$
$\langle proof \rangle$

lemma *closed-ball-nin*:
assumes $a \notin S$
shows *closed-ball* $a\ \varepsilon = \{\}$
$\langle proof \rangle$

lemma *open-ball-closed-ball*:
$open\text{-}ball\ a\ \varepsilon \subseteq closed\text{-}ball\ a\ \varepsilon$
⟨*proof*⟩

lemma *closed-ball-open-ball*:
assumes $e < f$
shows $closed\text{-}ball\ a\ e \subseteq open\text{-}ball\ a\ f$
⟨*proof*⟩

lemma *closed-ball-open-ball-un1*:
assumes $e > 0$
shows $open\text{-}ball\ a\ e \cup \{x{\in}S.\ dist\ a\ x = e\} = closed\text{-}ball\ a\ e$
⟨*proof*⟩

lemma *closed-ball-open-ball-un2*:
assumes $a \in S$
shows $open\text{-}ball\ a\ e \cup \{x{\in}S.\ dist\ a\ x = e\} = closed\text{-}ball\ a\ e$
⟨*proof*⟩

definition *mtopology* :: *'a topology* **where**
$mtopology = topology\ (\lambda U.\ U \subseteq S \wedge (\forall x{\in}U.\ \exists\varepsilon{>}0.\ open\text{-}ball\ x\ \varepsilon \subseteq U))$

lemma *mtopology-istopology*:
$istopology\ (\lambda U.\ U \subseteq S \wedge (\forall x{\in}U.\ \exists\varepsilon{>}0.\ open\text{-}ball\ x\ \varepsilon \subseteq U))$
⟨*proof*⟩

lemma *mtopology-openin-iff*:
$openin\ mtopology\ U \longleftrightarrow U \subseteq S \wedge (\forall x{\in}U.\ \exists\varepsilon{>}0.\ open\text{-}ball\ x\ \varepsilon \subseteq U)$
⟨*proof*⟩

lemma *mtopology-topspace*: $topspace\ mtopology = S$
⟨*proof*⟩

lemma *openin-S*[*simp*]: $openin\ mtopology\ S$
⟨*proof*⟩

lemma *mtopology-open-ball-in'*:
assumes $x \in open\text{-}ball\ a\ \varepsilon$
shows $\exists\varepsilon'{>}0.\ open\text{-}ball\ x\ \varepsilon' \subseteq open\text{-}ball\ a\ \varepsilon$
⟨*proof*⟩

lemma *mtopology-open-ball-in*:
assumes $a \in S$ **and** $\varepsilon > 0$
shows $openin\ mtopology\ (open\text{-}ball\ a\ \varepsilon)$
⟨*proof*⟩

lemma *openin-open-ball*: $openin\ mtopology\ (open\text{-}ball\ a\ \varepsilon)$
⟨*proof*⟩

lemma *closedin-closed-ball*: *closedin mtopology* (*closed-ball a* ε)
⟨*proof*⟩

lemma *mtopology-def2*:
mtopology = *topology-generated-by* {*open-ball a* ε | *a* ε. *a* ∈ *S* ∧ ε > *0*}
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

abbreviation *mtopology-subbasis* :: *'a set set* ⇒ *bool* **where**
mtopology-subbasis $\mathcal{O}$ ≡ *subbase-of mtopology* $\mathcal{O}$

lemma *mtopology-subbasis1*:
mtopology-subbasis {*open-ball a* ε | *a* ε. *a* ∈ *S* ∧ ε > *0*}
⟨*proof*⟩

abbreviation *mtopology-basis* :: *'a set set* ⇒ *bool* **where**
mtopology-basis $\mathcal{O}$ ≡ *base-of mtopology* $\mathcal{O}$

lemma *mtopology-basis-ball*:
mtopology-basis {*open-ball a* ε | *a* ε. *a* ∈ *S* ∧ ε > *0*}
⟨*proof*⟩

abbreviation *sequence* :: (*nat* ⇒ *'a*) *set* **where**
sequence ≡ *UNIV* → *S*

lemma *sequence-comp*:
xn ∈ *sequence* ⟹ (λ*n*. (*xn* (*a n*))) ∈ *sequence*
xn ∈ *sequence* ⟹ *xn* ∘ *an* ∈ *sequence*
⟨*proof*⟩

definition *converge-to-inS* :: [*nat* ⇒ *'a*, *'a*] ⇒ *bool* **where**
converge-to-inS f s ≡ *f* ∈ *sequence* ∧ *s* ∈ *S* ∧ (λ*n*. *dist* (*f n*) *s*) ⟶ *0*

lemma *converge-to-inS-const*:
assumes *x* ∈ *S*
shows *converge-to-inS* (λ*n*. *x*) *x*
⟨*proof*⟩

lemma *converge-to-inS-subseq*:
assumes *strict-mono a converge-to-inS f s*
shows *converge-to-inS* (*f* ∘ *a*) *s*
⟨*proof*⟩

lemma *converge-to-inS-ignore-initial*:
assumes *converge-to-inS xn x*
shows *converge-to-inS* (λ*n*. *xn* (*n* + *k*)) *x*
⟨*proof*⟩

lemma *converge-to-inS-offset*:

assumes *converge-to-inS* $(\lambda n.\ xn\ (n + k))\ x\ xn \in sequence$
shows *converge-to-inS xn x*
⟨*proof*⟩

lemma *converge-to-inS-def2*:
converge-to-inS f s $\longleftrightarrow (f \in sequence \wedge s \in S \wedge (\forall \varepsilon>0.\ \exists N.\ \forall n\geq N.\ dist\ (f\ n)\ s < \varepsilon))$
⟨*proof*⟩

lemma *converge-to-inS-def2'*:
converge-to-inS f s $\longleftrightarrow (f \in sequence \wedge s \in S \wedge (\forall \varepsilon>0.\ \exists N.\ \forall n\geq N.\ (f\ n) \in open\text{-}ball\ s\ \varepsilon))$
⟨*proof*⟩

lemma *converge-to-inS-unique*:
assumes *converge-to-inS f x converge-to-inS f y*
shows $x = y$
⟨*proof*⟩

lemma *mtopology-closedin-iff*: *closedin mtopology M* $\longleftrightarrow M \subseteq S \wedge (\forall f\in(UNIV \to M).\ \forall s.\ converge\text{-}to\text{-}inS\ f\ s \longrightarrow s \in M)$
⟨*proof*⟩

lemma *mtopology-closedin-iff2*: *closedin mtopology M* $\longleftrightarrow M \subseteq S \wedge (\forall x.\ x \in M \longleftrightarrow (\forall \varepsilon>0.\ open\text{-}ball\ x\ \varepsilon \cap M \neq \{\}))$
⟨*proof*⟩

lemma *mtopology-openin-iff2*:
openin mtopology A $\longleftrightarrow A \subseteq S \wedge (\forall f\ x.\ converge\text{-}to\text{-}inS\ f\ x \wedge x \in A \longrightarrow (\exists N.\ \forall n\geq N.\ f\ n \in A))$
⟨*proof*⟩

lemma *closure-of-mtopology*: *mtopology closure-of A* $= \{a.\ \forall \varepsilon>0.\ open\text{-}ball\ a\ \varepsilon \cap A \neq \{\}\}$
⟨*proof*⟩

lemma *closure-of-mtopology'*:
mtopology closure-of A $= \{a.\ \exists an\in UNIV \to A.\ converge\text{-}to\text{-}inS\ an\ a\}$
⟨*proof*⟩

lemma *closure-of-mtopology-an*:
assumes $a \in$ *mtopology closure-of A*
obtains *an* **where** $an\in UNIV \to A$ *converge-to-inS an a*
⟨*proof*⟩

lemma *closure-of-open-ball*: *mtopology closure-of open-ball a* $\varepsilon \subseteq$ *closed-ball a* ε
⟨*proof*⟩

lemma *interior-of-closed-ball*: *open-ball a e* $\subseteq$ *mtopology interior-of closed-ball a e*

⟨*proof*⟩

lemma *derived-set-of-mtopology*:
$mtopology\ derived\text{-}set\text{-}of\ A = \{a.\ \exists an{\in}UNIV \rightarrow A.\ (\forall n.\ an\ n \neq a) \wedge converge\text{-}to\text{-}inS\ an\ a\}$
⟨*proof*⟩

lemma *isolated-points-of-mtopology*:
$mtopology\ isolated\text{-}points\text{-}of\ A = \{a{\in}S{\cap}A.\ \forall an{\in}UNIV \rightarrow A.\ converge\text{-}to\text{-}inS\ an\ a \longrightarrow (\exists no.\ \forall n{\geq}no.\ an\ n = a)\}$
⟨*proof*⟩

lemma *perfect-set-open-ball-infinite*:
assumes *perfect-set mtopology A*
shows $closedin\ mtopology\ A \wedge (\forall a{\in}A.\ \forall \varepsilon{>}0.\ infinite\ (open\text{-}ball\ a\ \varepsilon))$
⟨*proof*⟩

lemma *nbh-subset*:
assumes A: $A \subseteq S$ **and** e: $e > 0$
shows $A \subseteq (\bigcup a{\in}A.\ open\text{-}ball\ a\ e)$
⟨*proof*⟩

lemma *nbh-decseq*:
assumes *decseq an*
shows $decseq\ (\lambda n.\ \bigcup a{\in}A.\ open\text{-}ball\ a\ (an\ n))$
⟨*proof*⟩

lemma *nbh-Int*:
assumes A: $A \neq \{\}\ A \subseteq S$
and an:$\bigwedge n.\ an\ n > 0\ decseq\ an\ an \longrightarrow 0$
shows $(\bigcap n.\ \bigcup a{\in}A.\ open\text{-}ball\ a\ (an\ n)) = mtopology\ closure\text{-}of\ A$
⟨*proof*⟩

lemma *nbh-add*: $(\bigcup b{\in}(\bigcup a{\in}A.\ open\text{-}ball\ a\ e).\ open\text{-}ball\ b\ f) \subseteq (\bigcup a{\in}A.\ open\text{-}ball\ a\ (e + f))$
⟨*proof*⟩

definition *convergent-inS* :: $(nat \Rightarrow 'a) \Rightarrow bool$ **where**
$convergent\text{-}inS\ f \equiv \exists s.\ converge\text{-}to\text{-}inS\ f\ s$

lemma *convergent-inS-const*:
assumes $x \in S$
shows $convergent\text{-}inS\ (\lambda n.\ x)$
⟨*proof*⟩

lemma *convergent-inS-ignore-initial*:
assumes *convergent-inS xn*
shows $convergent\text{-}inS\ (\lambda n.\ xn\ (n + k))$
⟨*proof*⟩

lemma *convergent-inS-offset*:
  assumes *convergent-inS* $(\lambda n.\ xn\ (n + k))$ $xn \in$ *sequence*
  shows *convergent-inS xn*
  ⟨*proof*⟩

definition *the-limit-of* :: $(nat \Rightarrow 'a) \Rightarrow 'a$ **where**
the-limit-of xn $\equiv$ *THE x. converge-to-inS xn x*

lemma *the-limit-if-converge*:
  assumes *convergent-inS xn*
  shows *converge-to-inS xn* (*the-limit-of xn*)
  ⟨*proof*⟩

lemma *the-limit-of-eq*:
  assumes *converge-to-inS xn x*
  shows *the-limit-of xn* $= x$
  ⟨*proof*⟩

lemma *the-limit-of-inS*:
  assumes *convergent-inS xn*
  shows *the-limit-of xn* $\in S$
  ⟨*proof*⟩

lemma *the-limit-of-const*:
  assumes $x \in S$
  shows *the-limit-of* $(\lambda n.\ x) = x$
  ⟨*proof*⟩

lemma *convergent-inS-dest1*:
  assumes *convergent-inS f*
  shows $f\ n \in S$
  ⟨*proof*⟩

definition *Cauchy-inS*:: $(nat \Rightarrow 'a) \Rightarrow bool$ **where**
Cauchy-inS f $\equiv f \in$ *sequence* $\wedge$ $(\forall \varepsilon>0.\ \exists N.\ \forall n \geq N.\ \forall m \geq N.\ dist\ (f\ n)\ (f\ m) < \varepsilon)$

lemma *Cauchy-inS-def2*:
 Cauchy-inS f $\longleftrightarrow f \in$ *sequence* $\wedge$ $(\forall \varepsilon>0.\ \exists N.\ \forall n \geq N.\ f\ n \in$ *open-ball* $(f\ N)\ \varepsilon)$
  ⟨*proof*⟩

lemma *Cauchy-inS-def2'*:
 Cauchy-inS f $\longleftrightarrow f \in$ *sequence* $\wedge$ $(\forall \varepsilon>0.\ \exists x \in S.\ \exists N.\ \forall n \geq N.\ f\ n \in$ *open-ball* $x\ \varepsilon)$
  ⟨*proof*⟩

lemma *Cauchy-inS-def2''*:
 Cauchy-inS f $\longleftrightarrow f \in$ *sequence* $\wedge$ $(\forall \varepsilon>0.\ \exists x \in S.\ \exists N.\ \forall n \geq N.\ dist\ x\ (f\ n) < \varepsilon)$

⟨*proof*⟩

lemma *Cauchy-inS-dest1*:
assumes *Cauchy-inS f*
shows $f\ n \in S$
⟨*proof*⟩

lemma *Cauchy-if-convergent-inS*:
assumes *convergent-inS f*
shows *Cauchy-inS f*
⟨*proof*⟩

corollary *Cauchy-inS-const*: $a \in S \Longrightarrow$ *Cauchy-inS* $(\lambda n.\ a)$
⟨*proof*⟩

lemma *converge-if-Cauchy-and-subconverge*:
assumes *strict-mono a converge-to-inS* $(f \circ a)$ *s Cauchy-inS f*
shows *converge-to-inS f s*
⟨*proof*⟩

lemma *subCauchy-Cahcuy*:
assumes *Cauchy-inS xn strict-mono a*
shows *Cauchy-inS* $(xn \circ a)$
⟨*proof*⟩

corollary *Cauchy-inS-ignore-initial*:
assumes *Cauchy-inS xn*
shows *Cauchy-inS* $(\lambda n.\ xn\ (n + k))$
⟨*proof*⟩

lemma *Cauchy-inS-dist-Cauchy*:
assumes *Cauchy-inS xn Cauchy-inS yn*
shows *Cauchy* $(\lambda n.\ dist\ (xn\ n)\ (yn\ n))$
⟨*proof*⟩

corollary *Cauchy-inS-dist-convergent*:
assumes *Cauchy-inS xn Cauchy-inS yn*
shows *convergent* $(\lambda n.\ dist\ (xn\ n)\ (yn\ n))$
⟨*proof*⟩

https://people.bath.ac.uk/mw2319/ma30252/sec-dense.html.

abbreviation *dense-set* $\equiv$ *dense-of mtopology*

lemma *dense-set-def*:
dense-set $U \longleftrightarrow U \subseteq S \wedge (\forall x{\in}S.\ \forall \varepsilon{>}0.\ open\text{-}ball\ x\ \varepsilon \cap U \neq \{\})$
⟨*proof*⟩

corollary *dense-set-balls-cover*:
assumes *dense-set U* **and** $e > 0$
shows $(\bigcup u{\in}U.\ \textit{open-ball}\ u\ e) = S$
⟨*proof*⟩

lemma *dense-set-empty-iff*: *dense-set* $\{\} \longleftrightarrow S = \{\}$
⟨*proof*⟩

lemma *dense-set-S*: *dense-set S*
⟨*proof*⟩

lemma *dense-set-def2*:
dense-set $U \longleftrightarrow U \subseteq S \wedge (\forall x{\in}S.\ \forall \varepsilon{>}0.\exists y{\in}U.\ \textit{dist}\ x\ y < \varepsilon)$
⟨*proof*⟩

lemma *dense-set-def2'*:
dense-set $U \longleftrightarrow U \subseteq S \wedge (\forall x{\in}S.\ \exists f{\in}\textit{UNIV} \to U.\ \textit{converge-to-inS}\ f\ x)$
⟨*proof*⟩

lemma *dense-set-infinite*:
assumes *infinite S dense-set U*
shows *infinite U*
⟨*proof*⟩

lemma *mtopology-Hausdorff*: *Hausdorff-space mtopology*
⟨*proof*⟩

Diameter

definition *diam* :: *'a set* ⇒ *ennreal* **where**
$\textit{diam}\ A \equiv \bigsqcup\ \{\textit{ennreal}\ (\textit{dist}\ x\ y) \mid x\ y.\ x \in A \cap S \wedge y \in A \cap S\}$

lemma *diam-empty*[*simp*]:
$\textit{diam}\ \{\} = 0$
⟨*proof*⟩

lemma *diam-def2*:
assumes $A \subseteq S$
shows $\textit{diam}\ A = \bigsqcup\ \{\textit{ennreal}\ (\textit{dist}\ x\ y) \mid x\ y.\ x \in A \wedge y \in A\}$
⟨*proof*⟩

lemma *diam-subset*:
assumes $A \subseteq B$
shows $\textit{diam}\ A \le \textit{diam}\ B$
⟨*proof*⟩

lemma *diam-cball-leq*: $\textit{diam}\ (\textit{closed-ball}\ a\ \varepsilon) \le \textit{ennreal}\ (2 * \varepsilon)$
⟨*proof*⟩

lemma *diam-ball-leq*:

$diam\ (open\text{-}ball\ a\ \varepsilon) \leq ennreal\ (2 * \varepsilon)$
⟨proof⟩

lemma *diam-is-sup*:
assumes $x \in A \cap S$ $y \in A \cap S$
shows $dist\ x\ y \leq diam\ A$
⟨proof⟩

lemma *diam-is-sup′*:
assumes $x \in A \cap S$ $y \in A \cap S$ $diam\ A \leq ennreal\ r$ $r \geq 0$
shows $dist\ x\ y \leq r$
⟨proof⟩

lemma *diam-le*:
assumes $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow dist\ x\ y \leq r$
shows $diam\ A \leq r$
⟨proof⟩

lemma *diam-eq-closure*: $diam\ (mtopology\ closure\text{-}of\ A) = diam\ A$
⟨proof⟩

definition *bounded-set* :: $'a\ set \Rightarrow bool$ **where**
$bounded\text{-}set\ A \longleftrightarrow diam\ A < \infty$

lemma *bounded-set-def2′*: $bounded\text{-}set\ A \longleftrightarrow (\exists\ e.\ \forall\ x{\in}A{\cap}S.\ \forall\ y{\in}A{\cap}S.\ dist\ x\ y < e)$
⟨proof⟩

lemma *bounded-set-def2*:
assumes $A \subseteq S$
shows $bounded\text{-}set\ A \longleftrightarrow (\exists\ e.\ \forall\ x{\in}A.\ \forall\ y{\in}A.\ dist\ x\ y < e)$
⟨proof⟩

lemma *bounded-set-def3′*:
assumes $S \neq \{\}$
shows $bounded\text{-}set\ A \longleftrightarrow (\exists\ e.\ \exists\ x{\in}S.\ \forall\ y{\in}A{\cap}S.\ dist\ x\ y < e)$
⟨proof⟩

lemma *bounded-set-def4′*:
$bounded\text{-}set\ A \longleftrightarrow (\exists\ x\ e.\ A \cap S \subseteq open\text{-}ball\ x\ e)$
⟨proof⟩

lemma *bounded-set-def4*:
assumes $A \subseteq S$
shows $bounded\text{-}set\ A \longleftrightarrow (\exists\ x\ e.\ A \subseteq open\text{-}ball\ x\ e)$
⟨proof⟩

Distance between a point and a set.

definition *dist-set* :: $'a\ set \Rightarrow\ 'a \Rightarrow real$ **where**

dist-set $A \equiv (\lambda x.\ \mathit{if}\ A = \{\}\ \mathit{then}\ 0\ \mathit{else}\ \mathit{Inf}\ \{\mathit{dist}\ x\ y\ |y.\ y \in A\})$

lemma *dist-set-geq0*:
$\mathit{dist\text{-}set}\ A\ x \geq 0$
⟨*proof*⟩

lemma *dist-set-bdd-below*[*simp*]:
$\mathit{bdd\text{-}below}\ \{\mathit{dist}\ x\ y\ |y.\ y \in A\}$
⟨*proof*⟩

lemma *dist-set-singleton*[*simp*]:
$\mathit{dist\text{-}set}\ \{y\}\ x = \mathit{dist}\ x\ y$
⟨*proof*⟩

lemma *dist-set-singleton'*[*simp*]:
$\mathit{dist\text{-}set}\ \{y\} = (\lambda x.\ \mathit{dist}\ x\ y)$
⟨*proof*⟩

lemma *dist-set-empty*[*simp*]:
$\mathit{dist\text{-}set}\ \{\}\ x = 0$
⟨*proof*⟩

lemma *dist-set-nsubset0*[*simp*]:
assumes $\neg (A \subseteq S)$
shows $\mathit{dist\text{-}set}\ A\ x = 0$
⟨*proof*⟩

lemma *dist-set-notin*[*simp*]:
assumes $x \notin S$
shows $\mathit{dist\text{-}set}\ A\ x = 0$
⟨*proof*⟩

lemma *dist-set-inA*:
assumes $x \in A$
shows $\mathit{dist\text{-}set}\ A\ x = 0$
⟨*proof*⟩

lemma *dist-set-nzeroD*:
assumes $\mathit{dist\text{-}set}\ A\ x \neq 0$
shows $A \subseteq S\ x \notin A$
⟨*proof*⟩

lemma *dist-set-antimono*:
assumes $A \subseteq B\ A \neq \{\}$
shows $\mathit{dist\text{-}set}\ B\ x \leq \mathit{dist\text{-}set}\ A\ x$
⟨*proof*⟩

lemma *dist-set-bounded*:
assumes $\bigwedge y.\ y \in A \Longrightarrow \mathit{dist}\ x\ y < K\ K > 0$

shows *dist-set A x* $<$ *K*
⟨*proof*⟩

lemma *dist-set-tr*:
assumes $x \in S$ $y \in S$
shows *dist-set A x* $\leq$ *dist x y* $+$ *dist-set A y*
⟨*proof*⟩

lemma *dist-set-abs-le*:
assumes $x \in S$ $y \in S$
shows $|$*dist-set A x* $-$ *dist-set A y*$|$ $\leq$ *dist x y*
⟨*proof*⟩

lemma *dist-set-inA-le*:
assumes $y \in A$
shows *dist-set A x* $\leq$ *dist x y*
⟨*proof*⟩

lemma *dist-set-ball-open*:
openin mtopology $\{x{\in}S.$ *dist-set A x* $< \varepsilon\}$
⟨*proof*⟩

lemma *dist-set-ball-empty*:
assumes $A \neq \{\}$ $A \subseteq S$ $e > 0$ $x \in S$ *open-ball x e* $\cap A = \{\}$
shows *dist-set A x* $\geq e$
⟨*proof*⟩

lemma *dist-set-closed-ge0*:
assumes *closedin mtopology A* $A \neq \{\}$ $x \in S$ $x \notin A$
shows *dist-set A x* > 0
⟨*proof*⟩

lemma *g-delta-of-closed*:
assumes *closedin mtopology M*
shows *g-delta-of mtopology M*
⟨*proof*⟩

Oscillation

definition *osc-on* :: $['b$ *set*, $'b$ *topology*, $'b \Rightarrow 'a$, $'b] \Rightarrow$ *ennreal* **where**
osc-on A X f $\equiv (\lambda y. \bigsqcap \{$*diam* $(f$ ‘ $(A \cap U))$ $|U.\ y \in U \wedge$ *openin X U*$\})$

abbreviation *osc X* $\equiv$ *osc-on* (*topspace X*) *X*

lemma *osc-def*: *osc X f* $= (\lambda y. \bigsqcap \{$*diam* $(f$ ‘ $U)$ $|U.\ y \in U \wedge$ *openin X U*$\})$
⟨*proof*⟩

lemma *osc-on-less-iff*:
osc-on A X f x $< t \longleftrightarrow (\exists\, v.\ x \in v \wedge$ *openin X v* $\wedge$ *diam* $(f$ ‘ $(A \cap v)) < t)$
⟨*proof*⟩

lemma *osc-less-iff*:
$osc\ X\ f\ x < t \longleftrightarrow (\exists\, v.\ x \in v \wedge openin\ X\ v \wedge diam\ (f\ `\ v) < t)$
$\langle proof \rangle$

definition *sequentially-compact* :: *bool* **where**
$sequentially\text{-}compact \longleftrightarrow (\forall\, xn{\in}sequence.\ \exists\, a.\ strict\text{-}mono\ a \wedge convergent\text{-}inS\ (xn \circ a))$

definition *eps-net-on* :: *'a set* $\Rightarrow$ *real* $\Rightarrow$ *'a set* $\Rightarrow$ *bool* **where**
$eps\text{-}net\text{-}on\ B\ \varepsilon\ A \longleftrightarrow \varepsilon > 0 \wedge finite\ A \wedge A \subseteq S \wedge B \subseteq (\bigcup a{\in}A.\ open\text{-}ball\ a\ \varepsilon)$

abbreviation *eps-net* $\equiv$ *eps-net-on S*

lemma *eps-net-def*: $eps\text{-}net\ \varepsilon\ A \longleftrightarrow \varepsilon > 0 \wedge finite\ A \wedge A \subseteq S \wedge S = \bigcup\ ((\lambda a.\ open\text{-}ball\ a\ \varepsilon)\ `\ A)$
$\langle proof \rangle$

lemma *eps-net-onD*:
assumes *eps-net-on B e A*
shows $e > 0$ *finite A* $A \subseteq S$ $B \subseteq (\bigcup a{\in}A.\ open\text{-}ball\ a\ e)$ $B \subseteq S$
$\langle proof \rangle$

lemma *eps-netD*:
assumes *eps-net* ε *A*
shows $\varepsilon > 0$ *finite A* $A \subseteq S$ $S = \bigcup\ ((\lambda a.\ open\text{-}ball\ a\ \varepsilon)\ `\ A)$
$\langle proof \rangle$

lemma *eps-net-le*:
assumes *eps-net e A* $e \le e'$
shows *eps-net* e' *A*
$\langle proof \rangle$

definition *totally-bounded-on* :: *'a set* $\Rightarrow$ *bool* **where**
$totally\text{-}bounded\text{-}on\ B \longleftrightarrow (\forall\, e{>}0.\ \exists\, A.\ eps\text{-}net\text{-}on\ B\ e\ A)$

abbreviation *totally-boundedS* $\equiv$ *totally-bounded-on S*

lemma *totally-boundedS-def*: $totally\text{-}boundedS \longleftrightarrow (\forall\, e{>}0.\ \exists\, A.\ eps\text{-}net\ e\ A)$
$\langle proof \rangle$

lemma *totally-bounded-onD-sub*:
assumes *totally-bounded-on B*
shows $B \subseteq S$
$\langle proof \rangle$

lemma *totally-bounded-onD*:
assumes *totally-bounded-on B* $e > 0$
obtains *A* **where** *finite A* $A \subseteq S$ $B \subseteq (\bigcup a{\in}A.\ open\text{-}ball\ a\ e)$

⟨*proof*⟩

lemma *totally-boundedSD*:
assumes *totally-boundedS* $e > 0$
obtains A **where** *finite* A $A \subseteq S$ $S = (\bigcup a{\in}A.\ \textit{open-ball}\ a\ e)$
⟨*proof*⟩

lemma *totally-bounded-on-iff*:
totally-bounded-on $B \longleftrightarrow B \subseteq S \wedge (\forall xn{\in}(\textit{UNIV} :: \textit{nat set}) \to B.\ \exists a.\ \textit{strict-mono}\ a \wedge \textit{Cauchy-inS}\ (xn \circ a))$
⟨*proof*⟩

corollary *totally-boundedS-iff*: *totally-boundedS* $\longleftrightarrow (\forall xn{\in}\textit{sequence}.\ \exists a.\ \textit{strict-mono}\ a \wedge \textit{Cauchy-inS}\ (xn \circ a))$
⟨*proof*⟩

Metric embedding

definition *embed-dist-on* :: $['b\ \textit{set}, 'b \Rightarrow 'a, 'b, 'b] \Rightarrow \textit{real}$ **where**
embed-dist-on $B\ f\ a\ b \equiv (\textit{if}\ a \in B \wedge b \in B\ \textit{then dist}\ (f\ a)\ (f\ b)\ \textit{else}\ 0)$

context
fixes $f\ B$
assumes f: $f \in B \to S$ *inj-on* $f\ B$
begin

abbreviation *ed* $\equiv$ *embed-dist-on* $B\ f$

lemma *embed-dist-dist*: *metric-set* B (*embed-dist-on* $B\ f$)
⟨*proof*⟩

interpretation *ed* : *metric-set* B *ed*
⟨*proof*⟩

lemma *embed-dist-open-ball*:
assumes $a \in B$
shows f ‘ (*ed.open-ball* $a\ e$) = *open-ball* $(f\ a)\ e \cap f$ ‘ B
⟨*proof*⟩

lemma *embed-dist-closed-ball*:
assumes $a \in B$
shows f ‘ (*ed.closed-ball* $a\ e$) = *closed-ball* $(f\ a)\ e \cap f$ ‘ B
⟨*proof*⟩

lemma *embed-dist-topology-homeomorphic-maps*:
assumes *g1*: $\bigwedge x.\ x \in B \Longrightarrow g\ (f\ x) = x$
shows *homeomorphic-maps ed.mtopology* (*subtopology mtopology* (f ‘ B)) $f\ g$
⟨*proof*⟩

lemma *embed-dist-topology-homeomorphic-map*:

homeomorphic-map ed.mtopology (subtopology mtopology (f ‘ B)) f
⟨*proof*⟩

corollary *embed-dist-topology-homeomorphic*:
ed.mtopology homeomorphic-space (subtopology mtopology (f ‘ B))
⟨*proof*⟩

corollary *embed-dist-topology-homeomorphic-map′*:
assumes f ‘ $B = S$
shows *homeomorphic-map ed.mtopology mtopology f*
⟨*proof*⟩

corollary *embed-dist-topology-homeomorphic′*:
assumes f ‘ $B = S$
shows *ed.mtopology homeomorphic-space mtopology*
⟨*proof*⟩

lemma *embed-dist-converge-to-inS-iff*:
ed.converge-to-inS xn x $\longleftrightarrow$ *xn* $\in$ *ed.sequence* $\wedge$ $x \in B$ $\wedge$ *converge-to-inS* ($\lambda n.$ *f* (*xn n*)) (*f x*)
⟨*proof*⟩

lemma *embed-dist-convergent-inS-iff*:
assumes *closedin mtopology* (*f* ‘ *B*)
shows *ed.convergent-inS xn* $\longleftrightarrow$ *xn* $\in$ *ed.sequence* $\wedge$ *convergent-inS* ($\lambda n.$ *f* (*xn n*))
⟨*proof*⟩

lemma *embed-dist-Cauchy-inS-iff*:
ed.Cauchy-inS xn $\longleftrightarrow$ *xn* $\in$ *ed.sequence* $\wedge$ *Cauchy-inS* ($\lambda n.$ *f* (*xn n*))
⟨*proof*⟩

end

end

Relations to elementary topology.

lemma *ball-def-set*: *ball a* ε = *metric-set.open-ball UNIV dist a* ε
⟨*proof*⟩

lemma *converge-to-def-set*:
fixes *xn* :: *nat* $\Rightarrow$ (*′a*::*metric-space*)
shows *xn* $\longrightarrow$ *x* $\longleftrightarrow$ *metric-set.converge-to-inS UNIV dist xn x*
⟨*proof*⟩

lemma *the-limit-of-limit*:
fixes *xn* :: *nat* $\Rightarrow$ (*′a*::*metric-space*)
shows *metric-set.the-limit-of UNIV dist xn* = *lim xn*
⟨*proof*⟩

lemma *convergent-def-set*:
fixes f :: *nat* $\Rightarrow$ (*'a::metric-space*)
shows *convergent f* $\longleftrightarrow$ *metric-set.convergent-inS UNIV dist f*
⟨*proof*⟩

lemma *Cahuchy-def-set*: *Cauchy f* $\longleftrightarrow$ *metric-set.Cauchy-inS UNIV dist f*
⟨*proof*⟩

lemma *open-openin-set*: *open U* $\longleftrightarrow$ *openin* (*metric-set.mtopology UNIV dist*) *U*
(**is** *?LHS* $\longleftrightarrow$ *?RHS*)
⟨*proof*⟩

lemma *topological-basis-set*: *topological-basis* $\mathcal{O}$ $\longleftrightarrow$ *metric-set.mtopology-basis UNIV dist* $\mathcal{O}$
(**is** *?LHS* = *?RHS*)
⟨*proof*⟩

lemma *euclidean-mtopology*: *metric-set.mtopology UNIV dist* = *euclidean*
⟨*proof*⟩

Distances generate the same topological space.

lemma *metric-generates-same-topology*:
assumes *metric-set S d metric-set S d'*
$\bigwedge x\ U.\ U \subseteq S \Longrightarrow (\forall y{\in}U.\ \exists\varepsilon{>}0.$ *metric-set.open-ball S d y* $\varepsilon \subseteq U) \Longrightarrow x \in U \Longrightarrow \exists\varepsilon{>}0.$ *metric-set.open-ball S d' x* $\varepsilon \subseteq U$
and $\bigwedge x\ U.\ U \subseteq S \Longrightarrow (\forall y{\in}U.\ \exists\varepsilon{>}0.$ *metric-set.open-ball S d' y* $\varepsilon \subseteq U) \Longrightarrow x \in U \Longrightarrow \exists\varepsilon{>}0.$ *metric-set.open-ball S d x* $\varepsilon \subseteq U$
shows *metric-set.mtopology S d* = *metric-set.mtopology S d'*
⟨*proof*⟩

lemma *metric-generates-same-topology-inverse*:
assumes *metric-set S d metric-set S d'*
and *metric-set.mtopology S d* = *metric-set.mtopology S d'*
shows $U \subseteq S \Longrightarrow (\forall y{\in}U.\ \exists\varepsilon{>}0.$ *metric-set.open-ball S d y* $\varepsilon \subseteq U) \Longrightarrow x \in U \Longrightarrow \exists\varepsilon{>}0.$ *metric-set.open-ball S d' x* $\varepsilon \subseteq U$
and $U \subseteq S \Longrightarrow (\forall y{\in}U.\ \exists\varepsilon{>}0.$ *metric-set.open-ball S d' y* $\varepsilon \subseteq U) \Longrightarrow x \in U \Longrightarrow \exists\varepsilon{>}0.$ *metric-set.open-ball S d x* $\varepsilon \subseteq U$
⟨*proof*⟩

lemma *metric-generates-same-topology-converges'*:
assumes *metric-set S d metric-set S d'*
metric-set.mtopology S d = *metric-set.mtopology S d'*
and *metric-set.converge-to-inS S d f x*
shows *metric-set.converge-to-inS S d' f x*
⟨*proof*⟩

lemma *metric-generates-same-topology-converges*:
assumes *metric-set S d metric-set S d'*

and *metric-set.mtopology* S d = *metric-set.mtopology* S d'
shows *metric-set.converge-to-inS* S d f x $\longleftrightarrow$ *metric-set.converge-to-inS* S d' f x
⟨*proof*⟩

lemma *metric-generates-same-topology-convergent*:
assumes *metric-set* S d *metric-set* S d'
and *metric-set.mtopology* S d = *metric-set.mtopology* S d'
shows *metric-set.convergent-inS* S d f $\longleftrightarrow$ *metric-set.convergent-inS* S d' f
⟨*proof*⟩

2.1.1 Sub-Metric Spaces

definition *submetric* :: $['a\ set, 'a \Rightarrow 'a \Rightarrow real] \Rightarrow 'a \Rightarrow 'a \Rightarrow real$ **where**
submetric S' d $\equiv$ ($\lambda x\ y.$ *if* $x \in S' \wedge y \in S'$ *then* $d\ x\ y$ *else* 0)

lemma(**in** *metric-set*) *submetric-metric-set*:
assumes $S' \subseteq S$
shows *metric-set* S' (*submetric* S' *dist*)
⟨*proof*⟩

lemma(**in** *metric-set*) *submetric-open-ball*:
assumes $S' \subseteq S$ **and** $a \in S'$
shows *open-ball* a $\varepsilon \cap S'$ = *metric-set.open-ball* S' (*submetric* S' *dist*) a ε
⟨*proof*⟩

lemma(**in** *metric-set*) *submetric-open-ball-subset*:
assumes $S' \subseteq S$
shows *metric-set.open-ball* S' (*submetric* S' *dist*) a $\varepsilon \subseteq$ *open-ball* a ε
⟨*proof*⟩

lemma(**in** *metric-set*) *submetric-subtopology*:
assumes $S' \subseteq S$
shows *subtopology mtopology* S' = *metric-set.mtopology* S' (*submetric* S' *dist*)
⟨*proof*⟩

lemma(**in** *metric-set*) *converge-to-insub-converge-to-inS*:
assumes $S' \subseteq S$ **and** *metric-set.converge-to-inS* S' (*submetric* S' *dist*) f x
shows *converge-to-inS* f x
⟨*proof*⟩

lemma(**in** *metric-set*) *convergent-insub-convergent-inS*:
assumes $S' \subseteq S$ **and** *metric-set.convergent-inS* S' (*submetric* S' *dist*) f
shows *convergent-inS* f
⟨*proof*⟩

lemma(**in** *metric-set*) *Cauchy-insub-Cauchy*:
assumes $S' \subseteq S$ **and** *metric-set.Cauchy-inS* S' (*submetric* S' *dist*) f
shows *Cauchy-inS* f

$\langle proof \rangle$

lemma(**in** *metric-set*) *Cauchy-insub-Cauchy-inverse*:
  assumes $S' \subseteq S$ $f \in UNIV \to S'$ *Cauchy-inS f*
  shows *metric-set.Cauchy-inS* S' (*submetric* S' *dist*) *f*
$\langle proof \rangle$

lemma(**in** *metric-set*) *convergent-insubmetric*:
  assumes $S' \subseteq S$ $f \in UNIV \to S'$ $x \in S'$ *converge-to-inS f x*
  shows *metric-set.converge-to-inS* S' (*submetric* S' *dist*) *f x*
$\langle proof \rangle$

lemma(**in** *metric-set*) *the-limit-of-submetric-eq*:
  assumes $S' \subseteq S$ *metric-set.convergent-inS* S' (*submetric* S' *dist*) *f*
  shows *metric-set.the-limit-of* S' (*submetric* S' *dist*) $f =$ *the-limit-of f*
  $\langle proof \rangle$

lemma *submetric-of-euclidean*:
 metric-set A (*submetric A dist*) *metric-set.mtopology A* (*submetric A dist*) $=$ *top-of-set A*
  $\langle proof \rangle$

lemma(**in** *metric-set*)
  assumes $B \subseteq S$
 shows *totally-bounded-on-submetric*: *totally-bounded-on B* $\longleftrightarrow$ *metric-set.totally-boundedS B* (*submetric B dist*)
$\langle proof \rangle$

Continuous functions

context
  fixes S :: $'a$ *set* **and** d
    and S':: $'b$ *set* **and** d'
  assumes *metric-set S d metric-set* S' d'
begin

interpretation *m1*: *metric-set S d* $\langle proof \rangle$
interpretation *m2*: *metric-set* S' d' $\langle proof \rangle$

lemma *metric-set-continuous-map-eq*:
  shows *continuous-map m1.mtopology m2.mtopology f*
        $\longleftrightarrow f \in S \to S' \wedge (\forall x{\in}S.\ \forall \varepsilon{>}0.\ \exists \delta{>}0.\ \forall y{\in}S.\ d\ x\ y < \delta \longrightarrow d'\ (f\ x)\ (f\ y) < \varepsilon)$
$\langle proof \rangle$

lemma *metric-set-continuous-map-eq'*:
  shows *continuous-map m1.mtopology m2.mtopology f*
        $\longleftrightarrow f \in S \to S' \wedge (\forall x\ z.$ *m1.converge-to-inS x z* $\longrightarrow$ *m2.converge-to-inS* $(\lambda n.\ f\ (x\ n))\ (f\ z))$
$\langle proof \rangle$

lemma *continuous-map-limit-of*:
  assumes *continuous-map m1.mtopology m2.mtopology f m1.convergent-inS xn*
  shows *m2.the-limit-of* $(\lambda n.\ f\ (xn\ n)) = f\ (m1.the\text{-}limit\text{-}of\ xn)$
  ⟨*proof*⟩

Uniform continuous functions.

definition *uniform-continuous-map* :: $('a \Rightarrow 'b) \Rightarrow bool$ **where**
uniform-continuous-map f $\longleftrightarrow f \in S \to S' \wedge (\forall \varepsilon>0.\ \exists \delta>0.\ \forall x\in S.\ \forall y\in S.\ d\ x\ y < \delta \longrightarrow d'\ (f\ x)\ (f\ y) < \varepsilon)$

lemma *uniform-continuous-map-const*:
  assumes $y \in S'$
  shows *uniform-continuous-map* $(\lambda x.\ y)$
  ⟨*proof*⟩

lemma *continuous-if-uniform-continuous*:
  assumes *uniform-continuous-map f*
  shows *continuous-map m1.mtopology m2.mtopology f*
  ⟨*proof*⟩

definition *converges-uniformly* :: $[nat \Rightarrow 'a \Rightarrow 'b, 'a \Rightarrow 'b] \Rightarrow bool$ **where**
converges-uniformly fn f $\longleftrightarrow (\forall n.\ fn\ n \in S \to S') \wedge (f \in S \to S') \wedge (\forall e>0.\ \exists N.\ \forall n\geq N.\ \forall x\in S.\ d'\ (fn\ n\ x)\ (f\ x) < e)$

lemma *converges-uniformly-continuous*:
  assumes $\bigwedge n.$ *continuous-map m1.mtopology m2.mtopology* $(fn\ n)$
    and *converges-uniformly fn f*
   shows *continuous-map m1.mtopology m2.mtopology f*
  ⟨*proof*⟩

Lemma related *osc-on*.

lemma *osc-on-inA-0*:
 assumes $x \in A \cap S$ *continuous-map* (*subtopology m1.mtopology* $(A \cap S)$) *m2.mtopology f*
  shows *m2.osc-on A m1.mtopology f x* $= 0$
⟨*proof*⟩

end

context *metric-set*
begin

interpretation *rnv*: *metric-set UNIV* :: $('b :: real\text{-}normed\text{-}vector)$ *set dist-typeclass*
  ⟨*proof*⟩

lemma *dist-set-uniform-continuous*:
 uniform-continuous-map S dist UNIV dist-typeclass (*dist-set A*)
  ⟨*proof*⟩

lemma *dist-set-continuous*: *continuous-map mtopology euclideanreal* (*dist-set A*)
⟨*proof*⟩

lemma *uniform-continuous-map-add*:
fixes f :: $'a \Rightarrow\ 'b$::*real-normed-vector*
assumes *uniform-continuous-map S dist UNIV dist-typeclass f uniform-continuous-map S dist UNIV dist-typeclass g*
shows *uniform-continuous-map S dist UNIV dist-typeclass* $(\lambda x.\ f\ x + g\ x)$
⟨*proof*⟩

lemma *uniform-continuous-map-real-devide*:
fixes f :: $'a \Rightarrow real$
assumes *uniform-continuous-map S dist UNIV dist-typeclass f uniform-continuous-map S dist UNIV dist-typeclass g*
and $\bigwedge x.\ x \in S \Longrightarrow g\ x \neq 0$ $\bigwedge x.\ x \in S \Longrightarrow |g\ x| \geq a$ $a > 0$ $\bigwedge x.\ x \in S \Longrightarrow |g\ x| < Kg$
and $\bigwedge x.\ x \in S \Longrightarrow |f\ x| < Kf$
shows *uniform-continuous-map S dist UNIV dist-typeclass* $(\lambda x.\ f\ x\ /\ g\ x)$
⟨*proof*⟩

lemma *the-limit-of-dist-converge*:
assumes *converge-to-inS xn x*
shows $(\lambda n.\ dist\ (xn\ n)\ y) \longlonglongrightarrow dist\ (\textit{the-limit-of}\ xn)\ y$
⟨*proof*⟩

lemma *the-limit-of-dist-converge'*:
assumes *converge-to-inS xn x* $\varepsilon > 0$
shows $\exists N.\ \forall n \geq N.\ |\ dist\ (xn\ n)\ y - dist\ (\textit{the-limit-of}\ xn)\ y\ | < \varepsilon$
⟨*proof*⟩

lemma *the-limit-of-dist*:
assumes *converge-to-inS xn x*
shows $lim\ (\lambda n.\ dist\ (xn\ n)\ y) = dist\ (\textit{the-limit-of}\ xn)\ y$
⟨*proof*⟩

Upper-semicontinuous functions.

lemma *upper-semicontinuous-map-def2*:
fixes f :: $'a \Rightarrow ('b$ :: {*complete-linorder*,*linorder-topology*})
shows *upper-semicontinuous-map mtopology* $f \longleftrightarrow (\forall x\ y.$ *converge-to-inS* $x\ y \longrightarrow limsup\ (\lambda n.\ f\ (x\ n)) \leq f\ y)$
⟨*proof*⟩

lemma *upper-semicontinuous-map-def2real*:
fixes f :: $'a \Rightarrow real$
shows *upper-semicontinuous-map mtopology* $f \longleftrightarrow (\forall x\ y.$ *converge-to-inS* $x\ y \longrightarrow limsup\ (\lambda n.\ f\ (x\ n)) \leq f\ y)$
⟨*proof*⟩

lemma *osc-upper-semicontinuous-map*:
upper-semicontinuous-map X (*osc X f*)
⟨*proof*⟩

end

Open maps.

lemma *metric-set-opem-map-from-dist*:
assumes *metric-set S d metric-set S′ d′ f* $\in S \to S'$
and $\bigwedge x\ \varepsilon.\ x \in S \Longrightarrow \varepsilon > 0 \Longrightarrow \exists\delta{>}0.\ \forall y{\in}S.\ d'\ (f\ x)\ (f\ y) < \delta \longrightarrow d\ x\ y < \varepsilon$
shows *open-map* (*metric-set.mtopology S d*) (*subtopology* (*metric-set.mtopology S′ d′*) (*f* ‘ *S*)) *f*
⟨*proof*⟩

2.1.2 Complete Metric Spaces

locale *complete-metric-set* = *metric-set* +
assumes *convergence*: $\bigwedge f.$ *Cauchy-inS f* $\Longrightarrow$ *convergent-inS f*

lemma *complete-space-complete-metric-set*:
complete-metric-set (*UNIV* :: *′a* :: *complete-space set*) *dist*
⟨*proof*⟩

lemma(**in** *complete-metric-set*) *submetric-complete-iff*:
assumes $M \subseteq S$
shows *complete-metric-set M* (*submetric M dist*) $\longleftrightarrow$ *closedin mtopology M*
⟨*proof*⟩

lemma(**in** *complete-metric-set*) *embed-dist-complete*:
assumes $f \in B \to S$ *inj-on f B closedin mtopology* (*f* ‘ *B*)
shows *complete-metric-set B* (*embed-dist-on B f*)
⟨*proof*⟩

lemma(**in** *metric-set*) *Cantor-intersection-theorem*:
complete-metric-set S dist $\longleftrightarrow$ ($\forall$ *Fn*. ($\forall$ *n*. *Fn n* $\neq$ {}) $\wedge$ ($\forall$ *n*. *closedin mtopology* (*Fn n*)) $\wedge$ *decseq Fn* $\wedge$ ($\forall\varepsilon > 0.\ \exists N.\ \forall n{\geq}N.$ *diam* (*Fn n*) $< \varepsilon$) $\longrightarrow$ ($\exists x{\in}S.\ \bigcap$ (*range Fn*) = {*x*}))
⟨*proof*⟩

lemma(**in** *complete-metric-set*) *closed-decseq-Inter′*:
assumes $\bigwedge n.$ *Fn n* $\neq$ {} $\bigwedge n.$ *closedin mtopology* (*Fn n*) *decseq Fn*
and $\bigwedge\varepsilon.\ \varepsilon > 0 \Longrightarrow \exists N.\ \forall n{\geq}N.$ *diam* (*Fn n*) $< \varepsilon$
shows $\exists x{\in}S.\ \bigcap$ (*range Fn*) = {*x*}
⟨*proof*⟩

lemma(**in** *complete-metric-set*) *closed-decseq-Inter*:
assumes $\bigwedge n.$ *Fn n* $\neq$ {} $\bigwedge n.$ *closedin mtopology* (*Fn n*) *decseq Fn*

and $\bigwedge\varepsilon.\ \varepsilon > 0 \Longrightarrow \exists N.\ \forall n{\geq}N.\ diam\ (Fn\ n) < ennreal\ \varepsilon$
shows $\exists x{\in}S.\ \bigcap\ (range\ Fn) = \{x\}$
⟨*proof*⟩

2.1.3 Separable Metric Spaces

locale *separable-metric-set* = *metric-set* +
assumes *separable*: $\exists U.\ countable\ U \wedge dense\text{-}set\ U$

lemma(**in** *metric-set*) *separable-if-countable*:
assumes *countable S*
shows *separable-metric-set S dist*
⟨*proof*⟩

lemma(**in** *metric-set*) *separable-iff-topological-separable*:
separable-metric-set S dist $\longleftrightarrow$ *separable mtopology*
⟨*proof*⟩

lemma(**in** *separable-metric-set*) *topological-separable-if-separable*:
separable mtopology
⟨*proof*⟩

lemma *separable-metric-setI*:
assumes *metric-set S d separable* (*metric-set.mtopology S d*)
shows *separable-metric-set S d*
⟨*proof*⟩

For a metric space X, X is separable iff X is second countable.

lemma(**in** *metric-set*) *generated-by-countable-balls*:
assumes *countable U* **and** *dense-set U*
shows $mtopology = topology\text{-}generated\text{-}by\ \{open\text{-}ball\ y\ (1\ /\ real\ n) \mid y\ n.\ y \in U\}$
⟨*proof*⟩

lemma(**in** *separable-metric-set*) *second-countable'*:
$\exists \mathcal{O}.\ countable\ \mathcal{O} \wedge mtopology\text{-}basis\ \mathcal{O}$
⟨*proof*⟩

lemma(**in** *separable-metric-set*) *second-countable*: *second-countable mtopology*
⟨*proof*⟩

lemma(**in** *metric-set*) *separable-if-second-countable*:
assumes *countable* $\mathcal{O}$ **and** *mtopology-basis* $\mathcal{O}$
shows *separable-metric-set S dist*
⟨*proof*⟩

lemma *metric-generates-same-topology-separable-if*:
assumes *metric-set S d metric-set S d'*
and *metric-set.mtopology S d = metric-set.mtopology S d'*
and *separable-metric-set S d*

shows *separable-metric-set S d′*
⟨*proof*⟩

lemma *metric-generates-same-topology-separable*:
assumes *metric-set S d metric-set S d′*
and *metric-set.mtopology S d* = *metric-set.mtopology S d′*
shows *separable-metric-set S d* ⟷ *separable-metric-set S d′*
⟨*proof*⟩

lemma(**in** *metric-set*) *separable-if-totally-bounded*:
assumes *totally-boundedS*
shows *separable-metric-set S dist*
⟨*proof*⟩

lemma *second-countable-metric-class-separable-set*:
separable-metric-set (*UNIV* :: *′a* ::{*metric-space*,*second-countable-topology*} *set*) *dist*
⟨*proof*⟩

lemma *second-countable-euclidean*[*simp*]:
second-countable (*euclidean* :: *′a* :: {*metric-space*,*second-countable-topology*} *topology*)
⟨*proof*⟩

lemma *separable-euclidean*[*simp*]:
separable (*euclidean* :: *′a* :: {*metric-space*,*second-countable-topology*} *topology*)
⟨*proof*⟩

lemma(**in** *separable-metric-set*) *submetric-separable*:
assumes $S' \subseteq S$
shows *separable-metric-set S′* (*submetric S′ dist*)
⟨*proof*⟩

lemma(**in** *separable-metric-set*) *Lindelof-diam*:
assumes $0 < e$
shows $\exists\, U.\ \mathit{countable}\ U \wedge \bigcup\ U = S \wedge (\forall\, u \in U.\ \mathit{diam}\ u < \mathit{ennreal}\ e)$
⟨*proof*⟩

2.1.4 Polish Metric Spaces

locale *polish-metric-set* = *complete-metric-set* + *separable-metric-set*

lemma *polish-class-polish-set*[*simp*]:
polish-metric-set (*UNIV* :: *′a* :: *polish-space set*) *dist*
⟨*proof*⟩

lemma(**in** *polish-metric-set*) *submetric-polish*:
assumes $M \subseteq S$ **and** *closedin mtopology M*
shows *polish-metric-set M* (*submetric M dist*)

⟨*proof*⟩

lemma *polish-metric-setI*:
assumes *complete-metric-set S d separable* (*metric-set.mtopology S d*)
shows *polish-metric-set S d*
⟨*proof*⟩

2.1.5 Compact Metric Spaces

locale *compact-metric-set* = *metric-set* +
assumes *mtopology-compact*:*compact-space mtopology*
begin

context
fixes S' :: *'b set* **and** *dist'*
assumes S'*-dist*: *metric-set* S' *dist'*
begin

interpretation *m'*: *metric-set* S' *dist'* ⟨*proof*⟩

lemma *continuous-map-is-uniform*:
assumes *continuous-map mtopology m'.mtopology f*
shows *uniform-continuous-map S dist* S' *dist' f*
⟨*proof*⟩

end

lemma *totally-bounded*: *totally-boundedS*
⟨*proof*⟩

lemma *sequentially-compact*: *sequentially-compact*
⟨*proof*⟩

lemma *polish*: *polish-metric-set S dist*
⟨*proof*⟩

sublocale *polish-metric-set*
⟨*proof*⟩

end

lemma(**in** *metric-set*) *ex-lebesgue-number*:
assumes $S \neq \{\}$ *sequentially-compact* $\bigwedge u.\ u \in U \Longrightarrow$ *openin mtopology u* $S \subseteq \bigcup U$
shows $\exists\, d{>}0.\ \forall\, a{\subseteq}S.\ diam\ a < ennreal\ d \longrightarrow (\exists\, u{\in}U.\ a \subseteq u)$
⟨*proof*⟩

lemma(**in** *metric-set*) *sequentially-compact-iff1*:

sequentially-compact $\longleftrightarrow$ *totally-boundedS* $\wedge$ *complete-metric-set S dist*
⟨*proof*⟩

lemma(**in** *metric-set*) *sequentially-compact-compact*:
assumes *sequentially-compact*
shows *compact-metric-set S dist*
⟨*proof*⟩

corollary(**in** *metric-set*) *compact-iff-sequentially-compact*:
compact-space mtopology $\longleftrightarrow$ *sequentially-compact*
⟨*proof*⟩

corollary(**in** *metric-set*) *compact-iff2*:
compact-space mtopology $\longleftrightarrow$ *totally-boundedS* $\wedge$ *complete-metric-set S dist*
⟨*proof*⟩

corollary(**in** *complete-metric-set*) *compactin-closed-iff*:
assumes *closedin mtopology C*
shows *compactin mtopology C* $\longleftrightarrow$ *totally-bounded-on C*
⟨*proof*⟩

2.1.6 Completion

context *metric-set*
begin

abbreviation *Cauchys* $\equiv$ *Collect Cauchy-inS*

definition *Cauchy-r* :: $((nat \Rightarrow 'a) \times (nat \Rightarrow 'a))$ *set* **where**
Cauchy-r $\equiv \{(xn,yn)|xn\ yn.$ *Cauchy-inS* $xn \wedge$ *Cauchy-inS* $yn \wedge (\lambda n.\ dist\ (xn\ n)\ (yn\ n)) \longrightarrow 0\}$

lemma *Cauchy-r-equiv*[*simp*]: *equiv Cauchys Cauchy-r*
⟨*proof*⟩

abbreviation *S-completion* :: $(nat \Rightarrow 'a)$ *set set* (S^*) **where**
S-completion $\equiv$ *Cauchys* // *Cauchy-r*

lemma *S-c-represent*:
assumes $X \in S^*$
obtains *xn* **where** $xn \in X$ *Cauchy-inS xn*
⟨*proof*⟩

lemma *Cauchy-inS-ignore-initial-eq*:
assumes *Cauchy-inS xn*
shows $(xn, (\lambda n.\ xn\ (n + k))) \in$ *Cauchy-r*
⟨*proof*⟩

corollary *Cauchy-inS-r*: $a \in S \Longrightarrow (\lambda n.\ a, \lambda n.\ a) \in$ *Cauchy-r*

$\langle proof \rangle$

abbreviation *dist-completion'* :: $[nat \Rightarrow 'a,\ nat \Rightarrow 'a] \Rightarrow real$ **where**
dist-completion' xn yn $\equiv$ *lim* $(\lambda n.\ dist\ (xn\ n)\ (yn\ n))$

lemma *dist-of-completion-congruent2*: *dist-completion' respects2 Cauchy-r*
$\langle proof \rangle$

definition *dist-completion* :: $[(nat \Rightarrow 'a)\ set,\ (nat \Rightarrow 'a)\ set] \Rightarrow real$ $(dist^*)$ **where**
$dist^*\ X\ Y \equiv (if\ X \in S^* \wedge Y \in S^*\ then\ dist\text{-}completion'\ (SOME\ xn.\ xn \in X)\ (SOME\ yn.\ yn \in Y)\ else\ 0)$

lemma *dist-c-def*:
assumes $xn \in X$ $yn \in Y$ $X \in S^*$ $Y \in S^*$
shows $dist^*\ X\ Y = dist\text{-}completion'\ xn\ yn$
$\langle proof \rangle$

lemma *completion-metric-set*: *metric-set* S^* $dist^*$
$\langle proof \rangle$

interpretation *c:metric-set* S^* $dist^*$
$\langle proof \rangle$

definition *into-S-c* :: $'a \Rightarrow (nat \Rightarrow 'a)\ set$ **where**
into-S-c a $\equiv$ *Cauchy-r* `` $\{(\lambda n.\ a)\}$

lemma *into-S-c-in*:
assumes $a \in S$
shows $(\lambda n.\ a) \in into\text{-}S\text{-}c\ a$
$\langle proof \rangle$

lemma *into-S-c-into*:
assumes $a \in S$
shows $into\text{-}S\text{-}c\ a \in S^*$
$\langle proof \rangle$

lemma *into-S-inj*: *inj-on into-S-c S*
$\langle proof \rangle$

lemma *dist-into-S-c*:
assumes $x \in S$ $y \in S$
shows $dist^*\ (into\text{-}S\text{-}c\ x)\ (into\text{-}S\text{-}c\ y) = dist\ x\ y$
$\langle proof \rangle$

lemma *S-c-isometry*:
c.ed into-S-c S = dist
$\langle proof \rangle$

corollary *mtopology-embedding-S-c-map*:
homeomorphic-map mtopology (*subtopology c.mtopology* (*into-S-c* ‘ *S*)) *into-S-c*
⟨*proof*⟩

corollary *mtopology-embedding-S-c*:
mtopology homeomorphic-space subtopology c.mtopology (*into-S-c* ‘ *S*)
⟨*proof*⟩

lemma *into-S-c-image-dense*: *c.dense-set* (*into-S-c* ‘ *S*)
⟨*proof*⟩

lemma *completion-complete*:*complete-metric-set* S^* $dist^*$
⟨*proof*⟩

lemma *dense-set-c-dense*:
assumes *dense-set U*
shows *c.dense-set* (*into-S-c* ‘ *U*)
⟨*proof*⟩

end

lemma(**in** *separable-metric-set*) *completion-polish*: *polish-metric-set* S^* $dist^*$
⟨*proof*⟩

2.2 Discrete Distance

definition *discrete-dist* :: *'a set* ⇒ *'a* ⇒ *'a* ⇒ *real* **where**
discrete-dist S ≡ (λ*a b*. *if a* ∈ *S* ∧ *b* ∈ *S* ∧ *a* ≠ *b then 1 else 0*)

lemma
assumes *a* ∈ *S* **and** *b* ∈ *S*
shows *discrete-dist-iff-1*: *discrete-dist S a b* = *1* ⟷ *a* ≠ *b*
and *discrete-dist-iff-0*: *discrete-dist S a b* = *0* ⟷ *a* = *b*
⟨*proof*⟩

lemma *discrete-dist-metric*:
metric-set S (*discrete-dist S*)
⟨*proof*⟩

lemma
shows *discrete-dist-ball-ge1*: *x* ∈ *S* ⟹ *1* < ε ⟹ *metric-set.open-ball S* (*discrete-dist S*) *x* ε = *S*
and *discrete-dist-ball-leq1*: *x* ∈ *S* ⟹ *0* < ε ⟹ ε ≤ *1* ⟹ *metric-set.open-ball S* (*discrete-dist S*) *x* ε = {*x*}
⟨*proof*⟩

lemma *discrete-dist-complete-metric*:
complete-metric-set S (*discrete-dist S*)

⟨*proof*⟩

lemma *discrete-dist-dense-set*:
metric-set.dense-set S (*discrete-dist* S) $U \longleftrightarrow S = U$
⟨*proof*⟩

lemma *discrete-dist-separable-iff*:
separable-metric-set S (*discrete-dist* S) $\longleftrightarrow$ *countable* S
⟨*proof*⟩

lemma *discrete-dist-polish-iff*: *polish-metric-set* S (*discrete-dist* S) $\longleftrightarrow$ *countable* S
⟨*proof*⟩

lemma *discrete-dist-topology-x*:
assumes $x \in S$
shows *openin* (*metric-set.mtopology* S (*discrete-dist* S)) $\{x\}$
⟨*proof*⟩

lemma *discrete-dist-topology*:
openin (*metric-set.mtopology* S (*discrete-dist* S)) $U \longleftrightarrow U \subseteq S$
⟨*proof*⟩

lemma *discrete-dist-topology'*:
metric-set.mtopology S (*discrete-dist* S) = *discrete-topology* S
⟨*proof*⟩

Empty space.

lemma *empty-metric-compact*: *compact-metric-set* {} ($\lambda x\ y.\ 0$)
⟨*proof*⟩

corollary
shows *empty-metric-polish*: *polish-metric-set* {} ($\lambda x\ y.\ 0$)
and *empty-metric-complete*: *complete-metric-set* {} ($\lambda x\ y.\ 0$)
and *empty-metric-separable*: *separable-metric-set* {} ($\lambda x\ y.\ 0$)
and *empty-metric*: *metric-set* {} ($\lambda x\ y.\ 0$)
⟨*proof*⟩

lemma *empty-metric-unique*:
assumes *metric-set* {} d
shows $d = (\lambda x\ y.\ 0)$
⟨*proof*⟩

lemma *empty-metric-mtopology*:
metric-set.mtopology {} ($\lambda x\ y.\ 0$) = *discrete-topology* {}
⟨*proof*⟩

Singleton space

lemma *singleton-metric-compact*:
compact-metric-set $\{a\}$ $(\lambda x\ y.\ 0)$
⟨*proof*⟩

corollary
shows *singleton-metric-polish*: *polish-metric-set* $\{a\}$ $(\lambda x\ y.\ 0)$
and *singleton-metric-complete*: *complete-metric-set* $\{a\}$ $(\lambda x\ y.\ 0)$
and *singleton-metric-separable*: *separable-metric-set* $\{a\}$ $(\lambda x\ y.\ 0)$
and *singleton-metric*: *metric-set* $\{a\}$ $(\lambda x\ y.\ 0)$
⟨*proof*⟩

lemma *singleton-metric-unique*:
assumes *metric-set* $\{a\}$ d
shows $d = (\lambda x\ y.\ 0)$
⟨*proof*⟩

lemma *singleton-metric-mtopology*:
metric-set.mtopology $\{a\}$ $(\lambda x\ y.\ 0)$ = *discrete-topology* $\{a\}$
⟨*proof*⟩

2.3 Binary Product Metric Spaces

We define the L^1-distance. L^1-distance and L^2 distance (Euclid distance) generate the same topological space.

definition *binary-distance* :: $['a\ set,\ 'a \Rightarrow 'a \Rightarrow real,\ 'b\ set,\ 'b \Rightarrow 'b \Rightarrow real] \Rightarrow 'a \times 'b \Rightarrow 'a \times 'b \Rightarrow real$ **where**
binary-distance $S\ d\ S'\ d' \equiv (\lambda(x,x')\ (y,y').\ if\ (x,x') \in S \times S' \wedge (y,y') \in S \times S'\ then\ d\ x\ y + d'\ x'\ y'\ else\ 0)$

context
fixes $S\ S'\ d\ d'$
assumes *metric-set* $S\ d$ *metric-set* $S'\ d'$
begin

interpretation *m1*: *metric-set* $S\ d$ ⟨*proof*⟩
interpretation *m2*: *metric-set* $S'\ d'$ ⟨*proof*⟩

lemma *binary-metric-set*:
metric-set $(S \times S')$ (*binary-distance* $S\ d\ S'\ d'$)
⟨*proof*⟩

interpretation *m*: *metric-set* $S \times S'$ *binary-distance* $S\ d\ S'\ d'$
⟨*proof*⟩

lemma *binary-distance-geq*:
assumes $x \in S$ $y \in S$ $x' \in S'$ $y' \in S'$
shows $d\ x\ y \leq$ *binary-distance* $S\ d\ S'\ d'\ (x,x')\ (y,y')$
$d'\ x'\ y' \leq$ *binary-distance* $S\ d\ S'\ d'\ (x,x')\ (y,y')$

$\langle proof \rangle$

lemma *binary-distance-ball*:
assumes $(x,x') \in m.open\text{-}ball\ (a,a')\ \varepsilon$
shows $x \in m1.open\text{-}ball\ a\ \varepsilon$
and $x' \in m2.open\text{-}ball\ a'\ \varepsilon$
$\langle proof \rangle$

lemma *binary-distance-ball'*:
assumes $z \in m.open\text{-}ball\ a\ \varepsilon$
shows $fst\ z \in m1.open\text{-}ball\ (fst\ a)\ \varepsilon$
and $snd\ z \in m2.open\text{-}ball\ (snd\ a)\ \varepsilon$
$\langle proof \rangle$

lemma *binary-distance-ball1'*:
assumes $a \in S\ \varepsilon > 0\ a' \in S'\ \varepsilon' > 0$
shows $\exists \varepsilon''>0.\ m.open\text{-}ball\ (a,a')\ \varepsilon'' \subseteq m1.open\text{-}ball\ a\ \varepsilon \times m2.open\text{-}ball\ a'\ \varepsilon'$
$\langle proof \rangle$

lemma *binary-distance-ball1*:
assumes $b \in m1.open\text{-}ball\ a\ \varepsilon\ b' \in m2.open\text{-}ball\ a'\ \varepsilon'$
shows $\exists \varepsilon''>0.\ m.open\text{-}ball\ (b,b')\ \varepsilon'' \subseteq m1.open\text{-}ball\ a\ \varepsilon \times m2.open\text{-}ball\ a'\ \varepsilon'$
$\langle proof \rangle$

lemma *binary-distance-ball2'*:
assumes $a \in S\ \varepsilon'' > 0\ a' \in S'$
shows $\exists \varepsilon>0.\ \exists \varepsilon'>0.\ m1.open\text{-}ball\ a\ \varepsilon \times m2.open\text{-}ball\ a'\ \varepsilon' \subseteq m.open\text{-}ball\ (a,a')\ \varepsilon''$
$\langle proof \rangle$

lemma *binary-distance-ball2*:
assumes $(b,b') \in m.open\text{-}ball\ (a,a')\ \varepsilon''$
shows $\exists \varepsilon>0.\ \exists \varepsilon'>0.\ m1.open\text{-}ball\ b\ \varepsilon \times m2.open\text{-}ball\ b'\ \varepsilon' \subseteq m.open\text{-}ball\ (a,a')\ \varepsilon''$
$\langle proof \rangle$

lemma *binary-distance-mtopology*:
$m.mtopology = prod\text{-}topology\ m1.mtopology\ m2.mtopology$
$\langle proof \rangle$

lemma *binary-distance-converge-to-inS-iff*:
$m.converge\text{-}to\text{-}inS\ zn\ (x,y) \longleftrightarrow m1.converge\text{-}to\text{-}inS\ (\lambda n.\ fst\ (zn\ n))\ x \land m2.converge\text{-}to\text{-}inS\ (\lambda n.\ snd\ (zn\ n))\ y$
$\langle proof \rangle$

lemma *binary-distance-converge-to-inS-iff'*:
$m.converge\text{-}to\text{-}inS\ zn\ z \longleftrightarrow m1.converge\text{-}to\text{-}inS\ (\lambda n.\ fst\ (zn\ n))\ (fst\ z) \land m2.converge\text{-}to\text{-}inS$

$(\lambda n.\ snd\ (zn\ n))\ (snd\ z)$
  ⟨*proof*⟩

corollary *binary-distance-convergent-inS-iff*:
 $m.convergent\text{-}inS\ zn \longleftrightarrow m1.convergent\text{-}inS\ (\lambda n.\ fst\ (zn\ n)) \wedge m2.convergent\text{-}inS\ (\lambda n.\ snd\ (zn\ n))$
  ⟨*proof*⟩

lemma *binary-distance-Cauchy-inS-iff*:
 $m.Cauchy\text{-}inS\ zn \longleftrightarrow m1.Cauchy\text{-}inS\ (\lambda n.\ fst\ (zn\ n)) \wedge m2.Cauchy\text{-}inS\ (\lambda n.\ snd\ (zn\ n))$
⟨*proof*⟩

end

lemma *binary-distance-separable*:
  assumes *separable-metric-set S d separable-metric-set S′ d′*
  shows *separable-metric-set* $(S \times S')$ *(binary-distance S d S′ d′)*
⟨*proof*⟩

lemma *binary-distance-complete*:
  assumes *complete-metric-set S d complete-metric-set S′ d′*
  shows *complete-metric-set* $(S \times S')$ *(binary-distance S d S′ d′)*
⟨*proof*⟩

lemma *binary-distance-polish*:
  assumes *polish-metric-set S d* **and** *polish-metric-set S′ d′*
  shows *polish-metric-set* $(S \times S')$ *(binary-distance S d S′ d′)*
  ⟨*proof*⟩

2.4 Sum Metric Spaces

locale *sum-metric* =
  fixes $I :: 'i\ set$
    and $Si :: 'i \Rightarrow 'a\ set$
    and $di :: 'i \Rightarrow 'a \Rightarrow 'a \Rightarrow real$
  assumes *disj-fam*: *disjoint-family-on Si I*
      and *d-nonneg*: $\bigwedge i\ x\ y.\ 0 \le di\ i\ x\ y$
      and *d-bounded*: $\bigwedge i\ x\ y.\ di\ i\ x\ y < 1$
      and *sd-metric*: $\bigwedge i.\ i \in I \Longrightarrow metric\text{-}set\ (Si\ i)\ (di\ i)$
begin

abbreviation $S \equiv \bigcup i{\in}I.\ Si\ i$

lemma *Si-inj-on*:
  assumes $i \in I\ j \in I\ a \in Si\ i\ a \in Si\ j$
  shows $i = j$
  ⟨*proof*⟩

definition *sum-dist* :: $['a, 'a] \Rightarrow$ *real* **where**
sum-dist x y $\equiv$ (*if* $x \in S \wedge y \in S$ *then* (*if* $\exists i \in I.\ x \in Si\ i \wedge y \in Si\ i$ *then di* (*THE* $i.\ i \in I \wedge x \in Si\ i \wedge y \in Si\ i$) *x y else 1*) *else 0*)

lemma *sum-dist-simps*:
  shows $\bigwedge i.\ [\![i \in I;\ x \in Si\ i;\ y \in Si\ i]\!] \Longrightarrow$ *sum-dist* $x\ y = di\ i\ x\ y$
    and $\bigwedge i\ j.\ [\![i \in I;\ j \in I;\ i \neq j;\ x \in Si\ i;\ y \in Si\ j]\!] \Longrightarrow$ *sum-dist* $x\ y = 1$
    and $\bigwedge i.\ [\![i \in I;\ y \in S;\ x \in Si\ i;\ y \notin Si\ i]\!] \Longrightarrow$ *sum-dist* $x\ y = 1$
    and $\bigwedge i.\ [\![i \in I;\ x \in S;\ y \in Si\ i;\ x \notin Si\ i]\!] \Longrightarrow$ *sum-dist* $x\ y = 1$
    and $x \notin S \Longrightarrow$ *sum-dist* $x\ y = 0$
⟨*proof*⟩

lemma *sum-dist-if-less1*:
  assumes $i \in I$ $x \in Si\ i$ $y \in S$ *sum-dist* $x\ y < 1$
  shows $y \in Si\ i$
  ⟨*proof*⟩

lemma *inS-cases*:
  assumes $x \in S$ $y \in S$
    and $\bigwedge i.\ [\![i \in I;\ x \in Si\ i;\ y \in Si\ i]\!] \Longrightarrow P\ x\ y$
    and $\bigwedge i\ j.\ [\![i \in I;\ j \in I;\ i \neq j;\ x \in Si\ i;\ y \in Si\ j;\ x \neq y]\!] \Longrightarrow P\ x\ y$
  shows $P\ x\ y$ ⟨*proof*⟩

sublocale *metric-set S sum-dist*
⟨*proof*⟩

lemma *sum-dist-le1*: *sum-dist* $x\ y \leq 1$
  ⟨*proof*⟩

lemma *sum-dist-ball-eq-ball*:
  assumes $i \in I$ $e \leq 1$ $x \in Si\ i$
  shows *metric-set.open-ball* (*Si i*) (*di i*) *x e* = *open-ball x e*
⟨*proof*⟩

lemma *ball-le-sum-dist-ball*:
  assumes $i \in I$
  shows *metric-set.open-ball* (*Si i*) (*di i*) *x e* $\subseteq$ *open-ball x e*
⟨*proof*⟩

lemma *openin-sum-mtopology-iff*:
openin mtopology U $\longleftrightarrow U \subseteq S \wedge (\forall i \in I.$ *openin* (*metric-set.mtopology* (*Si i*) (*di i*)) $(U \cap Si\ i))$
⟨*proof*⟩

corollary *openin-sum-mtopology-Si*:
  assumes $i \in I$
  shows *openin mtopology* (*Si i*)
  ⟨*proof*⟩

lemma *converge-to-inSi-converge-to-inS*:
  assumes $i \in I$ *metric-set.converge-to-inS* (*Si i*) (*di i*) *xn x*
  shows *converge-to-inS xn x*
⟨*proof*⟩

corollary *convergent-inSi-convergent-inS*:
  assumes $i \in I$ *metric-set.convergent-inS* (*Si i*) (*di i*) *xn*
  shows *convergent-inS xn*
  ⟨*proof*⟩

lemma *converge-to-inS-converge-to-inSi-off-set*:
  assumes *converge-to-inS xn x*
  shows $\exists\, n.\ \exists j{\in}I.$ *metric-set.converge-to-inS* (*Si j*) (*di j*) ($\lambda i.$ *xn* $(i + n)$) *x*
⟨*proof*⟩

corollary *convergent-inS-convergent-inSi-off-set*:
  assumes *convergent-inS xn*
  shows $\exists\, n.\ \exists j{\in}I.$ *metric-set.convergent-inS* (*Si j*) (*di j*) ($\lambda i.$ *xn* $(i + n)$)
  ⟨*proof*⟩

lemma *Cauchy-inSi-Cauchy-inS*:
  assumes $i \in I$ *metric-set.Cauchy-inS* (*Si i*) (*di i*)*xn*
  shows *Cauchy-inS xn*
⟨*proof*⟩

lemma *Cauchy-inS-Cauchy-inSi*:
  assumes *Cauchy-inS xn*
  shows $\exists\, n.\ \exists j{\in}I.$ *metric-set.Cauchy-inS* (*Si j*) (*di j*) ($\lambda i.$ *xn* $(i + n)$)
⟨*proof*⟩

end

lemma *sum-metricI*:
  fixes *Si*
  assumes *disjoint-family-on Si I*
    and $\bigwedge i\ x\ y.\ i \notin I \Longrightarrow 0 \le di\ i\ x\ y$
    and $\bigwedge i\ x\ y.\ di\ i\ x\ y < 1$
    and $\bigwedge i.\ i \in I \Longrightarrow$ *metric-set* (*Si i*) (*di i*)
   shows *sum-metric I Si di*
  ⟨*proof*⟩

locale *sum-separable-metric* = *sum-metric* +
  assumes *I*: *countable I*
    and *sd-separable-metric*: $\bigwedge i.\ i \in I \Longrightarrow$ *separable-metric-set* (*Si i*) (*di i*)
begin

sublocale *separable-metric-set S sum-dist*

⟨proof⟩

end

locale *sum-complete-metric* = *sum-metric* +
  assumes *sd-complete-metric*: $\bigwedge i.\ i \in I \Longrightarrow$ *complete-metric-set* (*Si i*) (*di i*)
begin

sublocale *complete-metric-set S sum-dist*
⟨proof⟩

end

locale *sum-polish-metric* = *sum-complete-metric* + *sum-separable-metric*
begin

sublocale *polish-metric-set S sum-dist*
  ⟨proof⟩

end

lemma *sum-polish-metricI*:
  fixes *Si*
  assumes *countable I*
    and *disjoint-family-on Si I*
    and $\bigwedge i\ x\ y.\ i \notin I \Longrightarrow 0 \le di\ i\ x\ y$
    and $\bigwedge i\ x\ y.\ di\ i\ x\ y < 1$
    and $\bigwedge i.\ i \in I \Longrightarrow$ *polish-metric-set* (*Si i*) (*di i*)
  shows *sum-polish-metric I Si di*
  ⟨proof⟩

end

2.5 Product Metric Spaces

theory *Set-Based-Metric-Product*
  imports *Set-Based-Metric-Space*
begin

lemma *nsum-of-r'*:
  fixes *r* :: *real*
  assumes *r*:$0 < r$ $r < 1$
  shows $(\sum n.\ r\widehat{}(n + k) * K) = r\widehat{}k\ /\ (1 - r) * K$
 (**is** *?lhs* = -)
⟨proof⟩

lemma *nsum-of-r-leq*:
  fixes *r* :: *real* **and** *a* :: *nat* ⇒ *real*
  assumes *r*:$0 < r$ $r < 1$

and *a*:$\bigwedge n.\ 0 \leq a\ n$ $\bigwedge n.\ a\ n \leq K$
shows $0 \leq (\sum n.\ r\widehat{\ }(n + k) * a\ (n + l))$ $(\sum n.\ r\widehat{\ }(n + k) * a\ (n + l)) \leq r\widehat{\ }k / (1 - r) * K$
⟨*proof*⟩

lemma *nsum-of-r-le*:
fixes r :: *real* **and** a :: *nat* ⇒ *real*
assumes *r*:$0 < r$ $r < 1$
and *a*:$\bigwedge n.\ 0 \leq a\ n$ $\bigwedge n.\ a\ n \leq K$ $\exists n' \geq l.\ a\ n' < K$
shows $(\sum n.\ r\widehat{\ }(n + k) * a\ (n + l)) < r\widehat{\ }k / (1 - r) * K$
⟨*proof*⟩

definition *product-dist'* :: [*real*, *'i set*, *nat* ⇒ *'i*, *'i* ⇒ *'a set*, *'i* ⇒ *'a* ⇒ *'a* ⇒ *real*] ⇒ (*'i* ⇒ *'a*) ⇒ (*'i* ⇒ *'a*) ⇒ *real* **where**
product-dist-def: *product-dist'* $r\ I\ g\ S\ d \equiv (\lambda x\ y.$ *if* $x \in (\Pi_E\ i{\in}I.\ S\ i) \wedge y \in (\Pi_E\ i{\in}I.\ S\ i)$ *then* $(\sum n.$ *if* $g\ n \in I$ *then* $r\widehat{\ }n * d\ (g\ n)\ (x\ (g\ n))\ (y\ (g\ n))$ *else* $0)$ *else* $0)$

$d(x,y) = \sum_{n\in\mathbb{N}} r^n * d_{g_I(i)}(x_{g_I(i)}, y_{g_I(i)})$.

locale *product-metric* =
fixes r :: *real*
and I :: *'i set*
and f :: *'i* ⇒ *nat*
and g :: *nat* ⇒ *'i*
and S :: *'i* ⇒ *'a set*
and d :: *'i* ⇒ *'a* ⇒ *'a* ⇒ *real*
and K :: *real*
assumes *r*: $0 < r$ $r < 1$
and *I*: *countable I*
and *gf-comp-id* : $\bigwedge i.\ i \in I \Longrightarrow g\ (f\ i) = i$
and *gf-if-finite*: *finite* $I \Longrightarrow$ *bij-betw* $f\ I$ {..< *card* I}
finite $I \Longrightarrow$ *bij-betw* g {..< *card* I} I
and *gf-if-infinite*: *infinite* $I \Longrightarrow$ *bij-betw* $f\ I$ *UNIV*
infinite $I \Longrightarrow$ *bij-betw* g *UNIV* I
$\bigwedge n.$ *infinite* $I \Longrightarrow f\ (g\ n) = n$
and *sd-metric*: $\bigwedge i.\ i \in I \Longrightarrow$ *metric-set* $(S\ i)\ (d\ i)$
and *d-nonneg*: $\bigwedge i\ x\ y.\ 0 \leq d\ i\ x\ y$
and *d-bound*: $\bigwedge i\ x\ y.\ d\ i\ x\ y \leq K$
and *K-pos* : $0 < K$

lemma *from-nat-into-to-nat-on-product-metric-pair*:
assumes *countable I*
shows $\bigwedge i.\ i \in I \Longrightarrow$ *from-nat-into* I (*to-nat-on* $I\ i$) $= i$
and *finite* $I \Longrightarrow$ *bij-betw* (*to-nat-on* I) I {..< *card* I}
and *finite* $I \Longrightarrow$ *bij-betw* (*from-nat-into* I) {..< *card* I} I
and *infinite* $I \Longrightarrow$ *bij-betw* (*to-nat-on* I) I *UNIV*
and *infinite* $I \Longrightarrow$ *bij-betw* (*from-nat-into* I) *UNIV* I
and $\bigwedge n.$ *infinite* $I \Longrightarrow$ *to-nat-on* I (*from-nat-into* $I\ n$) $= n$
⟨*proof*⟩

lemma *product-metric-pair-finite-nat*:
bij-betw id $\{..n\}$ $\{..<$ *card* $\{..n\}\}$ *bij-betw id* $\{..<$ *card* $\{..n\}\}$ $\{..n\}$
⟨*proof*⟩

lemma *product-metric-pair-finite-nat'*:
bij-betw id $\{..<n\}$ $\{..<$ *card* $\{..<n\}\}$ *bij-betw id* $\{..<$ *card* $\{..<n\}\}$ $\{..<n\}$
⟨*proof*⟩

context *product-metric*
begin

abbreviation *product-dist* $\equiv$ *product-dist'* $r\ I\ g\ S\ d$

lemma *nsum-of-rK*: $(\sum n.\ r\widehat{\ }(n + k)*K) = r\widehat{\ }k\ /\ (1 - r) * K$
⟨*proof*⟩

lemma *i-min*:
assumes $i \in I$ $g\ n = i$
shows $f\ i \leq n$
⟨*proof*⟩

lemma *g-surj*:
assumes $i \in I$
shows $\exists\, n.\ g\ n = i$
⟨*proof*⟩

lemma *product-dist-summable'*[*simp*]:
summable $(\lambda n.\ r\widehat{\ }n * d\ (g\ n)\ (x\ (g\ n))\ (y\ (g\ n)))$
⟨*proof*⟩

lemma *product-dist-summable*[*simp*]:
summable $(\lambda n.$ *if* $g\ n \in I$ *then* $r\widehat{\ }n * d\ (g\ n)\ (x\ (g\ n))\ (y\ (g\ n))$ *else* $0)$
⟨*proof*⟩

lemma *summable-rK*[*simp*]: *summable* $(\lambda n.\ r\widehat{\ }n * K)$
⟨*proof*⟩

lemma *product-dist-distance*: *metric-set* $(\Pi_E\ i{\in}I.\ S\ i)$ *product-dist*
⟨*proof*⟩

sublocale *metric-set* $\Pi_E\ i{\in}I.\ S\ i$ *product-dist*
⟨*proof*⟩

lemma *product-dist-leqr*: *product-dist* $x\ y \leq 1\ /\ (1 - r) * K$
⟨*proof*⟩

lemma *product-dist-geq*:
assumes $i \in I$ **and** $g\ n = i$ $x \in (\Pi_E\ i{\in}I.\ S\ i)$ $y \in (\Pi_E\ i{\in}I.\ S\ i)$

shows $d\ i\ (x\ i)\ (y\ i) \leq (1/r)\hat{}n * \textit{product-dist}\ x\ y$
(**is** $?lhs \leq\ ?rhs$)
⟨*proof*⟩

lemma *converge-to-iff*:
assumes $xn \in \textit{sequence}\ x \in (\Pi_E\ i{\in}I.\ S\ i)$
shows $\textit{converge-to-inS}\ xn\ x \longleftrightarrow (\forall\, i{\in}I.\ \textit{metric-set.converge-to-inS}\ (S\ i)\ (d\ i)\ (\lambda n.\ xn\ n\ i)\ (x\ i))$
⟨*proof*⟩

lemma *product-dist-mtopology*: $\textit{product-topology}\ (\lambda i.\ \textit{metric-set.mtopology}\ (S\ i)\ (d\ i))\ I = \textit{mtopology}$
⟨*proof*⟩

end

lemma *product-metricI*:
assumes $0 < r\ r < 1\ \textit{countable}\ I\ \bigwedge i.\ i \in I \Longrightarrow \textit{metric-set}\ (S\ i)\ (d\ i)$
and $\bigwedge i\ x\ y.\ 0 \leq d\ i\ x\ y\ \bigwedge i\ x\ y.\ d\ i\ x\ y \leq K\ 0 < K$
shows $\textit{product-metric}\ r\ I\ (\textit{to-nat-on}\ I)\ (\textit{from-nat-into}\ I)\ S\ d\ K$
⟨*proof*⟩

Case: all (S_i, d_i) are separable metric spaces.

locale *product-separable-metric* = *product-metric* +
assumes *sd-separable-metric*: $\bigwedge i.\ i \in I \Longrightarrow \textit{separable-metric-set}\ (S\ i)\ (d\ i)$
begin

sublocale *separable-metric-set* $\Pi_E\ i{\in}I.\ S\ i\ \textit{product-dist}$
⟨*proof*⟩

end

Case: all (S_i, d_i) are complete metric spaces.

locale *product-complete-metric* = *product-metric* +
assumes *sd-complete-metric*: $\bigwedge i.\ i \in I \Longrightarrow \textit{complete-metric-set}\ (S\ i)\ (d\ i)$
begin

lemma *product-dist-complete′*:
assumes $I \neq \{\}$
shows $\textit{complete-metric-set}\ (\Pi_E\ i{\in}I.\ S\ i)\ \textit{product-dist}$
⟨*proof*⟩

sublocale *complete-metric-set* $\Pi_E\ i{\in}I.\ S\ i\ \textit{product-dist}$
⟨*proof*⟩

end

lemma *product-complete-metricI*:
assumes $0 < r\ r < 1\ \textit{countable}\ I\ \bigwedge i.\ i \in I \Longrightarrow \textit{complete-metric-set}\ (S\ i)\ (d\ i)$

and $\bigwedge i\ x\ y.\ 0 \le d\ i\ x\ y$ $\bigwedge i\ x\ y.\ d\ i\ x\ y \le K$ $0 < K$
shows *product-complete-metric r I* (*to-nat-on I*) (*from-nat-into I*) *S d K*
⟨*proof*⟩

lemma *product-complete-metric-natI*:
assumes $0 < r$ $r < 1$ $\bigwedge n.$ *complete-metric-set* (*S n*) (*d n*)
and $\bigwedge i\ x\ y.\ 0 \le d\ i\ x\ y$ $\bigwedge i\ x\ y.\ d\ i\ x\ y \le K$ $0 < K$
shows *product-complete-metric r UNIV id id S d K*
⟨*proof*⟩

locale *product-polish-metric* = *product-complete-metric* + *product-separable-metric*
begin

sublocale *polish-metric-set* Π_E $i{\in}I.$ *S i product-dist*
⟨*proof*⟩

end

lemma *product-polish-metricI*:
assumes $0 < r$ $r < 1$ *countable I* $\bigwedge i.\ i \in I \Longrightarrow$ *polish-metric-set* (*S i*) (*d i*)
and $\bigwedge i\ x\ y.\ 0 \le d\ i\ x\ y$ $\bigwedge i\ x\ y.\ d\ i\ x\ y \le K$ $0 < K$
shows *product-polish-metric r I* (*to-nat-on I*) (*from-nat-into I*) *S d K*
⟨*proof*⟩

lemma *product-polish-metric-natI*:
assumes $0 < r$ $r < 1$ $\bigwedge n.$ *polish-metric-set* (*S n*) (*d n*)
and $\bigwedge i\ x\ y.\ 0 \le d\ i\ x\ y$ $\bigwedge i\ x\ y.\ d\ i\ x\ y \le K$ $0 < K$
shows *product-polish-metric r UNIV id id S d K*
⟨*proof*⟩

Define a bounded distance function from a distance function

definition *bounded-dist* :: $('a \Rightarrow 'a \Rightarrow real) \Rightarrow 'a \Rightarrow 'a \Rightarrow real$ **where**
bounded-dist d $\equiv (\lambda a\ b.\ d\ a\ b\ /\ (1 + d\ a\ b))$

lemma *bounded-dist-mono*:
fixes *r l* :: *real*
assumes $0 \le r$ $0 \le l$ **and** $r \le l$
shows $r\ /\ (1 + r) \le l\ /\ (1 + l)$
⟨*proof*⟩

lemma *bounded-dist-mono-strict*:
fixes *r l* :: *real*
assumes $0 \le r$ $0 \le l$ **and** $r < l$
shows $r\ /\ (1 + r) < l\ /\ (1 + l)$
⟨*proof*⟩

lemma *bounded-dist-mono-inverse*:
fixes *r l* :: *real*
assumes $0 \le r$ $0 \le l$ **and** $r\ /\ (1 + r) \le l\ /\ (1 + l)$

shows $r \leq l$
⟨*proof*⟩

lemma *bounded-dist-mono-strict-inverse*:
fixes *r l* :: *real*
assumes $0 \leq r$ $0 \leq l$ **and** $r / (1 + r) < l / (1 + l)$
shows $r < l$
⟨*proof*⟩

lemma *bounded-dist-inverse-comp*:
fixes ε :: *real*
assumes $0 < \varepsilon$ **and** $\varepsilon < 1$
shows $\varepsilon = (\varepsilon / (1 - \varepsilon)) / (1 + (\varepsilon / (1 - \varepsilon)))$
(**is** - = $?\varepsilon' / (1 + ?\varepsilon')$)
⟨*proof*⟩

lemma(**in** *metric-set*) *bounded-dist-dist*:
shows *metric-set S* (*bounded-dist dist*)
and *bounded-dist dist a b* < 1
⟨*proof*⟩

lemma(**in** *metric-set*) *bounded-dist-ball-eq*:
assumes $x \in S$ **and** $\varepsilon > 0$
shows *open-ball* x ε = *metric-set.open-ball S* (*bounded-dist dist*) x $(\varepsilon / (1 + \varepsilon))$
⟨*proof*⟩

lemma(**in** *metric-set*) *bounded-dist-ball-ge1*:
assumes $x \in S$ **and** $1 \leq \varepsilon$
shows *metric-set.open-ball S* (*bounded-dist dist*) x $\varepsilon = S$
⟨*proof*⟩

lemma(**in** *metric-set*) *bounded-dist-generate-same-topology*:
mtopology = *metric-set.mtopology S* (*bounded-dist dist*)
⟨*proof*⟩

lemma(**in** *metric-set*) *bounded-dist-converge-to-inS-iff*:
converge-to-inS xn x ⟷ *metric-set.converge-to-inS S* (*bounded-dist dist*) *xn x*
⟨*proof*⟩

lemma(**in** *metric-set*) *bounded-dist-Cauchy-eq*:
Cauchy-inS f ⟷ *metric-set.Cauchy-inS S* (*bounded-dist dist*) *f*
⟨*proof*⟩

lemma(**in** *complete-metric-set*) *bounded-dist-complete*:
complete-metric-set S (*bounded-dist dist*)
⟨*proof*⟩

lemma(**in** *polish-metric-set*) *bounded-dist-polish*:
polish-metric-set S (*bounded-dist dist*)

$\langle proof \rangle$

lemma(**in** *metric-set*) *uniform-continuous-map-bounded-dist-equiv*:
assumes *metric-set T f*
shows *uniform-continuous-map S dist T f* = *uniform-continuous-map S* (*bounded-dist dist*) *T f*
$\langle proof \rangle$

lemma(**in** *metric-set*) *uniform-continuous-map-bounded-dist-equiv'*:
assumes *metric-set T f*
shows *uniform-continuous-map S dist T f* = *uniform-continuous-map S* (*bounded-dist dist*) *T* (*bounded-dist f*)
$\langle proof \rangle$

lemma(**in** *metric-set*) *Urysohn-uniform*:
assumes *closedin mtopology T closedin mtopology U* $T \cap U = \{\}$ $\bigwedge x\ y.\ x \in T \Longrightarrow y \in U \Longrightarrow dist\ x\ y \geq e$ $e > 0$
obtains $f :: 'a \Rightarrow real$
where *uniform-continuous-map S dist UNIV dist-typeclass f*
$\bigwedge x.\ f\ x \geq 0$ $\bigwedge x.\ f\ x \leq 1$ $\bigwedge x.\ x \in T \Longrightarrow f\ x = 1$ $\bigwedge x.\ x \in U \Longrightarrow f\ x = 0$
$\langle proof \rangle$

lemma *product-metricI'*:
assumes $0 < r$ $r < 1$ *countable I* $\bigwedge i.\ i \in I \Longrightarrow$ *metric-set* (*S i*) (*d i*)
shows *product-metric r I* (*to-nat-on I*) (*from-nat-into I*) *S* ($\lambda i\ x\ y.$ *if* $i \in I$ *then bounded-dist* (*d i*) *x y else 0*) *1*
$\langle proof \rangle$

lemma *product-complete-metricI'*:
assumes $0 < r$ $r < 1$ *countable I* $\bigwedge i.\ i \in I \Longrightarrow$ *complete-metric-set* (*S i*) (*d i*)
shows *product-complete-metric r I* (*to-nat-on I*) (*from-nat-into I*) *S* ($\lambda i\ x\ y.$ *if* $i \in I$ *then bounded-dist* (*d i*) *x y else 0*) *1*
$\langle proof \rangle$

lemma *product-complete-metric-natI'*:
assumes $0 < r$ $r < 1$ $\bigwedge n.$ *complete-metric-set* (*S n*) (*d n*)
shows *product-complete-metric r UNIV id id S* ($\lambda n.$ *bounded-dist* (*d n*)) *1*
$\langle proof \rangle$

lemma *product-polish-metricI'*:
assumes $0 < r$ $r < 1$ *countable I* $\bigwedge i.\ i \in I \Longrightarrow$ *polish-metric-set* (*S i*) (*d i*)
shows *product-polish-metric r I* (*to-nat-on I*) (*from-nat-into I*) *S* ($\lambda i\ x\ y.$ *if* $i \in I$ *then bounded-dist* (*d i*) *x y else 0*) *1*
$\langle proof \rangle$

end

3 Abstract Metrizable Topology

theory *Abstract-Metrizable-Topology*
  imports *Set-Based-Metric-Product*
begin

3.1 Metrizable Spaces

locale *metrizable* =
  fixes S :: *'a topology*
  assumes *ex-metric*:$\exists\, \varrho.$ *metric-set* (*topspace* S) $\varrho \wedge S =$ *metric-set.mtopology* (*topspace* S) ϱ
begin

lemma *metric*:
  obtains ϱ **where** *metric-set* (*topspace* S) ϱ *metric-set.mtopology* (*topspace* S) ϱ $= S$
  ⟨*proof*⟩

lemma *bounded-metric*:
  obtains ϱ **where** *metric-set* (*topspace* S) ϱ *metric-set.mtopology* (*topspace* S) ϱ $= S$
    $\bigwedge x\ y.\ \varrho\ x\ y < 1$
⟨*proof*⟩

lemma *second-countable-if-separable*:
  assumes *separable* S
  shows *second-countable* S
⟨*proof*⟩

corollary *second-countable-iff-separable*: *second-countable* $S \longleftrightarrow$ *separable* S
  ⟨*proof*⟩

lemma *Hausdorff*: *Hausdorff-space* S
  ⟨*proof*⟩

lemma *subtopology*: *metrizable* (*subtopology* S X)
⟨*proof*⟩

lemma *g-delta-of-closedin*:
  assumes *closedin* S X
  shows *g-delta-of* S X
  ⟨*proof*⟩

lemma *closedin-singleton*:
  assumes $s \in$ *topspace* S
  shows *closedin* S $\{s\}$
⟨*proof*⟩

lemma *dense-of-infinite*:

assumes *infinite* (*topspace* S) *dense-of* S U
shows *infinite* U
⟨*proof*⟩

lemma *homeomorphic-metrizable*:
assumes S *homeomorphic-space* S'
shows *metrizable* S'
⟨*proof*⟩

end

lemma *euclidean-metrizable*: *metrizable* (*euclidean* :: ('*a* ::*metric-space*) *topology*)
⟨*proof*⟩

sublocale *metric-set* ⊆ *metrizable mtopology*
⟨*proof*⟩

lemma *metrizable-prod*:
assumes *metrizable* X *metrizable* Y
shows *metrizable* (*prod-topology* X Y)
⟨*proof*⟩

lemma *metrizable-product*:
assumes *countable* I $\bigwedge i.\ i \in I \Longrightarrow$ *metrizable* (X i)
shows *metrizable* (*product-topology* X I)
⟨*proof*⟩

3.2 Complete Metrizable Spaces

locale *complete-metrizable* =
fixes S :: '*a topology*
assumes *ex-cmetric*: $\exists\, \varrho$. *complete-metric-set* (*topspace* S) $\varrho \wedge S =$ *metric-set.mtopology* (*topspace* S) ϱ
begin

lemma *cmetric*:
obtains ϱ **where** *complete-metric-set* (*topspace* S) ϱ *metric-set.mtopology* (*topspace* S) $\varrho = S$
⟨*proof*⟩

lemma *bounded-cmetric*:
obtains ϱ **where** *complete-metric-set* (*topspace* S) ϱ *metric-set.mtopology* (*topspace* S) $\varrho = S$
$\bigwedge x\ y.\ \varrho\ x\ y < 1$
⟨*proof*⟩

lemma *metrizable*: *metrizable* S
⟨*proof*⟩

sublocale *metrizable*
⟨*proof*⟩

lemma *closedin-complete-metrizable*:
assumes *closedin S A*
shows *complete-metrizable* (*subtopology S A*)
⟨*proof*⟩

lemma *homeomorphic-complete-metrizable*:
assumes *S homeomorphic-space S'*
shows *complete-metrizable S'*
⟨*proof*⟩

end

lemma *euclidean-complete-metrizable*[*simp*]:
complete-metrizable (*euclidean* :: (*'a* ::*complete-space*) *topology*)
⟨*proof*⟩

sublocale *complete-metric-set* ⊆ *complete-metrizable mtopology*
⟨*proof*⟩

lemma *complete-metrizable-prod*:
assumes *complete-metrizable X complete-metrizable Y*
shows *complete-metrizable* (*prod-topology X Y*)
⟨*proof*⟩

lemma *complete-metrizable-product*:
assumes *countable I* $\bigwedge i.\ i \in I \Longrightarrow$ *complete-metrizable* (*X i*)
shows *complete-metrizable* (*product-topology X I*)
⟨*proof*⟩

lemma(**in** *complete-metrizable*) *g-delta-of-complete-metrizable*:
assumes *g-delta-of S B*
shows *complete-metrizable* (*subtopology S B*)
⟨*proof*⟩

corollary(**in** *complete-metrizable*) *openin-complete-metrizable*:
assumes *openin S u*
shows *complete-metrizable* (*subtopology S u*)
⟨*proof*⟩

3.3 Polish Spaces

locale *polish-topology* = *complete-metrizable* +
assumes *S-separable*:*separable S*
begin

lemma *S-second-countable*: *second-countable S*

⟨*proof*⟩

lemma *closedin-polish*:
assumes *closedin S A*
shows *polish-topology* (*subtopology S A*)
⟨*proof*⟩

lemma *g-delta-of-polish*:
assumes *g-delta-of S A*
shows *polish-topology* (*subtopology S A*)
⟨*proof*⟩

corollary *openin-polish*:
assumes *openin S A*
shows *polish-topology* (*subtopology S A*)
⟨*proof*⟩

lemma *homeomorphic-polish-topology*:
assumes *S homeomorphic-space S′*
shows *polish-topology S′*
⟨*proof*⟩

end

lemma *polish-topology-def2*:
polish-topology S ⟷ (∃ ϱ. *polish-metric-set* (*topspace S*) ϱ ∧ *S* = *metric-set.mtopology* (*topspace S*) ϱ)
⟨*proof*⟩

lemma(**in** *polish-topology*) *polish-metric*:
obtains *d* **where** *polish-metric-set* (*topspace S*) *d*
and *S* = *metric-set.mtopology* (*topspace S*) *d*
⟨*proof*⟩

lemma(**in** *polish-topology*) *bounded-polish-metric*:
obtains *d* **where** *polish-metric-set* (*topspace S*) *d*
and *S* = *metric-set.mtopology* (*topspace S*) *d*
and $\bigwedge x\ y.\ d\ x\ y < 1$
⟨*proof*⟩

sublocale *polish-metric-set* ⊆ *polish-topology mtopology*
⟨*proof*⟩

lemma *polish-topology-euclidean*[*simp*]: *polish-topology* (*euclidean* :: (′*a* :: *polish-space*) *topology*)
⟨*proof*⟩

lemma *polish-topology-countable*[*simp*]:
polish-topology (*euclidean* :: ′*a* :: {*countable*,*discrete-topology*} *topology*)

⟨*proof*⟩

lemma *polish-topology-prod*:
assumes *polish-topology S* **and** *polish-topology S′*
shows *polish-topology* (*prod-topology S S′*)
⟨*proof*⟩

lemma *polish-topology-product*:
assumes *countable I* **and** $\bigwedge i.\ i \in I \Longrightarrow$ *polish-topology* (*S i*)
shows *polish-topology* (*product-topology S I*)
⟨*proof*⟩

lemma *polish-topology-closedin-polish*:
assumes *polish-topology S* **and** *closedin S U*
shows *polish-topology* (*subtopology S U*)
⟨*proof*⟩

3.4 Compact Metrizable Spaces

locale *compact-metrizable* = *metrizable* +
assumes *compact*: *compact-space S*
begin

sublocale *polish-topology*
⟨*proof*⟩

lemma *compact-metric*:
obtains *d* **where** *compact-metric-set* (*topspace S*) *d metric-set.mtopology* (*topspace S*) *d* = *S*
⟨*proof*⟩

end

3.5 Continuous Embddings

abbreviation *Hilbert-cube-as-topology* :: (*nat* ⇒ *real*) *topology* **where**
Hilbert-cube-as-topology ≡ (*product-topology* (λ*n*. *top-of-set* {*0..1*}) *UNIV*)

lemma *topspace-Hilbert-cube*: *topspace Hilbert-cube-as-topology* = (Π_E *x*∈*UNIV*. {*0..1*})
⟨*proof*⟩

lemma *Hilbert-cube-Polish-topology*: *polish-topology Hilbert-cube-as-topology*
⟨*proof*⟩

abbreviation *Cantor-space-as-topology* :: (*nat* ⇒ *real*) *topology* **where**
Cantor-space-as-topology ≡ (*product-topology* (λ*n*. *top-of-set* {*0*,*1*}) *UNIV*)

lemma *topspace-Cantor-space*:
topspace Cantor-space-as-topology = (Π_E *x*∈*UNIV*. {*0*,*1*})

⟨*proof*⟩

lemma *Cantor-space-Polish-topology*:
polish-topology Cantor-space-as-topology
⟨*proof*⟩

Proposition 2.2.3 in [3]

lemma *continuous-map-metrizable-extension*:
assumes $A \subseteq$ *topspace* W *metrizable* W *complete-metrizable* Z *continuous-map* (*subtopology* W A) Z f
shows $\exists\, h$ *gd*. *g-delta-of* W *gd* $\wedge$ $(\forall\, a{\in}A.\ f\ a = h\ a)$ $\wedge$ $A \subseteq$ *gd* $\wedge$ *continuous-map* (*subtopology* W *gd*) Z h
⟨*proof*⟩

lemma *Lavrentiev-theorem*:
assumes *complete-metrizable* X *complete-metrizable* Y $A \subseteq$ *topspace* X $B \subseteq$ *topspace* Y *homeomorphic-map* (*subtopology* X A) (*subtopology* Y B) f
shows $\exists\, h$ *gda gdb*. *g-delta-of* X *gda* $\wedge$ *g-delta-of* Y *gdb* $\wedge$ $A \subseteq$ *gda* $\wedge$ $B \subseteq$ *gdb* $\wedge$ $(\forall\, x{\in}A.\ f\ x = h\ x)$ $\wedge$ *homeomorphic-map* (*subtopology* X *gda*) (*subtopology* Y *gdb*) h
⟨*proof*⟩

corollary(**in** *complete-metrizable*) *complete-metrizable-subtopology-is-g-delta*:
assumes $A \subseteq$ *topspace* S *complete-metrizable* (*subtopology* S A)
shows *g-delta-of* S A
⟨*proof*⟩

corollary(**in** *complete-metrizable*) *subtopology-complete-metrizable-iff*:
assumes $A \subseteq$ *topspace* S
shows *complete-metrizable* (*subtopology* S A) $\longleftrightarrow$ *g-delta-of* S A
⟨*proof*⟩

corollary *complete-metrizable-homeo-image-g-delta*:
assumes *complete-metrizable* X *complete-metrizable* Y $B \subseteq$ *topspace* Y X *homeomorphic-space subtopology* Y B
shows *g-delta-of* Y B
⟨*proof*⟩

lemma(**in** *metrizable*) *embedding-into-Hilbert-cube*:
assumes *separable* S
shows $\exists\, A \subseteq$ *topspace Hilbert-cube-as-topology*. S *homeomorphic-space* (*subtopology Hilbert-cube-as-topology* A)
⟨*proof*⟩

corollary(**in** *complete-metrizable*) *embedding-into-Hilbert-cube-g-delta-of*:
assumes *separable* S
shows $\exists\, A$. *g-delta-of Hilbert-cube-as-topology* A $\wedge$ S *homeomorphic-space* (*subtopology Hilbert-cube-as-topology* A)
⟨*proof*⟩

corollary(**in** *polish-topology*) *embedding-into-Hilbert-cube-g-delta-of*:
$\exists A$. *g-delta-of Hilbert-cube-as-topology A* $\wedge$ *S homeomorphic-space* (*subtopology Hilbert-cube-as-topology A*)
⟨*proof*⟩

lemma(**in** *polish-topology*) *uncountable-contains-Cantor-space'*:
assumes *uncountable* (*topspace S*)
shows $\exists A \subseteq$ *topspace S*. *Cantor-space-as-topology homeomorphic-space* (*subtopology S A*)
⟨*proof*⟩

lemma(**in** *polish-topology*) *uncountable-contains-Cantor-space*:
assumes *uncountable* (*topspace S*)
shows $\exists A$. *g-delta-of S A* $\wedge$ *Cantor-space-as-topology homeomorphic-space* (*subtopology S A*)
⟨*proof*⟩

3.6 Borel Spaces

Borel spaces generated from abstract topology

definition *borel-of* :: *'a topology* $\Rightarrow$ *'a measure* **where**
borel-of S $\equiv$ *sigma* (*topspace S*) $\{U$. *openin S U*$\}$

lemma *emeasure-borel-of*: *emeasure* (*borel-of S*) *A* = *0*
⟨*proof*⟩

lemma *borel-of-euclidean*: *borel-of euclidean* = *borel*
⟨*proof*⟩

lemma *space-borel-of*: *space* (*borel-of S*) = *topspace S*
⟨*proof*⟩

lemma *sets-borel-of*: *sets* (*borel-of S*) = *sigma-sets* (*topspace S*) $\{U$. *openin S U*$\}$
⟨*proof*⟩

lemma *sets-borel-of-closed*: *sets* (*borel-of S*) = *sigma-sets* (*topspace S*) $\{U$. *closedin S U*$\}$
⟨*proof*⟩

lemma *borel-of-open*:
assumes *openin S U*
shows $U \in$ *sets* (*borel-of S*)
⟨*proof*⟩

lemma *borel-of-closed*:
assumes *closedin S U*
shows $U \in$ *sets* (*borel-of S*)
⟨*proof*⟩

lemma(**in** *metric-set*) *nbh-sets*[*measurable*]: $(\bigcup a{\in}A.\ \textit{open-ball a e}) \in \textit{sets}\ (\textit{borel-of mtopology})$
⟨*proof*⟩

lemma *borel-of-g-delta-of*:
assumes *g-delta-of S U*
shows $U \in \textit{sets}\ (\textit{borel-of}\ S)$
⟨*proof*⟩

lemma *borel-of-subtopology*:
$\textit{borel-of}\ (\textit{subtopology}\ S\ U) = \textit{restrict-space}\ (\textit{borel-of}\ S)\ U$
⟨*proof*⟩

lemma(**in** *metrizable*) *sigma-sets-eq-cinter-dunion*:
$\textit{sigma-sets}\ (\textit{topspace}\ S)\ \{U.\ \textit{openin}\ S\ U\} = \textit{sigma-sets-cinter-dunion}\ (\textit{topspace}\ S)\ \{U.\ \textit{openin}\ S\ U\}$
⟨*proof*⟩

lemma(**in** *metrizable*) *sigma-sets-eq-cinter*:
$\textit{sigma-sets}\ (\textit{topspace}\ S)\ \{U.\ \textit{openin}\ S\ U\} = \textit{sigma-sets-cinter}\ (\textit{topspace}\ S)\ \{U.\ \textit{openin}\ S\ U\}$
⟨*proof*⟩

lemma *continuous-map-measurable*:
assumes *continuous-map X Y f*
shows $f \in \textit{borel-of}\ X \rightarrow_M \textit{borel-of}\ Y$
⟨*proof*⟩

lemma *open-map-preserves-sets*:
assumes *open-map S T f inj-on f* $(\textit{topspace}\ S)\ A \in \textit{sets}\ (\textit{borel-of}\ S)$
shows f ‘ $A \in \textit{sets}\ (\textit{borel-of}\ T)$
⟨*proof*⟩

lemma *open-map-preserves-sets′*:
assumes *open-map S* (*subtopology T* (*f* ‘ (*topspace S*))) *f inj-on f* (*topspace S*) *f* ‘ (*topspace S*) $\in$ *sets* (*borel-of T*) $A \in$ *sets* (*borel-of S*)
shows f ‘ $A \in \textit{sets}\ (\textit{borel-of}\ T)$
⟨*proof*⟩

Abstract topology version of *open* = *generate-topology ?X* $\Longrightarrow$ *borel* = *sigma UNIV ?X*.

lemma *borel-of-second-countable′*:
assumes *second-countable S* **and** *subbase-of S* $\mathcal{U}$
shows *borel-of S* = *sigma* (*topspace S*) $\mathcal{U}$
⟨*proof*⟩

Abstract topology version *borel* $\bigotimes_M$ *borel* = *borel*.

lemma *borel-of-prod*:
  assumes *second-countable S* **and** *second-countable S′*
  shows *borel-of S* $\bigotimes_M$ *borel-of S′* = *borel-of* (*prod-topology S S′*)
⟨*proof*⟩

lemma *product-borel-of-measurable*:
  assumes $i \in I$
  shows ($\lambda x.\ x\ i$) ∈ (*borel-of* (*product-topology S I*)) $\rightarrow_M$ *borel-of* (*S i*)
  ⟨*proof*⟩

Abstract topology version of *sets* (Pi_M *UNIV* (λ-. *borel*)) ⊆ *sets borel*

lemma *sets-PiM-subset-borel-of*:
  sets (Π_M $i{\in}I$. *borel-of* (*S i*)) ⊆ *sets* (*borel-of* (*product-topology S I*))
⟨*proof*⟩

Abstract topology version of *sets* (Pi_M *UNIV* (λ*i*. *borel*)) = *sets borel*.

lemma *sets-PiM-equal-borel-of*:
  assumes *countable I* **and** $\bigwedge i.\ i \in I \Longrightarrow$ *second-countable* (*S i*)
  shows *sets* (Π_M $i{\in}I$. *borel-of* (*S i*)) = *sets* (*borel-of* (*product-topology S I*))
⟨*proof*⟩

lemma *homeomorphic-map-borel-isomorphic*:
  assumes *homeomorphic-map X Y f*
  shows *measurable-isomorphic-map* (*borel-of X*) (*borel-of Y*) *f*
⟨*proof*⟩

lemma *homeomorphic-space-measurable-isomorphic*:
  assumes *S homeomorphic-space T*
  shows *borel-of S measurable-isomorphic borel-of T*
  ⟨*proof*⟩

lemma *measurable-isomorphic-borel-map*:
  assumes *sets M* = *sets* (*borel-of S*) **and** *f*: *measurable-isomorphic-map M N f*
  shows ∃ *S′*. *homeomorphic-map S S′ f* ∧ *sets N* = *sets* (*borel-of S′*)
⟨*proof*⟩

lemma *measurable-isomorphic-borels*:
  assumes *sets M* = *sets* (*borel-of S*) *M measurable-isomorphic N*
  shows ∃ *S′*. *S homeomorphic-space S′* ∧ *sets N* = *sets* (*borel-of S′*)
  ⟨*proof*⟩

lemma(**in** *polish-topology*) *closedin-clopen-topology*:
  assumes *closedin S a*
  shows ∃ *S′*. *polish-topology S′* ∧ (∀ *u*. *openin S u* ⟶ *openin S′ u*) ∧ *topspace S* = *topspace S′* ∧ *sets* (*borel-of S*) = *sets* (*borel-of S′*) ∧ *openin S′ a* ∧ *closedin S′ a*

$\langle proof \rangle$

lemma *polish-topology-union-polish*:
 fixes X :: *nat* $\Rightarrow$ *'a topology*
 assumes $\bigwedge n.$ *polish-topology* $(X\ n)$ $\bigwedge n.$ *topspace* $(X\ n) = Xt$ $\bigwedge x\ y.\ x \in Xt \Longrightarrow y \in Xt \Longrightarrow x \neq y \Longrightarrow \exists\, Ox\ Oy.\ (\forall\, n.\ openin\ (X\ n)\ Ox) \wedge (\forall\, n.\ openin\ (X\ n)\ Oy) \wedge x \in Ox \wedge y \in Oy \wedge disjnt\ Ox\ Oy$
 defines $Xun \equiv$ *topology-generated-by* $(\bigcup n.\ \{u.\ openin\ (X\ n)\ u\})$
 shows *polish-topology Xun*
$\langle proof \rangle$

lemma(**in** *polish-topology*) *sets-clopen-topology*:
 assumes $a \in sets\ (borel\text{-}of\ S)$
 shows $\exists\, S'.\ polish\text{-}topology\ S' \wedge (\forall\, u.\ openin\ S\ u \longrightarrow openin\ S'\ u) \wedge topspace\ S = topspace\ S' \wedge sets\ (borel\text{-}of\ S) = sets\ (borel\text{-}of\ S') \wedge openin\ S'\ a \wedge closedin\ S'\ a$
$\langle proof \rangle$

end

4 Standard Borel Spaces

4.1 Standard Borel Spaces

theory *StandardBorel*
 imports *Abstract-Metrizable-Topology*
begin

locale *standard-borel* =
 fixes M :: *'a measure*
 assumes *polish-topology*: $\exists\, S.\ polish\text{-}topology\ S \wedge sets\ M = sets\ (borel\text{-}of\ S)$
begin

lemma *singleton-sets*:
 assumes $x \in space\ M$
 shows $\{x\} \in sets\ M$
$\langle proof \rangle$

corollary *countable-sets*:
 assumes $A \subseteq space\ M$ *countable* A
 shows $A \in sets\ M$
 $\langle proof \rangle$

lemma *standard-borel-restrict-space*:
 assumes $A \in sets\ M$
 shows *standard-borel* (*restrict-space* M A)
$\langle proof \rangle$

end

locale *standard-borel-ne* = *standard-borel* +
  assumes *space-ne*: *space* $M \neq \{\}$
begin

lemma *standard-borel-ne-restrict-space*:
  assumes $A \in$ *sets* M $A \neq \{\}$
  shows *standard-borel-ne* (*restrict-space* M A)
  ⟨*proof*⟩

lemma *standard-borel*: *standard-borel* M
  ⟨*proof*⟩

end

lemma *standard-borel-sets*:
  assumes *standard-borel* M **and** *sets* M = *sets* N
  shows *standard-borel* N
  ⟨*proof*⟩

lemma *standard-borel-ne-sets*:
  assumes *standard-borel-ne* M **and** *sets* M = *sets* N
  shows *standard-borel-ne* N
  ⟨*proof*⟩

lemma *pair-standard-borel*:
  assumes *standard-borel* M *standard-borel* N
  shows *standard-borel* ($M \bigotimes_M N$)
⟨*proof*⟩

lemma *pair-standard-borel-ne*:
  assumes *standard-borel-ne* M *standard-borel-ne* N
  shows *standard-borel-ne* ($M \bigotimes_M N$)
  ⟨*proof*⟩

lemma *product-standard-borel*:
  assumes *countable* I
      and $\bigwedge i.\ i \in I \Longrightarrow$ *standard-borel* (M i)
    shows *standard-borel* (Π_M $i{\in}I.$ M i)
⟨*proof*⟩

lemma *product-standard-borel-ne*:
  assumes *countable* I
      and $\bigwedge i.\ i \in I \Longrightarrow$ *standard-borel-ne* (M i)
    shows *standard-borel-ne* (Π_M $i{\in}I.$ M i)
  ⟨*proof*⟩

lemma *closed-set-standard-borel*[*simp*]:
  fixes U :: *'a* :: *topological-space set*
  assumes *polish-topology* (*euclidean* :: *'a topology*) *closed* U

shows *standard-borel* (*restrict-space borel U*)
⟨*proof*⟩

lemma *closed-set-standard-borel-ne*[*simp*]:
fixes *U* :: *'a* :: *topological-space set*
assumes *polish-topology* (*euclidean* :: *'a topology*) *closed U* $U \neq \{\}$
shows *standard-borel-ne* (*restrict-space borel U*)
⟨*proof*⟩

lemma *open-set-standard-borel*[*simp*]:
fixes *U* :: *'a* :: *topological-space set*
assumes *polish-topology* (*euclidean* :: *'a topology*) *open U*
shows *standard-borel* (*restrict-space borel U*)
⟨*proof*⟩

lemma *open-set-standard-borel-ne*[*simp*]:
fixes *U* :: *'a* :: *topological-space set*
assumes *polish-topology* (*euclidean* :: *'a topology*) *open U* $U \neq \{\}$
shows *standard-borel-ne* (*restrict-space borel U*)
⟨*proof*⟩

lemma *standard-borel-ne-borel*[*simp*]: *standard-borel-ne* (*borel* :: (*'a* :: *polish-space*) *measure*)
and *standard-borel-ne-lborel*[*simp*]: *standard-borel-ne lborel*
⟨*proof*⟩

lemma *count-space-standard'*[*simp*]:
assumes *countable I*
shows *standard-borel* (*count-space I*)
⟨*proof*⟩

lemma *count-space-standard-ne*[*simp*]: *standard-borel-ne* (*count-space* (*UNIV* :: (- :: *countable*) *set*))
⟨*proof*⟩

corollary *measure-pmf-standard-borel-ne*[*simp*]: *standard-borel-ne* (*measure-pmf* (*p* :: (- :: *countable*) *pmf*))
⟨*proof*⟩

corollary *measure-spmf-standard-borel-ne*[*simp*]: *standard-borel-ne* (*measure-spmf* (*p* :: (- :: *countable*) *spmf*))
⟨*proof*⟩

corollary *countable-standard-ne*[*simp*]:
standard-borel-ne (*borel* :: *'a* :: {*countable*,*t2-space*} *measure*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *countable-discrete-space*:

assumes *countable* (*space M*)
shows *sets M* = *Pow* (*space M*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *measurable-isomorphic-standard*:
assumes *M measurable-isomorphic N*
shows *standard-borel N*
⟨*proof*⟩

lemma(**in** *standard-borel-ne*) *measurable-isomorphic-standard-ne*:
assumes *M measurable-isomorphic N*
shows *standard-borel-ne N*
⟨*proof*⟩

lemma *ereal-standard-ne*: *standard-borel-ne* (*borel* :: *ereal measure*)
⟨*proof*⟩

corollary *ennreal-stanndard-ne*: *standard-borel-ne* (*borel* :: *ennreal measure*)
⟨*proof*⟩

Cantor space $\mathscr{C}$

definition *Cantor-space* :: (*nat* ⇒ *real*) *measure* **where**
Cantor-space ≡ (Π_M *i*∈ *UNIV*. *restrict-space borel* {*0*,*1*})

lemma *Cantor-space-standard-ne*: *standard-borel-ne Cantor-space*
⟨*proof*⟩

lemma *Cantor-space-borel*:
sets (*borel-of Cantor-space-as-topology*) = *sets Cantor-space*
(**is** *?lhs* = -)
⟨*proof*⟩

Baire space

definition *Baire-space* :: (*nat* ⇒ *nat*) *measure* **where**
Baire-space ≡ (Π_M *i*∈ *UNIV*. *borel*)

lemma *Baire-space-standard*: *standard-borel-ne Baire-space*
⟨*proof*⟩

Hilbert cube $\mathscr{H}$

definition *Hilbert-cube* :: (*nat* ⇒ *real*) *measure* **where**
Hilbert-cube ≡ (Π_M *i*∈ *UNIV*. *restrict-space borel* {*0*..*1*})

lemma *Hilbert-cube-standard-ne*: *standard-borel-ne Hilbert-cube*
⟨*proof*⟩

lemma *Hilbert-cube-borel*:
sets (*borel-of Hilbert-cube-as-topology*) = *sets Hilbert-cube* (**is** *?lhs* = -)
⟨*proof*⟩

4.2 Isomorphism between $\mathscr{C}$ and $\mathscr{H}$

lemma *space-Cantor-space*: *space Cantor-space* = (Π_E *i*∈ *UNIV*. {*0*,*1*})
⟨*proof*⟩

lemma *space-Cantor-space-01*[*simp*]:
assumes *x* ∈ *space Cantor-space*
shows *0* ≤ *x n x n* ≤ *1 x n* ∈ {*0*,*1*}
⟨*proof*⟩

lemma *Cantor-minus-abs-cantor*:
assumes *x* ∈ *space Cantor-space y* ∈ *space Cantor-space*
shows (λ*n*. |*x n* − *y n*|) ∈ *space Cantor-space*
⟨*proof*⟩

Isomorphism between $\mathscr{C}$ and $[0, 1]$

definition *Cantor-to-01* :: (*nat* ⇒ *real*) ⇒ *real* **where**
Cantor-to-01 ≡ (λ*x*. (∑*n*. (*1*/*3*)^(*Suc n*)∗ *x n*))

Cantor-to-01 is a measurable injective embedding.

lemma *Cantor-to-01-summable'*[*simp*]:
assumes *x* ∈ *space Cantor-space*
shows *summable* (λ*n*. (*1*/*3*)^(*Suc n*)∗ *x n*)
⟨*proof*⟩

lemma *Cantor-to-01-summable*[*simp*]:
assumes *x* ∈ *space Cantor-space*
shows *summable* (λ*n*. (*1*/*3*)^ *n*∗ *x n*)
⟨*proof*⟩

lemma *Cantor-to-01-subst-summable*[*simp*]:
assumes *x* ∈ *space Cantor-space y* ∈ *space Cantor-space*
shows *summable* (λ*n*. (*1*/*3*)^ *n*∗ (*x n* − *y n*))
⟨*proof*⟩

lemma *Cantor-to-01-image*: *Cantor-to-01* ∈ *space Cantor-space* → {*0*..*1*}
⟨*proof*⟩

lemma *Cantor-to-01-measurable*: *Cantor-to-01* ∈ *Cantor-space* $\to_M$ *restrict-space borel* {*0*..*1*}
⟨*proof*⟩

lemma
shows *Cantor-to-01-inj*: *inj-on Cantor-to-01* (*space Cantor-space*)
and *Cantor-to-01-preserves-sets*: *A* ∈ *sets Cantor-space* ⟹ *Cantor-to-01* ‘ *A* ∈ *sets* (*restrict-space borel* {*0*..*1*})
⟨*proof*⟩

Next, we construct measurable embedding from $[0, 1]$ to $0, 1^{\mathbb{N}}$.

definition *to-Cantor-from-01* :: *real* ⇒ *nat* ⇒ *real* **where**
to-Cantor-from-01 ≡ (λ*r n*. *if r* = *1 then 1 else real-of-int* (⌊*2*^(*Suc n*) ∗ *r*⌋ *mod 2*))

to-Cantor-from-01 is a measurable injective embedding into Cantor space.

lemma *to-Cantor-from-01-image′*: *to-Cantor-from-01 r n* ∈ {*0*,*1*}
⟨*proof*⟩

lemma *to-Cantor-from-01-image″*: *0* ≤ *to-Cantor-from-01 r n to-Cantor-from-01 r n* ≤ *1*
⟨*proof*⟩

lemma *to-Cantor-from-01-image*: *to-Cantor-from-01* ∈ {*0..1*} → *space Cantor-space*
⟨*proof*⟩

lemma *to-Cantor-from-01-measurable*:
to-Cantor-from-01 ∈ *restrict-space borel* {*0..1*} $\to_M$ *Cantor-space*
⟨*proof*⟩

lemma *to-Cantor-from-01-summable*[*simp*]:
summable (λ*n*. (*1/2*)^*n* ∗ *to-Cantor-from-01 r n*)
⟨*proof*⟩

lemma *to-Cantor-from-sumn′*:
assumes *r* ∈ {*0..<1*}
shows (∑ *i<n*. (*1/2*)^(*Suc i*)∗*to-Cantor-from-01 r i*) ≤ *r*
and *r* − (∑ *i<n*. (*1/2*)^(*Suc i*)∗*to-Cantor-from-01 r i*) < (*1/2*)^*n*
and *to-Cantor-from-01 r n* = *1* ⟷ (*1/2*)^(*Suc n*) ≤ *r* − (∑ *i<n*. (*1/2*)^(*Suc i*)∗*to-Cantor-from-01 r i*)
and *to-Cantor-from-01 r n* = *0* ⟷ *r* − (∑ *i<n*. (*1/2*)^(*Suc i*)∗*to-Cantor-from-01 r i*) < (*1/2*)^(*Suc n*)
⟨*proof*⟩

lemma *to-Cantor-from-sumn*:
assumes *r* ∈ {*0..1*}
shows (∑ *i<n*. (*1/2*)^(*Suc i*)∗*to-Cantor-from-01 r i*) ≤ *r*
and *r* − (∑ *i<n*. (*1/2*)^(*Suc i*)∗*to-Cantor-from-01 r i*) ≤ (*1/2*)^*n*
and *to-Cantor-from-01 r n* = *1* ⟷ (*1/2*)^(*Suc n*) ≤ *r* − (∑ *i<n*. (*1/2*)^(*Suc i*)∗*to-Cantor-from-01 r i*)
and *to-Cantor-from-01 r n* = *0* ⟷ *r* − (∑ *i<n*. (*1/2*)^(*Suc i*)∗*to-Cantor-from-01 r i*) < (*1/2*)^(*Suc n*)
⟨*proof*⟩

lemma *to-Cantor-from-sum*:
assumes *r* ∈ {*0..1*}
shows (∑ *n*. (*1/2*)^(*Suc n*)∗*to-Cantor-from-01 r n*) = *r*
⟨*proof*⟩

lemma *to-Cantor-from-sum'*:
assumes $r \in \{0..1\}$
shows $(\sum i<n.\ (1/2)\widehat{\ }(Suc\ i)*to\text{-}Cantor\text{-}from\text{-}01\ r\ i) = r - (\sum m.\ (1/2)\widehat{\ }(Suc\ (m + n))*to\text{-}Cantor\text{-}from\text{-}01\ r\ (m + n))$
⟨*proof*⟩

lemma *to-Cantor-from-01-exist0*:
assumes $r \in \{0..<1\}$
shows $\forall n.\exists k\geq n.\ to\text{-}Cantor\text{-}from\text{-}01\ r\ k = 0$
⟨*proof*⟩

lemma *to-Cantor-from-01-if-exist0*:
assumes $\bigwedge n.\ a\ n \in \{0,1\}$ $\forall n.\exists k\geq n.\ a\ k = 0$
shows $to\text{-}Cantor\text{-}from\text{-}01\ (\sum n.\ (1\ /\ 2)\ \widehat{\ }\ Suc\ n * a\ n) = a$
⟨*proof*⟩

lemma *to-Cantor-from-01-sum-of-to-Cantor-from-01*:
assumes $r \in \{0..1\}$
shows $to\text{-}Cantor\text{-}from\text{-}01\ (\sum n.\ (1\ /\ 2)\ \widehat{\ }\ Suc\ n * to\text{-}Cantor\text{-}from\text{-}01\ r\ n) = to\text{-}Cantor\text{-}from\text{-}01\ r$
⟨*proof*⟩

lemma *to-Cantor-from-01-inj*: *inj-on to-Cantor-from-01* (*space* (*restrict-space borel* $\{0..1\}$))
⟨*proof*⟩

lemma *to-Cantor-from-01-preserves-sets*:
assumes $A \in$ *sets* (*restrict-space borel* $\{0..1\}$)
shows *to-Cantor-from-01* ` $A \in$ *sets Cantor-space*
⟨*proof*⟩

lemma *Cantor-space-isomorphic-to-01closed*:
Cantor-space measurable-isomorphic (*restrict-space borel* $\{0..1::real\}$)
⟨*proof*⟩

lemma *Cantor-space-isomorphic-to-Hilbert-cube*:
Cantor-space measurable-isomorphic Hilbert-cube
⟨*proof*⟩

lemma(**in** *standard-borel*) *embedding-into-Hilbert-cube*:
$\exists A \in$ *sets Hilbert-cube*. *M measurable-isomorphic* (*restrict-space Hilbert-cube A*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *uncountable-contains-Cantor-space*:
assumes *uncountable* (*space M*)
shows $\exists A \in$ *sets M*. *Cantor-space measurable-isomorphic* (*restrict-space M A*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *uncountable-isomorphic-to-Hilbert-cube*:

assumes *uncountable* (*space M*)
shows *Hilbert-cube measurable-isomorphic M*
⟨*proof*⟩

lemma(**in** *standard-borel*) *uncountable-isomorphic-to-real*:
assumes *uncountable* (*space M*)
shows *M measurable-isomorphic* (*borel* :: *real measure*)
⟨*proof*⟩

definition *to-real-on* :: *'a measure* ⇒ *'a* ⇒ *real* **where**
to-real-on M ≡ (*if uncountable* (*space M*) *then* (*SOME f. measurable-isomorphic-map M* (*borel* :: *real measure*) *f*) *else* (*real* ∘ *to-nat-on* (*space M*)))

definition *from-real-into* :: *'a measure* ⇒ *real* ⇒ *'a* **where**
from-real-into M ≡ (*if uncountable* (*space M*) *then the-inv-into* (*space M*) (*to-real-on M*) *else* (*λr. from-nat-into* (*space M*) (*nat* ⌊*r*⌋)))

context *standard-borel*
begin

abbreviation *to-real* ≡ *to-real-on M*
abbreviation *from-real* ≡ *from-real-into M*

lemma *to-real-def-countable*:
assumes *countable* (*space M*)
shows *to-real* = (*λr. real* (*to-nat-on* (*space M*) *r*))
⟨*proof*⟩

lemma *from-real-def-countable*:
assumes *countable* (*space M*)
shows *from-real* = (*λr. from-nat-into* (*space M*) (*nat* ⌊*r*⌋))
⟨*proof*⟩

lemma *from-real-to-real*[*simp*]:
assumes $x \in$ *space M*
shows *from-real* (*to-real x*) = *x*
⟨*proof*⟩

lemma *to-real-measurable*[*measurable*]:
to-real $\in M \to_M$ *borel*
⟨*proof*⟩

lemma *from-real-measurable'*:
assumes *space M* ≠ {}
shows *from-real* ∈ *borel* $\to_M M$
⟨*proof*⟩

lemma *countable-isomorphic-to-subset-real*:
assumes *countable* (*space M*)

obtains A :: *real set*
where *countable A A* $\in$ *sets borel M measurable-isomorphic* (*restrict-space borel A*)
⟨*proof*⟩

lemma *to-real-from-real*:
assumes *uncountable* (*space M*)
shows *to-real* (*from-real r*) = *r*
⟨*proof*⟩

end

lemma(**in** *standard-borel-ne*) *from-real-measurable*[*measurable*]: *from-real* $\in$ *borel* $\rightarrow_M$ *M*
⟨*proof*⟩

end

4.3 Example: The Metric Space of Continuous Functions

theory *Space-of-Continuous-Maps*
imports *StandardBorel*
begin

definition *cmaps* :: [*'a topology*, *'b topology*] $\Rightarrow$ (*'a* $\Rightarrow$ *'b*) *set* **where**
cmaps X Y $\equiv$ {*f*. *f* $\in$ *extensional* (*topspace X*) $\wedge$ *continuous-map X Y f*}

definition *cmaps-dist* :: [*'a topology*, *'b topology*, *'b* $\Rightarrow$ *'b* $\Rightarrow$ *real*, *'a* $\Rightarrow$ *'b*, *'a* $\Rightarrow$ *'b*] $\Rightarrow$ *real* **where**
cmaps-dist X Y d f g $\equiv$ *if f* $\in$ *cmaps X Y* $\wedge$ *g* $\in$ *cmaps X Y* $\wedge$ *topspace X* $\neq$ {}
then ($\bigsqcup$ {*d* (*f x*) (*g x*) |*x*. *x* $\in$ *topspace X*}) *else 0*

lemma *cmaps-X-empty*:
assumes *topspace X* = {}
shows *cmaps X Y* = {λ*x*. *undefined*}
⟨*proof*⟩

lemma *cmaps-Y-empty*:
assumes *topspace X* $\neq$ {} *topspace Y* = {}
shows *cmaps X Y* = {}
⟨*proof*⟩

lemma *cmaps-dist-X-empty*:
assumes *topspace X* = {}
shows *cmaps-dist X* = (λ*Y d f g*. *0*)
⟨*proof*⟩

lemma *cmaps-dist-Y-empty*:
assumes *topspace X* $\neq$ {} *topspace Y* = {}

shows *cmaps-dist X Y* = $(\lambda d\ f\ g.\ 0)$
⟨*proof*⟩

4.3.1 Definition

context *metric-set*
begin

context
fixes *K* **and** *X* :: *'b topology*
assumes *m-bounded* :$\bigwedge x\ y.\ dist\ x\ y \leq K$
begin

lemma *cm-dest*:
shows $\bigwedge f\ x.\ f \in (cmaps\ X\ mtopology) \Longrightarrow x \in topspace\ X \Longrightarrow f\ x \in S$
and $\bigwedge f\ x.\ f \in (cmaps\ X\ mtopology) \Longrightarrow x \notin topspace\ X \Longrightarrow f\ x = undefined$
and $\bigwedge f.\ f \in (cmaps\ X\ mtopology) \Longrightarrow continuous\text{-}map\ X\ mtopology\ f$
⟨*proof*⟩

lemma *cmaps-dist-bdd-above*[*simp*]: *bdd-above* $\{dist\ (f\ x)\ (g\ x)\ |x.\ x \in A\}$
⟨*proof*⟩

lemma *cmaps-metric-set*: *metric-set* (*cmaps X mtopology*) (*cmaps-dist X mtopology dist*)
⟨*proof*⟩

end

end

4.3.2 Topology of Uniform Convergence

locale *topology-of-uniform-convergence-c* = *complete-metric-set* + *compact-metrizable X* **for** *X*
+ **fixes** *K*
assumes *d-bounded*: $\bigwedge x\ y.\ dist\ x\ y \leq K$
begin

lemmas *cm-dist-bdd-above*[*simp*] = *cmaps-dist-bdd-above*[*OF d-bounded*]

abbreviation *cm* ≡ *cmaps X mtopology*
abbreviation *cm-dist* ≡ *cmaps-dist X mtopology dist*

lemma *cm-complete-metric-set*: *complete-metric-set cm cm-dist*
⟨*proof*⟩

end

locale *topology-of-uniform-convergence* = *polish-metric-set* + *compact-metrizable X* **for** *X*

+ **fixes** K
assumes *d-bounded*: $\bigwedge x\ y.\ dist\ x\ y \leq K$
begin

sublocale *topology-of-uniform-convergence-c*
⟨*proof*⟩

lemma *cm-polish-metric-set*: *polish-metric-set cm cm-dist*
⟨*proof*⟩

end

end

References

[1] Lecture note of math245b in UCLA. https://web.archive.org/web/20210506130459/https://www.math.ucla.edu/~biskup/245b.1.20w/, 2020. Accessed: June 27. 2023.

[2] K. Matsuzaka. **集合・位相入門**. Iwanami Shoten, 1968. written in Japanese.

[3] S. M. Srivastava. *A Course on Borel Sets*. Springer, 1998.