Standard Borel Spaces

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Abstract

This entry includes a formalization of standard Borel spaces and (a variant of) the Borel isomorphism theorem. A separable complete metrizable topological space is called a polish space and a measurable space generated from a polish space is called a standard Borel space. We formalize the notion of standard Borel spaces by establishing set-based metric spaces, and then prove (a variant of) the Borel isomorphism theorem. The theorem states that a standard Borel spaces is either a countable discrete space or isomorphic to $\mathbb{R}$.

Contents

We refer to the HOL-Analysis library, the textbooks by Matsuzaka [2] and Srivastava [3], and the lecture note by Biskup [1].

1 Lemmas

theory *Lemmas-StandardBorel*
imports *HOL−Probability.Probability*
begin

1.1 Lemmas for Abstract Topology

1.1.1 Generated By

lemma *topology-generated-by-sub*:
assumes $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow (openin\ X\ U)$
and $openin\ (topology\text{-}generated\text{-}by\ \mathcal{U})\ U$
shows $openin\ X\ U$
⟨*proof*⟩

lemma *topology-generated-by-open*:
$S = topology\text{-}generated\text{-}by\ \{U \mid U\ .\ openin\ S\ U\}$
⟨*proof*⟩

lemma *topology-generated-by-eq*:
assumes $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow (openin\ (topology\text{-}generated\text{-}by\ \mathcal{O})\ U)$
and $\bigwedge U.\ U \in \mathcal{O} \Longrightarrow (openin\ (topology\text{-}generated\text{-}by\ \mathcal{U})\ U)$
shows $topology\text{-}generated\text{-}by\ \mathcal{O} = topology\text{-}generated\text{-}by\ \mathcal{U}$
⟨*proof*⟩

lemma *topology-generated-by-homeomorphic-spaces*:
assumes $homeomorphic\text{-}map\ X\ Y\ f\ X = topology\text{-}generated\text{-}by\ \mathcal{O}$
shows $Y = topology\text{-}generated\text{-}by\ ((`)\ f\ `\ \mathcal{O})$

⟨*proof*⟩

lemma *open-map-generated-topo*:
assumes $\bigwedge u.\ u \in U \Longrightarrow$ *openin* S $(f\ `\ u)$ *inj-on* f (*topspace* (*topology-generated-by* U))
shows *open-map* (*topology-generated-by* U) S f
⟨*proof*⟩

lemma *subtopology-generated-by*:
subtopology (*topology-generated-by* $\mathcal{O}$) T = *topology-generated-by* $\{T \cap U \mid U.\ U \in \mathcal{O}\}$
⟨*proof*⟩

lemma *prod-topology-generated-by*:
topology-generated-by $\{\ U \times V \mid U\ V.\ U \in \mathcal{O} \wedge V \in \mathcal{U}\}$ = *prod-topology* (*topology-generated-by* $\mathcal{O}$) (*topology-generated-by* $\mathcal{U}$)
⟨*proof*⟩

lemma *prod-topology-generated-by-open*:
prod-topology S S' = *topology-generated-by* $\{U \times V \mid U\ V.$ *openin* S U $\wedge$ *openin* S' $V\}$
⟨*proof*⟩

lemma *product-topology-cong*:
assumes $\bigwedge i.\ i \in I \Longrightarrow S\ i = K\ i$
shows *product-topology* S I = *product-topology* K I
⟨*proof*⟩

lemma *topology-generated-by-without-empty*:
topology-generated-by $\mathcal{O}$ = *topology-generated-by* $\{\ U \in \mathcal{O}.\ U \neq \{\}\}$
⟨*proof*⟩

lemma *topology-from-bij*:
assumes *bij-betw* f A (*topspace* S)
shows *homeomorphic-map* (*pullback-topology* A f S) S f *topspace* (*pullback-topology* A f S) = A
⟨*proof*⟩

lemma *openin-pullback-topology'*:
assumes *bij-betw* f A (*topspace* S)
shows *openin* (*pullback-topology* A f S) $u \longleftrightarrow$ (*openin* S $(f\ `\ u)$) $\wedge\ u \subseteq A$
⟨*proof*⟩

1.1.2 Isolated Point

definition *isolated-points-of* :: *'a topology* ⇒ *'a set* ⇒ *'a set* (**infixr** ‹*isolated'-points'-of*› *80*) **where**
X *isolated-points-of* $A \equiv \{x{\in}$*topspace* $X \cap A.\ x \notin X$ *derived-set-of* $A\}$

lemma *isolated-points-of-eq*:
X *isolated-points-of* $A = \{x \in topspace\ X \cap A.\ \exists\, U.\ x \in U \wedge openin\ X\ U \wedge U \cap (A - \{x\}) = \{\}\}$
⟨*proof*⟩

lemma *in-isolated-points-of*:
$x \in X$ *isolated-points-of* $A \longleftrightarrow x \in topspace\ X \wedge x \in A \wedge (\exists\, U.\ x \in U \wedge openin\ X\ U \wedge U \cap (A - \{x\}) = \{\})$
⟨*proof*⟩

lemma *derived-set-of-eq*:
$x \in X$ *derived-set-of* $A \longleftrightarrow x \in X$ *closure-of* $(A - \{x\})$
⟨*proof*⟩

1.1.3 Perfect Set

definition *perfect-set* :: *'a topology* ⇒ *'a set* ⇒ *bool* **where**
perfect-set $X\ A \longleftrightarrow$ *closedin* $X\ A \wedge X$ *isolated-points-of* $A = \{\}$

abbreviation *perfect-space* $X \equiv$ *perfect-set* X (*topspace* X)

lemma *perfect-space-euclidean*: *perfect-space* (*euclidean* :: *'a* :: *perfect-space topology*)
⟨*proof*⟩

lemma *perfect-setI*:
assumes *closedin* $X\ A$
and $\bigwedge x\ T.\ [\![x \in A;\ x \in T;\ openin\ X\ T]\!] \Longrightarrow \exists\, y \neq x.\ y \in T \wedge y \in A$
shows *perfect-set* $X\ A$
⟨*proof*⟩

lemma *perfect-spaceI*:
assumes $\bigwedge x\ T.\ [\![x \in T;\ openin\ X\ T]\!] \Longrightarrow \exists\, y \neq x.\ y \in T$
shows *perfect-space* X
⟨*proof*⟩

lemma *perfect-setD*:
assumes *perfect-set* $X\ A$
shows *closedin* $X\ A$ $A \subseteq topspace\ X$ $\bigwedge x\ T.\ [\![x \in A;\ x \in T;\ openin\ X\ T]\!] \Longrightarrow \exists\, y \neq x.\ y \in T \wedge y \in A$
⟨*proof*⟩

lemma *perfect-space-perfect*:
perfect-set euclidean (*UNIV* :: *'a* :: *perfect-space set*)
⟨*proof*⟩

lemma *perfect-set-subtopology*:
assumes *perfect-set* $X\ A$
shows *perfect-space* (*subtopology* $X\ A$)

⟨*proof*⟩

1.1.4 Bases and Sub-Bases in Abstract Topology

definition *subbase-in* :: [′*a topology*, ′*a set set*] ⇒ *bool* **where**
subbase-in S $\mathcal{O}$ ⟷ *S* = *topology-generated-by* $\mathcal{O}$

definition *base-in* :: [′*a topology*, ′*a set set*] ⇒ *bool* **where**
base-in S $\mathcal{O}$ ⟷ (∀ *U*. *openin S U* ⟷ (∃$\mathcal{U}$. $U = \bigcup\mathcal{U} \wedge \mathcal{U} \subseteq \mathcal{O}$))

lemma *second-countable-base-in*: *second-countable S* ⟷ (∃$\mathcal{O}$. *countable* $\mathcal{O}$ ∧ *base-in S* $\mathcal{O}$)
⟨*proof*⟩

definition *zero-dimensional* :: ′*a topology* ⇒ *bool* **where**
zero-dimensional S ⟷ (∃$\mathcal{O}$. *base-in S* $\mathcal{O}$ ∧ (∀ *u*∈$\mathcal{O}$. *openin S u* ∧ *closedin S u*))

lemma *openin-base*:
assumes *base-in S* $\mathcal{O}$ $U = \bigcup\mathcal{U}$ **and** $\mathcal{U} \subseteq \mathcal{O}$
shows *openin S U*
⟨*proof*⟩

lemma *base-is-subbase*:
assumes *base-in S* $\mathcal{O}$
shows *subbase-in S* $\mathcal{O}$
⟨*proof*⟩

lemma *subbase-in-subset*:
assumes *subbase-in S* $\mathcal{O}$ **and** $U \in \mathcal{O}$
shows $U \subseteq$ *topspace S*
⟨*proof*⟩

lemma *subbase-in-openin*:
assumes *subbase-in S* $\mathcal{O}$ **and** $U \in \mathcal{O}$
shows *openin S U*
⟨*proof*⟩

lemma *base-in-subset*:
assumes *base-in S* $\mathcal{O}$ **and** $U \in \mathcal{O}$
shows $U \subseteq$ *topspace S*
⟨*proof*⟩

lemma *base-in-openin*:
assumes *base-in S* $\mathcal{O}$ **and** $U \in \mathcal{O}$
shows *openin S U*
⟨*proof*⟩

lemma *base-in-def2*:
assumes $\bigwedge U.\ U \in \mathcal{O} \Longrightarrow$ *openin S U*

shows *base-in* S $\mathcal{O}$ $\longleftrightarrow$ ($\forall$ U. *openin* S U $\longrightarrow$ ($\forall$ $x{\in}U$. $\exists$ $W{\in}\mathcal{O}$. $x \in W \wedge W \subseteq U$))
⟨*proof*⟩

lemma *base-in-def2'*:
base-in S $\mathcal{O}$ $\longleftrightarrow$ ($\forall$ $b{\in}\mathcal{O}$. *openin* S b) $\wedge$ ($\forall$ x. *openin* S x $\longrightarrow$ ($\exists$ $B'{\subseteq}\mathcal{O}$. $\bigcup$ $B' = x$))
⟨*proof*⟩

corollary *base-in-in-subset*:
assumes *base-in* S $\mathcal{O}$ *openin* S u $x \in u$
shows $\exists$ $v{\in}\mathcal{O}$. $x \in v \wedge v \subseteq u$
⟨*proof*⟩

lemma *base-in-without-empty*:
assumes *base-in* S $\mathcal{O}$
shows *base-in* S $\{U \in \mathcal{O}.\ U \neq \{\}\}$
⟨*proof*⟩

lemma *second-countable-ex-without-empty*:
assumes *second-countable* S
shows $\exists$ $\mathcal{O}$. *countable* $\mathcal{O}$ $\wedge$ *base-in* S $\mathcal{O}$ $\wedge$ ($\forall$ $U{\in}\mathcal{O}$. $U \neq \{\}$)
⟨*proof*⟩

lemma *subtopology-subbase-in*:
assumes *subbase-in* S $\mathcal{O}$
shows *subbase-in* (*subtopology* S T) $\{T \cap U \mid U.\ U \in \mathcal{O}\}$
⟨*proof*⟩

lemma *subtopology-base-in*:
assumes *base-in* S $\mathcal{O}$
shows *base-in* (*subtopology* S T) $\{T \cap U \mid U.\ U \in \mathcal{O}\}$
⟨*proof*⟩

lemma *second-countable-subtopology*:
assumes *second-countable* S
shows *second-countable* (*subtopology* S T)
⟨*proof*⟩

lemma *open-map-with-base*:
assumes *base-in* S $\mathcal{O}$ $\bigwedge A.\ A \in \mathcal{O} \Longrightarrow$ *openin* S' (f ‘ A)
shows *open-map* S S' f
⟨*proof*⟩

Construct a base from a subbase.

lemma *finite'-intersection-of-idempot* [*simp*]:
finite' *intersection-of* *finite'* *intersection-of* P = *finite'* *intersection-of* P
⟨*proof*⟩

lemma *finite′-intersection-of-countable*:
assumes *countable* $\mathcal{O}$
shows *countable* (*Collect* (*finite′ intersection-of* ($\lambda x.\ x \in \mathcal{O}$)))
⟨*proof*⟩

lemma *finite′-intersection-of-openin*:
assumes (*finite′ intersection-of* ($\lambda x.\ x \in \mathcal{O}$)) U
shows *openin* (*topology-generated-by* $\mathcal{O}$) U
⟨*proof*⟩

lemma *topology-generated-by-finite-intersections*:
topology-generated-by $\mathcal{O}$ = *topology-generated-by* (*Collect* (*finite′ intersection-of* ($\lambda x.\ x \in \mathcal{O}$)))
⟨*proof*⟩

lemma *base-from-subbase*:
assumes *subbase-in* S $\mathcal{O}$
shows *base-in* S (*Collect* (*finite′ intersection-of* ($\lambda x.\ x \in \mathcal{O}$)))
⟨*proof*⟩

lemma *countable-base-from-countable-subbase*:
assumes *countable* $\mathcal{O}$ **and** *subbase-in* S $\mathcal{O}$
shows *second-countable* S
⟨*proof*⟩

lemma *prod-topology-second-countable*:
assumes *second-countable* S **and** *second-countable* S'
shows *second-countable* (*prod-topology* S S')
⟨*proof*⟩

Abstract version of the theorem $\exists K.$ *topological-basis* K $\wedge$ *countable* K $\wedge$ ($\forall k \in K.\ \exists X.\ k = Pi_E$ *UNIV* X $\wedge$ ($\forall i.$ *open* ($X\ i$)) $\wedge$ *finite* $\{i.\ X\ i \neq$ *UNIV*$\}$).

lemma *product-topology-countable-base-in*:
assumes *countable* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *second-countable* ($S\ i$)
shows $\exists \mathcal{O}'.$ *countable* $\mathcal{O}'$ $\wedge$ *base-in* (*product-topology* S I) $\mathcal{O}'$ $\wedge$
($\forall k \in \mathcal{O}'.\ \exists X.\ k = (\Pi_E\ i \in I.\ X\ i) \wedge (\forall i.$ *openin* ($S\ i$) ($X\ i$)) $\wedge$ *finite* $\{i.\ X\ i \neq$ *topspace* ($S\ i$)$\}$ $\wedge$ $\{i.\ X\ i \neq$ *topspace* ($S\ i$)$\} \subseteq I$)
⟨*proof*⟩

lemma *product-topology-second-countable*:
assumes *countable* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *second-countable* ($S\ i$)
shows *second-countable* (*product-topology* S I)
⟨*proof*⟩

lemma *second-countable-euclidean*[*simp*]:
second-countable (*euclidean* :: ′*a* :: *second-countable-topology topology*)
⟨*proof*⟩

lemma *Cantor-Bendixon*:
assumes *second-countable X*
shows $\exists$ *U P*. *countable U* $\wedge$ *openin X U* $\wedge$ *perfect-set X P* $\wedge$ *U* $\cup$ *P* = *topspace X* $\wedge$ *U* $\cap$ *P* = {} $\wedge$ ($\forall$ *a*$\neq${}. *openin* (*subtopology X P*) *a* $\longrightarrow$ *uncountable a*)
⟨*proof*⟩

1.1.5 Separable Spaces

definition *dense-in* :: ['*a topology*, '*a set*] $\Rightarrow$ *bool* **where**
dense-in S U $\longleftrightarrow$ (*U* $\subseteq$ *topspace S* $\wedge$ ($\forall$ *V*. *openin S V* $\longrightarrow$ *V* $\neq$ {} $\longrightarrow$ *U* $\cap$ *V* $\neq$ {}))

lemma *dense-in-def2*:
dense-in S U $\longleftrightarrow$ (*U* $\subseteq$ *topspace S* $\wedge$ (*S closure-of U*) = *topspace S*)
⟨*proof*⟩

lemma *dense-in-topspace*[*simp*]: *dense-in S* (*topspace S*)
⟨*proof*⟩

lemma *dense-in-subset*:
assumes *dense-in S U*
shows *U* $\subseteq$ *topspace S*
⟨*proof*⟩

lemma *dense-in-nonempty*:
assumes *topspace S* $\neq$ {} *dense-in S U*
shows *U* $\neq$ {}
⟨*proof*⟩

lemma *dense-inI*:
assumes *U* $\subseteq$ *topspace S*
and $\bigwedge$*V*. *openin S V* $\Longrightarrow$ *V* $\neq$ {} $\Longrightarrow$ *U* $\cap$ *V* $\neq$ {}
shows *dense-in S U*
⟨*proof*⟩

lemma *dense-in-infinite*:
assumes *t1-space X infinite* (*topspace X*) *dense-in X U*
shows *infinite U*
⟨*proof*⟩

lemma *dense-in-prod*:
assumes *dense-in S U* **and** *dense-in S' U'*
shows *dense-in* (*prod-topology S S'*) (*U* × *U'*)
⟨*proof*⟩

lemma *separable-space-def2*:*separable-space S* $\longleftrightarrow$ ($\exists$ *U*. *countable U* $\wedge$ *dense-in S U*)
⟨*proof*⟩

lemma *countable-space-separable-space*:
assumes *countable* (*topspace* S)
shows *separable-space* S
⟨*proof*⟩

lemma *separable-space-prod*:
assumes *separable-space* S **and** *separable-space* S'
shows *separable-space* (*prod-topology* S S')
⟨*proof*⟩

lemma *dense-in-product*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *dense-in* (T i) (U i)
shows *dense-in* (*product-topology* T I) (Π_E $i{\in}I.$ U i)
⟨*proof*⟩

lemma *separable-countable-product*:
assumes *countable* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *separable-space* (T i)
shows *separable-space* (*product-topology* T I)
⟨*proof*⟩

lemma *separable-finite-product*:
assumes *finite* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *separable-space* (T i)
shows *separable-space* (*product-topology* T I)
⟨*proof*⟩

1.1.6 G_δ Set

lemma *gdelta-inD*:
assumes *gdelta-in* S A
shows $\exists \mathcal{U}.\ \mathcal{U} \neq \{\} \wedge$ *countable* $\mathcal{U}$ $\wedge$ ($\forall\, b{\in}\mathcal{U}.$ *openin* S b) $\wedge$ $A = \bigcap \mathcal{U}$
⟨*proof*⟩

lemma *gdelta-inD'*:
assumes *gdelta-in* S A
shows $\exists$ U. ($\forall$ n::*nat*. *openin* S (U n)) $\wedge$ $A = \bigcap$ (*range* U)
⟨*proof*⟩

lemma *gdelta-in-continuous-map*:
assumes *continuous-map* X Y f *gdelta-in* Y a
shows *gdelta-in* X (f −‘ a ∩ *topspace* X)
⟨*proof*⟩

lemma *g-delta-of-inj-open-map*:
assumes *open-map* X Y f *inj-on* f (*topspace* X) *gdelta-in* X a
shows *gdelta-in* Y (f ‘ a)
⟨*proof*⟩

lemma *gdelta-in-prod*:
assumes *gdelta-in* X A *gdelta-in* Y B

shows *gdelta-in* (*prod-topology* X Y) ($A \times B$)
⟨*proof*⟩

corollary *gdelta-in-prod1*:
assumes *gdelta-in* X A
shows *gdelta-in* (*prod-topology* X Y) ($A \times$ *topspace* Y)
⟨*proof*⟩

corollary *gdelta-in-prod2*:
assumes *gdelta-in* Y B
shows *gdelta-in* (*prod-topology* X Y) (*topspace* $X \times B$)
⟨*proof*⟩

lemma *continuous-map-imp-closed-graph'*:
assumes *continuous-map* X Y f *Hausdorff-space* Y
shows *closedin* (*prod-topology* Y X) (($\lambda x.$ (f x,x)) ‘ *topspace* X)
⟨*proof*⟩

1.1.7 Continuous Maps on First Countable Topology

Generalized version of *Metric-space ?M ?d* $\Longrightarrow$ *eventually ?P* (*atin* (*Metric-space.mtopology ?M ?d*) *?a*) = ($\forall\,\sigma.$ *range* $\sigma \subseteq$ *?M* $-$ {*?a*} $\wedge$ *limitin* (*Metric-space.mtopology ?M ?d*) σ *?a sequentially* $\longrightarrow$ ($\forall_F$ n *in sequentially. ?P* (σ n)))

lemma *eventually-atin-sequentially*:
assumes *first-countable* X
shows *eventually* P (*atin* X a) $\longleftrightarrow$ ($\forall\,\sigma.$ *range* $\sigma \subseteq$ *topspace* $X - \{a\} \wedge$ *limitin* X σ a *sequentially* $\longrightarrow$ *eventually* ($\lambda n.$ P (σ n)) *sequentially*)
⟨*proof*⟩

lemma *continuous-map-iff-limit-seq*:
assumes *first-countable* X
shows *continuous-map* X Y f $\longleftrightarrow$ ($\forall$ *xn* x. *limitin* X *xn* x *sequentially* $\longrightarrow$ *limitin* Y ($\lambda n.$ f (*xn* n)) (f x) *sequentially*)
⟨*proof*⟩

1.1.8 Upper-Semicontinuous Functions

definition *upper-semicontinuous-map* :: ['*a topology*, '*a* ⇒ '*b* :: *linorder-topology*] ⇒ *bool* **where**
upper-semicontinuous-map X f $\longleftrightarrow$ ($\forall\,a.$ *openin* X {x∈*topspace* X. f $x < a$})

lemma *continuous-upper-semicontinuous*:
assumes *continuous-map* X (*euclidean* :: ('*b* :: *linorder-topology*) *topology*) f
shows *upper-semicontinuous-map* X f
⟨*proof*⟩

lemma *upper-semicontinuous-map-iff-closed*:
upper-semicontinuous-map X f $\longleftrightarrow$ ($\forall\,a.$ *closedin* X {x∈*topspace* X. f $x \geq a$})
⟨*proof*⟩

lemma *upper-semicontinuous-map-real-iff*:
fixes f :: $'a \Rightarrow real$
shows *upper-semicontinuous-map* X f $\longleftrightarrow$ *upper-semicontinuous-map* X $(\lambda x.\ ereal\ (f\ x))$
⟨*proof*⟩

1.1.9 Lower-Semicontinuous Functions

definition *lower-semicontinuous-map* :: $['a\ topology,\ 'a \Rightarrow 'b :: linorder\text{-}topology] \Rightarrow bool$ **where**
lower-semicontinuous-map $X\ f \longleftrightarrow (\forall\, a.\ openin\ X\ \{x{\in}topspace\ X.\ a < f\ x\})$

lemma *continuous-lower-semicontinuous*:
assumes *continuous-map* X $(euclidean :: ('b :: linorder\text{-}topology)\ topology)\ f$
shows *lower-semicontinuous-map* $X\ f$
⟨*proof*⟩

lemma *lower-semicontinuous-map-iff-closed*:
lower-semicontinuous-map $X\ f \longleftrightarrow (\forall\, a.\ closedin\ X\ \{x{\in}topspace\ X.\ f\ x \leq a\})$
⟨*proof*⟩

lemma *lower-semicontinuous-map-real-iff*:
fixes f :: $'a \Rightarrow real$
shows *lower-semicontinuous-map* X f $\longleftrightarrow$ *lower-semicontinuous-map* X $(\lambda x.\ ereal\ (f\ x))$
⟨*proof*⟩

1.2 Lemmas for Measure Theory

1.2.1 Lemmas for Measurable Sets

lemma *measurable-preserve-sigma-sets*:
assumes $sets\ M = sigma\text{-}sets\ \Omega\ S$ $S \subseteq Pow\ \Omega$
$\bigwedge a.\ a \in S \Longrightarrow f\ `\ a \in sets\ N$ $inj\text{-}on\ f\ (space\ M)$ $f\ `\ space\ M \in sets\ N$
and $b \in sets\ M$
shows $f\ `\ b \in sets\ N$
⟨*proof*⟩

inductive-set *sigma-sets-cinter* :: $'a\ set \Rightarrow 'a\ set\ set \Rightarrow 'a\ set\ set$
for sp :: $'a\ set$ **and** A :: $'a\ set\ set$
where
Basic-c[*intro*, *simp*]: $a \in A \Longrightarrow a \in sigma\text{-}sets\text{-}cinter\ sp\ A$
| *Top-c*[*simp*]: $sp \in sigma\text{-}sets\text{-}cinter\ sp\ A$
| *Inter-c*: $(\bigwedge i{::}nat.\ a\ i \in sigma\text{-}sets\text{-}cinter\ sp\ A) \Longrightarrow (\bigcap i.\ a\ i) \in sigma\text{-}sets\text{-}cinter\ sp\ A$
| *Union-c*: $(\bigwedge i{::}nat.\ a\ i \in sigma\text{-}sets\text{-}cinter\ sp\ A) \Longrightarrow (\bigcup i.\ a\ i) \in sigma\text{-}sets\text{-}cinter\ sp\ A$

inductive-set *sigma-sets-cinter-dunion* :: $'a\ set \Rightarrow 'a\ set\ set \Rightarrow 'a\ set\ set$

for *sp* :: *'a set* **and** *A* :: *'a set set*
where
Basic-cd[*intro*, *simp*]: $a \in A \Longrightarrow a \in$ *sigma-sets-cinter-dunion sp A*
| *Top-cd*[*simp*]: $sp \in$ *sigma-sets-cinter-dunion sp A*
| *Inter-cd*: $(\bigwedge i::nat.\ a\ i \in$ *sigma-sets-cinter-dunion sp A*$) \Longrightarrow (\bigcap i.\ a\ i) \in$ *sigma-sets-cinter-dunion sp A*
| *Union-cd*: $(\bigwedge i::nat.\ a\ i \in$ *sigma-sets-cinter-dunion sp A*$) \Longrightarrow$ *disjoint-family a* $\Longrightarrow (\bigcup i.\ a\ i) \in$ *sigma-sets-cinter-dunion sp A*

lemma *sigma-sets-cinter-dunion-subset*: *sigma-sets-cinter-dunion sp A* $\subseteq$ *sigma-sets-cinter sp A*
⟨*proof*⟩

lemma *sigma-sets-cinter-into-sp*:
assumes $A \subseteq$ *Pow sp* $x \in$ *sigma-sets-cinter sp A*
shows $x \subseteq sp$
⟨*proof*⟩

lemma *sigma-sets-cinter-dunion-into-sp*:
assumes $A \subseteq$ *Pow sp* $x \in$ *sigma-sets-cinter-dunion sp A*
shows $x \subseteq sp$
⟨*proof*⟩

lemma *sigma-sets-cinter-int*:
assumes $a \in$ *sigma-sets-cinter sp A* $b \in$ *sigma-sets-cinter sp A*
shows $a \cap b \in$ *sigma-sets-cinter sp A*
⟨*proof*⟩

lemma *sigma-sets-cinter-dunion-int*:
assumes $a \in$ *sigma-sets-cinter-dunion sp A* $b \in$ *sigma-sets-cinter-dunion sp A*
shows $a \cap b \in$ *sigma-sets-cinter-dunion sp A*
⟨*proof*⟩

lemma *sigma-sets-cinter-un*:
assumes $a \in$ *sigma-sets-cinter sp A* $b \in$ *sigma-sets-cinter sp A*
shows $a \cup b \in$ *sigma-sets-cinter sp A*
⟨*proof*⟩

lemma *sigma-sets-eq-cinter-dunion*:
assumes *metrizable-space X*
shows *sigma-sets* (*topspace X*) $\{U.$ *openin X U*$\}$ = *sigma-sets-cinter-dunion* (*topspace X*) $\{U.$ *openin X U*$\}$
⟨*proof*⟩

lemma *sigma-sets-eq-cinter*:
assumes *metrizable-space X*
shows *sigma-sets* (*topspace X*) $\{U.$ *openin X U*$\}$ = *sigma-sets-cinter* (*topspace X*) $\{U.$ *openin X U*$\}$
⟨*proof*⟩

1.2.2 Measurable Isomorphisms

definition *measurable-isomorphic-map*::[$'a$ *measure*, $'b$ *measure*, $'a \Rightarrow 'b$] $\Rightarrow$ *bool*
where
measurable-isomorphic-map M N f $\longleftrightarrow$ *bij-betw* f (*space* M) (*space* N) $\wedge$ $f \in M \rightarrow_M N$ $\wedge$ *the-inv-into* (*space* M) $f \in N \rightarrow_M M$

lemma *measurable-isomorphic-map-sets-cong*:
assumes *sets* M = *sets* M' *sets* N = *sets* N'
shows *measurable-isomorphic-map* M N f $\longleftrightarrow$ *measurable-isomorphic-map* M' N' f
⟨*proof*⟩

lemma *measurable-isomorphic-map-surj*:
assumes *measurable-isomorphic-map* M N f
shows f ‘ *space* M = *space* N
⟨*proof*⟩

lemma *measurable-isomorphic-mapI*:
assumes *bij-betw* f (*space* M) (*space* N) $f \in M \rightarrow_M N$ *the-inv-into* (*space* M) $f \in N \rightarrow_M M$
shows *measurable-isomorphic-map* M N f
⟨*proof*⟩

lemma *measurable-isomorphic-map-byWitness*:
assumes $f \in M \rightarrow_M N$ $g \in N \rightarrow_M M$ $\bigwedge x.\ x \in$ *space* $M \Longrightarrow g\ (f\ x) = x$ $\bigwedge x.\ x \in$ *space* $N \Longrightarrow f\ (g\ x) = x$
shows *measurable-isomorphic-map* M N f
⟨*proof*⟩

lemma *measurable-isomorphic-map-restrict-space*:
assumes $f \in M \rightarrow_M N$ $\bigwedge A.\ A \in$ *sets* $M \Longrightarrow f$ ‘ $A \in$ *sets* N *inj-on* f (*space* M)
shows *measurable-isomorphic-map* M (*restrict-space* N (f ‘ *space* M)) f
⟨*proof*⟩

lemma *measurable-isomorphic-mapD'*:
assumes *measurable-isomorphic-map* M N f
shows $\bigwedge A.\ A \in$ *sets* $M \Longrightarrow f$ ‘ $A \in$ *sets* N $f \in M \rightarrow_M N$
$\exists g.$ *bij-betw* g (*space* N) (*space* M) $\wedge$ $g \in N \rightarrow_M M$ $\wedge$ ($\forall x \in$ *space* M. $g\ (f\ x) = x$) $\wedge$ ($\forall x\in$ *space* N. $f\ (g\ x) = x$) $\wedge$ ($\forall A\in$*sets* N. g ‘ $A \in$ *sets* M)
⟨*proof*⟩

lemma *measurable-isomorphic-map-inv*:
assumes *measurable-isomorphic-map* M N f
shows *measurable-isomorphic-map* N M (*the-inv-into* (*space* M) f)
⟨*proof*⟩

lemma *measurable-isomorphic-map-comp*:
assumes *measurable-isomorphic-map* M N f **and** *measurable-isomorphic-map* N L g

shows *measurable-isomorphic-map M L* $(g \circ f)$
⟨*proof*⟩

definition *measurable-isomorphic*::[$'a$ *measure*, $'b$ *measure*] ⇒ *bool* (**infixr** ‹*measurable′-isomorphic*› *50*) **where**
M measurable-isomorphic N ⟷ (∃*f*. *measurable-isomorphic-map M N f*)

lemma *measurable-isomorphic-sets-cong*:
assumes *sets M* = *sets M′ sets N* = *sets N′*
shows *M measurable-isomorphic N* ⟷ *M′ measurable-isomorphic N′*
⟨*proof*⟩

lemma *measurable-isomorphicD*:
assumes *M measurable-isomorphic N*
shows $\exists f\ g.\ f \in M \to_M N \wedge g \in N \to_M M \wedge (\forall x{\in}space\ M.\ g\ (f\ x) = x) \wedge (\forall y{\in}space\ N.\ f\ (g\ y) = y) \wedge (\forall A{\in}sets\ M.\ f\ `\ A \in sets\ N) \wedge (\forall A{\in}sets\ N.\ g\ `\ A \in sets\ M)$
⟨*proof*⟩

lemma *measurable-isomorphic-cardinality-eq*:
assumes *M measurable-isomorphic N*
shows *space M* ≈ *space N*
⟨*proof*⟩

lemma *measurable-isomorphic-count-spaces*: *count-space A measurable-isomorphic count-space B* ⟷ *A* ≈ *B*
⟨*proof*⟩

lemma *measurable-isomorphic-byWitness*:
assumes $f \in M \to_M N$ $\bigwedge x.\ x{\in}space\ M \Longrightarrow g\ (f\ x) = x$
and $g \in N \to_M M$ $\bigwedge y.\ y{\in}space\ N \Longrightarrow f\ (g\ y) = y$
shows *M measurable-isomorphic N*
⟨*proof*⟩

lemma *measurable-isomorphic-refl*:
M measurable-isomorphic M
⟨*proof*⟩

lemma *measurable-isomorphic-sym*:
assumes *M measurable-isomorphic N*
shows *N measurable-isomorphic M*
⟨*proof*⟩

lemma *measurable-isomorphic-trans*:
assumes *M measurable-isomorphic N* **and** *N measurable-isomorphic L*
shows *M measurable-isomorphic L*
⟨*proof*⟩

lemma *measurable-isomorphic-empty*:

assumes *space* $M = \{\}$ *space* $N = \{\}$
shows M *measurable-isomorphic* N
⟨*proof*⟩

lemma *measurable-isomorphic-empty1*:
assumes *space* $M = \{\}$ M *measurable-isomorphic* N
shows *space* $N = \{\}$
⟨*proof*⟩

lemma *measurable-ismorphic-empty2*:
assumes *space* $N = \{\}$ M *measurable-isomorphic* N
shows *space* $M = \{\}$
⟨*proof*⟩

lemma *measurable-lift-product*:
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i \in (M\ i) \rightarrow_M (N\ i)$
shows $(\lambda x\ i.\ \mathit{if}\ i \in I\ \mathit{then}\ f\ i\ (x\ i)\ \mathit{else\ undefined}) \in (\Pi_M\ i{\in}I.\ M\ i) \rightarrow_M (\Pi_M\ i{\in}I.\ N\ i)$
⟨*proof*⟩

lemma *measurable-isomorphic-map-lift-product*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *measurable-isomorphic-map* $(M\ i)\ (N\ i)\ (h\ i)$
shows *measurable-isomorphic-map* $(\Pi_M\ i{\in}I.\ M\ i)\ (\Pi_M\ i{\in}I.\ N\ i)\ (\lambda x\ i.\ \mathit{if}\ i \in I\ \mathit{then}\ h\ i\ (x\ i)\ \mathit{else\ undefined})$
⟨*proof*⟩

lemma *measurable-isomorphic-lift-product*:
assumes $\bigwedge i.\ i \in I \Longrightarrow (M\ i)$ *measurable-isomorphic* $(N\ i)$
shows $(\Pi_M\ i{\in}I.\ M\ i)$ *measurable-isomorphic* $(\Pi_M\ i{\in}I.\ N\ i)$
⟨*proof*⟩

https://math24.net/cantor-schroder-bernstein-theorem.html

lemma *Schroeder-Bernstein-measurable'*:
assumes f ‘ (*space* M) $\in$ *sets* N g ‘ (*space* N) $\in$ *sets* M
and *measurable-isomorphic-map* M (*restrict-space* N (f ‘ (*space* M))) f **and** *measurable-isomorphic-map* N (*restrict-space* M (g ‘ (*space* N))) g
shows $\exists h.$ *measurable-isomorphic-map* $M\ N\ h$
⟨*proof*⟩

lemma *Schroeder-Bernstein-measurable*:
assumes $f \in M \rightarrow_M N$ $\bigwedge A.\ A \in \mathit{sets}\ M \Longrightarrow f$ ‘ $A \in$ *sets* N *inj-on* f (*space* M)
and $g \in N \rightarrow_M M$ $\bigwedge A.\ A \in \mathit{sets}\ N \Longrightarrow g$ ‘ $A \in$ *sets* M *inj-on* g (*space* N)
shows $\exists h.$ *measurable-isomorphic-map* $M\ N\ h$
⟨*proof*⟩

lemma *measurable-isomorphic-from-embeddings*:
assumes M *measurable-isomorphic* (*restrict-space* $N\ B$) N *measurable-isomorphic* (*restrict-space* $M\ A$)
and $A \in$ *sets* M $B \in$ *sets* N

shows *M measurable-isomorphic N*
⟨*proof*⟩

lemma *measurable-isomorphic-antisym*:
assumes *B measurable-isomorphic* (*restrict-space C c*) *A measurable-isomorphic* (*restrict-space B b*)
and $c \in$ *sets C* $b \in$ *sets B C measurable-isomorphic A*
shows *C measurable-isomorphic B*
⟨*proof*⟩

lemma *countable-infinite-isomorphisc-to-nat-index*:
assumes *countable I* **and** *infinite I*
shows (Π_M $x{\in}I$. *M*) *measurable-isomorphic* (Π_M (x::*nat*)$\in$*UNIV*. *M*)
⟨*proof*⟩

lemma *PiM-PiM-isomorphic-to-PiM*:
(Π_M $i{\in}I$. Π_M $j{\in}J$. *M i j*) *measurable-isomorphic* (Π_M $(i,j){\in}I{\times}J$. *M i j*)
⟨*proof*⟩

lemma *measurable-isomorphic-map-sigma-sets*:
assumes *sets M* = *sigma-sets* (*space M*) *U measurable-isomorphic-map M N f*
shows *sets N* = *sigma-sets* (*space N*) ((') *f* ' *U*)
⟨*proof*⟩

1.2.3 Borel Spaces Genereted from Abstract Topologies

definition *borel-of* :: *'a topology* ⇒ *'a measure* **where**
borel-of X ≡ *sigma* (*topspace X*) {*U*. *openin X U*}

lemma *emeasure-borel-of*: *emeasure* (*borel-of X*) *A* = *0*
⟨*proof*⟩

lemma *borel-of-euclidean*: *borel-of euclidean* = *borel*
⟨*proof*⟩

lemma *space-borel-of*: *space* (*borel-of X*) = *topspace X*
⟨*proof*⟩

lemma *sets-borel-of*: *sets* (*borel-of X*) = *sigma-sets* (*topspace X*) {*U*. *openin X U*}
⟨*proof*⟩

lemma *sets-borel-of-closed*: *sets* (*borel-of X*) = *sigma-sets* (*topspace X*) {*U*. *closedin X U*}
⟨*proof*⟩

lemma *borel-of-open*:
assumes *openin X U*
shows $U \in$ *sets* (*borel-of X*)

⟨*proof*⟩

lemma *borel-of-closed*:
assumes *closedin X U*
shows $U \in$ *sets* (*borel-of X*)
⟨*proof*⟩

lemma(**in** *Metric-space*) *nbh-sets*[*measurable*]: ($\bigcup a \in A$. *mball a e*) $\in$ *sets* (*borel-of mtopology*)
⟨*proof*⟩

lemma *borel-of-gdelta-in*:
assumes *gdelta-in X U*
shows $U \in$ *sets* (*borel-of X*)
⟨*proof*⟩

lemma *borel-of-subtopology*:
borel-of (*subtopology X U*) = *restrict-space* (*borel-of X*) *U*
⟨*proof*⟩

lemma *sets-borel-of-discrete-topology*: *sets* (*borel-of* (*discrete-topology I*)) = *sets* (*count-space I*)
⟨*proof*⟩

lemma *continuous-map-measurable*:
assumes *continuous-map X Y f*
shows $f \in$ *borel-of X* $\rightarrow_M$ *borel-of Y*
⟨*proof*⟩

lemma *upper-semicontinuous-map-measurable*:
fixes *f* :: *'a* $\Rightarrow$ *'b* :: {*linorder-topology*, *second-countable-topology*}
assumes *upper-semicontinuous-map X f*
shows $f \in$ *borel-measurable* (*borel-of X*)
⟨*proof*⟩

lemma *lower-semicontinuous-map-measurable*:
fixes *f* :: *'a* $\Rightarrow$ *'b* :: {*linorder-topology*, *second-countable-topology*}
assumes *lower-semicontinuous-map X f*
shows $f \in$ *borel-measurable* (*borel-of X*)
⟨*proof*⟩

lemma *open-map-preserves-sets*:
assumes *open-map S T f inj-on f* (*topspace S*) $A \in$ *sets* (*borel-of S*)
shows *f* ‘ $A \in$ *sets* (*borel-of T*)
⟨*proof*⟩

lemma *open-map-preserves-sets'*:
assumes *open-map S* (*subtopology T* (*f* ‘ (*topspace S*))) *f inj-on f* (*topspace S*) *f* ‘ (*topspace S*) $\in$ *sets* (*borel-of T*) $A \in$ *sets* (*borel-of S*)

shows f ‘ $A \in$ *sets* (*borel-of* T)
⟨*proof*⟩

Abstract topology version of *open* = *generate-topology ?X* $\Longrightarrow$ *borel* = *sigma UNIV ?X*.

lemma *borel-of-second-countable*′:
assumes *second-countable* S **and** *subbase-in* S $\mathcal{U}$
shows *borel-of* S = *sigma* (*topspace* S) $\mathcal{U}$
⟨*proof*⟩

Abstract topology version *borel* $\bigotimes_M$ *borel* = *borel*.

lemma *borel-of-prod*:
assumes *second-countable* S **and** *second-countable* S'
shows *borel-of* S $\bigotimes_M$ *borel-of* S' = *borel-of* (*prod-topology* S S')
⟨*proof*⟩

lemma *product-borel-of-measurable*:
assumes $i \in I$
shows ($\lambda x.\ x\ i$) $\in$ (*borel-of* (*product-topology* S I)) $\rightarrow_M$ *borel-of* (S i)
⟨*proof*⟩

Abstract topology version of *sets* (Pi_M *UNIV* (λ-. *borel*)) $\subseteq$ *sets borel*

lemma *sets-PiM-subset-borel-of*:
sets (Π_M $i{\in}I$. *borel-of* (S i)) $\subseteq$ *sets* (*borel-of* (*product-topology* S I))
⟨*proof*⟩

Abstract topology version of *sets* (Pi_M *UNIV* (λi. *borel*)) = *sets borel*.

lemma *sets-PiM-equal-borel-of*:
assumes *countable* I **and** $\bigwedge i.\ i \in I \Longrightarrow$ *second-countable* (S i)
shows *sets* (Π_M $i{\in}I$. *borel-of* (S i)) = *sets* (*borel-of* (*product-topology* S I))
⟨*proof*⟩

lemma *homeomorphic-map-borel-isomorphic*:
assumes *homeomorphic-map* X Y f
shows *measurable-isomorphic-map* (*borel-of* X) (*borel-of* Y) f
⟨*proof*⟩

lemma *homeomorphic-space-measurable-isomorphic*:
assumes S *homeomorphic-space* T
shows *borel-of* S *measurable-isomorphic* *borel-of* T
⟨*proof*⟩

lemma *measurable-isomorphic-borel-map*:
assumes *sets* M = *sets* (*borel-of* S) **and** f: *measurable-isomorphic-map* M N f
shows $\exists S'$. *homeomorphic-map* S S' f $\wedge$ *sets* N = *sets* (*borel-of* S')
⟨*proof*⟩

lemma *measurable-isomorphic-borels*:

assumes *sets M* = *sets* (*borel-of S*) *M measurable-isomorphic N*
shows $\exists S'$. *S homeomorphic-space* $S' \wedge$ *sets N* = *sets* (*borel-of* S')
⟨*proof*⟩

end

1.3 Lemmas for Abstract Metric Spaces

theory *Set-Based-Metric-Space*
imports *Lemmas-StandardBorel*
begin

We prove additional lemmas related to set-based metric spaces.

1.3.1 Basic Lemmas

lemma
assumes *Metric-space M d* $\bigwedge x\ y.\ x \in M \Longrightarrow y \in M \Longrightarrow d\ x\ y = d'\ x\ y$
and $\bigwedge x\ y.\ d'\ x\ y = d'\ y\ x$ $\bigwedge x\ y.\ d'\ x\ y \geq 0$
shows *Metric-space-eq*: *Metric-space M* d'
and *Metric-space-eq-mtopology*: *Metric-space.mtopology M d* = *Metric-space.mtopology M* d'
and *Metric-space-eq-mcomplete*: *Metric-space.mcomplete M d* $\longleftrightarrow$ *Metric-space.mcomplete M* d'
⟨*proof*⟩

context *Metric-space*
begin

lemma *mtopology-base-in-balls*: *base-in mtopology* $\{mball\ a\ \varepsilon \mid a\ \varepsilon.\ a \in M \wedge \varepsilon > 0\}$
⟨*proof*⟩

lemma *closedin-metric2*: *closedin mtopology C* $\longleftrightarrow C \subseteq M \wedge (\forall x.\ x \in C \longleftrightarrow (\forall \varepsilon>0.\ mball\ x\ \varepsilon \cap C \neq \{\}))$
⟨*proof*⟩

lemma *openin-mtopology2*:
openin mtopology U $\longleftrightarrow U \subseteq M \wedge (\forall xn\ x.$ *limitin mtopology xn x sequentially* $\wedge\ x \in U \longrightarrow (\exists N.\ \forall n \geq N.\ xn\ n \in U))$
⟨*proof*⟩

lemma *closure-of-mball*: *mtopology closure-of mball a e* $\subseteq$ *mcball a e*
⟨*proof*⟩

lemma *interior-of-mcball*: *mball a e* $\subseteq$ *mtopology interior-of mcball a e*
⟨*proof*⟩

lemma *isolated-points-of-mtopology*:

mtopology isolated-points-of $A = \{x \in M \cap A.\ \forall xn.\ range\ xn \subseteq A \wedge limitin\ mtopology\ xn\ x\ sequentially \longrightarrow (\exists no.\ \forall n \geq no.\ xn\ n = x)\}$
⟨*proof*⟩

lemma *perfect-set-mball-infinite*:
assumes *perfect-set mtopology A* $a \in A$ $e > 0$
shows *infinite* (*mball a e*)
⟨*proof*⟩

lemma *MCauchy-dist-Cauchy*:
assumes *MCauchy xn MCauchy yn*
shows *Cauchy* ($\lambda n.\ d\ (xn\ n)\ (yn\ n)$)
⟨*proof*⟩

1.3.2 Dense in Metric Spaces

abbreviation *mdense* ≡ *dense-in mtopology*

https://people.bath.ac.uk/mw2319/ma30252/sec-dense.html

lemma *mdense-def*:
$mdense\ U \longleftrightarrow U \subseteq M \wedge (\forall x \in M.\ \forall \varepsilon > 0.\ mball\ x\ \varepsilon \cap U \neq \{\})$
⟨*proof*⟩

corollary *mdense-balls-cover*:
assumes *mdense U* **and** $e > 0$
shows $(\bigcup u \in U.\ mball\ u\ e) = M$
⟨*proof*⟩

lemma *mdense-empty-iff*: $mdense\ \{\} \longleftrightarrow M = \{\}$
⟨*proof*⟩

lemma *mdense-M*: *mdense M*
⟨*proof*⟩

lemma *mdense-def2*:
$mdense\ U \longleftrightarrow U \subseteq M \wedge (\forall x \in M.\ \forall \varepsilon > 0. \exists y \in U.\ d\ x\ y < \varepsilon)$
⟨*proof*⟩

lemma *mdense-def3*:
$mdense\ U \longleftrightarrow U \subseteq M \wedge (\forall x \in M.\ \exists xn.\ range\ xn \subseteq U \wedge limitin\ mtopology\ xn\ x\ sequentially)$
⟨*proof*⟩

Diameter

definition *mdiameter* :: *'a set* ⇒ *ennreal* **where**
$mdiameter\ A \equiv \bigsqcup \{ennreal\ (d\ x\ y) \mid x\ y.\ x \in A \cap M \wedge y \in A \cap M\}$

lemma *mdiameter-empty*[*simp*]:
$mdiameter\ \{\} = 0$

⟨*proof*⟩

lemma *mdiameter-def2*:
assumes $A \subseteq M$
shows $mdiameter\ A = \bigsqcup\ \{ennreal\ (d\ x\ y) \mid x\ y.\ x \in A \wedge y \in A\}$
⟨*proof*⟩

lemma *mdiameter-subset*:
assumes $A \subseteq B$
shows $mdiameter\ A \leq mdiameter\ B$
⟨*proof*⟩

lemma *mdiameter-cball-leq*: $mdiameter\ (mcball\ a\ \varepsilon) \leq ennreal\ (2 * \varepsilon)$
⟨*proof*⟩

lemma *mdiameter-ball-leq*:
$mdiameter\ (mball\ a\ \varepsilon) \leq ennreal\ (2 * \varepsilon)$
⟨*proof*⟩

lemma *mdiameter-is-sup*:
assumes $x \in A \cap M\ y \in A \cap M$
shows $d\ x\ y \leq mdiameter\ A$
⟨*proof*⟩

lemma *mdiameter-is-sup'*:
assumes $x \in A \cap M\ y \in A \cap M\ mdiameter\ A \leq ennreal\ r\ r \geq 0$
shows $d\ x\ y \leq r$
⟨*proof*⟩

lemma *mdiameter-le*:
assumes $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow d\ x\ y \leq r$
shows $mdiameter\ A \leq r$
⟨*proof*⟩

lemma *mdiameter-eq-closure*: $mdiameter\ (mtopology\ closure\text{-}of\ A) = mdiameter\ A$
⟨*proof*⟩

lemma *mbounded-finite-mdiameter*: $mbounded\ A \longleftrightarrow A \subseteq M \wedge mdiameter\ A < \infty$
⟨*proof*⟩

Distance between a point and a set.

definition *d-set* :: $'a\ set \Rightarrow 'a \Rightarrow real$ **where**
$d\text{-}set\ A \equiv (\lambda x.\ if\ A \neq \{\} \wedge A \subseteq M \wedge x \in M\ then\ Inf\ \{d\ x\ y\ |y.\ y \in A\}\ else\ 0)$

lemma *d-set-nonneg*[*simp*]:
$d\text{-}set\ A\ x \geq 0$
⟨*proof*⟩

lemma *d-set-bdd-below*[*simp*]:
bdd-below $\{d\ x\ y\ |y.\ y \in A\}$
⟨*proof*⟩

lemma *d-set-singleton*[*simp*]:
$x \in M \Longrightarrow y \in M \Longrightarrow$ *d-set* $\{y\}\ x = d\ x\ y$
⟨*proof*⟩

lemma *d-set-empty*[*simp*]:
d-set $\{\}\ x = 0$
⟨*proof*⟩

lemma *d-set-notin*:
$x \notin M \Longrightarrow$ *d-set* $A\ x = 0$
⟨*proof*⟩

lemma *d-set-inA*:
assumes $x \in A$
shows *d-set* $A\ x = 0$
⟨*proof*⟩

lemma *d-set-nzeroD*:
assumes *d-set* $A\ x \neq 0$
shows $A \subseteq M\ x \notin A\ A \neq \{\}$
⟨*proof*⟩

lemma *d-set-antimono*:
assumes $A \subseteq B\ A \neq \{\}\ B \subseteq M$
shows *d-set* $B\ x \leq$ *d-set* $A\ x$
⟨*proof*⟩

lemma *d-set-bounded*:
assumes $\bigwedge y.\ y \in A \Longrightarrow d\ x\ y < K\ K > 0$
shows *d-set* $A\ x < K$
⟨*proof*⟩

lemma *d-set-tr*:
assumes $x \in M\ y \in M$
shows *d-set* $A\ x \leq d\ x\ y +$ *d-set* $A\ y$
⟨*proof*⟩

lemma *d-set-abs-le*:
assumes $x \in M\ y \in M$
shows $|$*d-set* $A\ x -$ *d-set* $A\ y| \leq d\ x\ y$
⟨*proof*⟩

lemma *d-set-inA-le*:
assumes $y \in A$

shows *d-set A x* $\leq$ *d x y*
⟨*proof*⟩

lemma *d-set-ball-empty*:
assumes $A \neq \{\}$ $A \subseteq M$ $e > 0$ $x \in M$ *mball x e* $\cap$ $A = \{\}$
shows *d-set A x* $\geq$ *e*
⟨*proof*⟩

lemma *d-set-closed-pos*:
assumes *closedin mtopology A* $A \neq \{\}$ $x \in M$ $x \notin A$
shows *d-set A x* > 0
⟨*proof*⟩

lemma *gdelta-in-closed*:
assumes *closedin mtopology M*
shows *gdelta-in mtopology M*
⟨*proof*⟩

Oscillation

definition *osc-on* :: ['b set, 'b topology, 'b ⇒ 'a, 'b] ⇒ *ennreal* **where**
osc-on A X f ≡ (λ*y*. ⨅ {*mdiameter* (*f* ‘ (*A* ∩ *U*)) |*U*. *y* ∈ *U* ∧ *openin X U*})

abbreviation *osc X* ≡ *osc-on* (*topspace X*) *X*

lemma *osc-def*: *osc X f* = (λ*y*. ⨅ {*mdiameter* (*f* ‘ *U*) |*U*. *y* ∈ *U* ∧ *openin X U*})
⟨*proof*⟩

lemma *osc-on-less-iff*:
osc-on A X f x < *t* ⟷ (∃ *v*. *x* ∈ *v* ∧ *openin X v* ∧ *mdiameter* (*f* ‘ (*A* ∩ *v*)) < *t*)
⟨*proof*⟩

lemma *osc-less-iff*:
osc X f x < *t* ⟷ (∃ *v*. *x* ∈ *v* ∧ *openin X v* ∧ *mdiameter* (*f* ‘ *v*) < *t*)
⟨*proof*⟩

end

definition *mdist-set* :: 'a metric ⇒ 'a set ⇒ 'a ⇒ *real* **where**
mdist-set m ≡ *Metric-space.d-set* (*mspace m*) (*mdist m*)

lemma(**in** *Metric-space*) *mdist-set-Self*: *mdist-set Self* = *d-set*
⟨*proof*⟩

lemma *mdist-set-nonneg*[*simp*]: *mdist-set m A x* ≥ 0
⟨*proof*⟩

lemma *mdist-set-singleton*[*simp*]:
x ∈ *mspace m* ⟹ *y* ∈ *mspace m* ⟹ *mdist-set m* {*y*} *x* = *mdist m x y*
⟨*proof*⟩

lemma *mdist-set-empty*[*simp*]: *mdist-set m* {} *x* = *0*
⟨*proof*⟩

lemma *mdist-set-inA*:
assumes $x \in A$
shows *mdist-set m A x* = *0*
⟨*proof*⟩

lemma *mdist-set-nzeroD*:
assumes *mdist-set m A x* ≠ *0*
shows $A \subseteq$ *mspace m* $x \notin A$ $A \neq \{\}$
⟨*proof*⟩

lemma *mdist-set-antimono*:
assumes $A \subseteq B$ $A \neq \{\}$ $B \subseteq$ *mspace m*
shows *mdist-set m B x* ≤ *mdist-set m A x*
⟨*proof*⟩

lemma *mdist-set-bounded*:
assumes $\bigwedge y.\ y \in A \Longrightarrow$ *mdist m x y* < *K* *K* > *0*
shows *mdist-set m A x* < *K*
⟨*proof*⟩

lemma *mdist-set-tr*:
assumes $x \in$ *mspace m* $y \in$ *mspace m*
shows *mdist-set m A x* ≤ *mdist m x y* + *mdist-set m A y*
⟨*proof*⟩

lemma *mdist-set-abs-le*:
assumes $x \in$ *mspace m* $y \in$ *mspace m*
shows |*mdist-set m A x* − *mdist-set m A y*| ≤ *mdist m x y*
⟨*proof*⟩

lemma *mdist-set-inA-le*:
assumes $y \in A$
shows *mdist-set m A x* ≤ *mdist m x y*
⟨*proof*⟩

lemma *mdist-set-ball-empty*:
assumes $A \neq \{\}$ $A \subseteq$ *mspace m* *e* > *0* $x \in$ *mspace m* *mball-of m x e* ∩ *A* = {}
shows *mdist-set m A x* ≥ *e*
⟨*proof*⟩

lemma *mdist-set-closed-pos*:
assumes *closedin* (*mtopology-of m*) *A* $A \neq \{\}$ $x \in$ *mspace m* $x \notin A$
shows *mdist-set m A x* > *0*
⟨*proof*⟩

lemma *mdist-set-uniformly-continuous*: *uniformly-continuous-map m euclidean-metric* (*mdist-set m A*)
⟨*proof*⟩

lemma *uniformly-continuous-map-add*:
fixes $f :: 'a \Rightarrow 'b::real\text{-}normed\text{-}vector$
assumes *uniformly-continuous-map m euclidean-metric f uniformly-continuous-map m euclidean-metric g*
shows *uniformly-continuous-map m euclidean-metric* ($\lambda x.\ f\ x + g\ x$)
⟨*proof*⟩

lemma *uniformly-continuous-map-real-divide*:
fixes $f :: 'a \Rightarrow real$
assumes *uniformly-continuous-map m euclidean-metric f uniformly-continuous-map m euclidean-metric g*
and $\bigwedge x.\ x \in mspace\ m \Longrightarrow g\ x \neq 0$ $\bigwedge x.\ x \in mspace\ m \Longrightarrow |g\ x| \geq a$ $a > 0$ $\bigwedge x.\ x \in mspace\ m \Longrightarrow |g\ x| < Kg$
and $\bigwedge x.\ x \in mspace\ m \Longrightarrow |f\ x| < Kf$
shows *uniformly-continuous-map m euclidean-metric* ($\lambda x.\ f\ x\ /\ g\ x$)
⟨*proof*⟩

lemma
assumes $e > 0$
shows *uniformly-continuous-map-from-capped-metric*:*uniformly-continuous-map* (*capped-metric e m1*) *m2 f* $\longleftrightarrow$ *uniformly-continuous-map m1 m2 f* (**is** *?g1*)
and *uniformly-continuous-map-to-capped-metric*:*uniformly-continuous-map m1* (*capped-metric e m2*) *f* $\longleftrightarrow$ *uniformly-continuous-map m1 m2 f* (**is** *?g2*)
⟨*proof*⟩

lemma *Urysohn-lemma-uniform*:
assumes *closedin* (*mtopology-of m*) *T closedin* (*mtopology-of m*) *U* $T \cap U = \{\}$ $\bigwedge x\ y.\ x \in T \Longrightarrow y \in U \Longrightarrow mdist\ m\ x\ y \geq e$ $e > 0$
obtains $f :: 'a \Rightarrow real$
where *uniformly-continuous-map m euclidean-metric f*
$\bigwedge x.\ f\ x \geq 0$ $\bigwedge x.\ f\ x \leq 1$ $\bigwedge x.\ x \in T \Longrightarrow f\ x = 1$ $\bigwedge x.\ x \in U \Longrightarrow f\ x = 0$
⟨*proof*⟩

Open maps

lemma *Metric-space-open-map-from-dist*:
assumes $f \in mspace\ m1 \rightarrow mspace\ m2$
and $\bigwedge x\ \varepsilon.\ x \in mspace\ m1 \Longrightarrow \varepsilon > 0 \Longrightarrow \exists \delta>0.\ \forall y \in mspace\ m1.\ mdist\ m2\ (f\ x)\ (f\ y) < \delta \longrightarrow mdist\ m1\ x\ y < \varepsilon$
shows *open-map* (*mtopology-of m1*) (*subtopology* (*mtopology-of m2*) (*f ‘ mspace m1*)) *f*
⟨*proof*⟩

1.3.3 Separability in Metric Spaces

context *Metric-space*

begin

For a metric space M, M is separable iff M is second countable.

lemma *generated-by-countable-balls*:
assumes *countable U* **and** *mdense U*
shows *mtopology = topology-generated-by* {*mball y* (*1* / *real n*) | *y n. y* ∈ *U*}
⟨*proof*⟩

lemma *separable-space-imp-second-countable*:
assumes *separable-space mtopology*
shows *second-countable mtopology*
⟨*proof*⟩

corollary *separable-space-iff-second-countable*:
separable-space mtopology ⟷ *second-countable mtopology*
⟨*proof*⟩

lemma *Lindelof-mdiameter*:
assumes *separable-space mtopology 0* < *e*
shows ∃ *U. countable U* ∧ ⋃ *U* = *M* ∧ (∀ *u*∈*U. mdiameter u* < *ennreal e*)
⟨*proof*⟩

end

lemma *metrizable-space-separable-iff-second-countable*:
assumes *metrizable-space X*
shows *separable-space X* ⟷ *second-countable X*
⟨*proof*⟩

abbreviation *mdense-of m U* ≡ *dense-in* (*mtopology-of m*) *U*

lemma *mdense-of-def*: *mdense-of m U* ⟷ (*U* ⊆ *mspace m* ∧ (∀ *x*∈*mspace m*. ∀ *ε*>*0*. *mball-of m x ε* ∩ *U* ≠ {}))
⟨*proof*⟩

lemma *mdense-of-def2*: *mdense-of m U* ⟷ (*U* ⊆ *mspace m* ∧ (∀ *x*∈*mspace m*. ∀ *ε*>*0*. ∃ *y*∈*U. mdist m x y* < *ε*))
⟨*proof*⟩

lemma *mdense-of-def3*: *mdense-of m U* ⟷ (*U* ⊆ *mspace m* ∧ (∀ *x*∈*mspace m*. ∃ *xn. range xn* ⊆ *U* ∧ *limitin* (*mtopology-of m*) *xn x sequentially*))
⟨*proof*⟩

1.3.4 Compact Metric Spaces

context *Metric-space*
begin

lemma *mtotally-bounded-eq-compact-closedin*:

assumes *mcomplete closedin mtopology S*
shows *mtotally-bounded S* ⟷ *S* ⊆ *M* ∧ *compactin mtopology S*
⟨*proof*⟩

lemma *mtotally-bounded-def2*: *mtotally-bounded S* ⟷ (∀*ε*>*0*. ∃*K*. *finite K* ∧ *K* ⊆ *M* ∧ *S* ⊆ (⋃*x*∈*K*. *mball x ε*))
⟨*proof*⟩

lemma *compact-space-imp-separable*:
assumes *compact-space mtopology*
shows *separable-space mtopology*
⟨*proof*⟩

lemma *separable-space-cfunspace*:
assumes *separable-space mtopology mcomplete*
and *metrizable-space X compact-space X*
shows *separable-space* (*mtopology-of* (*cfunspace X Self*))
⟨*proof*⟩

end

context *Submetric*
begin

lemma *separable-sub*:
assumes *separable-space mtopology*
shows *separable-space sub.mtopology*
⟨*proof*⟩

end

1.3.5 Discrete Distance

lemma(**in** *discrete-metric*) *separable-space-iff*: *separable-space disc.mtopology* ⟷ *countable M*
⟨*proof*⟩

1.3.6 Binary Product Metric Spaces

We define the L^1-distance. L^1-distance and L^2 distance (Euclid distance) generate the same topological space.

definition *prod-dist-L1* ≡ λ*d1 d2* (*x*,*y*) (*x*′,*y*′). *d1 x x*′ + *d2 y y*′

context *Metric-space12*
begin

lemma *prod-L1-metric*: *Metric-space* (*M1* × *M2*) (*prod-dist-L1 d1 d2*)
⟨*proof*⟩

sublocale *Prod-metric-L1*: *Metric-space M1* $\times$ *M2 prod-dist-L1 d1 d2*
⟨*proof*⟩

lemma *prod-dist-L1-geq*:
shows *d1 x y* $\leq$ *prod-dist-L1 d1 d2* (x,x') (y,y')
d2 x' y' $\leq$ *prod-dist-L1 d1 d2* (x,x') (y,y')
⟨*proof*⟩

lemma *prod-dist-L1-ball*:
assumes $(x,x') \in$ *Prod-metric-L1.mball* (a,a') ε
shows $x \in$ *M1.mball a* ε
and $x' \in$ *M2.mball* a' ε
⟨*proof*⟩

lemma *prod-dist-L1-ball'*:
assumes $z \in$ *Prod-metric-L1.mball a* ε
shows *fst z* $\in$ *M1.mball* (*fst a*) ε
and *snd z* $\in$ *M2.mball* (*snd a*) ε
⟨*proof*⟩

lemma *prod-dist-L1-ball1'*: *Prod-metric-L1.mball* (*a1*,*a2*) (*min e1 e2*) $\subseteq$ *M1.mball a1 e1* $\times$ *M2.mball a2 e2*
⟨*proof*⟩

lemma *prod-dist-L1-ball1*:
assumes *b1* $\in$ *M1.mball a1 e1 b2* $\in$ *M2.mball a2 e2*
shows $\exists$ *e12*>*0*. *Prod-metric-L1.mball* (*b1*,*b2*) *e12* $\subseteq$ *M1.mball a1 e1* $\times$ *M2.mball a2 e2*
⟨*proof*⟩

lemma *prod-dist-L1-ball2'*:
M1.mball a1 e1 $\times$ *M2.mball a2 e2* $\subseteq$ *Prod-metric-L1.mball* (*a1*,*a2*) (*e1* + *e2*)
⟨*proof*⟩

lemma *prod-dist-L1-ball2*:
assumes (*b1*,*b2*) $\in$ *Prod-metric-L1.mball* (*a1*,*a2*) *e12*
shows $\exists$ *e1*>*0*. $\exists$ *e2*>*0*. *M1.mball b1 e1* $\times$ *M2.mball b2 e2* $\subseteq$ *Prod-metric-L1.mball* (*a1*,*a2*) *e12*
⟨*proof*⟩

lemma *prod-dist-L1-mtopology*:
Prod-metric-L1.mtopology = *prod-topology M1.mtopology M2.mtopology*
⟨*proof*⟩

lemma *prod-dist-L1-limitin-iff*: *limitin Prod-metric-L1.mtopology zn z sequentially* $\longleftrightarrow$ *limitin M1.mtopology* (λ*n*. *fst* (*zn n*)) (*fst z*) *sequentially* $\wedge$ *limitin M2.mtopology* (λ*n*. *snd* (*zn n*)) (*snd z*) *sequentially*
⟨*proof*⟩

lemma *prod-dist-L1-MCauchy-iff*: *Prod-metric-L1.MCauchy zn* $\longleftrightarrow$ *M1.MCauchy* $(\lambda n.\ fst\ (zn\ n)) \wedge$ *M2.MCauchy* $(\lambda n.\ snd\ (zn\ n))$
⟨*proof*⟩

end

1.3.7 Sum Metric Spaces

locale *Sum-metric* =
  fixes I :: $'i$ *set*
    and *Mi* :: $'i \Rightarrow 'a$ *set*
    and *di* :: $'i \Rightarrow 'a \Rightarrow 'a \Rightarrow$ *real*
  assumes *Mi-disj*: *disjoint-family-on Mi I*
      and *d-nonneg*: $\bigwedge i\ x\ y.\ 0 \leq di\ i\ x\ y$
      and *d-bounded*: $\bigwedge i\ x\ y.\ di\ i\ x\ y < 1$
      and *Md-metric*: $\bigwedge i.\ i \in I \Longrightarrow$ *Metric-space* $(Mi\ i)\ (di\ i)$
begin

abbreviation $M \equiv \bigcup i{\in}I.\ Mi\ i$

lemma *Mi-inj-on*:
  assumes $i \in I$ $j \in I$ $a \in Mi\ i$ $a \in Mi\ j$
  shows $i = j$
  ⟨*proof*⟩

definition *sum-dist* :: $['a, 'a] \Rightarrow real$ **where**
sum-dist $x\ y \equiv (if\ x \in M \wedge y \in M\ then\ (if\ \exists i{\in}I.\ x \in Mi\ i \wedge y \in Mi\ i\ then\ di\ (THE\ i.\ i \in I \wedge x \in Mi\ i \wedge y \in Mi\ i)\ x\ y\ else\ 1)\ else\ 0)$

lemma *sum-dist-simps*:
  shows $\bigwedge i.\ [\![i \in I;\ x \in Mi\ i;\ y \in Mi\ i]\!] \Longrightarrow sum\text{-}dist\ x\ y = di\ i\ x\ y$
    and $\bigwedge i\ j.\ [\![i \in I;\ j \in I;\ i \neq j;\ x \in Mi\ i;\ y \in Mi\ j]\!] \Longrightarrow sum\text{-}dist\ x\ y = 1$
    and $\bigwedge i.\ [\![i \in I;\ y \in M;\ x \in Mi\ i;\ y \notin Mi\ i]\!] \Longrightarrow sum\text{-}dist\ x\ y = 1$
    and $\bigwedge i.\ [\![i \in I;\ x \in M;\ y \in Mi\ i;\ x \notin Mi\ i]\!] \Longrightarrow sum\text{-}dist\ x\ y = 1$
    and $x \notin M \Longrightarrow sum\text{-}dist\ x\ y = 0$ $y \notin M \Longrightarrow sum\text{-}dist\ x\ y = 0$
⟨*proof*⟩

lemma *sum-dist-if-less1*:
  assumes $i \in I$ $x \in Mi\ i$ $y \in M$ *sum-dist* $x\ y < 1$
  shows $y \in Mi\ i$
  ⟨*proof*⟩

lemma *inM-cases*:
  assumes $x \in M$ $y \in M$
      and $\bigwedge i.\ [\![i \in I;\ x \in Mi\ i;\ y \in Mi\ i]\!] \Longrightarrow P\ x\ y$
      and $\bigwedge i\ j.\ [\![i \in I;\ j \in I;\ i \neq j;\ x \in Mi\ i;\ y \in Mi\ j;\ x \neq y]\!] \Longrightarrow P\ x\ y$
    shows $P\ x\ y$ ⟨*proof*⟩

sublocale *Sum-metric*: *Metric-space M sum-dist*

⟨proof⟩

lemma *sum-dist-le1*: *sum-dist x y* ≤ *1*
⟨proof⟩

lemma *sum-dist-ball-eq-ball*:
assumes *i* ∈ *I e* ≤ *1 x* ∈ *Mi i*
shows *Metric-space.mball* (*Mi i*) (*di i*) *x e* = *Sum-metric.mball x e*
⟨proof⟩

lemma *ball-le-sum-dist-ball*:
assumes *i* ∈ *I*
shows *Metric-space.mball* (*Mi i*) (*di i*) *x e* ⊆ *Sum-metric.mball x e*
⟨proof⟩

lemma *openin-mtopology-iff*:
openin Sum-metric.mtopology U ⟷ *U* ⊆ *M* ∧ (∀ *i*∈*I*. *openin* (*Metric-space.mtopology* (*Mi i*) (*di i*)) (*U* ∩ *Mi i*))
⟨proof⟩

corollary *openin-mtopology-Mi*:
assumes *i* ∈ *I*
shows *openin Sum-metric.mtopology* (*Mi i*)
⟨proof⟩

corollary *subtopology-mtopology-Mi*:
assumes *i* ∈ *I*
shows *subtopology Sum-metric.mtopology* (*Mi i*) = *Metric-space.mtopology* (*Mi i*) (*di i*)
⟨proof⟩

lemma *limitin-Mi-limitin-M*:
assumes *i* ∈ *I limitin* (*Metric-space.mtopology* (*Mi i*) (*di i*)) *xn x sequentially*
shows *limitin Sum-metric.mtopology xn x sequentially*
⟨proof⟩

lemma *limitin-M-limitin-Mi*:
assumes *limitin Sum-metric.mtopology xn x sequentially*
shows ∃ *i*∈*I*. *limitin* (*Metric-space.mtopology* (*Mi i*) (*di i*)) *xn x sequentially*
⟨proof⟩

lemma *MCauchy-Mi-MCauchy-M*:
assumes *i* ∈ *I Metric-space.MCauchy* (*Mi i*) (*di i*) *xn*
shows *Sum-metric.MCauchy xn*
⟨proof⟩

lemma *MCauchy-M-MCauchy-Mi*:
assumes *Sum-metric.MCauchy xn*

shows $\exists$ *m*. $\exists$ *i*$\in$*I*. *Metric-space.MCauchy* (*Mi i*) (*di i*) (λ*n*. *xn* (*n* + *m*))
⟨*proof*⟩

lemma *separable-Mi-separable-M*:
assumes *countable I* $\bigwedge$*i*. *i* $\in$ *I* $\Longrightarrow$ *separable-space* (*Metric-space.mtopology* (*Mi i*) (*di i*))
shows *separable-space Sum-metric.mtopology*
⟨*proof*⟩

lemma *separable-M-separable-Mi*:
assumes *separable-space Sum-metric.mtopology* $\bigwedge$*i*. *i* $\in$ *I*
shows *separable-space* (*Metric-space.mtopology* (*Mi i*) (*di i*))
⟨*proof*⟩

lemma *mcomplete-Mi-mcomplete-M*:
assumes $\bigwedge$*i*. *i* $\in$ *I* $\Longrightarrow$ *Metric-space.mcomplete* (*Mi i*) (*di i*)
shows *Sum-metric.mcomplete*
⟨*proof*⟩

lemma *mcomplete-M-mcomplete-Mi*:
assumes *Sum-metric.mcomplete i* $\in$ *I*
shows *Metric-space.mcomplete* (*Mi i*) (*di i*)
⟨*proof*⟩

end

lemma *sum-metricI*:
fixes *Si*
assumes *disjoint-family-on Si I*
and $\bigwedge$*i x y*. *i* $\notin$ *I* $\Longrightarrow$ *0* $\leq$ *di i x y*
and $\bigwedge$*i x y*. *di i x y* < *1*
and $\bigwedge$*i*. *i* $\in$ *I* $\Longrightarrow$ *Metric-space* (*Si i*) (*di i*)
shows *Sum-metric I Si di*
⟨*proof*⟩

end

1.3.8 Product Metric Spaces

theory *Set-Based-Metric-Product*
imports *Set-Based-Metric-Space*
begin

lemma *nsum-of-r'*:
fixes *r* :: *real*
assumes *r*:*0* < *r r* < *1*
shows ($\sum$*n*. *r*^(*n* + *k*) $*$ *K*) = *r*^*k* / (*1* − *r*) $*$ *K*

(**is** *?lhs* = -)
⟨*proof*⟩

lemma *nsum-of-r-leq*:
fixes r :: *real* **and** a :: *nat* ⇒ *real*
assumes r:$0 < r$ $r < 1$
and a:$\bigwedge n.\ 0 \leq a\ n$ $\bigwedge n.\ a\ n \leq K$
shows $0 \leq (\sum n.\ r\hat{}(n + k) * a\ (n + l))$ $(\sum n.\ r\hat{}(n + k) * a\ (n + l)) \leq r\hat{}k\ /\ (1 - r) * K$
⟨*proof*⟩

lemma *nsum-of-r-le*:
fixes r :: *real* **and** a :: *nat* ⇒ *real*
assumes r:$0 < r$ $r < 1$
and a:$\bigwedge n.\ 0 \leq a\ n$ $\bigwedge n.\ a\ n \leq K$ $\exists n' \geq l.\ a\ n' < K$
shows $(\sum n.\ r\hat{}(n + k) * a\ (n + l)) < r\hat{}k\ /\ (1 - r) * K$
⟨*proof*⟩

definition *product-dist'* :: [*real*, *'i set*, *nat* ⇒ *'i*, *'i* ⇒ *'a set*, *'i* ⇒ *'a* ⇒ *'a* ⇒ *real*] ⇒ (*'i* ⇒ *'a*) ⇒ (*'i* ⇒ *'a*) ⇒ *real* **where**
product-dist-def: *product-dist'* $r\ I\ g\ Mi\ di \equiv (\lambda x\ y.\ if\ x \in (\Pi_E\ i{\in}I.\ Mi\ i) \wedge y \in (\Pi_E\ i{\in}I.\ Mi\ i)\ then\ (\sum n.\ if\ g\ n \in I\ then\ r\hat{}n * di\ (g\ n)\ (x\ (g\ n))\ (y\ (g\ n))\ else\ 0)\ else\ 0)$

$d(x, y) = \sum_{n\in\mathbb{N}} r^n * d_{g_I(i)}(x_{g_I(i)}, y_{g_I(i)})$.

locale *Product-metric* =
fixes r :: *real*
and I :: *'i set*
and f :: *'i* ⇒ *nat*
and g :: *nat* ⇒ *'i*
and *Mi* :: *'i* ⇒ *'a set*
and *di* :: *'i* ⇒ *'a* ⇒ *'a* ⇒ *real*
and K :: *real*
assumes r: $0 < r$ $r < 1$
and I: *countable* I
and *gf-comp-id* : $\bigwedge i.\ i \in I \Longrightarrow g\ (f\ i) = i$
and *gf-if-finite*: *finite* $I \Longrightarrow$ *bij-betw* $f\ I$ {..< *card* I}
finite $I \Longrightarrow$ *bij-betw* g {..< *card* I} I
and *gf-if-infinite*: *infinite* $I \Longrightarrow$ *bij-betw* $f\ I$ *UNIV*
infinite $I \Longrightarrow$ *bij-betw* g *UNIV* I
$\bigwedge n.$ *infinite* $I \Longrightarrow f\ (g\ n) = n$
and *Md-metric*: $\bigwedge i.\ i \in I \Longrightarrow$ *Metric-space* (*Mi* i) (*di* i)
and *di-nonneg*: $\bigwedge i\ x\ y.\ 0 \leq di\ i\ x\ y$
and *di-bounded*: $\bigwedge i\ x\ y.\ di\ i\ x\ y \leq K$
and *K-pos* : $0 < K$

lemma *from-nat-into-to-nat-on-product-metric-pair*:
assumes *countable* I
shows $\bigwedge i.\ i \in I \Longrightarrow$ *from-nat-into* I (*to-nat-on* I i) = i

and *finite I* $\Longrightarrow$ *bij-betw* (*to-nat-on I*) *I* {..< *card I*}
and *finite I* $\Longrightarrow$ *bij-betw* (*from-nat-into I*) {..< *card I*} *I*
and *infinite I* $\Longrightarrow$ *bij-betw* (*to-nat-on I*) *I UNIV*
and *infinite I* $\Longrightarrow$ *bij-betw* (*from-nat-into I*) *UNIV I*
and $\bigwedge$*n. infinite I* $\Longrightarrow$ *to-nat-on I* (*from-nat-into I n*) = *n*
⟨*proof*⟩

lemma *product-metric-pair-finite-nat*:
bij-betw id {..*n*} {..< *card* {..*n*}} *bij-betw id* {..< *card* {..*n*}} {..*n*}
⟨*proof*⟩

lemma *product-metric-pair-finite-nat′*:
bij-betw id {..<*n*} {..< *card* {..<*n*}} *bij-betw id* {..< *card* {..<*n*}} {..<*n*}
⟨*proof*⟩

context *Product-metric*
begin

abbreviation *product-dist* ≡ *product-dist′ r I g Mi di*

lemma *nsum-of-rK*: ($\sum$ *n. r*^(*n* + *k*)∗*K*) = *r*^*k* / (*1* − *r*) ∗ *K*
⟨*proof*⟩

lemma *i-min*:
assumes *i* ∈ *I g n* = *i*
shows *f i* ≤ *n*
⟨*proof*⟩

lemma *g-surj*:
assumes *i* ∈ *I*
shows ∃ *n. g n* = *i*
⟨*proof*⟩

lemma *product-dist-summable′*[*simp*]:
summable (λ*n. r*^*n* ∗ *di* (*g n*) (*x* (*g n*)) (*y* (*g n*)))
⟨*proof*⟩

lemma *product-dist-summable*[*simp*]:
summable (λ*n. if g n* ∈ *I then r*^*n* ∗ *di* (*g n*) (*x* (*g n*)) (*y* (*g n*)) *else 0*)
⟨*proof*⟩

lemma *summable-rK*[*simp*]: *summable* (λ*n. r*^*n* ∗ *K*)
⟨*proof*⟩

lemma *Product-metric*: *Metric-space* (Π_E *i*∈*I. Mi i*) *product-dist*
⟨*proof*⟩

sublocale *Product-metric*: *Metric-space* Π_E *i*∈*I. Mi i product-dist*
⟨*proof*⟩

lemma *product-dist-leqr*: *product-dist x y* $\leq$ *1* / (*1* $-$ *r*) $*$ *K*
⟨*proof*⟩

lemma *product-dist-geq*:
assumes $i \in I$ **and** $g\ n = i$ $x \in (\Pi_E\ i{\in}I.\ Mi\ i)$ $y \in (\Pi_E\ i{\in}I.\ Mi\ i)$
shows *di i* (*x i*) (*y i*) $\leq$ (*1*/*r*)^*n* $*$ *product-dist x y*
(**is** *?lhs* $\leq$ *?rhs*)
⟨*proof*⟩

lemma *limitin-M-iff-limitin-Mi*:
shows *limitin Product-metric.mtopology xn x sequentially* $\longleftrightarrow$ ($\exists N.\ \forall n{\geq}N.$ ($\forall i{\in}I.$ *xn n i* $\in$ *Mi i*) $\wedge$ ($\forall i.\ i{\notin}I \longrightarrow$ *xn n i* = *undefined*)) $\wedge$ ($\forall i{\in}I.$ *limitin* (*Metric-space.mtopology* (*Mi i*) (*di i*)) ($\lambda n.$ *xn n i*) (*x i*) *sequentially*) $\wedge$ $x \in (\Pi_E\ i{\in}I.\ Mi\ i)$
⟨*proof*⟩

lemma *Product-metric-mtopology-eq*: *product-topology* ($\lambda i.$ *Metric-space.mtopology* (*Mi i*) (*di i*)) *I* = *Product-metric.mtopology*
⟨*proof*⟩

corollary *separable-Mi-separable-M*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *separable-space* (*Metric-space.mtopology* (*Mi i*) (*di i*))
shows *separable-space Product-metric.mtopology*
⟨*proof*⟩

lemma *mcomplete-Mi-mcomplete-M*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *Metric-space.mcomplete* (*Mi i*) (*di i*)
shows *Product-metric.mcomplete*
⟨*proof*⟩

end

lemma *product-metricI*:
assumes $0 < r$ $r < 1$ *countable I* $\bigwedge i.\ i \in I \Longrightarrow$ *Metric-space* (*Mi i*) (*di i*)
and $\bigwedge i\ x\ y.\ 0 \leq di\ i\ x\ y$ $\bigwedge i\ x\ y.\ di\ i\ x\ y \leq K$ $0 < K$
shows *Product-metric r I* (*to-nat-on I*) (*from-nat-into I*) *Mi di K*
⟨*proof*⟩

lemma *product-metric-natI*:
assumes $0 < r$ $r < 1$ $\bigwedge n.$ *Metric-space* (*Mi n*) (*di n*)
and $\bigwedge i\ x\ y.\ 0 \leq di\ i\ x\ y$ $\bigwedge i\ x\ y.\ di\ i\ x\ y \leq K$ $0 < K$
shows *Product-metric r UNIV id id Mi di K*
⟨*proof*⟩

end

2 Abstract Polish Spaces

theory *Abstract-Metrizable-Topology*
imports *Set-Based-Metric-Product*
begin

2.1 Polish Spaces

definition *Polish-space X* ≡ *completely-metrizable-space X* ∧ *separable-space X*

lemma(**in** *Metric-space*) *Polish-space-mtopology*:
assumes *mcomplete separable-space mtopology*
shows *Polish-space mtopology*
⟨*proof*⟩

lemma
assumes *Polish-space X*
shows *Polish-space-imp-completely-metrizable-space*: *completely-metrizable-space X*
and *Polish-space-imp-metrizable-space*: *metrizable-space X*
and *Polish-space-imp-second-countable*: *second-countable X*
and *Polish-space-imp-separable-space*: *separable-space X*
⟨*proof*⟩

lemma *Polish-space-closedin*:
assumes *Polish-space X closedin X A*
shows *Polish-space* (*subtopology X A*)
⟨*proof*⟩

lemma *Polish-space-gdelta-in*:
assumes *Polish-space X gdelta-in X A*
shows *Polish-space* (*subtopology X A*)
⟨*proof*⟩

corollary *Polish-space-openin*:
assumes *Polish-space X openin X A*
shows *Polish-space* (*subtopology X A*)
⟨*proof*⟩

lemma *homeomorphic-Polish-space-aux*:
assumes *Polish-space X X homeomorphic-space Y*
shows *Polish-space Y*
⟨*proof*⟩

corollary *homeomorphic-Polish-space*:
assumes *X homeomorphic-space Y*
shows *Polish-space X* ⟷ *Polish-space Y*
⟨*proof*⟩

lemma *Polish-space-euclidean*[*simp*]: *Polish-space* (*euclidean* :: (*'a* :: *polish-space*)

topology)
⟨*proof*⟩

lemma *Polish-space-countable*[*simp*]:
Polish-space (*euclidean* :: *'a* :: {*countable*,*discrete-topology*} *topology*)
⟨*proof*⟩

lemma *Polish-space-discrete-topology*: *Polish-space* (*discrete-topology I*) ⟷ *countable I*
⟨*proof*⟩

lemma *Polish-space-prod*:
assumes *Polish-space X* **and** *Polish-space Y*
shows *Polish-space* (*prod-topology X Y*)
⟨*proof*⟩

lemma *Polish-space-product*:
assumes *countable I* **and** $\bigwedge i.\ i \in I \Longrightarrow$ *Polish-space* (*S i*)
shows *Polish-space* (*product-topology S I*)
⟨*proof*⟩

lemma(**in** *Product-metric*) *Polish-spaceI*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *separable-space* (*Metric-space.mtopology* (*Mi i*) (*di i*))
and $\bigwedge i.\ i \in I \Longrightarrow$ *Metric-space.mcomplete* (*Mi i*) (*di i*)
shows *Polish-space Product-metric.mtopology*
⟨*proof*⟩

lemma(**in** *Sum-metric*) *Polish-spaceI*:
assumes *countable I*
and $\bigwedge i.\ i \in I \Longrightarrow$ *separable-space* (*Metric-space.mtopology* (*Mi i*) (*di i*))
and $\bigwedge i.\ i \in I \Longrightarrow$ *Metric-space.mcomplete* (*Mi i*) (*di i*)
shows *Polish-space Sum-metric.mtopology*
⟨*proof*⟩

lemma *compact-metrizable-imp-Polish-space*:
assumes *metrizable-space X compact-space X*
shows *Polish-space X*
⟨*proof*⟩

2.2 Extended Reals and Non-Negative Extended Reals

lemma *Polish-space-ereal*:*Polish-space* (*euclidean* :: *ereal topology*)
⟨*proof*⟩

corollary *Polish-space-ennreal*:*Polish-space* (*euclidean* :: *ennreal topology*)
⟨*proof*⟩

2.3 Continuous Embddings

abbreviation *Hilbert-cube-topology* :: (*nat* ⇒ *real*) *topology* **where**

Hilbert-cube-topology ≡ (*product-topology* (λ*n*. *top-of-set* {*0..1*}) *UNIV*)

lemma *topspace-Hilbert-cube*: *topspace Hilbert-cube-topology* = (Π_E *x*∈*UNIV*. {*0..1*})
⟨*proof*⟩

lemma *Polish-space-Hilbert-cube*: *Polish-space Hilbert-cube-topology*
⟨*proof*⟩

abbreviation *Cantor-space-topology* :: (*nat* ⇒ *real*) *topology* **where**
Cantor-space-topology ≡ (*product-topology* (λ*n*. *top-of-set* {*0,1*}) *UNIV*)

lemma *topspace-Cantor-space*:
topspace Cantor-space-topology = (Π_E *x*∈*UNIV*. {*0,1*})
⟨*proof*⟩

lemma *Polish-space-Cantor-space*: *Polish-space Cantor-space-topology*
⟨*proof*⟩

corollary *completely-metrizable-space-homeo-image-gdelta-in*:
assumes *completely-metrizable-space X completely-metrizable-space Y B* ⊆ *topspace Y X homeomorphic-space subtopology Y B*
shows *gdelta-in Y B*
⟨*proof*⟩

2.3.1 Embedding into Hilbert Cube

lemma *embedding-into-Hilbert-cube*:
assumes *metrizable-space X separable-space X*
shows ∃ *A* ⊆ *topspace Hilbert-cube-topology*. *X homeomorphic-space* (*subtopology Hilbert-cube-topology A*)
⟨*proof*⟩

corollary *embedding-into-Hilbert-cube-gdelta-in*:
assumes *Polish-space X*
shows ∃ *A*. *gdelta-in Hilbert-cube-topology A* ∧ *X homeomorphic-space* (*subtopology Hilbert-cube-topology A*)
⟨*proof*⟩

2.3.2 Embedding from Cantor Space

lemma *embedding-from-Cantor-space*:
assumes *Polish-space X uncountable* (*topspace X*)
shows ∃ *A*. *gdelta-in X A* ∧ *Cantor-space-topology homeomorphic-space* (*subtopology X A*)
⟨*proof*⟩

2.4 Borel Spaces generated from Polish Spaces

lemma *closedin-clopen-topology*:
assumes *Polish-space X closedin X a*

shows $\exists X'$. *Polish-space* $X' \wedge (\forall u.$ *openin* X $u \longrightarrow$ *openin* X' $u) \wedge$ *topspace* X $=$ *topspace* $X' \wedge$ *sets* (*borel-of* X) $=$ *sets* (*borel-of* X') $\wedge$ *openin* X' a $\wedge$ *closedin* X' a
⟨*proof*⟩

lemma *Polish-space-union-Polish*:
fixes X :: *nat* ⇒ *'a topology*
assumes $\bigwedge n$. *Polish-space* (X n) $\bigwedge n$. *topspace* (X n) $= Xt$ $\bigwedge x\ y.\ x \in Xt \Longrightarrow y \in Xt \Longrightarrow x \neq y \Longrightarrow \exists\, Ox\ Oy.$ ($\forall n.$ *openin* (X n) Ox) $\wedge$ ($\forall n.$ *openin* (X n) Oy) $\wedge$ $x \in Ox \wedge y \in Oy \wedge$ *disjnt* Ox Oy
defines $Xun \equiv$ *topology-generated-by* ($\bigcup n$. {u. *openin* (X n) u})
shows *Polish-space* *Xun*
⟨*proof*⟩

lemma *sets-clopen-topology*:
assumes *Polish-space* X $a \in$ *sets* (*borel-of* X)
shows $\exists X'$. *Polish-space* $X' \wedge (\forall u.$ *openin* X $u \longrightarrow$ *openin* X' $u) \wedge$ *topspace* X $=$ *topspace* $X' \wedge$ *sets* (*borel-of* X) $=$ *sets* (*borel-of* X') $\wedge$ *openin* X' a $\wedge$ *closedin* X' a
⟨*proof*⟩

end

3 Standard Borel Spaces

3.1 Standard Borel Spaces

theory *StandardBorel*
imports *Abstract-Metrizable-Topology*
begin

locale *standard-borel* =
fixes M :: *'a measure*
assumes *Polish-space*: $\exists S$. *Polish-space* $S \wedge$ *sets* M = *sets* (*borel-of* S)
begin

lemma *singleton-sets*:
assumes $x \in$ *space* M
shows $\{x\} \in$ *sets* M
⟨*proof*⟩

corollary *countable-sets*:
assumes $A \subseteq$ *space* M *countable* A
shows $A \in$ *sets* M
⟨*proof*⟩

lemma *standard-borel-restrict-space*:
assumes $A \in$ *sets* M
shows *standard-borel* (*restrict-space* M A)

⟨*proof*⟩

end

locale *standard-borel-ne* = *standard-borel* +
  assumes *space-ne*: *space* $M \neq \{\}$
begin

lemma *standard-borel-ne-restrict-space*:
  assumes $A \in$ *sets* M $A \neq \{\}$
  shows *standard-borel-ne* (*restrict-space* M A)
  ⟨*proof*⟩

lemma *standard-borel*: *standard-borel* M
  ⟨*proof*⟩

end

lemma *standard-borel-sets*:
  assumes *standard-borel* M **and** *sets* M = *sets* N
  shows *standard-borel* N
  ⟨*proof*⟩

lemma *standard-borel-ne-sets*:
  assumes *standard-borel-ne* M **and** *sets* M = *sets* N
  shows *standard-borel-ne* N
  ⟨*proof*⟩

lemma *pair-standard-borel*:
  assumes *standard-borel* M *standard-borel* N
  shows *standard-borel* ($M \bigotimes_M N$)
⟨*proof*⟩

lemma *pair-standard-borel-ne*:
  assumes *standard-borel-ne* M *standard-borel-ne* N
  shows *standard-borel-ne* ($M \bigotimes_M N$)
  ⟨*proof*⟩

lemma *product-standard-borel*:
  assumes *countable* I
      and $\bigwedge i.\ i \in I \Longrightarrow$ *standard-borel* ($M\ i$)
    shows *standard-borel* ($\Pi_M\ i {\in} I.\ M\ i$)
⟨*proof*⟩

lemma *product-standard-borel-ne*:
  assumes *countable* I
      and $\bigwedge i.\ i \in I \Longrightarrow$ *standard-borel-ne* ($M\ i$)
    shows *standard-borel-ne* ($\Pi_M\ i {\in} I.\ M\ i$)
  ⟨*proof*⟩

lemma *closed-set-standard-borel*[*simp*]:
fixes *U* :: *'a* :: *topological-space set*
assumes *Polish-space* (*euclidean* :: *'a topology*) *closed U*
shows *standard-borel* (*restrict-space borel U*)
⟨*proof*⟩

lemma *closed-set-standard-borel-ne*[*simp*]:
fixes *U* :: *'a* :: *topological-space set*
assumes *Polish-space* (*euclidean* :: *'a topology*) *closed U U* ≠ {}
shows *standard-borel-ne* (*restrict-space borel U*)
⟨*proof*⟩

lemma *open-set-standard-borel*[*simp*]:
fixes *U* :: *'a* :: *topological-space set*
assumes *Polish-space* (*euclidean* :: *'a topology*) *open U*
shows *standard-borel* (*restrict-space borel U*)
⟨*proof*⟩

lemma *open-set-standard-borel-ne*[*simp*]:
fixes *U* :: *'a* :: *topological-space set*
assumes *Polish-space* (*euclidean* :: *'a topology*) *open U U* ≠ {}
shows *standard-borel-ne* (*restrict-space borel U*)
⟨*proof*⟩

lemma *standard-borel-ne-borel*[*simp*]: *standard-borel-ne* (*borel* :: (*'a* :: *polish-space*) *measure*)
and *standard-borel-ne-lborel*[*simp*]: *standard-borel-ne lborel*
⟨*proof*⟩

lemma *count-space-standard'*[*simp*]:
assumes *countable I*
shows *standard-borel* (*count-space I*)
⟨*proof*⟩

lemma *count-space-standard-ne*[*simp*]: *standard-borel-ne* (*count-space* (*UNIV* :: (- :: *countable*) *set*))
⟨*proof*⟩

corollary *measure-pmf-standard-borel-ne*[*simp*]: *standard-borel-ne* (*measure-pmf* (*p* :: (- :: *countable*) *pmf*))
⟨*proof*⟩

corollary *measure-spmf-standard-borel-ne*[*simp*]: *standard-borel-ne* (*measure-spmf* (*p* :: (- :: *countable*) *spmf*))
⟨*proof*⟩

corollary *countable-standard-ne*[*simp*]:
standard-borel-ne (*borel* :: *'a* :: {*countable*,*t2-space*} *measure*)

⟨*proof*⟩

lemma(**in** *standard-borel*) *countable-discrete-space*:
assumes *countable* (*space M*)
shows *sets M* = *Pow* (*space M*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *measurable-isomorphic-standard*:
assumes *M measurable-isomorphic N*
shows *standard-borel N*
⟨*proof*⟩

lemma(**in** *standard-borel-ne*) *measurable-isomorphic-standard-ne*:
assumes *M measurable-isomorphic N*
shows *standard-borel-ne N*
⟨*proof*⟩

lemma(**in** *standard-borel*) *standard-borel-embed-measure*:
assumes*inj-on f* (*space M*)
shows *standard-borel* (*embed-measure M f*)
⟨*proof*⟩

corollary(**in** *standard-borel-ne*) *standard-borel-ne-embed-measure*:
assumes*inj-on f* (*space M*)
shows *standard-borel-ne* (*embed-measure M f*)
⟨*proof*⟩

lemma
shows *standard-ne-ereal*: *standard-borel-ne* (*borel* :: *ereal measure*)
and *standard-ne-ennreal*: *standard-borel-ne* (*borel* :: *ennreal measure*)
⟨*proof*⟩

Cantor space $\mathscr{C}$

definition *Cantor-space* :: (*nat* ⇒ *real*) *measure* **where**
Cantor-space ≡ (Π_M *i*∈ *UNIV*. *restrict-space borel* {*0*,*1*})

lemma *Cantor-space-standard-ne*: *standard-borel-ne Cantor-space*
⟨*proof*⟩

lemma *Cantor-space-borel*:
sets (*borel-of Cantor-space-topology*) = *sets Cantor-space*
(**is** *?lhs* = -)
⟨*proof*⟩

Hilbert cube $\mathscr{H}$

definition *Hilbert-cube* :: (*nat* ⇒ *real*) *measure* **where**
Hilbert-cube ≡ (Π_M *i*∈ *UNIV*. *restrict-space borel* {*0*..*1*})

lemma *Hilbert-cube-standard-ne*: *standard-borel-ne Hilbert-cube*

⟨*proof*⟩

lemma *Hilbert-cube-borel*:
sets (*borel-of Hilbert-cube-topology*) = *sets Hilbert-cube* (**is** *?lhs* = -)
⟨*proof*⟩

3.2 Isomorphism between $\mathscr{C}$ and $\mathscr{H}$

lemma *Cantor-space-isomorphic-to-Hilbert-cube*:
Cantor-space measurable-isomorphic Hilbert-cube
⟨*proof*⟩

3.3 Final Results

lemma(**in** *standard-borel*) *embedding-into-Hilbert-cube*:
$\exists$ *A* $\in$ *sets Hilbert-cube*. *M measurable-isomorphic* (*restrict-space Hilbert-cube A*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *embedding-from-Cantor-space*:
assumes *uncountable* (*space M*)
shows $\exists$ *A* $\in$ *sets M*. *Cantor-space measurable-isomorphic* (*restrict-space M A*)
⟨*proof*⟩

corollary(**in** *standard-borel*) *uncountable-isomorphic-to-Hilbert-cube*:
assumes *uncountable* (*space M*)
shows *Hilbert-cube measurable-isomorphic M*
⟨*proof*⟩

corollary(**in** *standard-borel*) *uncountable-isomorphic-to-real*:
assumes *uncountable* (*space M*)
shows *M measurable-isomorphic* (*borel* :: *real measure*)
⟨*proof*⟩

lemma(**in** *standard-borel*) *isomorphic-subset-real*:
assumes *A* $\in$ *sets* (*borel* :: *real measure*) *uncountable A*
obtains *B* **where** *B* $\in$ *sets borel B* $\subseteq$ *A M measurable-isomorphic restrict-space borel B*
⟨*proof*⟩

lemma(**in** *standard-borel*) *countable-isomorphic-to-subset-real*:
assumes *countable* (*space M*)
obtains *A* :: *real set*
where *countable A A* $\in$ *sets borel M measurable-isomorphic restrict-space borel A*
⟨*proof*⟩

theorem *Borel-isomorphism-theorem*:
assumes *standard-borel M standard-borel N*
shows *space M* $\approx$ *space N* $\longleftrightarrow$ *M measurable-isomorphic N*
⟨*proof*⟩

definition *to-real-on* :: *'a measure* ⇒ *'a* ⇒ *real* **where**
to-real-on M ≡ (*if uncountable* (*space M*) *then* (*SOME f*. *measurable-isomorphic-map M* (*borel* :: *real measure*) *f*) *else* (*real* ∘ *to-nat-on* (*space M*)))

definition *from-real-into* :: *'a measure* ⇒ *real* ⇒ *'a* **where**
from-real-into M ≡ (*if uncountable* (*space M*) *then the-inv-into* (*space M*) (*to-real-on M*) *else* (*λr*. *from-nat-into* (*space M*) (*nat* ⌊*r*⌋)))

context *standard-borel*
begin

abbreviation *to-real* ≡ *to-real-on M*
abbreviation *from-real* ≡ *from-real-into M*

lemma *to-real-def-countable*:
assumes *countable* (*space M*)
shows *to-real* = (*λr*. *real* (*to-nat-on* (*space M*) *r*))
⟨*proof*⟩

lemma *from-real-def-countable*:
assumes *countable* (*space M*)
shows *from-real* = (*λr*. *from-nat-into* (*space M*) (*nat* ⌊*r*⌋))
⟨*proof*⟩

lemma *from-real-to-real*[*simp*]:
assumes *x* ∈ *space M*
shows *from-real* (*to-real x*) = *x*
⟨*proof*⟩

lemma *to-real-measurable*[*measurable*]:
to-real ∈ *M* $\to_M$ *borel*
⟨*proof*⟩

lemma *from-real-measurable'*:
assumes *space M* ≠ {}
shows *from-real* ∈ *borel* $\to_M$ *M*
⟨*proof*⟩

lemma *to-real-from-real*:
assumes *uncountable* (*space M*)
shows *to-real* (*from-real r*) = *r*
⟨*proof*⟩

end

lemma(**in** *standard-borel-ne*) *from-real-measurable*[*measurable*]: *from-real* ∈ *borel* $\to_M$ *M*
⟨*proof*⟩

end

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