Stalnaker's Epistemic Logic

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Abstract

This work is a formalization of the knowledge fragment Stalnaker's epistemic logic with countably many agents and its soundness and completeness theorems [5, 6, 2], as well as the equivalence between the axiomatization of S4 available in the Epistemic Logic theory and the topological one [1]. It builds on the Epistemic Logic theory, as this includes the formalization of the completeness by canonicity proof for normal modal logics [3, 4].

Contents

theory *Stalnaker-Logic*
imports *Epistemic-Logic.Epistemic-Logic*

begin

1 Utility

1.1 Some properties of Normal Modal Logics

lemma *duality-taut*: ‹*tautology* $(((K\ i\ p) \longrightarrow K\ i\ (\neg q)) \longrightarrow ((L\ i\ q) \longrightarrow (\neg\ K\ i\ p)))$›
⟨*proof*⟩

lemma *K-imp-trans*:
assumes ‹$A \vdash (p \longrightarrow q)$› ‹$A \vdash (q \longrightarrow r)$›
shows ‹$A \vdash (p \longrightarrow r)$›
⟨*proof*⟩

lemma *K-imp-trans′*:
assumes ‹$A \vdash (q \longrightarrow r)$›
shows ‹$A \vdash ((p \longrightarrow q) \longrightarrow (p \longrightarrow r))$›
⟨*proof*⟩

lemma *K-imply-multi*:
assumes ‹$A \vdash (a \longrightarrow b)$› **and** ‹$A \vdash (a \longrightarrow c)$›
shows ‹$A \vdash (a \longrightarrow (b \wedge c))$›
⟨*proof*⟩

lemma *K-multi-imply*:
assumes ‹$A \vdash (a \longrightarrow b \longrightarrow c)$›
shows ‹$A \vdash ((a \wedge b) \longrightarrow c)$›
⟨*proof*⟩

lemma *K-thm*: ‹$A \vdash ((K\ i\ p) \wedge (L\ i\ q) \longrightarrow L\ i\ (p \wedge q))$›
⟨*proof*⟩

primrec *conjunct* :: ‹′*i fm list* ⇒ ′*i fm*› **where**
‹*conjunct* [] = ⊤›
| ‹*conjunct* (*p*#*ps*) = (*p* ∧ *conjunct ps*)›

lemma *imply-conjunct*: ‹*tautology* ((*imply G p*) ⟶ ((*conjunct G*) ⟶ *p*))›
⟨*proof*⟩

lemma *conjunct-imply*: ‹*tautology* (((*conjunct G*) ⟶ *p*) ⟶(*imply G p*))›
⟨*proof*⟩

lemma *K-imply-conjunct*:
assumes ‹*A* ⊢ *imply G p*›
shows ‹*A* ⊢ ((*conjunct G*) ⟶ *p*)›
⟨*proof*⟩

lemma *K-conjunct-imply*:
assumes ‹*A* ⊢ ((*conjunct G*) ⟶ *p*)›
shows ‹*A* ⊢ *imply G p*›
⟨*proof*⟩

lemma *K-conj-imply-factor*:
  fixes A :: ‹('i fm ⇒ bool)›
  shows ‹A ⊢ ((((K i p) ∧ (K i q)) ⟶ r) ⟶((K i (p ∧ q)) ⟶ r))›
⟨*proof*⟩

lemma *K-conjunction-in*: ‹A ⊢ (K i (p ∧ q) ⟶ ((K i p) ∧ K i q))›
⟨*proof*⟩

lemma *K-conjunction-in-mult*: ‹A ⊢ ((K i (conjunct G)) ⟶ conjunct (map (K i) G))›
⟨*proof*⟩

lemma *K-conjunction-out*: ‹A ⊢ ((K i p) ∧ (K i q) ⟶ K i (p ∧ q))›
⟨*proof*⟩

lemma *K-conjunction-out-mult*: ‹A ⊢ (conjunct (map (K i) G) ⟶ (K i (conjunct G)))›
⟨*proof*⟩

1.2 More on mcs's properties

lemma *mcs-conjunction*:
  assumes ‹consistent A V› **and** ‹maximal A V›
  shows ‹p ∈ V ∧ q ∈ V ⟶ (p ∧ q) ∈ V›
⟨*proof*⟩

lemma *mcs-conjunction-mult*:
 assumes ‹consistent A V› **and** ‹maximal A V›
  shows ‹(set (S :: ('i fm list)) ⊆ V ∧ finite (set S)) ⟶ (conjunct S) ∈ V›
⟨*proof*⟩

lemma *reach-dualK*:
  assumes ‹consistent A V› ‹maximal A V›
    and ‹consistent A W› ‹maximal A W› ‹W ∈ reach A i V›
  shows ‹∀ p. p ∈ W ⟶ (L i p) ∈ V›
⟨*proof*⟩

lemma *dual-reach*:
  assumes ‹consistent A V› ‹maximal A V›
  shows ‹(L i p) ∈ V ⟶ (∃ W. W ∈ reach A i V ∧ p ∈ W)›
⟨*proof*⟩

2 Ax.2

definition *weakly-directed* :: ‹('i, 's) kripke ⇒ bool› **where**
  ‹weakly-directed M ≡ ∀ i. ∀ s ∈ $\mathcal{W}$ M. ∀ t ∈ $\mathcal{W}$ M. ∀ r ∈ $\mathcal{W}$ M.
    (r ∈ $\mathcal{K}$ M i s ∧ t ∈ $\mathcal{K}$ M i s)⟶(∃ u ∈ $\mathcal{W}$ M. (u ∈ $\mathcal{K}$ M i r ∧ u ∈ $\mathcal{K}$ M i t))›

inductive *Ax-2* :: ‹′*i fm* ⇒ *bool*› **where**
‹*Ax-2* (¬ *K i* (¬ *K i p*) ⟶ *K i* (¬ *K i* (¬ *p*)))›

2.1 Soundness

theorem *weakly-directed*:
assumes ‹*weakly-directed M*› ‹*w* ∈ 𝒲 *M*›
shows ‹*M*, *w* ⊨ (*L i* (*K i p*) ⟶ *K i* (*L i p*))›
⟨*proof*⟩

lemma *soundness-Ax-2*: ‹*Ax-2 p* ⟹ *weakly-directed M* ⟹ *w* ∈ 𝒲 *M* ⟹ *M*, *w* ⊨ *p*›
⟨*proof*⟩

2.2 Imply completeness

lemma *Ax-2-weakly-directed*:
fixes *A* :: ‹′*i fm* ⇒ *bool*›
assumes ‹∀ *p*. *Ax-2 p* ⟶ *A p*› ‹*consistent A V*› ‹*maximal A V*›
and ‹*consistent A W*› ‹*maximal A W*› ‹*consistent A U*› ‹*maximal A U*›
and ‹*W* ∈ *reach A i V*› ‹*U* ∈ *reach A i V*›
shows ‹∃ *X*. (*consistent A X*) ∧ (*maximal A X*) ∧ *X* ∈ (*reach A i W*) ∩ (*reach A i U*)›
⟨*proof*⟩

lemma *mcs*$_{-2}$*-weakly-directed*:
fixes *A* :: ‹′*i fm* ⇒ *bool*›
assumes ‹∀ *p*. *Ax-2 p* ⟶ *A p*›
shows ‹*weakly-directed* ⦇𝒲 = *mcss A*, 𝒦 = *reach A*, π = *pi*⦈›
⟨*proof*⟩

lemma *imply-completeness-K-2*:
assumes *valid*: ‹∀ (*M* :: (′*i*, ′*i fm set*) *kripke*). ∀ *w* ∈ 𝒲 *M*.
weakly-directed M ⟶ (∀ *q* ∈ *G*. *M*, *w* ⊨ *q*) ⟶ *M*, *w* ⊨ *p*›
shows ‹∃ *qs*. *set qs* ⊆ *G* ∧ (*Ax-2* ⊢ *imply qs p*)›
⟨*proof*⟩

3 System S4.2

abbreviation *SystemS4-2* :: ‹′*i fm* ⇒ *bool*› (‹$\vdash_{S42}$ -› [*50*] *50*) **where**
‹$\vdash_{S42}$ *p* ≡ *AxT* ⊕ *Ax4* ⊕ *Ax-2* ⊢ *p*›

abbreviation *AxS4-2* :: ‹′*i fm* ⇒ *bool*› **where**
‹*AxS4-2* ≡ *AxT* ⊕ *Ax4* ⊕ *Ax-2*›

3.1 Soundness

abbreviation *w-directed-preorder* :: ‹(′*i*, ′*w*) *kripke* ⇒ *bool*› **where**
‹*w-directed-preorder M* ≡ *reflexive M* ∧ *transitive M* ∧ *weakly-directed M*›

lemma *soundness-AxS4-2*: ‹$AxS4\text{-}2\ p \Longrightarrow w\text{-}directed\text{-}preorder\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models p$›
⟨*proof*⟩

lemma $soundness_{S42}$: ‹$\vdash_{S42}\ p \Longrightarrow w\text{-}directed\text{-}preorder\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models p$›
⟨*proof*⟩

3.2 Completeness

lemma *imply-completeness-S4-2*:
assumes *valid*: ‹$\forall (M :: ('i,\ 'i\ fm\ set)\ kripke).\ \forall w \in \mathcal{W}\ M.\ w\text{-}directed\text{-}preorder\ M \longrightarrow (\forall q \in G.\ M,\ w \models q) \longrightarrow M,\ w \models p$›
shows ‹$\exists qs.\ set\ qs \subseteq G \wedge (AxS4\text{-}2 \vdash imply\ qs\ p)$›
⟨*proof*⟩

lemma $completeness_{S42}$:
assumes ‹$\forall (M :: ('i,\ 'i\ fm\ set)\ kripke).\ \forall w \in \mathcal{W}\ M.\ w\text{-}directed\text{-}preorder\ M \longrightarrow M,\ w \models p$›
shows ‹$\vdash_{S42}\ p$›
⟨*proof*⟩

abbreviation ‹$valid_{S42}\ p \equiv \forall (M :: (nat,\ nat\ fm\ set)\ kripke).\ \forall w \in \mathcal{W}\ M.\ w\text{-}directed\text{-}preorder\ M \longrightarrow M,\ w \models p$›

theorem $main_{S42}$: ‹$valid_{S42}\ p \longleftrightarrow \vdash_{S42}\ p$›
⟨*proof*⟩

corollary
assumes ‹$w\text{-}directed\text{-}preorder\ M$› ‹$w \in \mathcal{W}\ M$›
shows ‹$valid_{S42}\ p \longrightarrow M,\ w \models p$›
⟨*proof*⟩

4 Topological S4 axioms

abbreviation *DoubleImp* (**infixr** ‹$\longleftrightarrow$› *25*) **where**
‹$(p \longleftrightarrow q) \equiv ((p \longrightarrow q) \wedge (q \longrightarrow p))$›

inductive *System-topoS4* :: ‹$'i\ fm \Rightarrow bool$› (‹$\vdash_{Top}$ -› *[50] 50*) **where**
A1': ‹$tautology\ p \Longrightarrow \vdash_{Top}\ p$›
| *AR*: ‹$\vdash_{Top}\ ((K\ i\ (p \wedge q)) \longleftrightarrow ((K\ i\ p) \wedge K\ i\ q))$›
| *AT'*: ‹$\vdash_{Top}\ (K\ i\ p \longrightarrow p)$›
| *A4'*: ‹$\vdash_{Top}\ (K\ i\ p \longrightarrow K\ i\ (K\ i\ p))$›
| *AN*: ‹$\vdash_{Top}\ K\ i\ \top$›
| *R1'*: ‹$\vdash_{Top}\ p \Longrightarrow \vdash_{Top}\ (p \longrightarrow q) \Longrightarrow \vdash_{Top}\ q$›
| *RM*: ‹$\vdash_{Top}\ (p \longrightarrow q) \Longrightarrow \vdash_{Top}\ ((K\ i\ p) \longrightarrow K\ i\ q)$›

lemma *topoS4-trans*: ‹$\vdash_{Top}\ ((p \longrightarrow q) \longrightarrow (q \longrightarrow r) \longrightarrow p \longrightarrow r)$›

⟨*proof*⟩

lemma *topoS4-conjElim*: ‹$\vdash_{Top}$ $(p \wedge q \longrightarrow q)$›
⟨*proof*⟩

lemma *topoS4-AxK*: ‹$\vdash_{Top}$ $(K\ i\ p \wedge K\ i\ (p \longrightarrow q) \longrightarrow K\ i\ q)$›
⟨*proof*⟩

lemma *topoS4-NecR*:
assumes ‹$\vdash_{Top}$ p›
shows‹$\vdash_{Top}$ $K\ i\ p$›
⟨*proof*⟩

lemma *empty-S4*: $\{\} \vdash_{S4} p \longleftrightarrow AxT \oplus Ax4 \vdash p$
⟨*proof*⟩

lemma *S4-topoS4*: ‹$\{\} \vdash_{S4} p \Longrightarrow \vdash_{Top} p$›
⟨*proof*⟩

lemma *topoS4-S4*:
fixes p :: ‹'*i fm*›
shows ‹$\vdash_{Top}$ $p \Longrightarrow \{\} \vdash_{S4} p$›
⟨*proof*⟩

theorem *main*$_{S4}$': ‹$\{\} \Vdash_{S4} p \longleftrightarrow (\vdash_{Top}\ p)$›
⟨*proof*⟩

end

References

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