

Real-Valued Special Functions:
Upper and Lower Bounds

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Abstract

This development proves upper and lower bounds for several familiar real-valued functions. For \sin , \cos , \exp and the square root function, it defines and verifies infinite families of upper and lower bounds, mostly based on Taylor series expansions. For \tan^{-1} , \ln and \exp , it verifies a finite collection of upper and lower bounds, originally obtained from the functions' continued fraction expansions using the computer algebra system Maple. A common theme in these proofs is to take the difference between a function and its approximation, which should be zero at one point, and then consider the sign of the derivative.

The immediate purpose of this development is to verify axioms used by MetiTarski [1], an automatic theorem prover for real-valued special functions. Crucial to MetiTarski's operation is the provision of upper and lower bounds for each function of interest.

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Chapter 1

General Lemmas for Proving Function Inequalities

theory *Bounds-Lemmas*
imports *Complex-Main*

begin

These are for functions that are differentiable over a closed interval.

lemma *gen-lower-bound-increasing:*

fixes $a :: \text{real}$

assumes $a \leq x$

and $\bigwedge y. a \leq y \implies y \leq x \implies ((\lambda x. fl\ x - f\ x) \text{ has-real-derivative } g\ y) (at\ y)$

and $\bigwedge y. a \leq y \implies y \leq x \implies g\ y \leq 0$

and $fl\ a = f\ a$

shows $fl\ x \leq f\ x$

<proof>

lemma *gen-lower-bound-decreasing:*

fixes $a :: \text{real}$

assumes $x \leq a$

and $\bigwedge y. x \leq y \implies y \leq a \implies ((\lambda x. fl\ x - f\ x) \text{ has-real-derivative } g\ y) (at\ y)$

and $\bigwedge y. x \leq y \implies y \leq a \implies g\ y \geq 0$

and $fl\ a = f\ a$

shows $fl\ x \leq f\ x$

<proof>

lemma *gen-upper-bound-increasing:*

fixes $a :: \text{real}$

assumes $a \leq x$

and $\bigwedge y. a \leq y \implies y \leq x \implies ((\lambda x. fu\ x - f\ x) \text{ has-real-derivative } g\ y) (at\ y)$

and $\bigwedge y. a \leq y \implies y \leq x \implies g\ y \geq 0$

and $fu\ a = f\ a$

shows $f\ x \leq fu\ x$

<proof>

```
lemma gen-upper-bound-decreasing:  
  fixes  $a :: \text{real}$   
  assumes  $x \leq a$   
    and  $\bigwedge y. x \leq y \implies y \leq a \implies ((\lambda x. fu\ x - f\ x) \text{ has-real-derivative } g\ y) \text{ (at } y)$   
    and  $\bigwedge y. x \leq y \implies y \leq a \implies g\ y \leq 0$   
    and  $fu\ a = f\ a$   
  shows  $f\ x \leq fu\ x$   
  <proof>  
  
end
```

Chapter 2

Arctan Upper and Lower Bounds

```
theory Atan-CF-Bounds
imports Bounds-Lemmas
         HOL-Library.Sum-of-Squares
```

```
begin
```

Covers all bounds used in `arctan-upper.ax`, `arctan-lower.ax` and `arctan-extended.ax`, excepting only `arctan-extended2.ax`, which is used in two `atan-error-analysis` problems.

2.1 Upper Bound 1

```
definition arctan-upper-11 :: real  $\Rightarrow$  real
  where arctan-upper-11  $\equiv \lambda x. -(pi/2) - 1/x$ 
```

```
definition diff-delta-arctan-upper-11 :: real  $\Rightarrow$  real
  where diff-delta-arctan-upper-11  $\equiv \lambda x. 1 / (x^2 * (1 + x^2))$ 
```

```
lemma d-delta-arctan-upper-11:  $x \neq 0 \implies$ 
   $((\lambda x. \text{arctan-upper-11 } x - \text{arctan } x) \text{ has-field-derivative } \text{diff-delta-arctan-upper-11 } x)$  (at  $x$ )
  <proof>
```

```
lemma d-delta-arctan-upper-11-pos:  $x \neq 0 \implies \text{diff-delta-arctan-upper-11 } x > 0$ 
  <proof>
```

Different proof needed here: they coincide not at zero, but at $(-)$ infinity!

```
lemma arctan-upper-11:
  assumes  $x < 0$ 
  shows  $\text{arctan}(x) < \text{arctan-upper-11 } x$ 
  <proof>
```

definition *arctan-upper-12* :: *real* \Rightarrow *real*
where *arctan-upper-12* $\equiv \lambda x. 3*x / (x^2 + 3)$

definition *diff-delta-arctan-upper-12* :: *real* \Rightarrow *real*
where *diff-delta-arctan-upper-12* $\equiv \lambda x. -4*x^4 / ((x^2+3)^2 * (1+x^2))$

lemma *d-delta-arctan-upper-12*:
(($\lambda x. \text{arctan-upper-12 } x - \text{arctan } x$) *has-field-derivative* *diff-delta-arctan-upper-12*
x) (*at x*)
(*proof*)

Strict inequalities also possible

lemma *arctan-upper-12*:
assumes $x \leq 0$ **shows** $\text{arctan}(x) \leq \text{arctan-upper-12 } x$
(*proof*)

definition *arctan-upper-13* :: *real* \Rightarrow *real*
where *arctan-upper-13* $\equiv \lambda x. x$

definition *diff-delta-arctan-upper-13* :: *real* \Rightarrow *real*
where *diff-delta-arctan-upper-13* $\equiv \lambda x. x^2 / (1 + x^2)$

lemma *d-delta-arctan-upper-13*:
(($\lambda x. \text{arctan-upper-13 } x - \text{arctan } x$) *has-field-derivative* *diff-delta-arctan-upper-13*
x) (*at x*)
(*proof*)

lemma *arctan-upper-13*:
assumes $x \geq 0$ **shows** $\text{arctan}(x) \leq \text{arctan-upper-13 } x$
(*proof*)

definition *arctan-upper-14* :: *real* \Rightarrow *real*
where *arctan-upper-14* $\equiv \lambda x. \text{pi}/2 - 3*x / (1 + 3*x^2)$

definition *diff-delta-arctan-upper-14* :: *real* \Rightarrow *real*
where *diff-delta-arctan-upper-14* $\equiv \lambda x. -4 / ((1 + 3*x^2)^2 * (1+x^2))$

lemma *d-delta-arctan-upper-14*:
(($\lambda x. \text{arctan-upper-14 } x - \text{arctan } x$) *has-field-derivative* *diff-delta-arctan-upper-14*
x) (*at x*)
(*proof*)

lemma *d-delta-arctan-upper-14-neg*: *diff-delta-arctan-upper-14* $x < 0$
(*proof*)

lemma *lim14*: (($\lambda x::\text{real}. 3 * x / (1 + 3 * x^2)$) $\longrightarrow 0$) *at-infinity*
(*proof*)

Different proof needed here: they coincide not at zero, but at (+) infinity!

lemma *arctan-upper-14*:
assumes $x > 0$
shows $\arctan(x) < \text{arctan-upper-14 } x$
 $\langle \text{proof} \rangle$

2.2 Lower Bound 1

definition *arctan-lower-11* :: $\text{real} \Rightarrow \text{real}$
where $\text{arctan-lower-11} \equiv \lambda x. -(\pi/2) - 3*x / (1 + 3*x^2)$

lemma *arctan-lower-11*:
assumes $x < 0$
shows $\arctan(x) > \text{arctan-lower-11 } x$
 $\langle \text{proof} \rangle$

abbreviation *arctan-lower-12* $\equiv \text{arctan-upper-13}$

lemma *arctan-lower-12*:
assumes $x \leq 0$
shows $\arctan(x) \geq \text{arctan-lower-12 } x$
 $\langle \text{proof} \rangle$

abbreviation *arctan-lower-13* $\equiv \text{arctan-upper-12}$

lemma *arctan-lower-13*:
assumes $x \geq 0$
shows $\arctan(x) \geq \text{arctan-lower-13 } x$
 $\langle \text{proof} \rangle$

definition *arctan-lower-14* :: $\text{real} \Rightarrow \text{real}$
where $\text{arctan-lower-14} \equiv \lambda x. \pi/2 - 1/x$

lemma *arctan-lower-14*:
assumes $x > 0$
shows $\arctan(x) > \text{arctan-lower-14 } x$
 $\langle \text{proof} \rangle$

2.3 Upper Bound 3

definition *arctan-upper-31* :: $\text{real} \Rightarrow \text{real}$
where $\text{arctan-upper-31} \equiv \lambda x. -(\pi/2) - (64 + 735*x^2 + 945*x^4) / (15*x*(15 + 70*x^2 + 63*x^4))$

definition *diff-delta-arctan-upper-31* :: $\text{real} \Rightarrow \text{real}$
where $\text{diff-delta-arctan-upper-31} \equiv \lambda x. 64 / (x^2 * (15 + 70*x^2 + 63*x^4)^2 * (1 + x^2))$

lemma *d-delta-arctan-upper-31*:

assumes $x \neq 0$
shows $((\lambda x. \arctan\text{-upper-31 } x - \arctan x) \text{ has-field-derivative } \text{diff-delta-arctan-upper-31 } x) \text{ (at } x)$
 $\langle \text{proof} \rangle$

lemma $d\text{-delta-arctan-upper-31-pos}: x \neq 0 \implies \text{diff-delta-arctan-upper-31 } x > 0$
 $\langle \text{proof} \rangle$

lemma $\arctan\text{-upper-31}$:
assumes $x < 0$
shows $\arctan(x) < \arctan\text{-upper-31 } x$
 $\langle \text{proof} \rangle$

definition $\arctan\text{-upper-32} :: \text{real} \Rightarrow \text{real}$
where $\arctan\text{-upper-32} \equiv \lambda x. 7 * (33 * x^4 + 170 * x^2 + 165) * x / (5 * (5 * x^6 + 105 * x^4 + 315 * x^2 + 231))$

definition $\text{diff-delta-arctan-upper-32} :: \text{real} \Rightarrow \text{real}$
where $\text{diff-delta-arctan-upper-32} \equiv \lambda x. -256 * x^{12} / ((5 * x^6 + 105 * x^4 + 315 * x^2 + 231)^2 * (1 + x^2))$

lemma $d\text{-delta-arctan-upper-32}$:
 $((\lambda x. \arctan\text{-upper-32 } x - \arctan x) \text{ has-field-derivative } \text{diff-delta-arctan-upper-32 } x) \text{ (at } x)$
 $\langle \text{proof} \rangle$

lemma $\arctan\text{-upper-32}$:
assumes $x \leq 0$ **shows** $\arctan(x) \leq \arctan\text{-upper-32 } x$
 $\langle \text{proof} \rangle$

definition $\arctan\text{-upper-33} :: \text{real} \Rightarrow \text{real}$
where $\arctan\text{-upper-33} \equiv \lambda x. (64 * x^4 + 735 * x^2 + 945) * x / (15 * (15 * x^4 + 70 * x^2 + 63))$

definition $\text{diff-delta-arctan-upper-33} :: \text{real} \Rightarrow \text{real}$
where $\text{diff-delta-arctan-upper-33} \equiv \lambda x. 64 * x^{10} / ((15 * x^4 + 70 * x^2 + 63)^2 * (1 + x^2))$

lemma $d\text{-delta-arctan-upper-33}$:
 $((\lambda x. \arctan\text{-upper-33 } x - \arctan x) \text{ has-field-derivative } \text{diff-delta-arctan-upper-33 } x) \text{ (at } x)$
 $\langle \text{proof} \rangle$

lemma $\arctan\text{-upper-33}$:
assumes $x \geq 0$ **shows** $\arctan(x) \leq \arctan\text{-upper-33 } x$
 $\langle \text{proof} \rangle$

definition $\arctan\text{-upper-34} :: \text{real} \Rightarrow \text{real}$
where $\arctan\text{-upper-34} \equiv$
 $\lambda x. \pi/2 - (33 + 170 * x^2 + 165 * x^4) * 7 * x / (5 * (5 + 105 * x^2 + 315 * x^4 + 231 * x^6))$

definition *diff-delta-arctan-upper-34* :: *real* \Rightarrow *real*
where *diff-delta-arctan-upper-34* $\equiv \lambda x. -256 / ((5+105*x^2+315*x^4+231*x^6)^2*(1+x^2))$

lemma *d-delta-arctan-upper-34*:
 $((\lambda x. \arctan\text{-upper-34 } x - \arctan x)$ *has-field-derivative* *diff-delta-arctan-upper-34*
 $x)$ (*at* x)
<proof>

lemma *d-delta-arctan-upper-34-pos*: *diff-delta-arctan-upper-34* $x < 0$
<proof>

lemma *arctan-upper-34*:
assumes $x > 0$
shows $\arctan(x) < \arctan\text{-upper-34 } x$
<proof>

2.4 Lower Bound 3

definition *arctan-lower-31* :: *real* \Rightarrow *real*
where *arctan-lower-31* $\equiv \lambda x. -(\pi/2) - (33 + 170*x^2 + 165*x^4)*7*x /$
 $(5*(5 + 105*x^2 + 315*x^4 + 231*x^6))$

lemma *arctan-lower-31*:
assumes $x < 0$
shows $\arctan(x) > \arctan\text{-lower-31 } x$
<proof>

abbreviation *arctan-lower-32* $\equiv \arctan\text{-upper-33}$

lemma *arctan-lower-32*:
assumes $x \leq 0$
shows $\arctan(x) \geq \arctan\text{-lower-32 } x$
<proof>

abbreviation *arctan-lower-33* $\equiv \arctan\text{-upper-32}$

lemma *arctan-lower-33*:
assumes $x \geq 0$
shows $\arctan(x) \geq \arctan\text{-lower-33 } x$
<proof>

definition *arctan-lower-34* :: *real* \Rightarrow *real*
where *arctan-lower-34* $\equiv \lambda x. \pi/2 - (64 + 735*x^2 + 945*x^4) / (15*x*(15$
 $+ 70*x^2 + 63*x^4))$

lemma *arctan-lower-34*:
assumes $x > 0$
shows $\arctan(x) > \arctan\text{-lower-34 } x$
<proof>

2.5 Upper Bound 4

definition *arctan-upper-41* :: real \Rightarrow real

where *arctan-upper-41* \equiv

$$\lambda x. -(\pi/2) - (256 + 5943*x^2 + 19250*x^4 + 15015*x^6) / (35*x*(35 + 315*x^2 + 693*x^4 + 429*x^6))$$

definition *diff-delta-arctan-upper-41* :: real \Rightarrow real

where *diff-delta-arctan-upper-41* $\equiv \lambda x. 256 / (x^2*(35+315*x^2+693*x^4+429*x^6)^2*(1+x^2))$

lemma *d-delta-arctan-upper-41*:

assumes $x \neq 0$

shows $((\lambda x. \text{arctan-upper-41 } x - \text{arctan } x)$ has-field-derivative *diff-delta-arctan-upper-41* $x)$ (at x)
 ⟨proof⟩

lemma *d-delta-arctan-upper-41-pos*: $x \neq 0 \implies \text{diff-delta-arctan-upper-41 } x > 0$

⟨proof⟩

lemma *arctan-upper-41*:

assumes $x < 0$

shows $\text{arctan}(x) < \text{arctan-upper-41 } x$
 ⟨proof⟩

definition *arctan-upper-42* :: real \Rightarrow real

where *arctan-upper-42* \equiv

$$\lambda x. (15159*x^6 + 147455*x^4 + 345345*x^2 + 225225)*x / (35*(35*x^8 + 1260*x^6 + 6930*x^4 + 12012*x^2 + 6435))$$

definition *diff-delta-arctan-upper-42* :: real \Rightarrow real

where *diff-delta-arctan-upper-42* \equiv

$$\lambda x. -16384*x^16 / ((35*x^8 + 1260*x^6 + 6930*x^4 + 12012*x^2 + 6435)^2*(1+x^2))$$

lemma *d-delta-arctan-upper-42*:

$((\lambda x. \text{arctan-upper-42 } x - \text{arctan } x)$ has-field-derivative *diff-delta-arctan-upper-42* $x)$ (at x)
 ⟨proof⟩

lemma *arctan-upper-42*:

assumes $x \leq 0$ **shows** $\text{arctan}(x) \leq \text{arctan-upper-42 } x$
 ⟨proof⟩

definition *arctan-upper-43* :: real \Rightarrow real

where *arctan-upper-43* \equiv

$$\lambda x. (256*x^6 + 5943*x^4 + 19250*x^2 + 15015)*x / (35 * (35*x^6 + 315*x^4 + 693*x^2 + 429))$$

definition *diff-delta-arctan-upper-43* :: real \Rightarrow real

where *diff-delta-arctan-upper-43* $\equiv \lambda x. 256*x^14 / ((35*x^6 + 315*x^4 + 693*x^2 + 429)^2*(1+x^2))$

lemma *d-delta-arctan-upper-43*:

((λx. arctan-upper-43 x - arctan x) has-field-derivative diff-delta-arctan-upper-43 x) (at x)
<proof>

lemma *arctan-upper-43*:

assumes $x \geq 0$ **shows** $\arctan(x) \leq \text{arctan-upper-43 } x$
<proof>

definition *arctan-upper-44* :: *real* \Rightarrow *real*

where *arctan-upper-44* \equiv

$$\lambda x. \pi/2 - (15159 + 147455*x^2 + 345345*x^4 + 225225*x^6)*x / (35*(35 + 1260*x^2 + 6930*x^4 + 12012*x^6 + 6435*x^8))$$

definition *diff-delta-arctan-upper-44* :: *real* \Rightarrow *real*

where *diff-delta-arctan-upper-44* \equiv

$$\lambda x. -16384 / ((35 + 1260*x^2 + 6930*x^4 + 12012*x^6 + 6435*x^8)^2*(1 + x^2))$$

lemma *d-delta-arctan-upper-44*:

((λx. arctan-upper-44 x - arctan x) has-field-derivative diff-delta-arctan-upper-44 x) (at x)
<proof>

lemma *d-delta-arctan-upper-44-pos*: *diff-delta-arctan-upper-44 x < 0*

<proof>

lemma *arctan-upper-44*:

assumes $x > 0$
shows $\arctan(x) < \text{arctan-upper-44 } x$
<proof>

2.6 Lower Bound 4

definition *arctan-lower-41* :: *real* \Rightarrow *real*

where *arctan-lower-41* \equiv

$$\lambda x. -(pi/2) - (15159 + 147455*x^2 + 345345*x^4 + 225225*x^6)*x / (35*(35 + 1260*x^2 + 6930*x^4 + 12012*x^6 + 6435*x^8))$$

lemma *arctan-lower-41*:

assumes $x < 0$

shows $\arctan(x) > \text{arctan-lower-41 } x$

<proof>

abbreviation *arctan-lower-42* \equiv *arctan-upper-43*

lemma *arctan-lower-42*:

assumes $x \leq 0$

shows $\arctan(x) \geq \text{arctan-lower-42 } x$

<proof>

abbreviation $\text{arctan-lower-43} \equiv \text{arctan-upper-42}$

lemma arctan-lower-43 :

assumes $x \geq 0$

shows $\text{arctan}(x) \geq \text{arctan-lower-43 } x$

$\langle \text{proof} \rangle$

definition $\text{arctan-lower-44} :: \text{real} \Rightarrow \text{real}$

where $\text{arctan-lower-44} \equiv$

$$\lambda x. \pi/2 - (256 + 5943*x^2 + 19250*x^4 + 15015*x^6) / (35*x*(35 + 315*x^2 + 693*x^4 + 429*x^6))$$

lemma arctan-lower-44 :

assumes $x > 0$

shows $\text{arctan}(x) > \text{arctan-lower-44 } x$

$\langle \text{proof} \rangle$

end

Chapter 3

Exp Upper and Lower Bounds

```
theory Exp-Bounds
imports Bounds-Lemmas
         HOL-Library.Sum-of-Squares
         Sturm-Sequences.Sturm
```

```
begin
```

Covers all bounds used in `exp-upper.ax`, `exp-lower.ax` and `exp-extended.ax`.

3.1 Taylor Series Bounds

exp-positive is the theorem $0 \leq \exp ?x$

exp-lower-taylor-1 is the theorem $1 + ?x \leq \exp ?x$

All even approximants are lower bounds.

```
lemma exp-lower-taylor-even:
```

```
  fixes x::real
```

```
  shows even n  $\implies (\sum m < n. (x \wedge m) / (\text{fact } m)) \leq \exp x$ 
```

```
   $\langle \text{proof} \rangle$ 
```

```
lemma exp-upper-taylor-even:
```

```
  fixes x::real
```

```
  assumes n: even n
```

```
    and pos:  $(\sum m < n. ((-x) \wedge m) / (\text{fact } m)) > 0$  (is ?sum > 0)
```

```
  shows  $\exp x \leq \text{inverse } ?\text{sum}$ 
```

```
   $\langle \text{proof} \rangle$ 
```

3 if the previous lemma is expressed in terms of $(2::'a) * m$.

```
lemma exp-lower-taylor-3:
```

```
  fixes x::real
```

shows $1 + x + (1/2)*x^2 + (1/6)*x^3 + (1/24)*x^4 + (1/120)*x^5 \leq \exp x$
 <proof>

lemma *exp-lower-taylor-3-cubed*:

fixes $x::real$
shows $(1 + x/3 + (1/2)*(x/3)^2 + (1/6)*(x/3)^3 + (1/24)*(x/3)^4 + (1/120)*(x/3)^5)^3 \leq \exp x$
 <proof>

lemma *exp-lower-taylor-2*:

fixes $x::real$
shows $1 + x + (1/2)*x^2 + (1/6)*x^3 \leq \exp x$
 <proof>

lemma *exp-upper-bound-case-3*:

fixes $x::real$
assumes $x \leq 3.19$
shows $\exp x \leq 2304 / (-(x^3) + 6*x^2 - 24*x + 48)^2$
 <proof>

lemma *exp-upper-bound-case-5*:

fixes $x::real$
assumes $x \leq 6.36$
shows $\exp x \leq 21743271936 / (-(x^3) + 12*x^2 - 96*x + 384)^4$
 <proof>

3.2 Continued Fraction Bound 2

definition *exp-cf2* :: $real \Rightarrow real$

where $exp-cf2 \equiv \lambda x. (x^2 + 6*x + 12) / (x^2 - 6*x + 12)$

lemma *denom-cf2-pos*: **fixes** $x::real$ **shows** $x^2 - 6 * x + 12 > 0$
 <proof>

lemma *numer-cf2-pos*: **fixes** $x::real$ **shows** $x^2 + 6 * x + 12 > 0$
 <proof>

lemma *exp-cf2-pos*: $exp-cf2 x > 0$
 <proof>

definition *diff-delta-lnexp-cf2* :: $real \Rightarrow real$

where $diff-delta-lnexp-cf2 \equiv \lambda x. -(x^4) / ((x^2 - 6*x + 12) * (x^2 + 6*x + 12))$

lemma *d-delta-lnexp-cf2-nonpos*: $diff-delta-lnexp-cf2 x \leq 0$
 <proof>

lemma *d-delta-lnexp-cf2*:

$((\lambda x. \ln (\exp\text{-cf2 } x) - x) \text{ has-field-derivative } \text{diff-delta-lnexp-cf2 } x) \text{ (at } x)$
 ⟨proof⟩

Upper bound for non-positive x

lemma *ln-exp-cf2-upper-bound-neg*:
assumes $x \leq 0$
shows $x \leq \ln (\exp\text{-cf2 } x)$
 ⟨proof⟩

theorem *exp-cf2-upper-bound-neg*: $x \leq 0 \implies \exp(x) \leq \exp\text{-cf2 } x$
 ⟨proof⟩

Lower bound for non-negative x

lemma *ln-exp-cf2-lower-bound-pos*:
assumes $0 \leq x$
shows $\ln (\exp\text{-cf2 } x) \leq x$
 ⟨proof⟩

theorem *exp-cf2-lower-bound-pos*: $0 \leq x \implies \exp\text{-cf2 } x \leq \exp x$
 ⟨proof⟩

3.3 Continued Fraction Bound 3

This bound crosses the X-axis twice, causing complications.

definition *numer-cf3* :: *real* \Rightarrow *real*
where $\text{numer-cf3} \equiv \lambda x. x^3 + 12*x^2 + 60*x + 120$

definition *exp-cf3* :: *real* \Rightarrow *real*
where $\text{exp-cf3} \equiv \lambda x. \text{numer-cf3 } x / \text{numer-cf3 } (-x)$

lemma *numer-cf3-pos*: $-4.64 \leq x \implies \text{numer-cf3 } x > 0$
 ⟨proof⟩

lemma *exp-cf3-pos*: $\text{numer-cf3 } x > 0 \implies \text{numer-cf3 } (-x) > 0 \implies \text{exp-cf3 } x > 0$
 ⟨proof⟩

definition *diff-delta-lnexp-cf3* :: *real* \Rightarrow *real*
where $\text{diff-delta-lnexp-cf3} \equiv \lambda x. (x^6) / (\text{numer-cf3 } (-x) * \text{numer-cf3 } x)$

lemma *d-delta-lnexp-cf3-nonneg*: $\text{numer-cf3 } x > 0 \implies \text{numer-cf3 } (-x) > 0 \implies$
 $\text{diff-delta-lnexp-cf3 } x \geq 0$
 ⟨proof⟩

lemma *d-delta-lnexp-cf3*:
assumes $\text{numer-cf3 } x > 0 \text{ numer-cf3 } (-x) > 0$
shows $((\lambda x. \ln (\exp\text{-cf3 } x) - x) \text{ has-field-derivative } \text{diff-delta-lnexp-cf3 } x) \text{ (at } x)$
 ⟨proof⟩

lemma *numer-cf3-mono*: $y \leq x \implies \text{numer-cf3 } y \leq \text{numer-cf3 } x$
 ⟨proof⟩

Upper bound for non-negative x

lemma *ln-exp-cf3-upper-bound-nonneg*:
assumes $x0$: $0 \leq x$ **and** $xless$: $\text{numer-cf3 } (-x) > 0$
shows $x \leq \ln (\text{exp-cf3 } x)$
 ⟨proof⟩

theorem *exp-cf3-upper-bound-pos*: $0 \leq x \implies \text{numer-cf3 } (-x) > 0 \implies \text{exp } x \leq \text{exp-cf3 } x$
 ⟨proof⟩

corollary $0 \leq x \implies x \leq 4.64 \implies \text{exp } x \leq \text{exp-cf3 } x$
 ⟨proof⟩

Lower bound for negative x, provided $0 < \text{exp-cf3 } x$

lemma *ln-exp-cf3-lower-bound-neg*:
assumes $x0$: $x \leq 0$ **and** $xgtr$: $\text{numer-cf3 } x > 0$
shows $\ln (\text{exp-cf3 } x) \leq x$
 ⟨proof⟩

theorem *exp-cf3-lower-bound-pos*:
assumes $x \leq 0$ **shows** $\text{exp-cf3 } x \leq \text{exp } x$
 ⟨proof⟩

3.4 Continued Fraction Bound 4

Here we have the extended exp continued fraction bounds

definition *numer-cf4* :: $\text{real} \Rightarrow \text{real}$
where $\text{numer-cf4} \equiv \lambda x. x^4 + 20*x^3 + 180*x^2 + 840*x + 1680$

definition *exp-cf4* :: $\text{real} \Rightarrow \text{real}$
where $\text{exp-cf4} \equiv \lambda x. \text{numer-cf4 } x / \text{numer-cf4 } (-x)$

lemma *numer-cf4-pos*: **fixes** $x::\text{real}$ **shows** $\text{numer-cf4 } x > 0$
 ⟨proof⟩

lemma *exp-cf4-pos*: $\text{exp-cf4 } x > 0$
 ⟨proof⟩

definition *diff-delta-lnexp-cf4* :: $\text{real} \Rightarrow \text{real}$
where $\text{diff-delta-lnexp-cf4} \equiv \lambda x. -(x^8) / (\text{numer-cf4 } (-x) * \text{numer-cf4 } x)$

lemma *d-delta-lnexp-cf4-nonpos*: $\text{diff-delta-lnexp-cf4 } x \leq 0$
 ⟨proof⟩

lemma *d-delta-lnexp-cf4*:

$((\lambda x. \ln (\exp\text{-cf}_4 x) - x)$ has-field-derivative $\text{diff-delta-lnexp-cf}_4 x$) (at x)
 ⟨proof⟩

Upper bound for non-positive x

lemma $\text{ln-exp-cf}_4\text{-upper-bound-neg}$:
assumes $x \leq 0$
shows $x \leq \ln (\exp\text{-cf}_4 x)$
 ⟨proof⟩

theorem $\text{exp-cf}_4\text{-upper-bound-neg}$: $x \leq 0 \implies \exp(x) \leq \exp\text{-cf}_4 x$
 ⟨proof⟩

Lower bound for non-negative x

lemma $\text{ln-exp-cf}_4\text{-lower-bound-pos}$:
assumes $0 \leq x$
shows $\ln (\exp\text{-cf}_4 x) \leq x$
 ⟨proof⟩

theorem $\text{exp-cf}_4\text{-lower-bound-pos}$: $0 \leq x \implies \exp\text{-cf}_4 x \leq \exp x$
 ⟨proof⟩

3.5 Continued Fraction Bound 5

This bound crosses the X-axis twice, causing complications.

definition $\text{numer-cf}_5 :: \text{real} \Rightarrow \text{real}$
where $\text{numer-cf}_5 \equiv \lambda x. x^5 + 30*x^4 + 420*x^3 + 3360*x^2 + 15120*x + 30240$

definition $\text{exp-cf}_5 :: \text{real} \Rightarrow \text{real}$
where $\text{exp-cf}_5 \equiv \lambda x. \text{numer-cf}_5 x / \text{numer-cf}_5 (-x)$

lemma $\text{numer-cf}_5\text{-pos}$: $-7.293 \leq x \implies \text{numer-cf}_5 x > 0$
 ⟨proof⟩

lemma $\text{exp-cf}_5\text{-pos}$: $\text{numer-cf}_5 x > 0 \implies \text{numer-cf}_5 (-x) > 0 \implies \text{exp-cf}_5 x > 0$
 ⟨proof⟩

definition $\text{diff-delta-lnexp-cf}_5 :: \text{real} \Rightarrow \text{real}$
where $\text{diff-delta-lnexp-cf}_5 \equiv \lambda x. (x^{\wedge}10) / (\text{numer-cf}_5 (-x) * \text{numer-cf}_5 x)$

lemma $d\text{-delta-lnexp-cf}_5\text{-nonneg}$: $\text{numer-cf}_5 x > 0 \implies \text{numer-cf}_5 (-x) > 0 \implies \text{diff-delta-lnexp-cf}_5 x \geq 0$
 ⟨proof⟩

lemma $d\text{-delta-lnexp-cf}_5$:
assumes $\text{numer-cf}_5 x > 0$ $\text{numer-cf}_5 (-x) > 0$
shows $((\lambda x. \ln (\exp\text{-cf}_5 x) - x)$ has-field-derivative $\text{diff-delta-lnexp-cf}_5 x$) (at x)
 ⟨proof⟩

3.5.1 Proving monotonicity via a non-negative derivative

definition *numer-cf5-deriv* :: *real* \Rightarrow *real*

where *numer-cf5-deriv* $\equiv \lambda x. 5*x^4 + 120*x^3 + 1260*x^2 + 6720*x + 15120$

lemma *numer-cf5-deriv*:

shows (*numer-cf5* has-field-derivative *numer-cf5-deriv* *x*) (at *x*)

\langle *proof* \rangle

lemma *numer-cf5-deriv-pos*: *numer-cf5-deriv* *x* ≥ 0

\langle *proof* \rangle

lemma *numer-cf5-mono*: $y \leq x \implies \text{numer-cf5 } y \leq \text{numer-cf5 } x$

\langle *proof* \rangle

3.5.2 Results

Upper bound for non-negative *x*

lemma *ln-exp-cf5-upper-bound-nonneg*:

assumes *x0*: $0 \leq x$ **and** *xless*: *numer-cf5* $(-x) > 0$

shows $x \leq \ln (\text{exp-cf5 } x)$

\langle *proof* \rangle

theorem *exp-cf5-upper-bound-pos*: $0 \leq x \implies \text{numer-cf5 } (-x) > 0 \implies \text{exp } x \leq \text{exp-cf5 } x$

\langle *proof* \rangle

corollary $0 \leq x \implies x \leq 7.293 \implies \text{exp } x \leq \text{exp-cf5 } x$

\langle *proof* \rangle

Lower bound for negative *x*, provided $0 < \text{exp-cf5 } x$

lemma *ln-exp-cf5-lower-bound-neg*:

assumes *x0*: $x \leq 0$ **and** *xgtr*: *numer-cf5* $x > 0$

shows $\ln (\text{exp-cf5 } x) \leq x$

\langle *proof* \rangle

theorem *exp-cf5-lower-bound-pos*:

assumes $x \leq 0$ **shows** $\text{exp-cf5 } x \leq \text{exp } x$

\langle *proof* \rangle

3.6 Continued Fraction Bound 6

definition *numer-cf6* :: *real* \Rightarrow *real*

where *numer-cf6* $\equiv \lambda x. x^6 + 42*x^5 + 840*x^4 + 10080*x^3 + 75600*x^2 + 332640*x + 665280$

definition *exp-cf6* :: *real* \Rightarrow *real*

where *exp-cf6* $\equiv \lambda x. \text{numer-cf6 } x / \text{numer-cf6 } (-x)$

lemma *numer-cf6-pos*: **fixes** $x::real$ **shows** $numer-cf6\ x > 0$
 ⟨*proof*⟩

lemma *exp-cf6-pos*: $exp-cf6\ x > 0$
 ⟨*proof*⟩

definition *diff-delta-lnexp-cf6* :: $real \Rightarrow real$
where $diff-delta-lnexp-cf6 \equiv \lambda x. -(x^{12}) / (numer-cf6\ (-x) * numer-cf6\ x)$

lemma *d-delta-lnexp-cf6-nonpos*: $diff-delta-lnexp-cf6\ x \leq 0$
 ⟨*proof*⟩

lemma *d-delta-lnexp-cf6*:
 (($\lambda x. \ln (exp-cf6\ x) - x$) *has-field-derivative* $diff-delta-lnexp-cf6\ x$) (at x)
 ⟨*proof*⟩

Upper bound for non-positive x

lemma *ln-exp-cf6-upper-bound-neg*:
assumes $x \leq 0$
shows $x \leq \ln (exp-cf6\ x)$
 ⟨*proof*⟩

theorem *exp-cf6-upper-bound-neg*: $x \leq 0 \implies exp(x) \leq exp-cf6\ x$
 ⟨*proof*⟩

Lower bound for non-negative x

lemma *ln-exp-cf6-lower-bound-pos*:
assumes $0 \leq x$
shows $\ln (exp-cf6\ x) \leq x$
 ⟨*proof*⟩

theorem *exp-cf6-lower-bound-pos*: $0 \leq x \implies exp-cf6\ x \leq exp\ x$
 ⟨*proof*⟩

3.7 Continued Fraction Bound 7

This bound crosses the X-axis twice, causing complications.

definition *numer-cf7* :: $real \Rightarrow real$
where $numer-cf7 \equiv \lambda x. x^7 + 56*x^6 + 1512*x^5 + 25200*x^4 + 277200*x^3 + 1995840*x^2 + 8648640*x + 17297280$

definition *exp-cf7* :: $real \Rightarrow real$
where $exp-cf7 \equiv \lambda x. numer-cf7\ x / numer-cf7\ (-x)$

lemma *numer-cf7-pos*: $-9.943 \leq x \implies numer-cf7\ x > 0$
 ⟨*proof*⟩

lemma *exp-cf7-pos*: $numer-cf7\ x > 0 \implies numer-cf7\ (-x) > 0 \implies exp-cf7\ x > 0$

<proof>

definition *diff-delta-lnexp-cf7* :: *real* \Rightarrow *real*

where *diff-delta-lnexp-cf7* $\equiv \lambda x. (x^{14}) / (\text{numer-cf7 } (-x) * \text{numer-cf7 } x)$

lemma *d-delta-lnexp-cf7-nonneg*: *numer-cf7* $x > 0 \implies \text{numer-cf7 } (-x) > 0 \implies$
diff-delta-lnexp-cf7 $x \geq 0$

<proof>

lemma *d-delta-lnexp-cf7*:

assumes *numer-cf7* $x > 0$ *numer-cf7* $(-x) > 0$

shows $((\lambda x. \ln (\text{exp-cf7 } x) - x)$ *has-field-derivative* *diff-delta-lnexp-cf7* $x)$ (at x)
<proof>

3.7.1 Proving monotonicity via a non-negative derivative

definition *numer-cf7-deriv* :: *real* \Rightarrow *real*

where *numer-cf7-deriv* $\equiv \lambda x. 7*x^6 + 336*x^5 + 7560*x^4 + 100800*x^3 + 831600*x^2 + 3991680*x + 8648640$

lemma *numer-cf7-deriv*:

shows (*numer-cf7* *has-field-derivative* *numer-cf7-deriv* x) (at x)

<proof>

lemma *numer-cf7-deriv-pos*: *numer-cf7-deriv* $x \geq 0$

<proof>

lemma *numer-cf7-mono*: $y \leq x \implies \text{numer-cf7 } y \leq \text{numer-cf7 } x$

<proof>

3.7.2 Results

Upper bound for non-negative x

lemma *ln-exp-cf7-upper-bound-nonneg*:

assumes $x0: 0 \leq x$ **and** $xless: \text{numer-cf7 } (-x) > 0$

shows $x \leq \ln (\text{exp-cf7 } x)$

<proof>

theorem *exp-cf7-upper-bound-pos*: $0 \leq x \implies \text{numer-cf7 } (-x) > 0 \implies \text{exp } x \leq$
exp-cf7 x

<proof>

corollary $0 \leq x \implies x \leq 9.943 \implies \text{exp } x \leq \text{exp-cf7 } x$

<proof>

Lower bound for negative x, provided $0 < \text{exp-cf7 } x$

lemma *ln-exp-cf7-lower-bound-neg*:

assumes $x0: x \leq 0$ **and** $xgtr: \text{numer-cf7 } x > 0$

shows $\ln (\text{exp-cf7 } x) \leq x$

<proof>

theorem *exp-cf7-lower-bound-pos:*

assumes $x \leq 0$ **shows** $\exp\text{-cf7 } x \leq \exp x$
<proof>

end

Chapter 4

Log Upper and Lower Bounds

theory *Log-CF-Bounds*
imports *Bounds-Lemmas*

begin

theorem *ln-upper-1*: $0 < x \implies \ln(x::\text{real}) \leq x - 1$
<proof>

definition *ln-lower-1* :: *real* \Rightarrow *real*
where *ln-lower-1* $\equiv \lambda x. 1 - (\text{inverse } x)$

corollary *ln-lower-1*: $0 < x \implies \text{ln-lower-1 } x \leq \ln x$
<proof>

theorem *ln-lower-1-eq*: $0 < x \implies \text{ln-lower-1 } x = (x - 1)/x$
<proof>

4.1 Upper Bound 3

definition *ln-upper-3* :: *real* \Rightarrow *real*
where *ln-upper-3* $\equiv \lambda x. (x + 5) * (x - 1) / (2 * (2 * x + 1))$

definition *diff-delta-ln-upper-3* :: *real* \Rightarrow *real*
where *diff-delta-ln-upper-3* $\equiv \lambda x. (x - 1)^3 / ((2 * x + 1)^2 * x)$

lemma *d-delta-ln-upper-3*: $x > 0 \implies$
 $((\lambda x. \text{ln-upper-3 } x - \ln x) \text{ has-field-derivative } \text{diff-delta-ln-upper-3 } x) \text{ (at } x)$
<proof>

Strict inequalities also possible

lemma *ln-upper-3-pos*:
assumes $1 \leq x$ **shows** $\ln(x) \leq \text{ln-upper-3 } x$

<proof>

lemma *ln-upper-3-neg*:

assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \text{ln-upper-3 } x$
<proof>

theorem *ln-upper-3*: $0 < x \implies \ln(x) \leq \text{ln-upper-3 } x$
<proof>

definition *ln-lower-3* :: *real* \Rightarrow *real*
where *ln-lower-3* $\equiv \lambda x. - \text{ln-upper-3 } (inverse\ x)$

corollary *ln-lower-3*: $0 < x \implies \text{ln-lower-3 } x \leq \ln\ x$
<proof>

theorem *ln-lower-3-eq*: $0 < x \implies \text{ln-lower-3 } x = (1/2) * (1 + 5*x) * (x - 1) / (x * (2 + x))$
<proof>

4.2 Upper Bound 5

definition *ln-upper-5* :: *real* \Rightarrow *real*
where *ln-upper-5* $x \equiv (x^2 + 19*x + 10) * (x - 1) / (3 * (3*x^2 + 6*x + 1))$

definition *diff-delta-ln-upper-5* :: *real* \Rightarrow *real*
where *diff-delta-ln-upper-5* $\equiv \lambda x. (x - 1)^5 / ((3*x^2 + 6*x + 1)^2 * x)$

lemma *d-delta-ln-upper-5*: $x > 0 \implies$
 $((\lambda x. \text{ln-upper-5 } x - \ln\ x)$ *has-field-derivative* *diff-delta-ln-upper-5* $x)$ (at x)
<proof>

lemma *ln-upper-5-pos*:
assumes $1 \leq x$ **shows** $\ln(x) \leq \text{ln-upper-5 } x$
<proof>

lemma *ln-upper-5-neg*:
assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \text{ln-upper-5 } x$
<proof>

theorem *ln-upper-5*: $0 < x \implies \ln(x) \leq \text{ln-upper-5 } x$
<proof>

definition *ln-lower-5* :: *real* \Rightarrow *real*
where *ln-lower-5* $\equiv \lambda x. - \text{ln-upper-5 } (inverse\ x)$

corollary *ln-lower-5*: $0 < x \implies \text{ln-lower-5 } x \leq \ln\ x$
<proof>

theorem *ln-lower-5-eq*: $0 < x \implies$

ln-lower-5 $x = (1/3)*(10*x^2 + 19*x + 1)*(x - 1) / (x*(x^2 + 6*x + 3))$
 ⟨proof⟩

4.3 Upper Bound 7

definition *ln-upper-7* :: real ⇒ real

where *ln-upper-7* $x \equiv (3*x^3 + 131*x^2 + 239*x + 47)*(x - 1) / (12*(4*x^3 + 18*x^2 + 12*x + 1))$

definition *diff-delta-ln-upper-7* :: real ⇒ real

where *diff-delta-ln-upper-7* $\equiv \lambda x. (x - 1)^7 / ((4*x^3 + 18*x^2 + 12*x + 1)^2 * x)$

lemma *d-delta-ln-upper-7*: $x > 0 \implies$

$((\lambda x. \text{ln-upper-7 } x - \text{ln } x) \text{ has-field-derivative } \text{diff-delta-ln-upper-7 } x) \text{ (at } x)$
 ⟨proof⟩

lemma *ln-upper-7-pos*:

assumes $1 \leq x$ **shows** $\text{ln}(x) \leq \text{ln-upper-7 } x$
 ⟨proof⟩

lemma *ln-upper-7-neg*:

assumes $0 < x$ **and** $x \leq 1$ **shows** $\text{ln}(x) \leq \text{ln-upper-7 } x$
 ⟨proof⟩

theorem *ln-upper-7*: $0 < x \implies \text{ln}(x) \leq \text{ln-upper-7 } x$

⟨proof⟩

definition *ln-lower-7* :: real ⇒ real

where *ln-lower-7* $\equiv \lambda x. - \text{ln-upper-7 } (x^{-1})$

corollary *ln-lower-7*: $0 < x \implies \text{ln-lower-7 } x \leq \text{ln } x$

⟨proof⟩

theorem *ln-lower-7-eq*: $0 < x \implies$

$\text{ln-lower-7 } x = (1/12)*(47*x^3 + 239*x^2 + 131*x + 3)*(x - 1) / (x*(x^3 + 12*x^2 + 18*x + 4))$

⟨proof⟩

4.4 Upper Bound 9

definition *ln-upper-9* :: real ⇒ real

where *ln-upper-9* $x \equiv (6*x^4 + 481*x^3 + 1881*x^2 + 1281*x + 131)*(x - 1) /$

$$(30 * (5*x^4 + 40*x^3 + 60*x^2 + 20*x + 1))$$

definition *diff-delta-ln-upper-9* :: real ⇒ real

where *diff-delta-ln-upper-9* $\equiv \lambda x. (x - 1)^9 / (((5*x^4 + 40*x^3 + 60*x^2 + 20*x + 1) * 30))$

+ 20*x + 1)^2) * x)

lemma *d-delta-ln-upper-9*: $x > 0 \implies$

(($\lambda x. \ln\text{-upper-9 } x - \ln x$) has-field-derivative *diff-delta-ln-upper-9* x) (at x)
(proof)

lemma *ln-upper-9-pos*:

assumes $1 \leq x$ shows $\ln(x) \leq \ln\text{-upper-9 } x$
(proof)

lemma *ln-upper-9-neg*:

assumes $0 < x$ and $x1: x \leq 1$ shows $\ln(x) \leq \ln\text{-upper-9 } x$
(proof)

theorem *ln-upper-9*: $0 < x \implies \ln(x) \leq \ln\text{-upper-9 } x$

(proof)

definition *ln-lower-9* :: *real* \Rightarrow *real*

where *ln-lower-9* $\equiv \lambda x. - \ln\text{-upper-9 } (inverse\ x)$

corollary *ln-lower-9*: $0 < x \implies \ln\text{-lower-9 } x \leq \ln x$

(proof)

theorem *ln-lower-9-eq*: $0 < x \implies$

$\ln\text{-lower-9 } x = (1/30)*(6 + 481*x + 1881*x^2 + 1281*x^3 + 131*x^4)*(x - 1) /$
 $(x*(5 + 40*x + 60*x^2 + 20*x^3 + x^4))$

(proof)

4.5 Upper Bound 11

Extended bounds start here

definition *ln-upper-11* :: *real* \Rightarrow *real*

where *ln-upper-11* $x \equiv$
 $(5*x^5 + 647*x^4 + 4397*x^3 + 6397*x^2 + 2272*x + 142) * (x - 1) /$
 $(30*(6*x^5 + 75*x^4 + 200*x^3 + 150*x^2 + 30*x + 1))$

definition *diff-delta-ln-upper-11* :: *real* \Rightarrow *real*

where *diff-delta-ln-upper-11* $\equiv \lambda x. (x - 1)^{11} / ((6*x^5 + 75*x^4 + 200*x^3 + 150*x^2 + 30*x + 1)^2 * x)$

lemma *d-delta-ln-upper-11*: $x > 0 \implies$

(($\lambda x. \ln\text{-upper-11 } x - \ln x$) has-field-derivative *diff-delta-ln-upper-11* x) (at x)
(proof)

lemma *ln-upper-11-pos*:

assumes $1 \leq x$ **shows** $\ln(x) \leq \text{ln-upper-11 } x$
 ⟨proof⟩

lemma *ln-upper-11-neg*:

assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \text{ln-upper-11 } x$
 ⟨proof⟩

theorem *ln-upper-11*: $0 < x \implies \ln(x) \leq \text{ln-upper-11 } x$
 ⟨proof⟩

definition *ln-lower-11* :: *real* \Rightarrow *real*

where *ln-lower-11* $\equiv \lambda x. - \text{ln-upper-11 } (inverse\ x)$

corollary *ln-lower-11*: $0 < x \implies \text{ln-lower-11 } x \leq \ln\ x$
 ⟨proof⟩

theorem *ln-lower-11-eq*: $0 < x \implies$

$$\text{ln-lower-11 } x = (1/30) * (142 * x^5 + 2272 * x^4 + 6397 * x^3 + 4397 * x^2 + 647 * x + 5) * (x - 1) / (x * (x^5 + 30 * x^4 + 150 * x^3 + 200 * x^2 + 75 * x + 6))$$

⟨proof⟩

4.6 Upper Bound 13

definition *ln-upper-13* :: *real* \Rightarrow *real*

where *ln-upper-13* $x \equiv (353 + 8389 * x + 20149 * x^4 + 50774 * x^3 + 38524 * x^2 + 1921 * x^5 + 10 * x^6) * (x - 1) / (70 * (1 + 42 * x + 525 * x^4 + 700 * x^3 + 315 * x^2 + 126 * x^5 + 7 * x^6))$

definition *diff-delta-ln-upper-13* :: *real* \Rightarrow *real*

where *diff-delta-ln-upper-13* $\equiv \lambda x. (x - 1)^{13} / ((1 + 42 * x + 525 * x^4 + 700 * x^3 + 315 * x^2 + 126 * x^5 + 7 * x^6)^2 * x)$

lemma *d-delta-ln-upper-13*: $x > 0 \implies$

$((\lambda x. \text{ln-upper-13 } x - \ln\ x)$ *has-field-derivative* *diff-delta-ln-upper-13* $x)$ (at x)
 ⟨proof⟩

lemma *ln-upper-13-pos*:

assumes $1 \leq x$ **shows** $\ln(x) \leq \text{ln-upper-13 } x$
 ⟨proof⟩

lemma *ln-upper-13-neg*:

assumes $0 < x$ **and** $x1: x \leq 1$ **shows** $\ln(x) \leq \text{ln-upper-13 } x$
 ⟨proof⟩

theorem *ln-upper-13*: $0 < x \implies \ln(x) \leq \text{ln-upper-13 } x$
 ⟨proof⟩

definition *ln-lower-13* :: real \Rightarrow real

where *ln-lower-13* $\equiv \lambda x. - \text{ln-upper-13}$ (inverse x)

corollary *ln-lower-13*: $0 < x \implies \text{ln-lower-13 } x \leq \ln x$

<proof>

theorem *ln-lower-13-eq*: $0 < x \implies$

$$\begin{aligned} \text{ln-lower-13 } x &= (1/70) * (10 + 1921 * x + 20149 * x^2 + 50774 * x^3 + 38524 * x^4 \\ &+ 8389 * x^5 + 353 * x^6) * (x - 1) / \\ &(x * (7 + 126 * x + 525 * x^2 + 700 * x^3 + 315 * x^4 + 42 * x^5 + \\ &x^6)) \end{aligned}$$

<proof>

4.7 Upper Bound 15

definition *ln-upper-15* :: real \Rightarrow real

where *ln-upper-15* $x \equiv$

$$\begin{aligned} &(1487 + 49199 * x + 547235 * x^4 + 718735 * x^3 + 334575 * x^2 + \\ &141123 * x^5 + 35 * x^7 + 9411 * x^6) * (x - 1) / \\ &(280 * (1 + 56 * x + 2450 * x^4 + 1960 * x^3 + 588 * x^2 + 1176 * x^5 + \\ &8 * x^7 + 196 * x^6)) \end{aligned}$$

definition *diff-delta-ln-upper-15* :: real \Rightarrow real

where *diff-delta-ln-upper-15*

$$\equiv \lambda x. (x - 1)^{15} / ((1 + 56 * x + 2450 * x^4 + 1960 * x^3 + 588 * x^2 + 8 * x^7 + 196 * x^6 + 1176 * x^5)^2 * x)$$

lemma *d-delta-ln-upper-15*: $x > 0 \implies$

$((\lambda x. \text{ln-upper-15 } x - \ln x)$ has-field-derivative $\text{diff-delta-ln-upper-15 } x)$ (at x)

<proof>

lemma *ln-upper-15-pos*:

assumes $1 \leq x$ **shows** $\ln(x) \leq \text{ln-upper-15 } x$

<proof>

lemma *ln-upper-15-neg*:

assumes $0 < x$ **and** $x \leq 1$ **shows** $\ln(x) \leq \text{ln-upper-15 } x$

<proof>

theorem *ln-upper-15*: $0 < x \implies \ln(x) \leq \text{ln-upper-15 } x$

<proof>

definition *ln-lower-15* :: real \Rightarrow real

where *ln-lower-15* $\equiv \lambda x. - \text{ln-upper-15}$ (inverse x)

corollary *ln-lower-15*: $0 < x \implies \text{ln-lower-15 } x \leq \ln x$

<proof>

theorem *ln-lower-15-eq*: $0 < x \implies$

$$\begin{aligned} \text{ln-lower-15 } x = & (1/280) * (35 + 9411 * x + 141123 * x^2 + 547235 * x^3 + \\ & 718735 * x^4 + 334575 * x^5 + 49199 * x^6 + 1487 * x^7) * (x - 1) / \\ & (x * (8 + 196 * x + 1176 * x^2 + 2450 * x^3 + 1960 * x^4 + 588 * x^5 \\ & + 56 * x^6 + x^7)) \\ & \langle \text{proof} \rangle \end{aligned}$$

end

Chapter 5

Sine and Cosine Upper and Lower Bounds

```
theory Sin-Cos-Bounds
imports Bounds-Lemmas
```

```
begin
```

5.1 Simple base cases

Upper bound for $(0::'a) \leq x$

```
lemma sin-le-arg:
  fixes  $x :: real$ 
  shows  $0 \leq x \implies \sin x \leq x$ 
  <proof>
```

```
lemma cos-ge-1-arg:
  fixes  $x :: real$ 
  assumes  $0 \leq x$ 
  shows  $1 - x \leq \cos x$ 
  <proof>
```

lemmas *sin-Taylor-0-upper-bound-pos* = *sin-le-arg* — MetiTarski bound

```
lemma cos-Taylor-1-lower-bound:
  fixes  $x :: real$ 
  assumes  $0 \leq x$ 
  shows  $(1 - x^2 / 2) \leq \cos x$ 
  <proof>
```

```
lemma sin-Taylor-1-lower-bound:
  fixes  $x :: real$ 
  assumes  $0 \leq x$ 
  shows  $(x - x^3 / 6) \leq \sin x$ 
```

<proof>

5.2 Taylor series approximants

definition *sinpoly* :: [nat,real] \Rightarrow real
where *sinpoly* n = ($\lambda x. \sum k < n. \text{sin-coeff } k * x \wedge k$)

definition *cospoly* :: [nat,real] \Rightarrow real
where *cospoly* n = ($\lambda x. \sum k < n. \text{cos-coeff } k * x \wedge k$)

lemma *sinpoly-Suc*: *sinpoly* (Suc n) = ($\lambda x. \text{sinpoly } n x + \text{sin-coeff } n * x \wedge n$)
<proof>

lemma *cospoly-Suc*: *cospoly* (Suc n) = ($\lambda x. \text{cospoly } n x + \text{cos-coeff } n * x \wedge n$)
<proof>

lemma *sinpoly-minus* [simp]: *sinpoly* n (-x) = - *sinpoly* n x
<proof>

lemma *cospoly-minus* [simp]: *cospoly* n (-x) = *cospoly* n x
<proof>

lemma *d-sinpoly-cospoly*:
(*sinpoly* (Suc n) has-field-derivative *cospoly* n x) (at x)
<proof>

lemma *d-cospoly-sinpoly*:
(*cospoly* (Suc n) has-field-derivative -*sinpoly* n x) (at x)
<proof>

5.3 Inductive proof of sine inequalities

lemma *sinpoly-lb-imp-cospoly-ub*:
assumes *x0*: $0 \leq x$ and *k0*: $k > 0$ and $\bigwedge x. 0 \leq x \implies \text{sinpoly } (k - 1) x \leq \text{sin } x$
shows $\text{cos } x \leq \text{cospoly } k x$
<proof>

lemma *cospoly-ub-imp-sinpoly-ub*:
assumes *x0*: $0 \leq x$ and *k0*: $k > 0$ and $\bigwedge x. 0 \leq x \implies \text{cos } x \leq \text{cospoly } (k - 1) x$
shows $\text{sin } x \leq \text{sinpoly } k x$
<proof>

lemma *sinpoly-ub-imp-cospoly-lb*:
assumes *x0*: $0 \leq x$ and *k0*: $k > 0$ and $\bigwedge x. 0 \leq x \implies \text{sin } x \leq \text{sinpoly } (k - 1) x$
shows $\text{cospoly } k x \leq \text{cos } x$
<proof>

lemma *cospoly-lb-imp-sinpoly-lb*:

assumes $x0: 0 \leq x$ **and** $k0: k > 0$ **and** $\bigwedge x. 0 \leq x \implies \text{cospoly } (k - 1) x \leq \text{cos } x$
shows $\text{sinpoly } k x \leq \text{sin } x$
<proof>

lemma

assumes $0 \leq x$
shows *sinpoly-lower-nonneg*: $\text{sinpoly } (4 * \text{Suc } n) x \leq \text{sin } x$ (**is** *?th1*)
and *sinpoly-upper-nonneg*: $\text{sin } x \leq \text{sinpoly } (\text{Suc } (\text{Suc } (4 * n))) x$ (**is** *?th2*)
<proof>

5.4 Collecting the results

corollary *sinpoly-upper-nonpos*:

$x \leq 0 \implies \text{sin } x \leq \text{sinpoly } (4 * \text{Suc } n) x$
<proof>

corollary *sinpoly-lower-nonpos*:

$x \leq 0 \implies \text{sinpoly } (\text{Suc } (\text{Suc } (4 * n))) x \leq \text{sin } x$
<proof>

corollary *cospoly-lower-nonneg*:

$0 \leq x \implies \text{cospoly } (\text{Suc } (\text{Suc } (\text{Suc } (4 * n)))) x \leq \text{cos } x$
<proof>

lemma *cospoly-lower*:

$\text{cospoly } (\text{Suc } (\text{Suc } (\text{Suc } (4 * n)))) x \leq \text{cos } x$
<proof>

lemma *cospoly-upper-nonneg*:

assumes $0 \leq x$
shows $\text{cos } x \leq \text{cospoly } (\text{Suc } (4 * n)) x$
<proof>

lemma *cospoly-upper*:

$\text{cos } x \leq \text{cospoly } (\text{Suc } (4 * n)) x$
<proof>

end

Chapter 6

Square Root Upper and Lower Bounds

```
theory Sqrt-Bounds
imports Bounds-Lemmas
        HOL-Library.Sum-of-Squares
```

```
begin
```

Covers all bounds used in sqrt-upper.ax, sqrt-lower.ax and sqrt-extended.ax.

6.1 Upper bounds

```
primrec sqrtu :: [real,nat]  $\Rightarrow$  real where
  sqrtu x 0 = (x+1)/2
| sqrtu x (Suc n) = (sqrtu x n + x/sqrtu x n) / 2
```

```
lemma sqrtu-upper:  $x \leq \text{sqrtu } x \ n \ ^2$ 
<proof>
```

```
lemma sqrtu-numeral:
  sqrtu x (numeral n) = (sqrtu x (pred-numeral n) + x/sqrtu x (pred-numeral n))
/ 2
<proof>
```

```
lemma sqrtu-gt-0:  $x \geq 0 \implies \text{sqrtu } x \ n > 0$ 
<proof>
```

```
theorem gen-sqrt-upper:  $0 \leq x \implies \text{sqrt } x \leq \text{sqrtu } x \ n$ 
<proof>
```

```
lemma sqrt-upper-bound-0:
  assumes  $x \geq 0$  shows  $\text{sqrt } x \leq (x+1)/2$  (is -  $\leq$  ?rhs)
<proof>
```

lemma *sqrt-upper-bound-1*:

assumes $x \geq 0$ **shows** $\text{sqrt } x \leq (1/4)*(x^2+6*x+1) / (x+1)$ (**is - ≤ ?rhs**)
{proof}

lemma *sqrtu-2-eq*:

$\text{sqrtu } x^2 = (1/8)*(x^4 + 28*x^3 + 70*x^2 + 28*x + 1) / ((x + 1)*(x^2 + 6*x + 1))$
{proof}

lemma *sqrt-upper-bound-2*:

assumes $x \geq 0$
shows $\text{sqrt } x \leq (1/8)*(x^4 + 28*x^3 + 70*x^2 + 28*x + 1) / ((x + 1)*(x^2 + 6*x + 1))$
{proof}

lemma *sqrtu-4-eq*:

$x \geq 0 \implies$

$\text{sqrtu } x^4 = (1/32)*(225792840*x^6+64512240*x^5+601080390*x^8+471435600*x^7+496*x+1+35960$
 $/ ((x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7+$
{proof}

lemma *sqrt-upper-bound-4*:

assumes $x \geq 0$

shows $\text{sqrt } x \leq (1/32)*(225792840*x^6+64512240*x^5+601080390*x^8+471435600*x^7+496*x+1+35960$
 $/ ((x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7+$
{proof}

lemma *gen-sqrt-upper-scaled*:

assumes $0 \leq x < u$

shows $\text{sqrt } x \leq \text{sqrtu } (x*u^2) n / u$
{proof}

lemma *sqrt-upper-bound-2-small*:

assumes $0 \leq x$

shows $\text{sqrt } x \leq (1/32)*(65536*x^4 + 114688*x^3 + 17920*x^2 + 448*x + 1) / ((16*x + 1)*(256*x^2 + 96*x + 1))$
{proof}

lemma *sqrt-upper-bound-2-large*:

assumes $0 \leq x$

shows $\text{sqrt } x \leq (1/32)*(65536 + 114688*x + 17920*x^2 + 448*x^3 + x^4) / ((x + 16)*(256 + 96*x + x^2))$
{proof}

6.2 Lower bounds

lemma *sqrt-lower-bound-id*:

assumes $0 \leq x \leq 1$

shows $x \leq \text{sqrt } x$

<proof>

definition *sqrtn* :: [real,nat] \Rightarrow real **where**

sqrtn $x\ n = x / \text{sqrtn } x\ n$

lemma *sqrtn-lower*: $0 \leq x \implies \text{sqrtn } x\ n^2 \leq x$

<proof>

theorem *gen-sqrtn-lower*: $0 \leq x \implies \text{sqrtn } x\ n \leq \text{sqrtn } x$

<proof>

lemma *sqrtn-lower-bound-0*:

assumes $x \geq 0$ **shows** $2*x/(x+1) \leq \text{sqrtn } x$ (**is** ?lhs \leq -)

<proof>

lemma *sqrtn-lower-bound-1*:

assumes $x \geq 0$ **shows** $4*x*(x+1) / (x^2+6*x+1) \leq \text{sqrtn } x$ (**is** ?lhs \leq -)

<proof>

lemma *sqrtn-2-eq*: *sqrtn* $x\ 2 =$

$8*x*(x+1)*(x^2+6*x+1) / (x^4+28*x^3+70*x^2+28*x+1)$

<proof>

lemma *sqrtn-lower-bound-2*:

assumes $x \geq 0$

shows $8*x*(x+1)*(x^2+6*x+1) / (x^4+28*x^3+70*x^2+28*x+1) \leq \text{sqrtn } x$

<proof>

lemma *sqrtn-4-eq*: $x \geq 0 \implies$

sqrtn $x\ 4$

$= (32*x*(x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7-1) / (225792840*x^6+64512240*x^5+601080390*x^8+471435600*x^7+496*x+1+35960*x^2+906192*x^3-1)$

<proof>

lemma *sqrtn-lower-bound-4*:

assumes $x \geq 0$

shows $(32*x*(x+1)*(x^2+6*x+1)*(x^4+28*x^3+70*x^2+28*x+1)*(1820*x^6+8008*x^5+x^8+120*x^7-1) / (225792840*x^6+64512240*x^5+601080390*x^8+471435600*x^7+496*x+1+35960*x^2+906192*x^3-1) \leq \text{sqrtn } x$

<proof>

lemma *gen-sqrtn-lower-scaled*:

assumes $0 \leq x < u$

shows *sqrtn* $(x*u^2)\ n / u \leq \text{sqrtn } x$

<proof>

lemma *sqrtn-lower-bound-2-small*:

assumes $0 \leq x$

shows $32*x*(16*x + 1)*(256*x^2 + 96*x + 1) / (65536*x^4 + 114688*x^3 + 17920*x^2 + 448*x + 1) \leq \text{sqrt } x$
(proof)

lemma *sqrt-lower-bound-2-large*:

assumes $0 \leq x$

shows $32*x*(x + 16)*(x^2 + 96*x + 256) / (x^4 + 448*x^3 + 17920*x^2 + 114688*x + 65536) \leq \text{sqrt } x$
(proof)

end

Bibliography

- [1] B. Akbarpour and L. Paulson. MetiTarski: An automatic theorem prover for real-valued special functions. 44(3):175–205, Mar. 2010.