Real-Valued Special Functions: Upper and Lower Bounds

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Abstract

This development proves upper and lower bounds for several familiar real-valued functions. For sin, cos, exp and the square root function, it defines and verifies infinite families of upper and lower bounds, mostly based on Taylor series expansions. For $\tan^{-1}$, ln and exp, it verifies a finite collection of upper and lower bounds, originally obtained from the functions' continued fraction expansions using the computer algebra system Maple. A common theme in these proofs is to take the difference between a function and its approximation, which should be zero at one point, and then consider the sign of the derivative.

The immediate purpose of this development is to verify axioms used by MetiTarski [1], an automatic theorem prover for real-valued special functions. Crucial to MetiTarski's operation is the provision of upper and lower bounds for each function of interest.

Contents

Chapter 1

General Lemmas for Proving Function Inequalities

theory *Bounds-Lemmas*
imports *Complex-Main*

begin

These are for functions that are differentiable over a closed interval.

lemma *gen-lower-bound-increasing*:
  fixes $a :: real$
  assumes $a \leq x$
    and $\bigwedge y.\ a \leq y \Longrightarrow y \leq x \Longrightarrow ((\lambda x.\ fl\ x - f\ x)\ \mathit{has\text{-}real\text{-}derivative}\ g\ y)\ (at\ y)$
    and $\bigwedge y.\ a \leq y \Longrightarrow y \leq x \Longrightarrow g\ y \leq 0$
    and $fl\ a = f\ a$
  shows $fl\ x \leq f\ x$
⟨*proof*⟩

lemma *gen-lower-bound-decreasing*:
  fixes $a :: real$
  assumes $x \leq a$
    and $\bigwedge y.\ x \leq y \Longrightarrow y \leq a \Longrightarrow ((\lambda x.\ fl\ x - f\ x)\ \mathit{has\text{-}real\text{-}derivative}\ g\ y)\ (at\ y)$
    and $\bigwedge y.\ x \leq y \Longrightarrow y \leq a \Longrightarrow g\ y \geq 0$
    and $fl\ a = f\ a$
  shows $fl\ x \leq f\ x$
⟨*proof*⟩

lemma *gen-upper-bound-increasing*:
  fixes $a :: real$
  assumes $a \leq x$
    and $\bigwedge y.\ a \leq y \Longrightarrow y \leq x \Longrightarrow ((\lambda x.\ fu\ x - f\ x)\ \mathit{has\text{-}real\text{-}derivative}\ g\ y)\ (at\ y)$
    and $\bigwedge y.\ a \leq y \Longrightarrow y \leq x \Longrightarrow g\ y \geq 0$
    and $fu\ a = f\ a$
  shows $f\ x \leq fu\ x$
⟨*proof*⟩

lemma *gen-upper-bound-decreasing*:
fixes a :: *real*
assumes $x \leq a$
and $\bigwedge y.\ x \leq y \Longrightarrow y \leq a \Longrightarrow ((\lambda x.\ fu\ x - f\ x)$ *has-real-derivative* $g\ y)\ (at\ y)$
and $\bigwedge y.\ x \leq y \Longrightarrow y \leq a \Longrightarrow g\ y \leq 0$
and $fu\ a = f\ a$
shows $f\ x \leq fu\ x$
⟨*proof*⟩

end

Chapter 2

Arctan Upper and Lower Bounds

theory *Atan-CF-Bounds*
imports *Bounds-Lemmas*
HOL−Library.Sum-of-Squares

begin

Covers all bounds used in arctan-upper.ax, arctan-lower.ax and arctan-extended.ax, excepting only arctan-extended2.ax, which is used in two atan-error-analysis problems.

2.1 Upper Bound 1

definition *arctan-upper-11* :: *real* ⇒ *real*
where *arctan-upper-11* ≡ $\lambda x.\ -(pi/2) - 1/x$

definition *diff-delta-arctan-upper-11* :: *real* ⇒ *real*
where *diff-delta-arctan-upper-11* ≡ $\lambda x.\ 1\ /\ (x\hat{}2 * (1 + x\hat{}2))$

lemma *d-delta-arctan-upper-11*: $x \neq 0 \Longrightarrow$
(($\lambda x.$ *arctan-upper-11* x − *arctan* x) *has-field-derivative* *diff-delta-arctan-upper-11* x) (*at* x)
⟨*proof*⟩

lemma *d-delta-arctan-upper-11-pos*: $x \neq 0 \Longrightarrow$ *diff-delta-arctan-upper-11* $x > 0$
⟨*proof*⟩

Different proof needed here: they coincide not at zero, but at (-) infinity!

lemma *arctan-upper-11*:
assumes $x < 0$
shows *arctan*(x) < *arctan-upper-11* x
⟨*proof*⟩

definition *arctan-upper-12* :: *real* ⇒ *real*
where *arctan-upper-12* ≡ λ*x*. *3*∗*x* / (*x*^*2* + *3*)

definition *diff-delta-arctan-upper-12* :: *real* ⇒ *real*
where *diff-delta-arctan-upper-12* ≡ λ*x*. −*4*∗*x*^*4* / ((*x*^*2*+*3*)^*2* ∗ (*1*+*x*^*2*))

lemma *d-delta-arctan-upper-12*:
((λ*x*. *arctan-upper-12 x* − *arctan x*) *has-field-derivative diff-delta-arctan-upper-12 x*) (*at x*)
⟨*proof*⟩

Strict inequalities also possible

lemma *arctan-upper-12*:
assumes *x* ≤ *0* **shows** *arctan*(*x*) ≤ *arctan-upper-12 x*
⟨*proof*⟩

definition *arctan-upper-13* :: *real* ⇒ *real*
where *arctan-upper-13* ≡ λ*x*. *x*

definition *diff-delta-arctan-upper-13* :: *real* ⇒ *real*
where *diff-delta-arctan-upper-13* ≡ λ*x*. *x*^*2* / (*1* + *x*^*2*)

lemma *d-delta-arctan-upper-13*:
((λ*x*. *arctan-upper-13 x* − *arctan x*) *has-field-derivative diff-delta-arctan-upper-13 x*) (*at x*)
⟨*proof*⟩

lemma *arctan-upper-13*:
assumes *x* ≥ *0* **shows** *arctan*(*x*) ≤ *arctan-upper-13 x*
⟨*proof*⟩

definition *arctan-upper-14* :: *real* ⇒ *real*
where *arctan-upper-14* ≡ λ*x*. *pi*/*2* − *3*∗*x* / (*1* + *3*∗*x*^*2*)

definition *diff-delta-arctan-upper-14* :: *real* ⇒ *real*
where *diff-delta-arctan-upper-14* ≡ λ*x*. −*4* / ((*1* + *3*∗*x*^*2*)^*2* ∗ (*1*+*x*^*2*))

lemma *d-delta-arctan-upper-14*:
((λ*x*. *arctan-upper-14 x* − *arctan x*) *has-field-derivative diff-delta-arctan-upper-14 x*) (*at x*)
⟨*proof*⟩

lemma *d-delta-arctan-upper-14-neg*: *diff-delta-arctan-upper-14 x* < *0*
⟨*proof*⟩

lemma *lim14*: ((λ*x*::*real*. *3* ∗ *x* / (*1* + *3* ∗ x^2)) ⟶ *0*) *at-infinity*
⟨*proof*⟩

Different proof needed here: they coincide not at zero, but at (+) infinity!

lemma *arctan-upper-14*:
assumes $x > 0$
shows $arctan(x) < arctan\text{-}upper\text{-}14\ x$
⟨*proof*⟩

2.2 Lower Bound 1

definition *arctan-lower-11* :: *real* ⇒ *real*
where $arctan\text{-}lower\text{-}11 \equiv \lambda x.\ -(pi/2) - 3{*}x\ /\ (1 + 3{*}x\hat{}2)$

lemma *arctan-lower-11*:
assumes $x < 0$
shows $arctan(x) > arctan\text{-}lower\text{-}11\ x$
⟨*proof*⟩

abbreviation $arctan\text{-}lower\text{-}12 \equiv arctan\text{-}upper\text{-}13$

lemma *arctan-lower-12*:
assumes $x \leq 0$
shows $arctan(x) \geq arctan\text{-}lower\text{-}12\ x$
⟨*proof*⟩

abbreviation $arctan\text{-}lower\text{-}13 \equiv arctan\text{-}upper\text{-}12$

lemma *arctan-lower-13*:
assumes $x \geq 0$
shows $arctan(x) \geq arctan\text{-}lower\text{-}13\ x$
⟨*proof*⟩

definition *arctan-lower-14* :: *real* ⇒ *real*
where $arctan\text{-}lower\text{-}14 \equiv \lambda x.\ pi/2 - 1/x$

lemma *arctan-lower-14*:
assumes $x > 0$
shows $arctan(x) > arctan\text{-}lower\text{-}14\ x$
⟨*proof*⟩

2.3 Upper Bound 3

definition *arctan-upper-31* :: *real* ⇒ *real*
where $arctan\text{-}upper\text{-}31 \equiv \lambda x.\ -(pi/2) - (64 + 735{*}x\hat{}2 + 945{*}x\hat{}4)\ /\ (15{*}x{*}(15 + 70{*}x\hat{}2 + 63{*}x\hat{}4))$

definition *diff-delta-arctan-upper-31* :: *real* ⇒ *real*
where $diff\text{-}delta\text{-}arctan\text{-}upper\text{-}31 \equiv \lambda x.\ 64\ /\ (x\hat{}2 * (15 + 70{*}x\hat{}2 + 63{*}x\hat{}4)\hat{}2 * (1 + x\hat{}2))$

lemma *d-delta-arctan-upper-31*:

assumes $x \neq 0$
shows $((\lambda x.\ arctan\text{-}upper\text{-}31\ x - arctan\ x)\ has\text{-}field\text{-}derivative\ diff\text{-}delta\text{-}arctan\text{-}upper\text{-}31\ x)\ (at\ x)$
⟨*proof*⟩

lemma *d-delta-arctan-upper-31-pos*: $x \neq 0 \Longrightarrow diff\text{-}delta\text{-}arctan\text{-}upper\text{-}31\ x > 0$
⟨*proof*⟩

lemma *arctan-upper-31*:
assumes $x < 0$
shows $arctan(x) < arctan\text{-}upper\text{-}31\ x$
⟨*proof*⟩

definition *arctan-upper-32* :: *real* ⇒ *real*
where $arctan\text{-}upper\text{-}32 \equiv \lambda x.\ 7*(33*x\hat{}4 + 170*x\hat{}2 + 165)*x\ /\ (5*(5*x\hat{}6 + 105*x\hat{}4 + 315*x\hat{}2 + 231))$

definition *diff-delta-arctan-upper-32* :: *real* ⇒ *real*
where $diff\text{-}delta\text{-}arctan\text{-}upper\text{-}32 \equiv \lambda x.\ -256*x\hat{}12\ /\ ((5*x\hat{}6+105*x\hat{}4+315*x\hat{}2+231)\hat{}2*(1+x\hat{}2))$

lemma *d-delta-arctan-upper-32*:
$((\lambda x.\ arctan\text{-}upper\text{-}32\ x - arctan\ x)\ has\text{-}field\text{-}derivative\ diff\text{-}delta\text{-}arctan\text{-}upper\text{-}32\ x)\ (at\ x)$
⟨*proof*⟩

lemma *arctan-upper-32*:
assumes $x \leq 0$ **shows** $arctan(x) \leq arctan\text{-}upper\text{-}32\ x$
⟨*proof*⟩

definition *arctan-upper-33* :: *real* ⇒ *real*
where $arctan\text{-}upper\text{-}33 \equiv \lambda x.\ (64*x\hat{}4+735*x\hat{}2+945)*x\ /\ (15*(15*x\hat{}4+70*x\hat{}2+63))$

definition *diff-delta-arctan-upper-33* :: *real* ⇒ *real*
where $diff\text{-}delta\text{-}arctan\text{-}upper\text{-}33 \equiv \lambda x.\ 64*x\hat{}10\ /\ ((15*x\hat{}4+70*x\hat{}2+63)\hat{}2*(1+x\hat{}2))$

lemma *d-delta-arctan-upper-33*:
$((\lambda x.\ arctan\text{-}upper\text{-}33\ x - arctan\ x)\ has\text{-}field\text{-}derivative\ diff\text{-}delta\text{-}arctan\text{-}upper\text{-}33\ x)\ (at\ x)$
⟨*proof*⟩

lemma *arctan-upper-33*:
assumes $x \geq 0$ **shows** $arctan(x) \leq arctan\text{-}upper\text{-}33\ x$
⟨*proof*⟩

definition *arctan-upper-34* :: *real* ⇒ *real*
where $arctan\text{-}upper\text{-}34 \equiv$
$\lambda x.\ pi/2 - (33 + 170*x\hat{}2 + 165*x\hat{}4)*7*x\ /\ (5*(5 + 105*x\hat{}2 + 315*x\hat{}4 + 231*x\hat{}6))$

definition *diff-delta-arctan-upper-34* :: *real* ⇒ *real*
where *diff-delta-arctan-upper-34* ≡ $\lambda x.\ -256\ /\ ((5+105*x\hat{}2+315*x\hat{}4+231*x\hat{}6)\hat{}2*(1+x\hat{}2))$

lemma *d-delta-arctan-upper-34*:
((λ*x*. *arctan-upper-34 x* − *arctan x*) *has-field-derivative diff-delta-arctan-upper-34 x*) (*at x*)
⟨*proof*⟩

lemma *d-delta-arctan-upper-34-pos*: *diff-delta-arctan-upper-34 x* < *0*
⟨*proof*⟩

lemma *arctan-upper-34*:
assumes *x* > *0*
shows *arctan*(*x*) < *arctan-upper-34 x*
⟨*proof*⟩

2.4 Lower Bound 3

definition *arctan-lower-31* :: *real* ⇒ *real*
where *arctan-lower-31* ≡ $\lambda x.\ -(pi/2) - (33 + 170*x\hat{}2 + 165*x\hat{}4)*7*x\ /\ (5*(5 + 105*x\hat{}2 + 315*x\hat{}4 + 231*x\hat{}6))$

lemma *arctan-lower-31*:
assumes *x* < *0*
shows *arctan*(*x*) > *arctan-lower-31 x*
⟨*proof*⟩

abbreviation *arctan-lower-32* ≡ *arctan-upper-33*

lemma *arctan-lower-32*:
assumes *x* ≤ *0*
shows *arctan*(*x*) ≥ *arctan-lower-32 x*
⟨*proof*⟩

abbreviation *arctan-lower-33* ≡ *arctan-upper-32*

lemma *arctan-lower-33*:
assumes *x* ≥ *0*
shows *arctan*(*x*) ≥ *arctan-lower-33 x*
⟨*proof*⟩

definition *arctan-lower-34* :: *real* ⇒ *real*
where *arctan-lower-34* ≡ $\lambda x.\ pi/2 - (64 + 735*x\hat{}2 + 945*x\hat{}4)\ /\ (15*x*(15 + 70*x\hat{}2 + 63*x\hat{}4))$

lemma *arctan-lower-34*:
assumes *x* > *0*
shows *arctan*(*x*) > *arctan-lower-34 x*
⟨*proof*⟩

2.5 Upper Bound 4

definition *arctan-upper-41* :: *real* ⇒ *real*
  where *arctan-upper-41* ≡
    λx. −(pi/2) − (256 + 5943*x^2 + 19250*x^4 + 15015*x^6) /
      (35*x*(35 + 315*x^2 + 693*x^4 + 429*x^6))

definition *diff-delta-arctan-upper-41* :: *real* ⇒ *real*
 where *diff-delta-arctan-upper-41* ≡ λx. 256 / (x^2*(35+315*x^2+693*x^4+429*x^6)^2*(1+x^2))

lemma *d-delta-arctan-upper-41*:
  assumes x ≠ 0
   shows ((λx. *arctan-upper-41* x − *arctan* x) *has-field-derivative* *diff-delta-arctan-upper-41* x) (*at* x)
  ⟨*proof*⟩

lemma *d-delta-arctan-upper-41-pos*: x ≠ 0 ⟹ *diff-delta-arctan-upper-41* x > 0
⟨*proof*⟩

lemma *arctan-upper-41*:
  assumes x < 0
    shows *arctan*(x) < *arctan-upper-41* x
⟨*proof*⟩

definition *arctan-upper-42* :: *real* ⇒ *real*
  where *arctan-upper-42* ≡
    λx. (15159*x^6+147455*x^4+345345*x^2+225225)*x / (35*(35*x^8+1260*x^6+6930*x^4+12012*

definition *diff-delta-arctan-upper-42* :: *real* ⇒ *real*
  where *diff-delta-arctan-upper-42* ≡
    λx. −16384*x^16 / ((35*x^8+1260*x^6+6930*x^4+12012*x^2+6435)^2*(1+x^2))

lemma *d-delta-arctan-upper-42*:
  ((λx. *arctan-upper-42* x − *arctan* x) *has-field-derivative* *diff-delta-arctan-upper-42* x) (*at* x)
    ⟨*proof*⟩

lemma *arctan-upper-42*:
  assumes x ≤ 0 **shows** *arctan*(x) ≤ *arctan-upper-42* x
⟨*proof*⟩

definition *arctan-upper-43* :: *real* ⇒ *real*
  where *arctan-upper-43* ≡
    λx. (256*x^6+5943*x^4+19250*x^2+15015)*x /
      (35 * (35*x^6+315*x^4+693*x^2+429))

definition *diff-delta-arctan-upper-43* :: *real* ⇒ *real*
 where *diff-delta-arctan-upper-43* ≡ λx. 256*x^14 / ((35*x^6+315*x^4+693*x^2+429)^2*(1+x^2))

lemma *d-delta-arctan-upper-43*:
((λ*x*. *arctan-upper-43 x* − *arctan x*) *has-field-derivative diff-delta-arctan-upper-43 x*) (*at x*)
⟨*proof*⟩

lemma *arctan-upper-43*:
assumes *x* ≥ *0* **shows** *arctan*(*x*) ≤ *arctan-upper-43 x*
⟨*proof*⟩

definition *arctan-upper-44* :: *real* ⇒ *real*
where *arctan-upper-44* ≡
λ*x*. *pi/2* − (*15159*+*147455*∗*x*^*2*+*345345*∗*x*^*4*+*225225*∗*x*^*6*)∗*x* /
(*35*∗(*35*+*1260*∗*x*^*2*+*6930*∗*x*^*4*+*12012*∗*x*^*6*+*6435*∗*x*^*8*))

definition *diff-delta-arctan-upper-44* :: *real* ⇒ *real*
where *diff-delta-arctan-upper-44* ≡
λ*x*. −*16384* / ((*35*+*1260*∗*x*^*2*+*6930*∗*x*^*4*+*12012*∗*x*^*6*+*6435*∗*x*^*8*)^*2*∗(*1*+*x*^*2*))

lemma *d-delta-arctan-upper-44*:
((λ*x*. *arctan-upper-44 x* − *arctan x*) *has-field-derivative diff-delta-arctan-upper-44 x*) (*at x*)
⟨*proof*⟩

lemma *d-delta-arctan-upper-44-pos*: *diff-delta-arctan-upper-44 x* < *0*
⟨*proof*⟩

lemma *arctan-upper-44*:
assumes *x* > *0*
shows *arctan*(*x*) < *arctan-upper-44 x*
⟨*proof*⟩

2.6 Lower Bound 4

definition *arctan-lower-41* :: *real* ⇒ *real*
where *arctan-lower-41* ≡
λ*x*. −(*pi/2*) − (*15159*+*147455*∗*x*^*2*+*345345*∗*x*^*4*+*225225*∗*x*^*6*)∗*x* /
(*35*∗(*35*+*1260*∗*x*^*2*+*6930*∗*x*^*4*+*12012*∗*x*^*6*+*6435*∗*x*^*8*))

lemma *arctan-lower-41*:
assumes *x* < *0*
shows *arctan*(*x*) > *arctan-lower-41 x*
⟨*proof*⟩

abbreviation *arctan-lower-42* ≡ *arctan-upper-43*

lemma *arctan-lower-42*:
assumes *x* ≤ *0*
shows *arctan*(*x*) ≥ *arctan-lower-42 x*
⟨*proof*⟩

abbreviation *arctan-lower-43* ≡ *arctan-upper-42*

lemma *arctan-lower-43*:
 assumes $x \geq 0$
  shows $arctan(x) \geq arctan\text{-}lower\text{-}43\ x$
  ⟨*proof*⟩

definition *arctan-lower-44* :: *real* ⇒ *real*
 where *arctan-lower-44* ≡
  $\lambda x.\ pi/2 - (256+5943*x\hat{}2+19250*x\hat{}4+15015*x\hat{}6)\ /$
   $(35*x*(35+315*x\hat{}2+693*x\hat{}4+429*x\hat{}6))$

lemma *arctan-lower-44*:
 assumes $x > 0$
  shows $arctan(x) > arctan\text{-}lower\text{-}44\ x$
  ⟨*proof*⟩

end

Chapter 3

Exp Upper and Lower Bounds

theory *Exp-Bounds*
imports *Bounds-Lemmas*
    HOL−Library.Sum-of-Squares
    Sturm-Sequences.Sturm

begin

Covers all bounds used in exp-upper.ax, exp-lower.ax and exp-extended.ax.

3.1 Taylor Series Bounds

exp-positive is the theorem $0 \leq exp\ ?x$

exp-lower-taylor-1 is the theorem $1 + ?x \leq exp\ ?x$

All even approximants are lower bounds.

lemma *exp-lower-taylor-even*:
  fixes $x::real$
  shows $even\ n \Longrightarrow (\sum m{<}n.\ (x\ \hat{}\ m)\ /\ (fact\ m)) \leq exp\ x$
  ⟨*proof*⟩

lemma *exp-upper-taylor-even*:
  fixes $x::real$
  assumes n: $even\ n$
    and pos: $(\sum m{<}n.\ ((-x)\ \hat{}\ m)\ /\ (fact\ m)) > 0$ (**is** $?sum > 0$)
  shows $exp\ x \leq inverse\ ?sum$
  ⟨*proof*⟩

3 if the previous lemma is expressed in terms of $(2::'a) * m$.

lemma *exp-lower-taylor-3*:
  fixes $x::real$

shows *1 + x + (1/2)∗x^2 + (1/6)∗x^3 + (1/24)∗x^4 + (1/120)∗x^5 ≤ exp x*
⟨*proof*⟩

lemma *exp-lower-taylor-3-cubed*:
fixes *x::real*
shows *(1 + x/3 + (1/2)∗(x/3)^2 + (1/6)∗(x/3)^3 + (1/24)∗(x/3)^4 + (1/120)∗(x/3)^5)^3 ≤ exp x*
⟨*proof*⟩

lemma *exp-lower-taylor-2*:
fixes *x::real*
shows *1 + x + (1/2)∗x^2 + (1/6)∗x^3 ≤ exp x*
⟨*proof*⟩

lemma *exp-upper-bound-case-3*:
fixes *x::real*
assumes *x ≤ 3.19*
shows *exp x ≤ 2304 / (−(x^3) + 6∗x^2 − 24∗x + 48)^2*
⟨*proof*⟩

lemma *exp-upper-bound-case-5*:
fixes *x::real*
assumes *x ≤ 6.36*
shows *exp x ≤ 21743271936 / (−(x^3) + 12∗x^2 − 96∗x + 384)^4*
⟨*proof*⟩

3.2 Continued Fraction Bound 2

definition *exp-cf2 :: real ⇒ real*
where *exp-cf2 ≡ λx. (x^2 + 6∗x + 12) / (x^2 − 6∗x + 12)*

lemma *denom-cf2-pos*: **fixes** *x::real* **shows** $x^2 - 6 * x + 12 > 0$
⟨*proof*⟩

lemma *numer-cf2-pos*: **fixes** *x::real* **shows** $x^2 + 6 * x + 12 > 0$
⟨*proof*⟩

lemma *exp-cf2-pos*: *exp-cf2 x > 0*
⟨*proof*⟩

definition *diff-delta-lnexp-cf2 :: real ⇒ real*
where *diff-delta-lnexp-cf2 ≡ λx. − (x^4) / ((x^2 − 6∗x + 12) ∗ (x^2 + 6∗x + 12))*

lemma *d-delta-lnexp-cf2-nonpos*: *diff-delta-lnexp-cf2 x ≤ 0*
⟨*proof*⟩

lemma *d-delta-lnexp-cf2*:

$((\lambda x.\ ln\ (exp\text{-}cf2\ x) - x)$ *has-field-derivative diff-delta-lnexp-cf2* $x)$ $(at\ x)$
⟨*proof*⟩

Upper bound for non-positive x

lemma *ln-exp-cf2-upper-bound-neg*:
assumes $x \leq 0$
shows $x \leq ln\ (exp\text{-}cf2\ x)$
⟨*proof*⟩

theorem *exp-cf2-upper-bound-neg*: $x \leq 0 \Longrightarrow exp(x) \leq exp\text{-}cf2\ x$
⟨*proof*⟩

Lower bound for non-negative x

lemma *ln-exp-cf2-lower-bound-pos*:
assumes $0 \leq x$
shows $ln\ (exp\text{-}cf2\ x) \leq x$
⟨*proof*⟩

theorem *exp-cf2-lower-bound-pos*: $0 \leq x \Longrightarrow exp\text{-}cf2\ x \leq exp\ x$
⟨*proof*⟩

3.3 Continued Fraction Bound 3

This bound crosses the X-axis twice, causing complications.

definition *numer-cf3* :: *real* ⇒ *real*
where $numer\text{-}cf3 \equiv \lambda x.\ x\hat{\ }3 + 12{*}x\hat{\ }2 + 60{*}x + 120$

definition *exp-cf3* :: *real* ⇒ *real*
where $exp\text{-}cf3 \equiv \lambda x.\ numer\text{-}cf3\ x\ /\ numer\text{-}cf3\ (-x)$

lemma *numer-cf3-pos*: $-4.64 \leq x \Longrightarrow numer\text{-}cf3\ x > 0$
⟨*proof*⟩

lemma *exp-cf3-pos*: $numer\text{-}cf3\ x > 0 \Longrightarrow numer\text{-}cf3\ (-x) > 0 \Longrightarrow exp\text{-}cf3\ x > 0$
⟨*proof*⟩

definition *diff-delta-lnexp-cf3* :: *real* ⇒ *real*
where $diff\text{-}delta\text{-}lnexp\text{-}cf3 \equiv \lambda x.\ (x\hat{\ }6)\ /\ (numer\text{-}cf3\ (-x) * numer\text{-}cf3\ x)$

lemma *d-delta-lnexp-cf3-nonneg*: $numer\text{-}cf3\ x > 0 \Longrightarrow numer\text{-}cf3\ (-x) > 0 \Longrightarrow$ $diff\text{-}delta\text{-}lnexp\text{-}cf3\ x \geq 0$
⟨*proof*⟩

lemma *d-delta-lnexp-cf3*:
assumes $numer\text{-}cf3\ x > 0$ $numer\text{-}cf3\ (-x) > 0$
shows $((\lambda x.\ ln\ (exp\text{-}cf3\ x) - x)$ *has-field-derivative diff-delta-lnexp-cf3* $x)$ $(at\ x)$
⟨*proof*⟩

lemma *numer-cf3-mono*: $y \leq x \Longrightarrow$ *numer-cf3* $y \leq$ *numer-cf3* x
⟨*proof*⟩

Upper bound for non-negative x

lemma *ln-exp-cf3-upper-bound-nonneg*:
assumes *x0*: $0 \leq x$ **and** *xless*: *numer-cf3* $(-x) > 0$
shows $x \leq$ *ln* (*exp-cf3* x)
⟨*proof*⟩

theorem *exp-cf3-upper-bound-pos*: $0 \leq x \Longrightarrow$ *numer-cf3* $(-x) > 0 \Longrightarrow$ *exp* $x \leq$ *exp-cf3* x
⟨*proof*⟩

corollary $0 \leq x \Longrightarrow x \leq 4.64 \Longrightarrow$ *exp* $x \leq$ *exp-cf3* x
⟨*proof*⟩

Lower bound for negative x, provided $0 <$ *exp-cf3* x]

lemma *ln-exp-cf3-lower-bound-neg*:
assumes *x0*: $x \leq 0$ **and** *xgtr*: *numer-cf3* $x > 0$
shows *ln* (*exp-cf3* x) $\leq x$
⟨*proof*⟩

theorem *exp-cf3-lower-bound-pos*:
assumes $x \leq 0$ **shows** *exp-cf3* $x \leq$ *exp* x
⟨*proof*⟩

3.4 Continued Fraction Bound 4

Here we have the extended exp continued fraction bounds

definition *numer-cf4* :: *real* ⇒ *real*
where *numer-cf4* ≡ λ*x*. *x*^*4* + *20*∗*x*^*3* + *180*∗*x*^*2* + *840*∗*x* + *1680*

definition *exp-cf4* :: *real* ⇒ *real*
where *exp-cf4* ≡ λ*x*. *numer-cf4* *x* / *numer-cf4* (−*x*)

lemma *numer-cf4-pos*: **fixes** *x*::*real* **shows** *numer-cf4* $x > 0$
⟨*proof*⟩

lemma *exp-cf4-pos*: *exp-cf4* $x > 0$
⟨*proof*⟩

definition *diff-delta-lnexp-cf4* :: *real* ⇒ *real*
where *diff-delta-lnexp-cf4* ≡ λ*x*. − (*x*^*8*) / (*numer-cf4* (−*x*) ∗ *numer-cf4* *x*)

lemma *d-delta-lnexp-cf4-nonpos*: *diff-delta-lnexp-cf4* $x \leq 0$
⟨*proof*⟩

lemma *d-delta-lnexp-cf4*:

((λx. ln (exp-cf4 x) − x) has-field-derivative diff-delta-lnexp-cf4 x) (at x)
⟨proof⟩

Upper bound for non-positive x

lemma *ln-exp-cf4-upper-bound-neg*:
assumes $x \leq 0$
shows $x \leq ln\ (exp\text{-}cf4\ x)$
⟨proof⟩

theorem *exp-cf4-upper-bound-neg*: $x \leq 0 \Longrightarrow exp(x) \leq exp\text{-}cf4\ x$
⟨proof⟩

Lower bound for non-negative x

lemma *ln-exp-cf4-lower-bound-pos*:
assumes $0 \leq x$
shows $ln\ (exp\text{-}cf4\ x) \leq x$
⟨proof⟩

theorem *exp-cf4-lower-bound-pos*: $0 \leq x \Longrightarrow exp\text{-}cf4\ x \leq exp\ x$
⟨proof⟩

3.5 Continued Fraction Bound 5

This bound crosses the X-axis twice, causing complications.

definition *numer-cf5* :: *real* ⇒ *real*
where $numer\text{-}cf5 \equiv \lambda x.\ x\hat{}5 + 30*x\hat{}4 + 420*x\hat{}3 + 3360*x\hat{}2 + 15120*x + 30240$

definition *exp-cf5* :: *real* ⇒ *real*
where $exp\text{-}cf5 \equiv \lambda x.\ numer\text{-}cf5\ x\ /\ numer\text{-}cf5\ (-x)$

lemma *numer-cf5-pos*: $-7.293 \leq x \Longrightarrow numer\text{-}cf5\ x > 0$
⟨proof⟩

lemma *exp-cf5-pos*: $numer\text{-}cf5\ x > 0 \Longrightarrow numer\text{-}cf5\ (-x) > 0 \Longrightarrow exp\text{-}cf5\ x > 0$
⟨proof⟩

definition *diff-delta-lnexp-cf5* :: *real* ⇒ *real*
where $diff\text{-}delta\text{-}lnexp\text{-}cf5 \equiv \lambda x.\ (x\hat{}10)\ /\ (numer\text{-}cf5\ (-x) * numer\text{-}cf5\ x)$

lemma *d-delta-lnexp-cf5-nonneg*: $numer\text{-}cf5\ x > 0 \Longrightarrow numer\text{-}cf5\ (-x) > 0 \Longrightarrow diff\text{-}delta\text{-}lnexp\text{-}cf5\ x \geq 0$
⟨proof⟩

lemma *d-delta-lnexp-cf5*:
assumes $numer\text{-}cf5\ x > 0\ numer\text{-}cf5\ (-x) > 0$
shows ((λx. ln (exp-cf5 x) − x) has-field-derivative diff-delta-lnexp-cf5 x) (at x)
⟨proof⟩

3.5.1 Proving monotonicity via a non-negative derivative

definition *numer-cf5-deriv* :: *real* ⇒ *real*
where *numer-cf5-deriv* ≡ λ*x*. *5∗x^4 + 120∗x^3 + 1260∗x^2 + 6720∗x + 15120*

lemma *numer-cf5-deriv*:
shows (*numer-cf5 has-field-derivative numer-cf5-deriv x*) (*at x*)
⟨*proof*⟩

lemma *numer-cf5-deriv-pos*: *numer-cf5-deriv x* ≥ *0*
⟨*proof*⟩

lemma *numer-cf5-mono*: *y* ≤ *x* ⟹ *numer-cf5 y* ≤ *numer-cf5 x*
⟨*proof*⟩

3.5.2 Results

Upper bound for non-negative x

lemma *ln-exp-cf5-upper-bound-nonneg*:
assumes *x0*: *0* ≤ *x* **and** *xless*: *numer-cf5* (−*x*) > *0*
shows *x* ≤ *ln* (*exp-cf5 x*)
⟨*proof*⟩

theorem *exp-cf5-upper-bound-pos*: *0* ≤ *x* ⟹ *numer-cf5* (−*x*) > *0* ⟹ *exp x* ≤ *exp-cf5 x*
⟨*proof*⟩

corollary *0* ≤ *x* ⟹ *x* ≤ *7.293* ⟹ *exp x* ≤ *exp-cf5 x*
⟨*proof*⟩

Lower bound for negative x, provided *0* < *exp-cf5 x*]

lemma *ln-exp-cf5-lower-bound-neg*:
assumes *x0*: *x* ≤ *0* **and** *xgtr*: *numer-cf5 x* > *0*
shows *ln* (*exp-cf5 x*) ≤ *x*
⟨*proof*⟩

theorem *exp-cf5-lower-bound-pos*:
assumes *x* ≤ *0* **shows** *exp-cf5 x* ≤ *exp x*
⟨*proof*⟩

3.6 Continued Fraction Bound 6

definition *numer-cf6* :: *real* ⇒ *real*
where *numer-cf6* ≡ λ*x*. *x^6 + 42∗x^5 + 840∗x^4 + 10080∗x^3 + 75600∗x^2 + 332640∗x + 665280*

definition *exp-cf6* :: *real* ⇒ *real*
where *exp-cf6* ≡ λ*x*. *numer-cf6 x* / *numer-cf6* (−*x*)

lemma *numer-cf6-pos*: **fixes** *x*::*real* **shows** *numer-cf6 x* > *0*
⟨*proof*⟩

lemma *exp-cf6-pos*: *exp-cf6 x* > *0*
⟨*proof*⟩

definition *diff-delta-lnexp-cf6* :: *real* ⇒ *real*
where *diff-delta-lnexp-cf6* ≡ λ*x*. − (*x*^*12*) / (*numer-cf6* (−*x*) ∗ *numer-cf6 x*)

lemma *d-delta-lnexp-cf6-nonpos*: *diff-delta-lnexp-cf6 x* ≤ *0*
⟨*proof*⟩

lemma *d-delta-lnexp-cf6*:
((λ*x*. *ln* (*exp-cf6 x*) − *x*) *has-field-derivative diff-delta-lnexp-cf6 x*) (*at x*)
⟨*proof*⟩

Upper bound for non-positive x

lemma *ln-exp-cf6-upper-bound-neg*:
assumes *x* ≤ *0*
shows *x* ≤ *ln* (*exp-cf6 x*)
⟨*proof*⟩

theorem *exp-cf6-upper-bound-neg*: *x*≤*0* ⟹ *exp*(*x*) ≤ *exp-cf6 x*
⟨*proof*⟩

Lower bound for non-negative x

lemma *ln-exp-cf6-lower-bound-pos*:
assumes *0*≤*x*
shows *ln* (*exp-cf6 x*) ≤ *x*
⟨*proof*⟩

theorem *exp-cf6-lower-bound-pos*: *0*≤*x* ⟹ *exp-cf6 x* ≤ *exp x*
⟨*proof*⟩

3.7 Continued Fraction Bound 7

This bound crosses the X-axis twice, causing complications.

definition *numer-cf7* :: *real* ⇒ *real*
where *numer-cf7* ≡ λ*x*. *x*^*7* + *56*∗*x*^*6* + *1512*∗*x*^*5* + *25200*∗*x*^*4* + *277200*∗*x*^*3* + *1995840*∗*x*^*2* + *8648640*∗*x* + *17297280*

definition *exp-cf7* :: *real* ⇒ *real*
where *exp-cf7* ≡ λ*x*. *numer-cf7 x* / *numer-cf7* (−*x*)

lemma *numer-cf7-pos*: −*9.943* ≤ *x* ⟹ *numer-cf7 x* > *0*
⟨*proof*⟩

lemma *exp-cf7-pos*: *numer-cf7 x* > *0* ⟹ *numer-cf7* (−*x*) > *0* ⟹ *exp-cf7 x* > *0*

⟨*proof*⟩

definition *diff-delta-lnexp-cf7* :: *real* ⇒ *real*
where *diff-delta-lnexp-cf7* ≡ *λx. (x^14) / (numer-cf7 (−x) ∗ numer-cf7 x)*

lemma *d-delta-lnexp-cf7-nonneg*: *numer-cf7 x > 0* ⟹ *numer-cf7 (−x) > 0* ⟹ *diff-delta-lnexp-cf7 x ≥ 0*
⟨*proof*⟩

lemma *d-delta-lnexp-cf7*:
assumes *numer-cf7 x > 0 numer-cf7 (−x) > 0*
shows ((*λx. ln (exp-cf7 x) − x*) *has-field-derivative diff-delta-lnexp-cf7 x*) (*at x*)
⟨*proof*⟩

3.7.1 Proving monotonicity via a non-negative derivative

definition *numer-cf7-deriv* :: *real* ⇒ *real*
where *numer-cf7-deriv* ≡ *λx. 7∗x^6 + 336∗x^5 + 7560∗x^4 + 100800∗x^3 + 831600∗x^2 + 3991680∗x + 8648640*

lemma *numer-cf7-deriv*:
shows (*numer-cf7 has-field-derivative numer-cf7-deriv x*) (*at x*)
⟨*proof*⟩

lemma *numer-cf7-deriv-pos*: *numer-cf7-deriv x ≥ 0*
⟨*proof*⟩

lemma *numer-cf7-mono*: *y ≤ x* ⟹ *numer-cf7 y ≤ numer-cf7 x*
⟨*proof*⟩

3.7.2 Results

Upper bound for non-negative x

lemma *ln-exp-cf7-upper-bound-nonneg*:
assumes *x0*: *0 ≤ x* **and** *xless*: *numer-cf7 (−x) > 0*
shows *x ≤ ln (exp-cf7 x)*
⟨*proof*⟩

theorem *exp-cf7-upper-bound-pos*: *0 ≤ x* ⟹ *numer-cf7 (−x) > 0* ⟹ *exp x ≤ exp-cf7 x*
⟨*proof*⟩

corollary *0 ≤ x* ⟹ *x ≤ 9.943* ⟹ *exp x ≤ exp-cf7 x*
⟨*proof*⟩

Lower bound for negative x, provided *0 < exp-cf7 x*]

lemma *ln-exp-cf7-lower-bound-neg*:
assumes *x0*: *x ≤ 0* **and** *xgtr*: *numer-cf7 x > 0*
shows *ln (exp-cf7 x) ≤ x*

⟨*proof*⟩

theorem *exp-cf7-lower-bound-pos*:
assumes $x \le 0$ **shows** *exp-cf7* $x \le$ *exp* x
⟨*proof*⟩

end

Chapter 4

Log Upper and Lower Bounds

theory *Log-CF-Bounds*
imports *Bounds-Lemmas*

begin

theorem *ln-upper-1*: $0<x \Longrightarrow ln(x::real) \leq x - 1$
⟨*proof*⟩

definition *ln-lower-1* :: *real* ⇒ *real*
  where *ln-lower-1* ≡ $\lambda x.\ 1 - (inverse\ x)$

corollary *ln-lower-1*: $0<x \Longrightarrow ln\text{-}lower\text{-}1\ x \leq ln\ x$
  ⟨*proof*⟩

theorem *ln-lower-1-eq*: $0<x \Longrightarrow ln\text{-}lower\text{-}1\ x = (x - 1)/x$
  ⟨*proof*⟩

4.1 Upper Bound 3

definition *ln-upper-3* :: *real* ⇒ *real*
  where *ln-upper-3* ≡ $\lambda x.\ (x + 5)*(x - 1) / (2*(2*x + 1))$

definition *diff-delta-ln-upper-3* :: *real* ⇒ *real*
  where *diff-delta-ln-upper-3* ≡ $\lambda x.\ (x - 1)\hat{}3 / ((2*x + 1)\hat{}2 * x)$

lemma *d-delta-ln-upper-3*: $x > 0 \Longrightarrow$
  $((\lambda x.\ ln\text{-}upper\text{-}3\ x - ln\ x)$ *has-field-derivative* *diff-delta-ln-upper-3 x*) (*at x*)
⟨*proof*⟩

Strict inequalities also possible

lemma *ln-upper-3-pos*:
  assumes $1 \leq x$ **shows** $ln(x) \leq ln\text{-}upper\text{-}3\ x$

⟨*proof*⟩

lemma *ln-upper-3-neg*:
assumes $0 < x$ **and** *x1*: $x \leq 1$ **shows** $ln(x) \leq ln\text{-}upper\text{-}3\ x$
⟨*proof*⟩

theorem *ln-upper-3*: $0<x \Longrightarrow ln(x) \leq ln\text{-}upper\text{-}3\ x$
⟨*proof*⟩

definition *ln-lower-3* :: *real* ⇒ *real*
where $ln\text{-}lower\text{-}3 \equiv \lambda x.\ -\ ln\text{-}upper\text{-}3\ (inverse\ x)$

corollary *ln-lower-3*: $0<x \Longrightarrow ln\text{-}lower\text{-}3\ x \leq ln\ x$
⟨*proof*⟩

theorem *ln-lower-3-eq*: $0<x \Longrightarrow ln\text{-}lower\text{-}3\ x = (1/2)*(1 + 5*x)*(x - 1) / (x*(2 + x))$
⟨*proof*⟩

4.2 Upper Bound 5

definition *ln-upper-5* :: *real* ⇒ *real*
where $ln\text{-}upper\text{-}5\ x \equiv (x\hat{}2 + 19*x + 10)*(x - 1) / (3*(3*x\hat{}2 + 6*x + 1))$

definition *diff-delta-ln-upper-5* :: *real* ⇒ *real*
where $diff\text{-}delta\text{-}ln\text{-}upper\text{-}5 \equiv \lambda x.\ (x - 1)\hat{}5 / ((3*x\hat{}2 + 6*x + 1)\hat{}2*x)$

lemma *d-delta-ln-upper-5*: $x > 0 \Longrightarrow$
$((\lambda x.\ ln\text{-}upper\text{-}5\ x - ln\ x)\ has\text{-}field\text{-}derivative\ diff\text{-}delta\text{-}ln\text{-}upper\text{-}5\ x)\ (at\ x)$
⟨*proof*⟩

lemma *ln-upper-5-pos*:
assumes $1 \leq x$ **shows** $ln(x) \leq ln\text{-}upper\text{-}5\ x$
⟨*proof*⟩

lemma *ln-upper-5-neg*:
assumes $0 < x$ **and** *x1*: $x \leq 1$ **shows** $ln(x) \leq ln\text{-}upper\text{-}5\ x$
⟨*proof*⟩

theorem *ln-upper-5*: $0<x \Longrightarrow ln(x) \leq ln\text{-}upper\text{-}5\ x$
⟨*proof*⟩

definition *ln-lower-5* :: *real* ⇒ *real*
where $ln\text{-}lower\text{-}5 \equiv \lambda x.\ -\ ln\text{-}upper\text{-}5\ (inverse\ x)$

corollary *ln-lower-5*: $0<x \Longrightarrow ln\text{-}lower\text{-}5\ x \leq ln\ x$
⟨*proof*⟩

theorem *ln-lower-5-eq*: $0<x \Longrightarrow$

ln-lower-5 x = (*1/3*)∗(*10*∗*x*^*2* + *19*∗*x* + *1*)∗(*x* − *1*) / (*x*∗(*x*^*2* + *6*∗*x* + *3*))
⟨*proof*⟩

4.3 Upper Bound 7

definition *ln-upper-7* :: *real* ⇒ *real*
where *ln-upper-7 x* ≡ (*3*∗*x*^*3* + *131*∗*x*^*2* + *239*∗*x* + *47*)∗(*x* − *1*) / (*12*∗(*4*∗*x*^*3* + *18*∗*x*^*2* + *12*∗*x* + *1*))

definition *diff-delta-ln-upper-7* :: *real* ⇒ *real*
where *diff-delta-ln-upper-7* ≡ λ*x*. (*x* − *1*)^*7* / ((*4*∗*x*^*3* + *18*∗*x*^*2* + *12*∗*x* + *1*)^*2* ∗ *x*)

lemma *d-delta-ln-upper-7*: *x* > *0* ⟹
((λ*x*. *ln-upper-7 x* − *ln x*) *has-field-derivative diff-delta-ln-upper-7 x*) (*at x*)
⟨*proof*⟩

lemma *ln-upper-7-pos*:
assumes *1* ≤ *x* **shows** *ln*(*x*) ≤ *ln-upper-7 x*
⟨*proof*⟩

lemma *ln-upper-7-neg*:
assumes *0* < *x* **and** *x1*: *x* ≤ *1* **shows** *ln*(*x*) ≤ *ln-upper-7 x*
⟨*proof*⟩

theorem *ln-upper-7*: *0* < *x* ⟹ *ln*(*x*) ≤ *ln-upper-7 x*
⟨*proof*⟩

definition *ln-lower-7* :: *real* ⇒ *real*
where *ln-lower-7* ≡ λ*x*. − *ln-upper-7* (*inverse x*)

corollary *ln-lower-7*: *0* < *x* ⟹ *ln-lower-7 x* ≤ *ln x*
⟨*proof*⟩

theorem *ln-lower-7-eq*: *0* < *x* ⟹
ln-lower-7 x = (*1/12*)∗(*47*∗*x*^*3* + *239*∗*x*^*2* + *131*∗*x* + *3*)∗(*x* − *1*) / (*x*∗(*x*^*3* + *12*∗*x*^*2* + *18*∗*x* + *4*))
⟨*proof*⟩

4.4 Upper Bound 9

definition *ln-upper-9* :: *real* ⇒ *real*
where *ln-upper-9 x* ≡ (*6*∗*x*^*4* + *481*∗*x*^*3* + *1881*∗*x*^*2* + *1281*∗*x* + *131*)∗(*x* − *1*) /
(*30* ∗ (*5*∗*x*^*4* + *40*∗*x*^*3* + *60*∗*x*^*2* + *20*∗*x* + *1*))

definition *diff-delta-ln-upper-9* :: *real* ⇒ *real*
where *diff-delta-ln-upper-9* ≡ λ*x*. (*x* − *1*)^*9* / (((*5*∗*x*^*4* + *40*∗*x*^*3* + *60*∗*x*^*2*

+ 20*x + 1)^2) * x)

lemma *d-delta-ln-upper-9*: *x* > *0* ⟹
 ((λ*x*. *ln-upper-9 x* − *ln x*) *has-field-derivative diff-delta-ln-upper-9 x*) (*at x*)
⟨*proof*⟩

lemma *ln-upper-9-pos*:
 assumes *1* ≤ *x* **shows** *ln*(*x*) ≤ *ln-upper-9 x*
⟨*proof*⟩

lemma *ln-upper-9-neg*:
 assumes *0* < *x* **and** *x1*: *x* ≤ *1* **shows** *ln*(*x*) ≤ *ln-upper-9 x*
⟨*proof*⟩

theorem *ln-upper-9*: *0*<*x* ⟹ *ln*(*x*) ≤ *ln-upper-9 x*
⟨*proof*⟩

definition *ln-lower-9* :: *real* ⇒ *real*
 where *ln-lower-9* ≡ λ*x*. − *ln-upper-9* (*inverse x*)

corollary *ln-lower-9*: *0* < *x* ⟹ *ln-lower-9 x* ≤ *ln x*
 ⟨*proof*⟩

theorem *ln-lower-9-eq*: *0* < *x* ⟹
 ln-lower-9 x = (*1*/*30*)*(*6* + *481**x* + *1881**x*^*2* + *1281**x*^*3* + *131**x*^*4*)*(*x*
− *1*) /
 (*x**(*5* + *40**x* + *60**x*^*2* + *20**x*^*3* + *x*^*4*))
 ⟨*proof*⟩

4.5 Upper Bound 11

Extended bounds start here

definition *ln-upper-11* :: *real* ⇒ *real*
 where *ln-upper-11 x* ≡
 (*5**x*^*5* + *647**x*^*4* + *4397**x*^*3* + *6397**x*^*2* + *2272**x* + *142*) * (*x* −
1) /
 (*30**(*6**x*^*5* + *75**x*^*4* + *200**x*^*3* + *150**x*^*2* + *30**x* + *1*))

definition *diff-delta-ln-upper-11* :: *real* ⇒ *real*
 where *diff-delta-ln-upper-11* ≡ λ*x*. (*x* − *1*)^*11* / ((*6**x*^*5* + *75**x*^*4* + *200**x*^*3*
+ *150**x*^*2* + *30**x* + *1*)^*2* * *x*)

lemma *d-delta-ln-upper-11*: *x* > *0* ⟹
 ((λ*x*. *ln-upper-11 x* − *ln x*) *has-field-derivative diff-delta-ln-upper-11 x*) (*at x*)
⟨*proof*⟩

lemma *ln-upper-11-pos*:

assumes *1 ≤ x* **shows** *ln(x) ≤ ln-upper-11 x*
⟨*proof*⟩

lemma *ln-upper-11-neg*:
assumes *0 < x* **and** *x1*: *x ≤ 1* **shows** *ln(x) ≤ ln-upper-11 x*
⟨*proof*⟩

theorem *ln-upper-11*: *0<x ⟹ ln(x) ≤ ln-upper-11 x*
⟨*proof*⟩

definition *ln-lower-11* :: *real ⇒ real*
where *ln-lower-11 ≡ λx. − ln-upper-11 (inverse x)*

corollary *ln-lower-11*: *0<x ⟹ ln-lower-11 x ≤ ln x*
⟨*proof*⟩

theorem *ln-lower-11-eq*: *0<x ⟹*
ln-lower-11 x = (1/30)(142*x^5 + 2272*x^4 + 6397*x^3 + 4397*x^2 + 647*x + 5)*(x − 1) /*
(x(x^5 + 30*x^4 + 150*x^3 + 200*x^2 + 75*x + 6))*
⟨*proof*⟩

4.6 Upper Bound 13

definition *ln-upper-13* :: *real ⇒ real*
where *ln-upper-13 x ≡ (353 + 8389*x + 20149*x^4 + 50774*x^3 + 38524*x^2 + 1921*x^5 + 10*x^6) * (x − 1)*
/ (70(1 + 42*x + 525*x^4 + 700*x^3 + 315*x^2 + 126*x^5 + 7*x^6))*

definition *diff-delta-ln-upper-13* :: *real ⇒ real*
where *diff-delta-ln-upper-13 ≡ λx. (x − 1)^13 /*
*((1 + 42*x + 525*x^4 + 700*x^3 + 315*x^2 + 126*x^5 + 7*x^6)^2*x)*

lemma *d-delta-ln-upper-13*: *x > 0 ⟹*
((λx. ln-upper-13 x − ln x) has-field-derivative diff-delta-ln-upper-13 x) (at x)
⟨*proof*⟩

lemma *ln-upper-13-pos*:
assumes *1 ≤ x* **shows** *ln(x) ≤ ln-upper-13 x*
⟨*proof*⟩

lemma *ln-upper-13-neg*:
assumes *0 < x* **and** *x1*: *x ≤ 1* **shows** *ln(x) ≤ ln-upper-13 x*
⟨*proof*⟩

theorem *ln-upper-13*: *0<x ⟹ ln(x) ≤ ln-upper-13 x*
⟨*proof*⟩

definition *ln-lower-13* :: *real* ⇒ *real*
where *ln-lower-13* ≡ λ*x*. − *ln-upper-13* (*inverse x*)

corollary *ln-lower-13*: *0*<*x* ⟹ *ln-lower-13 x* ≤ *ln x*
⟨*proof*⟩

theorem *ln-lower-13-eq*: *0*<*x* ⟹
ln-lower-13 x = (*1*/*70*)∗(*10* + *1921*∗*x* + *20149*∗*x*^*2* + *50774*∗*x*^*3* + *38524*∗*x*^*4* + *8389*∗*x*^*5* + *353*∗*x*^*6*)∗(*x* − *1*) /
(*x*∗(*7* + *126*∗*x* + *525*∗*x*^*2* + *700*∗*x*^*3* + *315*∗*x*^*4* + *42*∗*x*^*5* + *x*^*6*))
⟨*proof*⟩

4.7 Upper Bound 15

definition *ln-upper-15* :: *real* ⇒ *real*
where *ln-upper-15 x* ≡
(*1487* + *49199*∗*x* + *547235*∗*x*^*4* + *718735*∗*x*^*3* + *334575*∗*x*^*2* + *141123*∗*x*^*5* + *35*∗*x*^*7* + *9411*∗*x*^*6*)∗(*x* − *1*) /
(*280*∗(*1* + *56*∗*x* + *2450*∗*x*^*4* + *1960*∗*x*^*3* + *588*∗*x*^*2* + *1176*∗*x*^*5* + *8*∗*x*^*7* + *196*∗*x*^*6*))

definition *diff-delta-ln-upper-15* :: *real* ⇒ *real*
where *diff-delta-ln-upper-15*
≡ λ*x*. (*x* − *1*)^*15* / ((*1*+*56*∗*x*+*2450*∗*x*^*4*+*1960*∗*x*^*3*+*588*∗*x*^*2*+*8*∗*x*^*7*+*196*∗*x*^*6*+*1176*∗*x*^*5*)^*2* ∗ *x*)

lemma *d-delta-ln-upper-15*: *x* > *0* ⟹
((λ*x*. *ln-upper-15 x* − *ln x*) *has-field-derivative diff-delta-ln-upper-15 x*) (*at x*)
⟨*proof*⟩

lemma *ln-upper-15-pos*:
assumes *1* ≤ *x* **shows** *ln*(*x*) ≤ *ln-upper-15 x*
⟨*proof*⟩

lemma *ln-upper-15-neg*:
assumes *0* < *x* **and** *x1*: *x* ≤ *1* **shows** *ln*(*x*) ≤ *ln-upper-15 x*
⟨*proof*⟩

theorem *ln-upper-15*: *0*<*x* ⟹ *ln*(*x*) ≤ *ln-upper-15 x*
⟨*proof*⟩

definition *ln-lower-15* :: *real* ⇒ *real*
where *ln-lower-15* ≡ λ*x*. − *ln-upper-15* (*inverse x*)

corollary *ln-lower-15*: *0*<*x* ⟹ *ln-lower-15 x* ≤ *ln x*
⟨*proof*⟩

theorem *ln-lower-15-eq*: $0<x \Longrightarrow$
ln-lower-15 x = (1/280)(35 + 9411*x + 141123*x^2 + 547235*x^3 + 718735*x^4 + 334575*x^5 + 49199*x^6 + 1487*x^7)*(x − 1) /*
(x(8 + 196*x + 1176*x^2 + 2450*x^3 + 1960*x^4 + 588*x^5 + 56*x^6 + x^7))*
⟨*proof*⟩

end

Chapter 5

Sine and Cosine Upper and Lower Bounds

theory *Sin-Cos-Bounds*
imports *Bounds-Lemmas*

begin

5.1 Simple base cases

Upper bound for $0 \leq x$

lemma *sin-le-arg*:
  fixes x :: *real*
    shows $0 \leq x \Longrightarrow \sin x \leq x$
  ⟨*proof*⟩

lemma *cos-ge-1-arg*:
  fixes x :: *real*
  assumes $0 \leq x$
    shows $1 - x \leq \cos x$
  ⟨*proof*⟩

lemmas *sin-Taylor-0-upper-bound-pos* = *sin-le-arg* — MetiTarski bound

lemma *cos-Taylor-1-lower-bound*:
  fixes x :: *real*
  assumes $0 \leq x$
    shows $(1 - x\hat{}2 / 2) \leq \cos x$
  ⟨*proof*⟩

lemma *sin-Taylor-1-lower-bound*:
  fixes x :: *real*
  assumes $0 \leq x$
    shows $(x - x \hat{} 3 / 6) \leq \sin x$

⟨*proof*⟩

5.2 Taylor series approximants

definition *sinpoly* :: [*nat*,*real*] ⇒ *real*
where *sinpoly n* = (λ*x*. ∑ *k*<*n*. *sin-coeff k* ∗ *x* ^ *k*)

definition *cospoly* :: [*nat*,*real*] ⇒ *real*
where *cospoly n* = (λ*x*. ∑ *k*<*n*. *cos-coeff k* ∗ *x* ^ *k*)

lemma *sinpoly-Suc*: *sinpoly* (*Suc n*) = (λ*x*. *sinpoly n x* + *sin-coeff n* ∗ *x* ^ *n*)
⟨*proof*⟩

lemma *cospoly-Suc*: *cospoly* (*Suc n*) = (λ*x*. *cospoly n x* + *cos-coeff n* ∗ *x* ^ *n*)
⟨*proof*⟩

lemma *sinpoly-minus* [*simp*]: *sinpoly n* (−*x*) = − *sinpoly n x*
⟨*proof*⟩

lemma *cospoly-minus* [*simp*]: *cospoly n* (−*x*) = *cospoly n x*
⟨*proof*⟩

lemma *d-sinpoly-cospoly*:
(*sinpoly* (*Suc n*) *has-field-derivative cospoly n x*) (*at x*)
⟨*proof*⟩

lemma *d-cospoly-sinpoly*:
(*cospoly* (*Suc n*) *has-field-derivative* −*sinpoly n x*) (*at x*)
⟨*proof*⟩

5.3 Inductive proof of sine inequalities

lemma *sinpoly-lb-imp-cospoly-ub*:
assumes *x0*: *0* ≤ *x* **and** *k0*: *k*>*0* **and** ⋀*x*. *0* ≤ *x* ⟹ *sinpoly* (*k* − *1*) *x* ≤ *sin x*
shows *cos x* ≤ *cospoly k x*
⟨*proof*⟩

lemma *cospoly-ub-imp-sinpoly-ub*:
assumes *x0*: *0* ≤ *x* **and** *k0*: *k*>*0* **and** ⋀*x*. *0* ≤ *x* ⟹ *cos x* ≤ *cospoly* (*k* − *1*) *x*
shows *sin x* ≤ *sinpoly k x*
⟨*proof*⟩

lemma *sinpoly-ub-imp-cospoly-lb*:
assumes *x0*: *0* ≤ *x* **and** *k0*: *k*>*0* **and** ⋀*x*. *0* ≤ *x* ⟹ *sin x* ≤ *sinpoly* (*k* − *1*) *x*
shows *cospoly k x* ≤ *cos x*
⟨*proof*⟩

lemma *cospoly-lb-imp-sinpoly-lb*:
assumes *x0*: $0 \leq x$ **and** *k0*: $k>0$ **and** $\bigwedge x.\ 0 \leq x \Longrightarrow cospoly\ (k - 1)\ x \leq cos\ x$
shows $sinpoly\ k\ x \leq sin\ x$
⟨*proof*⟩

lemma
assumes $0 \leq x$
shows *sinpoly-lower-nonneg*: $sinpoly\ (4 * Suc\ n)\ x \leq sin\ x$ (**is** *?th1*)
and *sinpoly-upper-nonneg*: $sin\ x \leq sinpoly\ (Suc\ (Suc\ (4{*}n)))\ x$ (**is** *?th2*)
⟨*proof*⟩

5.4 Collecting the results

corollary *sinpoly-upper-nonpos*:
$x \leq 0 \Longrightarrow sin\ x \leq sinpoly\ (4 * Suc\ n)\ x$
⟨*proof*⟩

corollary *sinpoly-lower-nonpos*:
$x \leq 0 \Longrightarrow sinpoly\ (Suc\ (Suc\ (4{*}n)))\ x \leq sin\ x$
⟨*proof*⟩

corollary *cospoly-lower-nonneg*:
$0 \leq x \Longrightarrow cospoly\ (Suc\ (Suc\ (Suc\ (4{*}n))))\ x \leq cos\ x$
⟨*proof*⟩

lemma *cospoly-lower*:
$cospoly\ (Suc\ (Suc\ (Suc\ (4{*}n))))\ x \leq cos\ x$
⟨*proof*⟩

lemma *cospoly-upper-nonneg*:
assumes $0 \leq x$
shows $cos\ x \leq cospoly\ (Suc\ (4{*}n))\ x$
⟨*proof*⟩

lemma *cospoly-upper*:
$cos\ x \leq cospoly\ (Suc\ (4{*}n))\ x$
⟨*proof*⟩

end

Chapter 6

Square Root Upper and Lower Bounds

theory *Sqrt-Bounds*
imports *Bounds-Lemmas*
HOL−Library.Sum-of-Squares

begin

Covers all bounds used in sqrt-upper.ax, sqrt-lower.ax and sqrt-extended.ax.

6.1 Upper bounds

primrec *sqrtu* :: [*real*,*nat*] ⇒ *real* **where**
sqrtu x 0 = (x+1)/2
| *sqrtu x (Suc n) = (sqrtu x n + x/sqrtu x n) / 2*

lemma *sqrtu-upper*: *x ≤ sqrtu x n ^ 2*
⟨*proof*⟩

lemma *sqrtu-numeral*:
sqrtu x (numeral n) = (sqrtu x (pred-numeral n) + x/sqrtu x (pred-numeral n))
/ 2
⟨*proof*⟩

lemma *sqrtu-gt-0*: *x ≥ 0 ⟹ sqrtu x n > 0*
⟨*proof*⟩

theorem *gen-sqrt-upper*: *0 ≤ x ⟹ sqrt x ≤ sqrtu x n*
⟨*proof*⟩

lemma *sqrt-upper-bound-0*:
assumes *x ≥ 0* **shows** *sqrt x ≤ (x+1)/2* (**is** - ≤ *?rhs*)
⟨*proof*⟩

lemma *sqrt-upper-bound-1*:
assumes $x \geq 0$ **shows** $sqrt\ x \leq (1/4)*(x\hat{}2+6*x+1) / (x+1)$ (**is** - $\leq$ *?rhs*)
⟨*proof*⟩

lemma *sqrtu-2-eq*:
$sqrtu\ x\ 2 = (1/8)*(x\hat{}4 + 28*x\hat{}3 + 70*x\hat{}2 + 28*x + 1) / ((x + 1)*(x\hat{}2 + 6*x + 1))$
⟨*proof*⟩

lemma *sqrt-upper-bound-2*:
assumes $x \geq 0$
shows $sqrt\ x \leq (1/8)*(x\hat{}4 + 28*x\hat{}3 + 70*x\hat{}2 + 28*x + 1) / ((x + 1)*(x\hat{}2 + 6*x + 1))$
⟨*proof*⟩

lemma *sqrtu-4-eq*:
$x \geq 0 \Longrightarrow$
$sqrtu\ x\ 4 = (1/32)*(225792840*x\hat{}6+64512240*x\hat{}5+601080390*x\hat{}8+471435600*x\hat{}7+496*x+1+35960$
$/ ((x+1)*(x\hat{}2+6*x+1)*(x\hat{}4+28*x\hat{}3+70*x\hat{}2+28*x+1)*(1820*x\hat{}6+8008*x\hat{}5+x\hat{}8+120*x\hat{}7+$
⟨*proof*⟩

lemma *sqrt-upper-bound-4*:
assumes $x \geq 0$
shows $sqrt\ x \leq (1/32)*(225792840*x\hat{}6+64512240*x\hat{}5+601080390*x\hat{}8+471435600*x\hat{}7+496*x+1+35$
$/ ((x+1)*(x\hat{}2+6*x+1)*(x\hat{}4+28*x\hat{}3+70*x\hat{}2+28*x+1)*(1820*x\hat{}6+8008*x\hat{}5+x\hat{}8+120*x\hat{}7+$
⟨*proof*⟩

lemma *gen-sqrt-upper-scaled*:
assumes $0 \leq x$ $0 < u$
shows $sqrt\ x \leq sqrtu\ (x*u\hat{}2)\ n / u$
⟨*proof*⟩

lemma *sqrt-upper-bound-2-small*:
assumes $0 \leq x$
shows $sqrt\ x \leq (1/32)*(65536*x\hat{}4 + 114688*x\hat{}3 + 17920*x\hat{}2 + 448*x + 1) / ((16*x + 1)*(256*x\hat{}2 + 96*x + 1))$
⟨*proof*⟩

lemma *sqrt-upper-bound-2-large*:
assumes $0 \leq x$
shows $sqrt\ x \leq (1/32)*(65536 + 114688*x + 17920*x\hat{}2 + 448*x\hat{}3 + x\hat{}4) / ((x + 16)*(256 + 96*x + x\hat{}2))$
⟨*proof*⟩

6.2 Lower bounds

lemma *sqrt-lower-bound-id*:
assumes $0 \leq x$ $x \leq 1$
shows $x \leq sqrt\ x$

⟨*proof*⟩

definition *sqrtl* :: [*real*,*nat*] ⇒ *real* **where**
sqrtl x n = x/sqrtu x n

lemma *sqrtl-lower*: *0 ≤ x ⟹ sqrtl x n ^ 2 ≤ x*
⟨*proof*⟩

theorem *gen-sqrt-lower*: *0 ≤ x ⟹ sqrtl x n ≤ sqrt x*
⟨*proof*⟩

lemma *sqrt-lower-bound-0*:
assumes *x ≥ 0* **shows** *2∗x/(x+1) ≤ sqrt x* (**is** *?lhs ≤ -*)
⟨*proof*⟩

lemma *sqrt-lower-bound-1*:
assumes *x ≥ 0* **shows** *4∗x∗(x+1) / (x^2+6∗x+1) ≤ sqrt x* (**is** *?lhs ≤ -*)
⟨*proof*⟩

lemma *sqrtl-2-eq*: *sqrtl x 2 =*
8∗x∗(x + 1)∗(x^2 + 6∗x + 1) / (x^4 + 28∗x^3 + 70∗x^2 + 28∗x + 1)
⟨*proof*⟩

lemma *sqrt-lower-bound-2*:
assumes *x ≥ 0*
shows *8∗x∗(x + 1)∗(x^2 + 6∗x + 1) / (x^4 + 28∗x^3 + 70∗x^2 + 28∗x +*
1) ≤ sqrt x
⟨*proof*⟩

lemma *sqrtl-4-eq*: *x ≥ 0 ⟹*
sqrtl x 4
= (32∗x∗(x+1)∗(x^2+6∗x+1)∗(x^4+28∗x^3+70∗x^2+28∗x+1)∗(1820∗x^6+8008∗x^5+x^8+120∗x^7+
*/ (225792840∗x^6+64512240∗x^5+601080390∗x^8+471435600∗x^7+496∗x+1+35960∗x^2+906192∗x^*
⟨*proof*⟩

lemma *sqrt-lower-bound-4*:
assumes *x ≥ 0*
shows *(32∗x∗(x+1)∗(x^2+6∗x+1)∗(x^4+28∗x^3+70∗x^2+28∗x+1)∗(1820∗x^6+8008∗x^5+x^8+120∗*
/ (225792840∗x^6+64512240∗x^5+601080390∗x^8+471435600∗x^7+496∗x+1+35960∗x^2+906192
≤ sqrt x
⟨*proof*⟩

lemma *gen-sqrt-lower-scaled*:
assumes *0 ≤ x 0 < u*
shows *sqrtl (x∗u^2) n / u ≤ sqrt x*
⟨*proof*⟩

lemma *sqrt-lower-bound-2-small*:
assumes *0 ≤ x*

shows *32*x*(16*x + 1)*(256*x^2 + 96*x + 1) / (65536*x^4 + 114688*x^3 + 17920*x^2 + 448*x + 1) ≤ sqrt x*
⟨*proof*⟩

lemma *sqrt-lower-bound-2-large*:
assumes *0 ≤ x*
shows *32*x*(x + 16)*(x^2 + 96*x + 256) / (x^4 + 448*x^3 + 17920*x^2 + 114688*x + 65536) ≤ sqrt x*
⟨*proof*⟩

end

Bibliography

[1] B. Akbarpour and L. Paulson. MetiTarski: An automatic theorem prover for real-valued special functions. 44(3):175–205, Mar. 2010.