

Exploring Simplified Variants of Gödel's Ontological Argument in Isabelle/HOL

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December 14, 2021

Abstract

Simplified variants of Gödel's ontological argument are explored. Among those is a particularly interesting simplified argument which is (i) valid already in basic modal logics K or KT, (ii) which does not suffer from modal collapse, and (iii) which avoids the rather complex predicates of essence (Ess.) and necessary existence (NE) as used by Gödel.

Whether the presented variants increase or decrease the attractiveness and persuasiveness of the ontological argument is a question I would like to pass on to philosophy and theology.

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1 Background Reading

The selected simplified variants of Gödel's ontological argument [5, 7] as presented in §3 have first been extracted from the insights gained in §6 of [4]. These variants are also influenced by the work presented in [1] and they significantly extend the findings from [3]. In §4 we additionally include the sources from [4].

2 Higher-Order Modal Logic in HOL (cf. [2] and Fig. 1 in [4]).

```
theory HOML imports Main
begin
nitpick-params[user-axioms,expect=genuine]
```

Type *i* is associated with possible worlds and type *e* with entities:

```
typedecl i — Possible worlds
typedecl e — Individuals
type-synonym  $\sigma = i \Rightarrow bool$  — World-lifted propositions
type-synonym  $\gamma = e \Rightarrow \sigma$  — Lifted predicates
type-synonym  $\mu = \sigma \Rightarrow \sigma$  — Unary modal connectives
type-synonym  $\nu = \sigma \Rightarrow \sigma \Rightarrow \sigma$  — Binary modal connectives
```

Logical connectives (operating on truth-sets):

```
abbreviation c1:: $\sigma$  ( $\perp$ ) where  $\perp \equiv \lambda w. False$ 
abbreviation c2:: $\sigma$  ( $\top$ ) where  $\top \equiv \lambda w. True$ 
abbreviation c3:: $\mu$  ( $\neg$ -[52]53) where  $\neg\varphi \equiv \lambda w. \neg(\varphi w)$ 
abbreviation c4:: $\nu$  (infix $\wedge$ 50) where  $\varphi \wedge \psi \equiv \lambda w. (\varphi w) \wedge (\psi w)$ 
abbreviation c5:: $\nu$  (infix $\vee$ 49) where  $\varphi \vee \psi \equiv \lambda w. (\varphi w) \vee (\psi w)$ 
abbreviation c6:: $\nu$  (infix $\rightarrow$ 48) where  $\varphi \rightarrow \psi \equiv \lambda w. (\varphi w) \rightarrow (\psi w)$ 
abbreviation c7:: $\nu$  (infix $\leftrightarrow$ 47) where  $\varphi \leftrightarrow \psi \equiv \lambda w. (\varphi w) \leftrightarrow (\psi w)$ 
consts R:: $i \Rightarrow i \Rightarrow bool$  (-r-) — Accessibility relation
abbreviation c8:: $\mu$  ( $\Box$ -[54]55) where  $\Box\varphi \equiv \lambda w. \forall v. (wr v) \rightarrow (\varphi v)$ 
abbreviation c9:: $\mu$  ( $\Diamond$ -[54]55) where  $\Diamond\varphi \equiv \lambda w. \exists v. (wr v) \wedge (\varphi v)$ 
abbreviation c10:: $\gamma \Rightarrow \gamma$  ( $\neg$ -[52]53) where  $\neg\Phi \equiv \lambda x. \lambda w. \neg(\Phi x w)$ 
abbreviation c11:: $\gamma \Rightarrow \gamma$  ( $\rightarrow$ -) where  $\rightarrow\Phi \equiv \lambda x. \lambda w. \neg(\Phi x w)$ 
abbreviation c12:: $e \Rightarrow e \Rightarrow \sigma$  ( $=$ -) where  $x=y \equiv \lambda w. (x=y)$ 
abbreviation c13:: $e \Rightarrow e \Rightarrow \sigma$  ( $\neq$ -) where  $x \neq y \equiv \lambda w. (x \neq y)$ 
```

Polymorphic possibilist quantification:

abbreviation $q1::('a\Rightarrow\sigma)\Rightarrow\sigma$ (\forall) **where** $\forall\Phi\equiv\lambda w.\forall x.(\Phi x w)$
abbreviation $q2$ (**binder** \forall [10] 11) **where** $\forall x.\varphi(x)\equiv\forall\varphi$
abbreviation $q3::('a\Rightarrow\sigma)\Rightarrow\sigma$ (\exists) **where** $\exists\Phi\equiv\lambda w.\exists x.(\Phi x w)$
abbreviation $q4$ (**binder** \exists [10] 11) **where** $\exists x.\varphi(x)\equiv\exists\varphi$

Actualist quantification for individuals/entities:

consts $existsAt::\gamma$ ($-\textcircled{-}$)
abbreviation $q5::\gamma\Rightarrow\sigma$ (\forall^E) **where** $\forall^E\Phi\equiv\lambda w.\forall x.(x\textcircled{w})\longrightarrow(\Phi x w)$
abbreviation $q6$ (**binder** \forall^E [8] 9) **where** $\forall^E x.\varphi(x)\equiv\forall^E\varphi$
abbreviation $q7::\gamma\Rightarrow\sigma$ (\exists^E) **where** $\exists^E\Phi\equiv\lambda w.\exists x.(x\textcircled{w})\wedge(\Phi x w)$
abbreviation $q8$ (**binder** \exists^E [8] 9) **where** $\exists^E x.\varphi(x)\equiv\exists^E\varphi$

Meta-logical predicate for global validity:

abbreviation $g1::\sigma\Rightarrow bool$ ($[-]$) **where** $[\psi]\equiv\forall w.\psi w$

Barcan and converse Barcan formula:

lemma *True nitpick*[*satisfy*] $\langle proof \rangle$
lemma $[(\forall^E x.\Box(\varphi x)) \rightarrow \Box(\forall^E x.(\varphi x))]$ **nitpick** $\langle proof \rangle$
lemma $[\Box(\forall^E x.(\varphi x)) \rightarrow (\forall^E x.\Box(\varphi x))]$ **nitpick** $\langle proof \rangle$
lemma $[(\forall x.\Box(\varphi x)) \rightarrow \Box(\forall x.\varphi x)]$ $\langle proof \rangle$
lemma $[\Box(\forall x.(\varphi x)) \rightarrow (\forall x.\Box(\varphi x))]$ $\langle proof \rangle$
end
theory *DisableKodkodScala*
imports *Main*
begin

Some of the nitpick invocation within this AFP entry do not work, if "Kodkod Scala" is enabled, i.e., if the box under Plugin Options — Isabelle — General — Miscellaneous Tools — Kodkod Scala is activated. Therefore, in this theory we explicitly disable this configuration option.

$\langle ML \rangle$

end

3 Selected Simplified Ontological Argument

theory *SimplifiedOntologicalArgument* **imports**
HOML
DisableKodkodScala
begin

Positive properties:

consts $posProp::\gamma\Rightarrow\sigma$ (\mathcal{P})

An entity x is God-like if it possesses all positive properties.

definition G (\mathcal{G}) **where** $\mathcal{G}(x)\equiv\forall\Phi.(\mathcal{P}(\Phi)\rightarrow\Phi(x))$

The axiom's of the simplified variant are presented next; these axioms are further motivated in [4, 1]).

Self-difference is not a positive property (possible alternative: the empty property is not a positive property).

axiomatization where CORO1: $[\neg(\mathcal{P}(\lambda x.(x \neq x)))]$

A property entailed by a positive property is positive.

axiomatization where CORO2: $[\forall \Phi \Psi. \mathcal{P}(\Phi) \wedge (\forall x. \Phi(x) \rightarrow \Psi(x)) \rightarrow \mathcal{P}(\Psi)]$

Being Godlike is a positive property.

axiomatization where AXIOM3: $[\mathcal{P} \mathcal{G}]$

3.1 Verifying the Selected Simplified Ontological Argument (version 1)

The existence of a non-exemplified positive property implies that self-difference (or, alternatively, the empty property) is a positive property.

lemma LEMMA1: $[(\exists \Phi. (\mathcal{P}(\Phi) \wedge \neg(\exists x. \Phi(x)))) \rightarrow \mathcal{P}(\lambda x.(x \neq x))]$
<proof>

A non-exemplified positive property does not exist.

lemma LEMMA2: $[\neg(\exists \Phi. (\mathcal{P}(\Phi) \wedge \neg(\exists x. \Phi(x))))]$
<proof>

Positive properties are exemplified.

lemma LEMMA3: $[\forall \Phi. (\mathcal{P}(\Phi) \rightarrow (\exists x. \Phi(x)))]$
<proof>

There exists a God-like entity.

theorem THEOREM3': $[\exists x. \mathcal{G}(x)]$
<proof>

Necessarily, there exists a God-like entity.

theorem THEOREM3: $[\Box(\exists x. \mathcal{G}(x))]$
<proof>

However, the possible existence of Godlike entity is not implied.

theorem CORO: $[\Diamond(\exists x. \mathcal{G}(x))]$
nitpick *<proof>*

3.2 Verifying the Selected Simplified Ontological Argument (version 2)

We switch to logic T.

axiomatization where T: $[\forall \varphi. \Box \varphi \rightarrow \varphi]$

lemma T': $[\forall \varphi. \varphi \rightarrow \Diamond \varphi]$ *<proof>*

Positive properties are possibly exemplified.

theorem THEOREM1: $[\forall \Phi. \mathcal{P}(\Phi) \rightarrow \Diamond(\exists x. \Phi(x))]$

<proof>

Possibly there exists a God-like entity.

theorem *CORO*: $[\diamond(\exists x. \mathcal{G}(x))]$

<proof>

The possible existence of a God-like entity implies the necessary existence of a God-like entity.

theorem *THEOREM2*: $[\diamond(\exists x. \mathcal{G}(x)) \rightarrow \Box(\exists x. \mathcal{G}(x))]$

<proof>

Necessarily, there exists a God-like entity.

theorem *THEO3*: $[\Box(\exists x. \mathcal{G}(x))]$

<proof>

There exists a God-like entity.

theorem *THEO3'*: $[\exists x. \mathcal{G}(x)]$

<proof>

Modal collapse is not implied; nitpick reports a countermodel.

lemma *MC*: $[\forall \Phi. \Phi \rightarrow \Box \Phi]$ **nitpick** *<proof>*

Consistency of the theory; nitpick reports a model.

lemma *True* **nitpick**[*satisfy*] *<proof>*

end

4 Presentation of All Variants as Studied in [4]

4.1 Preliminaries: Modal Ultrafilter (Fig. 2 in [4])

theory *MFilter* **imports** *HOML*

begin

Some abbreviations for auxiliary operations.

abbreviation *a*:: $\gamma \Rightarrow (\gamma \Rightarrow \sigma) \Rightarrow \sigma$ ($-\in-$) **where** $x \in S \equiv S x$

abbreviation *b*:: γ (\emptyset) **where** $\emptyset \equiv \lambda x. \perp$

abbreviation *c*:: γ (\mathbf{U}) **where** $\mathbf{U} \equiv \lambda x. \top$

abbreviation *d*:: $\gamma \Rightarrow \gamma \Rightarrow \sigma$ ($-\subseteq-$) **where** $\varphi \subseteq \psi \equiv \forall x. ((\varphi x) \rightarrow (\psi x))$

abbreviation *e*:: $\gamma \Rightarrow \gamma \Rightarrow \gamma$ ($-\sqcap-$) **where** $\varphi \sqcap \psi \equiv \lambda x. ((\varphi x) \wedge (\psi x))$

abbreviation *f*:: $\gamma \Rightarrow \gamma$ ($^{-1}-$) **where** $^{-1}\psi \equiv \lambda x. \neg(\psi x)$

Definition of modal filter.

abbreviation *g*:: $(\gamma \Rightarrow \sigma) \Rightarrow \sigma$ (*Filter*)

where *Filter* $\Phi \equiv (((\mathbf{U} \in \Phi) \wedge \neg(\emptyset \in \Phi))$

$\wedge (\forall \varphi \psi. (((\varphi \in \Phi) \wedge (\varphi \subseteq \psi)) \rightarrow (\psi \in \Phi))))$

$\wedge (\forall \varphi \psi. (((\varphi \in \Phi) \wedge (\psi \in \Phi)) \rightarrow ((\varphi \sqcap \psi) \in \Phi)))$

Definition of modal ultrafilter .

abbreviation $h::(\gamma \Rightarrow \sigma) \Rightarrow \sigma$ (*UFilter*) **where**
 $UFilter \Phi \equiv (Filter \Phi) \wedge (\forall \varphi. ((\varphi \in \Phi) \vee ((^{-1}\varphi) \in \Phi)))$

Modal filter and modal ultrafilter are consistent.

lemma $[\forall \Phi \varphi. ((UFilter \Phi) \rightarrow \neg((\varphi \in \Phi) \wedge ((^{-1}\varphi) \in \Phi)))]$ *<proof>*
end

4.2 Preliminaries: Further Basic Notions (Fig. 3 in [4])

theory *BaseDefs* **imports** *HOML*
begin

Positive properties.

consts $posProp::\gamma \Rightarrow \sigma$ (\mathcal{P})

Basic definitions for modal ontological argument.

abbreviation a ($-\sqsubseteq$) **where** $X \sqsubseteq Y \equiv \forall^E z. ((X z) \rightarrow (Y z))$

abbreviation b ($-\Rightarrow$) **where** $X \Rightarrow Y \equiv \Box(X \sqsubseteq Y)$

abbreviation c ($\mathcal{P}os$) **where** $\mathcal{P}os Z \equiv \forall X. ((Z X) \rightarrow (\mathcal{P} X))$

abbreviation d ($-\sqcap$) **where** $X \sqcap Z \equiv \Box(\forall^E u. ((X u) \leftrightarrow (\forall Y. ((Z Y) \rightarrow (Y u))))))$

Definition of Godlike.

definition G (\mathcal{G}) **where** $\mathcal{G} x \equiv \forall Y. ((\mathcal{P} Y) \rightarrow (Y x))$

Definitions of Essence and Necessary Existence.

definition E (\mathcal{E}) **where** $\mathcal{E} Y x \equiv (Y x) \wedge (\forall Z. ((Z x) \rightarrow (Y \Rightarrow Z)))$

definition NE ($\mathcal{N}\mathcal{E}$) **where** $\mathcal{N}\mathcal{E} x \equiv \forall Y. ((\mathcal{E} Y x) \rightarrow \Box(\exists^E Y))$

end

4.3 Ultrafilter Analysis of Scott's Variant (Fig. 3 in [4])

theory *ScottVariant* **imports**

HOML

MFilter

BaseDefs

begin

Axioms of Scott's variant.

axiomatization **where**

$A1: [\forall X. ((\neg(\mathcal{P} X)) \leftrightarrow (\mathcal{P}(\neg X)))]$ **and**

$A2: [\forall X Y. (((\mathcal{P} X) \wedge (X \Rightarrow Y)) \rightarrow (\mathcal{P} Y))]$ **and**

$A3: [\forall Z. ((\mathcal{P}os Z) \rightarrow (\forall X. ((X \sqcap Z) \rightarrow (\mathcal{P} X))))]$ **and**

$A4: [\forall X. ((\mathcal{P} X) \rightarrow \Box(\mathcal{P} X))]$ **and**

$A5: [\mathcal{P} \mathcal{N}\mathcal{E}]$ **and**

$B: [\forall \varphi. (\varphi \rightarrow \Box \Diamond \varphi)]$ — Logic KB

lemma $B': \forall x y. \neg(xry) \vee (yrx)$ *<proof>*

Necessary existence of a Godlike entity.

theorem T6: $[\Box(\exists^E \mathcal{G})]$
<proof>

Existence of a Godlike entity.

lemma $[\exists^E \mathcal{G}]$ *<proof>*

Consistency

lemma True nitpick*[satisfy] <proof>*

Modal collapse: holds.

lemma MC: $[\forall \Phi.(\Phi \rightarrow \Box \Phi)]$
<proof>

Analysis of positive properties using ultrafilters.

theorem U1: $[UFilter \mathcal{P}]$ — Proof found by sledgehammer
<proof>

lemma L1: $[\forall X Y.((X \Rightarrow Y) \rightarrow (X \sqsubseteq Y))]$ *<proof>*

lemma L2: $[\forall X Y.(((\mathcal{P} X) \wedge (X \sqsubseteq Y)) \rightarrow (\mathcal{P} Y))]$ *<proof>*

Set of supersets of X, we call this HF X.

abbreviation HF where $HF X \equiv \lambda Y.(X \sqsubseteq Y)$

$HF \mathcal{G}$ is a filter; hence, $HF \mathcal{G}$ is Hauptfilter of \mathcal{G} .

lemma F1: $[Filter (HF \mathcal{G})]$ *<proof>*

lemma F2: $[UFilter (HF \mathcal{G})]$ *<proof>*

T6 follows directly from F1.

theorem T6again: $[\Box(\exists^E \mathcal{G})]$ *<proof>*
end

4.4 Ultrafilter Variant (Fig. 5 in [4])

theory UFilterVariant imports

HOML

MFilter

BaseDefs

DisableKodkodScala

begin

Axiom's of ultrafilter variant.

axiomatization where

U1: $[UFilter \mathcal{P}]$ **and**

A2: $[\forall X Y.(((\mathcal{P} X) \wedge (X \Rightarrow Y)) \rightarrow (\mathcal{P} Y))]$ **and**

A3: $[\forall \mathcal{Z}.((\mathcal{P}_{os} \mathcal{Z}) \rightarrow (\forall X.((X \sqcap \mathcal{Z}) \rightarrow (\mathcal{P} X))))]$

Necessary existence of a Godlike entity.

theorem T6: $[\Box(\exists^E \mathcal{G})]$ — Proof also found by sledgehammer
<proof>

Checking for consistency.

lemma *True* **nitpick**[*satisfy*] \langle *proof* \rangle

Checking for modal collapse.

lemma *MC*: $[\forall \Phi.(\Phi \rightarrow \Box \Phi)]$ **nitpick** \langle *proof* \rangle
end

4.5 Simplified Variant (Fig. 6 in [4])

theory *SimpleVariant* **imports**

HOML

MFilter

BaseDefs

DisableKodkodScala

begin

Axiom's of new, simplified variant.

axiomatization **where**

A1': $[\neg(\mathcal{P}(\lambda x.(x \neq x)))]$ **and**

A2': $[\forall X Y.(((\mathcal{P} X) \wedge ((X \sqsubseteq Y) \vee (X \ni Y))) \rightarrow (\mathcal{P} Y))]$ **and**

A3': $[\forall \mathcal{Z}.((\mathcal{P}_{\text{os}} \mathcal{Z}) \rightarrow (\forall X.((X \sqcap \mathcal{Z}) \rightarrow (\mathcal{P} X))))]$

lemma *T2*: $[\mathcal{P} \mathcal{G}]$ \langle *proof* \rangle

lemma *L1*: $[\mathcal{P}(\lambda x.(x = x))]$ \langle *proof* \rangle

Necessary existence of a Godlike entity.

theorem *T6*: $[\Box(\exists^E \mathcal{G})]$ — Proof found by sledgehammer
 \langle *proof* \rangle

lemma *True* **nitpick**[*satisfy*] \langle *proof* \rangle

Modal collapse and monotheism: not implied.

lemma *MC*: $[\forall \Phi.(\Phi \rightarrow \Box \Phi)]$ **nitpick** \langle *proof* \rangle

lemma *MT*: $[\forall x y.(((\mathcal{G} x) \wedge (\mathcal{G} y)) \rightarrow (x = y))]$

nitpick \langle *proof* \rangle

Gödel's A1, A4, A5: not implied anymore.

lemma *A1*: $[\forall X.((\neg(\mathcal{P} X)) \leftrightarrow (\mathcal{P}(\neg X)))]$ **nitpick** \langle *proof* \rangle

lemma *A4*: $[\forall X.((\mathcal{P} X) \rightarrow \Box(\mathcal{P} X))]$ **nitpick** \langle *proof* \rangle

lemma *A5*: $[\mathcal{P} \mathcal{N}\mathcal{E}]$ **nitpick** \langle *proof* \rangle

Checking filter and ultrafilter properties.

theorem *F1*: $[\text{Filter } \mathcal{P}]$ \langle *proof* \rangle

theorem *U1*: $[\text{Ultrafilter } \mathcal{P}]$ **nitpick** \langle *proof* \rangle

end

4.6 Simplified Variant with Axiom T2 (Fig. 7 in [4])

theory *SimpleVariantPG* **imports**

HOML

MFilter

BaseDefs

DisableKodkodScala

begin

Axiom's of simplified variant with A3 replaced.

axiomatization where

A1': $\lceil \neg(\mathcal{P}(\lambda x.(x \neq x))) \rceil$ **and**

A2': $\lceil \forall X Y.((\mathcal{P} X) \wedge ((X \sqsubseteq Y) \vee (X \Rightarrow Y))) \rightarrow (\mathcal{P} Y) \rceil$ **and**

T2: $\lceil \mathcal{P} \mathcal{G} \rceil$

Necessary existence of a Godlike entity.

theorem *T6*: $\lceil \Box(\exists^E \mathcal{G}) \rceil$ — Proof found by sledgehammer
<proof>

lemma *True nitpick*[*satisfy*] *<proof>*

Modal collapse and Monotheism: not implied.

lemma *MC*: $\lceil \forall \Phi.(\Phi \rightarrow \Box \Phi) \rceil$ **nitpick** *<proof>*

lemma *MT*: $\lceil \forall x y.((\mathcal{G} x) \wedge (\mathcal{G} y)) \rightarrow (x=y) \rceil$ **nitpick** *<proof>*

end

4.7 Simplified Variant with Simple Entailment in Logic K (Fig. 8 in [4])

theory *SimpleVariantSE* **imports**

HOML

MFilter

BaseDefs

begin

Axiom's of new variant based on ultrafilters.

axiomatization where

A1': $\lceil \neg(\mathcal{P}(\lambda x.(x \neq x))) \rceil$ **and**

A2'': $\lceil \forall X Y.((\mathcal{P} X) \wedge (X \sqsubseteq Y)) \rightarrow (\mathcal{P} Y) \rceil$ **and**

T2: $\lceil \mathcal{P} \mathcal{G} \rceil$

Necessary existence of a Godlike entity.

theorem *T6*: $\lceil \Box(\exists^E \mathcal{G}) \rceil$ *<proof>*

theorem *T7*: $\lceil \exists^E \mathcal{G} \rceil$ *<proof>*

Possible existence of a Godlike: has counterodel.

lemma *T3*: $\lceil \Diamond(\exists^E \mathcal{G}) \rceil$ **nitpick** *<proof>*

lemma *T3'*: **assumes** *T*: $\lceil \forall \varphi.((\Box \varphi) \rightarrow \varphi) \rceil$

shows $[\Diamond(\exists^E \mathcal{G})]$
 $\langle proof \rangle$
end

4.8 Simplified Variant with Simple Entailment in Logic T (Fig. 9 in [4])

theory *SimpleVariantSEinT* **imports**

HOML

MFilter

BaseDefs

begin

Axiom's of new variant based on ultrafilters.

axiomatization where

$A1'$: $[\neg(\mathcal{P}(\lambda x.(x \neq x)))]$ **and**

$A2''$: $[\forall X Y.((\mathcal{P} X) \wedge (X \sqsubseteq Y)) \rightarrow (\mathcal{P} Y)]$ **and**

$T2$: $[\mathcal{P} \mathcal{G}]$

Modal Logic T.

axiomatization where T : $[\forall \varphi.((\Box \varphi) \rightarrow \varphi)]$

lemma T' : $[\forall \varphi.(\varphi \rightarrow (\Diamond \varphi))]$ $\langle proof \rangle$

Necessary existence of a Godlike entity.

theorem $T6$: $[\Box(\exists^E \mathcal{G})]$ — Proof found by sledgehammer
 $\langle proof \rangle$

T6 again, with an alternative, simpler proof.

theorem $T6again$: $[\Box(\exists^E \mathcal{G})]$

$\langle proof \rangle$

end

4.9 Hauptfiltervariant (Fig. 10 in [4])

theory *SimpleVariantHF* **imports**

HOML

MFilter

BaseDefs

DisableKodkodScala

begin

Definition: Set of supersets of X , we call this $\mathcal{HF} X$.

abbreviation $HF::\gamma \Rightarrow (\gamma \Rightarrow \sigma)$ **where** $HF X \equiv \lambda Y.(X \sqsubseteq Y)$

Postulate: $\mathcal{HF} \mathcal{G}$ is a filter; i.e., Hauptfilter of \mathcal{G} .

axiomatization where $F1$: $[Filter (HF \mathcal{G})]$

Necessary existence of a Godlike entity.

theorem $T6$: $[\Box(\exists^E \mathcal{G})]$ $\langle proof \rangle$

theorem *T6again*: $[\Box(\exists^E \mathcal{G})]$
 $\langle proof \rangle$

Possible existence of Godlike entity not implied.

lemma *T3*: $[\Diamond(\exists^E \mathcal{G})]$ **nitpick** $\langle proof \rangle$

Axiom T enforces possible existence of Godlike entity.

axiomatization

lemma *T3*: **assumes** *T*: $[\forall \varphi.((\Box \varphi) \rightarrow \varphi)]$
shows $[\Diamond(\exists^E \mathcal{G})]$ $\langle proof \rangle$

lemma *True* **nitpick**[*satisfy*] $\langle proof \rangle$

Modal collapse: not implied anymore.

lemma *MC*: $[\forall \Phi.(\Phi \rightarrow \Box \Phi)]$ **nitpick** $\langle proof \rangle$

lemma *MT*: $[\forall x y.(((\mathcal{G} x) \wedge (\mathcal{G} y)) \rightarrow (x=y))]$
nitpick $\langle proof \rangle$

end

4.10 Formal Study of Version No.2 of Gödel's Argument as Reported by Kanckos and Lethen, 2019 [6] (Fig. 11 in [4])

theory *KanckosLethenNo2Possibilist* **imports**

HOML

MFilter

BaseDefs

begin

Axioms of Version No. 2 [6].

abbreviation *delta* (Δ) **where** $\Delta A \equiv \lambda x.(\forall \psi. ((A \psi) \rightarrow (\psi x)))$

abbreviation *N* (\mathcal{N}) **where** $\mathcal{N} \varphi \equiv \lambda x.(\Box(\varphi x))$

axiomatization where

Axiom1: $[\forall \varphi \psi.(((\mathcal{P} \varphi) \wedge (\Box(\forall x. ((\varphi x) \rightarrow (\psi x)))))) \rightarrow (\mathcal{P} \psi)]$ **and** — The \Box can be omitted here; the proofs still work.

Axiom2: $[\forall A.(\Box((\forall \varphi.((A \varphi) \rightarrow (\mathcal{P} \varphi))) \rightarrow (\mathcal{P} (\Delta A))))]$ **and** — The \Box can be omitted here; the proofs still work.

Axiom3: $[\forall \varphi.((\mathcal{P} \varphi) \rightarrow (\mathcal{P} (\mathcal{N} \varphi)))]$ **and**

Axiom4: $[\forall \varphi.((\mathcal{P} \varphi) \rightarrow (\neg(\mathcal{P}(\neg\varphi))))]$ **and**

— Logic S5

axB: $[\forall \varphi.(\varphi \rightarrow \Box \Diamond \varphi)]$ **and** *axM*: $[\forall \varphi.((\Box \varphi) \rightarrow \varphi)]$ **and** *ax4*: $[\forall \varphi.((\Box \varphi) \rightarrow (\Box \Box \varphi))]$

Sahlqvist correspondences: they are better suited for proof automation.

lemma *axB'*: $\forall x y. \neg(x\mathbf{r}y) \vee (y\mathbf{r}x)$ $\langle proof \rangle$

lemma *axM'*: $\forall x. (x\mathbf{r}x)$ $\langle proof \rangle$

lemma *ax4'*: $\forall x y z. (((x\mathbf{r}y) \wedge (y\mathbf{r}z)) \rightarrow (x\mathbf{r}z))$ $\langle proof \rangle$

Proofs for all theorems for No.2 from [6].

theorem *Theorem0*: $[\forall \varphi \psi. ((\forall Q. ((Q \varphi) \rightarrow (Q \psi))) \rightarrow ((\mathcal{P} \varphi) \rightarrow (\mathcal{P} \psi)))]$
 $\langle proof \rangle$

theorem *Theorem1*: $[\mathcal{P} \mathcal{G}] \langle proof \rangle$

theorem *Theorem2*: $[\forall x. ((\mathcal{G} x) \rightarrow (\exists y. \mathcal{G} y))] \langle proof \rangle$

theorem *Theorem3a*: $[\mathcal{P} (\lambda x. (\exists y. \mathcal{G} y))] \langle proof \rangle$

theorem *Theorem3b*: $[\Box(\mathcal{P} (\lambda x. (\Box(\exists y. \mathcal{G} y))))] \langle proof \rangle$

theorem *Theorem4*: $[\forall x. \Box((\mathcal{G} x) \rightarrow ((\mathcal{P} (\lambda x. (\Box(\exists y. \mathcal{G} y)))) \rightarrow (\Box(\exists y. \mathcal{G} y))))]$
 $\langle proof \rangle$

theorem *Theorem5*: $[\forall x. \Box((\mathcal{G} x) \rightarrow (\Box(\exists y. \mathcal{G} y)))] \langle proof \rangle$

theorem *Theorem6*: $[\Box((\exists y. \mathcal{G} y) \rightarrow (\Box(\exists y. \mathcal{G} y)))] \langle proof \rangle$

theorem *Theorem7*: $[\Box((\Diamond(\exists y. \mathcal{G} y) \rightarrow (\Box(\exists y. \mathcal{G} y)))] \langle proof \rangle$

theorem *Theorem8*: $[\Box(\exists y. \mathcal{G} y)] \langle proof \rangle$

theorem *Theorem9*: $[\forall \varphi. ((\mathcal{P} \varphi) \rightarrow \Diamond(\exists x. \varphi x))] \langle proof \rangle$

Short proof of Theorem8; analogous to the one presented in Sec. 7 of Benzmler 2020.

theorem $[\Box(\exists y. \mathcal{G} y)]$ — Note: this version of the proof uses only axB' and axM' .
 $\langle proof \rangle$

theorem *T5*: $[(\Diamond(\exists y. \mathcal{G} y) \rightarrow \Box(\exists y. \mathcal{G} y))]$ — Obvious: If we can prove Theorem8, then we also have T5.
 $\langle proof \rangle$

Another short proof of Theorem8.

theorem $[\Box(\exists y. \mathcal{G} y)]$ — Note: fewer assumptions used in some cases than in [6].
 $\langle proof \rangle$

Are the axioms of the simplified versions implied?

Actualist version of the axioms.

lemma *A1'*: $[\neg(\mathcal{P}(\lambda x. (x \neq x)))] \langle proof \rangle$

lemma *A2'*: $[\forall X Y. (((\mathcal{P} X) \wedge ((X \sqsubseteq Y) \vee (X \Rightarrow Y))) \rightarrow (\mathcal{P} Y))] \text{ nitpick } \langle proof \rangle$

lemma *A3*: $[\forall \mathcal{Z}. ((\mathcal{P} \text{os } \mathcal{Z}) \rightarrow (\forall X. ((X \sqcap \mathcal{Z}) \rightarrow (\mathcal{P} X)))] \text{ nitpick } \langle proof \rangle$

Possibilist version of the axioms.

abbreviation *a* ($-\sqsubseteq^p$ -) **where** $X \sqsubseteq^p Y \equiv \forall z. ((X z) \rightarrow (Y z))$

abbreviation *b* ($-\Rightarrow^p$ -) **where** $X \Rightarrow^p Y \equiv \Box(X \sqsubseteq^p Y)$

abbreviation *d* ($-\sqcap^p$ -) **where** $X \sqcap^p \mathcal{Z} \equiv \Box(\forall u. ((X u) \leftrightarrow (\forall Y. ((\mathcal{Z} Y) \rightarrow (Y u)))))$

lemma *A1'P*: $[\neg(\mathcal{P}(\lambda x. (x \neq x)))] \langle proof \rangle$

lemma *A2'P*: $[\forall X Y. (((\mathcal{P} X) \wedge ((X \sqsubseteq^p Y) \vee (X \Rightarrow^p Y))) \rightarrow (\mathcal{P} Y))] \langle proof \rangle$

lemma *A2'aP*: $[\forall X Y. (((\mathcal{P} X) \wedge (X \Rightarrow^p Y)) \rightarrow (\mathcal{P} Y))] \langle proof \rangle$

lemma *A2'bP*: $[\forall X Y. (((\mathcal{P} X) \wedge (X \sqsubseteq^p Y)) \rightarrow (\mathcal{P} Y))] \langle proof \rangle$

lemma *A3P*: $[\forall \mathcal{Z}. ((\mathcal{P} \text{os } \mathcal{Z}) \rightarrow (\forall X. ((X \sqcap^p \mathcal{Z}) \rightarrow (\mathcal{P} X)))]$
 $\langle proof \rangle$

Are Axiom2 and A3 equivalent? Only when assuming Axiom1 and axiom M.

lemma $[\forall A . (\Box((\forall \varphi. ((A \varphi) \rightarrow (\mathcal{P} \varphi))) \rightarrow (\mathcal{P} (\Delta A))))] \equiv [\forall \mathcal{Z}. ((\mathcal{P}_{\text{os}} \mathcal{Z}) \rightarrow (\forall X. ((X \Box^{\mathcal{P}} \mathcal{Z}) \rightarrow (\mathcal{P} X)))]$

<proof>

end

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