# Exploring Simplified Variants of Gödel's Ontological Argument in Isabelle/HOL

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**Abstract**

Simplified variants of Gödel's ontological argument are explored. Among those is a particularly interesting simplified argument which is (i) valid already in basic modal logics K or KT, (ii) which does not suffer from modal collapse, and (iii) which avoids the rather complex predicates of essence (Ess.) and necessary existence (NE) as used by Gödel.

Whether the presented variants increase or decrease the attractiveness and persuasiveness of the ontological argument is a question I would like to pass on to philosophy and theology.

## Contents

# 1 Background Reading

The selected simplified variants of Gödel's ontological argument [5, 7] as presented in §3 have first been extracted from the insights gained in §6 of [4]. These variants are also influenced by the work presented in [1] and they significantly extend the findings from [3]. In §4 we additionally include the sources from [4].

# 2 Higher-Order Modal Logic in HOL (cf. [2] and Fig. 1 in [4]).

**theory** *HOML* **imports** *Main*
**begin**
**nitpick-params**[*user-axioms*,*expect*=*genuine*]

Type i is associated with possible worlds and type e with entities:

**typedecl** $i$ — Possible worlds
**typedecl** $e$ — Individuals
**type-synonym** $\sigma = i \Rightarrow bool$ — World-lifted propositions
**type-synonym** $\gamma = e \Rightarrow \sigma$ — Lifted predicates
**type-synonym** $\mu = \sigma \Rightarrow \sigma$ — Unary modal connectives
**type-synonym** $\nu = \sigma \Rightarrow \sigma \Rightarrow \sigma$ — Binary modal connectives

Logical connectives (operating on truth-sets):

**abbreviation** $c1{::}\sigma$ ($\bot$) **where** $\bot \equiv \lambda w.\ False$
**abbreviation** $c2{::}\sigma$ ($\top$) **where** $\top \equiv \lambda w.\ True$
**abbreviation** $c3{::}\mu$ ($\neg$-[52]53) **where** $\neg\varphi \equiv \lambda w.\neg(\varphi\ w)$
**abbreviation** $c4{::}\nu$ (**infix**$\wedge$50) **where** $\varphi\wedge\psi \equiv \lambda w.(\varphi\ w)\wedge(\psi\ w)$
**abbreviation** $c5{::}\nu$ (**infix**$\vee$49) **where** $\varphi\vee\psi \equiv \lambda w.(\varphi\ w)\vee(\psi\ w)$
**abbreviation** $c6{::}\nu$ (**infix**$\rightarrow$48) **where** $\varphi\rightarrow\psi \equiv \lambda w.(\varphi\ w)\longrightarrow(\psi\ w)$
**abbreviation** $c7{::}\nu$ (**infix**$\leftrightarrow$47) **where** $\varphi\leftrightarrow\psi \equiv \lambda w.(\varphi\ w)\longleftrightarrow(\psi\ w)$
**consts** $R{::}i\Rightarrow i\Rightarrow bool$ (-**r**-) — Accessibility relation
**abbreviation** $c8{::}\mu$ ($\Box$-[54]55) **where** $\Box\varphi \equiv \lambda w.\forall v.(w\mathbf{r}v)\longrightarrow(\varphi\ v)$
**abbreviation** $c9{::}\mu$ ($\Diamond$-[54]55) **where** $\Diamond\varphi \equiv \lambda w.\exists v.(w\mathbf{r}v)\wedge(\varphi\ v)$
**abbreviation** $c10{::}\gamma\Rightarrow\gamma$ ($\neg$-[52]53) **where** $\neg\Phi \equiv \lambda x.\lambda w.\neg(\Phi\ x\ w)$
**abbreviation** $c11{::}\gamma\Rightarrow\gamma$ ($\rightharpoondown$-) **where** $\rightharpoondown\Phi \equiv \lambda x.\lambda w.\neg(\Phi\ x\ w)$
**abbreviation** $c12{::}e\Rightarrow e\Rightarrow\sigma$ (-=-) **where** $x{=}y \equiv \lambda w.(x{=}y)$
**abbreviation** $c13{::}e\Rightarrow e\Rightarrow\sigma$ (-$\neq$-) **where** $x{\neq}y \equiv \lambda w.(x{\neq}y)$

Polymorphic possibilist quantification:

**abbreviation** *q1*::$('a{\Rightarrow}\sigma){\Rightarrow}\sigma$ ($\forall$ ) **where** $\forall\,\Phi \equiv \lambda w.\forall\,x.(\Phi\ x\ w)$
**abbreviation** *q2* (**binder**$\forall$ [*10*]*11*) **where** $\forall\,x.\ \varphi(x) \equiv \forall\,\varphi$
**abbreviation** *q3*::$('a{\Rightarrow}\sigma){\Rightarrow}\sigma$ ($\exists$ ) **where** $\exists\,\Phi \equiv \lambda w.\exists\,x.(\Phi\ x\ w)$
**abbreviation** *q4* (**binder**$\exists$ [*10*]*11*) **where** $\exists\,x.\ \varphi(x) \equiv \exists\,\varphi$

Actualist quantification for individuals/entities:

**consts** *existsAt*::$\gamma$ (-@-)
**abbreviation** *q5*::$\gamma{\Rightarrow}\sigma$ ($\forall^E$) **where** $\forall^E\Phi \equiv \lambda w.\forall\,x.(x@w){\longrightarrow}(\Phi\ x\ w)$
**abbreviation** *q6* (**binder**$\forall^E$[*8*]*9*) **where** $\forall^E x.\ \varphi(x) \equiv \forall^E\varphi$
**abbreviation** *q7*::$\gamma{\Rightarrow}\sigma$ ($\exists^E$) **where** $\exists^E\Phi \equiv \lambda w.\exists\,x.(x@w){\wedge}(\Phi\ x\ w)$
**abbreviation** *q8* (**binder**$\exists^E$[*8*]*9*) **where** $\exists^E x.\ \varphi(x) \equiv \exists^E\varphi$

Meta-logical predicate for global validity:

**abbreviation** *g1*::$\sigma{\Rightarrow}$*bool* ($\lfloor$-$\rfloor$) **where** $\lfloor\psi\rfloor \equiv\ \forall\,w.\ \psi\ w$

Barcan and converse Barcan formula:

**lemma** *True* **nitpick**[*satisfy*] ⟨*proof*⟩
**lemma** $\lfloor(\forall^E x.\Box(\varphi\ x)) \to \Box(\forall^E x.(\varphi\ x))\rfloor$ **nitpick** ⟨*proof*⟩
**lemma** $\lfloor\Box(\forall^E x.(\varphi\ x)) \to (\forall^E x.\Box(\varphi\ x))\rfloor$ **nitpick** ⟨*proof*⟩
**lemma** $\lfloor(\forall\,x.\Box(\varphi\ x)) \to \Box(\forall\,x.\ \varphi\ x)\rfloor$ ⟨*proof*⟩
**lemma** $\lfloor\Box(\forall\,x.(\varphi\ x)) \to (\forall\,x.\Box(\varphi\ x))\rfloor$ ⟨*proof*⟩
**end**

# 3 Selected Simplified Ontological Argument

**theory** *SimplifiedOntologicalArgument* **imports**
*HOML*
**begin**

Positive properties:

**consts** *posProp*::$\gamma{\Rightarrow}\sigma$ ($\mathcal{P}$)

An entity x is God-like if it possesses all positive properties.

**definition** *G* ($\mathcal{G}$) **where** $\mathcal{G}(x) \equiv \forall\,\Phi.(\mathcal{P}(\Phi) \to \Phi(x))$

The axiom's of the simplified variant are presented next; these axioms are further motivated in [4, 1]).

Self-difference is not a positive property (possible alternative: the empty property is not a positive property).

**axiomatization where** *CORO1*: $\lfloor\neg(\mathcal{P}(\lambda x.(x{\neq}x)))\rfloor$

A property entailed by a positive property is positive.

**axiomatization where** *CORO2*: $\lfloor\forall\,\Phi\ \Psi.\ \mathcal{P}(\Phi) \wedge (\forall\,x.\ \Phi(x) \to \Psi(x)) \to \mathcal{P}(\Psi)\rfloor$

Being Godlike is a positive property.

**axiomatization where** *AXIOM3*: $\lfloor\mathcal{P}\ \mathcal{G}\rfloor$

## 3.1 Verifying the Selected Simplified Ontological Argument (version 1)

The existence of a non-exemplified positive property implies that self-difference (or, alternatively, the empty property) is a positive property.

**lemma** *LEMMA1*: $\lfloor (\exists\,\Phi.(\mathcal{P}(\Phi) \wedge \neg(\exists\,x.\ \Phi(x)))) \rightarrow \mathcal{P}(\lambda x.(x{\neq}x))\rfloor$
$\langle proof \rangle$

A non-exemplified positive property does not exist.

**lemma** *LEMMA2*: $\lfloor \neg(\exists\,\Phi.(\mathcal{P}(\Phi) \wedge \neg(\exists\,x.\ \Phi(x))))\rfloor$
$\langle proof \rangle$

Positive properties are exemplified.

**lemma** *LEMMA3*: $\lfloor \forall\,\Phi.(\mathcal{P}(\Phi) \rightarrow (\exists\,x.\ \Phi(x)))\rfloor$
$\langle proof \rangle$

There exists a God-like entity.

**theorem** *THEOREM3'*: $\lfloor \exists\,x.\ \mathcal{G}(x)\rfloor$
$\langle proof \rangle$

Necessarily, there exists a God-like entity.

**theorem** *THEOREM3*: $\lfloor \Box(\exists\,x.\ \mathcal{G}(x))\rfloor$
$\langle proof \rangle$

However, the possible existence of Godlike entity is not implied.

**theorem** *CORO*: $\lfloor \Diamond(\exists\,x.\ \mathcal{G}(x))\rfloor$
**nitpick** $\langle proof \rangle$

## 3.2 Verifying the Selected Simplified Ontological Argument (version 2)

We switch to logic T.

**axiomatization where** *T*: $\lfloor \forall\,\varphi.\ \Box\varphi \rightarrow \varphi\rfloor$
**lemma** *T'*: $\lfloor \forall\,\varphi.\ \varphi \rightarrow \Diamond\varphi\rfloor$ $\langle proof \rangle$

Positive properties are possibly exemplified.

**theorem** *THEOREM1*: $\lfloor \forall\,\Phi.\ \mathcal{P}(\Phi) \rightarrow \Diamond(\exists\,x.\ \Phi(x))\rfloor$
$\langle proof \rangle$

Possibly there exists a God-like entity.

**theorem** *CORO*: $\lfloor \Diamond(\exists\,x.\ \mathcal{G}(x))\rfloor$
$\langle proof \rangle$

The possible existence of a God-like entity impplies the necessary existence of a God-like entity.

**theorem** *THEOREM2*: $\lfloor \Diamond(\exists\,x.\ \mathcal{G}(x)) \rightarrow \Box(\exists\,x.\ \mathcal{G}(x))\rfloor$
$\langle proof \rangle$

Necessarily, there exists a God-like entity.

**theorem** *THEO3*: $\lfloor \Box(\exists\, x.\ \mathcal{G}(x)) \rfloor$
$\langle proof \rangle$

There exists a God-like entity.

**theorem** *THEO3'*: $\lfloor \exists\, x.\ \mathcal{G}(x) \rfloor$
$\langle proof \rangle$

Modal collapse is not implied; nitpick reports a countermodel.

**lemma** *MC*: $\lfloor \forall\, \Phi.\ \Phi \to \Box\Phi \rfloor$ **nitpick** $\langle proof \rangle$

Consistency of the theory; nitpick reports a model.

**lemma** *True* **nitpick**[*satisfy*] $\langle proof \rangle$
**end**

# 4 Presentation of All Variants as Studied in [4]

## 4.1 Preliminaries: Modal Ultrafilter (Fig. 2 in [4])

**theory** *MFilter* **imports** *HOML*
**begin**

Some abbreviations for auxiliary operations.

**abbreviation** $a{::}\gamma{\Rightarrow}(\gamma{\Rightarrow}\sigma){\Rightarrow}\sigma$ (-$\in$-) **where** $x{\in}S \equiv S\ x$
**abbreviation** $b{::}\gamma$ ($\emptyset$) **where** $\emptyset \equiv \lambda x.\ \bot$
**abbreviation** $c{::}\gamma$ ($\mathbf{U}$) **where** $\mathbf{U} \equiv \lambda x.\ \top$
**abbreviation** $d{::}\gamma{\Rightarrow}\gamma{\Rightarrow}\sigma$ (-$\subseteq$-) **where** $\varphi{\subseteq}\psi \equiv \forall\, x.((\varphi\ x) \to (\psi\ x))$
**abbreviation** $e{::}\gamma{\Rightarrow}\gamma{\Rightarrow}\gamma$ (-$\sqcap$-) **where** $\varphi{\sqcap}\psi \equiv \lambda x.((\varphi\ x) \wedge (\psi\ x))$
**abbreviation** $f{::}\gamma{\Rightarrow}\gamma$ ($^{-1}$-) **where** $^{-1}\psi \equiv \lambda x.\ \neg(\psi\ x)$

Definition of modal filter.

**abbreviation** $g{::}(\gamma{\Rightarrow}\sigma){\Rightarrow}\sigma$ (*Filter*)
**where** $\mathit{Filter}\ \Phi \equiv (((\mathbf{U}{\in}\Phi) \wedge \neg(\emptyset{\in}\Phi))$
$\wedge\ (\forall\, \varphi\ \psi.\ (((\varphi{\in}\Phi) \wedge (\varphi{\subseteq}\psi)) \to (\psi{\in}\Phi))))$
$\wedge\ (\forall\, \varphi\ \psi.\ (((\varphi{\in}\Phi) \wedge (\psi{\in}\Phi)) \to ((\varphi{\sqcap}\psi){\in}\Phi)))$

Definition of modal ultrafilter .

**abbreviation** $h{::}(\gamma{\Rightarrow}\sigma){\Rightarrow}\sigma$ (*UFilter*) **where**
$\mathit{UFilter}\ \Phi \equiv (\mathit{Filter}\ \Phi){\wedge}(\forall\, \varphi.((\varphi{\in}\Phi) \vee ((^{-1}\varphi){\in}\Phi)))$

Modal filter and modal ultrafilter are consistent.

**lemma** $\lfloor \forall\, \Phi\ \varphi.((\mathit{UFilter}\ \Phi) \to \neg((\varphi{\in}\Phi) \wedge ((^{-1}\varphi){\in}\Phi))) \rfloor$ $\langle proof \rangle$
**end**

## 4.2 Preliminaries: Further Basic Notions (Fig. 3 in [4])

**theory** *BaseDefs* **imports** *HOML*
**begin**

Positive properties.

**consts** *posProp*::$\gamma \Rightarrow \sigma$ ($\mathcal{P}$)

Basic definitions for modal ontological argument.

**abbreviation** $a$ (-$\sqsubseteq$-) **where** $X \sqsubseteq Y \equiv \forall^E z.((X\ z) \rightarrow (Y\ z))$
**abbreviation** $b$ (-$\Rrightarrow$-) **where** $X \Rrightarrow Y \equiv \Box(X \sqsubseteq Y)$
**abbreviation** $c$ ($\mathcal{P}$os) **where** $\mathcal{P}\text{os}\ Z \equiv \forall X.((Z\ X) \rightarrow (\mathcal{P}\ X))$
**abbreviation** $d$ (-$\prod$-) **where** $X \prod \mathcal{Z} \equiv \Box(\forall^E u.((X\ u) \leftrightarrow (\forall Y.((\mathcal{Z}\ Y) \rightarrow (Y\ u)))))$

Definition of Godlike.

**definition** $G$ ($\mathcal{G}$) **where** $\mathcal{G}\ x \equiv \forall Y.((\mathcal{P}\ Y) \rightarrow (Y\ x))$

Definitions of Essence and Necessary Existence.

**definition** $E$ ($\mathcal{E}$) **where** $\mathcal{E}\ Y\ x \equiv (Y\ x) \wedge (\forall Z.((Z\ x) \rightarrow (Y \Rrightarrow Z)))$
**definition** $NE$ ($\mathcal{NE}$) **where** $\mathcal{NE}\ x \equiv \forall Y.((\mathcal{E}\ Y\ x) \rightarrow \Box(\exists^E\ Y))$
**end**

### 4.3 Ultrafilter Analysis of Scott's Variant (Fig. 3 in [4]))

**theory** *ScottVariant* **imports**
*HOML*
*MFilter*
*BaseDefs*
**begin**

Axioms of Scott's variant.

**axiomatization where**
*A1*: $\lfloor \forall X.((\neg(\mathcal{P}\ X)) \leftrightarrow (\mathcal{P}(\rightharpoondown X)))\rfloor$ **and**
*A2*: $\lfloor \forall X\ Y.(((\mathcal{P}\ X) \wedge (X \Rrightarrow Y)) \rightarrow (\mathcal{P}\ Y))\rfloor$ **and**
*A3*: $\lfloor \forall \mathcal{Z}.((\mathcal{P}\text{os}\ \mathcal{Z}) \rightarrow (\forall X.((X \prod \mathcal{Z}) \rightarrow (\mathcal{P}\ X))))\rfloor$ **and**
*A4*: $\lfloor \forall X.((\mathcal{P}\ X) \rightarrow \Box(\mathcal{P}\ X))\rfloor$ **and**
*A5*: $\lfloor \mathcal{P}\ \mathcal{NE}\rfloor$ **and**
*B*: $\lfloor \forall \varphi.(\varphi \rightarrow \Box\Diamond\varphi)\rfloor$ — Logic KB

**lemma** $B'$: $\forall x\ y.\ \neg(x\mathbf{r}y) \vee (y\mathbf{r}x)$ ⟨*proof*⟩

Necessary existence of a Godlike entity.

**theorem** *T6*: $\lfloor \Box(\exists^E\ \mathcal{G})\rfloor$
⟨*proof*⟩

Existence of a Godlike entity.

**lemma** $\lfloor \exists^E\ \mathcal{G}\rfloor$ ⟨*proof*⟩

Consistency

**lemma** *True* **nitpick**[*satisfy*] ⟨*proof*⟩

Modal collapse: holds.

**lemma** *MC*: $\lfloor \forall \Phi.(\Phi \rightarrow \Box\Phi)\rfloor$

⟨*proof*⟩

Analysis of positive properties using ultrafilters.

**theorem** *U1*: $\lfloor UFilter\ \mathcal{P}\rfloor$ — Proof found by sledgehammer
⟨*proof*⟩

**lemma** *L1*: $\lfloor\forall X\ Y.((X \Rrightarrow Y) \rightarrow (X \sqsubseteq Y))\rfloor$ ⟨*proof*⟩
**lemma** *L2*: $\lfloor\forall X\ Y.(((\mathcal{P}\ X) \wedge (X \sqsubseteq Y)) \rightarrow (\mathcal{P}\ Y))\rfloor$ ⟨*proof*⟩

Set of supersets of X, we call this HF X.

**abbreviation** *HF* **where** $HF\ X \equiv \lambda Y.(X \sqsubseteq Y)$

*HF* $\mathcal{G}$ is a filter; hence, *HF* $\mathcal{G}$ is Hauptfilter of $\mathcal{G}$.

**lemma** *F1*: $\lfloor Filter\ (HF\ \mathcal{G})\rfloor$ ⟨*proof*⟩
**lemma** *F2*: $\lfloor UFilter\ (HF\ \mathcal{G})\rfloor$ ⟨*proof*⟩

T6 follows directly from F1.

**theorem** *T6again*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$ ⟨*proof*⟩
**end**

### 4.4 Ultrafilter Variant (Fig. 5 in [4])

**theory** *UFilterVariant* **imports**
*HOML*
*MFilter*
*BaseDefs*
**begin**

Axiom's of ultrafilter variant.

**axiomatization where**
*U1*: $\lfloor UFilter\ \mathcal{P}\rfloor$ **and**
*A2*: $\lfloor\forall X\ Y.(((\mathcal{P}\ X) \wedge (X \Rrightarrow Y)) \rightarrow (\mathcal{P}\ Y))\rfloor$ **and**
*A3*: $\lfloor\forall \mathcal{Z}.((\mathcal{P}os\ \mathcal{Z}) \rightarrow (\forall X.((X \prod \mathcal{Z}) \rightarrow (\mathcal{P}\ X))))\rfloor$

Necessary existence of a Godlike entity.

**theorem** *T6*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$ — Proof also found by sledgehammer
⟨*proof*⟩

Checking for consistency.

**lemma** *True* **nitpick**[*satisfy*] ⟨*proof*⟩

Checking for modal collapse.

**lemma** *MC*: $\lfloor\forall \Phi.(\Phi \rightarrow \Box\Phi)\rfloor$ **nitpick** ⟨*proof*⟩
**end**

## 4.5 Simplified Variant (Fig. 6 in [4])

**theory** *SimpleVariant* **imports**
*HOML*
*MFilter*
*BaseDefs*
**begin**

Axiom's of new, simplified variant.

**axiomatization where**
*A1′*: $\lfloor\neg(\mathcal{P}(\lambda x.(x \neq x)))\rfloor$ **and**
*A2′*: $\lfloor\forall X\ Y.(((\mathcal{P}\ X) \wedge ((X \sqsubseteq Y) \vee (X \Rrightarrow Y))) \rightarrow (\mathcal{P}\ Y))\rfloor$ **and**
*A3*: $\lfloor\forall \mathcal{Z}.((\mathcal{P}os\ \mathcal{Z}) \rightarrow (\forall X.((X \sqcap \mathcal{Z}) \rightarrow (\mathcal{P}\ X))))\rfloor$

**lemma** *T2*: $\lfloor\mathcal{P}\ \mathcal{G}\rfloor$ ⟨*proof*⟩
**lemma** *L1*: $\lfloor\mathcal{P}(\lambda x.(x=x))\rfloor$ ⟨*proof*⟩

Necessary existence of a Godlike entity.

**theorem** *T6*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$ — Proof found by sledgehammer
⟨*proof*⟩

**lemma** *True* **nitpick**[*satisfy*] ⟨*proof*⟩

Modal collapse and monotheism: not implied.

**lemma** *MC*: $\lfloor\forall \Phi.(\Phi \rightarrow \Box\Phi)\rfloor$ **nitpick** ⟨*proof*⟩
**lemma** *MT*: $\lfloor\forall x\ y.(((\mathcal{G}\ x) \wedge (\mathcal{G}\ y)) \rightarrow (x=y))\rfloor$
**nitpick** ⟨*proof*⟩

Gödel's A1, A4, A5: not implied anymore.

**lemma** *A1*: $\lfloor\forall X.((\neg(\mathcal{P}\ X)) \leftrightarrow (\mathcal{P}(\rightharpoondown X)))\rfloor$ **nitpick** ⟨*proof*⟩
**lemma** *A4*: $\lfloor\forall X.((\mathcal{P}\ X) \rightarrow \Box(\mathcal{P}\ X))\rfloor$ **nitpick** ⟨*proof*⟩
**lemma** *A5*: $\lfloor\mathcal{P}\ \mathcal{N}\mathcal{E}\rfloor$ **nitpick** ⟨*proof*⟩

Checking filter and ultrafilter properties.

**theorem** *F1*: $\lfloor$*Filter* $\mathcal{P}\rfloor$ ⟨*proof*⟩
**theorem** *U1*: $\lfloor$*UFilter* $\mathcal{P}\rfloor$ **nitpick** ⟨*proof*⟩
**end**

## 4.6 Simplified Variant with Axiom T2 (Fig. 7 in [4])

**theory** *SimpleVariantPG* **imports**
*HOML*
*MFilter*
*BaseDefs*
**begin**

Axiom's of simplified variant with A3 replaced.

**axiomatization where**
*A1′*: $\lfloor\neg(\mathcal{P}(\lambda x.(x \neq x)))\rfloor$ **and**
*A2′*: $\lfloor\forall X\ Y.(((\mathcal{P}\ X) \wedge ((X \sqsubseteq Y) \vee (X \Rrightarrow Y))) \rightarrow (\mathcal{P}\ Y))\rfloor$ **and**

*T2*: $\lfloor \mathcal{P}\ \mathcal{G} \rfloor$

Necessary existence of a Godlike entity.

**theorem** *T6*: $\lfloor \Box(\exists^E\ \mathcal{G}) \rfloor$ — Proof found by sledgehammer
⟨*proof*⟩

**lemma** *True* **nitpick**[*satisfy*] ⟨*proof*⟩

Modal collapse and Monotheism: not implied.

**lemma** *MC*: $\lfloor \forall\,\Phi.(\Phi \rightarrow \Box\Phi) \rfloor$ **nitpick** ⟨*proof*⟩
**lemma** *MT*: $\lfloor \forall\, x\ y.(((\mathcal{G}\ x) \wedge (\mathcal{G}\ y)) \rightarrow (x{=}y)) \rfloor$ **nitpick** ⟨*proof*⟩
**end**

## 4.7 Simplified Variant with Simple Entailment in Logic K (Fig. 8 in [4])

**theory** *SimpleVariantSE* **imports**
*HOML*
*MFilter*
*BaseDefs*
**begin**

Axiom's of new variant based on ultrafilters.

**axiomatization where**
*A1′*: $\lfloor \neg(\mathcal{P}(\lambda x.(x{\neq}x))) \rfloor$ **and**
*A2″*: $\lfloor \forall\, X\ Y.(((\mathcal{P}\ X) \wedge (X{\sqsubseteq}Y)) \rightarrow (\mathcal{P}\ Y)) \rfloor$ **and**
*T2*: $\lfloor \mathcal{P}\ \mathcal{G} \rfloor$

Necessary existence of a Godlike entity.

**theorem** *T6*: $\lfloor \Box(\exists^E\ \mathcal{G}) \rfloor$ ⟨*proof*⟩
**theorem** *T7*: $\lfloor \exists^E\ \mathcal{G} \rfloor$ ⟨*proof*⟩

Possible existence of a Godlike: has counterodel.

**lemma** *T3*: $\lfloor \Diamond(\exists^E\ \mathcal{G}) \rfloor$ **nitpick** ⟨*proof*⟩

**lemma** *T3′*: **assumes** *T*: $\lfloor \forall\,\varphi.((\Box\varphi) \rightarrow \varphi) \rfloor$
**shows** $\lfloor \Diamond(\exists^E\ \mathcal{G}) \rfloor$
⟨*proof*⟩
**end**

## 4.8 Simplified Variant with Simple Entailment in Logic T (Fig. 9 in [4])

**theory** *SimpleVariantSEinT* **imports**
*HOML*
*MFilter*
*BaseDefs*
**begin**

Axiom's of new variant based on ultrafilters.

**axiomatization where**
$A1'$: $\lfloor\neg(\mathcal{P}(\lambda x.(x{\neq}x)))\rfloor$ **and**
$A2''$: $\lfloor\forall X\ Y.(((\mathcal{P}\ X) \wedge (X{\sqsubseteq}Y)) \rightarrow (\mathcal{P}\ Y))\rfloor$ **and**
$T2$: $\lfloor\mathcal{P}\ \mathcal{G}\rfloor$

Modal Logic T.

**axiomatization where** $T$: $\lfloor\forall\varphi.((\Box\varphi) \rightarrow \varphi)\rfloor$
**lemma** $T'$: $\lfloor\forall\varphi.(\varphi \rightarrow (\Diamond\varphi))\rfloor$ ⟨*proof*⟩

Necessary existence of a Godlike entity.

**theorem** *T6*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$ — Proof found by sledgehammer
⟨*proof*⟩

T6 again, with an alternative, simpler proof.

**theorem** *T6again*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$
⟨*proof*⟩
**end**

### 4.9 Hauptfiltervariant (Fig. 10 in [4])

**theory** *SimpleVariantHF* **imports**
*HOML*
*MFilter*
*BaseDefs*
**begin**

Definition: Set of supersets of $X$, we call this $\mathcal{HF}\ X$.

**abbreviation** $HF{::}\gamma{\Rightarrow}(\gamma{\Rightarrow}\sigma)$ **where** $HF\ X \equiv \lambda Y.(X{\sqsubseteq}Y)$

Postulate: $\mathcal{HF}\ \mathcal{G}$ is a filter; i.e., Hauptfilter of $\mathcal{G}$.

**axiomatization where** $F1$: $\lfloor Filter\ (HF\ \mathcal{G})\rfloor$

Necessary existence of a Godlike entity.

**theorem** *T6*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$ ⟨*proof*⟩

**theorem** *T6again*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$
⟨*proof*⟩

Possible existence of Godlike entity not implied.

**lemma** *T3*: $\lfloor\Diamond(\exists^E\ \mathcal{G})\rfloor$ **nitpick** ⟨*proof*⟩

Axiom T enforces possible existence of Godlike entity.

**axiomatization**
**lemma** *T3*: **assumes** $T$: $\lfloor\forall\varphi.((\Box\varphi) \rightarrow \varphi)\rfloor$
**shows** $\lfloor\Diamond(\exists^E\ \mathcal{G})\rfloor$ ⟨*proof*⟩

**lemma** *True* **nitpick**[*satisfy*] ⟨*proof*⟩

Modal collapse: not implied anymore.

**lemma** *MC*: $\lfloor\forall\,\Phi.(\Phi \rightarrow \Box\Phi)\rfloor$ **nitpick** $\langle proof\rangle$
**lemma** *MT*: $\lfloor\forall\,x\;y.(((\mathcal{G}\;x) \wedge (\mathcal{G}\;y)) \rightarrow (x{=}y))\rfloor$
**nitpick** $\langle proof\rangle$
**end**

### 4.10 Formal Study of Version No.2 of Gödel's Argument as Reported by Kanckos and Lethen, 2019 [6] (Fig. 11 in [4])

**theory** *KanckosLethenNo2Possibilist* **imports**
*HOML*
*MFilter*
*BaseDefs*
**begin**

Axioms of Version No. 2 [6].

**abbreviation** *delta* ($\Delta$) **where** $\Delta\;A \equiv \lambda x.(\forall\,\psi.\;((A\;\psi) \rightarrow (\psi\;x)))$
**abbreviation** *N* ($\mathcal{N}$) **where** $\mathcal{N}\;\varphi \equiv \lambda x.(\Box(\varphi\;x))$

**axiomatization where**
*Axiom1*: $\lfloor\forall\,\varphi\;\psi.(((\mathcal{P}\;\varphi) \wedge (\Box(\forall\,x.\;((\varphi\;x) \rightarrow (\psi\;x))))) \rightarrow (\mathcal{P}\;\psi))\rfloor$ **and** — The $\Box$ can be omitted here; the proofs still work.
*Axiom2*: $\lfloor\forall\,A\;.(\Box((\forall\,\varphi.((A\;\varphi) \rightarrow\;(\mathcal{P}\;\varphi))) \rightarrow (\mathcal{P}\;(\Delta\;A))))\rfloor$ **and** — The $\Box$ can be omitted here; the proofs still work.
*Axiom3*: $\lfloor\forall\,\varphi.((\mathcal{P}\;\varphi) \rightarrow (\mathcal{P}\;(\mathcal{N}\;\varphi)))\rfloor$ **and**
*Axiom4*: $\lfloor\forall\,\varphi.((\mathcal{P}\;\varphi) \rightarrow (\neg(\mathcal{P}(\rightharpoondown\varphi))))\rfloor$ **and**
— Logic S5
*axB*: $\lfloor\forall\,\varphi.(\varphi \rightarrow \Box\Diamond\varphi)\rfloor$ **and** *axM*: $\lfloor\forall\,\varphi.((\Box\varphi) \rightarrow \varphi)\rfloor$ **and** *ax4*: $\lfloor\forall\,\varphi.((\Box\varphi) \rightarrow (\Box\Box\varphi))\rfloor$

Sahlqvist correspondences: they are better suited for proof automation.

**lemma** *axB'*: $\forall\,x\;y.\;\neg(x\mathbf{r}y) \vee (y\mathbf{r}x)$ $\langle proof\rangle$
**lemma** *axM'*: $\forall\,x.\;(x\mathbf{r}x)$ $\langle proof\rangle$
**lemma** *ax4'*: $\forall\,x\;y\;z.\;(((x\mathbf{r}y) \wedge (y\mathbf{r}z)) \longrightarrow (x\mathbf{r}z))$ $\langle proof\rangle$

Proofs for all theorems for No.2 from [6].

**theorem** *Theorem0*: $\lfloor\forall\,\varphi\;\psi.((\forall\,Q.\;((Q\;\varphi)\;\rightarrow (Q\;\psi))) \rightarrow\;((\mathcal{P}\;\varphi) \rightarrow (\mathcal{P}\;\;\varphi)))\rfloor$ $\langle proof\rangle$
**theorem** *Theorem1*: $\lfloor\mathcal{P}\;\mathcal{G}\rfloor$ $\langle proof\rangle$
**theorem** *Theorem2*: $\lfloor\forall\,x.\;((\mathcal{G}\;x){\rightarrow}(\exists\,y.\;\mathcal{G}\;y))\rfloor$ $\langle proof\rangle$
**theorem** *Theorem3a*: $\lfloor\mathcal{P}\;(\lambda x.\;(\exists\,y.\;\mathcal{G}\;y))\rfloor$ $\langle proof\rangle$
**theorem** *Theorem3b*: $\lfloor\Box(\mathcal{P}\;(\lambda x.(\Box(\exists\,y.\;\mathcal{G}\;y))))\rfloor$ $\langle proof\rangle$
**theorem** *Theorem4*: $\lfloor\forall\,x.\;\Box((\mathcal{G}\;x) \rightarrow ((\mathcal{P}\;(\lambda x.(\Box(\exists\,y.\;\mathcal{G}\;y)))) \rightarrow\;(\Box(\exists\,y.\;\mathcal{G}\;y))))\rfloor$ $\langle proof\rangle$
**theorem** *Theorem5*: $\lfloor\forall\,x.\;\Box((\mathcal{G}\;x) \rightarrow (\Box(\exists\,y.\;\mathcal{G}\;y)))\rfloor$ $\langle proof\rangle$
**theorem** *Theorem6*: $\lfloor\Box((\exists\,y.\;\mathcal{G}\;y) \rightarrow (\Box(\exists\,y.\;\mathcal{G}\;y)))\rfloor$ $\langle proof\rangle$
**theorem** *Theorem7*: $\lfloor\Box((\Diamond(\exists\,y.\;\mathcal{G}\;y)) \rightarrow (\Box(\exists\,y.\;\mathcal{G}\;y)))\rfloor$ $\langle proof\rangle$
**theorem** *Theorem8*: $\lfloor\Box(\exists\,y.\;\mathcal{G}\;y)\rfloor$ $\langle proof\rangle$
**theorem** *Theorem9*: $\lfloor\forall\,\varphi.\;((\mathcal{P}\;\varphi) \rightarrow \Diamond(\exists\,x.\;\varphi\;x))\rfloor$ $\langle proof\rangle$

Short proof of Theorem8; analogous to the one presented in Sec. 7 of Benzmüller 2020.

**theorem** $\lfloor\Box(\exists\, y.\ \mathcal{G}\ y)\rfloor$ — Note: this version of the proof uses only $axB'$ and $axM'$.
⟨*proof*⟩

**theorem** *T5*: $\lfloor(\Diamond(\exists\, y.\ \mathcal{G}\ y)) \rightarrow \Box(\exists\, y.\ \mathcal{G}\ y)\rfloor$ — Obvious: If we can prove Theorem8, then we also have T5.
⟨*proof*⟩

Another short proof of Theorem8.

**theorem** $\lfloor\Box(\exists\, y.\ \mathcal{G}\ y)\rfloor$ — Note: fewer assumptions used in some cases than in [6].
⟨*proof*⟩

Are the axioms of the simplified versions implied?

Actualist version of the axioms.

**lemma** *A1'*: $\lfloor\neg(\mathcal{P}(\lambda x.(x{\neq}x)))\rfloor$ ⟨*proof*⟩
**lemma** *A2'*: $\lfloor\forall\, X\ Y.(((\mathcal{P}\ X) \wedge ((X{\sqsubseteq}Y){\vee}(X{\Rrightarrow}Y))) \rightarrow (\mathcal{P}\ Y))\rfloor$ **nitpick** ⟨*proof*⟩
**lemma** *A3*: $\lfloor\forall\, \mathcal{Z}.((\mathcal{P}\mathrm{os}\ \mathcal{Z}) \rightarrow (\forall\, X.((X{\textstyle\prod}\mathcal{Z}) \rightarrow (\mathcal{P}\ X))))\rfloor$ **nitpick** ⟨*proof*⟩

Possibilist version of the axioms.

**abbreviation** *a* $(\text{-}{\sqsubseteq}^p\text{-})$ **where** $X{\sqsubseteq}^p Y \equiv \forall\, z.((X\ z) \rightarrow (Y\ z))$
**abbreviation** *b* $(\text{-}{\Rrightarrow}^p\text{-})$ **where** $X{\Rrightarrow}^p Y \equiv \Box(X{\sqsubseteq}^p Y)$
**abbreviation** *d* $(\text{-}{\textstyle\prod}^p\text{-})$ **where** $X{\textstyle\prod}^p\mathcal{Z} \equiv \Box(\forall\, u.((X\ u) \leftrightarrow (\forall\, Y.((\mathcal{Z}\ Y) \rightarrow (Y\ u)))))$

**lemma** *A1'P*: $\lfloor\neg(\mathcal{P}(\lambda x.(x{\neq}x)))\rfloor$ ⟨*proof*⟩
**lemma** *A2'P*: $\lfloor\forall\, X\ Y.(((\mathcal{P}\ X) \wedge ((X{\sqsubseteq}^p Y){\vee}(X{\Rrightarrow}^p Y))) \rightarrow (\mathcal{P}\ Y))\rfloor$ ⟨*proof*⟩
**lemma** *A2'aP*: $\lfloor\forall\, X\ Y.(((\mathcal{P}\ X) \wedge (X{\Rrightarrow}^p Y)) \rightarrow (\mathcal{P}\ Y))\rfloor$ ⟨*proof*⟩
**lemma** *A2'bP*: $\lfloor\forall\, X\ Y.(((\mathcal{P}\ X) \wedge (X{\sqsubseteq}^p Y)) \rightarrow (\mathcal{P}\ Y))\rfloor$ ⟨*proof*⟩
**lemma** *A3P*: $\lfloor\forall\, \mathcal{Z}.((\mathcal{P}\mathrm{os}\ \mathcal{Z}) \rightarrow (\forall\, X.((X{\textstyle\prod}^p\mathcal{Z}) \rightarrow (\mathcal{P}\ X))))\rfloor$
⟨*proof*⟩

Are Axiom2 and A3 equivalent? Only when assuming Axiom1 and axiom M.

**lemma** $\lfloor\forall\, A\ .(\Box((\forall\, \varphi.((A\ \varphi) \rightarrow\ (\mathcal{P}\ \varphi))) \rightarrow (\mathcal{P}\ (\Delta\ A))))\rfloor \equiv \lfloor\forall\, \mathcal{Z}.((\mathcal{P}\mathrm{os}\ \mathcal{Z}) \rightarrow (\forall\, X.((X{\textstyle\prod}^p\mathcal{Z}) \rightarrow (\mathcal{P}\ X))))\rfloor$
⟨*proof*⟩
**end**

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