Exploring Simplified Variants of Gödel's Ontological Argument in Isabelle/HOL

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Abstract

Simplified variants of Gödel's ontological argument are explored. Among those is a particularly interesting simplified argument which is (i) valid already in basic modal logics K or KT, (ii) which does not suffer from modal collapse, and (iii) which avoids the rather complex predicates of essence (Ess.) and necessary existence (NE) as used by Gödel.

Whether the presented variants increase or decrease the attractiveness and persuasiveness of the ontological argument is a question I would like to pass on to philosophy and theology.

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1 Background Reading

The selected simplified variants of Gödel's ontological argument [5, 7] as presented in §3 have first been extracted from the insights gained in §6 of [4]. These variants are also influenced by the work presented in [1] and they significantly extend the findings from [3]. In §4 we additionally include the sources from [4].

2 Higher-Order Modal Logic in HOL (cf. [2] and Fig. 1 in [4]).

theory *HOML* **imports** *Main*
begin
nitpick-params[*user-axioms*,*expect=genuine*]

Type i is associated with possible worlds and type e with entities:

typedecl i — Possible worlds
typedecl e — Individuals
type-synonym $\sigma = i \Rightarrow bool$ — World-lifted propositions
type-synonym $\gamma = e \Rightarrow \sigma$ — Lifted predicates
type-synonym $\mu = \sigma \Rightarrow \sigma$ — Unary modal connectives
type-synonym $\nu = \sigma \Rightarrow \sigma \Rightarrow \sigma$ — Binary modal connectives

Logical connectives (operating on truth-sets):

abbreviation $c1::\sigma$ (‹⊥›) **where** $\bot \equiv \lambda w.\ False$
abbreviation $c2::\sigma$ (‹⊤›) **where** $\top \equiv \lambda w.\ True$
abbreviation $c3::\mu$ (‹¬-›[52]53) **where** $\neg\varphi \equiv \lambda w.\neg(\varphi\ w)$
abbreviation $c4::\nu$ (**infix**‹∧›50) **where** $\varphi\wedge\psi \equiv \lambda w.(\varphi\ w)\wedge(\psi\ w)$
abbreviation $c5::\nu$ (**infix**‹∨›49) **where** $\varphi\vee\psi \equiv \lambda w.(\varphi\ w)\vee(\psi\ w)$
abbreviation $c6::\nu$ (**infix**‹→›48) **where** $\varphi\rightarrow\psi \equiv \lambda w.(\varphi\ w)\longrightarrow(\psi\ w)$
abbreviation $c7::\nu$ (**infix**‹↔›47) **where** $\varphi\leftrightarrow\psi \equiv \lambda w.(\varphi\ w)\longleftrightarrow(\psi\ w)$
consts $R::i\Rightarrow i\Rightarrow bool$ (‹-**r**-›) — Accessibility relation
abbreviation $c8::\mu$ (‹□-›[54]55) **where** $\Box\varphi \equiv \lambda w.\forall v.(w\mathbf{r}v)\longrightarrow(\varphi\ v)$
abbreviation $c9::\mu$ (‹◇-›[54]55) **where** $\Diamond\varphi \equiv \lambda w.\exists v.(w\mathbf{r}v)\wedge(\varphi\ v)$
abbreviation $c10::\gamma\Rightarrow\gamma$ (‹¬-›[52]53) **where** $\neg\Phi \equiv \lambda x.\lambda w.\neg(\Phi\ x\ w)$
abbreviation $c11::\gamma\Rightarrow\gamma$ (‹⇁-›) **where** $⇁\Phi \equiv \lambda x.\lambda w.\neg(\Phi\ x\ w)$
abbreviation $c12::e\Rightarrow e\Rightarrow\sigma$ (‹-=-›) **where** $x{=}y \equiv \lambda w.(x{=}y)$
abbreviation $c13::e\Rightarrow e\Rightarrow\sigma$ (‹-≠-›) **where** $x{\neq}y \equiv \lambda w.(x{\neq}y)$

Polymorphic possibilist quantification:

abbreviation *q1*::('a⇒σ)⇒σ (‹∀›) **where** $\forall\,\Phi \equiv \lambda w.\forall\, x.(\Phi\ x\ w)$
abbreviation *q2* (**binder**‹∀›[*10*]*11*) **where** $\forall\, x.\ \varphi(x) \equiv \forall\, \varphi$
abbreviation *q3*::('a⇒σ)⇒σ (‹∃›) **where** $\exists\,\Phi \equiv \lambda w.\exists\, x.(\Phi\ x\ w)$
abbreviation *q4* (**binder**‹∃›[*10*]*11*) **where** $\exists\, x.\ \varphi(x) \equiv \exists\, \varphi$

Actualist quantification for individuals/entities:

consts *existsAt*::γ (‹-@-›)
abbreviation *q5*::γ⇒σ (‹$\forall^E$›) **where** $\forall^E \Phi \equiv \lambda w.\forall\, x.(x@w) \longrightarrow (\Phi\ x\ w)$
abbreviation *q6* (**binder**‹$\forall^E$›[*8*]*9*) **where** $\forall^E x.\ \varphi(x) \equiv \forall^E \varphi$
abbreviation *q7*::γ⇒σ (‹$\exists^E$›) **where** $\exists^E \Phi \equiv \lambda w.\exists\, x.(x@w) \wedge (\Phi\ x\ w)$
abbreviation *q8* (**binder**‹$\exists^E$›[*8*]*9*) **where** $\exists^E x.\ \varphi(x) \equiv \exists^E \varphi$

Meta-logical predicate for global validity:

abbreviation *g1*::σ⇒*bool* (‹⌊-⌋›) **where** $\lfloor\psi\rfloor \equiv\ \forall\, w.\ \psi\ w$

Barcan and converse Barcan formula:

lemma *True* **nitpick**[*satisfy*] ⟨*proof*⟩
lemma $\lfloor(\forall^E x.\Box(\varphi\ x)) \rightarrow \Box(\forall^E x.(\varphi\ x))\rfloor$ **nitpick** ⟨*proof*⟩
lemma $\lfloor\Box(\forall^E x.(\varphi\ x)) \rightarrow (\forall^E x.\Box(\varphi\ x))\rfloor$ **nitpick** ⟨*proof*⟩
lemma $\lfloor(\forall\, x.\Box(\varphi\ x)) \rightarrow \Box(\forall\, x.\ \varphi\ x)\rfloor$ ⟨*proof*⟩
lemma $\lfloor\Box(\forall\, x.(\varphi\ x)) \rightarrow (\forall\, x.\Box(\varphi\ x))\rfloor$ ⟨*proof*⟩
end

3 Selected Simplified Ontological Argument

theory *SimplifiedOntologicalArgument* **imports**
HOML
begin

Positive properties:

consts *posProp*::γ⇒σ (‹𝒫›)

An entity x is God-like if it possesses all positive properties.

definition *G* (‹𝒢›) **where** $\mathcal{G}(x) \equiv \forall\,\Phi.(\mathcal{P}(\Phi) \rightarrow \Phi(x))$

The axiom's of the simplified variant are presented next; these axioms are further motivated in [4, 1]).

Self-difference is not a positive property (possible alternative: the empty property is not a positive property).

axiomatization where *CORO1*: $\lfloor\neg(\mathcal{P}(\lambda x.(x\neq x)))\rfloor$

A property entailed by a positive property is positive.

axiomatization where *CORO2*: $\lfloor\forall\,\Phi\ \Psi.\ \mathcal{P}(\Phi) \wedge (\forall\, x.\ \Phi(x) \rightarrow \Psi(x)) \rightarrow \mathcal{P}(\Psi)\rfloor$

Being Godlike is a positive property.

axiomatization where *AXIOM3*: $\lfloor\mathcal{P}\ \mathcal{G}\rfloor$

3.1 Verifying the Selected Simplified Ontological Argument (version 1)

The existence of a non-exemplified positive property implies that self-difference (or, alternatively, the empty property) is a positive property.

lemma *LEMMA1*: $\lfloor(\exists\,\Phi.(\mathcal{P}(\Phi) \wedge \neg(\exists\,x.\ \Phi(x)))) \to \mathcal{P}(\lambda x.(x{\neq}x))\rfloor$
⟨*proof*⟩

A non-exemplified positive property does not exist.

lemma *LEMMA2*: $\lfloor\neg(\exists\,\Phi.(\mathcal{P}(\Phi) \wedge \neg(\exists\,x.\ \Phi(x))))\rfloor$
⟨*proof*⟩

Positive properties are exemplified.

lemma *LEMMA3*: $\lfloor\forall\,\Phi.(\mathcal{P}(\Phi) \to (\exists\,x.\ \Phi(x)))\rfloor$
⟨*proof*⟩

There exists a God-like entity.

theorem *THEOREM3'*: $\lfloor\exists\,x.\ \mathcal{G}(x)\rfloor$
⟨*proof*⟩

Necessarily, there exists a God-like entity.

theorem *THEOREM3*: $\lfloor\Box(\exists\,x.\ \mathcal{G}(x))\rfloor$
⟨*proof*⟩

However, the possible existence of Godlike entity is not implied.

theorem *CORO*: $\lfloor\Diamond(\exists\,x.\ \mathcal{G}(x))\rfloor$
nitpick ⟨*proof*⟩

3.2 Verifying the Selected Simplified Ontological Argument (version 2)

We switch to logic T.

axiomatization where *T*: $\lfloor\forall\,\varphi.\ \Box\varphi \to \varphi\rfloor$
lemma *T'*: $\lfloor\forall\,\varphi.\ \varphi \to \Diamond\varphi\rfloor$ ⟨*proof*⟩

Positive properties are possibly exemplified.

theorem *THEOREM1*: $\lfloor\forall\,\Phi.\ \mathcal{P}(\Phi) \to \Diamond(\exists\,x.\ \Phi(x))\rfloor$
⟨*proof*⟩

Possibly there exists a God-like entity.

theorem *CORO*: $\lfloor\Diamond(\exists\,x.\ \mathcal{G}(x))\rfloor$
⟨*proof*⟩

The possible existence of a God-like entity impplies the necessary existence of a God-like entity.

theorem *THEOREM2*: $\lfloor\Diamond(\exists\,x.\ \mathcal{G}(x)) \to \Box(\exists\,x.\ \mathcal{G}(x))\rfloor$
⟨*proof*⟩

Necessarily, there exists a God-like entity.

theorem *THEO3*: $\lfloor\Box(\exists\, x.\ \mathcal{G}(x))\rfloor$
⟨*proof*⟩

There exists a God-like entity.

theorem *THEO3′*: $\lfloor\exists\, x.\ \mathcal{G}(x)\rfloor$
⟨*proof*⟩

Modal collapse is not implied; nitpick reports a countermodel.

lemma *MC*: $\lfloor\forall\,\Phi.\ \Phi \rightarrow \Box\Phi\rfloor$ **nitpick** ⟨*proof*⟩

Consistency of the theory; nitpick reports a model.

lemma *True* **nitpick**[*satisfy*] ⟨*proof*⟩
end

4 Presentation of All Variants as Studied in [4]

4.1 Preliminaries: Modal Ultrafilter (Fig. 2 in [4])

theory *MFilter* **imports** *HOML*
begin

Some abbreviations for auxiliary operations.

abbreviation $a{::}\gamma{\Rightarrow}(\gamma{\Rightarrow}\sigma){\Rightarrow}\sigma$ (‹-∈-›) **where** $x{\in}S \equiv S\ x$
abbreviation $b{::}\gamma$ (‹∅›) **where** $\emptyset \equiv \lambda x.\ \bot$
abbreviation $c{::}\gamma$ (‹**U**›) **where** $\mathbf{U} \equiv \lambda x.\ \top$
abbreviation $d{::}\gamma{\Rightarrow}\gamma{\Rightarrow}\sigma$ (‹-⊆-›) **where** $\varphi{\subseteq}\psi \equiv \forall\, x.((\varphi\ x) \rightarrow (\psi\ x))$
abbreviation $e{::}\gamma{\Rightarrow}\gamma{\Rightarrow}\gamma$ (‹-⊓-›) **where** $\varphi{\sqcap}\psi \equiv \lambda x.((\varphi\ x) \wedge (\psi\ x))$
abbreviation $f{::}\gamma{\Rightarrow}\gamma$ (‹$^{-1}$-›) **where** $^{-1}\psi \equiv \lambda x.\ \neg(\psi\ x)$

Definition of modal filter.

abbreviation $g{::}(\gamma{\Rightarrow}\sigma){\Rightarrow}\sigma$ (‹*Filter*›)
 where $\mathit{Filter}\ \Phi \equiv (((\mathbf{U}{\in}\Phi) \wedge \neg(\emptyset{\in}\Phi))$
 $\wedge\ (\forall\,\varphi\ \psi.\ (((\varphi{\in}\Phi) \wedge (\varphi{\subseteq}\psi)) \rightarrow (\psi{\in}\Phi))))$
 $\wedge\ (\forall\,\varphi\ \psi.\ (((\varphi{\in}\Phi) \wedge (\psi{\in}\Phi)) \rightarrow ((\varphi{\sqcap}\psi){\in}\Phi)))$

Definition of modal ultrafilter .

abbreviation $h{::}(\gamma{\Rightarrow}\sigma){\Rightarrow}\sigma$ (‹*UFilter*›) **where**
 $\mathit{UFilter}\ \Phi \equiv (\mathit{Filter}\ \Phi){\wedge}(\forall\,\varphi.((\varphi{\in}\Phi) \vee ((^{-1}\varphi){\in}\Phi)))$

Modal filter and modal ultrafilter are consistent.

lemma $\lfloor\forall\,\Phi\ \varphi.((\mathit{UFilter}\ \Phi) \rightarrow \neg((\varphi{\in}\Phi) \wedge ((^{-1}\varphi){\in}\Phi)))\rfloor$ ⟨*proof*⟩
end

4.2 Preliminaries: Further Basic Notions (Fig. 3 in [4])

theory *BaseDefs* **imports** *HOML*
begin

Positive properties.

consts *posProp*::$\gamma\Rightarrow\sigma$ (‹$\mathcal{P}$›)

Basic definitions for modal ontological argument.

abbreviation a (‹-$\sqsubseteq$-›) **where** $X\sqsubseteq Y \equiv \forall^E z.((X\ z) \rightarrow (Y\ z))$
abbreviation b (‹-$\Rrightarrow$-›) **where** $X\Rrightarrow Y \equiv \Box(X\sqsubseteq Y)$
abbreviation c (‹$\mathcal{P}$os›) **where** $\mathcal{P}\text{os}\ Z \equiv \forall X.((Z\ X) \rightarrow (\mathcal{P}\ X))$
abbreviation d (‹-$\prod$-›) **where** $X\prod\mathcal{Z} \equiv \Box(\forall^E u.((X\ u) \leftrightarrow (\forall Y.((\mathcal{Z}\ \ Y) \rightarrow (Y\ u)))))$

Definition of Godlike.

definition G (‹$\mathcal{G}$›) **where** $\mathcal{G}\ x \equiv \forall Y.((\mathcal{P}\ \ Y) \rightarrow (Y\ x))$

Definitions of Essence and Necessary Existence.

definition E (‹$\mathcal{E}$›) **where** $\mathcal{E}\ \ Y\ x \equiv (Y\ x) \wedge (\forall Z.((Z\ x) \rightarrow (Y\Rrightarrow Z)))$
definition NE (‹$\mathcal{NE}$›) **where** $\mathcal{NE}\ x \equiv \forall Y.((\mathcal{E}\ \ Y\ x) \rightarrow \Box(\exists^E\ \ Y))$
end

4.3 Ultrafilter Analysis of Scott's Variant (Fig. 3 in [4]))

theory *ScottVariant* **imports**
HOML
MFilter
BaseDefs
begin

Axioms of Scott's variant.

axiomatization where
A1: $\lfloor\forall X.((\neg(\mathcal{P}\ \ X)) \leftrightarrow (\mathcal{P}(\rightharpoondown X)))\rfloor$ **and**
A2: $\lfloor\forall X\ Y.(((\mathcal{P}\ \ X) \wedge (X\Rrightarrow Y)) \rightarrow (\mathcal{P}\ \ Y))\rfloor$ **and**
A3: $\lfloor\forall\mathcal{Z}.((\mathcal{P}\text{os}\ \mathcal{Z}) \rightarrow (\forall X.((X\prod\mathcal{Z}) \rightarrow (\mathcal{P}\ \ X))))\rfloor$ **and**
A4: $\lfloor\forall X.((\mathcal{P}\ \ X) \rightarrow \Box(\mathcal{P}\ \ X))\rfloor$ **and**
A5: $\lfloor\mathcal{P}\ \mathcal{NE}\rfloor$ **and**
B: $\lfloor\forall\varphi.(\varphi \rightarrow \Box\Diamond\varphi)\rfloor$ — Logic KB

lemma B': $\forall x\ y.\ \neg(x\mathbf{r}y) \vee (y\mathbf{r}x)$ ⟨*proof*⟩

Necessary existence of a Godlike entity.

theorem *T6*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$
⟨*proof*⟩

Existence of a Godlike entity.

lemma $\lfloor\exists^E\ \mathcal{G}\rfloor$ ⟨*proof*⟩

Consistency

lemma *True* **nitpick**[*satisfy*] ⟨*proof*⟩

Modal collapse: holds.

lemma *MC*: $\lfloor\forall\Phi.(\Phi \rightarrow \Box\Phi)\rfloor$

$\langle proof \rangle$

Analysis of positive properties using ultrafilters.

theorem *U1*: $\lfloor UFilter\ \mathcal{P} \rfloor$ — Proof found by sledgehammer
$\langle proof \rangle$

lemma *L1*: $\lfloor \forall X\ Y.((X \Rrightarrow Y) \rightarrow (X \sqsubseteq Y)) \rfloor$ $\langle proof \rangle$
lemma *L2*: $\lfloor \forall X\ Y.(((\mathcal{P}\ X) \wedge (X \sqsubseteq Y)) \rightarrow (\mathcal{P}\ Y)) \rfloor$ $\langle proof \rangle$

Set of supersets of X, we call this HF X.

abbreviation *HF* **where** $HF\ X \equiv \lambda Y.(X \sqsubseteq Y)$

HF $\mathcal{G}$ is a filter; hence, *HF* $\mathcal{G}$ is Hauptfilter of $\mathcal{G}$.

lemma *F1*: $\lfloor Filter\ (HF\ \mathcal{G}) \rfloor$ $\langle proof \rangle$
lemma *F2*: $\lfloor UFilter\ (HF\ \mathcal{G}) \rfloor$ $\langle proof \rangle$

T6 follows directly from F1.

theorem *T6again*: $\lfloor \Box(\exists^E\ \mathcal{G}) \rfloor$ $\langle proof \rangle$
end

4.4 Ultrafilter Variant (Fig. 5 in [4])

theory *UFilterVariant* **imports**
HOML
MFilter
BaseDefs
begin

Axiom's of ultrafilter variant.

axiomatization where
U1: $\lfloor UFilter\ \mathcal{P} \rfloor$ **and**
A2: $\lfloor \forall X\ Y.(((\mathcal{P}\ X) \wedge (X \Rrightarrow Y)) \rightarrow (\mathcal{P}\ Y)) \rfloor$ **and**
A3: $\lfloor \forall \mathcal{Z}.((\mathcal{P}os\ \mathcal{Z}) \rightarrow (\forall X.((X \bigsqcap \mathcal{Z}) \rightarrow (\mathcal{P}\ X)))) \rfloor$

Necessary existence of a Godlike entity.

theorem *T6*: $\lfloor \Box(\exists^E\ \mathcal{G}) \rfloor$ — Proof also found by sledgehammer
$\langle proof \rangle$

Checking for consistency.

lemma *True* **nitpick**[*satisfy*] $\langle proof \rangle$

Checking for modal collapse.

lemma *MC*: $\lfloor \forall \Phi.(\Phi \rightarrow \Box\Phi) \rfloor$ **nitpick** $\langle proof \rangle$
end

4.5 Simplified Variant (Fig. 6 in [4])

theory *SimpleVariant* **imports**
HOML
MFilter
BaseDefs
begin

Axiom's of new, simplified variant.

axiomatization where
A1': $\lfloor\neg(\mathcal{P}(\lambda x.(x \neq x)))\rfloor$ **and**
A2': $\lfloor\forall X\ Y.(((\mathcal{P}\ X) \wedge ((X \sqsubseteq Y) \vee (X \Rrightarrow Y))) \rightarrow (\mathcal{P}\ Y))\rfloor$ **and**
A3: $\lfloor\forall \mathcal{Z}.((\mathcal{P}os\ \mathcal{Z}) \rightarrow (\forall X.((X \prod \mathcal{Z}) \rightarrow (\mathcal{P}\ X))))\rfloor$

lemma *T2*: $\lfloor\mathcal{P}\ \mathcal{G}\rfloor$ ⟨*proof*⟩
lemma *L1*: $\lfloor\mathcal{P}(\lambda x.(x=x))\rfloor$ ⟨*proof*⟩

Necessary existence of a Godlike entity.

theorem *T6*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$ — Proof found by sledgehammer
⟨*proof*⟩

lemma *True* **nitpick**[*satisfy*] ⟨*proof*⟩

Modal collapse and monotheism: not implied.

lemma *MC*: $\lfloor\forall \Phi.(\Phi \rightarrow \Box\Phi)\rfloor$ **nitpick** ⟨*proof*⟩
lemma *MT*: $\lfloor\forall x\ y.(((\mathcal{G}\ x) \wedge (\mathcal{G}\ y)) \rightarrow (x=y))\rfloor$
nitpick ⟨*proof*⟩

Gödel's A1, A4, A5: not implied anymore.

lemma *A1*: $\lfloor\forall X.((\neg(\mathcal{P}\ X)) \leftrightarrow (\mathcal{P}(\neg X)))\rfloor$ **nitpick** ⟨*proof*⟩
lemma *A4*: $\lfloor\forall X.((\mathcal{P}\ X) \rightarrow \Box(\mathcal{P}\ X))\rfloor$ **nitpick** ⟨*proof*⟩
lemma *A5*: $\lfloor\mathcal{P}\ \mathcal{NE}\rfloor$ **nitpick** ⟨*proof*⟩

Checking filter and ultrafilter properties.

theorem *F1*: $\lfloor$*Filter* $\mathcal{P}\rfloor$ ⟨*proof*⟩
theorem *U1*: $\lfloor$*UFilter* $\mathcal{P}\rfloor$ **nitpick** ⟨*proof*⟩
end

4.6 Simplified Variant with Axiom T2 (Fig. 7 in [4])

theory *SimpleVariantPG* **imports**
HOML
MFilter
BaseDefs
begin

Axiom's of simplified variant with A3 replaced.

axiomatization where
A1': $\lfloor\neg(\mathcal{P}(\lambda x.(x \neq x)))\rfloor$ **and**
A2': $\lfloor\forall X\ Y.(((\mathcal{P}\ X) \wedge ((X \sqsubseteq Y) \vee (X \Rrightarrow Y))) \rightarrow (\mathcal{P}\ Y))\rfloor$ **and**

$T2$: $\lfloor \mathcal{P}\ \mathcal{G} \rfloor$

Necessary existence of a Godlike entity.

theorem $T6$: $\lfloor \Box(\exists^E\ \mathcal{G}) \rfloor$ — Proof found by sledgehammer
⟨*proof*⟩

lemma *True* **nitpick**[*satisfy*] ⟨*proof*⟩

Modal collapse and Monotheism: not implied.

lemma MC: $\lfloor \forall\,\Phi.(\Phi \to \Box\Phi) \rfloor$ **nitpick** ⟨*proof*⟩
lemma MT: $\lfloor \forall\, x\ y.(((\mathcal{G}\ x) \wedge (\mathcal{G}\ y)) \to (x = y)) \rfloor$ **nitpick** ⟨*proof*⟩
end

4.7 Simplified Variant with Simple Entailment in Logic K (Fig. 8 in [4])

theory *SimpleVariantSE* **imports**
HOML
MFilter
BaseDefs
begin

Axiom's of new variant based on ultrafilters.

axiomatization where
$A1'$: $\lfloor \neg(\mathcal{P}(\lambda x.(x \neq x))) \rfloor$ **and**
$A2''$: $\lfloor \forall\, X\ Y.(((\mathcal{P}\ X) \wedge (X \sqsubseteq Y)) \to (\mathcal{P}\ Y)) \rfloor$ **and**
$T2$: $\lfloor \mathcal{P}\ \mathcal{G} \rfloor$

Necessary existence of a Godlike entity.

theorem $T6$: $\lfloor \Box(\exists^E\ \mathcal{G}) \rfloor$ ⟨*proof*⟩
theorem $T7$: $\lfloor \exists^E\ \mathcal{G} \rfloor$ ⟨*proof*⟩

Possible existence of a Godlike: has counterodel.

lemma $T3$: $\lfloor \Diamond(\exists^E\ \mathcal{G}) \rfloor$ **nitpick** ⟨*proof*⟩

lemma $T3'$: **assumes** T: $\lfloor \forall\,\varphi.((\Box\varphi) \to \varphi) \rfloor$
shows $\lfloor \Diamond(\exists^E\ \mathcal{G}) \rfloor$
⟨*proof*⟩
end

4.8 Simplified Variant with Simple Entailment in Logic T (Fig. 9 in [4])

theory *SimpleVariantSEinT* **imports**
HOML
MFilter
BaseDefs
begin

Axiom's of new variant based on ultrafilters.

axiomatization where
A1': $\lfloor\neg(\mathcal{P}(\lambda x.(x{\neq}x)))\rfloor$ **and**
A2'': $\lfloor\forall X\ Y.(((\mathcal{P}\ X) \wedge (X{\sqsubseteq}Y)) \rightarrow (\mathcal{P}\ Y))\rfloor$ **and**
T2: $\lfloor\mathcal{P}\ \mathcal{G}\rfloor$

Modal Logic T.

axiomatization where *T*: $\lfloor\forall\varphi.((\Box\varphi) \rightarrow \varphi)\rfloor$
lemma *T'*: $\lfloor\forall\varphi.(\varphi \rightarrow (\Diamond\varphi))\rfloor$ ⟨*proof*⟩

Necessary existence of a Godlike entity.

theorem *T6*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$ — Proof found by sledgehammer
⟨*proof*⟩

T6 again, with an alternative, simpler proof.

theorem *T6again*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$
⟨*proof*⟩
end

4.9 Hauptfiltervariant (Fig. 10 in [4])

theory *SimpleVariantHF* **imports**
HOML
MFilter
BaseDefs
begin

Definition: Set of supersets of X, we call this $\mathcal{HF}\ X$.

abbreviation *HF*::$\gamma\Rightarrow(\gamma\Rightarrow\sigma)$ **where** *HF* $X \equiv \lambda Y.(X{\sqsubseteq}Y)$

Postulate: $\mathcal{HF}\ \mathcal{G}$ is a filter; i.e., Hauptfilter of $\mathcal{G}$.

axiomatization where *F1*: $\lfloor$*Filter* (*HF* $\mathcal{G}$)$\rfloor$

Necessary existence of a Godlike entity.

theorem *T6*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$ ⟨*proof*⟩

theorem *T6again*: $\lfloor\Box(\exists^E\ \mathcal{G})\rfloor$
⟨*proof*⟩

Possible existence of Godlike entity not implied.

lemma *T3*: $\lfloor\Diamond(\exists^E\ \mathcal{G})\rfloor$ **nitpick** ⟨*proof*⟩

Axiom T enforces possible existence of Godlike entity.

axiomatization
lemma *T3*: **assumes** *T*: $\lfloor\forall\varphi.((\Box\varphi) \rightarrow \varphi)\rfloor$
shows $\lfloor\Diamond(\exists^E\ \mathcal{G})\rfloor$ ⟨*proof*⟩

lemma *True* **nitpick**[*satisfy*] ⟨*proof*⟩

Modal collapse: not implied anymore.

lemma *MC*: $\lfloor\forall\,\Phi.(\Phi \rightarrow \Box\Phi)\rfloor$ **nitpick** $\langle proof\rangle$
lemma *MT*: $\lfloor\forall\, x\, y.(((\mathcal{G}\ x) \wedge (\mathcal{G}\ y)) \rightarrow (x{=}y))\rfloor$
nitpick $\langle proof\rangle$
end

4.10 Formal Study of Version No.2 of Gödel's Argument as Reported by Kanckos and Lethen, 2019 [6] (Fig. 11 in [4])

theory *KanckosLethenNo2Possibilist* **imports**
HOML
MFilter
BaseDefs
begin

Axioms of Version No. 2 [6].

abbreviation *delta* (‹Δ›) **where** $\Delta\ A \equiv \lambda x.(\forall\,\psi.\ ((A\ \psi) \rightarrow (\psi\ x)))$
abbreviation *N* (‹$\mathcal{N}$›) **where** $\mathcal{N}\ \varphi \equiv \lambda x.(\Box(\varphi\ x))$

axiomatization where
Axiom1: $\lfloor\forall\,\varphi\ \psi.(((\mathcal{P}\ \varphi) \wedge (\Box(\forall\, x.\ ((\varphi\ x) \rightarrow (\psi\ x))))) \rightarrow (\mathcal{P}\ \psi))\rfloor$ **and** — The $\Box$ can be omitted here; the proofs still work.
Axiom2: $\lfloor\forall\, A\ .(\Box((\forall\,\varphi.((A\ \varphi) \rightarrow\ (\mathcal{P}\ \varphi))) \rightarrow (\mathcal{P}\ (\Delta\ A))))\rfloor$ **and** — The $\Box$ can be omitted here; the proofs still work.
Axiom3: $\lfloor\forall\,\varphi.((\mathcal{P}\ \varphi) \rightarrow (\mathcal{P}\ (\mathcal{N}\ \varphi)))\rfloor$ **and**
Axiom4: $\lfloor\forall\,\varphi.((\mathcal{P}\ \varphi) \rightarrow (\neg(\mathcal{P}(\rightharpoondown\varphi))))\rfloor$ **and**
— Logic S5
axB: $\lfloor\forall\,\varphi.(\varphi \rightarrow \Box\Diamond\varphi)\rfloor$ **and** *axM*: $\lfloor\forall\,\varphi.((\Box\varphi) \rightarrow \varphi)\rfloor$ **and** *ax4*: $\lfloor\forall\,\varphi.((\Box\varphi) \rightarrow (\Box\Box\varphi))\rfloor$

Sahlqvist correspondences: they are better suited for proof automation.

lemma *axB'*: $\forall\, x\ y.\ \neg(x\mathbf{r}y) \vee (y\mathbf{r}x)$ $\langle proof\rangle$
lemma *axM'*: $\forall\, x.\ (x\mathbf{r}x)$ $\langle proof\rangle$
lemma *ax4'*: $\forall\, x\ y\ z.\ (((x\mathbf{r}y) \wedge (y\mathbf{r}z)) \longrightarrow (x\mathbf{r}z))$ $\langle proof\rangle$

Proofs for all theorems for No.2 from [6].

theorem *Theorem0*: $\lfloor\forall\,\varphi\ \psi.((\forall\, Q.\ ((Q\ \varphi)\ \rightarrow (Q\ \psi))) \rightarrow\ ((\mathcal{P}\ \varphi) \rightarrow (\mathcal{P}\ \ \varphi)))\rfloor$ $\langle proof\rangle$
theorem *Theorem1*: $\lfloor\mathcal{P}\ \mathcal{G}\rfloor$ $\langle proof\rangle$
theorem *Theorem2*: $\lfloor\forall\, x.\ ((\mathcal{G}\ x){\rightarrow}(\exists\, y.\ \mathcal{G}\ y))\rfloor$ $\langle proof\rangle$
theorem *Theorem3a*: $\lfloor\mathcal{P}\ (\lambda x.\ (\exists\, y.\ \mathcal{G}\ y))\rfloor$ $\langle proof\rangle$
theorem *Theorem3b*: $\lfloor\Box(\mathcal{P}\ (\lambda x.(\Box(\exists\, y.\ \mathcal{G}\ y))))\rfloor$ $\langle proof\rangle$
theorem *Theorem4*: $\lfloor\forall\, x.\ \Box((\mathcal{G}\ x) \rightarrow ((\mathcal{P}\ (\lambda x.(\Box(\exists\, y.\ \mathcal{G}\ y)))) \rightarrow\ (\Box(\exists\, y.\ \mathcal{G}\ y))))\rfloor$ $\langle proof\rangle$
theorem *Theorem5*: $\lfloor\forall\, x.\ \Box((\mathcal{G}\ x) \rightarrow (\Box(\exists\, y.\ \mathcal{G}\ y)))\rfloor$ $\langle proof\rangle$
theorem *Theorem6*: $\lfloor\Box((\exists\, y.\ \mathcal{G}\ y) \rightarrow (\Box(\exists\, y.\ \mathcal{G}\ y)))\rfloor$ $\langle proof\rangle$
theorem *Theorem7*: $\lfloor\Box((\Diamond(\exists\, y.\ \mathcal{G}\ y)) \rightarrow (\Box(\exists\, y.\ \mathcal{G}\ y)))\rfloor$ $\langle proof\rangle$
theorem *Theorem8*: $\lfloor\Box(\exists\, y.\ \mathcal{G}\ y)\rfloor$ $\langle proof\rangle$
theorem *Theorem9*: $\lfloor\forall\,\varphi.\ ((\mathcal{P}\ \varphi) \rightarrow \Diamond(\exists\, x.\ \varphi\ x))\rfloor$ $\langle proof\rangle$

Short proof of Theorem8; analogous to the one presented in Sec. 7 of Benzmüller 2020.

theorem $\lfloor\Box(\exists\, y.\ \mathcal{G}\ y)\rfloor$ — Note: this version of the proof uses only axB' and axM'.
⟨*proof*⟩

theorem *T5*: $\lfloor(\Diamond(\exists\, y.\ \mathcal{G}\ y)) \rightarrow \Box(\exists\, y.\ \mathcal{G}\ y)\rfloor$ — Obvious: If we can prove Theorem8, then we also have T5.
⟨*proof*⟩

Another short proof of Theorem8.

theorem $\lfloor\Box(\exists\, y.\ \mathcal{G}\ y)\rfloor$ — Note: fewer assumptions used in some cases than in [6].
⟨*proof*⟩

Are the axioms of the simplified versions implied?

Actualist version of the axioms.

lemma *A1'*: $\lfloor\neg(\mathcal{P}(\lambda x.(x{\neq}x)))\rfloor$ ⟨*proof*⟩
lemma *A2'*: $\lfloor\forall\, X\ Y.(((\mathcal{P}\ X) \wedge ((X{\sqsubseteq}Y){\vee}(X{\Rrightarrow}Y))) \rightarrow (\mathcal{P}\ \ Y))\rfloor$ **nitpick** ⟨*proof*⟩
lemma *A3*: $\lfloor\forall\,\mathcal{Z}.((\mathcal{P}\text{os}\ \mathcal{Z}) \rightarrow (\forall\, X.((X{\textstyle\prod}\mathcal{Z}) \rightarrow (\mathcal{P}\ \ X))))\rfloor$ **nitpick** ⟨*proof*⟩

Possibilist version of the axioms.

abbreviation *a* $(\langle\text{-}{\sqsubseteq}^p\text{-}\rangle)$ **where** $X{\sqsubseteq}^p Y \equiv \forall\, z.((X\ z) \rightarrow (Y\ z))$
abbreviation *b* $(\langle\text{-}{\Rrightarrow}^p\text{-}\rangle)$ **where** $X{\Rrightarrow}^p Y \equiv \Box(X{\sqsubseteq}^p Y)$
abbreviation *d* $(\langle\text{-}{\textstyle\prod}^p\text{-}\rangle)$ **where** $X{\textstyle\prod}^p\mathcal{Z} \equiv \Box(\forall\, u.((X\ u) \leftrightarrow (\forall\, Y.((\mathcal{Z}\ \ Y) \rightarrow (Y\ u)))))$

lemma *A1'P*: $\lfloor\neg(\mathcal{P}(\lambda x.(x{\neq}x)))\rfloor$ ⟨*proof*⟩
lemma *A2'P*: $\lfloor\forall\, X\ Y.(((\mathcal{P}\ X) \wedge ((X{\sqsubseteq}^p Y){\vee}(X{\Rrightarrow}^p Y))) \rightarrow (\mathcal{P}\ \ Y))\rfloor$ ⟨*proof*⟩
lemma *A2'aP*: $\lfloor\forall\, X\ Y.(((\mathcal{P}\ X) \wedge (X{\Rrightarrow}^p Y)) \rightarrow (\mathcal{P}\ \ Y))\rfloor$ ⟨*proof*⟩
lemma *A2'bP*: $\lfloor\forall\, X\ Y.(((\mathcal{P}\ X) \wedge (X{\sqsubseteq}^p Y)) \rightarrow (\mathcal{P}\ \ Y))\rfloor$ ⟨*proof*⟩
lemma *A3P*: $\lfloor\forall\,\mathcal{Z}.((\mathcal{P}\text{os}\ \mathcal{Z}) \rightarrow (\forall\, X.((X{\textstyle\prod}^p\mathcal{Z}) \rightarrow (\mathcal{P}\ \ X))))\rfloor$
⟨*proof*⟩

Are Axiom2 and A3 equivalent? Only when assuming Axiom1 and axiom M.

lemma $\lfloor\forall\, A\ .(\Box((\forall\varphi.((A\ \varphi) \rightarrow\ (\mathcal{P}\ \varphi))) \rightarrow (\mathcal{P}\ (\Delta\ A))))\rfloor \equiv \lfloor\forall\,\mathcal{Z}.((\mathcal{P}\text{os}\ \mathcal{Z}) \rightarrow (\forall\, X.((X{\textstyle\prod}^p\mathcal{Z}) \rightarrow (\mathcal{P}\ \ X))))\rfloor$
⟨*proof*⟩
end

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