

# Arrow's General Possibility Theorem

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# 1 Overview

This is a fairly literal encoding of some of Armatya Sen's proofs [Sen70] in Isabelle/HOL. The author initially wrote it while learning to use the proof assistant, and some locutions remain naive. This work is somewhat complementary to the mechanisation of more recent proofs of Arrow's Theorem and the Gibbard-Satterthwaite Theorem by Tobias Nipkow [Nip08].

I strongly recommend Sen's book to anyone interested in social choice theory; his proofs are quite lucid and accessible, and he situates the theory quite well within the broader economic tradition.

## 2 General Lemmas

### 2.1 Extra Finite-Set Lemmas

Small variant of *Finite-Set.finite-subset-induct*: also assume  $F \subseteq A$  in the induction hypothesis.

```
lemma finite-subset-induct' [consumes 2, case-names empty insert]:
  assumes finite F and F ⊆ A
    and empty: P {}
    and insert: ∧ a F. [finite F; a ∈ A; F ⊆ A; a ∉ F; P F] ⇒ P (insert a F)
  shows P F
proof -
  from ⟨finite F⟩
  have F ⊆ A ⇒ ?thesis
  proof induct
    show P {} by fact
  next
    fix x F
    assume finite F and x ∉ F and
      P: F ⊆ A ⇒ P F and i: insert x F ⊆ A
    show P (insert x F)
    proof (rule insert)
      from i show x ∈ A by blast
      from i have F ⊆ A by blast
      with P show P F .
      show finite F by fact
      show x ∉ F by fact
      show F ⊆ A by fact
    qed
  qed
  with ⟨F ⊆ A⟩ show ?thesis by blast
qed
```

A slight improvement on *List.finite-list* - add *distinct*.

```
lemma finite-list: finite A ⇒ ∃ l. set l = A ∧ distinct l
proof (induct rule: finite-induct)
  case (insert x F)
  then obtain l where set l = F ∧ distinct l by auto
  with insert have set (x#l) = insert x F ∧ distinct (x#l) by auto
```

thus ?case by blast  
qed auto

## 2.2 Extra bijection lemmas

lemma *bij-betw-onto*:  $\text{bij-betw } f \ A \ B \implies f \ ' \ A = B$  **unfolding** *bij-betw-def* **by** *simp*

lemma *inj-on-UnI*:  $\llbracket \text{inj-on } f \ A; \text{inj-on } f \ B; f \ ' \ (A - B) \cap f \ ' \ (B - A) = \{\} \rrbracket \implies \text{inj-on } f \ (A \cup B)$   
**by** (*auto iff: inj-on-Un*)

lemma *card-compose-bij*:

**assumes** *bijf*: *bij-betw* *f* *A* *A*

**shows**  $\text{card } \{ a \in A. P \ (f \ a) \} = \text{card } \{ a \in A. P \ a \}$

**proof** –

**from** *bijf* **have**  $T: f \ ' \ \{ a \in A. P \ (f \ a) \} = \{ a \in A. P \ a \}$

**unfolding** *bij-betw-def* **by** *auto*

**from** *bijf* **have**  $\text{card } \{ a \in A. P \ (f \ a) \} = \text{card } (f \ ' \ \{ a \in A. P \ (f \ a) \})$

**unfolding** *bij-betw-def* **by** (*auto intro: subset-inj-on card-image[symmetric]*)

**with** *T* **show** ?thesis **by** *simp*

qed

lemma *card-eq-bij*:

**assumes** *cardAB*:  $\text{card } A = \text{card } B$

**and** *finiteA*: *finite* *A* **and** *finiteB*: *finite* *B*

**obtains** *f* **where** *bij-betw* *f* *A* *B*

**proof** –

**from** *finiteA* **obtain** *g* **where**  $G: \text{bij-betw } g \ A \ \{0..<\text{card } A\}$

**by** (*blast dest: ex-bij-betw-finite-nat*)

**from** *finiteB* **obtain** *h* **where**  $H: \text{bij-betw } h \ \{0..<\text{card } B\} \ B$

**by** (*blast dest: ex-bij-betw-nat-finite*)

**from** *G H cardAB* **have**  $I: \text{inj-on } (h \circ g) \ A$

**unfolding** *bij-betw-def* **by** – (*rule comp-inj-on, simp-all*)

**from** *G H cardAB* **have**  $(h \circ g) \ ' \ A = B$

**unfolding** *bij-betw-def* **by** (*simp add: image-comp [symmetric]*)

**with** *I* **have** *bij-betw*  $(h \circ g) \ A \ B$

**unfolding** *bij-betw-def* **by** *blast*

**thus** *thesis* ..

qed

lemma *bij-combine*:

**assumes** *ABCD*:  $A \subseteq B \ C \subseteq D$

**and** *bijf*: *bij-betw* *f* *A* *C*

**and** *bijg*: *bij-betw* *g*  $(B - A) \ (D - C)$

**obtains** *h*

**where** *bij-betw* *h* *B* *D*

**and**  $\bigwedge x. x \in A \implies h \ x = f \ x$

**and**  $\bigwedge x. x \in B - A \implies h \ x = g \ x$

**proof** –

**let** ?*h* =  $\lambda x. \text{if } x \in A \text{ then } f \ x \ \text{else } g \ x$

**have** *inj-on* ?*h*  $(A \cup (B - A))$

**proof**(*rule inj-on-UnI*)

**from** *bijf* **show** *inj-on* ?*h* *A*

**by** – (*rule inj-onI, auto dest: inj-onD bij-betw-imp-inj-on*)

**from** *bijg* **show** *inj-on ?h (B - A)*  
**by** - (*rule inj-onI, auto dest: inj-onD bij-betw-imp-inj-on*)  
**from** *bijf bijg* **show** *?h ' (A - (B - A)) ∩ ?h ' (B - A - A) = {}*  
**by** (*simp, blast dest: bij-betw-onto*)  
**qed**  
**with** *ABCD* **have** *inj-on ?h B* **by** (*auto iff: Un-absorb1*)  
**moreover**  
**have** *?h ' B = D*  
**proof** -  
**from** *ABCD* **have** *?h ' B = f ' A ∪ g ' (B - A)* **by** (*auto iff: image-Un Un-absorb1*)  
**also from** *ABCD bijf bijg* **have** *... = D* **by** (*blast dest: bij-betw-onto*)  
**finally show** *?thesis .*  
**qed**  
**ultimately have** *bij-betw ?h B D*  
**and**  $\bigwedge x. x \in A \implies ?h x = f x$   
**and**  $\bigwedge x. x \in B - A \implies ?h x = g x$   
**unfolding** *bij-betw-def* **by** *auto*  
**thus** *thesis ..*  
**qed**

**lemma** *bij-complete:*  
**assumes** *finiteC: finite C*  
**and** *ABC: A ⊆ C B ⊆ C*  
**and** *bijf: bij-betw f A B*  
**obtains** *f' where bij-betw f' C C*  
**and**  $\bigwedge x. x \in A \implies f' x = f x$   
**and**  $\bigwedge x. x \in C - A \implies f' x \in C - B$   
**proof** -  
**from** *finiteC ABC bijf* **have** *card B = card A*  
**unfolding** *bij-betw-def*  
**by** (*auto iff: inj-on-iff-eq-card [symmetric] intro: finite-subset*)  
**with** *finiteC ABC bijf* **have** *card (C - A) = card (C - B)*  
**by** (*auto iff: finite-subset card-Diff-subset*)  
**with** *finiteC* **obtain** *g where bijg: bij-betw g (C - A) (C - B)*  
**by** - (*drule card-eq-bij, auto*)  
**from** *ABC bijf bijg*  
**obtain** *f' where bijf': bij-betw f' C C*  
**and** *f'f:  $\bigwedge x. x \in A \implies f' x = f x$*   
**and** *f'g:  $\bigwedge x. x \in C - A \implies f' x = g x$*   
**by** - (*drule bij-combine, auto*)  
**from** *f'g bijg* **have**  $\bigwedge x. x \in C - A \implies f' x \in C - B$   
**by** (*blast dest: bij-betw-onto*)  
**with** *bijf' f'f* **show** *thesis ..*  
**qed**

**lemma** *card-greater:*  
**assumes** *finiteA: finite A*  
**and** *c: card { x ∈ A. P x } > card { x ∈ A. Q x }*  
**obtains** *C*  
**where** *card ({ x ∈ A. P x } - C) = card { x ∈ A. Q x }*  
**and** *C ≠ {}*  
**and** *C ⊆ { x ∈ A. P x }*  
**proof** -

```

let ?PA = { x ∈ A . P x }
let ?QA = { x ∈ A . Q x }
from finiteA obtain p where P: bij-betw p {0..<card ?PA} ?PA
  using ex-bij-betw-nat-finite[where M=?PA]
  by (blast intro: finite-subset)
let ?CN = {card ?QA..<card ?PA}
let ?C = p ' ?CN
have card ({ x ∈ A . P x } - ?C) = card ?QA
proof -
  have nat-add-sub-shuffle:  $\bigwedge x y z. \llbracket (x::nat) > y; x - y = z \rrbracket \implies x - z = y$  by simp
  from P have T: p ' {card ?QA..<card ?PA}  $\subseteq$  ?PA
    unfolding bij-betw-def by auto
  from P have card ?PA - card ?QA = card ?C
    unfolding bij-betw-def
    by (auto iff: card-image subset-inj-on[where A=?CN])
  with c have card ?PA - card ?C = card ?QA by (rule nat-add-sub-shuffle)
  with finiteA P T have card (?PA - ?C) = card ?QA
    unfolding bij-betw-def by (auto iff: finite-subset card-Diff-subset)
  thus ?thesis .
qed
moreover
from P c have ?C  $\neq$  {}
  unfolding bij-betw-def by auto
moreover
from P have ?C  $\subseteq$  { x ∈ A . P x }
  unfolding bij-betw-def by auto
ultimately show thesis ..
qed

```

### 2.3 Collections of witnesses: *hasw*, *has*

Given a set of cardinality at least  $n$ , we can find up to  $n$  distinct witnesses. The built-in *card* function unfortunately satisfies:

$$\text{Finite-Set.card-infinite: } \text{infinite } A \implies \text{card } A = 0$$

These lemmas handle the infinite case uniformly.

Thanks to Gerwin Klein suggesting this approach.

**definition** *hasw* :: 'a list  $\Rightarrow$  'a set  $\Rightarrow$  bool **where**  
*hasw* xs S  $\equiv$  set xs  $\subseteq$  S  $\wedge$  distinct xs

**definition** *has* :: nat  $\Rightarrow$  'a set  $\Rightarrow$  bool **where**  
*has* n S  $\equiv$   $\exists$  xs. *hasw* xs S  $\wedge$  length xs = n

**declare** *hasw-def*[simp]

**lemma** *hasI*[intro]: *hasw* xs S  $\implies$  *has* (length xs) S **by** (unfold *has-def*, auto)

**lemma** *card-has*:  
**assumes** *cardS*: card S = n  
**shows** *has* n S  
**proof**(cases n = 0)

```

case True thus ?thesis by (simp add: has-def)
next
case False
with cardS card-eq-0-iff [where  $A=S$ ] have finiteS: finite S by simp
show ?thesis
proof(rule ccontr)
  assume nhas:  $\neg$  has n S
  with distinct-card[symmetric]
  have nxs:  $\neg$  ( $\exists$  xs. set xs  $\subseteq$  S  $\wedge$  distinct xs  $\wedge$  card (set xs) = n)
    by (auto simp add: has-def)
  from finite-list finiteS
  obtain xs where  $S = \text{set } xs$  by blast
  with cardS nxs show False by auto
qed
qed

lemma card-has-rev:
  assumes finiteS: finite S
  shows has n S  $\implies$  card S  $\geq$  n (is ?lhs  $\implies$  ?rhs)
proof –
  assume ?lhs
  then obtain xs
    where set xs  $\subseteq$  S  $\wedge$  n = length xs
    and dxs: distinct xs by (unfold has-def hasw-def, blast)
  with card-mono[OF finiteS] distinct-card[OF dxs, symmetric]
  show ?rhs by simp
qed

lemma has-0: has 0 S by (simp add: has-def)

lemma has-suc-notempty: has (Suc n) S  $\implies$  {}  $\neq$  S
  by (clarsimp simp add: has-def)

lemma has-suc-subset: has (Suc n) S  $\implies$  {}  $\subset$  S
  by (rule psubsetI, (simp add: has-suc-notempty)+)

lemma has-notempty-1:
  assumes Sne: S  $\neq$  {}
  shows has 1 S
proof –
  from Sne obtain x where  $x \in S$  by blast
  hence set [x]  $\subseteq$  S  $\wedge$  distinct [x]  $\wedge$  length [x] = 1 by auto
  thus ?thesis by (unfold has-def hasw-def, blast)
qed

lemma has-le-has:
  assumes h: has n S
  and nn': n'  $\leq$  n
  shows has n' S
proof –
  from h obtain xs where hasw xs S length xs = n by (unfold has-def, blast)
  with nn' set-take-subset [where  $n=n'$  and  $xs=xs$ ]
  have hasw (take n' xs) S length (take n' xs) = n'

```

by (simp-all add: min-def, blast+)
   
 thus ?thesis by (unfold has-def, blast)
   
 qed

**lemma** *has-ge-has-not*:
   
 assumes  $h: \neg \text{has } n \ S$ 
  
   and  $nn': n \leq n'$ 
  
 shows  $\neg \text{has } n' \ S$ 
  
 using  $h \ nn'$  by (blast dest: has-le-has)

**lemma** *has-eq*:
   
 assumes  $h: \text{has } n \ S$ 
  
   and  $hn': \neg \text{has } (\text{Suc } n) \ S$ 
  
 shows  $\text{card } S = n$ 
  
**proof** –
   
 from  $h$  **obtain**  $xs$ 
  
   where  $xs: \text{hasw } xs \ S$  and  $lenxs: \text{length } xs = n$  by (unfold has-def, blast)
   
 have  $\text{set } xs = S$ 
  
**proof**
  
   from  $xs$  **show**  $\text{set } xs \subseteq S$  by simp
   
**next**
  
   **show**  $S \subseteq \text{set } xs$ 
  
   **proof**(rule ccontr)
   
     **assume**  $\neg S \subseteq \text{set } xs$ 
  
     **then obtain**  $x$  **where**  $x \in S \ x \notin \text{set } xs$  **by** blast
   
     **with**  $lenxs \ xs$  **have**  $\text{hasw } (x \ \# \ xs) \ S \ \text{length } (x \ \# \ xs) = \text{Suc } n$  **by** simp-all
   
     **with**  $hn'$  **show** *False* **by** (unfold has-def, blast)
   
   **qed**
  
**qed**
  
 with  $xs \ lenxs$  *distinct-card* **show**  $\text{card } S = n$  **by** auto
   
**qed**

**lemma** *has-extend-witness*:
   
 assumes  $h: \text{has } n \ S$ 
  
 shows  $\llbracket \text{set } xs \subseteq S; \text{length } xs < n \rrbracket \implies \text{set } xs \subset S$ 
  
**proof**(*induct*  $xs$ )
   
   **case** *Nil*
  
   with  $h$  *has-suc-notempty* **show** ?case **by** (cases  $n$ , auto)
   
**next**
  
   **case** (*Cons*  $x \ xs$ )
   
   **have**  $\text{set } (x \ \# \ xs) \neq S$ 
  
   **proof**
  
     **assume**  $Sxs: \text{set } (x \ \# \ xs) = S$ 
  
     **hence** *finiteS*: *finite*  $S$  **by** auto
   
     **from**  $h$  **obtain**  $xs'$ 
  
       **where**  $Sxs': \text{set } xs' \subseteq S$ 
  
       and  $dlxs': \text{distinct } xs' \wedge \text{length } xs' = n$ 
  
       **by** (unfold has-def hasw-def, blast)
   
     **with** *distinct-card* **have**  $\text{card } (\text{set } xs') = n$  **by** auto
   
     **with** *finiteS*  $Sxs'$  *card-mono* **have**  $\text{card } S \geq n$  **by** auto
   
     **moreover**
  
     **from**  $Sxs$  *Cons* *card-length*[**where**  $xs=x \ \# \ xs$ ]
   
     **have**  $\text{card } S < n$  **by** auto

ultimately show *False* by *simp*  
**qed**  
with *Cons* show *?case* by *auto*  
**qed**

**lemma** *has-extend-witness'*:  
 $\llbracket \text{has } n \ S; \text{hasw } xs \ S; \text{length } xs < n \rrbracket \implies \exists x. \text{hasw } (x \# xs) \ S$   
by (*simp*, *blast dest: has-extend-witness*)

**lemma** *has-witness-two*:  
**assumes** *hasnS*: *has n S*  
**and** *nn'*:  $2 \leq n$   
**shows**  $\exists x \ y. \text{hasw } [x,y] \ S$   
**proof** –  
**have** *has2S*: *has 2 S* by (*rule has-le-has[OF hasnS nn']*)  
**from** *has-extend-witness'*[*OF has2S*, **where** *xs=[]*]  
**obtain** *x* **where**  $x \in S$  by *auto*  
**with** *has-extend-witness'*[*OF has2S*, **where** *xs=[x]*]  
**show** *?thesis* by *auto*  
**qed**

**lemma** *has-witness-three*:  
**assumes** *hasnS*: *has n S*  
**and** *nn'*:  $3 \leq n$   
**shows**  $\exists x \ y \ z. \text{hasw } [x,y,z] \ S$   
**proof** –  
**from** *nn'* **obtain** *x y* **where**  $\text{hasw } [x,y] \ S$   
**using** *has-witness-two*[*OF hasnS*] by *auto*  
**with** *nn'* **show** *?thesis*  
**using** *has-extend-witness'*[*OF hasnS*, **where** *xs=[x,y]*] by *auto*  
**qed**

**lemma** *finite-set-singleton-contra*:  
**assumes** *finiteS*: *finite S*  
**and** *Sne*:  $S \neq \{\}$   
**and** *cardS*:  $\text{card } S > 1 \implies \text{False}$   
**shows**  $\exists j. S = \{j\}$   
**proof** –  
**from** *cardS Sne card-0-eq*[*OF finiteS*] **have** *Scard*:  $\text{card } S = 1$  by *auto*  
**from** *has-extend-witness*[**where** *xs=[]*, *OF card-has*[*OF this*]]  
**obtain** *j* **where**  $\{j\} \subseteq S$  by *auto*  
**from** *card-seteq*[*OF finiteS this*] *Scard* **show** *?thesis* by *auto*  
**qed**

### 3 Preliminaries

The auxiliary concepts defined here are standard [Rou79, Sen70, Tay05]. Throughout we make use of a fixed set  $A$  of alternatives, drawn from some arbitrary type  $'a$  of suitable size. Taylor [Tay05] terms this set an *agenda*. Similarly we have a type  $'i$  of individuals and a

population  $Is$ .

### 3.1 Rational Preference Relations (RPRs)

Definitions for rational preference relations (RPRs), which represent indifference or strict preference amongst some set of alternatives. These are also called *weak orders* or (ambiguously) *ballots*.

Unfortunately Isabelle's standard ordering operators and lemmas are typeclass-based, and as introducing new types is painful and we need several orders per type, we need to repeat some things.

**type-synonym**  $'a$  RPR = ( $'a * 'a$ ) set

**abbreviation**  $rpr\text{-}eq\text{-}syntax :: 'a \Rightarrow 'a$  RPR  $\Rightarrow 'a \Rightarrow bool$  ( $- \preceq - [50, 1000, 51] 50$ ) **where**  
 $x \preceq y \equiv (x, y) \in r$

**definition**  $indifferent\text{-}pref :: 'a \Rightarrow 'a$  RPR  $\Rightarrow 'a \Rightarrow bool$  ( $- \approx - [50, 1000, 51] 50$ ) **where**  
 $x \approx y \equiv (x \preceq y \wedge y \preceq x)$

**lemma**  $indifferent\text{-}prefI[intro]: \llbracket x \preceq y; y \preceq x \rrbracket \Longrightarrow x \approx y$   
**unfolding**  $indifferent\text{-}pref\text{-}def$  **by**  $simp$

**lemma**  $indifferent\text{-}prefD[dest]: x \approx y \Longrightarrow x \preceq y \wedge y \preceq x$   
**unfolding**  $indifferent\text{-}pref\text{-}def$  **by**  $simp$

**definition**  $strict\text{-}pref :: 'a \Rightarrow 'a$  RPR  $\Rightarrow 'a \Rightarrow bool$  ( $- \prec - [50, 1000, 51] 50$ ) **where**  
 $x \prec y \equiv (x \preceq y \wedge \neg(y \preceq x))$

**lemma**  $strict\text{-}pref\text{-}def\text{-}irrefl[simp]: \neg(x \prec x)$  **unfolding**  $strict\text{-}pref\text{-}def$  **by**  $blast$

**lemma**  $strict\text{-}prefI[intro]: \llbracket x \preceq y; \neg(y \preceq x) \rrbracket \Longrightarrow x \prec y$   
**unfolding**  $strict\text{-}pref\text{-}def$  **by**  $simp$

Traditionally,  $x \preceq y$  would be written  $x R y$ ,  $x \approx y$  as  $x I y$  and  $x \prec y$  as  $x P y$ , where the relation  $r$  is implicit, and profiles are indexed by subscripting.

*Complete* means that every pair of distinct alternatives is ranked. The "distinct" part is a matter of taste, as it makes sense to regard an alternative as as good as itself. Here I take reflexivity separately.

**definition**  $complete :: 'a$  set  $\Rightarrow 'a$  RPR  $\Rightarrow bool$  **where**  
 $complete\ A\ r \equiv (\forall x \in A. \forall y \in A - \{x\}. x \preceq y \vee y \preceq x)$

**lemma**  $completeI[intro]:$   
 $(\bigwedge x\ y. \llbracket x \in A; y \in A; x \neq y \rrbracket \Longrightarrow x \preceq y \vee y \preceq x) \Longrightarrow complete\ A\ r$   
**unfolding**  $complete\text{-}def$  **by**  $auto$

**lemma**  $completeD[dest]:$   
 $\llbracket complete\ A\ r; x \in A; y \in A; x \neq y \rrbracket \Longrightarrow x \preceq y \vee y \preceq x$   
**unfolding**  $complete\text{-}def$  **by**  $auto$

**lemma**  $complete\text{-}less\text{-}not: \llbracket complete\ A\ r; hasw\ [x,y]\ A; \neg x \prec y \rrbracket \Longrightarrow y \preceq x$   
**unfolding**  $complete\text{-}def\ strict\text{-}pref\text{-}def$  **by**  $auto$

**lemma** *complete-indiff-not*:  $\llbracket \text{complete } A \ r; \text{hasw } [x,y] \ A; \neg x \ r \approx y \rrbracket \Longrightarrow x \ r \prec y \vee y \ r \prec x$   
**unfolding** *complete-def indifferent-pref-def strict-pref-def* **by** *auto*

**lemma** *complete-exh*:  
**assumes** *complete*  $A \ r$   
**and** *hasw*  $[x,y] \ A$   
**obtains**  $(xPy) \ x \ r \prec y$   
|  $(yPx) \ y \ r \prec x$   
|  $(xIy) \ x \ r \approx y$   
**using** *assms* **unfolding** *complete-def strict-pref-def indifferent-pref-def* **by** *auto*

Use the standard *refl*. Also define *irreflexivity* analogously to how *refl* is defined in the standard library.

**declare** *refl-onI*[*intro*] *refl-onD*[*dest*]

**lemma** *complete-refl-on*:  
 $\llbracket \text{complete } A \ r; \text{refl-on } A \ r; x \in A; y \in A \rrbracket \Longrightarrow x \ r \preceq y \vee y \ r \preceq x$   
**unfolding** *complete-def* **by** *auto*

**definition** *irrefl* :: 'a set  $\Rightarrow$  'a RPR  $\Rightarrow$  bool **where**  
*irrefl*  $A \ r \equiv r \subseteq A \times A \wedge (\forall x \in A. \neg x \ r \preceq x)$

**lemma** *irreflI*[*intro*]:  $\llbracket r \subseteq A \times A; \bigwedge x. x \in A \Longrightarrow \neg x \ r \preceq x \rrbracket \Longrightarrow \text{irrefl } A \ r$   
**unfolding** *irrefl-def* **by** *simp*

**lemma** *irreflD*[*dest*]:  $\llbracket \text{irrefl } A \ r; (x, y) \in r \rrbracket \Longrightarrow \text{hasw } [x,y] \ A$   
**unfolding** *irrefl-def* **by** *auto*

**lemma** *irreflD'*[*dest*]:  
 $\llbracket \text{irrefl } A \ r; r \neq \{\} \rrbracket \Longrightarrow \exists x \ y. \text{hasw } [x,y] \ A \wedge (x, y) \in r$   
**unfolding** *irrefl-def* **by** *auto*

Rational preference relations, also known as weak orders and (I guess) complete pre-orders.

**definition** *rpr* :: 'a set  $\Rightarrow$  'a RPR  $\Rightarrow$  bool **where**  
*rpr*  $A \ r \equiv \text{complete } A \ r \wedge \text{refl-on } A \ r \wedge \text{trans } r$

**lemma** *rprI*[*intro*]:  $\llbracket \text{complete } A \ r; \text{refl-on } A \ r; \text{trans } r \rrbracket \Longrightarrow \text{rpr } A \ r$   
**unfolding** *rpr-def* **by** *simp*

**lemma** *rprD*:  $\text{rpr } A \ r \Longrightarrow \text{complete } A \ r \wedge \text{refl-on } A \ r \wedge \text{trans } r$   
**unfolding** *rpr-def* **by** *simp*

**lemma** *rpr-in-set*[*dest*]:  $\llbracket \text{rpr } A \ r; x \ r \preceq y \rrbracket \Longrightarrow \{x,y\} \subseteq A$   
**unfolding** *rpr-def refl-on-def* **by** *auto*

**lemma** *rpr-refl*[*dest*]:  $\llbracket \text{rpr } A \ r; x \in A \rrbracket \Longrightarrow x \ r \preceq x$   
**unfolding** *rpr-def* **by** *blast*

**lemma** *rpr-less-not*:  $\llbracket \text{rpr } A \ r; \text{hasw } [x,y] \ A; \neg x \ r \prec y \rrbracket \Longrightarrow y \ r \preceq x$   
**unfolding** *rpr-def* **by** (*auto simp add: complete-less-not*)

**lemma** *rpr-less-imp-le*[*simp*]:  $\llbracket x \ r \prec y \rrbracket \Longrightarrow x \ r \preceq y$

**unfolding** *strict-pref-def* **by** *simp*

**lemma** *rpr-less-imp-neq*[*simp*]:  $\llbracket x \prec y \rrbracket \implies x \neq y$   
**unfolding** *strict-pref-def* **by** *blast*

**lemma** *rpr-less-trans*[*trans*]:  $\llbracket x \prec y; y \prec z; rpr\ A\ r \rrbracket \implies x \prec z$   
**unfolding** *rpr-def* *strict-pref-def* *trans-def* **by** *blast*

**lemma** *rpr-le-trans*[*trans*]:  $\llbracket x \preceq y; y \preceq z; rpr\ A\ r \rrbracket \implies x \preceq z$   
**unfolding** *rpr-def* *trans-def* **by** *blast*

**lemma** *rpr-le-less-trans*[*trans*]:  $\llbracket x \preceq y; y \prec z; rpr\ A\ r \rrbracket \implies x \prec z$   
**unfolding** *rpr-def* *strict-pref-def* *trans-def* **by** *blast*

**lemma** *rpr-less-le-trans*[*trans*]:  $\llbracket x \prec y; y \preceq z; rpr\ A\ r \rrbracket \implies x \prec z$   
**unfolding** *rpr-def* *strict-pref-def* *trans-def* **by** *blast*

**lemma** *rpr-complete*:  $\llbracket rpr\ A\ r; x \in A; y \in A \rrbracket \implies x \preceq y \vee y \preceq x$   
**unfolding** *rpr-def* **by** (*blast* *dest*: *complete-refl-on*)

### 3.2 Profiles

A *profile* (also termed a collection of *ballots*) maps each individual to an RPR for that individual.

**type-synonym** (*'a*, *'i*) *Profile* = *'i*  $\Rightarrow$  *'a* *RPR*

**definition** *profile* :: *'a* *set*  $\Rightarrow$  *'i* *set*  $\Rightarrow$  (*'a*, *'i*) *Profile*  $\Rightarrow$  *bool* **where**  
*profile* *A* *Is* *P*  $\equiv$  *Is*  $\neq$   $\{\}$   $\wedge$  ( $\forall i \in Is. rpr\ A\ (P\ i)$ )

**lemma** *profileI*[*intro*]:  $\llbracket \bigwedge i. i \in Is \implies rpr\ A\ (P\ i); Is \neq \{\} \rrbracket \implies profile\ A\ Is\ P$   
**unfolding** *profile-def* **by** *simp*

**lemma** *profile-rprD*[*dest*]:  $\llbracket profile\ A\ Is\ P; i \in Is \rrbracket \implies rpr\ A\ (P\ i)$   
**unfolding** *profile-def* **by** *simp*

**lemma** *profile-non-empty*: *profile* *A* *Is* *P*  $\implies Is \neq \{\}$   
**unfolding** *profile-def* **by** *simp*

### 3.3 Choice Sets, Choice Functions

A *choice set* is the subset of *A* where every element of that subset is (weakly) preferred to every other element of *A* with respect to a given RPR. A *choice function* yields a non-empty choice set whenever *A* is non-empty.

**definition** *choiceSet* :: *'a* *set*  $\Rightarrow$  *'a* *RPR*  $\Rightarrow$  *'a* *set* **where**  
*choiceSet* *A* *r*  $\equiv$   $\{ x \in A . \forall y \in A. x \preceq y \}$

**definition** *choiceFn* :: *'a* *set*  $\Rightarrow$  *'a* *RPR*  $\Rightarrow$  *bool* **where**  
*choiceFn* *A* *r*  $\equiv$   $\forall A' \subseteq A. A' \neq \{\} \longrightarrow choiceSet\ A'\ r \neq \{\}$

**lemma** *choiceSetI*[intro]:

$\llbracket x \in A; \bigwedge y. y \in A \implies x \preceq y \rrbracket \implies x \in \text{choiceSet } A \ r$   
**unfolding** *choiceSet-def* **by** *simp*

**lemma** *choiceFnI*[intro]:

$(\bigwedge A'. \llbracket A' \subseteq A; A' \neq \{\} \rrbracket \implies \text{choiceSet } A' \ r \neq \{\}) \implies \text{choiceFn } A \ r$   
**unfolding** *choiceFn-def* **by** *simp*

If a complete and reflexive relation is also *quasi-transitive* it will yield a choice function.

**definition** *quasi-trans* :: 'a RPR  $\Rightarrow$  bool **where**

*quasi-trans*  $r \equiv \forall x \ y \ z. x \prec y \wedge y \prec z \longrightarrow x \prec z$

**lemma** *quasi-transI*[intro]:

$(\bigwedge x \ y \ z. \llbracket x \prec y; y \prec z \rrbracket \implies x \prec z) \implies \text{quasi-trans } r$   
**unfolding** *quasi-trans-def* **by** *blast*

**lemma** *quasi-transD*:  $\llbracket x \prec y; y \prec z; \text{quasi-trans } r \rrbracket \implies x \prec z$

**unfolding** *quasi-trans-def* **by** *blast*

**lemma** *trans-imp-quasi-trans*:  $\text{trans } r \implies \text{quasi-trans } r$

**by** (*rule quasi-transI, unfold strict-pref-def trans-def, blast*)

**lemma** *r-c-qt-imp-cf*:

**assumes** *finiteA*: *finite*  $A$

**and** *c*: *complete*  $A \ r$

**and** *qt*: *quasi-trans*  $r$

**and** *r*: *refl-on*  $A \ r$

**shows** *choiceFn*  $A \ r$

**proof**

**fix**  $B$  **assume**  $B: B \subseteq A \ B \neq \{\}$

**with** *finite-subset* *finiteA* **have** *finiteB*: *finite*  $B$  **by** *auto*

**from** *finiteB*  $B$  **show**  $\text{choiceSet } B \ r \neq \{\}$

**proof**(*induct rule: finite-subset-induct'*)

**case** *empty* **with**  $B$  **show** *?case* **by** *auto*

**next**

**case** (*insert*  $a \ B$ )

**hence** *finiteB*: *finite*  $B$

**and**  $aA$ :  $a \in A$

**and**  $AB$ :  $B \subseteq A$

**and**  $aB$ :  $a \notin B$

**and**  $cF$ :  $B \neq \{\} \implies \text{choiceSet } B \ r \neq \{\}$  **by**  $-$  *blast*

**show** *?case*

**proof**(*cases*  $B = \{\}$ )

**case** *True* **with**  $aA \ r$  **show** *?thesis*

**unfolding** *choiceSet-def* **by** *blast*

**next**

**case** *False*

**with**  $cF$  **obtain**  $b$  **where**  $bCF$ :  $b \in \text{choiceSet } B \ r$  **by** *blast*

**from**  $AB \ aA \ bCF$  *complete-refl-on*[*OF*  $c \ r$ ]

**have**  $a \prec b \vee b \preceq a$  **unfolding** *choiceSet-def* *strict-pref-def* **by** *blast*

**thus** *?thesis*

**proof**

**assume**  $ab$ :  $b \preceq a$

```

with bCF show ?thesis unfolding choiceSet-def by auto
next
assume ab: a r< b
have a ∈ choiceSet (insert a B) r
proof(rule ccontr)
  assume aCF: a ∉ choiceSet (insert a B) r
  from aB have ∧b. b ∈ B ⇒ a ≠ b by auto
  with aCF aA AB c r obtain b' where B: b' ∈ B b' r< a
    unfolding choiceSet-def complete-def strict-pref-def by blast
  with ab qt have b' r< b by (blast dest: quasi-transD)
  with bCF B show False unfolding choiceSet-def strict-pref-def by blast
qed
thus ?thesis by auto
qed
qed
qed
qed

```

```

lemma rpr-choiceFn: [ finite A; rpr A r ] ⇒ choiceFn A r
  unfolding rpr-def by (blast dest: trans-imp-quasi-trans r-c-qt-imp-cf)

```

### 3.4 Social Choice Functions (SCFs)

A *social choice function* (SCF), also called a *collective choice rule* by Sen [Sen70, p28], is a function that somehow aggregates society's opinions, expressed as a profile, into a preference relation.

**type-synonym** ('a, 'i) SCF = ('a, 'i) Profile ⇒ 'a RPR

The least we require of an SCF is that it be *complete* and some function of the profile. The latter condition is usually implied by other conditions, such as *via*.

**definition**

SCF :: ('a, 'i) SCF ⇒ 'a set ⇒ 'i set ⇒ ('a set ⇒ 'i set ⇒ ('a, 'i) Profile ⇒ bool) ⇒ bool

**where**

SCF scf A Is Pcond ≡ (∀ P. Pcond A Is P → (complete A (scf P)))

**lemma SCFI[*intro*]:**

**assumes** c: ∧P. Pcond A Is P ⇒ complete A (scf P)

**shows** SCF scf A Is Pcond

**unfolding** SCF-def **using** assms **by** blast

**lemma SCF-completeD[*dest*]:** [ SCF scf A Is Pcond; Pcond A Is P ] ⇒ complete A (scf P)

**unfolding** SCF-def **by** blast

### 3.5 Social Welfare Functions (SWFs)

A *Social Welfare Function* (SWF) is an SCF that expresses the society's opinion as a single RPR.

In some situations it might make sense to restrict the allowable profiles.

**definition**

SWF :: ('a, 'i) SCF ⇒ 'a set ⇒ 'i set ⇒ ('a set ⇒ 'i set ⇒ ('a, 'i) Profile ⇒ bool) ⇒ bool

**where**

$SWF\ swf\ A\ Is\ Pcond \equiv (\forall P. Pcond\ A\ Is\ P \longrightarrow rpr\ A\ (swf\ P))$

**lemma**  $SWF\text{-}rpr[dest]$ :  $\llbracket SWF\ swf\ A\ Is\ Pcond; Pcond\ A\ Is\ P \rrbracket \Longrightarrow rpr\ A\ (swf\ P)$   
**unfolding**  $SWF\text{-}def$  **by**  $simp$

### 3.6 General Properties of an SCF

An SCF has a *universal domain* if it works for all profiles.

**definition**  $universal\text{-}domain :: 'a\ set \Rightarrow 'i\ set \Rightarrow ('a, 'i)\ Profile \Rightarrow bool$  **where**  
 $universal\text{-}domain\ A\ Is\ P \equiv profile\ A\ Is\ P$

**declare**  $universal\text{-}domain\text{-}def[simp]$

An SCF is *weakly Pareto-optimal* if, whenever everyone strictly prefers  $x$  to  $y$ , the SCF does too.

**definition**

$weak\text{-}pareto :: ('a, 'i)\ SCF \Rightarrow 'a\ set \Rightarrow 'i\ set \Rightarrow ('a\ set \Rightarrow 'i\ set \Rightarrow ('a, 'i)\ Profile \Rightarrow bool) \Rightarrow bool$   
**where**  
 $weak\text{-}pareto\ scf\ A\ Is\ Pcond \equiv$   
 $(\forall P\ x\ y. Pcond\ A\ Is\ P \wedge x \in A \wedge y \in A \wedge (\forall i \in Is. x\ (P\ i) \prec y) \longrightarrow x\ (scf\ P) \prec y)$

**lemma**  $weak\text{-}paretoI[intro]$ :

$(\bigwedge P\ x\ y. \llbracket Pcond\ A\ Is\ P; x \in A; y \in A; \bigwedge i. i \in Is \Longrightarrow x\ (P\ i) \prec y \rrbracket \Longrightarrow x\ (scf\ P) \prec y)$   
 $\Longrightarrow weak\text{-}pareto\ scf\ A\ Is\ Pcond$

**unfolding**  $weak\text{-}pareto\text{-}def$  **by**  $simp$

**lemma**  $weak\text{-}paretoD$ :

$\llbracket weak\text{-}pareto\ scf\ A\ Is\ Pcond; Pcond\ A\ Is\ P; x \in A; y \in A;$   
 $(\bigwedge i. i \in Is \Longrightarrow x\ (P\ i) \prec y) \rrbracket \Longrightarrow x\ (scf\ P) \prec y$

**unfolding**  $weak\text{-}pareto\text{-}def$  **by**  $simp$

An SCF satisfies *independence of irrelevant alternatives* if, for two preference profiles  $P$  and  $P'$  where for all individuals  $i$ , alternatives  $x$  and  $y$  drawn from set  $S$  have the same order in  $P\ i$  and  $P'\ i$ , then alternatives  $x$  and  $y$  have the same order in  $scf\ P$  and  $scf\ P'$ .

**definition**  $iaa :: ('a, 'i)\ SCF \Rightarrow 'a\ set \Rightarrow 'i\ set \Rightarrow bool$  **where**

$iaa\ scf\ S\ Is \equiv$   
 $(\forall P\ P'\ x\ y. profile\ S\ Is\ P \wedge profile\ S\ Is\ P'$   
 $\wedge x \in S \wedge y \in S$   
 $\wedge (\forall i \in Is. ((x\ (P\ i) \preceq y) \longleftrightarrow (x\ (P'\ i) \preceq y)) \wedge ((y\ (P\ i) \preceq x) \longleftrightarrow (y\ (P'\ i) \preceq x)))$   
 $\longrightarrow ((x\ (scf\ P) \preceq y) \longleftrightarrow (x\ (scf\ P') \preceq y)))$

**lemma**  $iaaI[intro]$ :

$(\bigwedge P\ P'\ x\ y.$   
 $\llbracket profile\ S\ Is\ P; profile\ S\ Is\ P';$   
 $x \in S; y \in S;$   
 $\bigwedge i. i \in Is \Longrightarrow ((x\ (P\ i) \preceq y) \longleftrightarrow (x\ (P'\ i) \preceq y)) \wedge ((y\ (P\ i) \preceq x) \longleftrightarrow (y\ (P'\ i) \preceq x))$   
 $\rrbracket \Longrightarrow ((x\ (scf\ P) \preceq y) \longleftrightarrow (x\ (scf\ P') \preceq y)))$   
 $\Longrightarrow iaa\ scf\ S\ Is$

**unfolding**  $iaa\text{-}def$  **by**  $simp$

**lemma**  $iaaE$ :

$\llbracket$  *ia swf*  $S$   $Is$ ;  
 $\{x, y\} \subseteq S$ ;  
 $a \in \{x, y\}; b \in \{x, y\}$ ;  
 $\bigwedge i a b. \llbracket a \in \{x, y\}; b \in \{x, y\}; i \in Is \rrbracket \implies (a \text{ (} P' i \text{)} \preceq b) \longleftrightarrow (a \text{ (} P i \text{)} \preceq b)$ ;  
*profile*  $S$   $Is$   $P$ ; *profile*  $S$   $Is$   $P'$   $\rrbracket$   
 $\implies (a \text{ (} swf P \text{)} \preceq b) \longleftrightarrow (a \text{ (} swf P' \text{)} \preceq b)$   
**unfolding** *ia-def* **by** (*simp*, *blast*)

### 3.7 Decisiveness and Semi-decisiveness

This notion is the key to Arrow's Theorem, and hinges on the use of strict preference [Sen70, p42].

A coalition  $C$  of agents is *semi-decisive* for  $x$  over  $y$  if, whenever the coalition prefers  $x$  to  $y$  and all other agents prefer the converse, the coalition prevails.

**definition** *semidecisive*  $:: ('a, 'i) SCF \implies 'a \text{ set} \implies 'i \text{ set} \implies 'i \text{ set} \implies 'a \implies 'a \implies \text{bool}$  **where**  
*semidecisive scf*  $A$   $Is$   $C$   $x$   $y \equiv$   
 $C \subseteq Is \wedge (\forall P. \text{profile } A \text{ } Is \text{ } P \wedge (\forall i \in C. x \text{ (} P i \text{)} \prec y) \wedge (\forall i \in Is - C. y \text{ (} P i \text{)} \prec x)$   
 $\longrightarrow x \text{ (} scf P \text{)} \prec y$ )

**lemma** *semidecisiveI*[*intro*]:

$\llbracket C \subseteq Is$ ;  
 $\bigwedge P. \llbracket \text{profile } A \text{ } Is \text{ } P; \bigwedge i. i \in C \implies x \text{ (} P i \text{)} \prec y; \bigwedge i. i \in Is - C \implies y \text{ (} P i \text{)} \prec x \rrbracket$   
 $\implies x \text{ (} scf P \text{)} \prec y \rrbracket \implies \text{semidecisive scf } A \text{ } Is \text{ } C \text{ } x \text{ } y$   
**unfolding** *semidecisive-def* **by** *simp*

**lemma** *semidecisive-coalitionD*[*dest*]: *semidecisive scf*  $A$   $Is$   $C$   $x$   $y \implies C \subseteq Is$   
**unfolding** *semidecisive-def* **by** *simp*

**lemma** *sd-refl*:  $\llbracket C \subseteq Is; C \neq \{\} \rrbracket \implies \text{semidecisive scf } A \text{ } Is \text{ } C \text{ } x \text{ } x$   
**unfolding** *semidecisive-def strict-pref-def* **by** *blast*

A coalition  $C$  is *decisive* for  $x$  over  $y$  if, whenever the coalition prefers  $x$  to  $y$ , the coalition prevails.

**definition** *decisive*  $:: ('a, 'i) SCF \implies 'a \text{ set} \implies 'i \text{ set} \implies 'i \text{ set} \implies 'a \implies 'a \implies \text{bool}$  **where**  
*decisive scf*  $A$   $Is$   $C$   $x$   $y \equiv$   
 $C \subseteq Is \wedge (\forall P. \text{profile } A \text{ } Is \text{ } P \wedge (\forall i \in C. x \text{ (} P i \text{)} \prec y) \longrightarrow x \text{ (} scf P \text{)} \prec y)$

**lemma** *decisiveI*[*intro*]:

$\llbracket C \subseteq Is; \bigwedge P. \llbracket \text{profile } A \text{ } Is \text{ } P; \bigwedge i. i \in C \implies x \text{ (} P i \text{)} \prec y \rrbracket \implies x \text{ (} scf P \text{)} \prec y \rrbracket$   
 $\implies \text{decisive scf } A \text{ } Is \text{ } C \text{ } x \text{ } y$   
**unfolding** *decisive-def* **by** *simp*

**lemma** *d-imp-sd*: *decisive scf*  $A$   $Is$   $C$   $x$   $y \implies \text{semidecisive scf } A \text{ } Is \text{ } C \text{ } x \text{ } y$   
**unfolding** *decisive-def* **by** (*rule semidecisiveI*, *blast+*)

**lemma** *decisive-coalitionD*[*dest*]: *decisive scf*  $A$   $Is$   $C$   $x$   $y \implies C \subseteq Is$   
**unfolding** *decisive-def* **by** *simp*

Anyone is trivially decisive for  $x$  against  $x$ .

**lemma** *d-refl*:  $\llbracket C \subseteq Is; C \neq \{\} \rrbracket \implies \text{decisive scf } A \text{ } Is \text{ } C \text{ } x \text{ } x$

**unfolding** *decisive-def strict-pref-def* **by** *simp*

Agent  $j$  is a *dictator* if her preferences always prevail. This is the same as saying that she is decisive for all  $x$  and  $y$ .

**definition** *dictator* :: ('a, 'i) SCF  $\Rightarrow$  'a set  $\Rightarrow$  'i set  $\Rightarrow$  'i  $\Rightarrow$  bool **where**  
*dictator scf A Is j*  $\equiv j \in Is \wedge (\forall x \in A. \forall y \in A. \text{decisive scf A Is } \{j\} x y)$

**lemma** *dictatorI[intro]*:

$\llbracket j \in Is; \bigwedge x y. \llbracket x \in A; y \in A \rrbracket \implies \text{decisive scf A Is } \{j\} x y \rrbracket \implies \text{dictator scf A Is } j$

**unfolding** *dictator-def* **by** *simp*

**lemma** *dictator-individual[dest]*: *dictator scf A Is j*  $\implies j \in Is$

**unfolding** *dictator-def* **by** *simp*

## 4 Arrow's General Possibility Theorem

The proof falls into two parts: showing that a semi-decisive individual is in fact a dictator, and that a semi-decisive individual exists. I take them in that order.

It might be good to do some of this in a locale. The complication is untangling where various witnesses need to be quantified over.

### 4.1 Semi-decisiveness Implies Decisiveness

I follow [Sen70, Chapter 3\*] quite closely here. Formalising his appeal to the *ii*a assumption is the main complication here.

The witness for the first lemma: in the profile  $P'$ , special agent  $j$  strictly prefers  $x$  to  $y$  to  $z$ , and doesn't care about the other alternatives. Everyone else strictly prefers  $y$  to each of  $x$  to  $z$ , and inherits the relative preferences between  $x$  and  $z$  from profile  $P$ .

The model has to be specific about ordering all the other alternatives, but these are immaterial in the proof that uses this witness. Note also that the following lemma is used with different instantiations of  $x$ ,  $y$  and  $z$ , so we need to quantify over them here. This happens implicitly, but in a locale we would have to be more explicit.

This is just tedious.

**lemma** *decisive1-witness*:

**assumes** *has3A*: *hasw*  $[x,y,z]$   $A$

**and** *profileP*: *profile*  $A$   $Is$   $P$

**and** *jIs*:  $j \in Is$

**obtains**  $P'$

**where** *profile*  $A$   $Is$   $P'$

**and**  $x (P' j) \prec y \wedge y (P' j) \prec z$

**and**  $\bigwedge i. i \neq j \implies y (P' i) \prec x \wedge y (P' i) \prec z \wedge ((x (P' i) \preceq z) = (x (P i) \preceq z)) \wedge ((z (P' i) \preceq x) = (z (P i) \preceq x))$

**proof**

**let**  $?P' = \lambda i. (\text{if } i = j \text{ then } (\{ (x, u) \mid u. u \in A \} \cup \{ (y, u) \mid u. u \in A - \{x\} \}))$

$$\begin{aligned} & \cup \{ (z, u) \mid u. u \in A - \{x, y\} \} \\ \text{else } & \{ (y, u) \mid u. u \in A \} \\ & \cup \{ (x, u) \mid u. u \in A - \{y, z\} \} \\ & \cup \{ (z, u) \mid u. u \in A - \{x, y\} \} \\ & \cup (\text{if } x \text{ (P } i) \preceq z \text{ then } \{(x, z)\} \text{ else } \{\}) \\ & \cup (\text{if } z \text{ (P } i) \preceq x \text{ then } \{(z, x)\} \text{ else } \{\}) \\ & \cup (A - \{x, y, z\}) \times (A - \{x, y, z\}) \end{aligned}$$

**show** *profile* A Is ?P'  
**proof**  
**fix** *i* **assume** *i*Is: *i* ∈ Is  
**show** *rpr* A (?P' *i*)  
**proof**(*cases* *i* = *j*)  
**case** *True* **with** *has3A* **show** ?thesis  
**by** – (*rule* *rprI*, *simp-all* *add*: *trans-def*, *blast+*)  
**next**  
**case** *False* **hence** *ij*: *i* ≠ *j* .  
**show** ?thesis  
**proof**  
**from** *i*Is *profileP* **have** *complete* A (P *i*) **by** (*blast* *dest*: *rpr-complete*)  
**with** *ij* **show** *complete* A (?P' *i*) **by** (*simp* *add*: *complete-def*, *blast*)  
**from** *i*Is *profileP* **have** *refl-on* A (P *i*) **by** (*auto* *simp* *add*: *rpr-def*)  
**with** *has3A* *ij* **show** *refl-on* A (?P' *i*) **by** (*simp*, *blast*)  
**from** *ij* *has3A* **show** *trans* (?P' *i*) **by** (*clarsimp* *simp* *add*: *trans-def*)  
**qed**  
**qed**  
**next**  
**from** *profileP* **show** Is ≠ {} **by** (*rule* *profile-non-empty*)  
**qed**  
**from** *has3A*  
**show** *x* (?P' *j*) ≺ *y* ∧ *y* (?P' *j*) ≺ *z*  
**and** ∧*i. i* ≠ *j* ⇒ *y* (?P' *i*) ≺ *x* ∧ *y* (?P' *i*) ≺ *z* ∧ ((*x* (?P' *i*) ≼ *z*) = (*x* (P *i*) ≼ *z*)) ∧ ((*z* (?P' *i*) ≼ *x*) = (*z* (P *i*) ≼ *x*))  
**unfolding** *strict-pref-def* **by** *auto*  
**qed**

The key lemma: in the presence of Arrow's assumptions, an individual who is semi-decisive for  $x$  and  $y$  is actually decisive for  $x$  over any other alternative  $z$ . (This is where the quantification becomes important.)

**lemma** *decisive1*:

**assumes** *has3A*: *hasw* [*x*,*y*,*z*] A  
**and** *iaa*: *iaa* *swf* A Is  
**and** *swf*: *SWF* *swf* A Is *universal-domain*  
**and** *wp*: *weak-pareto* *swf* A Is *universal-domain*  
**and** *sd*: *semidecisive* *swf* A Is {*j*} *x* *y*  
**shows** *decisive* *swf* A Is {*j*} *x* *z*

**proof**

**from** *sd* **show** *j*Is: {*j*} ⊆ Is **by** *blast*  
**fix** P  
**assume** *profileP*: *profile* A Is P  
**and** *jxzP*: ∧*i. i* ∈ {*j*} ⇒ *x* (P *i*) ≺ *z*  
**from** *has3A* *profileP* *j*Is  
**obtain** P'

**where** *profileP'*: *profile A Is P'*  
**and** *jxyzP'*:  $x \prec_{(P' j)} y \prec_{(P' j)} z$   
**and** *ixyzP'*:  $\bigwedge i. i \neq j \rightarrow y \prec_{(P' i)} x \wedge y \prec_{(P' i)} z \wedge ((x \preceq_{(P' i)} z) = (x \preceq_{(P i)} z)) \wedge ((z \preceq_{(P' i)} x) = (z \preceq_{(P i)} x))$   
**by** – (*rule decisive1-witness, blast+*)  
**from** *iaa* **have**  $\bigwedge a b. \llbracket a \in \{x, z\}; b \in \{x, z\} \rrbracket \implies (a \preceq_{(swf P)} b) = (a \preceq_{(swf P')} b)$   
**proof**(*rule iiaE*)  
**from** *has3A* **show**  $\{x, z\} \subseteq A$  **by** *simp*  
**next**  
**fix** *i* **assume** *iIs*:  $i \in Is$   
**fix** *a b* **assume** *ab*:  $a \in \{x, z\} b \in \{x, z\}$   
**show**  $(a \preceq_{(P' i)} b) = (a \preceq_{(P i)} b)$   
**proof**(*cases i = j*)  
**case** *False*  
**with** *ab iIs ixyzP' profileP profileP' has3A*  
**show** *?thesis unfolding profile-def by auto*  
**next**  
**case** *True*  
**from** *profileP' jIs jxyzP'* **have**  $x \prec_{(P' j)} z$   
**by** (*auto dest: rpr-less-trans*)  
**with** *True ab iIs jxzP' profileP profileP' has3A*  
**show** *?thesis unfolding profile-def strict-pref-def by auto*  
**qed**  
**qed** (*simp-all add: profileP profileP'*)  
**moreover** **have**  $x \prec_{(swf P')} z$   
**proof** –  
**from** *profileP' sd jxyzP' ixyzP'* **have**  $x \prec_{(swf P')} y$  **by** (*simp add: semidecisive-def*)  
**moreover**  
**from** *jxyzP' ixyzP'* **have**  $\bigwedge i. i \in Is \implies y \prec_{(P' i)} z$  **by** (*case-tac i=j, auto*)  
**with** *wp profileP' has3A* **have**  $y \prec_{(swf P')} z$  **by** (*auto dest: weak-paretoD*)  
**moreover** **note** *SWF-rpr[OF swf] profileP'*  
**ultimately** **show**  $x \prec_{(swf P')} z$   
**unfolding** *universal-domain-def* **by** (*blast dest: rpr-less-trans*)  
**qed**  
**ultimately** **show**  $x \prec_{(swf P)} z$  **unfolding** *strict-pref-def* **by** *blast*  
**qed**

The witness for the second lemma: special agent  $j$  strictly prefers  $z$  to  $x$  to  $y$ , and everyone else strictly prefers  $z$  to  $x$  and  $y$  to  $x$ . (In some sense the last part is upside-down with respect to the first witness.)

**lemma** *decisive2-witness*:

**assumes** *has3A*: *hasw [x,y,z] A*  
**and** *profileP*: *profile A Is P*  
**and** *jIs*:  $j \in Is$   
**obtains** *P'*  
**where** *profile A Is P'*  
**and**  $z \prec_{(P' j)} x \wedge x \prec_{(P' j)} y$   
**and**  $\bigwedge i. i \neq j \implies z \prec_{(P' i)} x \wedge y \prec_{(P' i)} x \wedge ((y \preceq_{(P' i)} z) = (y \preceq_{(P i)} z)) \wedge ((z \preceq_{(P' i)} y) = (z \preceq_{(P i)} y))$   
**proof**

```

let ?P' = λi. (if i = j then ({ (z, u) | u. u ∈ A }
    ∪ { (x, u) | u. u ∈ A - {z} }
    ∪ { (y, u) | u. u ∈ A - {x,z} })
  else ({ (z, u) | u. u ∈ A - {y} }
    ∪ { (y, u) | u. u ∈ A - {z} }
    ∪ { (x, u) | u. u ∈ A - {y,z} }
    ∪ (if y (P i) ≼ z then {(y,z)} else {})
    ∪ (if z (P i) ≼ y then {(z,y)} else {})))
    ∪ (A - {x,y,z}) × (A - {x,y,z})
show profile A Is ?P'
proof
  fix i assume iIs: i ∈ Is
  show rpr A (?P' i)
  proof(cases i = j)
    case True with has3A show ?thesis
    by - (rule rprI, simp-all add: trans-def, blast+)
  next
    case False hence ij: i ≠ j .
    show ?thesis
    proof
      from iIs profileP have complete A (P i) by (auto simp add: rpr-def)
      with ij show complete A (?P' i) by (simp add: complete-def, blast)
      from iIs profileP have refl-on A (P i) by (auto simp add: rpr-def)
      with has3A ij show refl-on A (?P' i) by (simp, blast)
      from ij has3A show trans (?P' i) by (clarsimp simp add: trans-def)
    qed
  qed
  next
    show Is ≠ {} by (rule profile-non-empty[OF profileP])
  qed
  from has3A
  show z (?P' j) ≺ x ∧ x (?P' j) ≺ y
  and ∧i. i ≠ j ⇒ z (?P' i) ≺ x ∧ y (?P' i) ≺ x ∧ ((y (?P' i) ≼ z) = (y (P i) ≼ z)) ∧ ((z (?P' i) ≼
y) = (z (P i) ≼ y))
  unfolding strict-pref-def by auto
qed

```

**lemma** decisive2:

```

assumes has3A: hasw [x,y,z] A
  and iia: iia swf A Is
  and swf: SWF swf A Is universal-domain
  and wp: weak-pareto swf A Is universal-domain
  and sd: semidecisive swf A Is {j} x y
shows decisive swf A Is {j} z y
proof
  from sd show jIs: {j} ⊆ Is by blast
  fix P
  assume profileP: profile A Is P
  and jyzP: ∧i. i ∈ {j} ⇒ z (P i) ≺ y
  from has3A profileP jIs
  obtain P'
  where profileP': profile A Is P'

```

```

    and  $jxyzP'$ :  $z \prec_{(P' j)} x \wedge x \prec_{(P' j)} y$ 
    and  $ixyzP'$ :  $\bigwedge i. i \neq j \longrightarrow z \prec_{(P' i)} x \wedge y \prec_{(P' i)} x \wedge ((y \preceq_{(P' i)} z) = (y \preceq_{(P i)} z)) \wedge ((z \preceq_{(P' i)} y) = (z \preceq_{(P i)} y))$ 
    by - (rule decisive2-witness, blast+)
    from  $ia$  have  $\bigwedge a b. \llbracket a \in \{y, z\}; b \in \{y, z\} \rrbracket \implies (a \preceq_{(swf P)} b) = (a \preceq_{(swf P')} b)$ 
    proof(rule  $iaE$ )
    from  $has3A$  show  $\{y, z\} \subseteq A$  by simp
  next
    fix  $i$  assume  $iIs$ :  $i \in Is$ 
    fix  $a b$  assume  $ab$ :  $a \in \{y, z\} \wedge b \in \{y, z\}$ 
    show  $(a \preceq_{(P' i)} b) = (a \preceq_{(P i)} b)$ 
    proof(cases  $i = j$ )
    case False
    with  $ab iIs ixyzP'$  profileP profileP' has3A
    show ?thesis unfolding profile-def by auto
  next
    case True
    from profileP'  $jIs jxyzP'$  have  $z \prec_{(P' j)} y$ 
    by (auto dest: rpr-less-trans)
    with True  $ab iIs jyzP'$  profileP profileP' has3A
    show ?thesis unfolding profile-def strict-pref-def by auto
  qed
qed (simp-all add: profileP profileP')
moreover have  $z \prec_{(swf P')} y$ 
proof -
  from profileP'  $sd jxyzP'$   $ixyzP'$  have  $x \prec_{(swf P')} y$  by (simp add: semidecisive-def)
  moreover
  from  $jxyzP'$   $ixyzP'$  have  $\bigwedge i. i \in Is \implies z \prec_{(P' i)} x$  by (case-tac  $i=j$ , auto)
  with  $wp$  profileP' has3A have  $z \prec_{(swf P')} x$  by (auto dest: weak-paretoD)
  moreover note SWF-rpr[OF  $swf$ ] profileP'
  ultimately show  $z \prec_{(swf P')} y$ 
  unfolding universal-domain-def by (blast dest: rpr-less-trans)
qed
ultimately show  $z \prec_{(swf P)} y$  unfolding strict-pref-def by blast
qed

```

The following results permute  $x$ ,  $y$  and  $z$  to show how decisiveness can be obtained from semi-decisiveness in all cases. Again, quite tedious.

**lemma decisive3:**

```

assumes  $has3A$ :  $hasw [x, y, z] A$ 
    and  $ia$ :  $ia swf A Is$ 
    and  $swf$ :  $SWF swf A Is$  universal-domain
    and  $wp$ :  $weak-pareto swf A Is$  universal-domain
    and  $sd$ :  $semidecisive swf A Is \{j\} x z$ 
shows  $decisive swf A Is \{j\} y z$ 
using  $has3A$  decisive2[OF -  $ia swf wp sd$ ] by (simp, blast)

```

**lemma decisive4:**

```

assumes  $has3A$ :  $hasw [x, y, z] A$ 
    and  $ia$ :  $ia swf A Is$ 
    and  $swf$ :  $SWF swf A Is$  universal-domain

```

**and** *wp*: weak-pareto swf A Is universal-domain  
**and** *sd*: semidecisive swf A Is {j} y z  
**shows** decisive swf A Is {j} y x  
**using** has3A decisive1[OF - iia swf wp sd] **by** (simp, blast)

**lemma** *decisive5*:

**assumes** has3A: hasw [x,y,z] A  
**and** *iia*: iia swf A Is  
**and** *swf*: SWF swf A Is universal-domain  
**and** *wp*: weak-pareto swf A Is universal-domain  
**and** *sd*: semidecisive swf A Is {j} x y  
**shows** decisive swf A Is {j} y x

**proof** –

**from** *sd*  
**have** decisive swf A Is {j} x z **by** (rule decisive1[OF has3A iia swf wp])  
**hence** semidecisive swf A Is {j} x z **by** (rule d-imp-sd)  
**hence** decisive swf A Is {j} y z **by** (rule decisive3[OF has3A iia swf wp])  
**hence** semidecisive swf A Is {j} y z **by** (rule d-imp-sd)  
**thus** decisive swf A Is {j} y x **by** (rule decisive4[OF has3A iia swf wp])

**qed**

**lemma** *decisive6*:

**assumes** has3A: hasw [x,y,z] A  
**and** *iia*: iia swf A Is  
**and** *swf*: SWF swf A Is universal-domain  
**and** *wp*: weak-pareto swf A Is universal-domain  
**and** *sd*: semidecisive swf A Is {j} y x  
**shows** decisive swf A Is {j} y z decisive swf A Is {j} z x decisive swf A Is {j} x y

**proof** –

**from** has3A **have** has3A': hasw [y,x,z] A **by** auto  
**show** decisive swf A Is {j} y z **by** (rule decisive1[OF has3A' iia swf wp sd])  
**show** decisive swf A Is {j} z x **by** (rule decisive2[OF has3A' iia swf wp sd])  
**show** decisive swf A Is {j} x y **by** (rule decisive5[OF has3A' iia swf wp sd])

**qed**

**lemma** *decisive7*:

**assumes** has3A: hasw [x,y,z] A  
**and** *iia*: iia swf A Is  
**and** *swf*: SWF swf A Is universal-domain  
**and** *wp*: weak-pareto swf A Is universal-domain  
**and** *sd*: semidecisive swf A Is {j} x y  
**shows** decisive swf A Is {j} y z decisive swf A Is {j} z x decisive swf A Is {j} x y

**proof** –

**from** *sd*  
**have** decisive swf A Is {j} y x **by** (rule decisive5[OF has3A iia swf wp])  
**hence** semidecisive swf A Is {j} y x **by** (rule d-imp-sd)  
**thus** decisive swf A Is {j} y z decisive swf A Is {j} z x decisive swf A Is {j} x y  
**by** (rule decisive6[OF has3A iia swf wp])+

**qed**

**lemma** *j-decisive-xy*:

**assumes** has3A: hasw [x,y,z] A  
**and** *iia*: iia swf A Is

```

and swf: SWF swf A Is universal-domain
and wp: weak-pareto swf A Is universal-domain
and sd: semidecisive swf A Is {j} x y
and uv: hasw [u,v] {x,y,z}
shows decisive swf A Is {j} u v
using uv decisive1[OF has3A iia swf wp sd]
      decisive2[OF has3A iia swf wp sd]
      decisive5[OF has3A iia swf wp sd]
      decisive7[OF has3A iia swf wp sd]
by (simp, blast)

```

**lemma** *j-decisive*:

```

assumes has3A: has 3 A
and iia: iia swf A Is
and swf: SWF swf A Is universal-domain
and wp: weak-pareto swf A Is universal-domain
and xyA: hasw [x,y] A
and sd: semidecisive swf A Is {j} x y
and uv: hasw [u,v] A
shows decisive swf A Is {j} u v
proof –
from has-extend-witness'[OF has3A xyA]
obtain z where xyzA: hasw [x,y,z] A by auto
{
  assume ux: u = x and vy: v = y
  with xyzA iia swf wp sd have ?thesis by (auto intro: j-decisive-xy)
}
moreover
{
  assume ux: u = x and vNEy: v ≠ y
  with uv xyA iia swf wp sd have ?thesis by(auto intro: j-decisive-xy[of x y])
}
moreover
{
  assume uy: u = y and vx: v = x
  with xyzA iia swf wp sd have ?thesis by (auto intro: j-decisive-xy)
}
moreover
{
  assume uy: u = y and vNEx: v ≠ x
  with uv xyA iia swf wp sd have ?thesis by (auto intro: j-decisive-xy)
}
moreover
{
  assume uNExy: u ∉ {x,y} and vx: v = x
  with uv xyA iia swf wp sd have ?thesis by (auto intro: j-decisive-xy[of x y])
}
moreover
{
  assume uNExy: u ∉ {x,y} and vy: v = y
  with uv xyA iia swf wp sd have ?thesis by (auto intro: j-decisive-xy[of x y])
}
moreover

```

```

{
  assume  $uNExy: u \notin \{x,y\}$  and  $vNExy: v \notin \{x,y\}$ 
  with  $w xyA$  iia swf wp sd
  have decisive swf A Is {j} x u by (auto intro: j-decisive-xy[where  $x=x$  and  $z=u$ ])
  hence sdxu: semidecisive swf A Is {j} x u by (rule d-imp-sd)
  with  $uNExy vNExy w xyA$  iia swf wp have ?thesis by (auto intro: j-decisive-xy[of  $x$ ])
}
ultimately show ?thesis by blast
qed

```

The first result: if  $j$  is semidecisive for some alternatives  $u$  and  $v$ , then they are actually a dictator.

**lemma** *sd-imp-dictator*:

```

assumes has3A: has 3 A
  and iia: iia swf A Is
  and swf: SWF swf A Is universal-domain
  and wp: weak-pareto swf A Is universal-domain
  and uv: hasw [u,v] A
  and sd: semidecisive swf A Is {j} u v

```

shows *dictator swf A Is j*

**proof**

```

fix  $x y$  assume  $x: x \in A$  and  $y: y \in A$ 

```

```

show decisive swf A Is {j} x y

```

```

proof(cases  $x = y$ )

```

```

  case True with sd show decisive swf A Is {j} x y by (blast intro: d-refl)

```

```

next

```

```

  case False

```

```

    with  $x y$  iia swf wp has3A uv sd show decisive swf A Is {j} x y

```

```

    by (auto intro: j-decisive)

```

```

  qed

```

```

next

```

```

  from sd show  $j \in Is$  by blast

```

```

qed

```

## 4.2 The Existence of a Semi-decisive Individual

The second half of the proof establishes the existence of a semi-decisive individual. The required witness is essentially an encoding of the Condorcet paradox (aka "the paradox of voting" that shows we get tied up in knots if a certain agent didn't have dictatorial powers.

**lemma** *sd-exists-witness*:

```

assumes has3A: hasw [x,y,z] A

```

```

  and  $Vs: Is = V1 \cup V2 \cup V3$ 

```

```

     $\wedge V1 \cap V2 = \{\} \wedge V1 \cap V3 = \{\} \wedge V2 \cap V3 = \{\}$ 

```

```

  and  $Is: Is \neq \{\}$ 

```

```

obtains  $P$ 

```

```

  where profile A Is P

```

```

    and  $\forall i \in V1. x (P i) \prec y \wedge y (P i) \prec z$ 

```

```

    and  $\forall i \in V2. z (P i) \prec x \wedge x (P i) \prec y$ 

```

```

    and  $\forall i \in V3. y (P i) \prec z \wedge z (P i) \prec x$ 

```

**proof**

```

let  $?P =$ 

```

```

   $\lambda i. (if i \in V1 then \{(x, u) \mid u. u \in A\}$ 

```

$$\cup \{ (y, u) \mid u. u \in A \wedge u \neq x \}$$

$$\cup \{ (z, u) \mid u. u \in A \wedge u \neq x \wedge u \neq y \}$$

*else*

*if*  $i \in V2$  *then*  $(\{ (z, u) \mid u. u \in A \}$

$$\cup \{ (x, u) \mid u. u \in A \wedge u \neq z \}$$

$$\cup \{ (y, u) \mid u. u \in A \wedge u \neq x \wedge u \neq z \})$$

*else*  $(\{ (y, u) \mid u. u \in A \}$

$$\cup \{ (z, u) \mid u. u \in A \wedge u \neq y \}$$

$$\cup \{ (x, u) \mid u. u \in A \wedge u \neq y \wedge u \neq z \})$$

$$\cup \{ (u, v) \mid u v. u \in A - \{x, y, z\} \wedge v \in A - \{x, y, z\}\}$$

**show** *profile*  $A$  *Is*  $?P$

**proof**

**fix**  $i$  **assume**  $iIs: i \in Is$

**show** *rpr*  $A$   $(?P i)$

**proof**

**show** *complete*  $A$   $(?P i)$  **by**  $(simp\ add: complete-def, blast)$

**from** *has3A*  $iIs$  **show** *refl-on*  $A$   $(?P i)$  **by**  $- (simp, blast)$

**from** *has3A*  $iIs$  **show** *trans*  $(?P i)$  **by**  $(clarsimp\ simp\ add: trans-def)$

**qed**

**next**

**from**  $Is$  **show**  $Is \neq \{\}$  .

**qed**

**from** *has3A*  $Vs$

**show**  $\forall i \in V1. x$   $(?P i) \prec y \wedge y$   $(?P i) \prec z$

**and**  $\forall i \in V2. z$   $(?P i) \prec x \wedge x$   $(?P i) \prec y$

**and**  $\forall i \in V3. y$   $(?P i) \prec z \wedge z$   $(?P i) \prec x$

**unfolding** *strict-pref-def* **by** *auto*

**qed**

This proof is unfortunately long. Many of the statements rely on a lot of context, making it difficult to split it up.

**lemma** *sd-exists*:

**assumes** *has3A*: *has 3 A*

**and** *finiteIs*: *finite Is*

**and** *twoIs*: *has 2 Is*

**and** *iaa*: *iaa swf A Is*

**and** *swf*: *SWF swf A Is universal-domain*

**and** *wp*: *weak-pareto swf A Is universal-domain*

**shows**  $\exists j u v. hasw [u, v] A \wedge semidecisive\ swf\ A\ Is\ \{j\}\ u\ v$

**proof**  $-$

**let**  $?P = \lambda S. S \subseteq Is \wedge S \neq \{\} \wedge (\exists u v. hasw [u, v] A \wedge semidecisive\ swf\ A\ Is\ S\ u\ v)$

**obtain**  $u\ v$  **where**  $uvA: hasw [u, v] A$

**using** *has-witness-two[OF has3A]* **by** *auto*

$-$  The weak pareto requirement implies that the set of all individuals is decisive between any given alternatives.

**hence** *decisive swf A Is Is u v*

**by**  $- (rule, auto\ intro: weak-paretoD[OF wp])$

**hence** *semidecisive swf A Is Is u v* **by**  $(rule\ d-imp-sd)$

**with**  $uvA$  *twoIs has-suc-notempty[where n=1]* *nat-2[symmetric]*

**have**  $?P\ Is$  **by** *auto*

$-$  Obtain a minimally-sized semi-decisive set.

**from** *ex-has-least-nat[where P=?P and m=card, OF this]*

**obtain**  $V x y$  **where**  $VIs: V \subseteq Is$   
**and**  $Vnotempty: V \neq \{\}$   
**and**  $xyA: hasw [x,y] A$   
**and**  $Vsd: semidecisive swf A Is V x y$   
**and**  $Vmin: \bigwedge V'. ?P V' \implies card V \leq card V'$   
**by** *blast*  
**from**  $VIs$  *finiteIs* **have**  $Vfinite: finite V$  **by** (*rule finite-subset*)  
— Show that minimal set contains a single individual.  
**from**  $Vfinite$   $Vnotempty$  **have**  $\exists j. V = \{j\}$   
**proof**(*rule finite-set-singleton-contr*)  
**assume**  $Vcard: 1 < card V$   
**then obtain**  $j$  **where**  $jV: \{j\} \subseteq V$   
**using** *has-extend-witness*[**where**  $xs=[]$ , *OF card-has*[**where**  $n=card V$ ]] **by** *auto*  
— Split an individual from the "minimal" set.  
**let**  $?V1 = \{j\}$   
**let**  $?V2 = V - ?V1$   
**let**  $?V3 = Is - V$   
**from**  $jV$  *card-Diff-singleton*[*OF Vfinite*]  $Vcard$   
**have**  $V2card: card ?V2 > 0 \wedge card ?V2 < card V$  **by** *auto*  
**hence**  $V2notempty: \{\} \neq ?V2$  **by** *auto*  
**from**  $jV$   $VIs$   
**have**  $jV2V3: Is = ?V1 \cup ?V2 \cup ?V3 \wedge ?V1 \cap ?V2 = \{\} \wedge ?V1 \cap ?V3 = \{\} \wedge ?V2 \cap ?V3 =$   
 $\{\}$   
**by** *auto*  
— Show that that individual is semi-decisive for  $x$  over  $z$ .  
**from** *has-extend-witness*'[*OF has3A xyA*]  
**obtain**  $z$  **where** *threeDist*:  $hasw [x,y,z] A$  **by** *auto*  
**from** *sd-exists-witness*[*OF threeDist jV2V3*]  $VIs$   $Vnotempty$   
**obtain**  $P$  **where** *profileP*:  $profile A Is P$   
**and**  $V1xyzP: x (P j) \prec y \wedge y (P j) \prec z$   
**and**  $V2xyzP: \forall i \in ?V2. z (P i) \prec x \wedge x (P i) \prec y$   
**and**  $V3xyzP: \forall i \in ?V3. y (P i) \prec z \wedge z (P i) \prec x$   
**by** (*simp, blast*)  
**have**  $xPz: x (swf P) \prec z$   
**proof**(*rule rpr-less-le-trans*[**where**  $y=y$ ])  
**from** *profileP* *swf* **show** *rpr A (swf P)* **by** *auto*  
**next**  
—  $V2$  is semi-decisive, and everyone else opposes their choice. Ergo they prevail.  
**show**  $x (swf P) \prec y$   
**proof** —  
**from** *profileP*  $V3xyzP$   
**have**  $\forall i \in ?V3. y (P i) \prec x$  **by** (*blast dest: rpr-less-trans*)  
**with** *profileP*  $V1xyzP$   $V2xyzP$   $Vsd$   
**show** *?thesis unfolding semidecisive-def* **by** *auto*  
**qed**  
**next**  
— This result is unfortunately quite tortuous.  
**from** *SWF-rpr*[*OF swf*] **show**  $y (swf P) \preceq z$   
**proof**(*rule rpr-less-not*[*OF - - notI*])  
**from** *threeDist* **show**  $hasw [z, y] A$  **by** *auto*  
**next**  
**assume**  $zPy: z (swf P) \prec y$

**have** *semidecisive swf A Is ?V2 z y*  
**proof**  
**from** *VIs show  $V - \{j\} \subseteq Is$  by blast*  
**next**  
**fix** *P'*  
**assume** *profileP': profile A Is P'*  
**and** *V2yz':  $\bigwedge i. i \in ?V2 \implies z (P' i) \prec y$*   
**and** *nV2yz':  $\bigwedge i. i \in Is - ?V2 \implies y (P' i) \prec z$*   
**from** *iaa have  $\bigwedge a b. [a \in \{y, z\}; b \in \{y, z\}] \implies (a (swf P) \preceq b) = (a (swf P') \preceq b)$*   
**proof**(*rule iaaE*)  
**from** *threeDist show yzA:  $\{y, z\} \subseteq A$  by simp*  
**next**  
**fix** *i assume iIs:  $i \in Is$*   
**fix** *a b assume ab:  $a \in \{y, z\} b \in \{y, z\}$*   
**with** *VIs profileP V2xyzP*  
**have** *V2yzP:  $\forall i \in ?V2. z (P i) \prec y$  by (blast dest: rpr-less-trans)*  
**show** *( $a (P' i) \preceq b$ ) = ( $a (P i) \preceq b$ )*  
**proof**(*cases i  $\in ?V2$* )  
**case** *True*  
**with** *VIs profileP profileP' ab V2yz' V2yzP threeDist*  
**show** *?thesis unfolding strict-pref-def profile-def by auto*  
**next**  
**case** *False*  
**from** *V1xyzP V3xyzP*  
**have**  *$\forall i \in Is - ?V2. y (P i) \prec z$  by auto*  
**with** *iIs False VIs jV profileP profileP' ab nV2yz' threeDist*  
**show** *?thesis unfolding profile-def strict-pref-def by auto*  
**qed**  
**qed** (*simp-all add: profileP profileP'*)  
**with** *zPy show  $z (swf P') \prec y$  unfolding strict-pref-def by blast*  
**qed**  
**with** *VIs Vsd Vmin[where  $V' = ?V2$ ] V2card V2notempty threeDist show False*  
**by auto**  
**qed** (*simp add: profileP threeDist*)  
**qed**  
**have** *semidecisive swf A Is ?V1 x z*  
**proof**  
**from** *jV VIs show  $\{j\} \subseteq Is$  by blast*  
**next**  
— Use *iaa* to show the SWF must allow the individual to prevail.  
**fix** *P'*  
**assume** *profileP': profile A Is P'*  
**and** *V1yz':  $\bigwedge i. i \in ?V1 \implies x (P' i) \prec z$*   
**and** *nV1yz':  $\bigwedge i. i \in Is - ?V1 \implies z (P' i) \prec x$*   
**from** *iaa have  $\bigwedge a b. [a \in \{x, z\}; b \in \{x, z\}] \implies (a (swf P) \preceq b) = (a (swf P') \preceq b)$*   
**proof**(*rule iaaE*)  
**from** *threeDist show xzA:  $\{x, z\} \subseteq A$  by simp*  
**next**  
**fix** *i assume iIs:  $i \in Is$*   
**fix** *a b assume ab:  $a \in \{x, z\} b \in \{x, z\}$*   
**show** *( $a (P' i) \preceq b$ ) = ( $a (P i) \preceq b$ )*

```

proof(cases i ∈ ?V1)
  case True
    with jV VIs profileP V1xyzP
    have ∀ i ∈ ?V1. x (P i) < z by (blast dest: rpr-less-trans)
    with True jV VIs profileP profileP' ab V1yz' threeDist
    show ?thesis unfolding strict-pref-def profile-def by auto
  next
    case False
    from V2xyzP V3xyzP
    have ∀ i ∈ Is - ?V1. z (P i) < x by auto
    with iIs False VIs jV profileP profileP' ab nV1yz' threeDist
    show ?thesis unfolding strict-pref-def profile-def by auto
  qed
qed (simp-all add: profileP profileP')
with xPz show x (swf P') < z unfolding strict-pref-def by blast
qed
with jV VIs Vsd Vmin[where V'=?V1] V2card threeDist show False
  by auto
qed
with xyA Vsd show ?thesis by blast
qed

```

### 4.3 Arrow's General Possibility Theorem

Finally we conclude with the celebrated “possibility” result. Note that we assume the set of individuals is finite; [Rou79] relaxes this with some fancier set theory. Having an infinite set of alternatives doesn't matter, though the result is a bit more plausible if we assume finiteness [Sen70, p54].

```

theorem ArrowGeneralPossibility:
  assumes has3A: has 3 A
    and finiteIs: finite Is
    and has2Is: has 2 Is
    and iia: iia swf A Is
    and swf: SWF swf A Is universal-domain
    and wp: weak-pareto swf A Is universal-domain
  obtains j where dictator swf A Is j
  using sd-imp-dictator[OF has3A iia swf wp]
    sd-exists[OF has3A finiteIs has2Is iia swf wp]
  by blast

```

## 5 Sen's Liberal Paradox

### 5.1 Social Decision Functions (SDFs)

To make progress in the face of Arrow's Theorem, the demands placed on the social choice function need to be weakened. One approach is to only require that the set of alternatives that society ranks highest (and is otherwise indifferent about) be non-empty.

Following [Sen70, Chapter 4\*], a *Social Decision Function* (SDF) yields a choice function for every profile.

**definition**

$SDF :: ('a, 'i) SCF \Rightarrow 'a \text{ set} \Rightarrow 'i \text{ set} \Rightarrow ('a \text{ set} \Rightarrow 'i \text{ set} \Rightarrow ('a, 'i) \text{ Profile} \Rightarrow \text{bool}) \Rightarrow \text{bool}$

**where**

$SDF \text{ sdf } A \text{ Is } Pcond \equiv (\forall P. Pcond \ A \ \text{Is } P \longrightarrow \text{choiceFn } A \ (\text{sdf } P))$

**lemma** *SDFI[intro]*:

$(\bigwedge P. Pcond \ A \ \text{Is } P \Longrightarrow \text{choiceFn } A \ (\text{sdf } P)) \Longrightarrow SDF \ \text{sdf } A \ \text{Is } Pcond$

**unfolding** *SDF-def* **by** *simp*

**lemma** *SWF-SDF*:

**assumes** *finiteA*: *finite A*

**shows** *SWF scf A Is universal-domain*  $\Longrightarrow$  *SDF scf A Is universal-domain*

**unfolding** *SDF-def SWF-def* **by** (*blast dest: rpr-choiceFn[OF finiteA]*)

In contrast to SWFs, there are SDFs satisfying Arrow's (relevant) requirements. The lemma uses a witness to show the absence of a dictatorship.

**lemma** *SDF-nodictator-witness*:

**assumes** *has2A*: *hasw [x,y] A*

**and** *has2Is*: *hasw [i,j] Is*

**obtains** *P*

**where** *profile A Is P*

**and**  $x \ (P \ i) \prec y$

**and**  $y \ (P \ j) \prec x$

**proof**

**let**  $?P = \lambda k. (\text{if } k = i \text{ then } (\{ (x, u) \mid u. u \in A \} \cup \{ (y, u) \mid u. u \in A - \{x\} \})$   
 $\text{else } (\{ (y, u) \mid u. u \in A \} \cup \{ (x, u) \mid u. u \in A - \{y\} \}))$   
 $\cup (A - \{x, y\}) \times (A - \{x, y\})$

**show** *profile A Is ?P*

**proof**

**fix** *i* **assume** *iis*:  $i \in Is$

**from** *has2A* **show** *rpr A (?P i)*

**by**  $-$  (*rule rprI, simp-all add: trans-def, blast+*)

**next**

**from** *has2Is* **show**  $Is \neq \{\}$  **by** *auto*

**qed**

**from** *has2A has2Is*

**show**  $x \ (?P \ i) \prec y$

**and**  $y \ (?P \ j) \prec x$

**unfolding** *strict-pref-def* **by** *auto*

**qed**

**lemma** *SDF-possibility*:

**assumes** *finiteA*: *finite A*

**and** *has2A*: *has 2 A*

**and** *has2Is*: *has 2 Is*

**obtains** *sdf*

**where** *weak-pareto sdf A Is universal-domain*

**and** *iia sdf A Is*

**and**  $\neg(\exists j. \text{dictator } \text{sdf } A \text{ Is } j)$   
**and**  $SDF \text{ sdf } A \text{ Is universal-domain}$   
**proof** –  
**let**  $?sdf = \lambda P. \{ (x, y) . x \in A \wedge y \in A$   
 $\quad \wedge \neg((\forall i \in \text{Is}. y (P \ i) \preceq x)$   
 $\quad \wedge (\exists i \in \text{Is}. y (P \ i) \prec x)) \}$   
**have**  $\text{weak-pareto } ?sdf \ A \ \text{Is universal-domain}$   
**by** (*rule, unfold strict-pref-def, auto dest: profile-non-empty*)  
**moreover**  
**have**  $\text{ia } ?sdf \ A \ \text{Is unfolding strict-pref-def by auto}$   
**moreover**  
**have**  $\neg(\exists j. \text{dictator } ?sdf \ A \ \text{Is } j)$   
**proof**  
**assume**  $\exists j. \text{dictator } ?sdf \ A \ \text{Is } j$   
**then obtain**  $j \text{ where } j\text{Is}: j \in \text{Is}$   
**and**  $jD: \forall x \in A. \forall y \in A. \text{decisive } ?sdf \ A \ \text{Is } \{j\} \ x \ y$   
**unfolding** *dictator-def* *decisive-def* **by** *auto*  
**from**  $j\text{Is has-witness-two}[OF \ \text{has2Is}]$  **obtain**  $i \text{ where } ij\text{Is}: \text{hasw } [i, j] \ \text{Is}$   
**by** *auto*  
**from**  $\text{has-witness-two}[OF \ \text{has2A}]$  **obtain**  $x \ y \text{ where } xyA: \text{hasw } [x, y] \ A$  **by** *auto*  
**from**  $xyA \ ij\text{Is}$  **obtain**  $P$   
**where**  $\text{profile}P: \text{profile } A \ \text{Is } P$   
**and**  $yPix: x (P \ i) \prec y$   
**and**  $yPjx: y (P \ j) \prec x$   
**by** (*rule SDF-nodictator-witness*)  
**from**  $\text{profile}P \ jD \ j\text{Is} \ xyA \ yPjx$  **have**  $y (?sdf \ P) \prec x$   
**unfolding** *decisive-def* **by** *simp*  
**moreover**  
**from**  $ij\text{Is} \ xyA \ yPjx \ yPix$  **have**  $x (?sdf \ P) \preceq y$   
**unfolding** *strict-pref-def* **by** *auto*  
**ultimately show** *False*  
**unfolding** *strict-pref-def* **by** *blast*  
**qed**  
**moreover**  
**have**  $SDF \ ?sdf \ A \ \text{Is universal-domain}$   
**proof**  
**fix**  $P$  **assume**  $ud: \text{universal-domain } A \ \text{Is } P$   
**show**  $\text{choiceFn } A \ (?sdf \ P)$   
**proof**(*rule r-c-qt-imp-cf[OF finiteA]*)  
**show**  $\text{complete } A \ (?sdf \ P)$  **and**  $\text{refl-on } A \ (?sdf \ P)$   
**unfolding** *strict-pref-def* **by** *auto*  
**show**  $\text{quasi-trans } (?sdf \ P)$   
**proof**  
**fix**  $x \ y \ z$  **assume**  $xy: x (?sdf \ P) \prec y$  **and**  $yz: y (?sdf \ P) \prec z$   
**from**  $xy \ yz$  **have**  $xyzA: x \in A \ y \in A \ z \in A$   
**unfolding** *strict-pref-def* **by** *auto*  
**from**  $xy \ yz$  **have**  $AxRy: \forall i \in \text{Is}. x (P \ i) \preceq y$   
**and**  $ExPy: \exists i \in \text{Is}. x (P \ i) \prec y$   
**and**  $AyRz: \forall i \in \text{Is}. y (P \ i) \preceq z$   
**unfolding** *strict-pref-def* **by** *auto*  
**from**  $AxRy \ AyRz \ ud$  **have**  $AxRz: \forall i \in \text{Is}. x (P \ i) \preceq z$

```

    by – (unfold universal-domain-def, blast dest: rpr-le-trans)
  from ExPy AyRz ud have ExPz:  $\exists i \in Is. x (P i) \prec z$ 
    by – (unfold universal-domain-def, blast dest: rpr-less-le-trans)
  from xyzA AxRz ExPz show  $x (?sdf P) \prec z$  unfolding strict-pref-def by auto
qed
qed
qed
ultimately show thesis ..
qed

```

Sen makes several other stronger statements about SDFs later in the chapter. I leave these for future work.

## 5.2 Sen's Liberal Paradox

Having side-stepped Arrow's Theorem, Sen proceeds to other conditions one may ask of an SCF. His analysis of *liberalism*, mechanised in this section, has attracted much criticism over the years [AK96].

Following [Sen70, Chapter 6\*], a *liberal* social choice rule is one that, for each individual, there is a pair of alternatives that she is decisive over.

**definition** *liberal* :: ('a, 'i) SCF  $\Rightarrow$  'a set  $\Rightarrow$  'i set  $\Rightarrow$  bool **where**

```

liberal scf A Is  $\equiv$ 
  ( $\forall i \in Is. \exists x \in A. \exists y \in A. x \neq y$ 
    $\wedge$  decisive scf A Is {i} x y  $\wedge$  decisive scf A Is {i} y x)

```

**lemma** *liberalE*:

```

[[ liberal scf A Is; i  $\in$  Is ]]
 $\implies \exists x \in A. \exists y \in A. x \neq y$ 
   $\wedge$  decisive scf A Is {i} x y  $\wedge$  decisive scf A Is {i} y x
by (simp add: liberal-def)

```

This condition can be weakened to require just two such decisive individuals; if we required just one, we would allow dictatorships, which are clearly not liberal.

**definition** *minimally-liberal* :: ('a, 'i) SCF  $\Rightarrow$  'a set  $\Rightarrow$  'i set  $\Rightarrow$  bool **where**

```

minimally-liberal scf A Is  $\equiv$ 
  ( $\exists i \in Is. \exists j \in Is. i \neq j$ 
    $\wedge$  ( $\exists x \in A. \exists y \in A. x \neq y$ 
       $\wedge$  decisive scf A Is {i} x y  $\wedge$  decisive scf A Is {i} y x)
    $\wedge$  ( $\exists x \in A. \exists y \in A. x \neq y$ 
       $\wedge$  decisive scf A Is {j} x y  $\wedge$  decisive scf A Is {j} y x))

```

**lemma** *liberal-imp-minimally-liberal*:

**assumes** has2Is: has 2 Is

**and** L: liberal scf A Is

**shows** minimally-liberal scf A Is

**proof** –

```

from has-extend-witness[where xs=[], OF has2Is]
obtain i where i: i  $\in$  Is by auto
with has-extend-witness[where xs=[i], OF has2Is]
obtain j where j: j  $\in$  Is i  $\neq$  j by auto
from L i j show ?thesis

```

**unfolding** *minimally-liberal-def* **by** (*blast intro: liberalE*)  
**qed**

The key observation is that once we have at least two decisive individuals we can complete the Condorcet (paradox of voting) cycle using the weak Pareto assumption. The details of the proof don't give more insight.

Firstly we need three types of profile witnesses (one of which we saw previously). The main proof proceeds by case distinctions on which alternatives the two liberal agents are decisive for.

**lemmas** *liberal-witness-two = SDF-nodictator-witness*

**lemma** *liberal-witness-three:*

**assumes** *threeA: hasw [x,y,v] A*

**and** *twoIs: hasw [i,j] Is*

**obtains** *P*

**where** *profile A Is P*

**and**  $x (P\ i) \prec y$

**and**  $v (P\ j) \prec x$

**and**  $\forall i \in Is. y (P\ i) \prec v$

**proof** –

**let** *?P =*

*λa. if a = i then { (x, u) | u. u ∈ A }  
           ∪ { (y, u) | u. u ∈ A - {x} }  
           ∪ (A - {x,y}) × (A - {x,y})  
       else { (y, u) | u. u ∈ A }  
           ∪ { (v, u) | u. u ∈ A - {y} }  
           ∪ (A - {v,y}) × (A - {v,y})*

**have** *profile A Is ?P*

**proof**

**fix** *i* **assume** *iis: i ∈ Is*

**show** *rpr A (?P i)*

**proof**

**show** *complete A (?P i)* **by** (*simp, blast*)

**from** *threeA iis* **show** *refl-on A (?P i)* **by** (*simp, blast*)

**from** *threeA iis* **show** *trans (?P i)* **by** (*clarsimp simp add: trans-def*)

**qed**

**next**

**from** *twoIs* **show** *Is ≠ {}* **by** *auto*

**qed**

**moreover**

**from** *threeA twoIs* **have**  $x (?P\ i) \prec y$   $v (?P\ j) \prec x$   $\forall i \in Is. y (?P\ i) \prec v$

**unfolding** *strict-pref-def* **by** *auto*

**ultimately show** *?thesis ..*

**qed**

**lemma** *liberal-witness-four:*

**assumes** *fourA: hasw [x,y,u,v] A*

**and** *twoIs: hasw [i,j] Is*

**obtains** *P*

**where** *profile A Is P*

**and**  $x (P\ i) \prec y$

**and**  $u (P\ j) \prec v$

**and**  $\forall i \in Is. v (P i) \prec x \wedge y (P i) \prec u$   
**proof** –  
**let**  $?P =$   
 $\lambda a. \text{if } a = i \text{ then } \{ (v, w) \mid w. w \in A \}$   
 $\quad \cup \{ (x, w) \mid w. w \in A - \{v\} \}$   
 $\quad \cup \{ (y, w) \mid w. w \in A - \{v, x\} \}$   
 $\quad \cup (A - \{v, x, y\}) \times (A - \{v, x, y\})$   
**else**  $\{ (y, w) \mid w. w \in A \}$   
 $\quad \cup \{ (u, w) \mid w. w \in A - \{y\} \}$   
 $\quad \cup \{ (v, w) \mid w. w \in A - \{u, y\} \}$   
 $\quad \cup (A - \{u, v, y\}) \times (A - \{u, v, y\})$   
**have profile**  $A \text{ Is } ?P$   
**proof**  
**fix**  $i$  **assume**  $iis: i \in Is$   
**show**  $rpr A (?P i)$   
**proof**  
**show**  $complete A (?P i)$  **by**  $(simp, blast)$   
**from**  $fourA iis$  **show**  $refl-on A (?P i)$  **by**  $(simp, blast)$   
**from**  $fourA iis$  **show**  $trans (?P i)$  **by**  $(clarsimp simp add: trans-def)$   
**qed**  
**next**  
**from**  $twoIs$  **show**  $Is \neq \{\}$  **by**  $auto$   
**qed**  
**moreover**  
**from**  $fourA twoIs$  **have**  $x (?P i) \prec y \wedge u (?P j) \prec v \wedge \forall i \in Is. v (?P i) \prec x \wedge y (?P i) \prec u$   
**by**  $(unfold strict-pref-def, auto)$   
**ultimately show**  $thesis ..$   
**qed**

The Liberal Paradox: having two decisive individuals, an SDF and the weak pareto assumption is inconsistent.

**theorem** *LiberalParadox:*

**assumes**  $SDF: SDF \text{ sdf } A \text{ Is universal-domain}$   
**and**  $ml: minimally-liberal \text{ sdf } A \text{ Is}$   
**and**  $wp: weak-pareto \text{ sdf } A \text{ Is universal-domain}$   
**shows**  $False$

**proof** –  
**from**  $ml$  **obtain**  $i j x y u v$   
**where**  $i: i \in Is$  **and**  $j: j \in Is$  **and**  $ij: i \neq j$   
**and**  $x: x \in A$  **and**  $y: y \in A$  **and**  $u: u \in A$  **and**  $v: v \in A$   
**and**  $xy: x \neq y$   
**and**  $dixy: decisive \text{ sdf } A \text{ Is } \{i\} x y$   
**and**  $diy: decisive \text{ sdf } A \text{ Is } \{i\} y x$   
**and**  $uv: u \neq v$   
**and**  $djuv: decisive \text{ sdf } A \text{ Is } \{j\} u v$   
**and**  $djvu: decisive \text{ sdf } A \text{ Is } \{j\} v u$   
**by**  $(unfold minimally-liberal-def, auto)$   
**from**  $i j ij$  **have**  $twoIs: hasw [i,j] Is$  **by**  $simp$   
**{**  
**assume**  $xv: x = u$  **and**  $yv: y = v$   
**from**  $xy x y$  **have**  $twoA: hasw [x,y] A$  **by**  $simp$   
**obtain**  $P$

```

    where profile A Is P x (P i) < y y (P j) < x
    using liberal-witness-two[OF twoA twoIs] by blast
  with i j dixy djvu xu yv have False
  by (unfold decisive-def strict-pref-def, blast)
}
moreover
{
  assume xu: x = u and yv: y ≠ v
  with xy uv xu x y v have threeA: hasw [x,y,v] A by simp
  obtain P
  where profileP: profile A Is P
    and xPiy: x (P i) < y
    and vPjx: v (P j) < x
    and AyPv: ∀ i ∈ Is. y (P i) < v
  using liberal-witness-three[OF threeA twoIs] by blast
  from vPjx j djvu xu profileP have vPx: v (sdf P) < x
  by (unfold decisive-def strict-pref-def, auto)
  from xPiy i dixy profileP have xPy: x (sdf P) < y
  by (unfold decisive-def strict-pref-def, auto)
  from AyPv weak-paretoD[OF wp - y v] profileP have yPv: y (sdf P) < v
  by auto
  from threeA profileP SDF have choiceSet {x,y,v} (sdf P) ≠ {}
  by (simp add: SDF-def choiceFn-def)
  with vPx xPy yPv have False
  by (unfold choiceSet-def strict-pref-def, blast)
}
moreover
{
  assume xv: x = v and yu: y = u
  from xy x y have twoA: hasw [x,y] A by auto
  obtain P
  where profile A Is P x (P i) < y y (P j) < x
  using liberal-witness-two[OF twoA twoIs] by blast
  with i j dixy djvu xv yu have False
  by (unfold decisive-def strict-pref-def, blast)
}
moreover
{
  assume xv: x = v and yu: y ≠ u
  with xy uv u x y have threeA: hasw [x,y,u] A by simp
  obtain P
  where profileP: profile A Is P
    and xPiy: x (P i) < y
    and uPjx: u (P j) < x
    and AyPu: ∀ i ∈ Is. y (P i) < u
  using liberal-witness-three[OF threeA twoIs] by blast
  from uPjx j djvu xv profileP have uPx: u (sdf P) < x
  by (unfold decisive-def strict-pref-def, auto)
  from xPiy i dixy profileP have xPy: x (sdf P) < y
  by (unfold decisive-def strict-pref-def, auto)
  from AyPu weak-paretoD[OF wp - y u] profileP have yPu: y (sdf P) < u

```

```

    by auto
  from threeA profileP SDF have choiceSet {x,y,u} (sdf P) ≠ {}
    by (simp add: SDF-def choiceFn-def)
  with uPx xPy yPu have False
    by (unfold choiceSet-def strict-pref-def, blast)
}
moreover
{
  assume xu: x ≠ u and xv: x ≠ v and yu: y = u
  with v x y xy uv xu have threeA: hasw [y,x,v] A by simp
  obtain P
    where profileP: profile A Is P
      and yPix: y (P i) ≺ x
      and vPjy: v (P j) ≺ y
      and AxPv: ∀ i ∈ Is. x (P i) ≺ v
    using liberal-witness-three[OF threeA twoIs] by blast
  from yPix i diyx profileP have yPx: y (sdf P) ≺ x
    by (unfold decisive-def strict-pref-def, auto)
  from vPjy j djvu yu profileP have vPy: v (sdf P) ≺ y
    by (unfold decisive-def strict-pref-def, auto)
  from AxPv weak-paretoD[OF wp - x v] profileP have xPv: x (sdf P) ≺ v
    by auto
  from threeA profileP SDF have choiceSet {x,y,v} (sdf P) ≠ {}
    by (simp add: SDF-def choiceFn-def)
  with yPx vPy xPv have False
    by (unfold choiceSet-def strict-pref-def, blast)
}
moreover
{
  assume xu: x ≠ u and xv: x ≠ v and yv: y = v
  with u x y xy uv xu have threeA: hasw [y,x,u] A by simp
  obtain P
    where profileP: profile A Is P
      and yPix: y (P i) ≺ x
      and uPjy: u (P j) ≺ y
      and AxPu: ∀ i ∈ Is. x (P i) ≺ u
    using liberal-witness-three[OF threeA twoIs] by blast
  from yPix i diyx profileP have yPx: y (sdf P) ≺ x
    by (unfold decisive-def strict-pref-def, auto)
  from uPjy j djuv yv profileP have uPy: u (sdf P) ≺ y
    by (unfold decisive-def strict-pref-def, auto)
  from AxPu weak-paretoD[OF wp - x u] profileP have xPu: x (sdf P) ≺ u
    by auto
  from threeA profileP SDF have choiceSet {x,y,u} (sdf P) ≠ {}
    by (simp add: SDF-def choiceFn-def)
  with yPx uPy xPu have False
    by (unfold choiceSet-def strict-pref-def, blast)
}
moreover
{
  assume xu: x ≠ u and xv: x ≠ v and yu: y ≠ u and yv: y ≠ v

```

```

with  $u v x y xy uv xu$  have  $fourA: hasw [x,y,u,v] A$  by simp
obtain  $P$ 
  where  $profileP: profile A Is P$ 
    and  $xPiy: x (P i) \prec y$ 
    and  $uPjv: u (P j) \prec v$ 
    and  $AvPxAyPu: \forall i \in Is. v (P i) \prec x \wedge y (P i) \prec u$ 
  using liberal-witness-four[OF fourA twoIs] by blast
from  $xPiy i dixy profileP$  have  $xPy: x (sdf P) \prec y$ 
  by (unfold decisive-def strict-pref-def, auto)
from  $uPjv j djvw profileP$  have  $uPv: u (sdf P) \prec v$ 
  by (unfold decisive-def strict-pref-def, auto)
from  $AvPxAyPu weak-paretoD[OF wp] profileP x y u v$ 
have  $vPx: v (sdf P) \prec x$  and  $yPu: y (sdf P) \prec u$  by auto
from  $fourA profileP SDF$  have  $choiceSet \{x,y,u,v\} (sdf P) \neq \{\}$ 
  by (simp add: SDF-def choiceFn-def)
with  $xPy uPv vPx yPu$  have False
  by (unfold choiceSet-def strict-pref-def, blast)
}
ultimately show False by blast
qed

```

## 6 May's Theorem

May's Theorem [May52] provides a characterisation of majority voting in terms of four conditions that appear quite natural for *a priori* unbiased social choice scenarios. It can be seen as a refinement of some earlier work by Arrow [Arr63, Chapter V.1].

The following is a mechanisation of Sen's generalisation [Sen70, Chapter 5\*]; originally Arrow and May consider only two alternatives, whereas Sen's model maps profiles of full RPRs to a possibly intransitive relation that does at least generate a choice set that satisfies May's conditions.

### 6.1 May's Conditions

The condition of *anonymity* asserts that the individuals' identities are not considered by the choice rule. Rather than talk about permutations we just assert the result of the SCF is the same when the profile is composed with an arbitrary bijection on the set of individuals.

**definition** *anonymous* :: ( $'a, 'i$ ) *SCF*  $\Rightarrow 'a set \Rightarrow 'i set \Rightarrow bool$  **where**

*anonymous scf*  $A Is \equiv$

$(\forall P f x y. profile A Is P \wedge bij\text{-betw } f Is Is \wedge x \in A \wedge y \in A$

$\longrightarrow (x (scf P) \preceq y) = (x (scf (P \circ f)) \preceq y))$

**lemma** *anonymousI[intro]*:

$(\bigwedge P f x y. \llbracket profile A Is P; bij\text{-betw } f Is Is;$

$x \in A; y \in A \rrbracket \Longrightarrow (x (scf P) \preceq y) = (x (scf (P \circ f)) \preceq y))$

$\Longrightarrow anonymous\ scf A Is$

**unfolding** *anonymous-def* **by** *simp*

**lemma anonymousD:**

$\llbracket \text{anonymous scf } A \text{ Is}; \text{ profile } A \text{ Is } P; \text{ bij-betw } f \text{ Is Is}; x \in A; y \in A \rrbracket$   
 $\implies (x \text{ (scf } P) \preceq y) = (x \text{ (scf } (P \circ f)) \preceq y)$   
**unfolding anonymous-def by simp**

Similarly, an SCF is *neutral* if it is insensitive to the identity of the alternatives. This is Sen's characterisation [Sen70, p72].

**definition neutral :: ('a, 'i) SCF  $\Rightarrow$  'a set  $\Rightarrow$  'i set  $\Rightarrow$  bool where**

$\text{neutral scf } A \text{ Is} \equiv$   
 $(\forall P P' x y z w. \text{profile } A \text{ Is } P \wedge \text{profile } A \text{ Is } P' \wedge x \in A \wedge y \in A \wedge z \in A \wedge w \in A$   
 $\wedge (\forall i \in \text{Is}. x \text{ (P } i) \preceq y \longleftrightarrow z \text{ (P' } i) \preceq w) \wedge (\forall i \in \text{Is}. y \text{ (P } i) \preceq x \longleftrightarrow w \text{ (P' } i) \preceq z)$   
 $\longrightarrow ((x \text{ (scf } P) \preceq y \longleftrightarrow z \text{ (scf } P') \preceq w) \wedge (y \text{ (scf } P) \preceq x \longleftrightarrow w \text{ (scf } P') \preceq z)))$

**lemma neutralI[intro]:**

$(\bigwedge P P' x y z w.$   
 $\llbracket \text{profile } A \text{ Is } P; \text{ profile } A \text{ Is } P'; \{x, y, z, w\} \subseteq A;$   
 $\bigwedge i. i \in \text{Is} \implies x \text{ (P } i) \preceq y \longleftrightarrow z \text{ (P' } i) \preceq w;$   
 $\bigwedge i. i \in \text{Is} \implies y \text{ (P } i) \preceq x \longleftrightarrow w \text{ (P' } i) \preceq z \rrbracket$   
 $\implies ((x \text{ (scf } P) \preceq y \longleftrightarrow z \text{ (scf } P') \preceq w) \wedge (y \text{ (scf } P) \preceq x \longleftrightarrow w \text{ (scf } P') \preceq z)))$   
 $\implies \text{neutral scf } A \text{ Is}$   
**unfolding neutral-def by simp**

**lemma neutralD:**

$\llbracket \text{neutral scf } A \text{ Is};$   
 $\text{profile } A \text{ Is } P; \text{ profile } A \text{ Is } P'; \{x, y, z, w\} \subseteq A;$   
 $\bigwedge i. i \in \text{Is} \implies x \text{ (P } i) \preceq y \longleftrightarrow z \text{ (P' } i) \preceq w;$   
 $\bigwedge i. i \in \text{Is} \implies y \text{ (P } i) \preceq x \longleftrightarrow w \text{ (P' } i) \preceq z \rrbracket$   
 $\implies (x \text{ (scf } P) \preceq y \longleftrightarrow z \text{ (scf } P') \preceq w) \wedge (y \text{ (scf } P) \preceq x \longleftrightarrow w \text{ (scf } P') \preceq z)$   
**unfolding neutral-def by simp**

Neutrality implies independence of irrelevant alternatives.

**lemma neutral-ia: neutral scf A Is  $\implies$  ia scf A Is**

**unfolding neutral-def by (rule, auto)**

*Positive responsiveness* is a bit like non-manipulability: if one individual improves their opinion of  $x$ , then the result should shift in favour of  $x$ .

**definition positively-responsive :: ('a, 'i) SCF  $\Rightarrow$  'a set  $\Rightarrow$  'i set  $\Rightarrow$  bool where**

$\text{positively-responsive scf } A \text{ Is} \equiv$   
 $(\forall P P' x y. \text{profile } A \text{ Is } P \wedge \text{profile } A \text{ Is } P' \wedge x \in A \wedge y \in A$   
 $\wedge (\forall i \in \text{Is}. (x \text{ (P } i) \prec y \longrightarrow x \text{ (P' } i) \prec y) \wedge (x \text{ (P } i) \approx y \longrightarrow x \text{ (P' } i) \preceq y))$   
 $\wedge (\exists k \in \text{Is}. (x \text{ (P } k) \approx y \wedge x \text{ (P' } k) \prec y) \vee (y \text{ (P } k) \prec x \wedge x \text{ (P' } k) \preceq y))$   
 $\longrightarrow x \text{ (scf } P) \preceq y \longrightarrow x \text{ (scf } P') \prec y)$

**lemma positively-responsiveI[intro]:**

**assumes**  $I: \bigwedge P P' x y.$   
 $\llbracket \text{profile } A \text{ Is } P; \text{ profile } A \text{ Is } P'; x \in A; y \in A;$   
 $\bigwedge i. \llbracket i \in \text{Is}; x \text{ (P } i) \prec y \rrbracket \implies x \text{ (P' } i) \prec y;$   
 $\bigwedge i. \llbracket i \in \text{Is}; x \text{ (P } i) \approx y \rrbracket \implies x \text{ (P' } i) \preceq y;$   
 $\exists k \in \text{Is}. (x \text{ (P } k) \approx y \wedge x \text{ (P' } k) \prec y) \vee (y \text{ (P } k) \prec x \wedge x \text{ (P' } k) \preceq y);$

$x \text{ (scf } P) \preceq y \parallel$   
 $\implies x \text{ (scf } P') \prec y$   
**shows** *positively-responsive scf A Is*  
**unfolding** *positively-responsive-def*  
**by** (*blast intro: I*)

**lemma** *positively-responsiveD:*

$\parallel$  *positively-responsive scf A Is;*  
*profile A Is P; profile A Is P'; x ∈ A; y ∈ A;*  
 $\bigwedge i. \parallel i \in Is; x \text{ (P } i) \prec y \parallel \implies x \text{ (P' } i) \prec y;$   
 $\bigwedge i. \parallel i \in Is; x \text{ (P } i) \approx y \parallel \implies x \text{ (P' } i) \preceq y;$   
 $\exists k \in Is. (x \text{ (P } k) \approx y \wedge x \text{ (P' } k) \prec y) \vee (y \text{ (P } k) \prec x \wedge x \text{ (P' } k) \preceq y);$   
 $x \text{ (scf } P) \preceq y \parallel$   
 $\implies x \text{ (scf } P') \prec y$   
**unfolding** *positively-responsive-def*  
**apply** *clarsimp*  
**apply** (*erule alle[where x=P]*)  
**apply** (*erule alle[where x=P']*)  
**apply** (*erule alle[where x=x]*)  
**apply** (*erule alle[where x=y]*)  
**by** *auto*

## 6.2 The Method of Majority Decision satisfies May's conditions

The *method of majority decision* (MMD) says that if the number of individuals who strictly prefer  $x$  to  $y$  is larger than or equal to those who strictly prefer the converse, then  $x R y$ . Note that this definition only makes sense for a finite population.

**definition** *MMD :: 'i set ⇒ ('a, 'i) SCF where*

*MMD Is P ≡ { (x, y) . card { i ∈ Is. x (P i) < y } ≥ card { i ∈ Is. y (P i) < x } }*

The first part of May's Theorem establishes that the conditions are consistent, by showing that they are satisfied by MMD.

**lemma** *MMD-l2r:*

**fixes**  $A :: 'a \text{ set}$   
**and**  $Is :: 'i \text{ set}$   
**assumes** *finiteIs: finite Is*  
**shows** *SCF (MMD Is) A Is universal-domain*  
**and** *anonymous (MMD Is) A Is*  
**and** *neutral (MMD Is) A Is*  
**and** *positively-responsive (MMD Is) A Is*  
**proof** –  
**show** *SCF (MMD Is) A Is universal-domain*  
**proof**  
**fix**  $P$  **show** *complete A (MMD Is P)*  
**by** (*rule completeI, unfold MMD-def, simp, arith*)  
**qed**  
**show** *anonymous (MMD Is) A Is*  
**proof**  
**fix**  $P$   
**fix**  $x \ y :: 'a$   
**fix**  $f$  **assume** *bijf: bij-betw f Is Is*

**show**  $(x \text{ (MMD } Is \ P) \preceq y) = (x \text{ (MMD } Is \ (P \circ f)) \preceq y)$   
**using** *card-compose-bij*[*OF bijf*, **where**  $P = \lambda i. x \text{ (} P \ i) \prec y$ ]  
*card-compose-bij*[*OF bijf*, **where**  $P = \lambda i. y \text{ (} P \ i) \prec x$ ]  
**unfolding** *MMD-def* **by** *simp*  
**qed**  
**next**  
**show** *neutral*  $(\text{MMD } Is) \ A \ Is$   
**proof**  
**fix**  $P \ P'$   
**fix**  $x \ y \ z \ w$  **assume**  $xyzwA: \{x, y, z, w\} \subseteq A$   
**assume**  $xyzw: \bigwedge i. i \in Is \implies (x \text{ (} P \ i) \preceq y) = (z \text{ (} P' \ i) \preceq w)$   
**and**  $yxwz: \bigwedge i. i \in Is \implies (y \text{ (} P \ i) \preceq x) = (w \text{ (} P' \ i) \preceq z)$   
**from**  $xyzwA \ xyzw \ yxwz$   
**have**  $\{ i \in Is. x \text{ (} P \ i) \prec y \} = \{ i \in Is. z \text{ (} P' \ i) \prec w \}$   
**and**  $\{ i \in Is. y \text{ (} P \ i) \prec x \} = \{ i \in Is. w \text{ (} P' \ i) \prec z \}$   
**unfolding** *strict-pref-def* **by** *auto*  
**thus**  $(x \text{ (MMD } Is \ P) \preceq y) = (z \text{ (MMD } Is \ P') \preceq w) \wedge$   
 $(y \text{ (MMD } Is \ P) \preceq x) = (w \text{ (MMD } Is \ P') \preceq z)$   
**unfolding** *MMD-def* **by** *simp*  
**qed**  
**next**  
**show** *positively-responsive*  $(\text{MMD } Is) \ A \ Is$   
**proof**  
**fix**  $P \ P'$  **assume** *profileP*: *profile*  $A \ Is \ P$   
**fix**  $x \ y$  **assume**  $xyA: x \in A \ y \in A$   
**assume**  $xPy: \bigwedge i. \llbracket i \in Is; x \text{ (} P \ i) \prec y \rrbracket \implies x \text{ (} P' \ i) \prec y$   
**and**  $xIy: \bigwedge i. \llbracket i \in Is; x \text{ (} P \ i) \approx y \rrbracket \implies x \text{ (} P' \ i) \preceq y$   
**and**  $k: \exists k \in Is. x \text{ (} P \ k) \approx y \wedge x \text{ (} P' \ k) \prec y \vee y \text{ (} P \ k) \prec x \wedge x \text{ (} P' \ k) \preceq y$   
**and**  $xRSCFy: x \text{ (MMD } Is \ P) \preceq y$   
**from**  $k$  **obtain**  $k$   
**where**  $kIs: k \in Is$   
**and**  $kcond: (x \text{ (} P \ k) \approx y \wedge x \text{ (} P' \ k) \prec y) \vee (y \text{ (} P \ k) \prec x \wedge x \text{ (} P' \ k) \preceq y)$   
**by** *blast*  
**let**  $?xPy = \{ i \in Is. x \text{ (} P \ i) \prec y \}$   
**let**  $?xP'y = \{ i \in Is. x \text{ (} P' \ i) \prec y \}$   
**let**  $?yPx = \{ i \in Is. y \text{ (} P \ i) \prec x \}$   
**let**  $?yP'x = \{ i \in Is. y \text{ (} P' \ i) \prec x \}$   
**from** *profileP*  $xyA \ xPy \ xIy$  **have**  $yP'xyPx: ?yP'x \subseteq ?yPx$   
**unfolding** *strict-pref-def* *indifferent-pref-def*  
**by**  $(\text{blast dest: } rpr\text{-complete})$   
**with** *finiteIs* **have**  $yP'xyPxC: \text{card } ?yP'x \leq \text{card } ?yPx$   
**by**  $(\text{blast intro: } card\text{-mono } finite\text{-subset})$   
**from** *finiteIs*  $xPy$  **have**  $xPyxP'yC: \text{card } ?xPy \leq \text{card } ?xP'y$   
**by**  $(\text{blast intro: } card\text{-mono } finite\text{-subset})$   
**show**  $x \text{ (MMD } Is \ P') \prec y$   
**proof**  
**from**  $xRSCFy \ xPyxP'yC \ yP'xyPxC$  **show**  $x \text{ (MMD } Is \ P') \preceq y$   
**unfolding** *MMD-def* **by** *auto*  
**next**

```

{
  assume  $xIky: x (P k) \approx y$  and  $xP'ky: x (P' k) \prec y$ 
  have  $\text{card } ?xPy < \text{card } ?xP'y$ 
  proof -
    from  $xIky$  have  $knP: k \notin ?xPy$ 
    unfolding indifferent-pref-def strict-pref-def by blast
    from  $kIs xP'ky$  have  $kP': k \in ?xP'y$  by simp
    from finiteIs xPy knP kP' show ?thesis
    by (blast intro: psubset-card-mono finite-subset)
  qed
  with  $xRSCFy yP'xyPx C$  have  $\text{card } ?yP'x < \text{card } ?xP'y$ 
  unfolding MMD-def by auto
}
moreover
{
  assume  $yP'kx: y (P k) \prec x$  and  $xR'ky: x (P' k) \preceq y$ 
  have  $\text{card } ?yP'x < \text{card } ?yPx$ 
  proof -
    from  $kIs yP'kx$  have  $kP: k \in ?yPx$  by simp
    from  $kIs xR'ky$  have  $knP': k \notin ?yP'x$ 
    unfolding strict-pref-def by blast
    from  $yP'xyPx kP knP'$  have  $?yP'x \subset ?yPx$  by blast
    with finiteIs show ?thesis
    by (blast intro: psubset-card-mono finite-subset)
  qed
  with  $xRSCFy xPyxP'y C$  have  $\text{card } ?yP'x < \text{card } ?xP'y$ 
  unfolding MMD-def by auto
}
moreover note kcond
ultimately show  $\neg(y (MMD Is P') \preceq x)$ 
unfolding MMD-def by auto
qed
qed
qed

```

### 6.3 Everything satisfying May's conditions is the Method of Majority Decision

Now show that MMD is the only SCF that satisfies these conditions.

Firstly develop some theory about exchanging alternatives  $x$  and  $y$  in profile  $P$ .

**definition** *swapAlts* ::  $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a$  **where**  
 $\text{swapAlts } a b u \equiv \text{if } u = a \text{ then } b \text{ else if } u = b \text{ then } a \text{ else } u$

**lemma** *swapAlts-in-set-iff*:  $\{a, b\} \subseteq A \implies \text{swapAlts } a b u \in A \iff u \in A$   
 unfolding *swapAlts-def* by (*simp split: if-split*)

**definition** *swapAltsP* ::  $('a, 'i)$  Profile  $\Rightarrow 'a \Rightarrow 'a \Rightarrow ('a, 'i)$  Profile **where**  
 $\text{swapAltsP } P a b \equiv (\lambda i. \{ (u, v) . (\text{swapAlts } a b u, \text{swapAlts } a b v) \in P i \})$

**lemma** *swapAltsP-ab*:  $a (P i) \preceq b \iff b (\text{swapAltsP } P a b i) \preceq a b (P i) \preceq a \iff a (\text{swapAltsP } P a b i) \preceq b$

**unfolding** *swapAltsP-def swapAlts-def* **by** *simp-all*

**lemma** *profile-swapAltsP*:

**assumes** *profileP*: *profile A Is P*

**and** *abA*:  $\{a, b\} \subseteq A$

**shows** *profile A Is (swapAltsP P a b)*

**proof**(*rule profileI*)

**from** *profileP* **show**  $Is \neq \{\}$  **by** (*rule profile-non-empty*)

**next**

**fix** *i* **assume** *iIs*:  $i \in Is$

**show** *rpr A (swapAltsP P a b i)*

**proof**(*rule rprI*)

**show** *refl-on A (swapAltsP P a b i)*

**proof**(*rule refl-onI*)

**from** *profileP iIs abA* **show**  $swapAltsP P a b i \subseteq A \times A$

**unfolding** *swapAltsP-def* **by** (*blast dest: swapAlts-in-set-iff*)

**from** *profileP iIs abA* **show**  $\bigwedge x. x \in A \implies x (swapAltsP P a b i) \preceq x$

**unfolding** *swapAltsP-def swapAlts-def* **by** *auto*

**qed**

**next**

**from** *profileP iIs abA* **show** *complete A (swapAltsP P a b i)*

**unfolding** *swapAltsP-def*

**by** – (*rule completeI, simp, rule rpr-complete[where A=A],*

*auto iff: swapAlts-in-set-iff*)

**next**

**from** *profileP iIs* **show** *trans (swapAltsP P a b i)*

**unfolding** *swapAltsP-def* **by** (*blast dest: rpr-le-trans intro: transI*)

**qed**

**qed**

**lemma** *profile-bij-profile*:

**assumes** *profileP*: *profile A Is P*

**and** *bijf*: *bij-betw f Is Is*

**shows** *profile A Is (P o f)*

**using** *bij-betw-onto[OF bijf]* *profileP*

**by** – (*rule, auto dest: profile-non-empty*)

The locale keeps the conditions in scope for the next few lemmas. Note how weak the constraints on the sets of alternatives and individuals are; clearly there needs to be at least two alternatives and two individuals for conflict to occur, but it is pleasant that the proof uniformly handles the degenerate cases.

**locale** *May* =

**fixes** *A* :: *'a set*

**fixes** *Is* :: *'i set*

**assumes** *finiteIs*: *finite Is*

**fixes** *scf* :: (*'a, 'i*) *SCF*

**assumes** *SCF*: *SCF scf A Is universal-domain*

**and** *anonymous*: *anonymous scf A Is*

**and** *neutral*: *neutral scf A Is*

**and** *positively-responsive*: *positively-responsive scf A Is*

**begin**

Anonymity implies that, for any pair of alternatives, the social choice rule can only depend on the number of individuals who express any given preference between them. Note we also need *ia*, implied by neutrality, to restrict attention to alternatives  $x$  and  $y$ .

**lemma** *anonymous-card*:

**assumes** *profileP*: *profile*  $A$   $Is$   $P$

**and** *profileP'*: *profile*  $A$   $Is$   $P'$

**and** *xyA*: *hasw*  $[x,y]$   $A$

**and** *xytally*:  $\text{card } \{ i \in Is. x (P\ i) \prec y \} = \text{card } \{ i \in Is. x (P'\ i) \prec y \}$

**and** *yxtally*:  $\text{card } \{ i \in Is. y (P\ i) \prec x \} = \text{card } \{ i \in Is. y (P'\ i) \prec x \}$

**shows**  $x (scf\ P) \preceq y \longleftrightarrow x (scf\ P') \preceq y$

**proof** –

**let**  $?xPy = \{ i \in Is. x (P\ i) \prec y \}$

**let**  $?xP'y = \{ i \in Is. x (P'\ i) \prec y \}$

**let**  $?yPx = \{ i \in Is. y (P\ i) \prec x \}$

**let**  $?yP'x = \{ i \in Is. y (P'\ i) \prec x \}$

**have** *disjPxy*:  $(?xPy \cup ?yPx) - ?xPy = ?yPx$

**unfolding** *strict-pref-def* **by** *blast*

**have** *disjP'xy*:  $(?xP'y \cup ?yP'x) - ?xP'y = ?yP'x$

**unfolding** *strict-pref-def* **by** *blast*

**from** *finiteIs* *xytally*

**obtain**  $f$  **where** *bijf*: *bij-betw*  $f$   $?xPy$   $?xP'y$

**by** – (*drule* *card-eq-bij*, *auto*)

**from** *finiteIs* *yxtally*

**obtain**  $g$  **where** *bijg*: *bij-betw*  $g$   $?yPx$   $?yP'x$

**by** – (*drule* *card-eq-bij*, *auto*)

**from** *bijf* *bijg* *disjPxy* *disjP'xy*

**obtain**  $h$

**where** *bijh*: *bij-betw*  $h$   $(?xPy \cup ?yPx)$   $(?xP'y \cup ?yP'x)$

**and** *hf*:  $\bigwedge j. j \in ?xPy \implies h\ j = f\ j$

**and** *hg*:  $\bigwedge j. j \in (?xPy \cup ?yPx) - ?xPy \implies h\ j = g\ j$

**using** *bij-combine*[**where**  $f=f$  **and**  $g=g$  **and**  $A=?xPy$  **and**  $B=?xPy \cup ?yPx$  **and**  $C=?xP'y$  **and**  $D=?xP'y \cup ?yP'x$ ]

**by** *auto*

**from** *bijh* *finiteIs*

**obtain**  $h'$  **where** *bijh'*: *bij-betw*  $h'$   $Is$   $Is$

**and** *hh'*:  $\bigwedge j. j \in (?xPy \cup ?yPx) \implies h'\ j = h\ j$

**and** *hrest*:  $\bigwedge j. j \in Is - (?xPy \cup ?yPx) \implies h'\ j \in Is - (?xP'y \cup ?yP'x)$

**by** – (*drule* *bij-complete*, *auto*)

**from** *neutral-ia*[*OF* *neutral*]

**have**  $x (scf\ (P' \circ h')) \preceq y \longleftrightarrow x (scf\ P) \preceq y$

**proof**(*rule* *iaE*)

**from** *xyA* **show**  $\{x, y\} \subseteq A$  **by** *simp*

**next**

**fix**  $i$  **assume** *iIs*:  $i \in Is$

**fix**  $a\ b$  **assume** *ab*:  $a \in \{x, y\}$   $b \in \{x, y\}$

**from** *profileP* *iIs* **have** *completePi*: *complete*  $A$   $(P\ i)$  **by** (*auto* *dest*: *rprD*)

**from** *completePi* *xyA*

**show**  $(a (P\ i) \preceq b) \longleftrightarrow (a ((P' \circ h')\ i) \preceq b)$

**proof**(*cases* *rule*: *complete-exh*)

**case**  $xPy$  **with**  $profileP$   $profileP'$   $xyA$   $iIs$   $ab$   $hh'$   $hf$   $bijf$  **show**  $?thesis$   
**unfolding**  $strict-pref-def$   $bij-betw-def$  **by**  $(simp, blast)$   
**next**  
**case**  $yPx$  **with**  $profileP$   $profileP'$   $xyA$   $iIs$   $ab$   $hh'$   $hg$   $bijg$  **show**  $?thesis$   
**unfolding**  $strict-pref-def$   $bij-betw-def$  **by**  $(simp, blast)$   
**next**  
**case**  $xIy$  **with**  $profileP$   $profileP'$   $xyA$   $iIs$   $ab$   $hrest$  [**where**  $j=i$ ] **show**  $?thesis$   
**unfolding**  $indifferent-pref-def$   $strict-pref-def$   $bij-betw-def$   
**by**  $(simp, blast$   $dest: rpr-complete)$   
**qed**  
**qed**  $(simp-all$   $add: profileP$   $profile-bij-profile[OF$   $profileP'$   $bijh$   $])$   
**moreover**  
**from**  $anonymousD[OF$   $anonymous$   $profileP'$   $bijh$   $]$   $xyA$   
**have**  $x$   $(scf$   $P') \preceq y \iff x$   $(scf$   $(P' \circ h')) \preceq y$  **by**  $simp$   
**ultimately show**  $?thesis$  **by**  $simp$   
**qed**

Using the previous result and neutrality, it must be the case that if the tallies are tied for alternatives  $x$  and  $y$  then the social choice function is indifferent between those two alternatives.

**lemma** *anonymous-neutral-indifference*:

**assumes**  $profileP$ :  $profile$   $A$   $Is$   $P$   
**and**  $xyA$ :  $hasw$   $[x,y]$   $A$   
**and**  $tallyP$ :  $card$   $\{ i \in Is. x$   $(P$   $i) \prec y \} = card$   $\{ i \in Is. y$   $(P$   $i) \prec x \}$   
**shows**  $x$   $(scf$   $P) \approx y$   
**proof** –  
– Neutrality insists the results for  $P$  are symmetrical to those for  $swapAltsP$   $P$ .  
**from**  $xyA$   
**have**  $symPP'$ :  $(x$   $(scf$   $P) \preceq y \iff y$   $(scf$   $(swapAltsP$   $P$   $x$   $y)) \preceq x)$   
 $\wedge$   $(y$   $(scf$   $P) \preceq x \iff x$   $(scf$   $(swapAltsP$   $P$   $x$   $y)) \preceq y)$   
**by** –  $(rule$   $neutralD[OF$   $neutral$   $profileP$   $profile-swapAltsP[OF$   $profileP]$   $],$   
 $simp-all, (rule$   $swapAltsP-ab)+)$   
– Anonymity and neutrality insist the results for  $P$  are identical to those for  $swapAltsP$   $P$ .  
**from**  $xyA$   $tallyP$  **have**  $card$   $\{ i \in Is. x$   $(P$   $i) \prec y \} = card$   $\{ i \in Is. x$   $(swapAltsP$   $P$   $x$   $y$   $i) \prec y \}$   
**and**  $card$   $\{ i \in Is. y$   $(P$   $i) \prec x \} = card$   $\{ i \in Is. y$   $(swapAltsP$   $P$   $x$   $y$   $i) \prec x \}$   
**unfolding**  $swapAltsP-def$   $swapAlts-def$   $strict-pref-def$  **by**  $simp-all$   
**with**  $profileP$   $xyA$  **have**  $idPP'$ :  $x$   $(scf$   $P) \preceq y \iff x$   $(scf$   $(swapAltsP$   $P$   $x$   $y)) \preceq y$   
**and**  $y$   $(scf$   $P) \preceq x \iff y$   $(scf$   $(swapAltsP$   $P$   $x$   $y)) \preceq x$   
**by** –  $(rule$   $anonymous-card[OF$   $profileP$   $profile-swapAltsP], clarsimp+)+$   
**from**  $xyA$   $SCF-completeD[OF$   $SCF]$   $profileP$   $symPP'$   $idPP'$  **show**  $x$   $(scf$   $P) \approx y$  **by**  $(simp, blast)$   
**qed**

Finally, if the tallies are not equal then the social choice function must lean towards the one with the higher count due to positive responsiveness.

**lemma** *positively-responsive-prefer-witness*:

**assumes**  $profileP$ :  $profile$   $A$   $Is$   $P$   
**and**  $xyA$ :  $hasw$   $[x,y]$   $A$   
**and**  $tallyP$ :  $card$   $\{ i \in Is. x$   $(P$   $i) \prec y \} > card$   $\{ i \in Is. y$   $(P$   $i) \prec x \}$   
**obtains**  $P'$   $k$   
**where**  $profile$   $A$   $Is$   $P'$   
**and**  $\bigwedge i. \llbracket i \in Is; x$   $(P'$   $i) \prec y \rrbracket \implies x$   $(P$   $i) \prec y$

**and**  $\bigwedge i. \llbracket i \in Is; x (P' i) \approx y \rrbracket \implies x (P i) \preceq y$   
**and**  $k \in Is \wedge x (P' k) \approx y \wedge x (P k) \prec y$   
**and**  $\text{card} \{ i \in Is. x (P' i) \prec y \} = \text{card} \{ i \in Is. y (P' i) \prec x \}$

**proof** –

**from** *tallyP* **obtain**  $C$

**where** *tallyP'*:  $\text{card} (\{ i \in Is. x (P i) \prec y \} - C) = \text{card} \{ i \in Is. y (P i) \prec x \}$

**and**  $C: C \neq \{ \}$   $C \subseteq Is$

**and**  $CxPy: C \subseteq \{ i \in Is. x (P i) \prec y \}$

**by** – (*drule card-greater[OF finiteIs], auto*)

– Add  $(b, a)$  and close under transitivity.

**let**  $?P' = \lambda i. \text{if } i \in C$

*then*  $P i \cup \{ (y, x) \}$

$\cup \{ (y, u) \mid u. x (P i) \preceq u \}$

$\cup \{ (u, x) \mid u. u (P i) \preceq y \}$

$\cup \{ (v, u) \mid u v. x (P i) \preceq u \wedge v (P i) \preceq y \}$

*else*  $P i$

**have** *profile A Is ?P'*

**proof**

**fix**  $i$  **assume**  $i \in Is$

**show** *rpr A (?P' i)*

**proof**

**from** *profileP iIs* **show** *complete A (?P' i)*

**unfolding** *complete-def* **by** (*simp, blast dest: rpr-complete*)

**from** *profileP iIs xyA* **show** *refl-on A (?P' i)*

**by** – (*rule refl-onI, auto*)

**show** *trans (?P' i)*

**proof**(*cases i \in C*)

**case** *False* **with** *profileP iIs* **show** *?thesis*

**by** (*simp, blast dest: rpr-le-trans intro: transI*)

**next**

**case** *True* **with** *profileP iIs C CxPy xyA* **show** *?thesis*

**unfolding** *strict-pref-def*

**by** – (*rule transI, simp, blast dest: rpr-le-trans rpr-complete*)

**qed**

**qed**

**next**

**from**  $C$  **show**  $Is \neq \{ \}$  **by** *blast*

**qed**

**moreover**

**have**  $\bigwedge i. \llbracket i \in Is; x (?P' i) \prec y \rrbracket \implies x (P i) \prec y$

**unfolding** *strict-pref-def* **by** (*simp split: if-split-asm*)

**moreover**

**from** *profileP C xyA*

**have**  $\bigwedge i. \llbracket i \in Is; x (?P' i) \approx y \rrbracket \implies x (P i) \preceq y$

**unfolding** *indifferent-pref-def* **by** (*simp split: if-split-asm*)

**moreover**

**from**  $C$   $CxPy$  **obtain**  $k$  **where**  $kC: k \in C$  **and**  $xPky: x (P k) \prec y$  **by** *blast*

**hence**  $x (?P' k) \approx y$  **by** *auto*

**with**  $C$   $kC$   $xPky$  **have**  $k \in Is \wedge x (?P' k) \approx y \wedge x (P k) \prec y$  **by** *blast*

**moreover**

**have**  $\text{card} \{ i \in Is. x (?P' i) \prec y \} = \text{card} \{ i \in Is. y (?P' i) \prec x \}$

**proof** –  
**have**  $\{ i \in Is. x \text{ } (?P' i) \prec y \} = \{ i \in Is. x \text{ } (?P' i) \prec y \} - C$   
**proof** –  
**from**  $C$  **have**  $\bigwedge i. \llbracket i \in Is; x \text{ } (?P' i) \prec y \rrbracket \implies i \in Is - C$   
**unfolding** *indifferent-pref-def strict-pref-def* **by** *auto*  
**thus** *?thesis* **by** *blast*  
**qed**  
**also have**  $\dots = \{ i \in Is. x \text{ } (P i) \prec y \} - C$  **by** *auto*  
**finally have**  $\text{card } \{ i \in Is. x \text{ } (?P' i) \prec y \} = \text{card } (\{ i \in Is. x \text{ } (P i) \prec y \} - C)$   
**by** *simp*  
**with** *tallyP'* **have**  $\text{card } \{ i \in Is. x \text{ } (?P' i) \prec y \} = \text{card } \{ i \in Is. y \text{ } (P i) \prec x \}$   
**by** *simp*  
**also have**  $\dots = \text{card } \{ i \in Is. y \text{ } (?P' i) \prec x \}$  (**is**  $\text{card } ?lhs = \text{card } ?rhs$ )  
**proof** –  
**from** *profileP xyA* **have**  $\bigwedge i. \llbracket i \in Is; y \text{ } (?P' i) \prec x \rrbracket \implies y \text{ } (P i) \prec x$   
**unfolding** *strict-pref-def* **by** (*simp split: if-split-asm, blast dest: rpr-complete*)  
**hence**  $?rhs \subseteq ?lhs$  **by** *blast*  
**moreover**  
**from** *profileP xyA* **have**  $\bigwedge i. \llbracket i \in Is; y \text{ } (P i) \prec x \rrbracket \implies y \text{ } (?P' i) \prec x$   
**unfolding** *strict-pref-def* **by** *simp*  
**hence**  $?lhs \subseteq ?rhs$  **by** *blast*  
**ultimately show** *?thesis* **by** *simp*  
**qed**  
**finally show** *?thesis* .  
**qed**  
**ultimately show** *thesis* ..  
**qed**

**lemma** *positively-responsive-prefer*:  
**assumes** *profileP: profile A Is P*  
**and** *xyA: hasw [x,y] A*  
**and** *tallyP: card { i \in Is. x (P i) \prec y } > card { i \in Is. y (P i) \prec x }*  
**shows**  $x \text{ } (scf P) \prec y$   
**proof** –  
**from** *assms* **obtain**  $P' k$   
**where** *profileP': profile A Is P'*  
**and**  $F: \bigwedge i. \llbracket i \in Is; x \text{ } (P' i) \prec y \rrbracket \implies x \text{ } (P i) \prec y$   
**and**  $G: \bigwedge i. \llbracket i \in Is; x \text{ } (P' i) \approx y \rrbracket \implies x \text{ } (P i) \preceq y$   
**and** *pivot: k \in Is \wedge x (P' k) \approx y \wedge x (P k) \prec y*  
**and**  $\text{card}P': \text{card } \{ i \in Is. x \text{ } (P' i) \prec y \} = \text{card } \{ i \in Is. y \text{ } (P' i) \prec x \}$   
**by** – (*drule positively-responsive-prefer-witness, auto*)  
**from** *profileP' xyA cardP'* **have**  $x \text{ } (scf P') \approx y$   
**by** – (*rule anonymous-neutral-indifference, auto*)  
**with** *xyA F G pivot* **show** *?thesis*  
**by** – (*rule positively-responsiveD[OF positively-responsive profileP' profileP], auto*)  
**qed**

**lemma** *MMD-r2l*:  
**assumes** *profileP: profile A Is P*  
**and** *xyA: hasw [x,y] A*

```

shows  $x \text{ (scf } P) \preceq y \iff x \text{ (MMD } Is P) \preceq y$ 
proof(cases rule: linorder-cases)
  assume  $\text{card } \{ i \in Is. x \text{ (} P i) \prec y \} = \text{card } \{ i \in Is. y \text{ (} P i) \prec x \}$ 
  with profileP xyA show ?thesis
    using anonymous-neutral-indifference
    unfolding indifferent-pref-def MMD-def by simp
next
  assume  $\text{card } \{ i \in Is. x \text{ (} P i) \prec y \} > \text{card } \{ i \in Is. y \text{ (} P i) \prec x \}$ 
  with profileP xyA show ?thesis
    using positively-responsive-prefer
    unfolding strict-pref-def MMD-def by simp
next
  assume  $\text{card } \{ i \in Is. x \text{ (} P i) \prec y \} < \text{card } \{ i \in Is. y \text{ (} P i) \prec x \}$ 
  with profileP xyA show ?thesis
    using positively-responsive-prefer
    unfolding strict-pref-def MMD-def by clarsimp
qed

end

```

May's original paper [May52] goes on to show that the conditions are independent by exhibiting choice rules that differ from *MMD* and satisfy the conditions remaining after any particular one is removed. I leave this to future work.

May also wrote a later article [May53] where he shows that the conditions are completely independent, i.e. for every partition of the conditions into two sets, there is a voting rule that satisfies one and not the other.

There are many later papers that characterise *MMD* with different sets of conditions.

## 6.4 The Plurality Rule

Goodin and List [GL06] show that May's original result can be generalised to characterise plurality voting. The following shows that this result is a short step from Sen's much earlier generalisation.

*Plurality voting* is a choice function that returns the alternative that receives the most votes, or the set of such alternatives in the case of a tie. Profiles are restricted to those where each individual casts a vote in favour of a single alternative.

**type-synonym**  $( 'a, 'i) \text{ SVProfile} = 'i \Rightarrow 'a$

**definition** *svprofile* ::  $'a \text{ set} \Rightarrow 'i \text{ set} \Rightarrow ('a, 'i) \text{ SVProfile} \Rightarrow \text{bool}$  **where**  
*svprofile*  $A \text{ Is } F \equiv \text{Is} \neq \{ \} \wedge F ' \text{Is} \subseteq A$

**definition** *plurality-rule* ::  $'a \text{ set} \Rightarrow 'i \text{ set} \Rightarrow ('a, 'i) \text{ SVProfile} \Rightarrow 'a \text{ set}$  **where**  
*plurality-rule*  $A \text{ Is } F$   
 $\equiv \{ x \in A . \forall y \in A. \text{card } \{ i \in \text{Is} . F i = x \} \geq \text{card } \{ i \in \text{Is} . F i = y \} \}$

By translating single-vote profiles into RPRs in the obvious way, the choice function arising from *MMD* coincides with traditional plurality voting.

**definition** *MMD-plurality-rule* ::  $'a \text{ set} \Rightarrow 'i \text{ set} \Rightarrow ('a, 'i) \text{ Profile} \Rightarrow 'a \text{ set}$  **where**  
*MMD-plurality-rule*  $A \text{ Is } P \equiv \text{choiceSet } A \text{ (MMD } Is P)$

**definition** *single-vote-to-RPR* :: 'a set  $\Rightarrow$  'a  $\Rightarrow$  'a RPR **where**  
*single-vote-to-RPR* A a  $\equiv \{ (a, x) \mid x. x \in A \} \cup (A - \{a\}) \times (A - \{a\})$

**lemma** *single-vote-to-RPR-iff*:  
 $\llbracket a \in A; x \in A; a \neq x \rrbracket \Longrightarrow (a \text{ (single-vote-to-RPR } A \ b) \prec x) \longleftrightarrow (b = a)$   
**unfolding** *single-vote-to-RPR-def strict-pref-def* **by** *auto*

**lemma** *plurality-rule-equiv*:  
*plurality-rule* A Is F = *MMD-plurality-rule* A Is (single-vote-to-RPR A  $\circ$  F)

**proof** –

```
{
  fix x y
  have  $\llbracket x \in A; y \in A \rrbracket \Longrightarrow$ 
    (card {i  $\in$  Is. F i = y}  $\leq$  card {i  $\in$  Is. F i = x}) =
    (card {i  $\in$  Is. y (single-vote-to-RPR A (F i))  $\prec$  x}
      $\leq$  card {i  $\in$  Is. x (single-vote-to-RPR A (F i))  $\prec$  y})
  by (cases x=y, auto iff: single-vote-to-RPR-iff)
}
```

**thus** *?thesis*  
**unfolding** *plurality-rule-def MMD-plurality-rule-def choiceSet-def MMD-def*  
**by** *auto*

**qed**

Thus it is clear that Sen's generalisation of May's result applies to this case as well.

Their paper goes on to show how strengthening the anonymity condition gives rise to a characterisation of approval voting that strictly generalises May's original theorem. As this requires some rearrangement of the proof I leave it to future work.

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