

Secondary Sylow Theorems

Jakob von Raumer

December 14, 2021

Abstract

These theories extend the existent proof of the first sylow theorem (written by Florian Kammüller and L. C. Paulson) by what is often called the second, third and fourth sylow theorem. These theorems state propositions about the number of Sylow p -subgroups of a group and the fact that they are conjugate to each other. The proofs make use of an implementation of group actions and their properties.

Contents

1	Group Actions	1
1.1	Preliminaries and Definition	1
1.2	The orbit relation	4
1.3	Stabilizer and fixed points	6
1.4	The Orbit-Stabilizer Theorem	7
1.5	Some Examples for Group Actions	13
2	Conjugation of Subgroups and Cosets	16
2.1	Definitions and Preliminaries	16
2.2	Conjugation is a group action	17
2.3	Properties of the Conjugation Action	23
3	The Secondary Sylow Theorems	24
3.1	Preliminaries	24
3.2	Extending the Sylow Locale	25
3.3	Every p -group is Contained in a conjugate of a p -Sylow-Group	26
3.4	Every p -Group is Contained in a p -Sylow-Group	27
3.5	p -Sylow-Groups are conjugates of each other	28
3.6	Counting Sylow-Groups	29

```
theory GroupAction
imports
  HOL-Algebra.Bij
```

HOL-Algebra.Sylow
begin

1 Group Actions

This is an implementation of group actions based on the group implementation of HOL-Algebra. An action a group G on a set M is represented by a group homomorphism between G and the group of bijections on M

1.1 Preliminaries and Definition

First, we need two theorems about singletons and sets of singletons which unfortunately are not included in the library.

theorem *singleton-intersection:*

assumes $A: \text{card } A = 1$

assumes $B: \text{card } B = 1$

assumes *noteq*: $A \neq B$

shows $A \cap B = \{\}$

using *assms* **by** (*auto simp: card-Suc-eq*)

theorem *card-singleton-set:*

assumes *cardOne*: $\forall x \in A. (\text{card } x = 1)$

shows $\text{card } (\bigcup A) = \text{card } A$

proof –

have $\text{card } (\bigcup A) = (\sum_{x \in A} \text{card } x)$

proof (*rule card-Union-disjoint*)

from *cardOne* **show** $\bigwedge a. a \in A \implies \text{finite } a$ **by** (*auto intro: card-ge-0-finite*)

next

show *pairwise disjnt* A

unfolding *pairwise-def disjnt-def*

proof (*clarify*)

fix $x y$

assume $x: x \in A$ **and** $y: y \in A$ **and** $x \neq y$

with *cardOne* **have** $\text{card } x = 1$ $\text{card } y = 1$ **by** *auto*

with $\langle x \neq y \rangle$ **show** $x \cap y = \{\}$ **by** (*metis singleton-intersection*)

qed

qed

also from *cardOne* **have** $\dots = \text{card } A$ **by** *simp*

finally show *?thesis*.

qed

Intersecting Cosets are equal:

lemma (*in subgroup*) *repr-independence2:*

assumes *group*: *group* G

assumes $U: U \in \text{rcosets}_G H$

assumes $g: g \in U$

shows $U = H \#> g$

proof –
from U **obtain** h **where** $h: h \in \text{carrier } G \ U = H \ \#> \ h$ **unfolding** *RCOSETS-def*
by *auto*
with g **have** $g \in H \ \#> \ h$ **by** *simp*
with *group* h **show** $U = H \ \#> \ g$ **by** (*metis group.repr-independence is-subgroup*)
qed

locale *group-action* = *group* +
fixes $\varphi \ M$
assumes *grouphom*: *group-hom* $G \ (\text{BijGroup } M) \ \varphi$

context *group-action*
begin

lemma *is-group-action*: *group-action* $G \ \varphi \ M$.

The action of $\mathbf{1}$ has no effect:

lemma *one-is-id*:
assumes $m \in M$
shows $(\varphi \ \mathbf{1}) \ m = m$
proof –
from *grouphom* **have** $(\varphi \ \mathbf{1}) \ m = \mathbf{1}_{(\text{BijGroup } M)} \ m$ **by** (*metis group-hom.hom-one*)
also **have** $\dots = (\lambda x \in M. x) \ m$ **unfolding** *BijGroup-def* **by** (*metis monoid.select-convs(2)*)
also **from** *assms* **have** $\dots = m$ **by** *simp*
finally **show** *?thesis*.
qed

lemma *action-closed*:
assumes $m: m \in M$
assumes $g: g \in \text{carrier } G$
shows $\varphi \ g \ m \in M$
using *assms* *grouphom* *group-hom.hom-closed* **unfolding** *BijGroup-def* *Bij-def* *bij-betw-def*
by *fastforce*

lemma *img-in-bij*:
assumes $g \in \text{carrier } G$
shows $(\varphi \ g) \in \text{Bij } M$
using *assms* *grouphom* **unfolding** *BijGroup-def* **by** (*auto dest: group-hom.hom-closed*)

The action of *inv* g reverts the action of g

lemma *group-inv-rel*:
assumes $g: g \in \text{carrier } G$
assumes $mn: m \in M \ n \in M$
assumes $\text{phi}: (\varphi \ g) \ n = m$
shows $(\varphi \ (\text{inv } g)) \ m = n$
proof –
from g **have** $\text{bij}: (\varphi \ g) \in \text{Bij } M$ **unfolding** *BijGroup-def* **by** (*metis img-in-bij*)
with g *grouphom* **have** $\varphi \ (\text{inv } g) = \text{restrict } (\text{inv-into } M \ (\varphi \ g)) \ M$ **by** (*metis inv-BijGroup group-hom.hom-inv*)

hence $\varphi (\text{inv } g) m = (\text{restrict } (\text{inv-into } M (\varphi g)) M) m$ **by** *simp*
 also from *mn* have $\dots = (\text{inv-into } M (\varphi g)) m$ **by** (*metis restrict-def*)
 also from *g phi* have $\dots = (\text{inv-into } M (\varphi g)) ((\varphi g) n)$ **by** *simp*
 also from $\langle \varphi g \in \text{Bij } M \rangle$ *Bij-def* have *bij-betw* $(\varphi g) M M$ **by** *auto*
 hence *inj-on* $(\varphi g) M$ **by** (*metis bij-betw-imp-inj-on*)
 with *g mn* have $(\text{inv-into } M (\varphi g)) ((\varphi g) n) = n$ **by** (*metis inv-into-f-f*)
 finally show $\varphi (\text{inv } g) m = n$.
qed

lemma *images-are-bij*:

assumes $g: g \in \text{carrier } G$
 shows *bij-betw* $(\varphi g) M M$

proof –

from *g* have *bij*: $(\varphi g) \in \text{Bij } M$ **unfolding** *BijGroup-def* **by** (*metis img-in-bij*)
 with *Bij-def* show *bij-betw* $(\varphi g) M M$ **by** *auto*

qed

lemma *action-mult*:

assumes $g: g \in \text{carrier } G$
 assumes $h: h \in \text{carrier } G$
 assumes $m: m \in M$
 shows $(\varphi g) ((\varphi h) m) = (\varphi (g \otimes h)) m$

proof –

from *g* have $\varphi g: (\varphi g) \in \text{Bij } M$ **unfolding** *BijGroup-def* **by** (*rule img-in-bij*)
 from *h* have $\varphi h: (\varphi h) \in \text{Bij } M$ **unfolding** *BijGroup-def* **by** (*rule img-in-bij*)
 from *h* have *bij-betw* $(\varphi h) M M$ **by** (*rule images-are-bij*)
 hence $(\varphi h) \text{ ` } M = M$ **by** (*metis bij-betw-def*)
 with *m* have *hm*: $(\varphi h) m \in M$ **by** (*metis imageI*)
 from *group-hom* *g h* have $(\varphi (g \otimes h)) = ((\varphi g) \otimes_{(\text{BijGroup } M)} (\varphi h))$ **by** (*rule group-hom.hom-mult*)
 hence $\varphi (g \otimes h) m = ((\varphi g) \otimes_{(\text{BijGroup } M)} (\varphi h)) m$ **by** *simp*
 also from $\varphi g \varphi h$ have $\dots = (\text{compose } M (\varphi g) (\varphi h)) m$ **unfolding** *BijGroup-def*
by *simp*
 also from $\varphi g \varphi h \text{ hm}$ have $\dots = (\varphi g) ((\varphi h) m)$ **by** (*metis compose-eq m*)
 finally show $(\varphi g) ((\varphi h) m) = (\varphi (g \otimes h)) m$.
qed

1.2 The orbit relation

The following describes the relation containing the information whether two elements of M lie in the same orbit of the action

definition *same-orbit-rel*

where *same-orbit-rel* = $\{p \in M \times M. \exists g \in \text{carrier } G. (\varphi g) (\text{snd } p) = (\text{fst } p)\}$

Use the library about equivalence relations to define the set of orbits and the map assigning to each element of M its orbit

definition *orbits*

where *orbits* = $M // \text{same-orbit-rel}$

definition *orbit* :: 'c ⇒ 'c set
where *orbit* m = *same-orbit-rel* “ {m}

Next, we define a more easy-to-use characterization of an orbit.

lemma *orbit-char*:
assumes *m*:*m* ∈ *M*
shows *orbit* m = {*n*. ∃ *g*. *g* ∈ *carrier* *G* ∧ (φ *g*) *m* = *n*}
using *assms* **unfolding** *orbit-def* *Image-def* *same-orbit-rel-def*
proof(*auto*)
fix *x g*
assume *g*:*g* ∈ *carrier* *G* **and** φ *g* *x* ∈ *M* *x* ∈ *M*
hence φ (*inv* *g*) (φ *g* *x*) = *x* **by** (*metis* *group-inv-rel*)
moreover from *g* **have** *inv* *g* ∈ *carrier* *G* **by** (*rule* *inv-closed*)
ultimately show ∃ *h*. *h* ∈ *carrier* *G* ∧ φ *h* (φ *g* *x*) = *x* **by** *auto*
next
fix *g*
assume *g*:*g* ∈ *carrier* *G*
with *m* **show** φ *g* *m* ∈ *M* **by** (*metis* *action-closed*)
with *m g* **have** φ (*inv* *g*) (φ *g* *m*) = *m* **by** (*metis* *group-inv-rel*)
moreover from *g* **have** *inv* *g* ∈ *carrier* *G* **by** (*rule* *inv-closed*)
ultimately show ∃ *h* ∈ *carrier* *G*. φ *h* (φ *g* *m*) = *m* **by** *auto*
qed

lemma *same-orbit-char*:
assumes *m* ∈ *M* *n* ∈ *M*
shows (*m*, *n*) ∈ *same-orbit-rel* = (∃ *g* ∈ *carrier* *G*. ((φ *g*) *n* = *m*))
unfolding *same-orbit-rel-def* **using** *assms* **by** *auto*

Now we show that the relation we’ve defined is, indeed, an equivalence relation:

lemma *same-orbit-is-equiv*:
shows *equiv* *M* *same-orbit-rel*
proof(*rule* *equivI*)
show *refl-on* *M* *same-orbit-rel*
proof(*rule* *refl-onI*)
show *same-orbit-rel* ⊆ *M* × *M* **unfolding** *same-orbit-rel-def* **by** *auto*
next
fix *m*
assume *m* ∈ *M*
hence (φ **1**) *m* = *m* **by**(*rule* *one-is-id*)
with ⟨*m* ∈ *M*⟩ **show** (*m*, *m*) ∈ *same-orbit-rel* **unfolding** *same-orbit-rel-def*
by (*auto simp:same-orbit-char*)
qed
next
show *sym* *same-orbit-rel*
proof(*rule* *symI*)
fix *m n*
assume *mn*:(*m*, *n*) ∈ *same-orbit-rel*

```

then obtain  $g$  where  $g:g \in \text{carrier } G$   $\varphi g n = m$  unfolding same-orbit-rel-def
by auto
  hence  $\text{inv}g:\text{inv } g \in \text{carrier } G$  by (metis inv-closed)
  from  $mn$  have  $(m, n) \in M \times M$  unfolding same-orbit-rel-def by simp
  hence  $mn2:m \in M$   $n \in M$  by auto
  from  $g$   $mn2$  have  $\varphi (\text{inv } g) m = n$  by (metis group-inv-rel)
  with  $\text{inv}g$   $mn2$  show  $(n, m) \in \text{same-orbit-rel}$  unfolding same-orbit-rel-def by
auto
qed
next
  show trans same-orbit-rel
  proof(rule transI)
    fix  $x y z$ 
    assume  $xy:(x, y) \in \text{same-orbit-rel}$ 
    then obtain  $g$  where  $g:g \in \text{carrier } G$  and  $grel:(\varphi g) y = x$  unfolding
same-orbit-rel-def by auto
    assume  $yz:(y, z) \in \text{same-orbit-rel}$ 
    then obtain  $h$  where  $h:h \in \text{carrier } G$  and  $hrel:(\varphi h) z = y$  unfolding
same-orbit-rel-def by auto
    from  $g$   $h$  have  $gh:g \otimes h \in \text{carrier } G$  by simp
    from  $xy$   $yz$  have  $x \in M$   $z \in M$  unfolding same-orbit-rel-def by auto
    with  $g$   $h$  have  $\varphi (g \otimes h) z = (\varphi g) ((\varphi h) z)$  by (metis action-mult)
    also from  $hrel$   $grel$  have  $\dots = x$  by simp
    finally have  $\varphi (g \otimes h) z = x$ .
    with  $gh \langle x \in M \rangle \langle z \in M \rangle$  show  $(x, z) \in \text{same-orbit-rel}$  unfolding same-orbit-rel-def
by auto
  qed
qed

```

1.3 Stabilizer and fixed points

The following definition models the stabilizer of a group action:

definition *stabilizer* :: $'c \Rightarrow -$
where $\text{stabilizer } m = \{g \in \text{carrier } G. (\varphi g) m = m\}$

This shows that the stabilizer of m is a subgroup of G .

lemma *stabilizer-is-subgroup*:

```

assumes  $m:m \in M$ 
shows subgroup (stabilizer m) G
proof(rule subgroupI)
  show  $\text{stabilizer } m \subseteq \text{carrier } G$  unfolding stabilizer-def by auto
next
  from  $m$  have  $(\varphi \mathbf{1}) m = m$  by (rule one-is-id)
  hence  $\mathbf{1} \in \text{stabilizer } m$  unfolding stabilizer-def by simp
  thus  $\text{stabilizer } m \neq \{\}$  by auto
next
  fix  $g$ 
  assume  $g:g \in \text{stabilizer } m$ 
  hence  $g \in \text{carrier } G$   $(\varphi g) m = m$  unfolding stabilizer-def by simp+

```

```

with  $m$  have  $g \cdot (\varphi (inv\ g))\ m = m$  by (metis group-inv-rel)
from  $\langle g \in carrier\ G \rangle$  have  $inv\ g \in carrier\ G$  by (metis inv-closed)
with  $g$  show  $(inv\ g) \in stabilizer\ m$  unfolding stabilizer-def by simp
next
  fix  $g\ h$ 
  assume  $g : g \in stabilizer\ m$ 
  hence  $g^2 : g \in carrier\ G$  unfolding stabilizer-def by simp
  assume  $h : h \in stabilizer\ m$ 
  hence  $h^2 : h \in carrier\ G$  unfolding stabilizer-def by simp
  with  $g^2$  have  $gh : g \otimes h \in carrier\ G$  by (rule m-closed)
  from  $g^2\ h^2\ m$  have  $\varphi (g \otimes h)\ m = (\varphi\ g)\ ((\varphi\ h)\ m)$  by (metis action-mult)
  also from  $g\ h$  have  $\dots = m$  unfolding stabilizer-def by simp
  finally have  $\varphi (g \otimes h)\ m = m$ .
  with  $gh$  show  $g \otimes h \in stabilizer\ m$  unfolding stabilizer-def by simp
qed

```

Next, we define and characterize the fixed points of a group action.

```

definition fixed-points :: 'c set
  where fixed-points =  $\{m \in M. carrier\ G \subseteq stabilizer\ m\}$ 

```

```

lemma fixed-point-char:
  assumes  $m \in M$ 
  shows  $(m \in fixed-points) = (\forall g \in carrier\ G. \varphi\ g\ m = m)$ 
using assms unfolding fixed-points-def stabilizer-def by force

```

```

lemma orbit-contains-rep:
  assumes  $m : m \in M$ 
  shows  $m \in orbit\ m$ 
unfolding orbit-def using assms by (metis equiv-class-self same-orbit-is-equiv)

```

```

lemma singleton-orbit-eq-fixed-point:
  assumes  $m : m \in M$ 
  shows  $(card (orbit\ m) = 1) = (m \in fixed-points)$ 
proof
  assume  $card (orbit\ m) = 1$ 
  from  $m$  have  $m \in orbit\ m$  by (rule orbit-contains-rep)
  from  $m$  show  $m \in fixed-points$  unfolding fixed-points-def
  proof(auto)
    fix  $g$ 
    assume  $gG : g \in carrier\ G$ 
    with  $m$  have  $\varphi\ g\ m \in orbit\ m$  by (auto dest: orbit-char)
    with  $\langle m \in orbit\ m \rangle$  card have  $\varphi\ g\ m = m$  by (auto simp add: card-Suc-eq)
    with  $gG$  show  $g \in stabilizer\ m$  unfolding stabilizer-def by simp
  qed

```

```

next
  assume  $m \in fixed-points$ 
  hence  $fixed : carrier\ G \subseteq stabilizer\ m$  unfolding fixed-points-def by simp
  from  $m$  have  $orbit\ m = \{m\}$ 
  proof(auto simp add: orbit-contains-rep)

```

```

fix n
assume n ∈ orbit m
with m obtain g where g: g ∈ carrier G φ g m = n by (auto dest: orbit-char)
moreover with fixed have φ g m = m unfolding stabilizer-def by auto
ultimately show n = m by simp
qed
thus card (orbit m) = 1 by simp
qed

```

1.4 The Orbit-Stabilizer Theorem

This section contains some theorems about orbits and their quotient groups. The first one is the well-known orbit-stabilizer theorem which establishes a bijection between the the quotient group of the an element's stabilizer and its orbit.

theorem orbit-thm:

```

assumes m: m ∈ M
assumes rep:  $\bigwedge U. U \in (\text{carrier } (G \text{ Mod } (\text{stabilizer } m))) \implies \text{rep } U \in U$ 
shows bij-betw ( $\lambda H. (\varphi (\text{inv } (\text{rep } H)) m) (\text{carrier } (G \text{ Mod } (\text{stabilizer } m))) (\text{orbit } m)$ )
proof(auto simp add: bij-betw-def)
show inj-on ( $\lambda H. \varphi (\text{inv } (\text{rep } H)) m) (\text{carrier } (G \text{ Mod } \text{stabilizer } m))$ 
proof(rule inj-onI)
  fix U V
  assume U: U ∈ carrier (G Mod (stabilizer m))
  assume V: V ∈ carrier (G Mod (stabilizer m))
  define h where h = rep V
  define g where g = rep U
  have stabSubset: (stabilizer m) ⊆ carrier G unfolding stabilizer-def by auto
  from m have stabSubgroup: subgroup (stabilizer m) G by (metis stabilizer-is-subgroup)
  from V rep have hV: h ∈ V unfolding h-def by simp
  from V stabSubset have V ⊆ carrier G unfolding FactGroup-def RCOSETS-def
  r-coset-def by auto
  with hV have hG: h ∈ carrier G by auto
  hence hinvg: inv h ∈ carrier G by (metis inv-closed)
  from U rep have gU: g ∈ U unfolding g-def by simp
  from U stabSubset have U ⊆ carrier G unfolding FactGroup-def RCOSETS-def
  r-coset-def by auto
  with gU have gG: g ∈ carrier G by auto
  hence ginvg: inv g ∈ carrier G by (metis inv-closed)
  from gG hinvg have ginvhG: g ⊗ inv h ∈ carrier G by (metis m-closed)
  assume reps: φ (inv rep U) m = φ (inv rep V) m
  hence gh: φ (inv g) m = φ (inv h) m unfolding g-def h-def.
  from gG hinvg m have φ (g ⊗ (inv h)) m = φ g (φ (inv h) m) by (metis
  action-mult)
  also from gh ginvg gG m have ... = φ (g ⊗ inv g) m by (metis action-mult)
  also from m gG have ... = m by (auto simp: one-is-id)
  finally have φ (g ⊗ inv h) m = m.

```


with $g \otimes \text{inv } h \in \text{stabilizer } m$
unfolding *stabilizer-def* **by** *simp*
hence $(\text{stabilizer } m) \#> (g \otimes \text{inv } h) = (\text{stabilizer } m) \#> \mathbf{1}$
by (*metis coset-join2 coset-mult-one m stabSubset stabilizer-is-subgroup subgroup.mem-carrier*)
with $\text{hinvg } hG \text{ } gG \text{ } \text{stabSubset}$ **have** $\text{stabgstabh}:(\text{stabilizer } m) \#> g = (\text{stabilizer } m) \#> h$
by (*metis coset-mult-invl group.coset-mult-one is-group*)
from *stabSubgroup is-group U gU* **have** $U = (\text{stabilizer } m) \#> g$
unfolding *FactGroup-def* **by** (*simp add:subgroup.repr-independence2*)
also from *stabgstabh is-group stabSubgroup V hV subgroup.repr-independence2*
have $\dots = V$
unfolding *FactGroup-def* **by** *force*
finally show $U = V$.
qed
next
have $\text{stabSubset}:\text{stabilizer } m \subseteq \text{carrier } G$ **unfolding** *stabilizer-def* **by** *auto*
fix H
assume $H:H \in \text{carrier } (G \text{ Mod } \text{stabilizer } m)$
with *rep* **have** $\text{rep } H \in H$ **by** *simp*
moreover with $H \text{ } \text{stabSubset}$ **have** $H \subseteq \text{carrier } G$ **unfolding** *FactGroup-def*
RCOSETS-def r-coset-def **by** *auto*
ultimately have $\text{rep } H \in \text{carrier } G$.
hence $\text{inv rep } H \in \text{carrier } G$ **by** (*rule inv-closed*)
with m **show** $\varphi (\text{inv rep } H) m \in \text{orbit } m$ **by** (*auto dest:orbit-char*)
next
fix n
assume $n \in \text{orbit } m$
with m **obtain** g **where** $g:g \in \text{carrier } G \text{ } \varphi g m = n$ **by** (*auto dest:orbit-char*)
hence $\text{invg}:\text{inv } g \in \text{carrier } G$ **by** *simp*
hence $\text{stabinvg}:(\text{stabilizer } m) \#> (\text{inv } g) \in \text{carrier } (G \text{ Mod } \text{stabilizer } m)$ **un-**
folding *FactGroup-def RCOSETS-def* **by** *auto*
hence $\text{rep } ((\text{stabilizer } m) \#> (\text{inv } g)) \in (\text{stabilizer } m) \#> (\text{inv } g)$ **by** (*metis*
rep)
then obtain h **where** $h:h \in \text{stabilizer } m \text{ } \text{rep } ((\text{stabilizer } m) \#> (\text{inv } g)) = h \otimes$
 $(\text{inv } g)$ **unfolding** *r-coset-def* **by** *auto*
with g **have** $\varphi (\text{inv rep } ((\text{stabilizer } m) \#> (\text{inv } g))) m = \varphi (\text{inv } (h \otimes (\text{inv } g)))$
 m **by** *simp*
also from h **have** $hG:h \in \text{carrier } G$ **unfolding** *stabilizer-def* **by** *simp*
with g **have** $\varphi (\text{inv } (h \otimes (\text{inv } g))) m = \varphi (g \otimes (\text{inv } h)) m$ **by** (*metis inv-closed*
inv-inv inv-mult-group)
also from $g \text{ } hG \text{ } m$ **have** $\dots = \varphi g (\varphi (\text{inv } h) m)$ **by** (*metis action-mult inv-closed*)
also from $h \text{ } m$ **have** $\text{inv } h \in \text{stabilizer } m$ **by** (*metis stabilizer-is-subgroup sub-*
group.m-inv-closed)
hence $\varphi g (\varphi (\text{inv } h) m) = \varphi g m$ **unfolding** *stabilizer-def* **by** *simp*
also from g **have** $\dots = n$ **by** *simp*
finally have $n = \varphi (\text{inv rep } ((\text{stabilizer } m) \#> (\text{inv } g))) m$.
with *stabinvg* **show** $n \in (\lambda H. \varphi (\text{inv rep } H) m) \text{ 'carrier } (G \text{ Mod } \text{stabilizer } m)$
by *simp*

qed

In the case of G being finite, the last theorem can be reduced to a statement about the cardinality of orbit and stabilizer:

corollary *orbit-size:*

assumes $fin:finite$ ($carrier\ G$)

assumes $m:m \in M$

shows $order\ G = card\ (orbit\ m) * card\ (stabilizer\ m)$

proof –

define rep **where** $rep = (\lambda U \in (carrier\ (G\ Mod\ (stabilizer\ m))).\ SOME\ x.\ x \in U)$

have $\bigwedge U.\ U \in (carrier\ (G\ Mod\ (stabilizer\ m))) \implies rep\ U \in U$

proof –

fix U

assume $U:U \in carrier\ (G\ Mod\ stabilizer\ m)$

then obtain g **where** $g \in carrier\ G\ U = (stabilizer\ m) \#> g$ **unfolding** *FactGroup-def RCOSETS-def* **by** *auto*

with m **have** $(SOME\ x.\ x \in U) \in U$ **by** (*metis rcos-self stabilizer-is-subgroup someI-ex*)

with U **show** $rep\ U \in U$ **unfolding** *rep-def* **by** *simp*

qed

with m **have** $bij:card\ (carrier\ (G\ Mod\ (stabilizer\ m))) = card\ (orbit\ m)$ **by** (*metis bij-betw-same-card orbit-thm*)

from $fin\ m$ **have** $card\ (carrier\ (G\ Mod\ (stabilizer\ m))) * card\ (stabilizer\ m) = order\ G$ **unfolding** *FactGroup-def* **by** (*simp add: stabilizer-is-subgroup lagrange*)

with bij **show** *?thesis* **by** *simp*

qed

lemma *orbit-not-empty:*

assumes $fin:finite\ M$

assumes $A:A \in orbits$

shows $card\ A > 0$

proof –

from A **obtain** m **where** $m \in M\ A = orbit\ m$ **unfolding** *orbits-def quotient-def orbit-def* **by** *auto*

hence $m \in A$ **by** (*metis orbit-contains-rep*)

hence $A \neq \{\}$ **unfolding** *orbits-def* **by** *auto*

moreover from $fin\ A$ **have** $finite\ A$ **unfolding** *orbits-def quotient-def Image-def same-orbit-rel-def* **by** *auto*

ultimately show *?thesis* **by** *auto*

qed

lemma *fin-set-imp-fin-orbits:*

assumes $finM:finite\ M$

shows $finite\ orbits$

using *assms* **unfolding** *orbits-def quotient-def* **by** *simp*

lemma *singleton-orbits:*

```

shows  $\bigcup \{N \in \text{orbits}. \text{card } N = 1\} = \text{fixed-points}$ 
proof
show  $\bigcup \{N \in \text{orbits}. \text{card } N = 1\} \subseteq \text{fixed-points}$ 
proof
  fix  $x$ 
  assume  $a: x \in \bigcup \{N \in \text{orbits}. \text{card } N = 1\}$ 
  hence  $x \in M$  unfolding orbits-def quotient-def Image-def same-orbit-rel-def
by auto
  from  $a$  obtain  $N$  where  $N: N \in \text{orbits} \text{ card } N = 1 \ x \in N$  by auto
  then obtain  $y$  where  $\text{Norbit}: N = \text{orbit } y \ y \in M$  unfolding orbits-def quo-
tient-def orbit-def by auto
  hence  $y \in N$  by (metis orbit-contains-rep)
  with  $N$  have  $\text{Nsing}: N = \{x\} \ N = \{y\}$  by (auto simp: card-Suc-eq)
  hence  $x = y$  by simp
  with  $\text{Norbit}$  have  $\text{Norbit2}: N = \text{orbit } x$  by simp
  have  $\{g \in \text{carrier } G. \varphi \ g \ x = x\} = \text{carrier } G$ 
  proof(auto)
    fix  $g$ 
    assume  $g \in \text{carrier } G$ 
    with  $\langle x \in M \rangle$  have  $\varphi \ g \ x \in \text{orbit } x$  by (auto dest: orbit-char)
    with  $\text{Nsing}$  show  $\varphi \ g \ x = x$  by (metis Norbit2 singleton-iff)
  qed
  with  $\langle x \in M \rangle$  show  $x \in \text{fixed-points}$  unfolding fixed-points-def stabilizer-def
by simp
qed
next
show  $\text{fixed-points} \subseteq \bigcup \{N \in \text{orbits}. \text{card } N = 1\}$ 
proof
  fix  $m$ 
  assume  $m: m \in \text{fixed-points}$ 
  hence  $mM: m \in M$  unfolding fixed-points-def by simp
  hence  $\text{orbit}: \text{orbit } m \in \text{orbits}$  unfolding orbits-def quotient-def orbit-def by
auto
  from  $mM \ m$  have  $\text{card } (\text{orbit } m) = 1$  by (metis singleton-orbit-eq-fixed-point)
  with  $\text{orbit}$  have  $\text{orbit } m \in \{N \in \text{orbits}. \text{card } N = 1\}$  by simp
  with  $mM$  show  $m \in \bigcup \{N \in \text{orbits}. \text{card } N = 1\}$  by (auto dest: orbit-contains-rep)
qed
qed

```

If G is a p -group acting on a finite set, a given orbit is either a singleton or p divides its cardinality.

```

lemma p-dvd-orbit-size:
  assumes  $\text{order}G: \text{order } G = p \wedge a$ 
  assumes  $\text{prime}: \text{prime } p$ 
  assumes  $\text{fin}M: \text{finite } M$ 
  assumes  $\text{Norbit}: N \in \text{orbits}$ 
  assumes  $\text{card } N > 1$ 
  shows  $p \text{ dvd } \text{card } N$ 
proof –

```

```

from Norbit obtain m where m:m ∈ M N = orbit m unfolding orbits-def
quotient-def orbit-def by auto
from prime have 0 < p ^ a by (simp add: prime-gt-0-nat)
with orderG have finite (carrier G) unfolding order-def by (metis card.infinite
less-nat-zero-code)
with m have order G = card (orbit m) * card (stabilizer m) by (metis orbit-size)
with orderG m have p ^ a = card N * card (stabilizer m) by simp
with ⟨card N > 1⟩ show ?thesis
by (metis dvd-mult2 dvd-mult-cancel1 nat-dvd-not-less nat-mult-1 prime
prime-dvd-power-nat prime-factor-nat prime-nat-iff zero-less-one)
qed

```

As a result of the last lemma the only orbits that count modulo p are the fixed points

lemma *fixed-point-congruence*:

```

assumes order G = p ^ a
assumes prime p
assumes finM:finite M
shows card M mod p = card fixed-points mod p
proof –
define big-orbits where big-orbits = {N∈orbits. card N > 1}
from finM have orbit-part:orbits = big-orbits ∪ {N∈orbits. card N = 1} un-
folding big-orbits-def by (auto dest:orbit-not-empty)
have orbit-disj:big-orbits ∩ {N∈orbits. card N = 1} = {} unfolding big-orbits-def
by auto
from finM have orbits-fin:finite orbits by (rule fin-set-imp-fin-orbits)
hence fin-parts:finite big-orbits finite {N∈orbits. card N = 1} unfolding big-orbits-def
by simp+
from assms have  $\bigwedge N. N \in \text{big-orbits} \implies p \text{ dvd } \text{card } N$  unfolding big-orbits-def
by (auto simp: p-dvd-orbit-size)
hence orbit-div: $\bigwedge N. N \in \text{big-orbits} \implies \text{card } N = (\text{card } N \text{ div } p) * p$  by (metis
dvd-mult-div-cancel mult.commute)
have card M = card (∪ orbits) unfolding orbits-def by (metis Union-quotient
same-orbit-is-equiv)
also have card (∪ orbits) = (∑ N∈orbits. card N) unfolding orbits-def
proof (rule card-Union-disjoint)
show pairwise disjnt (M // same-orbit-rel)
unfolding pairwise-def disjnt-def by(metis same-orbit-is-equiv quotient-disj)
show  $\bigwedge A. A \in M // \text{same-orbit-rel} \implies \text{finite } A$ 
using finM same-orbit-rel-def by (auto dest:finite-equiv-class)
qed
also from orbit-part orbit-disj fin-parts have ... = (∑ N∈big-orbits. card N) +
(∑ N∈{N'∈orbits. card N' = 1}. card N) by (metis (lifting) sum.union-disjoint)
also from assms orbit-div fin-parts have ... = (∑ N∈big-orbits. (card N div p)
* p) + card (∪ {N'∈orbits. card N' = 1}) by (auto simp: card-singleton-set)
also have ... = (∑ N∈big-orbits. card N div p) * p + card fixed-points using
singleton-orbits by (auto simp:sum-distrib-right)
finally have card M = (∑ N∈big-orbits. card N div p) * p + card fixed-points.
hence card M mod p = ((∑ N∈big-orbits. card N div p) * p + card fixed-points)

```

```

mod p by simp
  also have ... = (card fixed-points) mod p by (metis mod-mult-self3)
  finally show ?thesis.
qed

```

We can restrict any group action to the action of a subgroup:

```

lemma subgroup-action:
  assumes H:subgroup H G
  shows group-action (G \carrier := H) \varphi M
unfolding group-action-def group-action-axioms-def group-hom-def group-hom-axioms-def
hom-def
using assms
proof (auto simp add: is-group subgroup.subgroup-is-group group-BijGroup)
  fix x
  assume x \in H
  with H have x \in carrier G by (metis subgroup.mem-carrier)
  with group-hom show \varphi x \in carrier (BijGroup M) by (metis group-hom.hom-closed)
next
  fix x y
  assume x:x \in H and y:y \in H
  with H have x \in carrier G y \in carrier G by (metis subgroup.mem-carrier)+
  with group-hom show \varphi (x \otimes y) = \varphi x \otimes_{BijGroup M} \varphi y by (simp add:
group-hom.hom-mult)
qed

end

```

1.5 Some Examples for Group Actions

```

lemma (in group) right-mult-is-bij:
  assumes h:h \in carrier G
  shows (\lambda g \in carrier G. h \otimes g) \in Bij (carrier G)
proof (auto simp add: Bij-def bij-betw-def inj-on-def)
  fix x y
  assume x:x \in carrier G and y:y \in carrier G and h \otimes x = h \otimes y
  with h show x = y
  by simp
next
  fix x
  assume x:x \in carrier G
  with h show h \otimes x \in carrier G by (metis m-closed)
  from x h have inv h \otimes x \in carrier G by (metis m-closed inv-closed)
  moreover from x h have h \otimes (inv h \otimes x) = x by (metis inv-closed r-inv
m-assoc l-one)
  ultimately show x \in (\otimes) h ^ carrier G by force
qed

```

```

lemma (in group) right-mult-group-action:
  shows group-action G (\lambda h. \lambda g \in carrier G. h \otimes g) (carrier G)

```

unfolding *group-action-def group-action-axioms-def group-hom-def group-hom-axioms-def hom-def*
proof(*auto simp add:is-group group-BijGroup*)
fix *h*
assume *h ∈ carrier G*
thus $(\lambda g \in \text{carrier } G. h \otimes g) \in \text{carrier } (\text{BijGroup } (\text{carrier } G))$ **unfolding**
BijGroup-def **by** (*auto simp:right-mult-is-bij*)
next
fix *x y*
assume *x:x ∈ carrier G and y:y ∈ carrier G*
define *multx multy*
where *multx = (λg∈carrier G. x ⊗ g)*
and *multy = (λg∈carrier G. y ⊗ g)*
with *x y* **have** *multx ∈ (Bij (carrier G)) multy ∈ (Bij (carrier G))* **by** (*metis right-mult-is-bij*)
hence $\text{multx} \otimes_{\text{BijGroup } (\text{carrier } G)} \text{multy} = (\lambda g \in \text{carrier } G. \text{multx } (\text{multy } g))$
unfolding *BijGroup-def* **by** (*auto simp: compose-def*)
also have $\dots = (\lambda g \in \text{carrier } G. (x \otimes y) \otimes g)$ **unfolding** *multx-def multy-def*
proof(*rule restrict-ext*)
fix *g*
assume *g:g ∈ carrier G*
with *x y* **have** $x \otimes y \in \text{carrier } G \ y \otimes g \in \text{carrier } G$ **by** *simp+*
with *x y g* **show** $(\lambda g \in \text{carrier } G. x \otimes g) ((\lambda g \in \text{carrier } G. y \otimes g) g) = x \otimes y \otimes g$
by (*auto simp:m-assoc*)
qed
finally show $(\lambda g \in \text{carrier } G. (x \otimes y) \otimes g) = (\lambda g \in \text{carrier } G. x \otimes g) \otimes_{\text{BijGroup } (\text{carrier } G)} (\lambda g \in \text{carrier } G. y \otimes g)$ **unfolding** *multx-def multy-def* **by** *simp*
qed

lemma (*in group*) *rcosets-closed*:
assumes *HG:subgroup H G*
assumes *g:g ∈ carrier G*
assumes *M:M ∈ rcosets H*
shows $M \#> g \in \text{rcosets } H$
proof –
from *M* **obtain** *h* **where** $h:h \in \text{carrier } G \ M = H \#> h$ **unfolding** *RCOSETS-def*
by *auto*
with *g HG* **have** $M \#> g = H \#> (h \otimes g)$ **by** (*metis coset-mult-assoc subgroup.subset*)
with *HG g h* **show** $M \#> g \in \text{rcosets } H$ **by** (*metis rcosetsI subgroup.m-closed subgroup.subset subgroup-self*)
qed

lemma (*in group*) *inv-mult-on-rcosets-is-bij*:
assumes *HG:subgroup H G*
assumes *g:g ∈ carrier G*
shows $(\lambda U \in \text{rcosets } H. U \#> \text{inv } g) \in \text{Bij } (\text{rcosets } H)$
proof(*auto simp add:Bij-def bij-betw-def inj-on-def*)
fix *M*

```

assume  $M \in \text{rcosets } H$ 
with  $HG \ g$  show  $M \#> \text{inv } g \in \text{rcosets } H$  by (metis inv-closed rcosets-closed)
next
fix  $M$ 
assume  $M:M \in \text{rcosets } H$ 
with  $HG \ g$  have  $M \#> g \in \text{rcosets } H$  by (rule rcosets-closed)
moreover from  $M \ HG \ g$  have  $M \#> g \#> \text{inv } g = M$  by (metis coset-mult-assoc
coset-mult-inv2 inv-closed is-group subgroup.rcosets-carrier)
ultimately show  $M \in (\lambda U. U \#> \text{inv } g) \text{ ' } (\text{rcosets } H)$  by auto
next
fix  $M \ N \ x$ 
assume  $M:M \in \text{rcosets } H$  and  $N:N \in \text{rcosets } H$  and  $M \#> \text{inv } g = N \#>$ 
inv g
hence  $(M \#> \text{inv } g) \#> g = (N \#> \text{inv } g) \#> g$  by simp
with  $HG \ M \ N \ g$  have  $M \#> (\text{inv } g \otimes g) = N \#> (\text{inv } g \otimes g)$  by (metis
coset-mult-assoc is-group subgroup.m-inv-closed subgroup.rcosets-carrier subgroup-self)
with  $HG \ M \ N \ g$  have  $a1:M = N$  by (metis l-inv coset-mult-one is-group sub-
group.rcosets-carrier)
{
  assume  $x \in M$ 
  with  $a1$  show  $x \in N$  by simp
}
{
  assume  $x \in N$ 
  with  $a1$  show  $x \in M$  by simp
}
qed

```

lemma (*in group*) *inv-mult-on-rcosets-action*:

```

assumes  $HG:\text{subgroup } H \ G$ 
shows group-action  $G \ (\lambda g. \lambda U \in \text{rcosets } H. U \#> \text{inv } g) \ (\text{rcosets } H)$ 
unfolding group-action-def group-action-axioms-def group-hom-def group-hom-axioms-def
hom-def
proof(auto simp add:is-group group-BijGroup)
fix  $h$ 
assume  $h \in \text{carrier } G$ 
with  $HG$  show  $(\lambda U \in \text{rcosets } H. U \#> \text{inv } h) \in \text{carrier } (\text{BijGroup } (\text{rcosets } H))$ 
unfolding BijGroup-def by (auto simp:inv-mult-on-rcosets-is-bij)
next
fix  $x \ y$ 
assume  $x:x \in \text{carrier } G$  and  $y:y \in \text{carrier } G$ 
define  $\text{cos } x \ \text{cos } y$ 
where  $\text{cos } x = (\lambda U \in \text{rcosets } H. U \#> \text{inv } x)$ 
and  $\text{cos } y = (\lambda U \in \text{rcosets } H. U \#> \text{inv } y)$ 
with  $x \ y \ HG$  have  $\text{cos } x \in (\text{Bij } (\text{rcosets } H))$   $\text{cos } y \in (\text{Bij } (\text{rcosets } H))$ 
by (metis inv-mult-on-rcosets-is-bij)+
hence  $\text{cos } x \otimes_{\text{BijGroup } (\text{rcosets } H)} \text{cos } y = (\lambda U \in \text{rcosets } H. \text{cos } x \ (\text{cos } y \ U))$ 
unfolding BijGroup-def by (auto simp:compose-def)

```

```

also have ... = ( $\lambda U \in \text{rcosets } H. U \#> \text{inv } (x \otimes y)$ ) unfolding cosx-def cosy-def
proof(rule restrict-ext)
  fix  $U$ 
  assume  $U:U \in \text{rcosets } H$ 
  with  $HG\ y$  have  $U \#> \text{inv } y \in \text{rcosets } H$  by (metis inv-closed rcosets-closed)
  with  $x\ y\ HG\ U$  have ( $\lambda U \in \text{rcosets } H. U \#> \text{inv } x$ ) ( $(\lambda U \in \text{rcosets } H. U \#>$ 
 $\text{inv } y) U) = U \#> \text{inv } y \#> \text{inv } x$ 
    by auto
  also from  $x\ y\ U\ HG$  have ... =  $U \#> \text{inv } (x \otimes y)$ 
    by (metis inv-mult-group coset-mult-assoc inv-closed is-group subgroup.rcosets-carrier)
  finally show ( $\lambda U \in \text{rcosets } H. U \#> \text{inv } x$ ) ( $(\lambda U \in \text{rcosets } H. U \#> \text{inv } y) U$ )
=  $U \#> \text{inv } (x \otimes y)$ .
  qed
  finally show ( $\lambda U \in \text{rcosets } H. U \#> \text{inv } (x \otimes y)$ ) = ( $\lambda U \in \text{rcosets } H. U \#> \text{inv } x$ )
 $\otimes \text{BijGroup } (\text{rcosets } H) (\lambda U \in \text{rcosets } H. U \#> \text{inv } y)$ 
    unfolding cosx-def cosy-def by simp
qed
end

```

```

theory SubgroupConjugation
imports GroupAction
begin

```

2 Conjugation of Subgroups and Cosets

This theory examines properties of the conjugation of subgroups of a fixed group as a group action

2.1 Definitions and Preliminaries

We define the set of all subgroups of G which have a certain cardinality. G will act on those sets. Afterwards some theorems which are already available for right cosets are dualized into statements about left cosets.

```

lemma (in subgroup) subgroup-of-subset:
  assumes  $G:\text{group } G$ 
  assumes  $PH:H \subseteq K$ 
  assumes  $KG:\text{subgroup } K\ G$ 
  shows subgroup  $H (G(\text{carrier} := K))$ 
using assms subgroup-def group.m-inv-consistent m-inv-closed by fastforce

```

```

context group
begin

```

```

definition subgroups-of-size ::  $\text{nat} \Rightarrow -$ 
  where subgroups-of-size  $p = \{H. \text{subgroup } H\ G \wedge \text{card } H = p\}$ 

```


lemma *lcosI*: $[[h \in H; H \subseteq \text{carrier } G; x \in \text{carrier } G]] \implies x \otimes h \in x <\# H$
by (*auto simp add: l-coset-def*)

lemma *lcoset-join2*:

assumes $H:\text{subgroup } H G$

assumes $g:g \in H$

shows $g <\# H = H$

proof *auto*

fix x

assume $x:x \in g <\# H$

then obtain h **where** $h:h \in H x = g \otimes h$ **unfolding** *l-coset-def* **by** *auto*

with $g H$ **show** $x \in H$ **by** (*metis subgroup.m-closed*)

next

fix x

assume $x:x \in H$

with $g H$ **have** $\text{inv } g \otimes x \in H$ **by** (*metis subgroup.m-closed subgroup.m-inv-closed*)

with $x g H$ **show** $x \in g <\# H$ **by** (*metis is-group subgroup.lcos-module-rev subgroup.mem-carrier*)

qed

lemma *cardeq-rcoset*:

assumes *finite* (*carrier* G)

assumes $M \subseteq \text{carrier } G$

assumes $g \in \text{carrier } G$

shows $\text{card } (M \#> g) = \text{card } M$

proof $-$

have $M \#> g \in \text{rcosets } M$ **by** (*metis assms(2) assms(3) rcosetsI*)

thus $\text{card } (M \#> g) = \text{card } M$

using *assms(2) card-rcosets-equal* **by** *auto*

qed

lemma *cardeq-lcoset*:

assumes *finite* (*carrier* G)

assumes $M:M \subseteq \text{carrier } G$

assumes $g:g \in \text{carrier } G$

shows $\text{card } (g <\# M) = \text{card } M$

proof $-$

have *bij-betw* $(\lambda m. g \otimes m) M (g <\# M)$

proof(*auto simp add: bij-betw-def*)

show *inj-on* $((\otimes) g) M$

proof(*rule inj-onI*)

from g **have** *inv* $g:\text{inv } g \in \text{carrier } G$ **by** (*rule inv-closed*)

fix $x y$

assume $x:x \in M$ **and** $y:y \in M$

with M **have** $xG:x \in \text{carrier } G$ **and** $yG:y \in \text{carrier } G$ **by** *auto*

assume $g \otimes x = g \otimes y$

hence $(\text{inv } g) \otimes (g \otimes x) = (\text{inv } g) \otimes (g \otimes y)$ **by** *simp*

with $g \text{ inv } g xG yG$ **have** $(\text{inv } g \otimes g) \otimes x = (\text{inv } g \otimes g) \otimes y$ **by** (*metis*)

```

m-assoc)
  with g invg xG yG show x = y by simp
qed
next
fix x
assume x ∈ M
thus g ⊗ x ∈ g <# M unfolding l-coset-def by auto
next
fix x
assume x:x ∈ g <# M
then obtain m where x = g ⊗ m m ∈ M unfolding l-coset-def by auto
thus x ∈ (⊗) g ' M by simp
qed
thus card (g <# M) = card M by (metis bij-betw-same-card)
qed

```

2.2 Conjugation is a group action

We will now prove that conjugation acts on the subgroups of a certain group. A large part of this proof consists of showing that the conjugation of a subgroup with a group element is, again, a subgroup.

```

lemma conjugation-subgroup:
  assumes HG:subgroup H G
  assumes gG:g ∈ carrier G
  shows subgroup (g <# (H #> inv g)) G
proof
  from gG have inv g ∈ carrier G by (rule inv-closed)
  with HG have (H #> inv g) ⊆ carrier G by (metis r-coset-subset-G subgroup.subset)
  with gG show g <# (H #> inv g) ⊆ carrier G by (metis l-coset-subset-G)
next
  from gG have invgG:inv g ∈ carrier G by (metis inv-closed)
  with HG have lcosSubset:(H #> inv g) ⊆ carrier G by (metis r-coset-subset-G subgroup.subset)
  fix x y
  assume x:x ∈ g <# (H #> inv g) and y:y ∈ g <# (H #> inv g)
  then obtain x' y' where x':x' ∈ H #> inv g x = g ⊗ x' and y':y' ∈ H #> inv g y = g ⊗ y' unfolding l-coset-def by auto
  then obtain hx hy where hx:hx ∈ H x' = hx ⊗ inv g and hy:hy ∈ H y' = hy ⊗ inv g unfolding r-coset-def by auto
  with x' y' have x2:x = g ⊗ (hx ⊗ inv g) and y2:y = g ⊗ (hy ⊗ inv g) by auto
  hence x ⊗ y = (g ⊗ (hx ⊗ inv g)) ⊗ (g ⊗ (hy ⊗ inv g)) by simp
  also from hx hy HG have hxG:hx ∈ carrier G and hyG:hy ∈ carrier G by (metis subgroup.mem-carrier)+
  with gG hy x2 invgG have (g ⊗ (hx ⊗ inv g)) ⊗ (g ⊗ (hy ⊗ inv g)) = g ⊗ hx ⊗ (inv g ⊗ g) ⊗ hy ⊗ inv g by (metis m-assoc m-closed)
  also from invgG gG have ... = g ⊗ hx ⊗ 1 ⊗ hy ⊗ inv g by simp
  also from gG hxG have ... = g ⊗ hx ⊗ hy ⊗ inv g by (metis m-closed r-one)

```

also from gG hxG $invG$ **have** $\dots = g \otimes ((hx \otimes hy) \otimes inv\ g)$ **by** (*metis* gG hxG hyG $invG$ *m-assoc* *m-closed*)
finally have $xy:x \otimes y = g \otimes (hx \otimes hy \otimes inv\ g)$.
from hx hy HG **have** $hx \otimes hy \in H$ **by** (*metis* *subgroup.m-closed*)
with $invG$ HG **have** $(hx \otimes hy) \otimes inv\ g \in H \#> inv\ g$ **by** (*metis* *rcosI* *subgroup.subset*)
with gG *lcosSubset* **have** $g \otimes (hx \otimes hy \otimes inv\ g) \in g <\# (H \#> inv\ g)$ **by** (*metis* *lcosI*)
with xy **show** $x \otimes y \in g <\# (H \#> inv\ g)$ **by** *simp*
next
from gG **have** $invG:inv\ g \in carrier\ G$ **by** (*metis* *inv-closed*)
with HG **have** *lcosSubset*: $(H \#> inv\ g) \subseteq carrier\ G$ **by** (*metis* *r-coset-subset-G* *subgroup.subset*)
from HG **have** $\mathbf{1} \in H$ **by** (*rule* *subgroup.one-closed*)
with $invG$ HG **have** $\mathbf{1} \otimes inv\ g \in H \#> inv\ g$ **by** (*metis* *rcosI* *subgroup.subset*)
with gG *lcosSubset* **have** $g \otimes (\mathbf{1} \otimes inv\ g) \in g <\# (H \#> inv\ g)$ **by** (*metis* *lcosI*)
with gG $invG$ **show** $\mathbf{1} \in g <\# (H \#> inv\ g)$ **by** *simp*
next
from gG **have** $invG:inv\ g \in carrier\ G$ **by** (*metis* *inv-closed*)
with HG **have** *lcosSubset*: $(H \#> inv\ g) \subseteq carrier\ G$ **by** (*metis* *r-coset-subset-G* *subgroup.subset*)
fix x
assume $x \in g <\# (H \#> inv\ g)$
then obtain x' **where** $x':x' \in H \#> inv\ g$ $x = g \otimes x'$ **unfolding** *l-coset-def*
by *auto*
then obtain hx **where** $hx:hx \in H$ $x' = hx \otimes inv\ g$ **unfolding** *r-coset-def* **by** *auto*
with HG **have** *invhx*: $inv\ hx \in H$ **by** (*metis* *subgroup.m-inv-closed*)
from x' hx **have** $inv\ x = inv\ (g \otimes (hx \otimes inv\ g))$ **by** *simp*
also from x' hx HG gG $invG$ **have** $\dots = inv\ (inv\ g) \otimes inv\ hx \otimes inv\ g$ **by** (*metis* *calculation* *in-mono* *inv-mult-group* *lcosSubset* *subgroup.mem-carrier*)
also from gG **have** $\dots = g \otimes inv\ hx \otimes inv\ g$ **by** *simp*
also from gG $invG$ *invhx* HG **have** $\dots = g \otimes (inv\ hx \otimes inv\ g)$ **by** (*metis* *m-assoc* *subgroup.mem-carrier*)
finally have *invx*: $inv\ x = g \otimes (inv\ hx \otimes inv\ g)$.
with *invhx* $invG$ HG **have** $(inv\ hx) \otimes inv\ g \in H \#> inv\ g$ **by** (*metis* *rcosI* *subgroup.subset*)
with gG *lcosSubset* **have** $g \otimes (inv\ hx \otimes inv\ g) \in g <\# (H \#> inv\ g)$ **by** (*metis* *lcosI*)
with *invx* **show** $inv\ x \in g <\# (H \#> inv\ g)$ **by** *simp*
qed

definition *conjugation-action*:: $nat \Rightarrow -$

where *conjugation-action* $p = (\lambda g \in carrier\ G. \lambda P \in subgroups-of-size\ p. g <\# (P \#> inv\ g))$

lemma *conjugation-is-size-invariant*:

assumes *fin*:*finite* (*carrier* G)

assumes $P:P \in \text{subgroups-of-size } p$
assumes $g:g \in \text{carrier } G$
shows $\text{conjugation-action } p \ g \ P \in \text{subgroups-of-size } p$
proof –
from g **have** $\text{inv}g:\text{inv } g \in \text{carrier } G$ **by** $(\text{metis } \text{inv-closed})$
from P **have** $PG:\text{subgroup } P \ G$ **and** $\text{card}:\text{card } P = p$ **unfolding** $\text{subgroups-of-size-def}$
by $\text{simp}+$
hence $P\text{sub}G:P \subseteq \text{carrier } G$ **by** $(\text{metis } \text{subgroup.subset})$
hence $P\text{inv}g\text{sub}G:P \ \#\> \ \text{inv } g \subseteq \text{carrier } G$ **by** $(\text{metis } \text{inv } r\text{-coset-subset-}G)$
have $g \ <\# \ (P \ \#\> \ \text{inv } g) \in \text{subgroups-of-size } p$
proof $(\text{auto } \text{simp } \text{add}:\text{subgroups-of-size-def})$
show $\text{subgroup } (g \ <\# \ (P \ \#\> \ \text{inv } g)) \ G$ **by** $(\text{metis } g \ PG \ \text{conjugation-subgroup})$
next
from $\text{card } P\text{sub}G \ \text{fin } \text{inv}g$ **have** $\text{card } (P \ \#\> \ \text{inv } g) = p$ **by** $(\text{metis } \text{cardeq-r coset})$
with $g \ P\text{inv}g\text{sub}G \ \text{fin}$ **show** $\text{card } (g \ <\# \ (P \ \#\> \ \text{inv } g)) = p$ **by** $(\text{metis } \text{cardeq-l coset})$
qed
with $P \ g$ **show** $?thesis$ **unfolding** $\text{conjugation-action-def}$ **by** simp
qed

lemma $\text{conjugation-is-Bij}$:
assumes $\text{fin}:\text{finite } (\text{carrier } G)$
assumes $g:g \in \text{carrier } G$
shows $\text{conjugation-action } p \ g \in \text{Bij } (\text{subgroups-of-size } p)$
proof –
from g **have** $\text{inv}g:\text{inv } g \in \text{carrier } G$ **by** $(\text{rule } \text{inv-closed})$
from g **have** $\text{conjugation-action } p \ g \in \text{extensional } (\text{subgroups-of-size } p)$ **unfolding**
 $\text{conjugation-action-def}$ **by** simp
moreover **have** $\text{bij-betw } (\text{conjugation-action } p \ g) \ (\text{subgroups-of-size } p) \ (\text{subgroups-of-size } p)$
proof $(\text{auto } \text{simp } \text{add}:\text{bij-betw-def})$
show $\text{inj-on } (\text{conjugation-action } p \ g) \ (\text{subgroups-of-size } p)$
proof $(\text{rule } \text{inj-onI})$
fix $U \ V$
assume $U:U \in \text{subgroups-of-size } p$ **and** $V:V \in \text{subgroups-of-size } p$
hence $\text{subset}G:U \subseteq \text{carrier } G \ V \subseteq \text{carrier } G$ **unfolding** $\text{subgroups-of-size-def}$
by $(\text{metis } (\text{lifting}) \ \text{mem-Collect-eq } \text{subgroup.subset})+$
hence $\text{subset}L:U \ \#\> \ \text{inv } g \subseteq \text{carrier } G \ V \ \#\> \ \text{inv } g \subseteq \text{carrier } G$ **by** $(\text{metis } \text{inv}g \ r\text{-coset-subset-}G)+$
assume $\text{conjugation-action } p \ g \ U = \text{conjugation-action } p \ g \ V$
with $g \ U \ V$ **have** $g \ <\# \ (U \ \#\> \ \text{inv } g) = g \ <\# \ (V \ \#\> \ \text{inv } g)$ **unfolding**
 $\text{conjugation-action-def}$ **by** simp
hence $(\text{inv } g) \ <\# \ (g \ <\# \ (U \ \#\> \ \text{inv } g)) = (\text{inv } g) \ <\# \ (g \ <\# \ (V \ \#\> \ \text{inv } g))$ **by** simp
hence $(\text{inv } g \ \otimes \ g) \ <\# \ (U \ \#\> \ \text{inv } g) = (\text{inv } g \ \otimes \ g) \ <\# \ (V \ \#\> \ \text{inv } g)$ **by**
 $(\text{metis } g \ \text{inv}g \ \text{lcos-m-assoc } r\text{-coset-subset-}G \ \text{subset}G)$
hence $\mathbf{1} \ <\# \ (U \ \#\> \ \text{inv } g) = \mathbf{1} \ <\# \ (V \ \#\> \ \text{inv } g)$ **by** $(\text{metis } g \ l\text{-inv})$
hence $U \ \#\> \ \text{inv } g = V \ \#\> \ \text{inv } g$ **by** $(\text{metis } \text{subsetL } \text{lcos-mult-one})$
hence $(U \ \#\> \ \text{inv } g) \ \#\> \ g = (V \ \#\> \ \text{inv } g) \ \#\> \ g$ **by** simp

hence $U \#> (inv\ g \otimes g) = V \#> (inv\ g \otimes g)$ **by** (*metis coset-mult-assoc g inv-closed subsetG*)
hence $U \#> \mathbf{1} = V \#> \mathbf{1}$ **by** (*metis g l-inv*)
thus $U = V$ **by** (*metis coset-mult-one subsetG*)
qed
next
fix P
assume $P \in \text{subgroups-of-size } p$
thus *conjugation-action* $p\ g\ P \in \text{subgroups-of-size } p$ **by** (*metis fin g conjugation-is-size-invariant*)
next
fix P
assume $P:P \in \text{subgroups-of-size } p$
with *invg* **have** *conjugation-action* $p\ (inv\ g)\ P \in \text{subgroups-of-size } p$ **by** (*metis fin invg conjugation-is-size-invariant*)
with *invg* P **have** $(inv\ g) <\# (P \#> (inv\ (inv\ g))) \in \text{subgroups-of-size } p$
unfolding *conjugation-action-def* **by** *simp*
hence $1:(inv\ g) <\# (P \#> g) \in \text{subgroups-of-size } p$ **by** (*metis g inv-inv*)
have $g <\# (((inv\ g) <\# (P \#> g)) \#> inv\ g) = (\bigcup p \in P. \{g \otimes (inv\ g \otimes (p \otimes g) \otimes inv\ g)\})$ **unfolding** *r-coset-def l-coset-def* **by** (*simp add:m-assoc*)
also from P **have** $PG:P \subseteq \text{carrier } G$ **unfolding** *subgroups-of-size-def* **by** (*auto simp add:subgroup.subset*)
have $\forall p \in P. g \otimes (inv\ g \otimes (p \otimes g) \otimes inv\ g) = p$
proof(*auto*)
fix p
assume $p \in P$
with PG **have** $p:p \in \text{carrier } G..$
with $g\ invg$ **have** $g \otimes (inv\ g \otimes (p \otimes g) \otimes inv\ g) = (g \otimes inv\ g) \otimes p \otimes (g \otimes inv\ g)$ **by** (*metis m-assoc m-closed*)
also with $g\ invg\ g\ p$ **have** $\dots = p$ **by** (*metis l-one r-inv r-one*)
finally show $g \otimes (inv\ g \otimes (p \otimes g) \otimes inv\ g) = p.$
qed
hence $(\bigcup p \in P. \{g \otimes (inv\ g \otimes (p \otimes g) \otimes inv\ g)\}) = P$ **by** *simp*
finally have $g <\# (((inv\ g) <\# (P \#> g)) \#> inv\ g) = P.$
with 1 **have** $P \in (\lambda P. g <\# (P \#> inv\ g))$ ‘*subgroups-of-size p*’ **by** *auto*
with $P\ g$ **show** $P \in \text{conjugation-action } p\ g$ ‘*subgroups-of-size p*’ **unfolding** *conjugation-action-def* **by** *simp*
qed
ultimately show *?thesis* **unfolding** *BijGroup-def Bij-def* **by** *simp*
qed

lemma *lr-coset-assoc*:

assumes $g:g \in \text{carrier } G$
assumes $h:h \in \text{carrier } G$
assumes $P:P \subseteq \text{carrier } G$
shows $g <\# (P \#> h) = (g <\# P) \#> h$
proof(*auto*)
fix x
assume $x \in g <\# (P \#> h)$

```

then obtain  $p$  where  $p \in P$  and  $p:x = g \otimes (p \otimes h)$  unfolding  $l\text{-coset-def}$ 
 $r\text{-coset-def}$  by  $auto$ 
with  $P$  have  $p \in \text{carrier } G$  by  $auto$ 
with  $g \ h \ p$  have  $x = (g \otimes p) \otimes h$  by ( $metis \ m\text{-assoc}$ )
with  $\langle p \in P \rangle$  show  $x \in (g \langle \# \ P \rangle \# \# \ h)$  unfolding  $l\text{-coset-def}$   $r\text{-coset-def}$  by
 $auto$ 
next
fix  $x$ 
assume  $x \in (g \langle \# \ P \rangle \# \# \ h)$ 
then obtain  $p$  where  $p \in P$  and  $p:x = (g \otimes p) \otimes h$  unfolding  $l\text{-coset-def}$ 
 $r\text{-coset-def}$  by  $auto$ 
with  $P$  have  $p \in \text{carrier } G$  by  $auto$ 
with  $g \ h \ p$  have  $x = g \otimes (p \otimes h)$  by ( $metis \ m\text{-assoc}$ )
with  $\langle p \in P \rangle$  show  $x \in g \langle \# \ (P \ \# \# \ h)$  unfolding  $l\text{-coset-def}$   $r\text{-coset-def}$  by
 $auto$ 
qed

```

theorem $acts\text{-on-subsets}$:

```

assumes  $fin:finite$  ( $\text{carrier } G$ )
shows  $group\text{-action } G$  ( $\text{conjugation-action } p$ ) ( $\text{subgroups-of-size } p$ )
unfolding  $group\text{-action-def}$   $group\text{-action-axioms-def}$   $group\text{-hom-def}$   $group\text{-hom-axioms-def}$ 
 $hom-def$ 
apply( $auto \ simp \ add:is\text{-group} \ group\text{-BijGroup}$ )
proof  $-$ 
fix  $g$ 
assume  $g:g \in \text{carrier } G$ 
with  $fin$  show  $\text{conjugation-action } p \ g \in \text{carrier } (BijGroup \ (\text{subgroups-of-size } p))$ 
unfolding  $BijGroup\text{-def}$  by ( $metis \ conjugation\text{-is-Bij} \ partial\text{-object.select-conv}(1)$ )
next
fix  $x \ y$ 
assume  $x:x \in \text{carrier } G$  and  $y:y \in \text{carrier } G$ 
hence  $invx:inv \ x \in \text{carrier } G$  and  $invy:inv \ y \in \text{carrier } G$  by ( $metis \ inv\text{-closed}$ )+
from  $x \ y$  have  $xyG:x \otimes y \in \text{carrier } G$  by ( $metis \ m\text{-closed}$ )
define  $conjx$  where  $conjx = \text{conjugation-action } p \ x$ 
define  $conjy$  where  $conjy = \text{conjugation-action } p \ y$ 
from  $fin \ x$  have  $xBij:conjx \in Bij$  ( $\text{subgroups-of-size } p$ ) unfolding  $conjx\text{-def}$  by
( $metis \ conjugation\text{-is-Bij}$ )
from  $fin \ y$  have  $yBij:conjy \in Bij$  ( $\text{subgroups-of-size } p$ ) unfolding  $conjy\text{-def}$  by
( $metis \ conjugation\text{-is-Bij}$ )
have  $conjx \otimes BijGroup \ (\text{subgroups-of-size } p) \ conjy$ 
 $= (\lambda g \in Bij \ (\text{subgroups-of-size } p). \text{restrict } (\text{compose } (\text{subgroups-of-size } p) \ g) \ (Bij$ 
 $(\text{subgroups-of-size } p))) \ conjx \ conjy$  unfolding  $BijGroup\text{-def}$  by  $simp$ 
also from  $xBij \ yBij$  have  $\dots = \text{compose } (\text{subgroups-of-size } p) \ conjx \ conjy$  by
 $simp$ 
also have  $\dots = (\lambda P \in \text{subgroups-of-size } p. \ conjx \ (conjy \ P))$  by ( $metis \ compose\text{-def}$ )
also have  $\dots = (\lambda P \in \text{subgroups-of-size } p. \ x \otimes y \langle \# \ (P \ \# \# \ inv \ (x \otimes y))$ )
proof( $rule \ restrict\text{-ext}$ )
fix  $P$ 
assume  $P:P \in \text{subgroups-of-size } p$ 

```

hence $PG:P \subseteq \text{carrier } G$ **unfolding** *subgroups-of-size-def* **by** (*auto simp:subgroup.subset*)
with y **have** $yPG:y <\# P \subseteq \text{carrier } G$ **by** (*metis l-coset-subset-G*)
from $x y$ **have** $\text{inv } xyG:\text{inv } (x \otimes y) \in \text{carrier } G$ **and** $xyG:x \otimes y \in \text{carrier } G$
using *inv-closed m-closed* **by** *auto*
from $yBij$ **have** $\text{conj } y$ ' *subgroups-of-size p = subgroups-of-size p* **unfolding**
Bij-def bij-betw-def **by** *simp*
with P **have** $\text{conj } P:\text{conj } P \in \text{subgroups-of-size } p$ **unfolding** *Bij-def bij-betw-def*
by (*metis (full-types) imageI*)
with $x y P$ **have** $\text{conj } x (\text{conj } P) = x <\# ((y <\# (P \#> \text{inv } y)) \#> \text{inv } x)$
unfolding *conjy-def conjx-def conjugation-action-def* **by** *simp*
also from y $\text{inv } y PG$ **have** $\dots = x <\# (((y <\# P) \#> \text{inv } y) \#> \text{inv } x)$ **by**
(*metis lr-coset-assoc*)
also from PG $\text{inv } x \text{inv } y$ **have** $\dots = x <\# ((y <\# P) \#> (\text{inv } y \otimes \text{inv } x))$
by (*metis coset-mult-assoc yPG*)
also from $x y$ **have** $\dots = x <\# ((y <\# P) \#> \text{inv } (x \otimes y))$ **by** (*metis*
inv-mult-group)
also from $\text{inv } xyG x yPG$ **have** $\dots = (x <\# (y <\# P)) \#> \text{inv } (x \otimes y)$ **by**
(*metis lr-coset-assoc*)
also from $x y PG$ **have** $\dots = ((x \otimes y) <\# P) \#> \text{inv } (x \otimes y)$ **by** (*metis*
lcos-m-assoc)
also from $xyG \text{inv } xyG PG$ **have** $\dots = (x \otimes y) <\# (P \#> \text{inv } (x \otimes y))$ **by**
(*metis lr-coset-assoc*)
finally show $\text{conj } x (\text{conj } P) = x \otimes y <\# (P \#> \text{inv } (x \otimes y))$.
qed
finally have $\text{conj } x \otimes_{BijGroup} (\text{subgroups-of-size } p) \text{ conj } y = (\lambda P \in \text{subgroups-of-size}$
 $p. x \otimes y <\# (P \#> \text{inv } (x \otimes y)))$.
with xyG **show** *conjugation-action p (x \otimes y)*
 $= \text{conjugation-action } p x \otimes_{BijGroup} (\text{subgroups-of-size } p) \text{ conjugation-action } p y$
unfolding *conjx-def conjy-def conjugation-action-def* **by** *simp*
qed

2.3 Properties of the Conjugation Action

lemma *stabilizer-contains-P:*

assumes $\text{fin:finite } (\text{carrier } G)$

assumes $P:P \in \text{subgroups-of-size } p$

shows $P \subseteq \text{group-action.stabilizer } G (\text{conjugation-action } p) P$

proof

from P **have** $PG:\text{subgroup } P G$ **unfolding** *subgroups-of-size-def* **by** *simp*

from fin **interpret** $\text{conj:group-action } G (\text{conjugation-action } p)$ (*subgroups-of-size*
 p) **by** (*rule acts-on-subsets*)

fix x

assume $x:x \in P$

with PG **have** $\text{inv } x \in P$ **by** (*metis subgroup.m-inv-closed*)

from $x P$ **have** $xG:x \in \text{carrier } G$ **unfolding** *subgroups-of-size-def subgroup-def*
by *auto*

with P **have** $\text{conjugation-action } p x P = x <\# (P \#> \text{inv } x)$ **unfolding** *con-*
jugation-action-def **by** *simp*

also from $\langle \text{inv } x \in P \rangle PG$ **have** $\dots = x <\# P$ **by** (*metis coset-join2 sub-*

group.mem-carrier)
also from PG x **have** $\dots = P$ **by** (*rule lcoset-join2*)
finally have *conjugation-action* p x $P = P$.
with xG **show** $x \in \text{group-action.stabilizer } G$ (*conjugation-action* p) P **unfolding**
conj.stabilizer-def **by** *simp*
qed

corollary *stabilizer-supergrp-P*:
assumes $fin:finite$ (*carrier* G)
assumes $P:P \in \text{subgroups-of-size } p$
shows *subgroup* P ($G(\backslash\text{carrier} := \text{group-action.stabilizer } G$ (*conjugation-action*
 p) P))
proof –
from *assms* **have** $P \subseteq \text{group-action.stabilizer } G$ (*conjugation-action* p) P **by**
(*rule stabilizer-contains-P*)
moreover from P **have** *subgroup* P G **unfolding** *subgroups-of-size-def* **by** *simp*
moreover from P fin **have** *subgroup* ($\text{group-action.stabilizer } G$ (*conjugation-action*
 p) P) G **by** (*metis acts-on-subsets group-action.stabilizer-is-subgroup*)
ultimately show *?thesis* **by** (*metis is-group subgroup.subgroup-of-subset*)
qed

lemma (*in group*) *P-fixed-point-of-P-conj*:
assumes $fin:finite$ (*carrier* G)
assumes $P:P \in \text{subgroups-of-size } p$
shows $P \in \text{group-action.fixed-points } (G(\backslash\text{carrier} := P))$ (*conjugation-action* p)
(*subgroups-of-size* p)
proof –
from fin **interpret** *conjG*: *group-action* G *conjugation-action* p *subgroups-of-size*
 p **by** (*rule acts-on-subsets*)
from P **have** *subgroup* P G **unfolding** *subgroups-of-size-def* **by** *simp*
with fin **interpret** *conjP*: *group-action* $G(\backslash\text{carrier} := P)$ (*conjugation-action* p)
(*subgroups-of-size* p) **by** (*metis acts-on-subsets group-action.subgroup-action*)
from fin P **have** $P \subseteq \text{conjG.stabilizer } P$ **by** (*rule stabilizer-contains-P*)
hence $P \subseteq \text{conjP.stabilizer } P$ **using** *conjG.stabilizer-def* *conjP.stabilizer-def* **by**
auto
with P **show** $P \in \text{conjP.fixed-points}$ **unfolding** *conjP.fixed-points-def* **by** *auto*
qed

lemma *conj-wo-inv*:
assumes $QG:\text{subgroup } Q$ G
assumes $PG:\text{subgroup } P$ G
assumes $g:g \in \text{carrier } G$
assumes $\text{conj:inv } g <\# (Q \#> g) = P$
shows $Q \#> g = g <\# P$
proof –
from g **have** $\text{invg:inv } g \in \text{carrier } G$ **by** (*metis inv-closed*)
from conj **have** $g <\# (\text{inv } g <\# (Q \#> g)) = g <\# P$ **by** *simp*
with QG g invg **have** $(g \otimes \text{inv } g) <\# (Q \#> g) = g <\# P$ **by** (*metis lcos-m-assoc*
r-coset-subset-G subgroup.subset)


```

with  $g \text{ invg}$  have  $1 <\# (Q \#> g) = g <\# P$  by (metis r-inv)
with  $QG \text{ } g$  show  $Q \#> g = g <\# P$  by (metis lcos-mult-one r-coset-subset-G
subgroup.subset)
qed

end

end

```

```

theory SndSylow
imports SubgroupConjugation
begin

```

```

no-notation Multiset.subset-mset (infix  $<\#$  50)

```

3 The Secondary Sylow Theorems

3.1 Preliminaries

```

lemma singletonI:
  assumes  $\bigwedge x. x \in A \implies x = y$ 
  assumes  $y \in A$ 
  shows  $A = \{y\}$ 
using assms by fastforce

```

```

context group
begin

```

```

lemma set-mult-inclusion:
  assumes  $H:\text{subgroup } H \ G$ 
  assumes  $Q:P \subseteq \text{carrier } G$ 
  assumes  $PQ:H <\#> P \subseteq H$ 
  shows  $P \subseteq H$ 
proof
  fix  $x$ 
  from  $H$  have  $1 \in H$  by (rule subgroup.one-closed)
  moreover assume  $x \in P$ 
  ultimately have  $1 \otimes x \in H <\#> P$  unfolding set-mult-def by auto
  with  $PQ$  have  $1 \otimes x \in H$  by auto
  with  $H \ Q \ x$  show  $x \in H$  by (metis in-mono l-one)
qed

```

```

lemma card-subgrp-dvd:
  assumes  $\text{subgroup } H \ G$ 
  shows  $\text{card } H \ \text{dvd} \ \text{order } G$ 
proof(cases finite (carrier G))
  case True
  with assms have  $\text{card} (\text{rcosets } H) * \text{card } H = \text{order } G$  by (metis lagrange)

```

```

thus ?thesis by (metis dvd-triv-left mult.commute)
next
  case False
  hence order  $G = 0$  unfolding order-def by (metis card.infinite)
  thus ?thesis by (metis dvd-0-right)
qed

```

```

lemma subgroup-finite:
  assumes subgroup:subgroup  $H G$ 
  assumes finite:finite (carrier  $G$ )
  shows finite  $H$ 
by (metis finite finite-subset subgroup subgroup.subset)

```

end

3.2 Extending the Sylow Locale

This locale extends the originale *syLOW* locale by adding the constraint that the p must not divide the remainder m , i.e. p^a is the maximal size of a p -subgroup of G .

```

locale snd-syLOW = syLOW +
  assumes pNotDvdM:  $\neg (p \text{ dvd } m)$ 

```

```

context snd-syLOW
begin

```

```

lemma pa-not-zero:  $p \wedge a \neq 0$ 
  by (simp add: prime-gt-0-nat prime-p)

```

```

lemma syLOW-greater-zero:
  shows card (subgroups-of-size  $(p \wedge a)$ )  $> 0$ 
proof –
  obtain  $P$  where PG:subgroup  $P G$  and cardP:card  $P = p \wedge a$  by (metis syLOW-thm)
  hence  $P \in$  subgroups-of-size  $(p \wedge a)$  unfolding subgroups-of-size-def by auto
  hence subgroups-of-size  $(p \wedge a) \neq \{\}$  by auto
  moreover from finite- $G$  have finite (subgroups-of-size  $(p \wedge a)$ ) unfolding subgroups-of-size-def subgroup-def by auto
  ultimately show ?thesis by auto
qed

```

```

lemma is-snd-syLOW: snd-syLOW  $G p a m$  by (rule snd-syLOW-axioms)

```

3.3 Every p -group is Contained in a conjugate of a p -Sylow-Group

```

lemma ex-conj-syLOW-group:
  assumes  $H:H \in$  subgroups-of-size  $(p \wedge b)$ 

```

assumes $Psize:P \in \text{subgroups-of-size } (p \wedge a)$
obtains g **where** $g \in \text{carrier } G \ H \subseteq g <\#\> (P \ \#\> \text{inv } g)$
proof –
from H **have** $HsubG:\text{subgroup } H \ G$ **unfolding** $\text{subgroups-of-size-def}$ **by** auto
hence $HG:H \subseteq \text{carrier } G$ **unfolding** $\text{subgroups-of-size-def}$ **by** $(\text{simp add:subgroup.subset})$
from $Psize$ **have** $PG:\text{subgroup } P \ G$ **and** $\text{card}P:\text{card } P = p \wedge a$ **unfolding** $\text{subgroups-of-size-def}$ **by** auto
define H' **where** $H' = G(\text{carrier} := H)$
from $HsubG$ **interpret** $Hgroup:\text{group } H'$ **unfolding** $H'\text{-def}$ **by** $(\text{metis subgroup-imp-group})$
from H **have** $\text{order}H':\text{order } H' = p \wedge b$ **unfolding** $H'\text{-def}$ $\text{subgroups-of-size-def}$ **order-def** **by** simp
define φ **where** $\varphi = (\lambda g. \lambda U \in \text{rcosets } P. U \ \#\> \text{inv } g)$
with PG **interpret** $Gact:\text{group-action } G \ \varphi \ \text{rcosets } P$ **unfolding** $\varphi\text{-def}$ **by** $(\text{metis inv-mult-on-rcosets-action})$
from H **interpret** $H'act:\text{group-action } H' \ \varphi \ \text{rcosets } P$ **unfolding** $H'\text{-def}$ $\text{subgroups-of-size-def}$ **by** $(\text{metis (mono-tags) Gact.subgroup-action mem-Collect-eq})$
from $\text{finite-}G \ PG$ **have** $\text{finite } (\text{rcosets } P)$ **unfolding** RCOSETS-def $r\text{-coset-def}$ **by** $(\text{metis (lifting) finite.emptyI finite-UN-I finite-insert})$
with $\text{order}H' \ \text{syLOW-axioms} \ \text{card}P$ **have** $\text{card } H'act.\text{fixed-points mod } p = \text{card } (\text{rcosets } P) \text{ mod } p$ **unfolding** syLOW-def syLOW-axioms-def **by** $(\text{metis } H'act.\text{fixed-point-congruence})$
moreover **from** $\text{finite-}G \ PG \ \text{order-}G \ \text{card}P$ **have** $\text{card } (\text{rcosets } P) * p \wedge a = p \wedge a * m$ **by** (metis lagrange)
with $\text{prime-}p$ **have** $\text{card } (\text{rcosets } P) = m$ **by** $(\text{metis less-nat-zero-code mult-cancel2 mult-is-0 mult commute order-}G \ \text{zero-less-o-}G)$
hence $\text{card } (\text{rcosets } P) \text{ mod } p = m \text{ mod } p$ **by** simp
moreover **from** $p \text{NotDvd}m \ \text{prime-}p$ **have** $\dots \neq 0$ **by** $(\text{metis dvd-eq-mod-eq-0})$
ultimately **have** $\text{card } H'act.\text{fixed-points} \neq 0$ **by** (metis mod-0)
then **obtain** N **where** $N:N \in H'act.\text{fixed-points}$ **by** fastforce
hence $N\text{coset}:N \in \text{rcosets } P$ **unfolding** $H'act.\text{fixed-points-def}$ **by** simp
then **obtain** g **where** $g:g \in \text{carrier } G \ N = P \ \#\> g$ **unfolding** RCOSETS-def **by** auto
hence $\text{inv}g:\text{inv } g \in \text{carrier } G$ **by** $(\text{metis inv-closed})$
hence $\text{invinv}g:\text{inv } (\text{inv } g) \in \text{carrier } G$ **by** $(\text{metis inv-closed})$
from N **have** $\text{carrier } H' \subseteq H'act.\text{stabilizer } N$ **unfolding** $H'act.\text{fixed-points-def}$ **by** simp
hence $\forall h \in H. \varphi h N = N$ **unfolding** $H'act.\text{stabilizer-def}$ **using** $H'\text{-def}$ **by** auto
with $HG \ N\text{coset}$ **have** $a1:\forall h \in H. N \ \#\> \text{inv } h \subseteq N$ **unfolding** $\varphi\text{-def}$ **by** simp
have $N <\#\> H \subseteq N$ **unfolding** set-mult-def $r\text{-coset-def}$
proof(auto)
fix $n \ h$
assume $n:n \in N$ **and** $h:h \in H$
with H **have** $\text{inv } h \in H$ **by** $(\text{metis (mono-tags) mem-Collect-eq subgroup.m-inv-closed subgroups-of-size-def})$
with $n \ HG \ PG \ a1$ **have** $n \otimes \text{inv } (\text{inv } h) \in N$ **unfolding** $r\text{-coset-def}$ **by** auto
with $HG \ h$ **show** $n \otimes h \in N$ **by** $(\text{metis in-mono inv-inv})$
qed
with g **have** $((P \ \#\> g) <\#\> H) \ \#\> \text{inv } g \subseteq (P \ \#\> g) \ \#\> \text{inv } g$ **unfolding** $r\text{-coset-def}$ **by** auto

with $PG\ g\ invg$ **have** $((P \#> g) <\#> H) \#> inv\ g \subseteq P$ **by** (*metis coset-mult-assoc coset-mult-one r-inv subgroup.subset*)
with $g\ HG\ PG\ invg$ **have** $P <\#> (g <\# H \#> inv\ g) \subseteq P$ **by** (*metis lr-coset-assoc r-coset-subset-G rcos-assoc-lcos setmult-rcos-assoc subgroup.subset*)
with $PG\ HG\ g\ invg$ **have** $g <\# H \#> inv\ g \subseteq P$ **by** (*metis l-coset-subset-G r-coset-subset-G set-mult-inclusion*)
with g **have** $(g <\# H \#> inv\ g) \#> inv\ (inv\ g) \subseteq P \#> inv\ (inv\ g)$ **unfolding** *r-coset-def* **by** *auto*
with $HG\ g\ invg\ invinv\ g$ **have** $g <\# H \subseteq P \#> inv\ (inv\ g)$ **by** (*metis coset-mult-assoc coset-mult-invg2 l-coset-subset-G*)
with g **have** $(inv\ g) <\# (g <\# H) \subseteq inv\ g <\# (P \#> inv\ (inv\ g))$ **unfolding** *l-coset-def* **by** *auto*
with $HG\ g\ invg\ invinv\ g$ **have** $H \subseteq inv\ g <\# (P \#> inv\ (inv\ g))$ **by** (*metis inv-inv lcos-m-assoc lcos-mult-one r-inv*)
with $invg$ **show** *thesis* **by** (*auto dest:that*)
qed

3.4 Every p -Group is Contained in a p -Sylow-Group

theorem *syLOW-contained-in-syLOW-group*:

assumes $H:H \in \text{subgroups-of-size } (p \wedge b)$

obtains S **where** $H \subseteq S$ **and** $S \in \text{subgroups-of-size } (p \wedge a)$

proof –

from H **have** $HG:H \subseteq \text{carrier } G$ **unfolding** *subgroups-of-size-def* **by** (*simp add:subgroup.subset*)

obtain P **where** $PG:\text{subgroup } P\ G$ **and** $\text{card } P:\text{card } P = p \wedge a$ **by** (*metis syLOW-thm*)

hence $Psize:P \in \text{subgroups-of-size } (p \wedge a)$ **unfolding** *subgroups-of-size-def* **by** *simp*

with H **obtain** g **where** $g:g \in \text{carrier } G\ H \subseteq g <\# (P \#> inv\ g)$ **by** (*metis ex-conj-syLOW-group*)

moreover **note** $Psize\ g$

moreover **with** $finite-G$ **have** *conjugation-action* $(p \wedge a)\ g\ P \in \text{subgroups-of-size } (p \wedge a)$ **by** (*metis conjugation-is-size-invariant*)

ultimately **show** *thesis* **unfolding** *conjugation-action-def* **by** (*auto dest:that*)

qed

3.5 p -Sylow-Groups are conjugates of each other

theorem *syLOW-conjugate*:

assumes $P:P \in \text{subgroups-of-size } (p \wedge a)$

assumes $Q:Q \in \text{subgroups-of-size } (p \wedge a)$

obtains g **where** $g \in \text{carrier } G\ Q = g <\# (P \#> inv\ g)$

proof –

from P **have** $\text{card } P = p \wedge a$ **unfolding** *subgroups-of-size-def* **by** *simp*

from Q **have** $Q\text{card}:\text{card } Q = p \wedge a$ **unfolding** *subgroups-of-size-def* **by** *simp*

from $Q\ P$ **obtain** g **where** $g:g \in \text{carrier } G\ Q \subseteq g <\# (P \#> inv\ g)$ **by** (*rule ex-conj-syLOW-group*)

moreover **with** $P\ finite-G$ **have** *conjugation-action* $(p \wedge a)\ g\ P \in \text{subgroups-of-size } (p \wedge a)$ **by** (*metis conjugation-is-size-invariant*)

moreover from $g P$ **have** *conjugation-action* $(p \hat{ } a) g P = g <\# (P \#> inv g)$ **unfolding** *conjugation-action-def* **by** *simp*
ultimately have $conjSize:g <\# (P \#> inv g) \in subgroups-of-size (p \hat{ } a)$ **unfolding** *conjugation-action-def* **by** *simp*
with $Qcard$ **have** $card:card (g <\# (P \#> inv g)) = card Q$ **unfolding** *subgroups-of-size-def* **by** *simp*
from $conjSize$ *finite-G* **have** *finite* $(g <\# (P \#> inv g))$ **by** $(metis (mono-tags) finite-subset mem-Collect-eq subgroup.subset subgroups-of-size-def)$
with $g card$ **have** $Q = g <\# (P \#> inv g)$ **by** $(metis card-subset-eq)$
with g **show** *thesis* **by** $(metis that)$
qed

corollary *syllow-conj-orbit-rel*:

assumes $P:P \in subgroups-of-size (p \hat{ } a)$
assumes $Q:Q \in subgroups-of-size (p \hat{ } a)$
shows $(P,Q) \in group-action.same-orbit-rel G (conjugation-action (p \hat{ } a)) (subgroups-of-size (p \hat{ } a))$
unfolding *group-action.same-orbit-rel-def*
proof –
from $Q P$ **obtain** g **where** $g:g \in carrier G P = g <\# (Q \#> inv g)$ **by** $(rule syllow-conjugate)$
with $Q P$ **have** $g':conjugation-action (p \hat{ } a) g Q = P$ **unfolding** *conjugation-action-def* **by** *simp*
from *finite-G* **interpret** $conj: group-action G (conjugation-action (p \hat{ } a)) (subgroups-of-size (p \hat{ } a))$ **by** $(rule acts-on-subsets)$
have $conj.same-orbit-rel = \{X \in (subgroups-of-size (p \hat{ } a) \times subgroups-of-size (p \hat{ } a)). \exists g \in carrier G. ((conjugation-action (p \hat{ } a)) g) (snd X) = (fst X)\}$ **by** $(rule conj.same-orbit-rel-def)$
with $g g' P Q$ **show** *?thesis* **by** *auto*
qed

3.6 Counting Sylow-Groups

The number of sylow groups is the orbit size of one of them:

theorem *num-eq-card-orbit*:

assumes $P:P \in subgroups-of-size (p \hat{ } a)$
shows $subgroups-of-size (p \hat{ } a) = group-action.orbit G (conjugation-action (p \hat{ } a)) (subgroups-of-size (p \hat{ } a)) P$
proof $(auto)$
from *finite-G* **interpret** $conj: group-action G (conjugation-action (p \hat{ } a)) (subgroups-of-size (p \hat{ } a))$ **by** $(rule acts-on-subsets)$
have $group-action.orbit G (conjugation-action (p \hat{ } a)) (subgroups-of-size (p \hat{ } a)) P = group-action.same-orbit-rel G (conjugation-action (p \hat{ } a)) (subgroups-of-size (p \hat{ } a)) \{P\}$ **by** $(rule conj.orbit-def)$
fix Q
{
assume $Q:Q \in subgroups-of-size (p \hat{ } a)$
from $P Q$ **obtain** g **where** $g:g \in carrier G Q = g <\# (P \#> inv g)$ **by** $(rule syllow-conjugate)$

with P *conj.orbit-char* **show** $Q \in \text{group-action.orbit } G (\text{conjugation-action } (p \wedge a)) (\text{subgroups-of-size } (p \wedge a)) P$
unfolding *conjugation-action-def* **by** *auto*
} **{**
assume $Q \in \text{group-action.orbit } G (\text{conjugation-action } (p \wedge a)) (\text{subgroups-of-size } (p \wedge a)) P$
with P *conj.orbit-char* **obtain** g **where** $g: g \in \text{carrier } G \ Q = \text{conjugation-action } (p \wedge a) \ g \ P$ **by** *auto*
with *finite-G P* **show** $Q \in \text{subgroups-of-size } (p \wedge a)$ **by** (*metis conjugation-is-size-invariant*)
}
qed

theorem *num-sylow-normalizer*:

assumes $Psize: P \in \text{subgroups-of-size } (p \wedge a)$
shows $\text{card } (rcosets_G (\text{carrier} := \text{group-action.stabilizer } G (\text{conjugation-action } (p \wedge a)) P)) P) * p \wedge a = \text{card } (\text{group-action.stabilizer } G (\text{conjugation-action } (p \wedge a)) P)$

proof –

from *finite-G* **interpret** *conj: group-action G (conjugation-action (p ^ a)) (subgroups-of-size (p ^ a))* **by** (*rule acts-on-subsets*)

from $Psize$ **have** $PG: \text{subgroup } P \ G$ **and** $\text{card} P: \text{card } P = p \wedge a$ **unfolding** *subgroups-of-size-def* **by** *auto*

with *finite-G* **have** $\text{order } G = \text{card } (\text{conj.orbit } P) * \text{card } (\text{conj.stabilizer } P)$ **by** (*metis Psize acts-on-subsets group-action.orbit-size*)

with *order-G Psize* **have** $p \wedge a * m = \text{card } (\text{subgroups-of-size } (p \wedge a)) * \text{card } (\text{conj.stabilizer } P)$ **by** (*metis num-eq-card-orbit*)

moreover from $Psize$ **interpret** *stabGroup: group G (carrier := conj.stabilizer P)* **by** (*metis conj.stabilizer-is-subgroup subgroup-imp-group*)

from *finite-G Psize* **have** $PStab: \text{subgroup } P \ (G (\text{carrier} := \text{conj.stabilizer } P))$ **by** (*rule stabilizer-supergp-P*)

from *finite-G Psize* **have** *finite (conj.stabilizer P)* **by** (*metis card.infinite conj.stabilizer-is-subgroup less-nat-zero-code subgroup.finite-imp-card-positive*)

with *finite-G PStab stabGroup.lagrange* **have** $\text{card } (rcosets_G (\text{carrier} := \text{conj.stabilizer } P)) P) * \text{card } P = \text{order } (G (\text{carrier} := \text{conj.stabilizer } P))$ **by** *force*

with $\text{card} P$ **show** *?thesis* **unfolding** *order-def* **by** *auto*

qed

theorem (*in snd-sylow*) *num-sylow-dvd-remainder*:

shows $\text{card } (\text{subgroups-of-size } (p \wedge a)) \text{ dvd } m$

proof –

from *finite-G* **interpret** *conj: group-action G (conjugation-action (p ^ a)) (subgroups-of-size (p ^ a))* **by** (*rule acts-on-subsets*)

obtain P **where** $PG: \text{subgroup } P \ G$ **and** $\text{card} P: \text{card } P = p \wedge a$ **by** (*metis sylow-thm*)

hence $Psize: P \in \text{subgroups-of-size } (p \wedge a)$ **unfolding** *subgroups-of-size-def* **by** *simp*

with *finite-G* **have** $\text{order } G = \text{card } (\text{conj.orbit } P) * \text{card } (\text{conj.stabilizer } P)$ **by** (*metis Psize acts-on-subsets group-action.orbit-size*)

with *order-G Psize* **have** $\text{orderEq}: p \wedge a * m = \text{card } (\text{subgroups-of-size } (p \wedge a))$

```

* card (conj.stabilizer P) by (metis num-eq-card-orbit)
  define k where k = card (rcosets G(\carrier := conj.stabilizer P) P)
  with Psize have k * p ^ a = card (conj.stabilizer P) by (metis num-sylow-normalizer)
  with orderEq have p ^ a * m = card (subgroups-of-size (p ^ a)) * p ^ a * k by
(auto simp:mult.assoc mult.commute)
  hence p ^ a * m = p ^ a * card (subgroups-of-size (p ^ a)) * k by auto
  then have m = card (subgroups-of-size (p ^ a)) * k
    using pa-not-zero by auto
  then show ?thesis ..
qed

```

We can restrict this locale to refer to a subgroup of order at least p^a :

```

lemma (in snd-sylow) restrict-locale:
  assumes subgrp:subgroup P G
  assumes card:p ^ a dvd card P
  shows snd-sylow (G(\carrier := P)) p a ((card P) div (p ^ a))
proof -
  from subgrp interpret groupP: group G(\carrier := P) by (metis subgroup-imp-group)
  define k where k = (card P) div (p ^ a)
  with card have cardP:card P = p ^ a * k by auto
  hence orderP:order (G(\carrier := P)) = p ^ a * k unfolding order-def by
simp
  from cardP subgrp order-G have p ^ a * k dvd p ^ a * m by (metis card-subgrp-dvd)
  hence k dvd m
    by (metis nat-mult-dvd-cancel-disj pa-not-zero)
  with pNotDvd m have ndvd:¬ p dvd k
    by (blast intro: dvd-trans)
  define PcalM where PcalM = {s. s ⊆ carrier (G(\carrier := P)) ∧ card s = p
^ a}
  define PRelM where PRelM = {(N1, N2). N1 ∈ PcalM ∧ N2 ∈ PcalM ∧
(∃ g ∈ carrier (G(\carrier := P)). N1 = N2 #> G(\carrier := P) g)}
  from subgrp finite-G have finite-groupP:finite (carrier (G(\carrier := P))) by
(auto simp:subgroup-finite)
  interpret NsyLOW: snd-sylow G(\carrier := P) p a k PcalM PRelM
    unfolding snd-sylow-def snd-sylow-axioms-def sylow-def sylow-axioms-def k-def
    using groupP.is-group prime-p orderP finite-groupP ndvd PcalM-def PRelM-def
k-def by fastforce+
  show ?thesis using k-def by (metis NsyLOW.is-snd-sylow)
qed

```

```

theorem (in snd-sylow) p-sylow-mod-p:
  shows card (subgroups-of-size (p ^ a)) mod p = 1
proof -
  obtain P where PG:subgroup P G and cardP:card P = p ^ a by (metis sylow-thm)
  hence orderP:order (G(\carrier := P)) = p ^ a unfolding order-def by auto
  from PG have PsubG:P ⊆ carrier G by (metis subgroup.subset)
  from PG cardP have PSize:P ∈ subgroups-of-size (p ^ a) unfolding sub-
groups-of-size-def by auto

```

from PG **interpret** $groupP:group (G(\text{carrier} := P))$ **by** (rule $subgroup\text{-}imp\text{-}group$)
from $cardP$ **have** $PSize2:P \in groupP.subgroups\text{-}of\text{-}size (p \wedge a)$ **using** $groupP.subgroups\text{-}of\text{-}size\text{-}def$
 $groupP.subgroup\text{-}self$ **by** $auto$
from $finite\text{-}G$ **interpret** $conjG: group\text{-}action G conjugation\text{-}action (p \wedge a) sub\text{-}groups\text{-}of\text{-}size (p \wedge a)$ **by** (rule $acts\text{-}on\text{-}subsets$)
from PG **interpret** $conjP: group\text{-}action G(\text{carrier} := P) conjugation\text{-}action (p \wedge a) subgroups\text{-}of\text{-}size (p \wedge a)$ **by** (rule $conjG.subgroup\text{-}action$)
from $finite\text{-}G$ **have** $finite (subgroups\text{-}of\text{-}size (p \wedge a))$ **unfolding** $subgroups\text{-}of\text{-}size\text{-}def$
 $subgroup\text{-}def$ **by** $auto$
with $orderP$ $prime\text{-}p$ **have** $card (subgroups\text{-}of\text{-}size (p \wedge a)) \bmod p = card conjP.fixed\text{-}points \bmod p$ **by** (rule $conjP.fixed\text{-}point\text{-}congruence$)
also **have** $\dots = 1$
proof $-$
have $\bigwedge Q. Q \in conjP.fixed\text{-}points \implies Q = P$
proof $-$
fix Q
assume $Qfixed:Q \in conjP.fixed\text{-}points$
hence $Qsize:Q \in subgroups\text{-}of\text{-}size (p \wedge a)$ **unfolding** $conjP.fixed\text{-}points\text{-}def$
by $simp$
hence $cardQ:card Q = p \wedge a$ **unfolding** $subgroups\text{-}of\text{-}size\text{-}def$ **by** $simp$
 $-$ The normalizer of Q in G
 $-$ Let's first show some basic properties of N
define N **where** $N = conjG.stabilizer Q$
define k **where** $k = (card N) \text{ div } (p \wedge a)$
from $N\text{-}def$ $Qsize$ **have** $NG:subgroup N G$ **by** (metis $conjG.stabilizer\text{-}is\text{-}subgroup$)
then **interpret** $groupN: group G(\text{carrier} := N)$ **by** (metis $subgroup\text{-}imp\text{-}group$)
from $Qsize$ $N\text{-}def$ **have** $QN:subgroup Q (G(\text{carrier} := N))$ **using** $stabilizer\text{-}supergroup\text{-}P$ **by** $auto$
 $-$ The following proposition is used to show that $P = Q$ later
from $Qsize$ **have** $NfixesQ:\forall g \in N. conjugation\text{-}action (p \wedge a) g Q = Q$ **unfolding** $N\text{-}def$ $conjG.stabilizer\text{-}def$ **by** $auto$
from $Qfixed$ **have** $PfixesQ:\forall g \in P. conjugation\text{-}action (p \wedge a) g Q = Q$
unfolding $conjP.fixed\text{-}points\text{-}def$ $conjP.stabilizer\text{-}def$ **by** $auto$
with $PsubG$ **have** $P \subseteq N$ **unfolding** $N\text{-}def$ $conjG.stabilizer\text{-}def$ **by** $auto$
with PG $N\text{-}def$ $Qsize$ **have** $PN:subgroup P (G(\text{carrier} := N))$ **by** (metis $conjG.stabilizer\text{-}is\text{-}subgroup$ $is\text{-}group$ $subgroup.subgroup\text{-}of\text{-}subset$)
with $cardP$ **have** $p \wedge a \text{ dvd } order (G(\text{carrier} := N))$ **using** $groupN.card\text{-}subgroup\text{-}dvd$
by $force$
hence $p \wedge a \text{ dvd } card N$ **unfolding** $order\text{-}def$ **by** $simp$
with NG **have** $smaller\text{-}syllow:snd\text{-}syllow (G(\text{carrier} := N)) p a k$ **unfolding** $k\text{-}def$ **by** (rule $restrict\text{-}locale$)
 $-$ Instantiate the $snd\text{-}syllow$ Locale a second time for the normalizer of Q
define $NcalM$ **where** $NcalM = \{s. s \subseteq carrier (G(\text{carrier} := N)) \wedge card s = p \wedge a\}$
define $NRelM$ **where** $NRelM = \{(N1, N2). N1 \in NcalM \wedge N2 \in NcalM \wedge (\exists g \in carrier (G(\text{carrier} := N)). N1 = N2 \#>_{G(\text{carrier} := N)} g)\}$
interpret $Nsyllow: snd\text{-}syllow G(\text{carrier} := N) p a k NcalM NRelM$
unfolding $NcalM\text{-}def$ $NRelM\text{-}def$ **using** $smaller\text{-}syllow$.
 $-$ P and Q are conjugate in N :

from *cardP PN* **have** *PsizeN:P ∈ groupN.subgroups-of-size (p ^ a)* **unfolding**
groupN.subgroups-of-size-def **by** *auto*
from *cardQ QN* **have** *QsizeN:Q ∈ groupN.subgroups-of-size (p ^ a)* **unfolding**
groupN.subgroups-of-size-def **by** *auto*
from *QsizeN PsizeN* **obtain** *g* **where** *g:g ∈ carrier (G(|carrier := N)) P*
 $= g <\#_{G(|carrier := N)} (Q \#>_{G(|carrier := N)} \text{inv}_{G(|carrier := N)} g)$ **by** (*rule*
Nsylvow.sylvow-conjugate)
with *NG* **have** $P = g <\# (Q \#> \text{inv } g)$ **unfolding** *r-coset-def l-coset-def*
by (*auto simp:m-inv-consistent*)
with *NG g Qsize* **have** *conjugation-action (p ^ a) g Q = P* **unfolding**
conjugation-action-def **using** *subgroup.subset* **by** *force*
with *g NfixesQ* **show** $Q = P$ **by** *auto*
qed
moreover from *finite-G PSize* **have** $P \in \text{conj}P.\text{fixed-points}$ **using** *P-fixed-point-of-P-conj*
by *auto*
ultimately have $\text{conj}P.\text{fixed-points} = \{P\}$ **by** *fastforce*
hence $\text{one:card conj}P.\text{fixed-points} = 1$ **by** (*auto simp: card-Suc-eq*)
with *prime-p* **have** $\text{card conj}P.\text{fixed-points} < p$ **unfolding** *prime-nat-iff* **by**
auto
with *one* **show** *?thesis* **using** *mod-pos-pos-trivial* **by** *auto*
qed
finally show *?thesis*.
qed
end
end