Geometric Axioms for Minkowski Spacetime

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December 14, 2021

Abstract

This is a formalisation of Schutz' system of axioms for Minkowski spacetime [1], as well as the results in his third chapter ("Temporal Order on a Path"), with the exception of the second part of Theorem 12. Many results are proven here that cannot be found in Schutz, either preceding the theorem they are needed for, or in their own thematic section.

Contents

theory *TernaryOrdering*
  imports *Util*

begin

Definition of chains using an ordering on sets of events based on natural numbers, plus some proofs.

1 Totally ordered chains

Based on page 110 of Phil Scott's thesis and the following HOL Light definition:

```
let ORDERING = new_definition
  `ORDERING f X <=> (!n. (FINITE X ==> n < CARD X) ==> f n IN X)
                    /\ (!x. x IN X ==> ?n. (FINITE X ==> n < CARD X)
                        /\ f n = x)
                    /\ !n n' n''. (FINITE X ==> n'' < CARD X)
                          /\ n < n' /\ n' < n''
                          ==> between (f n) (f n') (f n'')`;;
```

I've made it strict for simplicity, and because that's how Schutz's ordering is. It could be made more generic by taking in the function corresponding to $<$ as a paramater. Main difference to Schutz: he has local order, not total (cf Theorem 2 and *ordering2*).

definition *ordering* :: $(nat \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow bool$
where

$$ordering\ f\ ord\ X \equiv (\forall n.\ (finite\ X \longrightarrow n < card\ X) \longrightarrow f\ n \in X) \wedge (\forall x{\in}X.\ (\exists n.\ (finite\ X \longrightarrow n < card\ X) \wedge f\ n = x)) \wedge (\forall n\ n'\ n''.\ (finite\ X \longrightarrow n'' < card\ X) \wedge n < n' \wedge n' < n'' \longrightarrow ord\ (f\ n)\ (f\ n')\ (f\ n''))$$

lemma *ordering-ord-ijk*:
  assumes *ordering f ord X*
    and $i < j \wedge j < k \wedge (finite\ X \longrightarrow k < card\ X)$
  shows $ord\ (f\ i)\ (f\ j)\ (f\ k)$
⟨*proof*⟩

lemma *empty-ordering* [*simp*]: $\exists f.\ ordering\ f\ ord\ \{\}$
⟨*proof*⟩

lemma *singleton-ordering* [*simp*]: $\exists f.\ ordering\ f\ ord\ \{a\}$
⟨*proof*⟩

lemma *two-ordering* [*simp*]: $\exists f.$ *ordering f ord* $\{a,\ b\}$
$\langle proof \rangle$

lemma *card-le2-ordering*:
assumes *finiteX*: *finite X*
and *card-le2*: *card* $X \leq 2$
shows $\exists f.$ *ordering f ord X*
$\langle proof \rangle$

lemma *ord-ordered*:
assumes *abc*: *ord a b c*
and *abc-neq*: $a \neq b \wedge a \neq c \wedge b \neq c$
shows $\exists f.$ *ordering f ord* $\{a,b,c\}$
$\langle proof \rangle$

lemma *overlap-ordering*:
assumes *abc*: *ord a b c*
and *bcd*: *ord b c d*
and *abd*: *ord a b d*
and *acd*: *ord a c d*
and *abc-neq*: $a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d$
shows $\exists f.$ *ordering f ord* $\{a,b,c,d\}$
$\langle proof \rangle$

lemma *overlap-ordering-alt1*:
assumes *abc*: *ord a b c*
and *bcd*: *ord b c d*
and *abc-bcd-abd*: $\forall\ a\ b\ c\ d.$ *ord a b c* $\wedge$ *ord b c d* $\longrightarrow$ *ord a b d*
and *abc-bcd-acd*: $\forall\ a\ b\ c\ d.$ *ord a b c* $\wedge$ *ord b c d* $\longrightarrow$ *ord a c d*
and *ord-distinct*: $\forall\, a\ b\ c.$ (*ord a b c* $\longrightarrow a \neq b \wedge a \neq c \wedge b \neq c$)
shows $\exists f.$ *ordering f ord* $\{a,b,c,d\}$
$\langle proof \rangle$

lemma *overlap-ordering-alt2*:
assumes *abc*: *ord a b c*
and *bcd*: *ord b c d*
and *abd*: *ord a b d*
and *acd*: *ord a c d*
and *ord-distinct*: $\forall\, a\ b\ c.$ (*ord a b c* $\longrightarrow a \neq b \wedge a \neq c \wedge b \neq c$)
shows $\exists f.$ *ordering f ord* $\{a,b,c,d\}$
$\langle proof \rangle$

lemma *overlap-ordering-alt*:
assumes *abc*: *ord a b c*
and *bcd*: *ord b c d*
and *abc-bcd-abd*: $\forall\ a\ b\ c\ d.$ *ord a b c* $\wedge$ *ord b c d* $\longrightarrow$ *ord a b d*
and *abc-bcd-acd*: $\forall\ a\ b\ c\ d.$ *ord a b c* $\wedge$ *ord b c d* $\longrightarrow$ *ord a c d*
and *abc-neq*: $a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d$
shows $\exists f.$ *ordering f ord* $\{a,b,c,d\}$

$\langle proof \rangle$

The lemmas below are easy to prove for $X = \{\}$, and if I included that case then I would have to write a conditional definition in place of $\{0..|X| - 1\}$.

lemma *finite-ordering-img*: $[\![X \neq \{\}; \textit{finite } X; \textit{ordering } f \textit{ ord } X]\!] \Longrightarrow f \text{ ‘ } \{0..\textit{card } X - 1\} = X$
$\langle proof \rangle$

lemma *inf-ordering-img*: $[\![\textit{infinite } X; \textit{ordering } f \textit{ ord } X]\!] \Longrightarrow f \text{ ‘ } \{0..\} = X$
$\langle proof \rangle$

lemma *finite-ordering-inv-img*: $[\![X \neq \{\}; \textit{finite } X; \textit{ordering } f \textit{ ord } X]\!] \Longrightarrow f -\text{‘ } X = \{0..\textit{card } X - 1\}$
$\langle proof \rangle$

lemma *inf-ordering-inv-img*: $[\![\textit{infinite } X; \textit{ordering } f \textit{ ord } X]\!] \Longrightarrow f -\text{‘ } X = \{0..\}$
$\langle proof \rangle$

lemma *inf-ordering-img-inv-img*: $[\![\textit{infinite } X; \textit{ordering } f \textit{ ord } X]\!] \Longrightarrow f \text{ ‘ } f -\text{‘ } X = X$
$\langle proof \rangle$

lemma *finite-ordering-inj-on*: $[\![\textit{finite } X; \textit{ordering } f \textit{ ord } X]\!] \Longrightarrow \textit{inj-on } f \{0..\textit{card } X - 1\}$
$\langle proof \rangle$

lemma *finite-ordering-bij*:
  assumes *orderingX*: *ordering f ord X*
    and *finiteX*: *finite X*
    and *non-empty*: $X \neq \{\}$
  shows *bij-betw f* $\{0..\textit{card } X - 1\}$ *X*
$\langle proof \rangle$

lemma *inf-ordering-inj'*:
  assumes *infX*: *infinite X*
    and *f-ord*: *ordering f ord X*
    and *ord-distinct*: $\forall\, a\ b\ c.\ (\textit{ord } a\ b\ c \longrightarrow a \neq b \land a \neq c \land b \neq c)$
    and *f-eq*: $f\ m = f\ n$
  shows $m = n$

$\langle proof \rangle$

lemma *inf-ordering-inj*:
  assumes *infinite X*
    and *ordering f ord X*
    and $\forall\, a\ b\ c.\ (\textit{ord } a\ b\ c \longrightarrow a \neq b \land a \neq c \land b \neq c)$
  shows *inj f*

⟨*proof*⟩

The finite case is a little more difficult as I can't just choose some other natural number to form the third part of the betweenness relation and the initial simplification isn't as nice. Note that I cannot prove *inj f* (over the whole type that *f* is defined on, i.e. natural numbers), because I need to capture the *m* and *n* that obey specific requirements for the finite case. In order to prove *inj f*, I would have to extend the definition for ordering to include *m* and *n* beyond *card X*, such that it is still injective. That would probably not be very useful.

lemma *finite-ordering-inj*:
  assumes *finiteX*: *finite X*
    and *f-ord*: *ordering f ord X*
    and *ord-distinct*: $\forall\, a\ b\ c.\ (ord\ a\ b\ c \longrightarrow a \neq b \land a \neq c \land b \neq c)$
    and *m-less-card*: $m < card\ X$
    and *n-less-card*: $n < card\ X$
    and *f-eq*: $f\ m = f\ n$
  shows $m = n$
⟨*proof*⟩

lemma *ordering-inj*:
  assumes *ordering f ord X*
    and $\forall\, a\ b\ c.\ (ord\ a\ b\ c \longrightarrow a \neq b \land a \neq c \land b \neq c)$
    and $finite\ X \longrightarrow m < card\ X$
    and $finite\ X \longrightarrow n < card\ X$
    and $f\ m = f\ n$
  shows $m = n$
  ⟨*proof*⟩

lemma *ordering-sym*:
  assumes *ord-sym*: $\bigwedge a\ b\ c.\ ord\ a\ b\ c \Longrightarrow ord\ c\ b\ a$
    and *finite X*
    and *ordering f ord X*
  shows $ordering\ (\lambda n.\ f\ (card\ X - 1 - n))\ ord\ X$
⟨*proof*⟩

lemma *zero-into-ordering*:
  assumes *ordering f betw X*
  and $X \neq \{\}$
  shows $(f\ 0) \in X$
    ⟨*proof*⟩

2 Locally ordered chains

Definitions for Schutz-like chains, with local order only.

definition *ordering2* :: $(nat \Rightarrow {}'a) \Rightarrow ({}'a \Rightarrow {}'a \Rightarrow {}'a \Rightarrow bool) \Rightarrow {}'a\ set \Rightarrow bool$
where

$ordering2\ f\ ord\ X \equiv (\forall\, n.\ (finite\ X \longrightarrow n < card\ X) \longrightarrow f\ n \in X)$
$\wedge\ (\forall\, x{\in}X.\ (\exists\, n.\ (finite\ X \longrightarrow n < card\ X) \wedge f\ n = x))$
$\wedge\ (\forall\, n\ n'\ n''.\ (finite\ X \longrightarrow n'' < card\ X) \wedge Suc\ n = n' \wedge Suc\ n'$
$= n''$
$\longrightarrow ord\ (f\ n)\ (f\ n')\ (f\ n''))$

Analogue to *ordering-ord-ijk*, which is quicker to use in sledgehammer than the definition.

lemma *ordering2-ord-ijk*:
assumes $ordering2\ f\ ord\ X$
and $Suc\ i = j \wedge Suc\ j = k \wedge (finite\ X \longrightarrow k < card\ X)$
shows $ord\ (f\ i)\ (f\ j)\ (f\ k)$
$\langle proof \rangle$

end

theory *Minkowski*
imports *Main TernaryOrdering*
begin

Primitives and axioms as given in [1, pp. 9-17].

I've tried to do little to no proofs in this file, and keep that in other files. So, this is mostly locale and other definitions, except where it is nice to prove something about definitional equivalence and the like (plus the intermediate lemmas that are necessary for doing so).

Minkowski spacetime = $(\mathcal{E}, \mathcal{P}, [\ldots])$ except in the notation here I've used $[[\ldots]]$ for $[\ldots]$ as Isabelle uses $[\ldots]$ for lists.
Except where stated otherwise all axioms are exactly as they appear in Schutz97. It is the independent axiomatic system provided in the main body of the book. The axioms O1-O6 are the axioms of order, and largely concern properties of the betweenness relation. I1-I7 are the axioms of incidence. I1-I3 are similar to axioms found in systems for Euclidean geometry. As compared to Hilbert's Foundations (HIn), our incidence axioms (In) are loosely identifiable as I1 $\rightarrow$ HI3, HI8; I2 $\rightarrow$ HI1; I3 $\rightarrow$ HI2. I4 fixes the dimension of the space. I5-I7 are what makes our system non-Galilean, and lead (I think) to Lorentz transforms (together with S?) and the ultimate speed limit. Axioms S and C and the axioms of symmetry and continuity, where the latter is what makes the system second order. Symmetry replaces all of Hilbert's axioms of congruence, when considered in the context of I5-I7.

3 MinkowskiPrimitive: I1-I3

Events $\mathcal{E}$, paths $\mathcal{P}$, and sprays. Sprays only need to refer to $\mathcal{E}$ and $\mathcal{P}$. Axiom *in-path-event* is covered in English by saying "a path is a set of events", but is necessary to have explicitly as an axiom as the types do not force it to be the case.

I think part of why Schutz has I1, together with the trickery $[\![\ \mathcal{E}{\neq}\{\}\ ]\!] \Longrightarrow \ldots$ in I4, is that then I4 talks *only* about dimension, and results such as *no-empty-paths* can be proved using only existence of elements and unreachable sets. In our case, it's also a question of ordering the sequence of axiom introductions: dimension should really go at the end, since it is not needed for quite a while; but many earlier proofs rely on the set of events being non-empty. It may be nice to have the existence of paths as a separate axiom too, which currently still relies on the axiom of dimension (Schutz has no such axiom either).

locale *MinkowskiPrimitive* =

fixes $\mathcal{E}$ :: *'a set*
and $\mathcal{P}$ :: *('a set) set*
assumes *in-path-event* [*simp*]: $[\![Q \in \mathcal{P};\ a \in Q]\!] \Longrightarrow a \in \mathcal{E}$

and *nonempty-events* [*simp*]: $\mathcal{E} \neq \{\}$

and *events-paths*: $[\![a \in \mathcal{E};\ b \in \mathcal{E};\ a \neq b]\!] \Longrightarrow \exists R{\in}\mathcal{P}.\ \exists S{\in}\mathcal{P}.\ a \in R \wedge b \in S \wedge R \cap S \neq \{\}$

and *eq-paths* [*intro*]: $[\![P \in \mathcal{P};\ Q \in \mathcal{P};\ a \in P;\ b \in P;\ a \in Q;\ b \in Q;\ a \neq b]\!] \Longrightarrow P = Q$
begin

This should be ensured by the additional axiom.

lemma *path-sub-events*:
$Q \in \mathcal{P} \Longrightarrow Q \subseteq \mathcal{E}$
⟨*proof*⟩

lemma *paths-sub-power*:
$\mathcal{P} \subseteq Pow\ \mathcal{E}$
⟨*proof*⟩

For more terse statements. $a \neq b$ because a and b are being used to identify the path, and $a = b$ would not do that.

abbreviation *path* :: *'a set* ⇒ *'a* ⇒ *'a* ⇒ *bool* **where**
$path\ ab\ a\ b \equiv ab \in \mathcal{P} \wedge a \in ab \wedge b \in ab \wedge a \neq b$

abbreviation *path-ex* :: *'a* ⇒ *'a* ⇒ *bool* **where**
$path\text{-}ex\ a\ b \equiv \exists\, Q.\ path\ Q\ a\ b$

lemma *path-permute*:
$path\ ab\ a\ b = path\ ab\ b\ a$
⟨*proof*⟩

abbreviation *path-of* :: *'a* ⇒ *'a* ⇒ *'a set* **where**
$path\text{-}of\ a\ b \equiv THE\ ab.\ path\ ab\ a\ b$

lemma *path-of-ex*: $path\ (path\text{-}of\ a\ b)\ a\ b \longleftrightarrow path\text{-}ex\ a\ b$
⟨*proof*⟩

lemma *path-unique*:
assumes *path ab a b* **and** *path ab' a b*
shows $ab = ab'$
⟨*proof*⟩

4 Primitives: Unreachable Subset (from an Event)

The $Q \in \mathcal{P} \wedge b \in \mathcal{E}$ constraints are necessary as the types as not expressive enough to do it on their own. Schutz's notation is: $Q(b, \emptyset)$.

definition *unreachable-subset* :: $'a\ set \Rightarrow\ 'a \Rightarrow\ 'a\ set$ ($\emptyset$ - - [*100*, *100*] *100*) **where**
unreachable-subset $Q\ b \equiv \{x{\in}Q.\ Q \in \mathcal{P} \wedge b \in \mathcal{E} \wedge b \notin Q \wedge \neg(\textit{path-ex}\ b\ x)\}$

5 Primitives: Kinematic Triangle

definition *kinematic-triangle* :: $'a \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$ ($\triangle$ - - - [*100*, *100*, *100*] *100*) **where**

$$
\begin{aligned}
&\textit{kinematic-triangle}\ a\ b\ c \equiv \\
&\quad a \in \mathcal{E} \wedge b \in \mathcal{E} \wedge c \in \mathcal{E} \wedge a \neq b \wedge a \neq c \wedge b \neq c \\
&\quad \wedge (\exists Q{\in}\mathcal{P}.\ \exists R{\in}\mathcal{P}.\ Q \neq R \wedge (\exists S{\in}\mathcal{P}.\ Q \neq S \wedge R \neq S \\
&\qquad\qquad \wedge a \in Q \wedge b \in Q \\
&\qquad\qquad \wedge a \in R \wedge c \in R \\
&\qquad\qquad \wedge b \in S \wedge c \in S))
\end{aligned}
$$

A fuller, more explicit equivalent of $\triangle$, to show that the above definition is sufficient.

lemma *tri-full*:

$$
\begin{aligned}
&\triangle\ a\ b\ c = (a \in \mathcal{E} \wedge b \in \mathcal{E} \wedge c \in \mathcal{E} \wedge a \neq b \wedge a \neq c \wedge b \neq c \\
&\quad \wedge (\exists Q{\in}\mathcal{P}.\ \exists R{\in}\mathcal{P}.\ Q \neq R \wedge (\exists S{\in}\mathcal{P}.\ Q \neq S \wedge R \neq S \\
&\qquad\qquad \wedge a \in Q \wedge b \in Q \wedge c \notin Q \\
&\qquad\qquad \wedge a \in R \wedge c \in R \wedge b \notin R \\
&\qquad\qquad \wedge b \in S \wedge c \in S \wedge a \notin S)))
\end{aligned}
$$

⟨*proof*⟩

6 Primitives: SPRAY

It's okay to not require $x \in \mathcal{E}$ because if $x \notin \mathcal{E}$ the *SPRAY* will be empty anyway, and if it's nonempty then $x \in \mathcal{E}$ is derivable.

definition *SPRAY* :: $'a \Rightarrow ('a\ set)\ set$ **where**
$\textit{SPRAY}\ x \equiv \{R{\in}\mathcal{P}.\ x \in R\}$

definition *spray* :: $'a \Rightarrow\ 'a\ set$ **where**
$\textit{spray}\ x \equiv \{y.\ \exists R{\in}\textit{SPRAY}\ x.\ y \in R\}$

definition *is-SPRAY* :: $('a\ set)\ set \Rightarrow bool$ **where**
$\textit{is-SPRAY}\ S \equiv \exists x{\in}\mathcal{E}.\ S = \textit{SPRAY}\ x$

definition *is-spray* :: $'a\ set \Rightarrow bool$ **where**
$\textit{is-spray}\ S \equiv \exists x{\in}\mathcal{E}.\ S = \textit{spray}\ x$

Some very simple SPRAY and spray lemmas below.

lemma *SPRAY-event*:
$SPRAY\ x \neq \{\} \Longrightarrow x \in \mathcal{E}$
⟨*proof*⟩

lemma *SPRAY-nonevent*:
$x \notin \mathcal{E} \Longrightarrow SPRAY\ x = \{\}$
⟨*proof*⟩

lemma *SPRAY-path*:
$P \in SPRAY\ x \Longrightarrow P \in \mathcal{P}$
⟨*proof*⟩

lemma *in-SPRAY-path*:
$P \in SPRAY\ x \Longrightarrow x \in P$
⟨*proof*⟩

lemma *source-in-SPRAY*:
$SPRAY\ x \neq \{\} \Longrightarrow \exists P \in SPRAY\ x.\ x \in P$
⟨*proof*⟩

lemma *spray-event*:
$spray\ x \neq \{\} \Longrightarrow x \in \mathcal{E}$
⟨*proof*⟩

lemma *spray-nonevent*:
$x \notin \mathcal{E} \Longrightarrow spray\ x = \{\}$
⟨*proof*⟩

lemma *in-spray-event*:
$y \in spray\ x \Longrightarrow y \in \mathcal{E}$
⟨*proof*⟩

lemma *source-in-spray*:
$spray\ x \neq \{\} \Longrightarrow x \in spray\ x$
⟨*proof*⟩

7 Primitives: Path (In)dependence

"A subset of three paths of a SPRAY is dependent if there is a path which does not belong to the SPRAY and which contains one event from each of the three paths: we also say any one of the three paths is dependent on the other two. Otherwise the subset is independent." [Schutz97]

The definition of *SPRAY* constrains x, Q, R, S to be in $\mathcal{E}$ and $\mathcal{P}$.

definition *dep3-event* :: *'a set* ⇒ *'a set* ⇒ *'a set* ⇒ *'a* ⇒ *bool* **where**
$dep3\text{-}event\ Q\ R\ S\ x \equiv Q \neq R \wedge Q \neq S \wedge R \neq S \wedge Q \in SPRAY\ x \wedge R \in SPRAY\ x \wedge S \in SPRAY\ x$

$\wedge$ ($\exists\, T{\in}\mathcal{P}.\ T \notin SPRAY\ x \wedge (\exists\, y{\in}Q.\ y \in T) \wedge (\exists\, y{\in}R.\ y \in T) \wedge (\exists\, y{\in}S.\ y \in T)$)

definition *dep3-spray* :: *'a set* ⇒ *'a set* ⇒ *'a set* ⇒ (*'a set*) *set* ⇒ *bool* **where**
dep3-spray Q R S SPR ≡ ∃ *x*. *SPRAY x* = *SPR* ∧ *dep3-event Q R S x*

definition *dep3* :: *'a set* ⇒ *'a set* ⇒ *'a set* ⇒ *bool* **where**
dep3 Q R S ≡ ∃ *x*. *dep3-event Q R S x*

Some very simple lemmas related to *dep3-event*.

lemma *dep3-nonspray*:
assumes *dep3-event Q R S x*
shows $\exists\, P{\in}\mathcal{P}.\ P \notin SPRAY\ x$
⟨*proof*⟩

lemma *dep3-path*:
assumes *dep3-QRSx*: *dep3-event Q R S x*
shows $Q \in \mathcal{P}$ $R \in \mathcal{P}$ $S \in \mathcal{P}$
⟨*proof*⟩

lemma *dep3-is-event*:
dep3-event Q R S x ⟹ $x \in \mathcal{E}$
⟨*proof*⟩

lemma *dep3-event-permute* [*no-atp*]:
assumes *dep3-event Q R S x*
shows *dep3-event Q S R x* *dep3-event R Q S x* *dep3-event R S Q x*
dep3-event S Q R x *dep3-event S R Q x*
⟨*proof*⟩

"We next give recursive definitions of dependence and independence which will be used to characterize the concept of dimension. A path T is dependent on the set of n paths (where $n \geq 3$)

$$S = \{Q_i : i = 1, 2, ..., n; Q_i \in \mathrm{SPRAY}x\}$$

if it is dependent on two paths S_1 and S_2, where each of these two paths is dependent on some subset of $n - 1$ paths from the set S." [Schutz97]

inductive *dep-path* :: *'a set* ⇒ (*'a set*) *set* ⇒ *'a* ⇒ *bool* **where**
dep-two: *dep3-event T A B x* ⟹ *dep-path T* {*A*, *B*} *x*
| *dep-n*: ⟦*S* ⊆ *SPRAY x*; *card S* ≥ *3*; *dep-path T* {*S1*, *S2*} *x*;
S' ⊆ *S*; *S''* ⊆ *S*; *card S'* = *card S* − *1*; *card S''* = *card S* − *1*;
dep-path S1 S' x; *dep-path S2 S'' x*⟧ ⟹ *dep-path T S x*

"We also say that the set of $n{+}1$ paths $S{\cup}\{T\}$ is a dependent set." [Schutz97] Starting from this constructive definition, the below gives an analytical one.

definition *dep-set* :: (*'a set*) *set* ⇒ *bool* **where**
dep-set S ≡ ∃ *x*. ∃ *S'*⊆*S*. ∃ *P*∈(*S*−*S'*). *dep-path P S' x*

lemma *dependent-superset*:
assumes *dep-set* A **and** $A \subseteq B$
shows *dep-set* B
⟨*proof*⟩

lemma *path-in-dep-set*:
assumes *dep3-event* $P\ Q\ R\ x$
shows *dep-set* $\{P,Q,R\}$
⟨*proof*⟩

lemma *path-in-dep-set2*:
assumes *dep3-event* $P\ Q\ R\ x$
shows *dep-path* $P\ \{P,Q,R\}\ x$
⟨*proof*⟩

definition *indep-set* :: $('a\ set)\ set \Rightarrow bool$ **where**
indep-set $S \equiv \neg(\exists\ T \subseteq S.\ dep\text{-}set\ T)$

8 Primitives: 3-SPRAY

"We now make the following definition which enables us to specify the dimensions of Minkowski space-time. A SPRAY is a 3-SPRAY if: i) it contains four independent paths, and ii) all paths of the SPRAY are dependent on these four paths." [Schutz97]

definition *three-SPRAY* :: $'a \Rightarrow bool$ **where**
$three\text{-}SPRAY\ x \equiv \exists\ S1 \in \mathcal{P}.\ \exists\ S2 \in \mathcal{P}.\ \exists\ S3 \in \mathcal{P}.\ \exists\ S4 \in \mathcal{P}.$
$S1 \neq S2 \land S1 \neq S3 \land S1 \neq S4 \land S2 \neq S3 \land S2 \neq S4 \land S3 \neq S4$
$\land\ S1 \in SPRAY\ x \land S2 \in SPRAY\ x \land S3 \in SPRAY\ x \land S4 \in SPRAY\ x$
$\land\ (indep\text{-}set\ \{S1, S2, S3, S4\})$
$\land\ (\forall\ S \in SPRAY\ x.\ dep\text{-}path\ S\ \{S1,S2,S3,S4\}\ x)$

Lemma *is-three-SPRAY* says "this set of sets of elements is a set of paths which is a 3-SPRAY". Lemma *three-SPRAY-ge4* just extracts a bit of the definition.

definition *is-three-SPRAY* :: $('a\ set)\ set \Rightarrow bool$ **where**
$is\text{-}three\text{-}SPRAY\ SPR \equiv \exists\ x.\ SPR = SPRAY\ x \land three\text{-}SPRAY\ x$

lemma *three-SPRAY-ge4*:
assumes *three-SPRAY* x
shows $\exists\ Q1 \in \mathcal{P}.\ \exists\ Q2 \in \mathcal{P}.\ \exists\ Q3 \in \mathcal{P}.\ \exists\ Q4 \in \mathcal{P}.\ Q1 \neq Q2 \land Q1 \neq Q3 \land Q1 \neq Q4 \land Q2 \neq Q3 \land Q2 \neq Q4 \land Q3 \neq Q4$
⟨*proof*⟩

end

9 MinkowskiBetweenness: O1-O5

In O4, I have removed the requirement that $a \neq d$ in order to prove negative betweenness statements as Schutz does. For example, if we have [abc] and [bca] we want to conclude [aba] and claim "contradiction!", but we can't as long as we mandate that $a \neq d$.

locale *MinkowskiBetweenness* = *MinkowskiPrimitive* +
fixes *betw* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ ([[- - -]])

assumes *abc-ex-path*: $[[a\ b\ c]] \Longrightarrow \exists\, Q{\in}\mathcal{P}.\ a \in Q \wedge b \in Q \wedge c \in Q$

and *abc-sym*: $[[a\ b\ c]] \Longrightarrow [[c\ b\ a]]$

and *abc-ac-neq*: $[[a\ b\ c]] \Longrightarrow a \neq c$

and *abc-bcd-abd* [*intro*]: $[\![[[a\ b\ c]]; [[b\ c\ d]]]\!] \Longrightarrow [[a\ b\ d]]$

and *some-betw*: $[\![Q \in \mathcal{P};\ a \in Q;\ b \in Q;\ c \in Q;\ a \neq b;\ a \neq c;\ b \neq c]\!]$
$\Longrightarrow [[a\ b\ c]] \vee [[b\ c\ a]] \vee [[c\ a\ b]]$

begin

The next few lemmas either provide the full axiom from the text derived from a new simpler statement, or provide some very simple fundamental additions which make sense to prove immediately before starting, usually related to set-level things that should be true which fix the type-level ambiguity of 'a.

lemma *betw-events*:
assumes *abc*: $[[a\ b\ c]]$
shows $a \in \mathcal{E} \wedge b \in \mathcal{E} \wedge c \in \mathcal{E}$
⟨*proof*⟩

This shows the shorter version of O5 is equivalent.

lemma *O5-still-O5* [*no-atp*]:
$((Q \in \mathcal{P} \wedge \{a,b,c\} \subseteq Q \wedge a \in \mathcal{E} \wedge b \in \mathcal{E} \wedge c \in \mathcal{E} \wedge a \neq b \wedge a \neq c \wedge b \neq c)$
$\longrightarrow [[a\ b\ c]] \vee [[b\ c\ a]] \vee [[c\ a\ b]])$
$=$
$((Q \in \mathcal{P} \wedge \{a,b,c\} \subseteq Q \wedge a \in \mathcal{E} \wedge b \in \mathcal{E} \wedge c \in \mathcal{E} \wedge a \neq b \wedge a \neq c \wedge b \neq c)$
$\longrightarrow [[a\ b\ c]] \vee [[b\ c\ a]] \vee [[c\ a\ b]] \vee [[c\ b\ a]] \vee [[a\ c\ b]] \vee [[b\ a\ c]])$
⟨*proof*⟩

lemma *some-betw-xor*:
$[\![Q \in \mathcal{P};\ a \in Q;\ b \in Q;\ c \in Q;\ a \neq b;\ a \neq c;\ b \neq c]\!]$
$\Longrightarrow ([[a\ b\ c]] \wedge \neg\ [[b\ c\ a]] \wedge \neg\ [[c\ a\ b]])$
$\vee ([[b\ c\ a]] \wedge \neg\ [[a\ b\ c]] \wedge \neg\ [[c\ a\ b]])$
$\vee ([[c\ a\ b]] \wedge \neg\ [[a\ b\ c]] \wedge \neg\ [[b\ c\ a]])$
⟨*proof*⟩

The lemma *abc-abc-neq* is the full O3 as stated by Schutz.

lemma *abc-abc-neq*:

assumes *abc*: $[[a\ b\ c]]$
shows $a \neq b \wedge a \neq c \wedge b \neq c$
$\langle proof \rangle$

lemma *abc-bcd-acd*:
assumes *abc*: $[[a\ b\ c]]$
and *bcd*: $[[b\ c\ d]]$
shows $[[a\ c\ d]]$
$\langle proof \rangle$

lemma *abc-only-cba*:
assumes $[[a\ b\ c]]$
shows $\neg\ [[b\ a\ c]]$ $\neg\ [[a\ c\ b]]$ $\neg\ [[b\ c\ a]]$ $\neg\ [[c\ a\ b]]$
$\langle proof \rangle$

10 Betweenness: Unreachable Subset Via a Path

definition *unreachable-subset-via* :: $'a\ set \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow 'a \Rightarrow 'a\ set$
($\emptyset$ *- from - via - at -* [*100*, *100*, *100*, *100*] *100*) **where**
unreachable-subset-via $Q\ Qa\ R\ x \equiv \{Qy.\ [[x\ Qy\ Qa]] \wedge (\exists Rw{\in}R.\ Qa \in \emptyset\ Q\ Rw \wedge Qy \in \emptyset\ Q\ Rw)\}$

11 Betweenness: Chains

11.1 Totally ordered chains with indexing

definition *short-ch* :: $'a\ set \Rightarrow bool$ **where**
short-ch $X \equiv$
— EITHER two distinct events connected by a path
$\exists x{\in}X.\ \exists y{\in}X.\ path\text{-}ex\ x\ y \wedge \neg(\exists z{\in}X.\ z{\neq}x \wedge z{\neq}y)$

Infinite sets have card 0, because card gives a natural number always.

definition *long-ch-by-ord* :: $(nat \Rightarrow 'a) \Rightarrow 'a\ set \Rightarrow bool$ **where**
long-ch-by-ord $f\ X \equiv$
— OR at least three events such that any three events are ordered
$\exists x{\in}X.\ \exists y{\in}X.\ \exists z{\in}X.\ x{\neq}y \wedge y{\neq}z \wedge x{\neq}z \wedge ordering\ f\ betw\ X$

Does this restrict chains to lie on paths? Proven in Ch3's Interlude!

definition *ch-by-ord* :: $(nat \Rightarrow 'a) \Rightarrow 'a\ set \Rightarrow bool$ **where**
ch-by-ord $f\ X \equiv short\text{-}ch\ X \vee long\text{-}ch\text{-}by\text{-}ord\ f\ X$

definition *ch* :: $'a\ set \Rightarrow bool$ **where**
ch $X \equiv \exists f.\ ch\text{-}by\text{-}ord\ f\ X$

Since $f(0)$ is always in the chain, and plays a special role particularly for infinite chains (as the 'endpoint', the non-finite edge) let us fix it straight in the definition. Notice we require both *infinite X* and *long-ch-by-ord*, thus circumventing infinite Isabelle sets having cardinality 0.

definition *semifin-chain*:: $(nat \Rightarrow 'a) \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow bool$ ([-[- ..]-]) **where**
  semifin-chain f x Q ≡
    infinite Q ∧ *long-ch-by-ord f Q*
    ∧ *f 0* = *x*

definition *fin-long-chain*:: $(nat \Rightarrow 'a) \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow bool$
  ([-[- .. - .. -]-]) **where**
  fin-long-chain f x y z Q ≡
    x≠*y* ∧ *x*≠*z* ∧ *y*≠*z*
    ∧ *finite Q* ∧ *long-ch-by-ord f Q*
    ∧ *f 0* = *x* ∧ *y*∈*Q* ∧ *f* (*card Q* − *1*) = *z*

lemma *index-middle-element*:
  assumes [*f*[*a*..*b*..*c*]*X*]
  shows ∃ *n*. *0*<*n* ∧ *n*<(*card X* − *1*) ∧ *f n* = *b*
⟨*proof*⟩

lemma *fin-ch-betw*:
  assumes [*f*[*a*..*b*..*c*]*X*]
  shows [[*a b c*]]
⟨*proof*⟩

lemma *chain-sym-obtain*:
  assumes [*f*[*a*..*b*..*c*]*X*]
  obtains *g* **where** [*g*[*c*..*b*..*a*]*X*] **and** *g*=(λ*n*. *f* (*card X* − *1* − *n*))
⟨*proof*⟩

lemma *chain-sym*:
  assumes [*f*[*a*..*b*..*c*]*X*]
    shows [λ*n*. *f* (*card X* − *1* − *n*)[*c*..*b*..*a*]*X*]
  ⟨*proof*⟩

definition *fin-long-chain-2*:: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow bool$ **where**
  fin-long-chain-2 x y z Q ≡ ∃ *f*. [*f*[*x*..*y*..*z*]*Q*]

definition *fin-chain*:: $(nat \Rightarrow 'a) \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow bool$ ([-[- .. -]-]) **where**
  fin-chain f x y Q ≡
    (*short-ch Q* ∧ *x*∈*Q* ∧ *y*∈*Q* ∧ *x*≠*y*)
    ∨ (∃ *z*∈*Q*. [*f*[*x*..*z*..*y*]*Q*])

lemma *points-in-chain*:
  assumes [*f*[*x*..*y*..*z*]*Q*]
  shows *x*∈*Q* ∧ *y*∈*Q* ∧ *z*∈*Q*
⟨*proof*⟩

lemma *ch-long-if-card-ge3*:
  assumes *ch X*

and *card* $X \geq 3$
shows $\exists f.$ *long-ch-by-ord* f X
⟨*proof*⟩

11.2 Locally ordered chains with indexing

Definition for Schutz-like chains, with local order only.

definition *long-ch-by-ord2* :: $(nat \Rightarrow {'a}) \Rightarrow {'a}\ set \Rightarrow bool$ **where**
long-ch-by-ord2 f $X \equiv$
$\exists x{\in}X.\ \exists y{\in}X.\ \exists z{\in}X.\ x{\neq}y \wedge y{\neq}z \wedge x{\neq}z \wedge$ *ordering2* f *betw* X

11.3 Chains using betweenness

Old definitions of chains. Shown equivalent to *fin-long-chain-2* in TemporalOrderOnPath.thy.

definition *chain-with* :: ${'a} \Rightarrow {'a} \Rightarrow {'a} \Rightarrow {'a}\ set \Rightarrow bool$ ([[.. - .. - .. - ..]-]) **where**
chain-with $x\ y\ z\ X \equiv [[x\ y\ z]] \wedge x \in X \wedge y \in X \wedge z \in X \wedge (\exists f.$ *ordering* f *betw* $X)$
definition *finite-chain-with3* :: ${'a} \Rightarrow {'a} \Rightarrow {'a} \Rightarrow {'a}\ set \Rightarrow bool$ ([[- .. - .. -]-]) **where**
finite-chain-with3 $x\ y\ z\ X \equiv [[..x..y..z..]X] \wedge \neg(\exists w{\in}X.\ [[w\ x\ y]] \vee [[y\ z\ w]])$

lemma *long-chain-betw*: $[[..a..b..c..]X] \Longrightarrow [[a\ b\ c]]$
⟨*proof*⟩

lemma *finite-chain3-betw*: $[[a..b..c]X] \Longrightarrow [[a\ b\ c]]$
⟨*proof*⟩

definition *finite-chain-with2* :: ${'a} \Rightarrow {'a} \Rightarrow {'a}\ set \Rightarrow bool$ ([[- .. -]-]) **where**
finite-chain-with2 $x\ z\ X \equiv \exists y{\in}X.\ [[x..y..z]X]$

lemma *finite-chain2-betw*: $[[a..c]X] \Longrightarrow \exists b.\ [[a\ b\ c]]$
⟨*proof*⟩

12 Betweenness: Rays and Intervals

"Given any two distinct events a, b of a path we define the segment $(ab) = \{x : [a\ x\ b],\ x \in ab\}$" [Schutz97] Our version is a little different, because it is defined for any a, b of type ${'a}$. Thus we can have empty set segments, while Schutz can prove (once he proves path density) that segments are never empty.

definition *segment* :: ${'a} \Rightarrow {'a} \Rightarrow {'a}\ set$
where *segment* $a\ b \equiv \{x{::}{'a}.\ \exists ab.\ [[a\ x\ b]] \wedge x{\in}ab \wedge$ *path* $ab\ a\ b\}$

abbreviation *is-segment* :: ${'a}\ set \Rightarrow bool$
where *is-segment* $ab \equiv (\exists a\ b.\ ab =$ *segment* $a\ b)$

definition *interval* :: $'a \Rightarrow 'a \Rightarrow 'a$ *set*
  where *interval a b* $\equiv$ *insert b* (*insert a* (*segment a b*))

abbreviation *is-interval* :: $'a$ *set* $\Rightarrow$ *bool*
  where *is-interval ab* $\equiv$ ($\exists$ *a b*. *ab* = *interval a b*)

definition *prolongation* :: $'a \Rightarrow 'a \Rightarrow 'a$ *set*
  where *prolongation a b* $\equiv$ $\{x::'a.\ \exists ab.\ [[a\ b\ x]] \wedge x \in ab \wedge path\ ab\ a\ b\}$

abbreviation *is-prolongation* :: $'a$ *set* $\Rightarrow$ *bool*
  where *is-prolongation ab* $\equiv$ $\exists$ *a b*. *ab* = *prolongation a b*

I think this is what Schutz actually meant, maybe there is a typo in the text? Notice that $b \in$ *ray a b* for any a, always. Cf the comment on *segment-def*. Thus $\exists$ *ray a b* $\neq \{\}$ is no guarantee that a path ab exists.

definition *ray* :: $'a \Rightarrow 'a \Rightarrow 'a$ *set*
  where *ray a b* $\equiv$ *insert b* (*segment a b* $\cup$ *prolongation a b*)

abbreviation *is-ray* :: $'a$ *set* $\Rightarrow$ *bool*
  where *is-ray R* $\equiv$ $\exists$ *a b*. *R* = *ray a b*

definition *is-ray-on* :: $'a$ *set* $\Rightarrow 'a$ *set* $\Rightarrow$ *bool*
  where *is-ray-on R P* $\equiv$ $P \in \mathcal{P} \wedge R \subseteq P \wedge$ *is-ray R*

This is as in Schutz. Notice b is not in the ray through b?

definition *ray-Schutz* :: $'a \Rightarrow 'a \Rightarrow 'a$ *set*
  where *ray-Schutz a b* $\equiv$ *insert a* (*segment a b* $\cup$ *prolongation a b*)

lemma *ends-notin-segment*: $a \notin$ *segment a b* $\wedge$ $b \notin$ *segment a b*
  ⟨*proof*⟩

lemma *ends-in-int*: $a \in$ *interval a b* $\wedge$ $b \in$ *interval a b*
  ⟨*proof*⟩

lemma *seg-betw*: $x \in$ *segment a b* $\longleftrightarrow$ $[[a\ x\ b]]$
  ⟨*proof*⟩

lemma *pro-betw*: $x \in$ *prolongation a b* $\longleftrightarrow$ $[[a\ b\ x]]$
  ⟨*proof*⟩

lemma *seg-sym*: *segment a b* = *segment b a*
  ⟨*proof*⟩

lemma *empty-segment*: *segment a a* = $\{\}$
  ⟨*proof*⟩

lemma *int-sym*: *interval a b* = *interval b a*
  ⟨*proof*⟩

lemma *seg-path*:
  assumes $x \in$ *segment a b*
  obtains *ab* **where** *path ab a b segment a b* $\subseteq$ *ab*
$\langle proof \rangle$

lemma *seg-path2*:
  assumes *segment a b* $\neq \{\}$
  obtains *ab* **where** *path ab a b segment a b* $\subseteq$ *ab*
  $\langle proof \rangle$

Path density (theorem 17) will extend this by weakening the assumptions to *segment a b* $\neq \{\}$.

lemma *seg-endpoints-on-path*:
  assumes *card* (*segment a b*) $\geq$ *2 segment a b* $\subseteq$ *P* $P \in \mathcal{P}$
  shows *path P a b*
$\langle proof \rangle$

lemma *pro-path*:
  assumes $x \in$ *prolongation a b*
  obtains *ab* **where** *path ab a b prolongation a b* $\subseteq$ *ab*
$\langle proof \rangle$

lemma *ray-cases*:
  assumes $x \in$ *ray a b*
  shows $[[a\ x\ b]] \vee [[a\ b\ x]] \vee x = b$
$\langle proof \rangle$

lemma *ray-path*:
  assumes $x \in$ *ray a b* $x \neq b$
  obtains *ab* **where** *path ab a b* $\wedge$ *ray a b* $\subseteq$ *ab*
$\langle proof \rangle$

end

13 MinkowskiChain: O6

O6 supposedly serves the same purpose as Pasch's axiom.

locale *MinkowskiChain* = *MinkowskiBetweenness* +
  assumes *O6*: $[\![Q \in \mathcal{P};\ R \in \mathcal{P};\ S \in \mathcal{P};\ T \in \mathcal{P};\ Q \neq R;\ Q \neq S;\ R \neq S;\ a \in Q \cap R \wedge b \in Q \cap S \wedge c \in R \cap S;$
  $\exists\, d \in S.\ [[b\ c\ d]] \wedge (\exists\, e \in R.\ d \in T \wedge e \in T \wedge [[c\ e\ a]])]\!]$
  $\Longrightarrow \exists\, f \in T \cap Q.\ \exists\, X.\ [[a..f..b]X]$
begin

14 Chains: (Closest) Bounds

definition *is-bound-f* :: $'a \Rightarrow\ 'a\ set \Rightarrow (nat \Rightarrow 'a) \Rightarrow bool$ **where**

is-bound-f Q_b Q f $\equiv$
$\forall\, i\ j$ ::*nat*. $[f[(f\ 0)..]Q] \wedge (i{<}j \longrightarrow [[(f\ i)\ (f\ j)\ Q_b]])$

definition *is-bound* :: $'a \Rightarrow 'a\ set \Rightarrow bool$ **where**
is-bound Q_b Q $\equiv$
$\exists\, f$::$(nat{\Rightarrow}'a)$. *is-bound-f* Q_b Q f

Q_b has to be on the same path as the chain Q. This is left implicit in the betweenness condition (as is $Q_b \in \mathcal{E}$). So this is equivalent to Schutz only if we also assume his axioms, i.e. the statement of the continuity axiom is no longer independent of other axioms.

definition *all-bounds* :: $'a\ set \Rightarrow 'a\ set$ **where**
all-bounds $Q = \{Q_b.$ *is-bound* $Q_b\ Q\}$

definition *bounded* :: $'a\ set \Rightarrow bool$ **where**
bounded $Q \equiv \exists\ Q_b$. *is-bound* Q_b Q

Just to make sure Continuity is not too strong.

lemma *bounded-imp-inf*:
assumes *bounded Q*
shows *infinite Q*
⟨*proof*⟩

definition *closest-bound-f* :: $'a \Rightarrow 'a\ set \Rightarrow (nat{\Rightarrow}'a) \Rightarrow bool$ **where**
closest-bound-f Q_b Q f $\equiv$
~~Q is an infinite chain indexed by f bound by Q_b~~
is-bound-f Q_b Q f $\wedge$
~~Any other bound must be further from the start of the chain than the closest bound~~
$(\forall\ Q_b'.\ ($*is-bound* $Q_b'\ Q \wedge Q_b' \neq Q_b) \longrightarrow [[(f\ 0)\ Q_b\ Q_b']])$

definition *closest-bound* :: $'a \Rightarrow 'a\ set \Rightarrow bool$ **where**
closest-bound Q_b Q $\equiv$
~~Q is an infinite chain indexed by f bound by Q_b~~
$\exists\, f$. *is-bound-f* Q_b Q f $\wedge$
~~Any other bound must be further from the start of the chain than the closest bound~~
$(\forall\ Q_b'.\ ($*is-bound* $Q_b'\ Q \wedge Q_b' \neq Q_b) \longrightarrow [[(f\ 0)\ Q_b\ Q_b']])$

end

15 MinkowskiUnreachable: I5-I7

locale *MinkowskiUnreachable* = *MinkowskiChain* +

assumes *two-in-unreach*: $[\![Q \in \mathcal{P};\ b \in \mathcal{E};\ b \notin Q]\!] \Longrightarrow \exists\, x{\in}\emptyset\ Q\ b.\ \exists\, y{\in}\emptyset\ Q\ b.\ x \neq y$

and *I6*: $[\![Q \in \mathcal{P};\ b \notin Q;\ b \in \mathcal{E};\ Qx \in (\emptyset\ Q\ b);\ Qz \in (\emptyset\ Q\ b);\ Qx{\neq}Qz]\!]$
$\Longrightarrow \exists\, X.\ \exists f.\ \textit{ch-by-ord}\ f\ X \wedge f\ 0 = Qx \wedge f\ (\textit{card}\ X - 1) = Qz$
$\wedge\ (\forall\, i{\in}\{1\ ..\ \textit{card}\ X - 1\}.\ (f\ i) \in \emptyset\ Q\ b$
$\wedge\ (\forall\, Qy{\in}\mathcal{E}.\ [[(f(i{-}1))\ Qy\ (f\ i)]] \longrightarrow Qy \in \emptyset\ Q\ b))$
$\wedge\ (\textit{short-ch}\ X \longrightarrow Qx{\in}X \wedge Qz{\in}X \wedge (\forall\, Qy{\in}\mathcal{E}.\ [[Qx\ Qy\ Qz]]$
$\longrightarrow Qy \in \emptyset\ Q\ b))$
and *I7*: $[\![Q \in \mathcal{P};\ b \notin Q;\ b \in \mathcal{E};\ Qx \in Q - \emptyset\ Q\ b;\ Qy \in \emptyset\ Q\ b]\!]$
$\Longrightarrow \exists\, g\ X\ Qn.\ [g[Qx..Qy..Qn]X] \wedge Qn \in Q - \emptyset\ Q\ b$
begin

lemma *card-unreach-geq-2*:
assumes $Q{\in}\mathcal{P}\ \ b{\in}\mathcal{E}{-}Q$
shows $2 \leq \textit{card}\ (\emptyset\ Q\ b) \vee (\textit{infinite}\ (\emptyset\ Q\ b))$
$\langle \textit{proof} \rangle$

end

16 MinkowskiSymmetry: Symmetry

locale *MinkowskiSymmetry* = *MinkowskiUnreachable* +
assumes *Symmetry*: $[\![Q \in \mathcal{P};\ R \in \mathcal{P};\ S \in \mathcal{P};\ Q \neq R;\ Q \neq S;\ R \neq S;$
$x \in Q{\cap}R{\cap}S;\ Q_a \in Q;\ Q_a \neq x;$
$\emptyset\ Q\ \textit{from}\ Q_a\ \textit{via}\ R\ \textit{at}\ x = \emptyset\ Q\ \textit{from}\ Q_a\ \textit{via}\ S\ \textit{at}\ x]\!]$
$\Longrightarrow \exists\, \vartheta{::}'a{\Rightarrow}'a.$ ~~i) there is a map $\vartheta{:}\mathcal{E}{\Rightarrow}\mathcal{E}$~~
$\textit{bij-betw}\ (\lambda P.\ \{\vartheta\ y \mid y.\ y{\in}P\})\ \mathcal{P}\ \mathcal{P}$ ~~ii) which induces a bijection~~
~~Θ~~
$\wedge\ (y{\in}Q \longrightarrow \vartheta\ y = y)$ ~~iii) ϑ leaves Q invariant~~
$\wedge\ (\lambda P.\ \{\vartheta\ y \mid y.\ y{\in}P\})\ R = S$ ~~iv) Θ maps R to S~~

17 MinkowskiContinuity: Continuity

locale *MinkowskiContinuity* = *MinkowskiSymmetry* +
assumes *Continuity*: $\textit{bounded}\ Q \Longrightarrow (\exists\, Q_b.\ \textit{closest-bound}\ Q_b\ Q)$

18 MinkowskiSpacetime: Dimension (I4)

locale *MinkowskiSpacetime* = *MinkowskiContinuity* +

assumes *ex-3SPRAY* [*simp*]: $[\![\mathcal{E} \neq \{\}]\!] \Longrightarrow \exists\, x{\in}\mathcal{E}.\ \textit{three-SPRAY}\ x$
begin

There exists an event by *nonempty-events*, and by *ex-3SPRAY* there is a three-SPRAY, which by *three-SPRAY-ge4* means that there are at least four paths.

lemma *four-paths*:

$\exists\, Q1 \in \mathcal{P}.\ \exists\, Q2 \in \mathcal{P}.\ \exists\, Q3 \in \mathcal{P}.\ \exists\, Q4 \in \mathcal{P}.\ Q1 \neq Q2 \wedge Q1 \neq Q3 \wedge Q1 \neq Q4 \wedge Q2 \neq Q3 \wedge Q2 \neq Q4 \wedge Q3 \neq Q4$
$\langle proof \rangle$

end

end

theory *TemporalOrderOnPath*
imports *Main Minkowski TernaryOrdering*
begin

In Schutz [1, pp. 18-30], this is "Chapter 3: Temporal order on a path". All theorems are from Schutz, all lemmas are either parts of the Schutz proofs extracted, or additional lemmas which needed to be added, with the exception of the three transitivity lemmas leading to Theorem 9, which are given by Schutz as well. Much of what we'd like to prove about chains with respect to injectivity, surjectivity, bijectivity, is proved in *TernaryOrdering.thy*. Some more things are proved in interlude sections.

Disable list syntax.

no-translations
  $[x, xs] == x\#[xs]$
  $[x] == x\#[]$
no-syntax
  — list Enumeration
  -list :: *args* => *'a list* ([(-)])
no-notation *Cons* (**infixr** # *65*)
no-notation *Nil* ([])

19 Preliminary Results for Primitives

First some proofs that belong in this section but aren't proved in the book or are covered but in a different form or off-handed remark.

context *MinkowskiPrimitive* **begin**

lemma *three-in-set3*:
  assumes *card* $X \geq 3$
  obtains $x\ y\ z$ **where** $x{\in}X$ **and** $y{\in}X$ **and** $z{\in}X$ **and** $x{\neq}y$ **and** $x{\neq}z$ **and** $y{\neq}z$
  ⟨*proof*⟩

lemma *paths-cross-once*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *path-R*: $R \in \mathcal{P}$
    and *Q-neq-R*: $Q \neq R$
    and *QR-nonempty*: $Q{\cap}R \neq \{\}$
  shows $\exists!a{\in}\mathcal{E}.\ Q{\cap}R = \{a\}$
⟨*proof*⟩

lemma *cross-once-notin*:
  assumes $Q \in \mathcal{P}$
    and $R \in \mathcal{P}$
    and $a \in Q$
    and $b \in Q$

and $b \in R$
and $a \neq b$
and $Q \neq R$
shows $a \notin R$
⟨*proof*⟩

lemma *paths-cross-at*:
assumes *path-Q*: $Q \in \mathcal{P}$ **and** *path-R*: $R \in \mathcal{P}$
and *Q-neq-R*: $Q \neq R$
and *QR-nonempty*: $Q \cap R \neq \{\}$
and *x-inQ*: $x \in Q$ **and** *x-inR*: $x \in R$
shows $Q \cap R = \{x\}$
⟨*proof*⟩

lemma *events-distinct-paths*:
assumes *a-event*: $a \in \mathcal{E}$
and *b-event*: $b \in \mathcal{E}$
and *a-neq-b*: $a \neq b$
shows $\exists R{\in}\mathcal{P}.\ \exists S{\in}\mathcal{P}.\ a \in R \wedge b \in S \wedge (R \neq S \longrightarrow (\exists! c{\in}\mathcal{E}.\ R \cap S = \{c\}))$
⟨*proof*⟩

end
context *MinkowskiBetweenness* **begin**

lemma assumes $[[a\ b\ c]]$ **shows** $\exists f.\ \textit{long-ch-by-ord}\ f\ \{a,b,c\}$
⟨*proof*⟩

lemma *between-chain*: $[[a\ b\ c]] \Longrightarrow ch\ \{a,b,c\}$
⟨*proof*⟩

lemma *overlap-chain*: $[\![[[a\ b\ c]];\ [[b\ c\ d]]]\!] \Longrightarrow ch\ \{a,b,c,d\}$
⟨*proof*⟩

end

20 3.1 Order on a finite chain

context *MinkowskiBetweenness* **begin**

20.1 Theorem 1

See *Minkowski.abc-only-cba*. Proving it again here to show it can be done following the prose in Schutz.

theorem *theorem1* [*no-atp*]:
assumes *abc*: $[[a\ b\ c]]$
shows $[[c\ b\ a]] \wedge \neg\ [[b\ c\ a]] \wedge \neg\ [[c\ a\ b]]$
⟨*proof*⟩

20.2 Theorem 2

The lemma *abc-bcd-acd*, equal to the start of Schutz's proof, is given in *Minkowski* in order to prove some equivalences. Splitting it up into the proof of: "there is a betweenness relation for each ordered triple", and "all events of a chain are distinct" The first part is obvious with total chains (using *ordering*), and will be proved using the local definition as well (*ordering2*), following Schutz' proof. The second part is proved as injectivity of the indexing function (see *index-injective*).

For the case of two-element chains: the elements are distinct by definition, and the statement on ordering is void (respectively, *False* $\Longrightarrow$ P for any P).

theorem *order-finite-chain*:
  assumes *chX*: *long-ch-by-ord f X*
    and *finiteX*: *finite X*
    and *ordered-nats*: $0 \leq (i::nat) \wedge i < j \wedge j < l \wedge l < card\ X$
  shows $[[(f\ i)\ (f\ j)\ (f\ l)]]$
  ⟨*proof*⟩

lemma *thm2-ind1*:
  assumes *chX*: *long-ch-by-ord2 f X*
    and *finiteX*: *finite X*
  shows $\forall j\ i.\ ((i::nat) < j \wedge j < card\ X - 1) \longrightarrow [[(f\ i)\ (f\ j)\ (f\ (j + 1))]]$
⟨*proof*⟩

lemma *thm2-ind2*:
  assumes *chX*: *long-ch-by-ord2 f X*
    and *finiteX*: *finite X*
  shows $\forall m\ l.\ (0<(l-m) \wedge (l-m) < l \wedge l < card\ X) \longrightarrow [[(f\ ((l-m)-1))\ (f\ (l-m))\ (f\ l)]]$
⟨*proof*⟩

lemma *thm2-ind2b*:
  assumes *chX*: *long-ch-by-ord2 f X*
    and *finiteX*: *finite X*
    and *ordered-nats*: $0<k \wedge k<l \wedge l < card\ X$
  shows $[[(f\ (k-1))\ (f\ k)\ (f\ l)]]$
  ⟨*proof*⟩

This is Theorem 2 properly speaking, except for the "chain elements are distinct" part (which is proved as injectivity of the index later). Follows Schutz fairly well! The statement Schutz proves under (i) is given in *Minkowski-Betweenness.abc-bcd-acd* instead.

theorem *order-finite-chain2*:
  assumes *chX*: *long-ch-by-ord2 f X*
    and *finiteX*: *finite X*
    and *ordered-nats*: $0 \leq (i::nat) \wedge i < j \wedge j < l \wedge l < card\ X$

shows $[[(f\ i)\ (f\ j)\ (f\ l)]]$
⟨*proof*⟩

lemma *three-in-long-chain2*:
assumes *long-ch-by-ord2 f X*
obtains $x\ y\ z$ **where** $x\in X$ **and** $y\in X$ **and** $z\in X$ **and** $x\neq y$ **and** $x\neq z$ **and** $y\neq z$
⟨*proof*⟩

lemma *short-ch-card-2*:
assumes *ch-by-ord f X*
shows *short-ch X* $\longleftrightarrow$ *card X* $= 2$
⟨*proof*⟩

lemma *long-chain2-card-geq*:
assumes *long-ch-by-ord2 f X* **and** *fin*: *finite X*
shows *card X* ≥ 3
⟨*proof*⟩

lemma *fin-chain-card-geq-2*:
assumes $[f[a..b]X]$
shows *card X* ≥ 2
⟨*proof*⟩

theorem *index-injective*:
fixes i::*nat* **and** j::*nat*
assumes *chX*: *long-ch-by-ord2 f X*
and *finiteX*: *finite X*
and *indices*: $i<j$ $j<$ *card X*
shows $f\ i \neq f\ j$
⟨*proof*⟩

end

21 Finite chain equivalence: local ↔ global

context *MinkowskiBetweenness* **begin**

lemma *ch-equiv1*:
assumes *long-ch-by-ord f X finite X*
shows *long-ch-by-ord2 f X*
⟨*proof*⟩

lemma *ch-equiv2*:
  assumes *long-ch-by-ord2 f X finite X*
  shows *long-ch-by-ord f X*
  ⟨*proof*⟩

lemma *ch-equiv*:
  assumes *finite X*
  shows *long-ch-by-ord f X* $\longleftrightarrow$ *long-ch-by-ord2 f X*
  ⟨*proof*⟩

end

22 Preliminary Results for Kinematic Triangles and Paths/Betweenness

Theorem 3 (collinearity) First we prove some lemmas that will be very helpful.

context *MinkowskiPrimitive* **begin**

lemma *triangle-permutes* [*no-atp*]:
  assumes $\triangle\ a\ b\ c$
  shows $\triangle\ a\ c\ b\ \triangle\ b\ a\ c\ \triangle\ b\ c\ a\ \triangle\ c\ a\ b\ \triangle\ c\ b\ a$
  ⟨*proof*⟩

lemma *triangle-paths* [*no-atp*]:
  assumes *tri-abc*: $\triangle\ a\ b\ c$
  shows *path-ex a b path-ex a c path-ex b c*
⟨*proof*⟩

lemma *triangle-paths-unique*:
  assumes *tri-abc*: $\triangle\ a\ b\ c$
  shows $\exists!ab.$ *path ab a b*
  ⟨*proof*⟩

The definition of the kinematic triangle says that there exist paths that a and b pass through, and a and c pass through etc that are not equal. But we can show there is a *unique* ab that a and b pass through, and assuming there is a path abc that a, b, c pass through, it must be unique. Therefore $ab = abc$ and $ac = abc$, but $ab \neq ac$, therefore *False*. Lemma *tri-three-paths* is not in the books but might simplify some path obtaining.

lemma *triangle-diff-paths*:
  assumes *tri-abc*: $\triangle\ a\ b\ c$
  shows $\neg\ (\exists\ Q \in \mathcal{P}.\ a \in Q \land b \in Q \land c \in Q)$

⟨*proof*⟩

lemma *tri-three-paths* [*elim*]:
  assumes *tri-abc*: $\triangle\ a\ b\ c$
  shows $\exists\, ab\ bc\ ca.\ path\ ab\ a\ b \wedge path\ bc\ b\ c \wedge path\ ca\ c\ a \wedge ab \neq bc \wedge ab \neq ca \wedge bc \neq ca$
⟨*proof*⟩

lemma *triangle-paths-neq*:
  assumes *tri-abc*: $\triangle\ a\ b\ c$
    and *path-ab*: $path\ ab\ a\ b$
    and *path-ac*: $path\ ac\ a\ c$
  shows $ab \neq ac$
⟨*proof*⟩

end
context *MinkowskiBetweenness* **begin**

lemma *abc-ex-path-unique*:
  assumes *abc*: $[[a\ b\ c]]$
  shows $\exists!Q{\in}\mathcal{P}.\ a \in Q \wedge b \in Q \wedge c \in Q$
⟨*proof*⟩

lemma *betw-c-in-path*:
  assumes *abc*: $[[a\ b\ c]]$
    and *path-ab*: $path\ ab\ a\ b$
  shows $c \in ab$

⟨*proof*⟩

lemma *betw-b-in-path*:
  assumes *abc*: $[[a\ b\ c]]$
    and *path-ab*: $path\ ac\ a\ c$
  shows $b \in ac$
⟨*proof*⟩

lemma *betw-a-in-path*:
  assumes *abc*: $[[a\ b\ c]]$
    and *path-ab*: $path\ bc\ b\ c$
  shows $a \in bc$
⟨*proof*⟩

lemma *triangle-not-betw-abc*:
  assumes *tri-abc*: $\triangle\ a\ b\ c$
  shows $\neg\ [[a\ b\ c]]$
⟨*proof*⟩

lemma *triangle-not-betw-acb*:
  assumes *tri-abc*: $\triangle\ a\ b\ c$

shows ¬ [[*a* *c* *b*]]
⟨*proof*⟩

lemma *triangle-not-betw-bac*:
assumes *tri-abc*: △ *a* *b* *c*
shows ¬ [[*b* *a* *c*]]
⟨*proof*⟩

lemma *triangle-not-betw-any*:
assumes *tri-abc*: △ *a* *b* *c*
shows ¬ (∃ *d*∈{*a*,*b*,*c*}. ∃ *e*∈{*a*,*b*,*c*}. ∃ *f*∈{*a*,*b*,*c*}. [[*d* *e* *f*]])
⟨*proof*⟩

end

23 3.2 First collinearity theorem

theorem (**in** *MinkowskiChain*) *collinearity-alt2*:
assumes *tri-abc*: △ *a* *b* *c*
and *path-de*: *path de d e*

and *path-ab*: *path ab a b*
and *bcd*: [[*b* *c* *d*]]
and *cea*: [[*c* *e* *a*]]
shows ∃*f*∈*de*∩*ab*. [[*a* *f* *b*]]
⟨*proof*⟩

theorem (**in** *MinkowskiChain*) *collinearity-alt*:
assumes *tri-abc*: △ *a* *b* *c*
and *path-de*: *path de d e*
and *bcd*: [[*b* *c* *d*]]
and *cea*: [[*c* *e* *a*]]
shows ∃ *ab*. *path ab a b* ∧ (∃*f*∈*de*∩*ab*. [[*a* *f* *b*]])
⟨*proof*⟩

theorem (**in** *MinkowskiChain*) *collinearity*:
assumes *tri-abc*: △ *a* *b* *c*
and *path-de*: *path de d e*
and *bcd*: [[*b* *c* *d*]]
and *cea*: [[*c* *e* *a*]]
shows (∃*f*∈*de*∩(*path-of a b*). [[*a* *f* *b*]])
⟨*proof*⟩

24 Additional results for Paths and Unreachables

context *MinkowskiPrimitive* **begin**

The degenerate case.

lemma *big-bang*:
  assumes *no-paths*: $\mathcal{P} = \{\}$
  shows $\exists\, a.\ \mathcal{E} = \{a\}$
⟨*proof*⟩

lemma *two-events-then-path*:
  assumes *two-events*: $\exists\, a\in\mathcal{E}.\ \exists\, b\in\mathcal{E}.\ a \neq b$
  shows $\exists\, Q.\ Q \in \mathcal{P}$
⟨*proof*⟩

lemma *paths-are-events*: $\forall\, Q\in\mathcal{P}.\ \forall\, a\in Q.\ a \in \mathcal{E}$
  ⟨*proof*⟩

lemma *same-empty-unreach*:
  $[\![Q \in \mathcal{P};\ a \in Q]\!] \Longrightarrow \emptyset\ Q\ a = \{\}$
⟨*proof*⟩

lemma *same-path-reachable*:
  $[\![Q \in \mathcal{P};\ a \in Q;\ b \in Q]\!] \Longrightarrow a \in Q - \emptyset\ Q\ b$
⟨*proof*⟩

If we have two paths crossing and a is on the crossing point, and b is on one of the paths, then a is in the reachable part of the path b is on.

lemma *same-path-reachable2*:
  $[\![Q \in \mathcal{P};\ R \in \mathcal{P};\ a \in Q;\ a \in R;\ b \in Q]\!] \Longrightarrow a \in R - \emptyset\ R\ b$
  ⟨*proof*⟩

lemma *cross-in-reachable*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *a-inQ*: $a \in Q$
    and *b-inQ*: $b \in Q$
    and *b-inR*: $b \in R$
  shows $b \in R - \emptyset\ R\ a$
⟨*proof*⟩

lemma *reachable-path*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *b-event*: $b \in \mathcal{E}$
    and *a-reachable*: $a \in Q - \emptyset\ Q\ b$
  shows $\exists\, R\in\mathcal{P}.\ a \in R \wedge b \in R$
⟨*proof*⟩

end
context *MinkowskiUnreachable* **begin**

First some basic facts about the primitive notions, which seem to belong here. I don't think any/all of these are explicitly proved in Schutz.

lemma *no-empty-paths* [*simp*]:
  assumes $Q \in \mathcal{P}$
  shows $Q \neq \{\}$
⟨*proof*⟩

lemma *events-ex-path*:
  assumes *ge1-path*: $\mathcal{P} \neq \{\}$
  shows $\forall x \in \mathcal{E}.\ \exists Q \in \mathcal{P}.\ x \in Q$
⟨*proof*⟩

lemma *unreach-ge2-then-ge2*:
  assumes $\exists x \in \emptyset\ Q\ b.\ \exists y \in \emptyset\ Q\ b.\ x \neq y$
  shows $\exists x \in Q.\ \exists y \in Q.\ x \neq y$
⟨*proof*⟩

This lemma just proves that the chain obtained to bound the unreachable set of a path is indeed on that path. Extends I6; requires Theorem 2; used in Theorem 13. Seems to be assumed in Schutz' chain notation in I6.

lemma *chain-on-path-I6*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *event-b*: $b \notin Q\ b \in \mathcal{E}$
    and *unreach*: $Q_x \in \emptyset\ Q\ b\ Q_z \in \emptyset\ Q\ b\ Q_x \neq Q_z$
    and *X-def*: $ch\text{-}by\text{-}ord\ f\ X\ f\ 0 = Q_x\ f\ (card\ X - 1) = Q_z$
      $(\forall i \in \{1\ ..\ card\ X - 1\}.\ (f\ i) \in \emptyset\ Q\ b \land (\forall Q_y \in \mathcal{E}.\ [[(f(i-1))\ Q_y\ (f\ i)]] \longrightarrow Q_y \in \emptyset\ Q\ b))$
      $(short\text{-}ch\ X \longrightarrow Q_x \in X \land Q_z \in X \land (\forall Q_y \in \mathcal{E}.\ [[Q_x\ Q_y\ Q_z]] \longrightarrow Q_y \in \emptyset\ Q\ b))$
  shows $X \subseteq Q$

⟨*proof*⟩

end

25 Results about Paths as Sets

Note several of the following don't need MinkowskiPrimitive, they are just Set lemmas; nevertheless I'm naming them and writing them this way for clarity.

context *MinkowskiPrimitive* **begin**

lemma *distinct-paths*:
  assumes $Q \in \mathcal{P}$
    and $R \in \mathcal{P}$
    and $d \notin Q$
    and $d \in R$
  shows $R \neq Q$

$\langle proof \rangle$

lemma *distinct-paths2*:
  assumes $Q \in \mathcal{P}$
    and $R \in \mathcal{P}$
    and $\exists\, d.\ d \notin Q \wedge d \in R$
  shows $R \neq Q$
$\langle proof \rangle$

lemma *external-events-neq*:
  $[\![Q \in \mathcal{P};\ a \in Q;\ b \in \mathcal{E};\ b \notin Q]\!] \Longrightarrow a \neq b$
$\langle proof \rangle$

lemma *notin-cross-events-neq*:
  $[\![Q \in \mathcal{P};\ R \in \mathcal{P};\ Q \neq R;\ a \in Q;\ b \in R;\ a \notin R{\cap}Q]\!] \Longrightarrow a \neq b$
$\langle proof \rangle$

lemma *nocross-events-neq*:
  $[\![Q \in \mathcal{P};\ R \in \mathcal{P};\ a \in Q;\ b \in R;\ R{\cap}Q = \{\}]\!] \Longrightarrow a \neq b$
$\langle proof \rangle$

Given a nonempty path Q, and an external point d, we can find another path R passing through d (by I2 aka *events-paths*). This path is distinct from Q, as it passes through a point external to it.

lemma *external-path*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *a-inQ*: $a \in Q$
    and *d-notinQ*: $d \notin Q$
    and *d-event*: $d \in \mathcal{E}$
  shows $\exists\, R{\in}\mathcal{P}.\ d \in R$
$\langle proof \rangle$

lemma *distinct-path*:
  assumes $Q \in \mathcal{P}$
    and $a \in Q$
    and $d \notin Q$
    and $d \in \mathcal{E}$
  shows $\exists\, R{\in}\mathcal{P}.\ R \neq Q$
$\langle proof \rangle$

lemma *external-distinct-path*:
  assumes $Q \in \mathcal{P}$
    and $a \in Q$
    and $d \notin Q$
    and $d \in \mathcal{E}$
  shows $\exists\, R{\in}\mathcal{P}.\ R \neq Q \wedge d \in R$
  $\langle proof \rangle$

end

26 3.3 Boundedness of the unreachable set

26.1 Theorem 4 (boundedness of the unreachable set)

The same assumptions as I7, different conclusion. This doesn't just give us boundedness, it gives us another event outside of the unreachable set, as long as we have one already. I7 conclusion: $\exists\, X\ Q0\ Qm\ Qn.\ [[Q0\ ..\ Qm\ ..\ Qn]X] \wedge Q0 = ?Qx \wedge Qm = ?Qy \wedge Qn \in ?Q - \emptyset\ ?Q\ ?b$

theorem (**in** *MinkowskiUnreachable*) *unreachable-set-bounded*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *b-nin-Q*: $b \notin Q$
    and *b-event*: $b \in \mathcal{E}$
    and *Qx-reachable*: $Qx \in Q - \emptyset\ Q\ b$
    and *Qy-unreachable*: $Qy \in \emptyset\ Q\ b$
  shows $\exists\, Qz{\in}Q - \emptyset\ Q\ b.\ [[Qx\ Qy\ Qz]] \wedge Qx \neq Qz$
  ⟨*proof*⟩

26.2 Theorem 5 (first existence theorem)

The lemma below is used in the contradiction in *external-event*, which is the essential part to Theorem 5(i).

lemma (**in** *MinkowskiUnreachable*) *only-one-path*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *all-inQ*: $\forall\, a{\in}\mathcal{E}.\ a \in Q$
    and *path-R*: $R \in \mathcal{P}$
  shows $R = Q$
⟨*proof*⟩

context *MinkowskiSpacetime* **begin**

Unfortunately, we cannot assume that a path exists without the axiom of dimension.

lemma *external-event*:
  assumes *path-Q*: $Q \in \mathcal{P}$
  shows $\exists\, d{\in}\mathcal{E}.\ d \notin Q$
⟨*proof*⟩

Now we can prove the first part of the theorem's conjunction. This follows pretty much exactly the same pattern as the book, except it relies on more intermediate lemmas.

theorem *ge2-events*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *a-inQ*: $a \in Q$
  shows $\exists\, b{\in}Q.\ b \neq a$
⟨*proof*⟩

Simple corollary which is easier to use when we don't have one event on a path yet. Anything which uses this implicitly used *no-empty-paths* on top of *ge2-events*.

lemma *ge2-events-lax*:
  assumes *path-Q*: $Q \in \mathcal{P}$
  shows $\exists\, a{\in}Q.\ \exists\, b{\in}Q.\ a \neq b$
$\langle proof \rangle$

lemma *ex-crossing-path*:
  assumes *path-Q*: $Q \in \mathcal{P}$
  shows $\exists\, R{\in}\mathcal{P}.\ R \neq Q \wedge (\exists\, c.\ c \in R \wedge c \in Q)$
$\langle proof \rangle$

If we have two paths Q and R with a on Q and b at the intersection of Q and R, then by *two-in-unreach* (I5) and Theorem 4 (boundedness of the unreachable set), there is an unreachable set from a on one side of b on R, and on the other side of that there is an event which is reachable from a by some path, which is the path we want.

lemma *path-past-unreach*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *path-R*: $R \in \mathcal{P}$
    and *a-inQ*: $a \in Q$
    and *b-inQ*: $b \in Q$
    and *b-inR*: $b \in R$
    and *Q-neq-R*: $Q \neq R$
    and *a-neq-b*: $a \neq b$
  shows $\exists\, S{\in}\mathcal{P}.\ S \neq Q \wedge a \in S \wedge (\exists\, c.\ c \in S \wedge c \in R)$
$\langle proof \rangle$

theorem *ex-crossing-at*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *a-inQ*: $a \in Q$
  shows $\exists\, ac{\in}\mathcal{P}.\ ac \neq Q \wedge (\exists\, c.\ c \notin Q \wedge a \in ac \wedge c \in ac)$
$\langle proof \rangle$

lemma *ex-crossing-at-alt*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *a-inQ*: $a \in Q$
  shows $\exists\, ac.\ \exists\, c.\ path\ ac\ a\ c \wedge ac \neq Q \wedge c \notin Q$
$\langle proof \rangle$

end

27 3.4 Prolongation

context *MinkowskiSpacetime* **begin**

lemma (**in** *MinkowskiPrimitive*) *unreach-on-path*:

$a \in \emptyset\ Q\ b \Longrightarrow a \in Q$
⟨*proof*⟩

lemma (**in** *MinkowskiUnreachable*) *unreach-equiv*:
$[\![Q \in \mathcal{P};\ R \in \mathcal{P};\ a \in Q;\ b \in R;\ a \in \emptyset\ Q\ b]\!] \Longrightarrow b \in \emptyset\ R\ a$
⟨*proof*⟩

theorem *prolong-betw*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
and *b-inQ*: $b \in Q$
and *ab-neq*: $a \neq b$
shows $\exists\, c{\in}\mathcal{E}.\ [[a\ b\ c]]$
⟨*proof*⟩

lemma (**in** *MinkowskiSpacetime*) *prolong-betw2*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
and *b-inQ*: $b \in Q$
and *ab-neq*: $a \neq b$
shows $\exists\, c{\in}Q.\ [[a\ b\ c]]$
⟨*proof*⟩

lemma (**in** *MinkowskiSpacetime*) *prolong-betw3*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
and *b-inQ*: $b \in Q$
and *ab-neq*: $a \neq b$
shows $\exists\, c{\in}Q.\ \exists\, d{\in}Q.\ [[a\ b\ c]] \wedge [[a\ b\ d]] \wedge c{\neq}d$
⟨*proof*⟩

lemma *finite-path-has-ends*:
assumes $Q \in \mathcal{P}$
and $X \subseteq Q$
and *finite* X
and *card* $X \geq 3$
shows $\exists\, a{\in}X.\ \exists\, b{\in}X.\ a \neq b \wedge (\forall\, c{\in}X.\ a \neq c \wedge b \neq c \longrightarrow [[a\ c\ b]])$
⟨*proof*⟩

lemma *obtain-fin-path-ends*:
assumes *path-X*: $X{\in}\mathcal{P}$
and *fin-Q*: *finite* Q
and *card-Q*: *card* $Q \geq 3$
and *events-Q*: $Q{\subseteq}X$
obtains $a\ b$ **where** $a{\neq}b$ **and** $a{\in}Q$ **and** $b{\in}Q$ **and** $\forall\, c{\in}Q.\ (a{\neq}c \wedge b{\neq}c) \longrightarrow [[a$
$c\ b]]$
⟨*proof*⟩

lemma *path-card-nil*:
  assumes $Q \in \mathcal{P}$
  shows $card\ Q = 0$
$\langle proof \rangle$

theorem *infinite-paths*:
  assumes $P \in \mathcal{P}$
  shows $infinite\ P$
$\langle proof \rangle$

end

28 3.5 Second collinearity theorem

We start with a useful betweenness lemma.

lemma (**in** *MinkowskiBetweenness*) *some-betw2*:
  assumes *path-Q*: $Q \in \mathcal{P}$
    and *a-inQ*: $a \in Q$
    and *b-inQ*: $b \in Q$
    and *c-inQ*: $c \in Q$
  shows $a = b \lor a = c \lor b = c \lor [[a\ b\ c]] \lor [[b\ c\ a]] \lor [[c\ a\ b]]$
  $\langle proof \rangle$

lemma (**in** *MinkowskiPrimitive*) *paths-tri*:
  assumes *path-ab*: $path\ ab\ a\ b$
    and *path-bc*: $path\ bc\ b\ c$
    and *path-ca*: $path\ ca\ c\ a$
    and *a-notin-bc*: $a \notin bc$
  shows $\triangle\ a\ b\ c$
$\langle proof \rangle$

lemma (**in** *MinkowskiPrimitive*) *paths-tri2*:
  assumes *path-ab*: $path\ ab\ a\ b$
    and *path-bc*: $path\ bc\ b\ c$
    and *path-ca*: $path\ ca\ c\ a$
    and *ab-neq-bc*: $ab \neq bc$
  shows $\triangle\ a\ b\ c$
$\langle proof \rangle$

Schutz states it more like $[\![tri\text{-}abc;\ bcd;\ cea]\!] \Longrightarrow (path\ de\ d\ e \longrightarrow \exists f \in de.\ [[a\ f\ b]] \land [[d\ e\ f]])$. Equivalent up to usage of *impI*.

theorem (**in** *MinkowskiChain*) *collinearity2*:
  assumes *tri-abc*: $\triangle\ a\ b\ c$
    and *bcd*: $[[b\ c\ d]]$
    and *cea*: $[[c\ e\ a]]$

and *path-de*: *path de d e*
shows $\exists f \in de.\ [[a\ f\ b]] \wedge [[d\ e\ f]]$
$\langle proof \rangle$

29 3.6 Order on a path - Theorems 8 and 9

context *MinkowskiSpacetime* **begin**

29.1 Theorem 8 (as in Veblen (1911) Theorem 6)

Note $a'b'c'$ don't necessarily form a triangle, as there still needs to be paths between them.

theorem (**in** *MinkowskiChain*) *tri-betw-no-path*:
assumes *tri-abc*: $\triangle\ a\ b\ c$
and *ab'c*: $[[a\ b'\ c]]$
and *bc'a*: $[[b\ c'\ a]]$
and *ca'b*: $[[c\ a'\ b]]$
shows $\neg\ (\exists\ Q \in \mathcal{P}.\ a' \in Q \wedge b' \in Q \wedge c' \in Q)$
$\langle proof \rangle$

29.2 Theorem 9

We now begin working on the transitivity lemmas needed to prove Theorem 9. Multiple lemmas below obtain primed variables (e.g. d'). These are starred in Schutz (e.g. $d*$), but that notation is already reserved in Isabelle.

lemma *unreachable-bounded-path-only*:
assumes *d'-def*: $d' \notin \emptyset\ ab\ e\ d' \in ab\ d' \neq e$
and *e-event*: $e \in \mathcal{E}$
and *path-ab*: $ab \in \mathcal{P}$
and *e-notin-S*: $e \notin ab$
shows $\exists\ d'e.\ path\ d'e\ d'\ e$
$\langle proof \rangle$

lemma *unreachable-bounded-path*:
assumes *S-neq-ab*: $S \neq ab$
and *a-inS*: $a \in S$
and *e-inS*: $e \in S$
and *e-neq-a*: $e \neq a$
and *path-S*: $S \in \mathcal{P}$
and *path-ab*: *path ab a b*
and *path-be*: *path be b e*
and *no-de*: $\neg(\exists\ de.\ path\ de\ d\ e)$
and *abd*: $[[a\ b\ d]]$
obtains $d'\ d'e$ **where** $d' \in ab \wedge path\ d'e\ d'\ e \wedge [[b\ d\ d']]$
$\langle proof \rangle$

This lemma collects the first three paragraphs of Schutz' proof of Theorem 9 - Lemma 1. Several case splits need to be considered, but have no further importance outside of this lemma: thus we parcel them away from the main proof.

lemma *exist-c′d′-alt*:
assumes *abc*: $[[a\ b\ c]]$
and *abd*: $[[a\ b\ d]]$
and *dbc*: $[[d\ b\ c]]$
and *c-neq-d*: $c \neq d$
and *path-ab*: *path ab a b*
and *path-S*: $S \in \mathcal{P}$
and *a-inS*: $a \in S$
and *e-inS*: $e \in S$
and *e-neq-a*: $e \neq a$
and *S-neq-ab*: $S \neq ab$
and *path-be*: *path be b e*
shows $\exists\, c'\ d'.\ \exists\, d'e\ c'e.\ c' \in ab \wedge d' \in ab$
$\wedge [[a\ b\ d']] \wedge [[c'\ b\ a]] \wedge [[c'\ b\ d']]$
$\wedge$ *path* $d'e\ d'\ e \wedge$ *path* $c'e\ c'\ e$
⟨*proof*⟩

lemma *exist-c′d′*:
assumes *abc*: $[[a\ b\ c]]$
and *abd*: $[[a\ b\ d]]$
and *dbc*: $[[d\ b\ c]]$
and *path-S*: *path S a e*
and *path-be*: *path be b e*
and *S-neq-ab*: $S \neq$ *path-of a b*
shows $\exists\, c'\ d'.\ [[a\ b\ d']] \wedge [[c'\ b\ a]] \wedge [[c'\ b\ d']] \wedge$
path-ex $d'\ e \wedge$ *path-ex* $c'\ e$
⟨*proof*⟩

lemma *exist-f′-alt*:
assumes *path-ab*: *path ab a b*
and *path-S*: $S \in \mathcal{P}$
and *a-inS*: $a \in S$
and *e-inS*: $e \in S$
and *e-neq-a*: $e \neq a$
and *f-def*: $[[e\ c'\ f]]\ f \in c'e$
and *S-neq-ab*: $S \neq ab$
and *c′d′-def*: $c' \in ab \wedge d' \in ab$
$\wedge [[a\ b\ d']] \wedge [[c'\ b\ a]] \wedge [[c'\ b\ d']]$
$\wedge$ *path* $d'e\ d'\ e \wedge$ *path* $c'e\ c'\ e$
shows $\exists f'.\ \exists f'b.\ [[e\ c'\ f']] \wedge$ *path* $f'b\ f'\ b$
⟨*proof*⟩

lemma *exist-f′*:
assumes *path-ab*: *path ab a b*
and *path-S*: *path S a e*
and *f-def*: $[[e\ c'\ f]]$

and *S-neq-ab*: $S \neq ab$
and *c'd'-def*: $[[a\ b\ d']]$ $[[c'\ b\ a]]$ $[[c'\ b\ d']]$
path $d'e$ d' e *path* $c'e$ c' e
shows $\exists f'.\ [[e\ c'\ f']] \wedge$ *path-ex* f' b
⟨*proof*⟩

lemma *abc-abd-bcdbdc*:
assumes *abc*: $[[a\ b\ c]]$
and *abd*: $[[a\ b\ d]]$
and *c-neq-d*: $c \neq d$
shows $[[b\ c\ d]] \vee [[b\ d\ c]]$
⟨*proof*⟩

lemma *abc-abd-acdadc*:
assumes *abc*: $[[a\ b\ c]]$
and *abd*: $[[a\ b\ d]]$
and *c-neq-d*: $c \neq d$
shows $[[a\ c\ d]] \vee [[a\ d\ c]]$
⟨*proof*⟩

lemma *abc-acd-bcd*:
assumes *abc*: $[[a\ b\ c]]$
and *acd*: $[[a\ c\ d]]$
shows $[[b\ c\ d]]$
⟨*proof*⟩

A few lemmas that don't seem to be proved by Schutz, but can be proven now, after Lemma 3. These sometimes avoid us having to construct a chain explicitly.

lemma *abd-bcd-abc*:
assumes *abd*: $[[a\ b\ d]]$
and *bcd*: $[[b\ c\ d]]$
shows $[[a\ b\ c]]$
⟨*proof*⟩

lemma *abc-acd-abd*:
assumes *abc*: $[[a\ b\ c]]$
and *acd*: $[[a\ c\ d]]$
shows $[[a\ b\ d]]$
⟨*proof*⟩

lemma *abd-acd-abcacb*:
assumes *abd*: $[[a\ b\ d]]$
and *acd*: $[[a\ c\ d]]$
and *bc*: $b \neq c$

shows $[[a\ b\ c]] \lor [[a\ c\ b]]$
⟨*proof*⟩

lemma *abe-ade-bcd-ace*:
assumes *abe*: $[[a\ b\ e]]$
and *ade*: $[[a\ d\ e]]$
and *bcd*: $[[b\ c\ d]]$
shows $[[a\ c\ e]]$
⟨*proof*⟩

Now we start on Theorem 9. Based on Veblen (1904) Lemma 2 p357.

lemma (**in** *MinkowskiBetweenness*) *chain3*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *a-inQ*: $a \in Q$
and *b-inQ*: $b \in Q$
and *c-inQ*: $c \in Q$
and *abc-neq*: $a \neq b \land a \neq c \land b \neq c$
shows *ch* $\{a,b,c\}$
⟨*proof*⟩

The book introduces Theorem 9 before the above three lemmas but can only complete the proof once they are proven. This doesn't exactly say it the same way as the book, as the book gives the ordering (abcd) explicitly (for arbitrarily named events), but is equivalent.

theorem *chain4*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *inQ*: $a \in Q\ b \in Q\ c \in Q\ d \in Q$
and *abcd-neq*: $a \neq b \land a \neq c \land a \neq d \land b \neq c \land b \neq d \land c \neq d$
shows *ch* $\{a,b,c,d\}$
⟨*proof*⟩

end

30 Interlude - Chains and Equivalences

This section is meant for our alternative definitions of chains, and proofs of equivalence. If we want to regain full independence of our axioms, we probably need to shuffle a few things around. Some of this may be redundant, but is kept for compatibility with legacy proofs.

Three definitions are given (cf 'Betweenness: Chains' in Minkowski.thy): - one relying on explicit betweenness conditions - one relying on a total ordering and explicit indexing - one equivalent to the above except for use of the weaker, local-only ordering2

context *MinkowskiChain* **begin**

30.1 Proofs for totally ordered index-chains

30.1.1 General results

lemma *inf-chain-is-long*:
  assumes *semifin-chain f x X*
  shows *long-ch-by-ord f X* $\wedge$ *f 0* = *x* $\wedge$ *infinite X*
⟨*proof*⟩

A reassurance that the starting point x is implied.

lemma *long-inf-chain-is-semifin*:
  assumes *long-ch-by-ord f X* $\wedge$ *infinite X*
  shows $\exists$ *x*. [*f*[*x*..]*X*]
  ⟨*proof*⟩

lemma *endpoint-in-semifin*:
  assumes *semifin-chain f x X*
    shows *x*∈*X*
  ⟨*proof*⟩

lemma *three-in-long-chain*:
  assumes *long-ch-by-ord f X* **and** *fin*: *finite X*
  obtains *x y z* **where** *x*∈*X* **and** *y*∈*X* **and** *z*∈*X* **and** *x*≠*y* **and** *x*≠*z* **and** *y*≠*z*
    ⟨*proof*⟩

30.1.2 Index-chains lie on paths

lemma *all-aligned-on-semifin-chain*:
  assumes [*f*[*x*..]*X*]
  and *a*: *y*∈*X* **and** *b*:*z*∈*X* **and** *xy*: *x*≠*y* **and** *xz*: *x*≠*z* **and** *yz*: *y*≠*z*
  shows [[*x y z*]] $\vee$ [[*x z y*]]
⟨*proof*⟩

lemma *semifin-chain-on-path*:
  assumes [*f*[*x*..]*X*]
  shows $\exists$ $P \in \mathcal{P}$. $X \subseteq P$
⟨*proof*⟩

lemma *card2-either-elt1-or-elt2*:
  assumes *card X* = *2* **and** *x*∈*X* **and** *y*∈*X* **and** *x*≠*y*
    and *z*∈*X* **and** *z*≠*x*
  shows *z*=*y*
⟨*proof*⟩

lemma *short-chain-on-path*:
  assumes *short-ch X*
  shows $\exists$ $P \in \mathcal{P}$. $X \subseteq P$
⟨*proof*⟩

lemma *all-aligned-on-long-chain*:
assumes *long-ch-by-ord f X* **and** *finite X*
and *a*: $x \in X$ **and** *b*: $y \in X$ **and** *c*:$z \in X$ **and** *xy*: $x \neq y$ **and** *xz*: $x \neq z$ **and** *yz*: $y \neq z$
shows $[[x\ y\ z]] \vee [[x\ z\ y]] \vee [[z\ x\ y]]$
⟨*proof*⟩

lemma *long-chain-on-path*:
assumes *long-ch-by-ord f X* **and** *finite X*
shows $\exists P \in \mathcal{P}.\ X \subseteq P$
⟨*proof*⟩

Notice that this whole proof would be unnecessary if including path-belongingness in the definition, as Schutz does. This would also keep path-belongingness independent of axiom O1 and O4, thus enabling an independent statement of axiom O6, which perhaps we now lose. In exchange, our definition is slightly weaker (for *card X* $\geq$ *3* and *infinite X*).

lemma *chain-on-path*:
assumes *ch-by-ord f X*
shows $\exists P \in \mathcal{P}.\ X \subseteq P$
⟨*proof*⟩

30.1.3 More general results

lemma *ch-some-betw*: $[\![x \in X;\ y \in X;\ z \in X;\ x \neq y;\ x \neq z;\ y \neq z;\ ch\ X]\!]$
$\Longrightarrow [[x\ y\ z]] \vee [[y\ x\ z]] \vee [[y\ z\ x]]$
⟨*proof*⟩

lemma *ch-all-betw-f*:
assumes $[f[x..yy..z]X]$ **and** $y \in X$ **and** $y \neq x$ **and** $y \neq z$
shows $[[x\ y\ z]]$
⟨*proof*⟩

lemma *get-fin-long-ch-bounds*:
assumes *long-ch-by-ord f X*
and *finite X*
shows $\exists x \in X.\ \exists y \in X.\ \exists z \in X.\ [f[x..y..z]X]$
⟨*proof*⟩

lemma *get-fin-long-ch-bounds2*:
assumes *long-ch-by-ord f X*
and *finite X*
obtains $x\ y\ z\ n_x\ n_y\ n_z$
where $x \in X \wedge y \in X \wedge z \in X \wedge [f[x..y..z]X] \wedge f\ n_x = x \wedge f\ n_y = y \wedge f\ n_z = z$
⟨*proof*⟩

lemma *long-ch-card-ge3*:
  assumes *ch-by-ord f X finite X*
  shows *long-ch-by-ord f X* $\longleftrightarrow$ *card X* $\geq$ *3*
⟨*proof*⟩

lemma *chain-bounds-unique*:
  assumes $[f[a..b..c]X]$ $[g[x..y..z]X]$
  shows $(a=x \wedge c=z) \vee (a=z \wedge c=x)$
⟨*proof*⟩

lemma *chain-bounds-unique2*:
  assumes $[f[a..c]X]$ $[g[x..z]X]$ *card X* $\geq$ *3*
  shows $(a=x \wedge c=z) \vee (a=z \wedge c=x)$
  ⟨*proof*⟩

30.2 Chain Equivalences

30.2.1 Betweenness-chains and strong index-chains

lemma *equiv-chain-1a*:
  assumes $[[..a..b..c..]X]$
  shows $\exists f.$ *ch-by-ord f X* $\wedge\ a{\in}X \wedge b{\in}X \wedge c{\in}X \wedge a{\neq}b \wedge a{\neq}c \wedge b{\neq}c$
⟨*proof*⟩

lemma *equiv-chain-1b*:
  assumes *ch-by-ord f X* $\wedge\ a{\in}X \wedge b{\in}X \wedge c{\in}X \wedge a{\neq}b \wedge a{\neq}c \wedge b{\neq}c \wedge [[a\ b\ c]]$
  shows $[[..a..b..c..]X]$
  ⟨*proof*⟩

lemma *equiv-chain-1*:
  $[[..a..b..c..]X] \longleftrightarrow (\exists f.$ *ch-by-ord f X* $\wedge\ a{\in}X \wedge b{\in}X \wedge c{\in}X \wedge a{\neq}b \wedge a{\neq}c \wedge b{\neq}c$
$\wedge\ [[a\ b\ c]])$
  ⟨*proof*⟩

lemma *index-order*:
  assumes *chain-with x y z X*
    and *ch-by-ord f X* **and** $f\ a = x$ **and** $f\ b = y$ **and** $f\ c = z$
    and *finite X* $\longrightarrow a<$*card X* **and** *finite X* $\longrightarrow b<$*card X* **and** *finite X* $\longrightarrow$
$c<$*card X*
   shows $(a<b \wedge b<c) \vee (c<b \wedge b<a)$
⟨*proof*⟩

lemma *old-fin-chain-finite*:
  assumes *finite-chain-with3 x y z X*
  shows *finite X*

⟨*proof*⟩

lemma *index-from-with3*:
  assumes *finite-chain-with3 a b c X*
  shows $\exists f.\ (f\ 0 = a \lor f\ 0 = c) \land ch\text{-}by\text{-}ord\ f\ X$
⟨*proof*⟩

lemma (**in** *MinkowskiSpacetime*) *with3-and-index-is-fin-chain*:
  assumes $f\ 0 = a$ **and** *ch-by-ord f X* **and** *finite-chain-with3 a b c X*
  shows $[f[a..b..c]X]$
⟨*proof*⟩

lemma (**in** *MinkowskiSpacetime*) *g-from-with3*:
  assumes *finite-chain-with3 a b c X*
  obtains *g* **where** $[g[a..b..c]X] \lor [g[c..b..a]X]$
⟨*proof*⟩

lemma (**in** *MinkowskiSpacetime*) *equiv-chain-2a*:
  assumes *finite-chain-with3 a b c X*
  obtains *f* **where** $[f[a..b..c]X]$
⟨*proof*⟩

lemma *equiv-chain-2b*:
  assumes $[f[a..b..c]X]$
  shows *finite-chain-with3 a b c X*
⟨*proof*⟩

lemma (**in** *MinkowskiSpacetime*) *equiv-chain-2*:
  $\exists f.\ [f[a..b..c]X] \longleftrightarrow [[a..b..c]X]$
  ⟨*proof*⟩

end

31 Results for segments, rays and chains

context *MinkowskiChain* **begin**

lemma *inside-not-bound*:
  assumes $[f[a..b..c]X]$
    and $j<card\ X$
  shows $j>0 \Longrightarrow f\ j \neq a\ j<card\ X - 1 \Longrightarrow f\ j \neq c$
⟨*proof*⟩

lemma *some-betw2*:
  assumes $[f[a..b..c]X]$
    and $j<card\ X$ $j>0$ $f\ j \neq b$
  shows $[[a\ b\ (f\ j)]] \vee [[a\ (f\ j)\ b]]$
⟨*proof*⟩

lemma *i-le-j-events-neq1*:
  assumes $[f[a..b..c]X]$
    and $i<j$ $j<card\ X$ $f\ j \neq b$
  shows $f\ i \neq f\ j$
⟨*proof*⟩

lemma *i-le-j-events-neq*:
  assumes $[f[a..b..c]X]$
    and $i<j$ $j<card\ X$
  shows $f\ i \neq f\ j$
⟨*proof*⟩

lemma *indices-neq-imp-events-neq*:
  assumes $[f[a..b..c]X]$
    and $i \neq j$ $j<card\ X$ $i<card\ X$
  shows $f\ i \neq f\ j$
  ⟨*proof*⟩

lemma *index-order2*:
  assumes $[f[x..y..z]X]$ **and** $f\ a = x$ **and** $f\ b = y$ **and** $f\ c = z$
    and $finite\ X \longrightarrow a < card\ X$ **and** $finite\ X \longrightarrow b < card\ X$ **and** $finite\ X \longrightarrow c < card\ X$
  shows $(a<b \wedge b<c) \vee (c<b \wedge b<a)$
  ⟨*proof*⟩

lemma *index-order3*:
  assumes $[[x\ y\ z]]$ **and** $f\ a = x$ **and** $f\ b = y$ **and** $f\ c = z$ **and** *long-ch-by-ord* $f\ X$
    and $finite\ X \longrightarrow a < card\ X$ **and** $finite\ X \longrightarrow b < card\ X$ **and** $finite\ X \longrightarrow c < card\ X$
  shows $(a<b \wedge b<c) \vee (c<b \wedge b<a)$
  ⟨*proof*⟩

end

context *MinkowskiSpacetime* **begin**

lemma *bound-on-path*:
  assumes $Q \in \mathcal{P}$ $[f[(f\ 0)..]X]$ $X \subseteq Q$ *is-bound-f* $b\ X\ f$
  shows $b \in Q$
⟨*proof*⟩

lemma *pro-basis-change*:
  assumes $[[a\ b\ c]]$
  shows *prolongation a c = prolongation b c* (**is** *?ac=?bc*)
⟨*proof*⟩

lemma *adjoining-segs-exclusive*:
  assumes $[[a\ b\ c]]$
  shows *segment a b* ∩ *segment b c* = {}
⟨*proof*⟩

end

32 3.6 Order on a path - Theorems 10 and 11

context *MinkowskiSpacetime* **begin**

32.1 Theorem 10 (based on Veblen (1904) theorem 10).

lemma (**in** *MinkowskiBetweenness*) *two-event-chain*:
  assumes *finiteX*: *finite X*
    and *path-Q*: $Q \in \mathcal{P}$
    and *events-X*: $X \subseteq Q$
    and *card-X*: *card X = 2*
   shows *ch X*
⟨*proof*⟩

lemma (**in** *MinkowskiBetweenness*) *three-event-chain*:
  assumes *finiteX*: *finite X*
    and *path-Q*: $Q \in \mathcal{P}$
    and *events-X*: $X \subseteq Q$
    and *card-X*: *card X = 3*
   shows *ch X*
⟨*proof*⟩

This is case (i) of the induction in Theorem 10.

lemma *chain-append-at-left-edge*:
  assumes *long-ch-Y*: $[f[a_1..a..a_n]Y]$
    and *bY*: $[[b\ a_1\ a_n]]$
   fixes *g* **defines** *g-def*: $g \equiv (\lambda j{::}nat.\ if\ j{\geq}1\ then\ f\ (j{-}1)\ else\ b)$
   shows $[g[b\ ..\ a_1\ ..\ a_n](insert\ b\ Y)]$
⟨*proof*⟩

This is case (iii) of the induction in Theorem 10. Schutz says merely "The proof for this case is similar to that for Case (i)." Thus I feel free to use a result on symmetry, rather than going through the pain of Case (i) (*chain-append-at-left-edge*) again.

lemma *chain-append-at-right-edge*:
  assumes *long-ch-Y*: $[f[a_1..a..a_n]Y]$

and Yb: $[[a_1\ \ a_n\ \ b]]$
fixes g **defines** $g\text{-}def$: $g \equiv (\lambda j{::}nat.\ if\ j \leq (card\ Y - 1)\ then\ f\ j\ else\ b)$
shows $[g[a_1\ ..\ a_n\ ..\ b](insert\ b\ Y)]$
⟨*proof*⟩

lemma *S-is-dense*:
assumes $long\text{-}ch\text{-}Y$: $[f[a_1..a..a_n]\,Y]$
and $S\text{-}def$: $S = \{k{::}nat.\ [[a_1\ (f\ k)\ b]] \wedge k < card\ Y\}$
and $k\text{-}def$: $S{\neq}\{\}\ k = Max\ S$
and $k'\text{-}def$: $k'{>}0\ k'{<}k$
shows $k' \in S$
⟨*proof*⟩

lemma *smallest-k-ex*:
assumes $long\text{-}ch\text{-}Y$: $[f[a_1..a..a_n]\,Y]$
and $Y\text{-}def$: $b{\notin}Y$
and Yb: $[[a_1\ b\ a_n]]$
shows $\exists\,k{>}0.\ [[a_1\ b\ (f\ k)]] \wedge k < card\ Y \wedge \neg(\exists\,k'{<}k.\ [[a_1\ b\ (f\ k')]])$
⟨*proof*⟩

lemma *greatest-k-ex*:
assumes $long\text{-}ch\text{-}Y$: $[f[a_1..a..a_n]\,Y]$
and $Y\text{-}def$: $b{\notin}Y$
and Yb: $[[a_1\ b\ a_n]]$
shows $\exists\,k.\ [[(f\ k)\ b\ a_n]] \wedge k < card\ Y - 1 \wedge \neg(\exists\,k'{<}card\ Y.\ k'{>}k \wedge [[(f\ k')\ b\ a_n]])$
⟨*proof*⟩

lemma *get-closest-chain-events*:
assumes $long\text{-}ch\text{-}Y$: $[f[a_0..a..a_n]\,Y]$
and $x\text{-}def$: $x{\notin}Y\ [[a_0\ x\ a_n]]$
obtains $n_b\ n_c\ b\ c$
where $b{=}f\ n_b\ c{=}f\ n_c\ [[b\ x\ c]]\ b{\in}Y\ c{\in}Y\ n_b = n_c - 1\ n_c{<}card\ Y\ n_c{>}0$
$\neg(\exists\,k < card\ Y.\ [[(f\ k)\ x\ a_n]] \wedge k{>}n_b)\ \neg(\exists\,k{<}n_c.\ [[a_0\ x\ (f\ k)]])$
⟨*proof*⟩

This is case (ii) of the induction in Theorem 10.

lemma *chain-append-inside*:
assumes $long\text{-}ch\text{-}Y$: $[f[a_1..a..a_n]\,Y]$
and $Y\text{-}def$: $b{\notin}Y$
and Yb: $[[a_1\ b\ a_n]]$
and $k\text{-}def$: $[[a_1\ b\ (f\ k)]]\ k < card\ Y\ \neg(\exists\,k'.\ (0{::}nat){<}k' \wedge k'{<}k \wedge [[a_1\ b\ (f\ k')]])$
fixes g

defines *g-def*: $g \equiv (\lambda j{::}nat.\ if\ (j{\le}k{-}1)\ then\ f\ j\ else\ (if\ (j{=}k)\ then\ b\ else\ f\ (j{-}1)))$
  shows $[g[a_1\ ..\ b\ ..\ a_n]insert\ b\ Y]$
$\langle proof \rangle$

lemma *card4-eq*:
assumes $card\ X = 4$
shows $\exists\, a\ b\ c\ d.\ a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d \wedge X = \{a, b, c, d\}$
$\langle proof \rangle$

theorem *path-finsubset-chain*:
assumes $Q \in \mathcal{P}$
  and $X \subseteq Q$
  and $card\ X \geq 2$
shows $ch\ X$
$\langle proof \rangle$

lemma *path-finsubset-chain2*:
assumes $Q \in \mathcal{P}$ **and** $X \subseteq Q$ **and** $card\ X \geq 2$
obtains $f\ a\ b$ **where** $[f[a..b]X]$
$\langle proof \rangle$

32.2 Theorem 11

Notice this case is so simple, it doesn't even require the path density larger sets of segments rely on for fixing their cardinality.

lemma *segmentation-ex-N2*:
assumes *path-P*: $P{\in}\mathcal{P}$
  and *Q-def*: $finite\ (Q{::}'a\ set)\ card\ Q = N\ Q{\subseteq}P\ N{=}2$
  and *f-def*: $[f[a..b]Q]$
  and *S-def*: $S = \{segment\ a\ b\}$
  and *P1-def*: $P1 = prolongation\ b\ a$
  and *P2-def*: $P2 = prolongation\ a\ b$
  shows $P = ((\bigcup S) \cup P1 \cup P2 \cup Q) \wedge$
    $card\ S = (N{-}1) \wedge (\forall\, x{\in}S.\ is\text{-}segment\ x) \wedge$
    $P1{\cap}P2{=}\{\} \wedge (\forall\, x{\in}S.\ (x{\cap}P1{=}\{\} \wedge x{\cap}P2{=}\{\} \wedge (\forall\, y{\in}S.\ x{\neq}y \longrightarrow x{\cap}y{=}\{\})))$
$\langle proof \rangle$

lemma *int-split-to-segs*:
assumes *f-def*: $[f[a..b..c]Q]$
fixes S **defines** *S-def*: $S \equiv \{segment\ (f\ i)\ (f(i{+}1)) \mid i.\ i{<}card\ Q{-}1\}$
shows $interval\ a\ c = (\bigcup S) \cup Q$
$\langle proof \rangle$

lemma *path-is-union*:
  assumes *path-P*: $P \in \mathcal{P}$
    and *Q-def*: *finite* $(Q::'a\ set)$ *card* $Q = N$ $Q \subseteq P$ $N \geq 3$
    and *f-def*: $a \in Q \wedge b \in Q \wedge c \in Q$ $[f[a..b..c]Q]$
    and *S-def*: $S = \{s.\ \exists\, i<(N-1).\ s = segment\ (f\ i)\ (f\ (i+1))\}$
    and *P1-def*: $P1 = prolongation\ b\ a$
    and *P2-def*: $P2 = prolongation\ b\ c$
  shows $P = ((\bigcup S) \cup P1 \cup P2 \cup Q)$
$\langle proof \rangle$

lemma *inseg-axc*:
  assumes *path-P*: $P \in \mathcal{P}$
    and *Q-def*: *finite* $(Q::'a\ set)$ *card* $Q = N$ $Q \subseteq P$ $N \geq 3$
    and *f-def*: $a \in Q \wedge b \in Q \wedge c \in Q$ $[f[a..b..c]Q]$
    and *S-def*: $S = \{s.\ \exists\, i<(N-1).\ s = segment\ (f\ i)\ (f\ (i+1))\}$
    and *x-def*: $x \in s$ $s \in S$
  shows $[[a\ x\ c]]$
$\langle proof \rangle$

lemma *disjoint-segmentation*:
  assumes *path-P*: $P \in \mathcal{P}$
    and *Q-def*: *finite* $(Q::'a\ set)$ *card* $Q = N$ $Q \subseteq P$ $N \geq 3$
    and *f-def*: $a \in Q \wedge b \in Q \wedge c \in Q$ $[f[a..b..c]Q]$
    and *S-def*: $S = \{s.\ \exists\, i<(N-1).\ s = segment\ (f\ i)\ (f\ (i+1))\}$
    and *P1-def*: $P1 = prolongation\ b\ a$
    and *P2-def*: $P2 = prolongation\ b\ c$
    shows $P1 \cap P2=\{\} \wedge (\forall\, x \in S.\ (x \cap P1=\{\} \wedge x \cap P2=\{\} \wedge (\forall\, y \in S.\ x \neq y \longrightarrow x \cap y=\{\})))$
$\langle proof \rangle$

lemma *segmentation-ex-Nge3*:
  assumes *path-P*: $P \in \mathcal{P}$
    and *Q-def*: *finite* $(Q::'a\ set)$ *card* $Q = N$ $Q \subseteq P$ $N \geq 3$
    and *f-def*: $a \in Q \wedge b \in Q \wedge c \in Q$ $[f[a..b..c]Q]$
    and *S-def*: $S = \{s.\ \exists\, i<(N-1).\ s = segment\ (f\ i)\ (f\ (i+1))\}$
    and *P1-def*: $P1 = prolongation\ b\ a$
    and *P2-def*: $P2 = prolongation\ b\ c$
  shows $P = ((\bigcup S) \cup P1 \cup P2 \cup Q) \wedge$
      $(\forall\, x \in S.\ is\text{-}segment\ x) \wedge$
    $P1 \cap P2=\{\} \wedge (\forall\, x \in S.\ (x \cap P1=\{\} \wedge x \cap P2=\{\} \wedge (\forall\, y \in S.\ x \neq y \longrightarrow x \cap y=\{\})))$
$\langle proof \rangle$

We define *disjoint* to be the same as in HOL-Library.DisjointSets. This saves importing a lot of baggage we don't need. The two lemmas below are just for safety.

abbreviation *disjoint*
  where $disjoint\ A \equiv (\forall a{\in}A.\ \forall b{\in}A.\ a \neq b \longrightarrow a \cap b = \{\})$

lemma
  fixes S:: (*'a set*) *set* **and** $P1$:: *'a set* **and** $P2$:: *'a set*
  assumes $\forall x{\in}S.\ (x{\cap}P1{=}\{\} \wedge x{\cap}P2{=}\{\} \wedge (\forall y{\in}S.\ x{\neq}y \longrightarrow x{\cap}y{=}\{\}))\ P1{\cap}P2{=}\{\}$
  shows $disjoint\ (S{\cup}\{P1,P2\})$
⟨*proof*⟩
lemma
  fixes S:: (*'a set*) *set* **and** $P1$:: *'a set* **and** $P2$:: *'a set*
  assumes $disjoint\ (S{\cup}\{P1,P2\})\ P1{\notin}S\ P2{\notin}S\ P1{\neq}P2$
  shows $\forall x{\in}S.\ (x{\cap}P1{=}\{\} \wedge x{\cap}P2{=}\{\} \wedge (\forall y{\in}S.\ x{\neq}y \longrightarrow x{\cap}y{=}\{\}))\ P1{\cap}P2{=}\{\}$
⟨*proof*⟩

Schutz says "As in the proof of the previous theorem [...]" - does he mean to imply that this should really be proved as induction? I can see that quite easily, induct on N, and add a segment by either splitting up a segment or taking a piece out of a prolongation. But I think that might be too much trouble.

theorem *show-segmentation*:
  assumes *path-P*: $P{\in}\mathcal{P}$
    and *Q-def*: $Q{\subseteq}P$
    and *f-def*: $[f[a..b]Q]$
   fixes $P1$ **defines** *P1-def*: $P1 \equiv prolongation\ b\ a$
   fixes $P2$ **defines** *P2-def*: $P2 \equiv prolongation\ a\ b$
   fixes S **defines** *S-def*: $S \equiv if\ card\ Q{=}2\ then\ \{segment\ a\ b\}$
     $else\ \{segment\ (f\ i)\ (f\ (i{+}1))\ |\ i.\ i{<}card\ Q{-}1\}$
   shows $P = ((\bigcup S) \cup P1 \cup P2 \cup Q)\ (\forall x{\in}S.\ is\text{-}segment\ x)$
    $disjoint\ (S{\cup}\{P1,P2\})\ P1{\neq}P2\ P1{\notin}S\ P2{\notin}S$
⟨*proof*⟩

theorem *segmentation*:
  assumes *path-P*: $P{\in}\mathcal{P}$
    and *Q-def*: $card\ Q{\geq}2\ Q{\subseteq}P$
   shows $\exists S\ P1\ P2.\ P = ((\bigcup S) \cup P1 \cup P2 \cup Q) \wedge$
     $disjoint\ (S{\cup}\{P1,P2\}) \wedge P1{\neq}P2 \wedge P1{\notin}S \wedge P2{\notin}S \wedge$
     $(\forall x{\in}S.\ is\text{-}segment\ x) \wedge is\text{-}prolongation\ P1 \wedge is\text{-}prolongation\ P2$
⟨*proof*⟩

end

33 Chains are unique up to reversal

lemma (**in** *MinkowskiSpacetime*) *chain-remove-at-right-edge*:
  assumes $[f[a..c]X]\ f\ (card\ X\ -\ 2) = p\ 3 \leq card\ X\ X = insert\ c\ Y\ c{\notin}Y$

shows $[f[a..p]\,Y]$
$\langle proof \rangle$

lemma (**in** *MinkowskiChain*) *fin-long-ch-imp-fin-ch*:
assumes $[f[a..b..c]X]$
shows $[f[a..c]X]$
$\langle proof \rangle$

If we ever want to have chains less strongly identified by endpoints, this result should generalise - a, c, x, z are only used to identify reversal/no-reversal cases.

lemma (**in** *MinkowskiSpacetime*) *chain-unique-induction-ax*:
assumes $card\ X \geq 3$
and $i < card\ X$
and $[f[a..c]X]$
and $[g[x..z]X]$
and $a = x \vee c = z$
shows $f\ i = g\ i$
$\langle proof \rangle$

I'm really impressed *sledgehammer/smt* can solve this if I just tell them "Use symmetry!".

lemma (**in** *MinkowskiSpacetime*) *chain-unique-induction-cx*:
assumes $card\ X \geq 3$
and $i < card\ X$
and $[f[a..c]X]$
and $[g[x..z]X]$
and $c = x \vee a = z$
shows $f\ i = g\ (card\ X - i - 1)$
$\langle proof \rangle$

This lemma has to exclude two-element chains again, because no order exists within them. Alternatively, the result is trivial: any function that assigns one element to index 0 and the other to 1 can be replaced with the (unique) other assignment, without destroying any (trivial, since ternary) "ordering" of the chain. This could be made generic over the ordering similar to *chain-sym* relying on *ordering-sym*.

lemma (**in** *MinkowskiSpacetime*) *chain-unique-upto-rev-cases*:
assumes *ch-f*: $[f[a..c]X]$
and *ch-g*: $[g[x..z]X]$
and *card-X*: $card\ X \geq 3$
and *valid-index*: $i < card\ X$
shows $((a{=}x \vee c{=}z) \longrightarrow (f\ i = g\ i))\ ((a{=}z \vee c{=}x) \longrightarrow (f\ i = g\ (card\ X - i - 1)))$
$\langle proof \rangle$

lemma (**in** *MinkowskiSpacetime*) *chain-unique-upto-rev*:

assumes $[f[a..c]X]$ $[g[x..z]X]$ *card* $X \geq 3$ $i <$ *card* X
shows $f\ i = g\ i \vee f\ i = g$ (*card* $X - i - 1$) $a{=}x \wedge c{=}z \vee c{=}x \wedge a{=}z$
⟨*proof*⟩

34 Subchains

context *MinkowskiSpacetime* **begin**

lemma *f-img-is-subset*:
assumes $[f[(f\ 0)\ ..]X]$ $i{\geq}0$ $j{>}i$ $Y{=}f`\{i..j\}$
shows $Y{\subseteq}X$
⟨*proof*⟩

lemma *f-inj-on-index-subset*:
assumes $[f[(f\ 0)\ ..]X]$ $i{\geq}0$ $j{>}i$ $Y{=}f`\{i..j\}$
shows *inj-on* f $\{i..j\}$
⟨*proof*⟩

lemma *f-bij-on-index-subset*:
assumes $[f[(f\ 0)\ ..]X]$ $i{\geq}0$ $j{>}i$ $Y{=}f`\{i..j\}$
shows *bij-betw* f $\{i..j\}$ Y
⟨*proof*⟩

lemma *only-one-index*:
assumes $[f[(f\ 0)\ ..]X]$ $i{\geq}0$ $j{>}i$ $Y{=}f`\{i..j\}$ $f\ n \in Y$
shows $n{\in}\{i..j\}$
⟨*proof*⟩

lemma *f-one-to-one-on-index-subset*:
assumes $[f[(f\ 0)\ ..]X]$ $i{\geq}0$ $j{>}i$ $Y{=}f`\{i..j\}$ $y{\in}Y$
shows $\exists!k{\in}\{i..j\}.\ f\ k = y$ $f\ k = y \longrightarrow k{\in}\{i..j\}$
⟨*proof*⟩

lemma *card-of-subchain*:
assumes $[f[(f\ 0)\ ..]X]$ $i{\geq}0$ $j{>}i$ $Y{=}f`\{i..j\}$
shows *card* Y = *card* $\{i..j\}$ *card* $Y = j{-}i{+}1$
⟨*proof*⟩

lemma *fin-long-subchain-of-semifin*:
assumes $[f[(f\ 0)\ ..]X]$ $i{\geq}0$ $j{>}i{+}1$ $Y{=}f`\{i..j\}$
$g = (\lambda n.\ f(n{+}i))$
shows $[g[(f\ i)..(f\ j)]Y]$

⟨*proof*⟩

end

35 Extensions of results to infinite chains

context *MinkowskiSpacetime* **begin**

lemma *i-neq-j-imp-events-neq-inf*:
  assumes $[f[(f\ 0)..]X]$ $i{\neq}j$
  shows $f\ i \neq f\ j$
⟨*proof*⟩

lemma *i-neq-j-imp-events-neq*:
  assumes *long-ch-by-ord f X* $i{\neq}j$ *finite* $X \longrightarrow (i{<}card\ X \wedge j{<}card\ X)$
  shows $f\ i \neq f\ j$
  ⟨*proof*⟩

lemma *inf-chain-origin-unique*:
  assumes $[f[f\ 0..]X]$ $[g[g\ 0..]X]$
  shows $f\ 0 = g\ 0$
⟨*proof*⟩

lemma *inf-chain-unique*:
  assumes $[f[f\ 0..]X]$ $[g[g\ 0..]X]$
  shows $\forall\ i{::}nat.\ f\ i = g\ i$
⟨*proof*⟩

end

36 Interlude: betw4 and WLOG

36.1 betw4 - strict and non-strict, basic lemmas

context *MinkowskiBetweenness* **begin**

Define additional notation for non-strict ordering - cf Schutz' monograph [1, p. 27].

abbreviation *nonstrict-betw-right* $:: 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ ([[- - -⟧) **where**
  nonstrict-betw-right $a\ b\ c \equiv [[a\ b\ c]] \vee b = c$

abbreviation *nonstrict-betw-left* $:: 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ (⟦- - -]]) **where**
  nonstrict-betw-left $a\ b\ c \equiv [[a\ b\ c]] \vee b = a$

abbreviation *nonstrict-betw-both* $:: 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ **where**

nonstrict-betw-both a b c ≡ *nonstrict-betw-left a b c* ∨ *nonstrict-betw-right a b c*

abbreviation *betw4* :: ′*a* ⇒ ′*a* ⇒ ′*a* ⇒ ′*a* ⇒ *bool* ([[- - - -]]) **where**
betw4 a b c d ≡ [[*a b c*]] ∧ [[*b c d*]]

abbreviation *nonstrict-betw-right4* :: ′*a* ⇒ ′*a* ⇒ ′*a* ⇒ ′*a* ⇒ *bool* ([[- - - -⟧) **where**
nonstrict-betw-right4 a b c d ≡ *betw4 a b c d* ∨ *c* = *d*

abbreviation *nonstrict-betw-left4* :: ′*a* ⇒ ′*a* ⇒ ′*a* ⇒ ′*a* ⇒ *bool* (⟦- - - -]]) **where**
nonstrict-betw-left4 a b c d ≡ *betw4 a b c d* ∨ *a* = *b*

abbreviation *nonstrict-betw-both4* :: ′*a* ⇒ ′*a* ⇒ ′*a* ⇒ ′*a* ⇒ *bool* **where**
nonstrict-betw-both4 a b c d ≡ *nonstrict-betw-left4 a b c d* ∨ *nonstrict-betw-right4 a b c d*

lemma *betw4-strong*:
assumes *betw4 a b c d*
shows [[*a b d*]] ∧ [[*a c d*]]
⟨*proof*⟩

lemma *betw4-imp-neq*:
assumes *betw4 a b c d*
shows *a*≠*b* ∧ *a*≠*c* ∧ *a*≠*d* ∧ *b*≠*c* ∧ *b*≠*d* ∧ *c*≠*d*
⟨*proof*⟩

end
context *MinkowskiSpacetime* **begin**

lemma *betw4-weak*:
fixes *a b c d* :: ′*a*
assumes [[*a b c*]] ∧ [[*a c d*]]
∨ [[*a b c*]] ∧ [[*b c d*]]
∨ [[*a b d*]] ∧ [[*b c d*]]
∨ [[*a b d*]] ∧ [[*b c d*]]
shows *betw4 a b c d*
⟨*proof*⟩

lemma *betw4-sym*:
fixes *a*::′*a* **and** *b*::′*a* **and** *c*::′*a* **and** *d*::′*a*
shows *betw4 a b c d* ⟷ *betw4 d c b a*
⟨*proof*⟩

lemma *abcd-dcba-only*:
fixes *a*::′*a* **and** *b*::′*a* **and** *c*::′*a* **and** *d*::′*a*
assumes *betw4 a b c d*
shows ¬*betw4 a b d c* ¬*betw4 a c b d* ¬*betw4 a c d b* ¬*betw4 a d b c* ¬*betw4 a d c b*

$\neg betw4\ b\ a\ c\ d$ $\neg betw4\ b\ a\ d\ c$ $\neg betw4\ b\ c\ a\ d$ $\neg betw4\ b\ c\ d\ a$ $\neg betw4\ b\ d\ c\ a$ $\neg betw4\ b\ d\ a\ c$
$\neg betw4\ c\ a\ b\ d$ $\neg betw4\ c\ a\ d\ b$ $\neg betw4\ c\ b\ a\ d$ $\neg betw4\ c\ b\ d\ a$ $\neg betw4\ c\ d\ a\ b$ $\neg betw4\ c\ d\ b\ a$
$\neg betw4\ d\ a\ b\ c$ $\neg betw4\ d\ a\ c\ b$ $\neg betw4\ d\ b\ a\ c$ $\neg betw4\ d\ b\ c\ a$ $\neg betw4\ d\ c\ a\ b$
⟨*proof*⟩

lemma *some-betw4a*:
fixes $a::'a$ **and** $b::'a$ **and** $c::'a$ **and** $d::'a$ **and** P
assumes $P\in\mathcal{P}$ $a\in P$ $b\in P$ $c\in P$ $d\in P$ $a\neq b \wedge a\neq c \wedge a\neq d \wedge b\neq c \wedge b\neq d \wedge c\neq d$
and $\neg(betw4\ a\ b\ c\ d \vee betw4\ a\ b\ d\ c \vee betw4\ a\ c\ b\ d \vee betw4\ a\ c\ d\ b \vee betw4\ a\ d\ b\ c \vee betw4\ a\ d\ c\ b)$
shows $betw4\ b\ a\ c\ d \vee betw4\ b\ a\ d\ c \vee betw4\ b\ c\ a\ d \vee betw4\ b\ d\ a\ c \vee betw4\ c\ a\ b\ d \vee betw4\ c\ b\ a\ d$
⟨*proof*⟩

lemma *some-betw4b*:
fixes $a::'a$ **and** $b::'a$ **and** $c::'a$ **and** $d::'a$ **and** P
assumes $P\in\mathcal{P}$ $a\in P$ $b\in P$ $c\in P$ $d\in P$ $a\neq b \wedge a\neq c \wedge a\neq d \wedge b\neq c \wedge b\neq d \wedge c\neq d$
and $\neg(betw4\ b\ a\ c\ d \vee betw4\ b\ a\ d\ c \vee betw4\ b\ c\ a\ d \vee betw4\ b\ d\ a\ c \vee betw4\ c\ a\ b\ d \vee betw4\ c\ b\ a\ d)$
shows $betw4\ a\ b\ c\ d \vee betw4\ a\ b\ d\ c \vee betw4\ a\ c\ b\ d \vee betw4\ a\ c\ d\ b \vee betw4\ a\ d\ b\ c \vee betw4\ a\ d\ c\ b$
⟨*proof*⟩

lemma *abd-acd-abcdacbd*:
fixes $a::'a$ **and** $b::'a$ **and** $c::'a$ **and** $d::'a$
assumes *abd*: $[[a\ b\ d]]$ **and** *acd*: $[[a\ c\ d]]$ **and** $b\neq c$
shows $betw4\ a\ b\ c\ d \vee betw4\ a\ c\ b\ d$
⟨*proof*⟩

end

36.2 WLOG for two general symmetric relations of two elements on a single path

context *MinkowskiBetweenness* **begin**

This first one is really just trying to get a hang of how to write these things. If you have a relation that does not care which way round the "endpoints" (if Q is the interval-relation) go, then anything you want to prove about both undistinguished endpoints, follows from a proof involving a single endpoint.

lemma *wlog-sym-element*:
assumes *symmetric-rel*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
and *one-endpoint*: $\bigwedge a\ b\ x\ I.\ [\![Q\ I\ a\ b;\ x=a]\!] \Longrightarrow P\ x\ I$
shows *other-endpoint*: $\bigwedge a\ b\ x\ I.\ [\![Q\ I\ a\ b;\ x=b]\!] \Longrightarrow P\ x\ I$
⟨*proof*⟩

This one gives the most pertinent case split: a proof involving e.g. an element of an interval must consider the edge case and the inside case.

lemma *wlog-element*:
  assumes *symmetric-rel*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
    and *one-endpoint*: $\bigwedge a\ b\ x\ I.\ [\![Q\ I\ a\ b;\ x{=}a]\!] \Longrightarrow P\ x\ I$
    and *neither-endpoint*: $\bigwedge a\ b\ x\ I.\ [\![Q\ I\ a\ b;\ x{\in}I;\ (x{\neq}a \wedge x{\neq}b)]\!] \Longrightarrow P\ x\ I$
  shows *any-element*: $\bigwedge x\ I.\ [\![x{\in}I;\ (\exists\, a\ b.\ Q\ I\ a\ b)]\!] \Longrightarrow P\ x\ I$
  ⟨*proof*⟩

Summary of the two above. Use for early case splitting in proofs. Doesn't need P to be symmetric - the context in the conclusion is explicitly symmetric.

lemma *wlog-two-sets-element*:
  assumes *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
    and *case-split*: $\bigwedge a\ b\ c\ d\ x\ I\ J.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d]\!] \Longrightarrow$
      $(x{=}a \vee x{=}c \longrightarrow P\ x\ I\ J) \wedge (\neg(x{=}a \vee x{=}b \vee x{=}c \vee x{=}d) \longrightarrow P\ x\ I\ J)$
  shows $\bigwedge x\ I\ J.\ [\![\exists\, a\ b.\ Q\ I\ a\ b;\ \exists\, a\ b.\ Q\ J\ a\ b]\!] \Longrightarrow P\ x\ I\ J$
  ⟨*proof*⟩

Now we start on the actual result of interest. First we assume the events are all distinct, and we deal with the degenerate possibilities after.

lemma *wlog-endpoints-distinct1*:
  assumes *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
    and $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ betw4\ a\ b\ c\ d]\!] \Longrightarrow P\ I\ J$
  shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
      $betw4\ b\ a\ c\ d \vee betw4\ a\ b\ d\ c \vee betw4\ b\ a\ d\ c \vee betw4\ d\ c\ b\ a]\!] \Longrightarrow P\ I\ J$
  ⟨*proof*⟩

lemma *wlog-endpoints-distinct2*:
  assumes *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
    and $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ betw4\ a\ c\ b\ d]\!] \Longrightarrow P\ I\ J$
  shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
      $betw4\ b\ c\ a\ d \vee betw4\ a\ d\ b\ c \vee betw4\ b\ d\ a\ c \vee betw4\ d\ b\ c\ a]\!] \Longrightarrow P\ I\ J$
  ⟨*proof*⟩

lemma *wlog-endpoints-distinct3*:
  assumes *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
    and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\, a\ b.\ Q\ I\ a\ b;\ \exists\, a\ b.\ Q\ J\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
    and $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ betw4\ a\ c\ d\ b]\!] \Longrightarrow P\ I\ J$
  shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
      $betw4\ a\ d\ c\ b \vee betw4\ b\ c\ d\ a \vee betw4\ b\ d\ c\ a \vee betw4\ c\ a\ b\ d]\!] \Longrightarrow P\ I\ J$
  ⟨*proof*⟩

lemma (**in** *MinkowskiSpacetime*) *wlog-endpoints-distinct4*:
  fixes Q:: $('a\ set) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$
    and P:: $('a\ set) \Rightarrow ('a\ set) \Rightarrow bool$
    and A:: $('a\ set)$
  assumes *path-A*: $A{\in}\mathcal{P}$

and *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
and *Q-implies-path*: $\bigwedge a\ b\ I.\ [\![I{\subseteq}A;\ Q\ I\ a\ b]\!] \Longrightarrow b{\in}A \wedge a{\in}A$
and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\, a\ b.\ Q\ I\ a\ b;\ \exists\, a\ b.\ Q\ J\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
and $\bigwedge I\ J\ a\ b\ c\ d.$
$[\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;\ \mathit{betw4}\ a\ b\ c\ d \vee \mathit{betw4}\ a\ c\ b\ d \vee \mathit{betw4}\ a\ c\ d$
$b]\!] \Longrightarrow P\ I\ J$
shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;$
$a{\neq}b \wedge a{\neq}c \wedge a{\neq}d \wedge b{\neq}c \wedge b{\neq}d \wedge c{\neq}d]\!] \Longrightarrow P\ I\ J$
⟨*proof*⟩

lemma (**in** *MinkowskiSpacetime*) *wlog-endpoints-distinct'*:
assumes $A \in \mathcal{P}$
and $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
and $\bigwedge a\ b\ I.\ [\![I \subseteq A;\ Q\ I\ a\ b]\!] \Longrightarrow a \in A$
and $\bigwedge I\ J.\ [\![\exists\, a\ b.\ Q\ I\ a\ b;\ \exists\, a\ b.\ Q\ J\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
and $\bigwedge I\ J\ a\ b\ c\ d.$
$[\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;\ \mathit{betw4}\ a\ b\ c\ d \vee \mathit{betw4}\ a\ c\ b\ d \vee \mathit{betw4}\ a\ c\ d$
$b]\!] \Longrightarrow P\ I\ J$
and $Q\ I\ a\ b$
and $Q\ J\ c\ d$
and $I \subseteq A$
and $J \subseteq A$
and $a \neq b\ a \neq c\ a \neq d\ b \neq c\ b \neq d\ c \neq d$
shows $P\ I\ J$
⟨*proof*⟩

lemma (**in** *MinkowskiSpacetime*) *wlog-endpoints-distinct*:
assumes *path-A*: $A{\in}\mathcal{P}$
and *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
and *Q-implies-path*: $\bigwedge a\ b\ I.\ [\![I{\subseteq}A;\ Q\ I\ a\ b]\!] \Longrightarrow b{\in}A \wedge a{\in}A$
and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\, a\ b.\ Q\ I\ a\ b;\ \exists\, a\ b.\ Q\ J\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
and $\bigwedge I\ J\ a\ b\ c\ d.$
$[\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;\ \mathit{betw4}\ a\ b\ c\ d \vee \mathit{betw4}\ a\ c\ b\ d \vee \mathit{betw4}\ a\ c\ d$
$b]\!] \Longrightarrow P\ I\ J$
shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;$
$a{\neq}b \wedge a{\neq}c \wedge a{\neq}d \wedge b{\neq}c \wedge b{\neq}d \wedge c{\neq}d]\!] \Longrightarrow P\ I\ J$
⟨*proof*⟩

lemma *wlog-endpoints-degenerate1*:
assumes *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\, a\ b.\ Q\ I\ a\ b;\ \exists\, a\ b.\ Q\ I\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$

and *two*: $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
$(a{=}b \wedge b{=}c \wedge c{=}d) \vee (a{=}b \wedge b{\neq}c \wedge c{=}d)]\!] \Longrightarrow P\ I\ J$

and *one*: $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
$(a{=}b \wedge b{=}c \wedge c{\neq}d) \vee (a{=}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d)]\!] \Longrightarrow P\ I\ J$

and *no*: $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;$
$(a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{=}d) \vee (a{\neq}b \wedge b{=}c \wedge c{\neq}d \wedge a{=}d)]\!] \Longrightarrow P\ I\ J$
shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ \neg(a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d \wedge a{\neq}c \wedge b{\neq}d)]\!] \Longrightarrow P\ I\ J$
$\langle proof \rangle$

lemma *wlog-endpoints-degenerate2*:
assumes *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
and *Q-implies-path*: $\bigwedge a\ b\ I\ A.\ [\![I{\subseteq}A;\ A{\in}\mathcal{P};\ Q\ I\ a\ b]\!] \Longrightarrow b{\in}A \wedge a{\in}A$
and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\ a\ b.\ Q\ I\ a\ b;\ \exists\ a\ b.\ Q\ J\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
and $\bigwedge I\ J\ a\ b\ c\ d\ A.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;\ A{\in}\mathcal{P};$
$[[a\ b\ c]] \wedge a{=}d]\!] \Longrightarrow P\ I\ J$
and $\bigwedge I\ J\ a\ b\ c\ d\ A.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;\ A{\in}\mathcal{P};$
$[[b\ a\ c]] \wedge a{=}d]\!] \Longrightarrow P\ I\ J$
shows $\bigwedge I\ J\ a\ b\ c\ d\ A.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;\ A{\in}\mathcal{P};$
$a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{=}d]\!] \Longrightarrow P\ I\ J$
$\langle proof \rangle$

lemma *wlog-endpoints-degenerate*:
assumes *path-A*: $A{\in}\mathcal{P}$
and *symmetric-Q*: $\bigwedge a\ b\ I.\ Q\ I\ a\ b \Longrightarrow Q\ I\ b\ a$
and *Q-implies-path*: $\bigwedge a\ b\ I.\ [\![I{\subseteq}A;\ Q\ I\ a\ b]\!] \Longrightarrow b{\in}A \wedge a{\in}A$
and *symmetric-P*: $\bigwedge I\ J.\ [\![\exists\ a\ b.\ Q\ I\ a\ b;\ \exists\ a\ b.\ Q\ J\ a\ b;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
and $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A]\!]$
$\Longrightarrow ((a{=}b \wedge b{=}c \wedge c{=}d) \longrightarrow P\ I\ J) \wedge ((a{=}b \wedge b{\neq}c \wedge c{=}d) \longrightarrow P\ I\ J)$
$\wedge ((a{=}b \wedge b{=}c \wedge c{\neq}d) \longrightarrow P\ I\ J) \wedge ((a{=}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d) \longrightarrow P\ I\ J)$
$\wedge ((a{\neq}b \wedge b{=}c \wedge c{\neq}d \wedge a{=}d) \longrightarrow P\ I\ J)$
$\wedge ((([[a\ b\ c]] \wedge a{=}d) \longrightarrow P\ I\ J) \wedge ((([[b\ a\ c]] \wedge a{=}d) \longrightarrow P\ I\ J)$
shows $\bigwedge I\ J\ a\ b\ c\ d.\ [\![Q\ I\ a\ b;\ Q\ J\ c\ d;\ I{\subseteq}A;\ J{\subseteq}A;$
$\neg(a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d \wedge a{\neq}c \wedge b{\neq}d)]\!] \Longrightarrow P\ I\ J$
$\langle proof \rangle$

end

36.3 WLOG for two intervals

context *MinkowskiBetweenness* **begin**

This section just specifies the results for a generic relation Q in the previous section to the interval relation.

lemma *wlog-two-interval-element*:
assumes $\bigwedge x\ I\ J.\ [\![\textit{is-interval}\ I;\ \textit{is-interval}\ J;\ P\ x\ J\ I]\!] \Longrightarrow P\ x\ I\ J$
and $\bigwedge a\ b\ c\ d\ x\ I\ J.\ [\![I = \textit{interval}\ a\ b;\ J = \textit{interval}\ c\ d]\!] \Longrightarrow$
$(x{=}a \vee x{=}c \longrightarrow P\ x\ I\ J) \wedge (\neg(x{=}a \vee x{=}b \vee x{=}c \vee x{=}d) \longrightarrow P\ x\ I\ J)$
shows $\bigwedge x\ I\ J.\ [\![\textit{is-interval}\ I;\ \textit{is-interval}\ J]\!] \Longrightarrow P\ x\ I\ J$

⟨*proof*⟩

lemma (**in** *MinkowskiSpacetime*) *wlog-interval-endpoints-distinct*:
assumes $\bigwedge I\ J.\ [\![\textit{is-interval}\ I;\ \textit{is-interval}\ J;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
$\bigwedge I\ J\ a\ b\ c\ d.\ [\![I = \textit{interval}\ a\ b;\ J = \textit{interval}\ c\ d]\!]$
$\Longrightarrow (\textit{betw4}\ a\ b\ c\ d \longrightarrow P\ I\ J) \wedge (\textit{betw4}\ a\ c\ b\ d \longrightarrow P\ I\ J) \wedge (\textit{betw4}\ a\ c\ d$
$b \longrightarrow P\ I\ J)$
shows $\bigwedge I\ J\ Q\ a\ b\ c\ d.\ [\![I = \textit{interval}\ a\ b;\ J = \textit{interval}\ c\ d;\ I{\subseteq}Q;\ J{\subseteq}Q;\ Q{\in}\mathcal{P};$
$a{\neq}b \wedge a{\neq}c \wedge a{\neq}d \wedge b{\neq}c \wedge b{\neq}d \wedge c{\neq}d]\!] \Longrightarrow P\ I\ J$
⟨*proof*⟩

lemma *wlog-interval-endpoints-degenerate*:
assumes *symmetry*: $\bigwedge I\ J.\ [\![\textit{is-interval}\ I;\ \textit{is-interval}\ J;\ P\ I\ J]\!] \Longrightarrow P\ J\ I$
and $\bigwedge I\ J\ a\ b\ c\ d\ Q.\ [\![I = \textit{interval}\ a\ b;\ J = \textit{interval}\ c\ d;\ I{\subseteq}Q;\ J{\subseteq}Q;\ Q{\in}\mathcal{P}]\!]$
$\Longrightarrow ((a{=}b \wedge b{=}c \wedge c{=}d) \longrightarrow P\ I\ J) \wedge ((a{=}b \wedge b{\neq}c \wedge c{=}d) \longrightarrow P\ I\ J)$
$\wedge ((a{=}b \wedge b{=}c \wedge c{\neq}d) \longrightarrow P\ I\ J) \wedge ((a{=}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d) \longrightarrow$
$P\ I\ J)$
$\wedge ((a{\neq}b \wedge b{=}c \wedge c{\neq}d \wedge a{=}d) \longrightarrow P\ I\ J)$
$\wedge (([[a\ b\ c]] \wedge a{=}d) \longrightarrow P\ I\ J) \wedge (([[b\ a\ c]] \wedge a{=}d) \longrightarrow P\ I\ J)$
shows $\bigwedge I\ J\ a\ b\ c\ d\ Q.\ [\![I = \textit{interval}\ a\ b;\ J = \textit{interval}\ c\ d;\ I{\subseteq}Q;\ J{\subseteq}Q;\ Q{\in}\mathcal{P};$
$\neg(a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d \wedge a{\neq}c \wedge b{\neq}d)]\!] \Longrightarrow P\ I\ J$
⟨*proof*⟩

end

37 Interlude: Intervals, Segments, Connectedness

context *MinkowskiSpacetime* **begin**

In this secion, we apply the WLOG lemmas from the previous section in order to reduce the number of cases we need to consider when thinking about two arbitrary intervals on a path. This is used to prove that the (countable) intersection of intervals is an interval. These results cannot be found in Schutz, but he does use them (without justification) in his proof of Theorem 12 (even for uncountable intersections).

lemma *int-of-ints-is-interval-neq*:
assumes $I1 = \textit{interval}\ a\ b\ I2 = \textit{interval}\ c\ d\ I1{\subseteq}P\ I2{\subseteq}P\ P{\in}\mathcal{P}\ I1{\cap}I2 \neq \{\}$
and *events-neq*: $a{\neq}b\ a{\neq}c\ a{\neq}d\ b{\neq}c\ b{\neq}d\ c{\neq}d$
shows *is-interval* $(I1 \cap I2)$
⟨*proof*⟩

lemma *int-of-ints-is-interval-deg*:
assumes $I = \textit{interval}\ a\ b\ J = \textit{interval}\ c\ d\ I{\cap}J \neq \{\}\ I{\subseteq}P\ J{\subseteq}P\ P{\in}\mathcal{P}$
and *events-deg*: $\neg(a{\neq}b \wedge b{\neq}c \wedge c{\neq}d \wedge a{\neq}d \wedge a{\neq}c \wedge b{\neq}d)$
shows *is-interval* $(I \cap J)$

⟨*proof*⟩

lemma *int-of-ints-is-interval*:
  assumes $\mathit{is\text{-}interval}\ I\ \mathit{is\text{-}interval}\ J\ I{\subseteq}P\ J{\subseteq}P\ P{\in}\mathcal{P}\ \ I{\cap}J \neq \{\}$
  shows $\mathit{is\text{-}interval}\ (I \cap J)$
  ⟨*proof*⟩

lemma *int-of-ints-is-interval2*:
  assumes $\forall x{\in}S.\ (\mathit{is\text{-}interval}\ x \wedge x{\subseteq}P)\ P{\in}\mathcal{P}\ \bigcap S \neq \{\}\ \mathit{finite}\ S\ S{\neq}\{\}$
  shows $\mathit{is\text{-}interval}\ (\bigcap S)$
⟨*proof*⟩

end

38 3.7 Continuity and the monotonic sequence property

context *MinkowskiSpacetime* **begin**

This section only includes a proof of the first part of Theorem 12, as well as some results that would be useful in proving part (ii).

theorem *two-rays*:
  assumes *path-Q*: $Q{\in}\mathcal{P}$
    and *event-a*: $a{\in}Q$
   shows $\exists R\ L.\ (\mathit{is\text{-}ray\text{-}on}\ R\ Q \wedge \mathit{is\text{-}ray\text{-}on}\ L\ Q$
      $\wedge\ Q{-}\{a\} \subseteq (R \cup L)$ ~~events of Q excl. a belong to two rays~~
      $\wedge\ (\forall r{\in}R.\ \forall l{\in}L.\ [[l\ a\ r]])$ ~~a is betw any a of one ray and any a of the other~~
      $\wedge\ (\forall x{\in}R.\ \forall y{\in}R.\ \neg\ [[x\ a\ y]])$ ~~but a is not betw any two events of the same ray~~
      $\wedge\ (\forall x{\in}L.\ \forall y{\in}L.\ \neg\ [[x\ a\ y]]))$
⟨*proof*⟩

The definition *closest-to* in prose: Pick any $r \in R$. The closest event c is such that there is no closer event in L, i.e. all other events of L are further away from r. Thus in L, c is the element closest to R.

definition $\mathit{closest\text{-}to} :: ('a\ \mathit{set}) \Rightarrow 'a \Rightarrow ('a\ \mathit{set}) \Rightarrow \mathit{bool}$
  where $\mathit{closest\text{-}to}\ L\ c\ R \equiv c{\in}L \wedge (\forall r{\in}R.\ \forall l{\in}L{-}\{c\}.\ [[l\ c\ r]])$

lemma *int-on-path*:
  assumes $l{\in}L\ r{\in}R\ Q{\in}\mathcal{P}$
    and *partition*: $L{\subseteq}Q\ L{\neq}\{\}\ R{\subseteq}Q\ R{\neq}\{\}\ L{\cup}R{=}Q$
   shows $\mathit{interval}\ l\ r \subseteq Q$
⟨*proof*⟩

lemma *ray-of-bounds1*:
assumes $Q \in \mathcal{P}$ $[f[(f\ 0)..]X]$ $X \subseteq Q$ *closest-bound* c X *is-bound-f* b X f $b \neq c$
assumes *is-bound-f* x X f
shows $x=b \vee x=c \vee [[c\ x\ b]] \vee [[c\ b\ x]]$
⟨*proof*⟩

lemma *ray-of-bounds2*:
assumes $Q \in \mathcal{P}$ $[f[(f\ 0)..]X]$ $X \subseteq Q$ *closest-bound-f* c X f *is-bound-f* b X f $b \neq c$
assumes $x=b \vee x=c \vee [[c\ x\ b]] \vee [[c\ b\ x]]$
shows *is-bound-f* x X f
⟨*proof*⟩

lemma *ray-of-bounds3*:
assumes $Q \in \mathcal{P}$ $[f[(f\ 0)..]X]$ $X \subseteq Q$ *closest-bound-f* c X f *is-bound-f* b X f $b \neq c$
shows *all-bounds* X = *insert* c (*ray* c b)
⟨*proof*⟩

lemma *ray-of-bounds*:
assumes $[f[(f\ 0)..]X]$ *closest-bound-f* c X f *is-bound-f* b X f $b \neq c$
shows *all-bounds* X = *insert* c (*ray* c b)
⟨*proof*⟩

lemma *int-in-closed-ray*:
assumes *path* ab a b
shows *interval* a b $\subset$ *insert* a (*ray* a b)
⟨*proof*⟩

lemma *bound-any-f*:
assumes $Q \in \mathcal{P}$ $[f[(f\ 0)..]X]$ $X \subseteq Q$ *is-bound* c X
shows *is-bound-f* c X f
⟨*proof*⟩

lemma *closest-bound-any-f*:
assumes $Q \in \mathcal{P}$ $[f[(f\ 0)..]X]$ $X \subseteq Q$ *closest-bound* c X
shows *closest-bound-f* c X f
⟨*proof*⟩

end

39 3.8 Connectedness of the unreachable set

context *MinkowskiSpacetime* **begin**

39.1 Theorem 13 (Connectedness of the Unreachable Set)

theorem *unreach-connected*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *event-b*: $b \notin Q$ $b \in \mathcal{E}$
and *unreach*: $Q_x \in \emptyset\ Q\ b$ $Q_z \in \emptyset\ Q\ b$ $Q_x \neq Q_z$
and *xyz*: $[[Q_x\ Q_y\ Q_z]]$
shows $Q_y \in \emptyset\ Q\ b$
⟨*proof*⟩

39.2 Theorem 14 (Second Existence Theorem)

lemma *union-of-bounded-sets-is-bounded*:
assumes $\forall x \in A.\ [[a\ x\ b]]$ $\forall x \in B.\ [[c\ x\ d]]$ $A \subseteq Q$ $B \subseteq Q$ $Q \in \mathcal{P}$
card $A > 1 \vee$ *infinite* A *card* $B > 1 \vee$ *infinite* B
shows $\exists l \in Q.\ \exists u \in Q.\ \forall x \in A \cup B.\ [[l\ x\ u]]$
⟨*proof*⟩

lemma *union-of-bounded-sets-is-bounded2*:
assumes $\forall x \in A.\ [[a\ x\ b]]$ $\forall x \in B.\ [[c\ x\ d]]$ $A \subseteq Q$ $B \subseteq Q$ $Q \in \mathcal{P}$
$1 <$ *card* $A \vee$ *infinite* A $1 <$ *card* $B \vee$ *infinite* B
shows $\exists l \in Q - (A \cup B).\ \exists u \in Q - (A \cup B).\ \forall x \in A \cup B.\ [[l\ x\ u]]$
⟨*proof*⟩

Schutz proves a mildly stronger version of this theorem than he states. Namely, he gives an additional condition that has to be fulfilled by the bounds y, z in the proof ($y,z \notin \emptyset\ Q\ ab$). This condition is trivial given *abc-abc-neq*. His stating it in the proof makes me wonder whether his (strictly speaking) undefined notion of bounded set is somehow weaker than the version using strict betweenness in his theorem statement and used here in Isabelle. This would make sense, given the obvious analogy with sets on the real line.

theorem *second-existence-thm-1*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *events*: $a \notin Q$ $b \notin Q$
and *reachable*: *path-ex a q1 path-ex b q2* $q1 \in Q$ $q2 \in Q$
shows $\exists y \in Q.\ \exists z \in Q.\ (\forall x \in \emptyset\ Q\ a.\ [[y\ x\ z]]) \wedge (\forall x \in \emptyset\ Q\ b.\ [[y\ x\ z]])$
⟨*proof*⟩

theorem *second-existence-thm-2*:
assumes *path-Q*: $Q \in \mathcal{P}$
and *events*: $a \notin Q$ $b \notin Q$ $c \in Q$ $d \in Q$ $c \neq d$
and *reachable*: $\exists P \in \mathcal{P}.\ \exists q \in Q.$ *path* $P\ a\ q$ $\exists P \in \mathcal{P}.\ \exists q \in Q.$ *path* $P\ b\ q$

shows $\exists\, e{\in}Q.\ \exists\, ae{\in}\mathcal{P}.\ \exists\, be{\in}\mathcal{P}.\ path\ ae\ a\ e \wedge path\ be\ b\ e \wedge [[c\ d\ e]]$
⟨*proof*⟩

The assumption $Q{\neq}R$ in Theorem 14(iii) is somewhat implicit in Schutz. If $Q{=}R$, $\emptyset\ Q\ a$ is empty, so the third conjunct of the conclusion is meaningless.

theorem *second-existence-thm-3*:
assumes *paths*: $Q{\in}\mathcal{P}\ R{\in}\mathcal{P}\ Q{\neq}R$
and *events*: $x{\in}Q\ x{\in}R\ a{\in}R\ a{\neq}x\ b{\notin}Q$
and *reachable*: $\exists\, P{\in}\mathcal{P}.\ \exists\, q{\in}Q.\ path\ P\ b\ q$
shows $\exists\, e{\in}\mathcal{E}.\ \exists\, ae{\in}\mathcal{P}.\ \exists\, be{\in}\mathcal{P}.\ path\ ae\ a\ e \wedge path\ be\ b\ e \wedge (\forall\, y{\in}\emptyset\ Q\ a.\ [[x\ y\ e]])$
⟨*proof*⟩

end

40 Theorem 11 - with path density assumed

locale *MinkowskiDense* = *MinkowskiSpacetime* +
assumes *path-dense*: $path\ ab\ a\ b \Longrightarrow \exists\, x.\ [[a\ x\ b]]$
begin

Path density: if a and b are connected by a path, then the segment between them is nonempty. Since Schutz insists on the number of segments in his segmentation (Theorem 11), we prove it here, showcasing where his missing assumption of path density fits in (it is used three times in *number-of-segments*, once in each separate meaningful ordering case).

lemma *segment-nonempty*:
assumes $path\ ab\ a\ b$
obtains x **where** $x \in segment\ a\ b$
⟨*proof*⟩

lemma *number-of-segments*:
assumes *path-P*: $P{\in}\mathcal{P}$
and *Q-def*: $Q{\subseteq}P$
and *f-def*: $[f[a..b..c]Q]$
shows $card\ \{segment\ (f\ i)\ (f\ (i{+}1)) \mid i.\ i{<}(card\ Q{-}1)\} = card\ Q - 1$
⟨*proof*⟩

theorem *segmentation-card*:
assumes *path-P*: $P{\in}\mathcal{P}$
and *Q-def*: $Q{\subseteq}P$
and *f-def*: $[f[a..b]Q]$
fixes *P1* **defines** *P1-def*: $P1 \equiv prolongation\ b\ a$
fixes *P2* **defines** *P2-def*: $P2 \equiv prolongation\ a\ b$
fixes *S* **defines** *S-def*: $S \equiv (if\ card\ Q{=}2\ then\ \{segment\ a\ b\}\ else\ \{segment\ (f\ i)\ (f\ (i{+}1)) \mid i.\ i{<}card\ Q{-}1\})$
shows $P = ((\bigcup S) \cup P1 \cup P2 \cup Q)$

$card\ S = (card\ Q{-}1) \land (\forall\, x{\in}S.\ is\text{-}segment\ x)$

$disjoint\ (S{\cup}\{P1,P2\})\ P1{\neq}P2\ P1{\notin}S\ P2{\notin}S$

$\langle proof \rangle$

end

end

References

[1] J. W. Schutz. *Independent Axioms for Minkowski Space-Time.* CRC Press, Oct. 1997.