Sauer-Shelah Lemma

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Abstract

The Sauer-Shelah Lemma is a fundamental result in extremal set theory and combinatorics, that guarantees the existence of a set T of size k which is shattered by a family of sets $\mathcal{F}$, if the cardinality of the family is greater than some bound dependent on k. A set T is said to be shattered by a family $\mathcal{F}$ if every subset of T can be obtained as an intersection of T with some set $S \in \mathcal{F}$. The Sauer-Shelah Lemma has found use in diverse fields such as computational geometry, approximation algorithms and machine learning. In this entry we formalize the notion of shattering and prove the generalized and standard versions of the Sauer-Shelah Lemma.

Contents

1 Introduction

The goal of this entry is to formalize the Sauer-Shelah Lemma. The result was first published by Sauer [2] and Shelah [3] independently from one another. The proof presented in this entry is based on an article by Kalai [1].

The lemma has a wide range of applications. Vapnik and Červonenkis [4] reproved and used the lemma in the context of statistical learning theory. For instance, the VC-dimension of a family of sets is defined as the size of the largest set the family shatters. In this context, the Sauer-Shelah Lemma is a result tying the VC-dimension of a family to the number of sets in the family.

2 Definitions and lemmas about shattering

In this section, we introduce the predicate *shatters* and the term for the family of sets that a family shatters *shattered-by*.

theory *Shattering*
  imports *Main*
begin

2.1 Intersection of a family of sets with a set

abbreviation $IntF :: 'a\ set\ set \Rightarrow 'a\ set \Rightarrow 'a\ set\ set$ (**infixl** $\cap*$ *60*)
  where $F \cap* S \equiv ((\cap)\ S)$ ‘ F

lemma *idem-IntF*:
  assumes $\bigcup A \subseteq Y$
  shows $A \cap* Y = A$
⟨*proof*⟩

lemma *subset-IntF*:
  assumes $A \subseteq B$
  shows $A \cap* X \subseteq B \cap* X$
  ⟨*proof*⟩

lemma *Int-IntF*: $(A \cap* Y) \cap* X = A \cap* (Y \cap X)$
⟨*proof*⟩

insert distributes over $(\cap*)$

lemma *insert-IntF*:
  shows *insert* x ‘ $(H \cap* S) = ($*insert* x ‘ $H) \cap* ($*insert* $x\ S)$
⟨*proof*⟩

2.2 Definition of *shatters*, *VC-dim* and *shattered-by*

abbreviation $shatters :: 'a\ set\ set \Rightarrow 'a\ set \Rightarrow bool$ (**infixl** *shatters 70*)
  where H *shatters* $A \equiv H \cap* A = Pow\ A$

definition $VC\text{-}dim :: 'a\ set\ set \Rightarrow nat$
  where $VC\text{-}dim\ F = Sup\ \{card\ S \mid S.\ F\ shatters\ S\}$

definition $shattered\text{-}by :: 'a\ set\ set \Rightarrow 'a\ set\ set$
  where $shattered\text{-}by\ F \equiv \{A.\ F\ shatters\ A\}$

lemma *shattered-by-in-Pow*:
  shows *shattered-by* $F \subseteq$ *Pow* $(\bigcup F)$
  ⟨*proof*⟩

lemma *subset-shatters*:
  assumes $A \subseteq B$ **and** A *shatters* X
  shows B *shatters* X
⟨*proof*⟩

lemma *supset-shatters*:
  assumes $Y \subseteq X$ **and** A *shatters* X
  shows A *shatters* Y
⟨*proof*⟩

lemma *shatters-empty*:
  assumes $F \neq \{\}$
  shows F *shatters* $\{\}$
⟨*proof*⟩

lemma *subset-shattered-by*:
  assumes $A \subseteq B$
  shows *shattered-by* $A \subseteq$ *shattered-by* B
⟨*proof*⟩

lemma *finite-shattered-by*:
  assumes *finite* $(\bigcup F)$
  shows *finite* (*shattered-by* F)
  ⟨*proof*⟩

The following example shows that requiring finiteness of a family of sets is not enough, to ensure that *shattered-by* also stays finite.

lemma $\exists F$::*nat set set*. *finite* $F \wedge$ *infinite* (*shattered-by* F)
⟨*proof*⟩

end

3 Lemmas involving the cardinality of sets

In this section, we prove some lemmas that make use of the term *card* or provide bounds for it.

theory *Card-Lemmas*
  imports *Main*
begin

lemma *card-Int-copy*:
  assumes *finite* X **and** $A \cup B \subseteq X$ **and** $\exists f.$ *inj-on* f $(A \cap B) \wedge (A \cup B) \cap (f$ ‘ $(A \cap B)) = \{\} \wedge f$ ‘ $(A \cap B) \subseteq X$

shows *card* A + *card* $B \leq$ *card* X
⟨*proof*⟩

lemma *finite-diff-not-empty*:
assumes *finite* Y **and** *card* $Y <$ *card* X
shows $X - Y \neq \{\}$
⟨*proof*⟩

lemma *obtain-difference-element*:
fixes $F :: {}'a$ *set set*
assumes $2 \leq$ *card* F
obtains x **where** $x \in \bigcup F$ $x \notin \bigcap F$
⟨*proof*⟩

end

4 Lemmas involving the binomial coefficient

In this section, we prove lemmas that use the term for the binomial coefficient *choose*.

theory *Binomial-Lemmas*
imports *Main*
begin

lemma *choose-mono*:
assumes $x \leq y$
shows x *choose* $n \leq y$ *choose* n
⟨*proof*⟩

lemma *choose-row-sum-set*:
assumes *finite* $(\bigcup F)$
shows *card* $\{S.\ S \subseteq \bigcup F \wedge$ *card* $S \leq k\} = (\sum i \leq k.$ *card* $(\bigcup F)$ *choose* $i)$
⟨*proof*⟩

end

5 Sauer-Shelah Lemma

theory *Sauer-Shelah-Lemma*
imports *Shattering Card-Lemmas Binomial-Lemmas*
begin

5.1 Generalized Sauer-Shelah Lemma

To prove the Sauer-Shelah Lemma, we will first prove a slightly stronger fact that every family F shatters at least as many sets as *card F*. We first fix an element $x \in \bigcup F$ and consider the subfamily *F0* of sets in the family not containing it. By induction, *F0* shatters at least as many elements of *F* as *card F0*. Next, we consider the subfamily *F1* of sets in the family that contain *x*. Again, by induction, *F1* shatters as many elements of *F* as its cardinality. The number of elements of *F* shattered by *F0* and *F1* sum up to at least *card F0* + *card F1* = *card F*. When a set $S \in F$ is shattered by only one of the two subfamilies, say *F0*, it contributes one unit to the set *shattered-by F0* and to *shattered-by F*. However, when the set is shattered by both subfamilies, both *S* and $S \cup \{x\}$ are in *shattered-by F*, so *S* contributes two units to *shattered-by F0* ∪ *shattered-by F1*. Therefore, the cardinality of *shattered-by F* is at least equal to the cardinality of *shattered-by F0* ∪ *shattered-by F1*, which is at least *card F*.

lemma *sauer-shelah-0*:
  fixes *F* :: *'a set set*
  shows *finite* (⋃ *F*) ⟹ *card F* ≤ *card* (*shattered-by F*)
⟨*proof*⟩

5.2 Sauer-Shelah Lemma

The generalized version immediately implies the Sauer-Shelah Lemma, because only ($\sum$ *i*≤*k*. *n choose i*) of the subsets of an *n*-item universe have cardinality less than *k* + *1*. Thus, when ($\sum$ *i*≤*k*. *n choose i*) < *card F*, there are not enough sets to be shattered, so one of the shattered sets must have cardinality at least *k* + *1*.

corollary *sauer-shelah*:
  fixes *F* :: *'a set set*
  assumes *finite* (⋃*F*) **and** ($\sum$ *i*≤*k*. *card* (⋃*F*) *choose i*) < *card F*
  shows ∃*S*. (*F shatters S* ∧ *card S* = *k* + *1*)
⟨*proof*⟩

5.3 Sauer-Shelah Lemma for hypergraphs

If we designate X to be the set of hyperedges and S the set of vertices, we can also formulate the Sauer-Shelah Lemma in terms of hypergraphs. In this form, the statement provides a sufficient condition for the existence of an hyperedge of a given cardinality which is shattered by the set of edges.

corollary *sauer-shelah-2*:
  fixes *X* :: *'a set set* **and** *S* :: *'a set*
  assumes *finite S* **and** *X* ⊆ *Pow S* **and** ($\sum$ *i*≤*k*. *card S choose i*) < *card X*
  shows ∃*Y*. (*X shatters Y* ∧ *card Y* = *k* + *1*)
⟨*proof*⟩

5.4 Alternative statement of the Sauer-Shelah Lemma

We can also state the Sauer-Shelah Lemma in terms of the *VC-dim*. If the VC-dimension of *F* is *k* then *F* can consist at most of ($\sum$ *i*≤*k*. *card* (⋃*F*) *choose i*) sets which is in $\mathcal{O}$(*card* (⋃*F*) ^*k*).

corollary *sauer-shelah-alt*:
  assumes *finite* $(\bigcup F)$ **and** *VC-dim* $F = k$
  shows *card* $F \le (\sum i{\le}k.\ card\ (\bigcup F)\ choose\ i)$
⟨*proof*⟩

end

References

[1] G. Kalai. Extremal combinatorics iii: Some basic theorems, Sep 2008.

[2] N. Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13(1):145–147, 1972.

[3] S. Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific Journal of Mathematics*, 41(1):247 – 261, 1972.

[4] V. N. Vapnik and A. J. Červonenkis. The uniform convergence of frequencies of the appearance of events to their probabilities. *Teor. Verojatnost. i Primenen.*, 16:264–279, 1971.