

# A Formally Verified Checker of the Safe Distance Traffic Rules for Autonomous Vehicles

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## Abstract

The Vienna Convention on Road Traffic defines the safe distance traffic rules informally. This could make autonomous vehicle liable for safe-distance-related accidents because there is no clear definition of how large a safe distance is. We provide a formally proven prescriptive definition of a safe distance, and checkers which can decide whether an autonomous vehicle is obeying the safe distance rule. Not only does our work apply to the domain of law, but it also serves as a specification for autonomous vehicle manufacturers and for online verification of path planners. This formalization accompanies our paper "A Formally Verified Checker of the Safe Distance Traffic Rules for Autonomous Vehicles". [1]

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# 1 Safe Distance

```

theory Safe-Distance
imports
  HOL-Analysis.Multivariate-Analysis
  HOL-Decision-Procs.Approximation
  Sturm-Sequences.Sturm
begin

```

This theory is about formalising the safe distance rule. The safe distance rule is obtained from Vienna Convention which basically states the following thing.

“The car at all times must maintain a safe distance towards the vehicle in front of it, such that whenever the vehicle in front and the ego vehicle apply maximum deceleration, there will not be a collision.”

To formalise this safe distance rule we have to define first what is a safe distance. To define this safe distance, we have to model the physics of the movement of the vehicle. The following model is sufficient.

$$s = s_0 + v_0 * t + 1 / 2 * a_0 * t^2$$

Assumptions in this model are :

- Both vehicles are assumed to be point mass. The exact location of the ego vehicle is the front-most occupancy of the ego vehicle. Similarly for the other vehicle, its exact location is the rearmost occupancy of the other vehicle.
- Both cars can never drive backward.

```

lemmas [simp del] = div-mult-self1 div-mult-self2 div-mult-self3 div-mult-self4

```

## 1.1 Quadratic Equations

```

lemma discriminant:  $a * x^2 + b * x + c = (0::real) \implies 0 \leq b^2 - 4 * a * c$ 
by (sos ((( $A < 0 * R < 1$ ) + ( $R < 1 * (R < 1 * [2 * a * x + b]^2)$ ))))

```

```

lemma quadratic-eq-factoring:

```

**assumes**  $D : D = b^2 - 4 * a * c$   
**assumes**  $nn : 0 \leq D$   
**assumes**  $x1 : x_1 = (-b + \text{sqrt } D) / (2 * a)$   
**assumes**  $x2 : x_2 = (-b - \text{sqrt } D) / (2 * a)$   
**assumes**  $a : a \neq 0$   
**shows**  $a * x^2 + b * x + c = a * (x - x_1) * (x - x_2)$   
**using**  $nn$   
**by** (*simp add: D x1 x2*)  
*(simp add: asms power2-eq-square power3-eq-cube field-split-simps)*

**lemma** *quadratic-eq-zeroes-iff*:

**assumes**  $D : D = b^2 - 4 * a * c$   
**assumes**  $x1 : x_1 = (-b + \text{sqrt } D) / (2 * a)$   
**assumes**  $x2 : x_2 = (-b - \text{sqrt } D) / (2 * a)$   
**assumes**  $a : a \neq 0$   
**shows**  $a * x^2 + b * x + c = 0 \iff (D \geq 0 \wedge (x = x_1 \vee x = x_2))$  (**is**  $?z \iff -$ )  
**using** *quadratic-eq-factoring[OF D - x1 x2 a, of x] discriminant[of a x b c] a*  
**by** (*auto simp: D*)

## 1.2 Convexity Condition

**lemma** *p-convex*:

**fixes**  $a b c x y z :: \text{real}$   
**assumes**  $p\text{-def} : p = (\lambda x. a * x^2 + b * x + c)$   
**assumes**  $less : x < y \wedge y < z$   
**and**  $ge : p x > p y \wedge p y \leq p z$   
**shows**  $a > 0$   
**using**  $less$  **ge** **unfolding**  $p\text{-def}$   
**by** (*sos* ((( $(A < 0 * (A < 1 * A < 2)) * R < 1$ ) + ((( $A < 2 * R < 1$ ) \* ( $R < 1 / 4 * [y + \sim 1 * z]^{\wedge 2}$ )) + ((( $A <= 1 * R < 1$ ) \* ( $R < 1 * [x + \sim 1 * y]^{\wedge 2}$ )) + ((( $A <= 1 * (A < 0 * (A < 1 * R < 1))$ ) \* ( $R < 1 / 4 * [1]^{\wedge 2}$ )) + ((( $A <= 0 * R < 1$ ) \* ( $R < 1 / 4 * [\sim 1 * y]^{\wedge 2} + x * y + \sim 1 * x * z + y * z]^{\wedge 2}$ )) + ((( $A <= 0 * (A < 0 * (A < 1 * R < 1))$ ) \* ( $R < 1 * [x + \sim 1 / 2 * y + \sim 1 / 2 * z]^{\wedge 2}$ ))))))))))

**definition** *root-in* ::  $\text{real} \Rightarrow \text{real} \Rightarrow (\text{real} \Rightarrow \text{real}) \Rightarrow \text{bool}$  **where**

*root-in*  $m M f = (\exists x \in \{m .. M\}. f x = 0)$

**definition** *quadroot-in* ::  $\text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{bool}$  **where**

*quadroot-in*  $m M a b c = \text{root-in } m M (\lambda x. a * x^2 + b * x + c)$

**lemma** *card-iff-exists*:  $0 < \text{card } X \iff \text{finite } X \wedge (\exists x. x \in X)$

**by** (*auto simp: card-gt-0-iff*)

**lemma** *quadroot-in-sturm*[code]:

*quadroot-in*  $m M a b c \iff (a = 0 \wedge b = 0 \wedge c = 0 \wedge m \leq M) \vee$

$(m \leq M \wedge \text{poly } [:c, b, a:] m = 0) \vee$

*count-roots-between*  $[:c, b, a:] m M > 0$

**apply** (*cases*  $a = 0 \wedge b = 0 \wedge c = 0 \wedge m \leq M$ )

**apply** (*force simp: quadroot-in-def root-in-def*)  
**apply** (*cases  $m \leq M \wedge \text{poly } [:c, b, a:] m = 0$* )  
**apply** (*force simp: quadroot-in-def root-in-def algebra-simps power2-eq-square count-roots-between-correct card-iff-exists*)  
**proof** –  
**assume**  $H: \neg (a = 0 \wedge b = 0 \wedge c = 0 \wedge m \leq M) \neg (m \leq M \wedge \text{poly } [:c, b, a:] m = 0)$   
**hence**  $\text{poly } [:c, b, a:] m \neq 0 \vee m > M$   
**by** *auto*  
**then have**  $\text{quadroot-in } m M a b c \longleftrightarrow 0 < \text{count-roots-between } [:c, b, a:] m M$   
**proof** (*rule disjE*)  
**assume**  $pnz: \text{poly } [:c, b, a:] m \neq 0$   
**then have**  $nz: [:c, b, a:] \neq 0$  **by** *auto*  
**show** *?thesis*  
**unfolding** *count-roots-between-correct card-iff-exists*  
**apply** *safe*  
**apply** (*rule finite-subset*[**where**  $B = \{x. \text{poly } [:c, b, a:] x = 0\}$ ])  
**apply** *force*  
**apply** (*rule poly-roots-finite*)  
**apply** (*rule nz*)  
**using** *pnz*  
**apply** (*auto simp add: count-roots-between-correct quadroot-in-def root-in-def card-iff-exists algebra-simps power2-eq-square*)  
**apply** (*case-tac  $x = m$* )  
**apply** (*force simp: algebra-simps*)  
**apply** *force*  
**done**  
**qed** (*auto simp: quadroot-in-def count-roots-between-correct root-in-def card-eq-0-iff*)  
**then show**  $\text{quadroot-in } m M a b c = (a = 0 \wedge b = 0 \wedge c = 0 \wedge m \leq M \vee m \leq M \wedge \text{poly } [:c, b, a:] m = 0 \vee 0 < \text{count-roots-between } [:c, b, a:] m M)$   
**using**  $H$  **by** *metis*  
**qed**

**lemma** *check-quadroot-linear:*

**fixes**  $a b c :: \text{real}$   
**assumes**  $a = 0$   
**shows**  $\neg \text{quadroot-in } m M a b c \longleftrightarrow ((b = 0 \wedge c = 0 \wedge M < m) \vee (b = 0 \wedge c \neq 0) \vee (b \neq 0 \wedge (\text{let } x = -c / b \text{ in } m > x \vee x > M)))$   
**proof** –  
**have**  $\text{quadroot-in } m M a b c \longleftrightarrow (b = 0 \longrightarrow \text{quadroot-in } m M a b c) \wedge (b \neq 0 \longrightarrow \text{quadroot-in } m M a b c)$   
**by** *auto*  
**also have**  $(b = 0 \longrightarrow \text{quadroot-in } m M a b c) \longleftrightarrow ((b = 0 \longrightarrow c = 0 \longrightarrow m \leq M) \wedge (b \neq 0 \vee c = 0))$   
**by** (*auto simp: quadroot-in-def Let-def root-in-def assms field-split-simps intro!: bexI*[**where**  $x = -c / b$ ])

**also have**  $(b \neq 0 \longrightarrow \text{quadroot-in } m \ M \ a \ b \ c) \longleftrightarrow (b = 0 \vee (\text{let } x = -c / b \text{ in } m \leq x \wedge x \leq M))$   
**apply** (*auto simp: quadroot-in-def Let-def root-in-def assms field-split-simps*  
*intro!: bexI[where x=-c / b]*)  
**by** (*metis mult.commute mult-le-cancel-left-neg add-eq-0-iff mult-le-cancel-iff2*)+  
**finally show** ?thesis  
**by** (*simp add: Let-def not-less not-le*)  
**qed**

**lemma** *check-quadroot-nonlinear:*

**assumes**  $a \neq 0$   
**shows**  $\text{quadroot-in } m \ M \ a \ b \ c =$   
 $(\text{let } D = b^2 - 4 * a * c \text{ in } D \geq 0 \wedge$   
 $((\text{let } x = (-b + \text{sqr } D) / (2 * a) \text{ in } m \leq x \wedge x \leq M) \vee$   
 $(\text{let } x = (-b - \text{sqr } D) / (2 * a) \text{ in } m \leq x \wedge x \leq M)))$   
**by** (*auto simp: quadroot-in-def Let-def root-in-def quadratic-eq-zeroes-iff[OF refl refl assms]*)

**lemma** *ncheck-quadroot:*

**shows**  $\neg \text{quadroot-in } m \ M \ a \ b \ c \longleftrightarrow$   
 $(a = 0 \longrightarrow \neg \text{quadroot-in } m \ M \ a \ b \ c) \wedge$   
 $(a = 0 \vee \neg \text{quadroot-in } m \ M \ a \ b \ c)$   
**by** *auto*

### 1.3 Movement

**locale** *movement =*  
**fixes**  $a \ v \ s_0 :: \text{real}$   
**begin**

Function to compute the distance using the equation

$$s(t) = s_0 + v_0 * t + 1 / 2 * a_0 * t^2$$

Input parameters:

- $s_0$ : initial distance
- $v_0$ : initial velocity (positive means forward direction and the converse is true)
- $a$ : acceleration (positive for increasing and negative for decreasing)
- $t$ : time

For the time  $t < 0$ , we assume the output of the function is  $s_0$ . Otherwise, the output is calculated according to the equation above.

### 1.3.1 Continuous Dynamics

**definition**  $p :: \text{real} \Rightarrow \text{real}$  **where**

$$p\ t = s0 + v * t + 1/2 * a * t^2$$

**lemma**  $p$ -all-zeroes:

**assumes**  $D: D = v^2 - 2 * a * s0$

**shows**  $p\ t = 0 \iff ((a \neq 0 \wedge 0 \leq D \wedge ((t = (-v + \text{sqrt } D) / a) \vee t = (-v - \text{sqrt } D) / a)) \vee$

$(a = 0 \wedge v = 0 \wedge s0 = 0) \vee (a = 0 \wedge v \neq 0 \wedge t = (-s0 / v))$ )

**using**  $\text{quadratic-eq-zeroes-iff}[OF\ \text{refl}\ \text{refl}\ \text{refl},\ \text{of } a / 2\ t\ v\ s0]$

**by**  $(\text{auto}\ \text{simp}: \text{movement.p-def } D\ \text{power2-eq-square}\ \text{field-split-simps})$

**lemma**  $p$ -zero[ $\text{simp}$ ]:  $p\ 0 = s0$

**by**  $(\text{simp}\ \text{add}: p\text{-def})$

**lemma**  $p$ -continuous[ $\text{continuous-intros}$ ]:  $\text{continuous-on } T\ p$

**unfolding**  $p$ -def **by**  $(\text{auto}\ \text{intro}!: \text{continuous-intros})$

**lemma**  $\text{isCont-p}$ [ $\text{continuous-intros}$ ]:  $\text{isCont } p\ x$

**using**  $p$ -continuous[ $\text{of UNIV}$ ]

**by**  $(\text{auto}\ \text{simp}: \text{continuous-on-eq-continuous-at})$

**definition**  $p' :: \text{real} \Rightarrow \text{real}$  **where**

$$p'\ t = v + a * t$$

**lemma**  $p'$ -zero:  $p'\ 0 = v$

**by**  $(\text{simp}\ \text{add}: p'\text{-def})$

**lemma**  $p$ -has-vector-derivative[ $\text{derivative-intros}$ ]:  $(p\ \text{has-vector-derivative } p'\ t)$  (at  $t$  within  $s$ )

**by**  $(\text{auto}\ \text{simp}: p\text{-def}[abs-def]\ p'\text{-def}\ \text{has-vector-derivative-def}\ \text{algebra-simps}\ \text{intro}!: \text{derivative-eq-intros})$

**lemma**  $p$ -has-real-derivative[ $\text{derivative-intros}$ ]:  $(p\ \text{has-real-derivative } p'\ t)$  (at  $t$  within  $s$ )

**using**  $p$ -has-vector-derivative

**by**  $(\text{simp}\ \text{add}: \text{has-field-derivative-iff-has-vector-derivative})$

**definition**  $p'' :: \text{real} \Rightarrow \text{real}$  **where**

$$p''\ t = a$$

**lemma**  $p'$ -has-vector-derivative[ $\text{derivative-intros}$ ]:  $(p'\ \text{has-vector-derivative } p''\ t)$  (at  $t$  within  $s$ )

**by**  $(\text{auto}\ \text{simp}: p'\text{-def}[abs-def]\ p''\text{-def}\ \text{has-vector-derivative-def}\ \text{algebra-simps}\ \text{intro}!: \text{derivative-eq-intros})$

**lemma**  $p'$ -has-real-derivative[ $\text{derivative-intros}$ ]:  $(p'\ \text{has-real-derivative } p''\ t)$  (at  $t$  within  $s$ )

**using**  $p'$ -has-vector-derivative

**by** (*simp add: has-field-derivative-iff-has-vector-derivative*)

**definition** *t-stop* :: *real* **where**

*t-stop* = - *v* / *a*

**lemma** *p'-stop-zero*: *p' t-stop* = (if *a* = 0 then *v* else 0) **by** (*auto simp: p'-def t-stop-def*)

**lemma** *p'-pos-iff*:  $p' x > 0 \iff (if\ a > 0\ then\ x > -v / a\ else\ if\ a < 0\ then\ x < -v / a\ else\ v > 0)$

**by** (*auto simp: p'-def field-split-simps*)

**lemma** *le-t-stop-iff*:  $a \neq 0 \implies x \leq t-stop \iff (if\ a < 0\ then\ p' x \geq 0\ else\ p' x \leq 0)$

**by** (*auto simp: p'-def field-split-simps t-stop-def*)

**lemma** *p'-continuous*[*continuous-intros*]: *continuous-on* *T p'*

**unfolding** *p'-def* **by** (*auto intro: continuous-intros*)

**lemma** *isCont-p'*[*continuous-intros*]: *isCont* *p' x*

**using** *p'-continuous*[*of UNIV*] **by** (*auto simp: continuous-on-eq-continuous-at*)

**definition** *p-max* :: *real* **where**

*p-max* = *p t-stop*

**lemmas** *p-t-stop* = *p-max-def*[*symmetric*]

**lemma** *p-max-eq*:  $p-max = s0 - v^2 / a / 2$

**by** (*auto simp: p-max-def p-def t-stop-def field-split-simps power2-eq-square*)

### 1.3.2 Hybrid Dynamics

**definition** *s* :: *real*  $\Rightarrow$  *real* **where**

*s t* = ( if *t*  $\leq$  0 then *s0*  
else if *t*  $\leq$  *t-stop* then *p t*  
else *p-max*)

**definition** *q* :: *real*  $\Rightarrow$  *real* **where**

*q t* = *s0* + *v* \* *t*

**definition** *q'* :: *real*  $\Rightarrow$  *real* **where**

*q' t* = *v*

**lemma** *init-q*: *q 0* = *s0* **unfolding** *q-def* **by** *auto*

**lemma** *q-continuous*[*continuous-intros*]: *continuous-on* *T q*

**unfolding** *q-def* **by** (*auto intro: continuous-intros*)

**lemma** *isCont-q*[*continuous-intros*]: *isCont* *q x*

```

using q-continuous[of UNIV]
by (auto simp: continuous-on-eq-continuous-at)

lemma q-has-vector-derivative[derivative-intros]: (q has-vector-derivative q' t) (at t within u)
by (auto simp: q-def[abs-def] q'-def has-vector-derivative-def algebra-simps intro!: derivative-eq-intros)

lemma q-has-real-derivative[derivative-intros]: (q has-real-derivative q' t) (at t within u)
using q-has-vector-derivative
by (simp add: has-field-derivative-iff-has-vector-derivative)

lemma s-cond-def:
   $t \leq 0 \implies s\ t = s\ 0$ 
   $0 \leq t \implies t \leq t\text{-stop} \implies s\ t = p\ t$ 
by (simp-all add: s-def)

end

locale braking-movement = movement +
  assumes decel:  $a < 0$ 
  assumes nonneg-vel:  $v \geq 0$ 
begin

lemma t-stop-nonneg:  $0 \leq t\text{-stop}$ 
using decel nonneg-vel
by (auto simp: t-stop-def divide-simps)

lemma t-stop-pos:
assumes  $v \neq 0$ 
shows  $0 < t\text{-stop}$ 
using decel nonneg-vel assms
by (auto simp: t-stop-def divide-simps)

lemma t-stop-zero:
assumes  $t\text{-stop} = 0$ 
shows  $v = 0$ 
using assms decel
by (auto simp: t-stop-def)

lemma t-stop-zero-not-moving:  $t\text{-stop} = 0 \implies q\ t = s\ 0$ 
unfolding q-def using t-stop-zero by auto

abbreviation s-stop  $\equiv s\ t\text{-stop}$ 

lemma s-t-stop:  $s\text{-stop} = p\text{-max}$ 
using t-stop-nonneg
by (auto simp: s-def t-stop-def p-max-def p-def)

```



```

lemma s0-le-s-stop: s0 ≤ s-stop
proof (rule subst[where t=s-stop and s=p-max])
  show p-max = s-stop by (rule sym[OF s-t-stop])
next
show s0 ≤ p-max
proof (rule subst[where t=p-max and s=s0 - v2 / a / 2])
  show s0 - v2 / a / 2 = p-max using p-max-eq by auto
next
have 0 ≤ - v2 / a / 2 using decel zero-le-square[of v]
proof -
  have f1: a ≤ 0
    using ⟨a < 0⟩ by linarith
  have (- 1 * v2 ≤ 0) = (0 ≤ v2)
    by auto
  then have 0 ≤ - 1 * v2 / a
    using f1 by (meson zero-le-divide-iff zero-le-power2)
  then show ?thesis
    by force
qed
thus s0 ≤ s0 - v2 / a / 2 by auto
qed
qed

```

```

lemma p-mono: x ≤ y ⇒ y ≤ t-stop ⇒ p x ≤ p y
using decel
proof -
  assume x ≤ y and y ≤ t-stop and a < 0
  hence x + y ≤ - 2 * v / a
    unfolding t-stop-def by auto
  hence -1 / 2 * a * (x + y) ≤ v (is ?lhs0 ≤ ?rhs0)
    using ⟨a < 0⟩ by (auto simp add: field-simps)
  hence ?lhs0 * (x - y) ≥ ?rhs0 * (x - y)
    using ⟨x ≤ y⟩ by sos
  hence v * x + 1 / 2 * a * x2 ≤ v * y + 1 / 2 * a * y2
    by (auto simp add: field-simps power-def)
  thus p x ≤ p y
    unfolding p-max-def p-def t-stop-def by auto
qed

```

```

lemma p-antimono: x ≤ y ⇒ t-stop ≤ x ⇒ p y ≤ p x
using decel
proof -
  assume x ≤ y and t-stop ≤ x and a < 0
  hence - 2 * v / a ≤ x + y
    unfolding t-stop-def by auto
  hence v ≤ - 1 / 2 * a * (x + y)
    using ⟨a < 0⟩ by (auto simp add: field-simps)
  hence v * (x - y) ≥ -1 / 2 * a * (x2 - y2)

```

**using**  $\langle x \leq y \rangle$  **by** *sos*  
**hence**  $v * y + 1/2 * a * y^2 \leq v * x + 1/2 * a * x^2$   
**by** (*auto simp add: field-simps*)  
**thus**  $p y \leq p x$   
**unfolding** *p-max-def p-def t-stop-def* **by** *auto*  
**qed**

**lemma** *p-strict-mono*:  $x < y \implies y \leq t\text{-stop} \implies p x < p y$   
**using** *decel*

**proof** –  
**assume**  $x < y$  **and**  $y \leq t\text{-stop}$  **and**  $a < 0$   
**hence**  $x + y < -2 * v / a$   
**unfolding** *t-stop-def* **by** *auto*  
**hence**  $-1 / 2 * a * (x + y) < v$  (**is**  $?lhs0 < ?rhs0$ )  
**using**  $\langle a < 0 \rangle$  **by** (*auto simp add: field-simps*)  
**hence**  $?lhs0 * (x - y) > ?rhs0 * (x - y)$   
**using**  $\langle x < y \rangle$  **by** *sos*  
**hence**  $v * x + 1 / 2 * a * x^2 < v * y + 1 / 2 * a * y^2$   
**by** (*auto simp add: field-simps power-def*)  
**thus**  $p x < p y$   
**unfolding** *p-max-def p-def t-stop-def* **by** *auto*  
**qed**

**lemma** *p-strict-antimono*:  $x < y \implies t\text{-stop} \leq x \implies p y < p x$   
**using** *decel*

**proof** –  
**assume**  $x < y$  **and**  $t\text{-stop} \leq x$  **and**  $a < 0$   
**hence**  $-2 * v / a < x + y$   
**unfolding** *t-stop-def* **by** *auto*  
**hence**  $v < -1 / 2 * a * (x + y)$   
**using**  $\langle a < 0 \rangle$  **by** (*auto simp add: field-simps*)  
**hence**  $v * (x - y) > -1/2 * a * (x^2 - y^2)$   
**using**  $\langle x < y \rangle$  **by** *sos*  
**hence**  $v * y + 1/2 * a * y^2 < v * x + 1/2 * a * x^2$   
**by** (*auto simp add: field-simps*)  
**thus**  $p y < p x$   
**unfolding** *p-max-def p-def t-stop-def* **by** *auto*  
**qed**

**lemma** *p-max*:  $p x \leq p\text{-max}$   
**unfolding** *p-max-def*  
**by** (*cases x ≤ t-stop*) (*auto intro: p-mono p-antimono*)

**lemma** *continuous-on-s[continuous-intros]*: *continuous-on T s*  
**unfolding** *s-def[abs-def]*  
**using** *t-stop-nonneg*  
**by** (*intro continuous-on-subset[where t=T and s = {.. 0} ∪ ({0 .. t-stop} ∪ {t-stop ..})]*) *continuous-on-If*  
(*auto simp: p-continuous p-max-def antisym-conv[where x=0]*)

**lemma** *isCont-s[continuous-intros]*: *isCont s x*  
**using** *continuous-on-s[of UNIV]*  
**by** (*auto simp: continuous-on-eq-continuous-at*)

**definition**  $s' :: \text{real} \Rightarrow \text{real}$  **where**  
 $s' t = (\text{if } t \leq t\text{-stop} \text{ then } p' t \text{ else } 0)$

**lemma** *s-has-real-derivative*:

**assumes**  $t \geq 0 \vee a \leq 0 \wedge a \neq 0$   
**shows** (*s has-real-derivative s' t*) (*at t within {0..}*)

**proof** –

**from** *assms* **have**  $*: t \leq t\text{-stop} \iff t \in \{0 .. t\text{-stop}\}$  **by** *simp*  
**from** *assms* **have**  $0 \leq t\text{-stop}$  **by** (*auto simp: t-stop-def*)  
**have** ( $(\lambda t. \text{if } t \in \{0 .. t\text{-stop}\} \text{ then } p' t \text{ else } p\text{-max})$  *has-real-derivative*  
(*if t ∈ {0..t-stop} then p' t else 0*)) (*at t within {0..}*)  
**unfolding** *s-def[abs-def]* *s'-def*  
*has-field-derivative-iff-has-vector-derivative*  
**apply** (*rule has-vector-derivative-If-within-closures[where T = {t-stop ..}]*)  
**using**  $\langle 0 \leq t\text{-stop} \rangle \langle a \neq 0 \rangle$   
**by** (*auto simp: assms p'-stop-zero p-t-stop max-def insert-absorb*  
*intro!: p-has-vector-derivative*)  
**from** - - *this* **show** ?*thesis*  
**unfolding** *has-vector-derivative-def has-field-derivative-iff-has-vector-derivative*  
*s'-def s-def[abs-def]* \*  
**by** (*rule has-derivative-transform*)  
(*auto simp: assms s-def p-max-def t-stop-def*)

**qed**

**lemma** *s-has-vector-derivative[derivative-intros]*:

**assumes**  $t \geq 0 \vee a \leq 0 \wedge a \neq 0$   
**shows** (*s has-vector-derivative s' t*) (*at t within {0..}*)  
**using** *s-has-real-derivative[OF assms]*  
**by** (*simp add: has-field-derivative-iff-has-vector-derivative*)

**lemma** *s-has-field-derivative[derivative-intros]*:

**assumes**  $t \geq 0 \vee a \leq 0 \wedge a \neq 0$   
**shows** (*s has-field-derivative s' t*) (*at t within {0..}*)  
**using** *s-has-vector-derivative[OF assms]*  
**by** (*simp add: has-field-derivative-iff-has-vector-derivative*)

**lemma** *s-has-real-derivative-at*:

**assumes**  $0 < x \leq v \wedge a < 0$   
**shows** (*s has-real-derivative s' x*) (*at x*)

**proof** –

**from** *assms* **have** (*s has-real-derivative s' x*) (*at x within {0 ..}*)  
**by** (*intro s-has-real-derivative*) (*auto intro!: divide-nonneg-nonpos*)  
**then** **have** (*s has-real-derivative s' x*) (*at x within {0<..}*)  
**by** (*rule DERIV-subset*) *auto*

**then show**  $(s \text{ has-real-derivative } s' x) \text{ (at } x)$  **using** *assms*  
**by**  $(\text{subst } (asm) \text{ at-within-open}) \text{ auto}$   
**qed**

**lemma** *s-delayed-has-field-derivative*[*derivative-intros*]:  
**assumes**  $\delta < t \ 0 \leq v \ a < 0$   
**shows**  $((\lambda x. s (x - \delta)) \text{ has-field-derivative } s' (t - \delta)) \text{ (at } t \text{ within } \{\delta < ..\})$   
**proof** –  
**from** *assms* **have**  $((\lambda x. s (x + - \delta)) \text{ has-real-derivative } s' (t - \delta)) \text{ (at } t)$   
**using** *DERIV-shift*[*of s (s' (t - \delta)) t - \delta*] *s-has-real-derivative-at*  
**by** *auto*  
  
**thus**  $((\lambda x. s (x - \delta)) \text{ has-field-derivative } s' (t - \delta)) \text{ (at } t \text{ within } \{\delta < ..\})$   
**using** *has-field-derivative-at-within* **by** *auto*  
**qed**

**lemma** *s-delayed-has-vector-derivative*[*derivative-intros*]:  
**assumes**  $\delta < t \ 0 \leq v \ a < 0$   
**shows**  $((\lambda x. s (x - \delta)) \text{ has-vector-derivative } s' (t - \delta)) \text{ (at } t \text{ within } \{\delta < ..\})$   
**using** *s-delayed-has-field-derivative*[*OF assms*]  
**by**  $(\text{simp add:has-field-derivative-iff-has-vector-derivative})$

**lemma** *s'-nonneg*:  $0 \leq v \implies a \leq 0 \implies 0 \leq s' x$   
**by**  $(\text{auto simp: } s'\text{-def } p'\text{-def } t\text{-stop-def } \text{field-split-simps})$

**lemma** *s'-pos*:  $0 \leq x \implies x < t\text{-stop} \implies 0 \leq v \implies a \leq 0 \implies 0 < s' x$   
**by**  $(\text{intro } \text{le-neq-trans } s'\text{-nonneg})$   
 $(\text{auto simp: } s'\text{-def } p'\text{-def } t\text{-stop-def } \text{field-split-simps})$

### 1.3.3 Monotonicity of Movement

**lemma** *s-mono*:  
**assumes**  $t \geq u \ u \geq 0$   
**shows**  $s t \geq s u$   
**using** *p-mono*[*of u t*] *assms* *p-max*[*of u*] **by**  $(\text{auto simp: } s\text{-def})$

**lemma** *s-strict-mono*:  
**assumes**  $u < t \ t \leq t\text{-stop} \ u \geq 0$   
**shows**  $s u < s t$   
**using** *p-strict-mono*[*of u t*] *assms* *p-max*[*of u*] **by**  $(\text{auto simp: } s\text{-def})$

**lemma** *s-antimono*:  
**assumes**  $x \leq y$   
**assumes**  $t\text{-stop} \leq x$   
**shows**  $s y \leq s x$   
**proof** –  
**from** *assms* **have**  $t\text{-stop} \leq y$  **by** *auto*  
**hence**  $s y \leq p\text{-max}$  **unfolding** *s-def* *p-max-eq*  
**using** *p-max-def* *p-max-eq* *s0-le-s-stop* *s-t-stop* **by** *auto*

also have  $\dots \leq s x$   
 using  $\langle t\text{-stop} \leq x \rangle$  *s-mono s-t-stop t-stop-nonneg* by *fastforce*  
 ultimately show  $s y \leq s x$  by *auto*  
 qed

lemma *init-s* :  $t \leq 0 \implies s t = s 0$   
 using *decel nonneg-vel* by (*simp add: divide-simps s-def*)

lemma *q-min*:  $0 \leq t \implies s 0 \leq q t$   
 unfolding *q-def* using *nonneg-vel* by *auto*

lemma *q-mono*:  $x \leq y \implies q x \leq q y$   
 unfolding *q-def* using *nonneg-vel* by (*auto simp: mult-left-mono*)

### 1.3.4 Maximum at Stopping Time

lemma *s-max*:  $s x \leq s\text{-stop}$   
 using *p-max[of x] p-max[of 0]* unfolding *s-t-stop* by (*auto simp: s-def*)

lemma *s-eq-s-stop*: *NO-MATCH*  $t\text{-stop } x \implies x \geq t\text{-stop} \implies s x = s\text{-stop}$   
 using *t-stop-nonneg* by (*auto simp: s-def p-max-def*)

end

## 1.4 Safe Distance

locale *safe-distance* =  
 fixes  $a_e v_e s_e :: \text{real}$   
 fixes  $a_o v_o s_o :: \text{real}$   
 assumes *nonneg-vel-ego* :  $0 \leq v_e$   
 assumes *nonneg-vel-other* :  $0 \leq v_o$   
 assumes *decelerate-ego* :  $a_e < 0$   
 assumes *decelerate-other* :  $a_o < 0$   
 assumes *in-front* :  $s_e < s_o$   
 begin

lemmas *hyps* =  
*nonneg-vel-ego*  
*nonneg-vel-other*  
*decelerate-ego*  
*decelerate-other*  
*in-front*

sublocale *ego*: *braking-movement*  $a_e v_e s_e$  by (*unfold-locales; rule hyps*)  
 sublocale *other*: *braking-movement*  $a_o v_o s_o$  by (*unfold-locales; rule hyps*)  
 sublocale *ego-other*: *movement*  $a_o - a_e v_o - v_e s_o - s_e$  by *unfold-locales*

### 1.4.1 Collision

definition *collision* :: *real set*  $\Rightarrow$  *bool* where

$collision\ time\ set \equiv (\exists t \in time\ set. ego.s\ t = other.s\ t)$

**abbreviation**  $no\ collision :: real\ set \Rightarrow bool$  **where**  
 $no\ collision\ time\ set \equiv \neg collision\ time\ set$

**lemma**  $no\ collision\ initially : no\ collision \{.. 0\}$   
**using**  $decelerate\ ego\ nonneg\ vel\ ego$   
**using**  $decelerate\ other\ nonneg\ vel\ other\ in\ front$   
**by**  $(auto\ simp: divide\_simps\ collision\ def\ ego.s\ def\ other.s\ def)$

**lemma**  $no\ collisionI:$   
 $(\bigwedge t. t \in S \Longrightarrow ego.s\ t \neq other.s\ t) \Longrightarrow no\ collision\ S$   
**unfolding**  $collision\ def$  **by**  $blast$

**theorem**  $cond\ 1: ego.s\ stop < s_o \Longrightarrow no\ collision \{0..\}$   
**proof**  $(rule\ no\ collisionI, simp)$

**fix**  $t::real$   
**assume**  $t \geq 0$   
**have**  $ego.s\ t \leq ego.s\ stop$   
**by**  $(rule\ ego.s\ max)$   
**also assume**  $\dots < s_o$   
**also have**  $\dots = other.s\ 0$   
**by**  $(simp\ add: other.init\ s)$   
**also have**  $\dots \leq other.s\ t$   
**using**  $\langle 0 \leq t \rangle hyps$   
**by**  $(intro\ other.s\ mono) auto$   
**finally show**  $ego.s\ t \neq other.s\ t$   
**by**  $simp$   
**qed**

**lemma**  $ego\ other\ strict\ ivt:$   
**assumes**  $ego.s\ t > other.s\ t$   
**shows**  $collision \{0 .. < t\}$

**proof**  $cases$   
**have**  $[simp]: s_e < s_o \Longrightarrow ego.s\ 0 \leq other.s\ 0$   
**by**  $(simp\ add: movement.s\ def)$   
**assume**  $0 \leq t$   
**with**  $assms\ in\ front$   
**have**  $\exists x \geq 0. x \leq t \wedge other.s\ x - ego.s\ x = 0$   
**by**  $(intro\ IVT2, auto\ simp: continuous\ diff\ other.isCont\ s\ ego.isCont\ s)$   
**then show**  $?thesis$   
**using**  $assms$   
**by**  $(auto\ simp\ add: algebra\_simps\ collision\ def\ Bex\ def\ order.order\ iff\ strict)$   
**qed**  $(insert\ assms\ hyps, auto\ simp: collision\ def\ ego.init\ s\ other.init\ s\ intro!: bexI[where\ x=0])$

**lemma**  $collision\ subset: collision\ s \Longrightarrow s \subseteq t \Longrightarrow collision\ t$   
**by**  $(auto\ simp: collision\ def)$

**lemma** *ego-other-ivt*:  
**assumes**  $ego.s\ t \geq other.s\ t$   
**shows** *collision*  $\{0 \ ..\ t\}$   
**proof** *cases*  
**assume**  $ego.s\ t > other.s\ t$   
**from** *ego-other-strict-ivt*[*OF this*]  
**show** *?thesis*  
**by** (*rule collision-subset*) *auto*  
**qed** (*insert hyps assms; cases t ≥ 0; force simp: collision-def ego.init-s other.init-s*)

**theorem** *cond-2*:  
**assumes**  $ego.s-stop \geq other.s-stop$   
**shows** *collision*  $\{0 \ ..\}$   
**using** *assms*  
**apply** (*intro collision-subset*[**where**  $t = \{0 \ ..\}$  **and**  $s = \{0 \ ..\ max\ ego.t-stop\ other.t-stop\}$ ])  
**apply** (*intro ego-other-ivt*[**where**  $t = max\ ego.t-stop\ other.t-stop$ ])  
**apply** (*auto simp: ego.s-eq-s-stop other.s-eq-s-stop*)  
**done**

**abbreviation**  $D2 :: real\ where$   
 $D2 \equiv (v_o - v_e)^2 - 2 * (a_o - a_e) * (s_o - s_e)$

**abbreviation**  $t_D' :: real\ where$   
 $t_D' \equiv sqrt\ (2 * (ego.s-stop - other.s-stop) / a_o)$

**lemma** *pos-via-half-dist*:  
 $dist\ a\ b < b / 2 \implies b > 0 \implies a > 0$   
**by** (*auto simp: dist-real-def abs-real-def split: if-splits*)

**lemma** *collision-within-p*:  
**assumes**  $s_o \leq ego.s-stop\ ego.s-stop < other.s-stop$   
**shows** *collision*  $\{0..\}$   $\longleftrightarrow (\exists t \geq 0. ego.p\ t = other.p\ t \wedge t < ego.t-stop \wedge t < other.t-stop)$   
**proof** (*auto simp: collision-def, goal-cases*)  
**case** (2  $t$ )  
**then show** *?case*  
**by** (*intro bexI*[**where**  $x = t$ ]) (*auto simp: ego.s-def other.s-def*)  
**next**  
**case** (1  $t$ )  
**then show** *?case* **using** *assms hyps ego.t-stop-nonneg other.t-stop-nonneg*  
**apply** (*auto simp: ego.s-def other.s-def ego.s-t-stop other.s-t-stop ego.p-t-stop other.p-t-stop not-le split: if-splits*)  
**defer**  
**proof** *goal-cases*  
**case** 1  
**from** 1 **have**  $le: ego.t-stop \leq other.t-stop$  **by** *auto*  
**from** 1 **have**  $ego.t-stop < t$  **by** *simp*

```

from other.s-strict-mono[OF this] 1
have other.s ego.t-stop < other.s t
  by auto
also have  $\dots = \text{ego.s ego.t-stop}$ 
  using ego.s-t-stop ego.t-stop-nonneg 1 other.s-def by auto
finally have other.s ego.t-stop < ego.s ego.t-stop .
from ego-other-strict-ivt[OF this] le in-front
show ?case
  by (auto simp add: collision-def) (auto simp: movement.s-def split: if-splits)
next
case 2
from 2 have other.p-max = ego.p t by simp
also have  $\dots \leq \text{ego.p ego.t-stop}$ 
  using 2
  by (intro ego.p-mono) auto
also have  $\dots = \text{ego.p-max}$ 
  by (simp add: ego.p-t-stop)
also note  $\langle \dots < \text{other.p-max} \rangle$ 
finally show ?case by arith
next
case 3
thus  $\exists t \geq 0. \text{ego.p } t = \text{other.p } t \wedge t < \text{ego.t-stop} \wedge t < \text{other.t-stop}$ 
  apply (cases t = other.t-stop)
  apply (simp add: other.p-t-stop)
  apply (metis (no-types) ego.p-max not-le)
  apply (cases t = ego.t-stop)
  apply (simp add: ego.p-t-stop)
  defer
  apply force
proof goal-cases
case (1)
let ?d =  $\lambda t. \text{other.p}' t - \text{ego.p}' t$ 
define d' where d' = ?d ego.t-stop / 2
have d-cont: isCont ?d ego.t-stop
  unfolding ego.t-stop-def other.p'-def ego.p'-def by simp
have ?d ego.t-stop > 0
  using 1
  by (simp add: ego.p'-stop-zero other.p'-pos-iff) (simp add: ego.t-stop-def
other.t-stop-def)
then have d' > 0 by (auto simp: d'-def)
from d-cont[unfolded continuous-at-eps-delta, THEN spec, rule-format, OF
 $\langle d' > 0 \rangle$ ]
obtain e where e: e > 0  $\wedge x. \text{dist } x \text{ ego.t-stop} < e \implies ?d x > 0$ 
  unfolding d'-def
  using  $\langle ?d \text{ ego.t-stop} > 0 \rangle$  pos-via-half-dist
  by force
define t' where t' = ego.t-stop - min (ego.t-stop / 2) (e / 2)
have  $0 < \text{ego.t-stop}$  using 1 by auto
have other.p t' - ego.p t' < other.p ego.t-stop - ego.p ego.t-stop

```



```

apply (rule DERIV-pos-imp-increasing[of  $t'$ ])
  apply (force simp: t'-def e min-def  $\langle 0 < \text{ego.t-stop} \rangle$ )
apply (auto intro!: exI[where  $x = ?d x$  for  $x$ ] intro!: derivative-intros e)
using  $\langle e > 0 \rangle$ 
apply (auto simp: t'-def dist-real-def algebra-simps)
done
also have  $\dots = 0$  using 1 by (simp add: ego.p-t-stop)
finally have less: other.p t' < ego.p t' by simp
have  $t' > 0$ 
  using 1 by (auto simp: t'-def algebra-simps min-def)
have  $t' < \text{ego.t-stop}$  by (auto simp: t'-def  $\langle e > 0 \rangle$   $\langle \text{ego.t-stop} > 0 \rangle$ )
from less-le-trans[OF  $\langle t' < \text{ego.t-stop} \rangle$   $\langle \text{ego.t-stop} \leq \text{other.t-stop} \rangle$ ]
have  $t' < \text{other.t-stop}$  .
from ego-other-strict-ivt[of  $t'$ ] less
have collision  $\{0..<t'\}$ 
  using  $\langle t' > 0 \rangle$   $\langle t' < \text{ego.t-stop} \rangle$   $\langle t' < \text{other.t-stop} \rangle$ 
  by (auto simp: other.s-def ego.s-def split: if-splits)
thus ?case
  using  $\langle t' > 0 \rangle$   $\langle t' < \text{ego.t-stop} \rangle$   $\langle t' < \text{other.t-stop} \rangle$ 
  apply (auto simp: collision-def ego.s-def other.s-def movement.p-def
    split: if-splits)
  apply (rule-tac  $x = t$  in exI)
  apply (auto simp: movement.p-def)[]
done
qed
qed
qed

```

**lemma** *collision-within-eq*:

```

assumes  $s_o \leq \text{ego.s-stop}$   $\text{ego.s-stop} < \text{other.s-stop}$ 
shows collision  $\{0..\}$   $\longleftrightarrow$  collision  $\{0 ..< \min \text{ego.t-stop other.t-stop}\}$ 
unfolding collision-within-p[OF assms]
unfolding collision-def
by (safe; force
  simp: ego.s-def other.s-def movement.p-def ego.t-stop-def other.t-stop-def
  split: if-splits)

```

**lemma** *collision-excluded*:  $(\bigwedge t. t \in T \implies \text{ego.s } t \neq \text{other.s } t) \implies \text{collision } S \longleftrightarrow \text{collision } (S - T)$

```

by (auto simp: collision-def)

```

**lemma** *collision-within-less*:

```

assumes  $s_o \leq \text{ego.s-stop}$   $\text{ego.s-stop} < \text{other.s-stop}$ 
shows collision  $\{0..\}$   $\longleftrightarrow$  collision  $\{0 <..< \min \text{ego.t-stop other.t-stop}\}$ 
proof –
  note collision-within-eq[OF assms]
  also have collision  $\{0 ..< \min \text{ego.t-stop other.t-stop}\} \longleftrightarrow$ 
    collision  $(\{0 ..< \min \text{ego.t-stop other.t-stop}\} - \{0\})$ 
  using hyps assms

```

by (intro collision-excluded) (auto simp: ego.s-def other.s-def)  
 also have  $\{0 \dots < \min \text{ego.t-stop other.t-stop}\} - \{0\} = \{0 < \dots < \min \text{ego.t-stop other.t-stop}\}$   
 by auto  
 finally show ?thesis  
 unfolding collision-def  
 by (safe;  
 force  
 simp: ego.s-def other.s-def movement.p-def ego.t-stop-def other.t-stop-def  
 split: if-splits)  
 qed

**theorem cond-3:**

assumes  $s_o \leq \text{ego.s-stop} \text{ego.s-stop} < \text{other.s-stop}$   
 shows collision  $\{0..\} \longleftrightarrow (a_o > a_e \wedge v_o < v_e \wedge 0 \leq D2 \wedge \text{sqrt } D2 > v_e - a_e / a_o * v_o)$

**proof** –

have  $v_o \neq 0$   
 using assms(1) assms(2) movement.s-def movement.t-stop-def by auto  
 with hyps have  $v_o > 0$  by auto  
 note hyps = hyps this  
 define t1 where  $t1 = (- (v_o - v_e) + \text{sqrt } D2) / (a_o - a_e)$   
 define t2 where  $t2 = (- (v_o - v_e) - \text{sqrt } D2) / (a_o - a_e)$   
 define bounded where bounded  $\equiv \lambda t. (0 \leq t \wedge t \leq \text{ego.t-stop} \wedge t \leq \text{other.t-stop})$   
 have ego-other-conv:  
 $\bigwedge t. \text{bounded } t \implies \text{ego.p } t = \text{other.p } t \longleftrightarrow \text{ego-other.p } t = 0$   
 by (auto simp: movement.p-def field-split-simps)  
 let ?r =  $\{0 < \dots < \min \text{ego.t-stop other.t-stop}\}$   
 have D2:  $D2 = (v_o - v_e)^2 - 4 * ((a_o - a_e) / 2) * (s_o - s_e)$  by simp  
 define D where  $D = D2$   
 note D = D-def[symmetric]  
 define x1 where  $x1 \equiv (- (v_o - v_e) + \text{sqrt } D2) / (2 * ((a_o - a_e) / 2))$   
 define x2 where  $x2 \equiv (- (v_o - v_e) - \text{sqrt } D2) / (2 * ((a_o - a_e) / 2))$   
 have x2:  $x2 = (- (v_o - v_e) - \text{sqrt } D2) / (a_o - a_e)$   
 by (simp add: x2-def field-split-simps)  
 have x1:  $x1 = (- (v_o - v_e) + \text{sqrt } D2) / (a_o - a_e)$   
 by (simp add: x1-def field-split-simps)  
 from collision-within-less[OF assms]  
 have coll-eq: collision  $\{0..\} = \text{collision } ?r$   
 by (auto simp add: bounded-def)  
 also have  $\dots \longleftrightarrow (a_o > a_e \wedge v_o < v_e \wedge 0 \leq D2 \wedge \text{sqrt } D2 > v_e - a_e / a_o * v_o)$

**proof safe**

assume H:  $a_e < a_o \ v_o < v_e \ 0 \leq D2$   
 assume sqrt:  $\text{sqrt } D2 > v_e - a_e / a_o * v_o$   
 have nz:  $(a_o - a_e) / 2 \neq 0$  using  $\langle a_e < a_o \rangle$  by simp  
 note sol = quadratic-eq-zeroes-iff[OF D2 x1-def[THEN meta-eq-to-obj-eq] x2-def[THEN meta-eq-to-obj-eq] nz]  
 from sol[of x2]  $\langle 0 \leq D2 \rangle$

```

have other.p x2 = ego.p x2
  by (auto simp: ego.p-def other.p-def field-split-simps)
moreover
have x2 > 0
proof (rule ccontr)
  assume ¬ 0 < x2
  then have ego-other.p x2 ≥ ego-other.p 0
    using H hyps
  by (intro DERIV-nonpos-imp-nonincreasing[of x2])
    (auto intro!: exI[where x=ego-other.p' x for x] derivative-eq-intros
      simp: ego-other.p'-def add-nonpos-nonpos mult-nonneg-nonpos)
  also have ego-other.p 0 > 0 using hyps by (simp add: ego-other.p-def)
  finally (xtrans) show False using ⟨other.p x2 = ego.p x2⟩
    by (simp add: movement.p-def field-split-simps power2-eq-square)
qed
moreover
have x2 < other.t-stop
  using sqrt H hyps
  by (auto simp: x2 other.t-stop-def field-split-simps power2-eq-square)

ultimately
show collision {0 < .. < min ego.t-stop other.t-stop}
proof (cases x2 < ego.t-stop, goal-cases)
  case 2
  then have other.s x2 = other.p x2
    by (auto simp: other.s-def)
  also from 2 have ... ≤ ego.p ego.t-stop
    by (auto intro!: ego.p-antimono)
  also have ... = ego.s x2
    using 2 by (auto simp: ego.s-def ego.p-t-stop)
  finally have other.s x2 ≤ ego.s x2 .
  from ego-other-ivt[OF this]
  show ?thesis
    unfolding coll-eq[symmetric]
    by (rule collision-subset) auto
qed (auto simp: collision-def ego.s-def other.s-def not-le intro!: bexI[where
x=x2])
next
let ?max = max ego.t-stop other.t-stop
let ?min = min ego.t-stop other.t-stop
assume collision ?r
then obtain t where t: ego.p t = other.p t 0 < t t < ?min
  by (auto simp: collision-def ego.s-def other.s-def)
then have t < - (ve / ae) t < - (vo / ao) t < other.t-stop
  by (simp-all add: ego.t-stop-def other.t-stop-def)
from t have ego-other.p t = 0
  by (auto simp: movement.p-def field-split-simps)
from t have t < ?max by auto
from hyps assms have 0 < ego-other.p 0

```

```

    by simp
  from ego-other.p-def[abs-def, THEN meta-eq-to-obj-eq]
  have eop-eq: ego-other.p = ( $\lambda t. 1 / 2 * (a_o - a_e) * t^2 + (v_o - v_e) * t + (s_o - s_e)$ )
  by (simp add: algebra-simps)
  show  $a_o > a_e$ 
  proof -
    have ego.p other.t-stop  $\leq$  ego.p-max
    by (rule ego.p-max)
    also have ...  $\leq$  other.p other.t-stop using hyps assms
    by (auto simp: other.s-def ego.s-def ego.p-t-stop split:if-splits)
    finally have  $0 \leq$  ego-other.p other.t-stop
    by (auto simp add: movement.p-def field-simps)
    from p-convex[OF eop-eq, of  $0 t$  other.t-stop, simplified  $\langle$ ego-other.p  $t = 0$  $\rangle$ ,
      OF  $\langle 0 < t \rangle \langle t < \text{other.t-stop} \rangle \langle 0 < \text{ego-other.p } 0 \rangle \langle 0 \leq \text{ego-other.p other.t-stop} \rangle$ ]
    show  $a_o > a_e$  by (simp add: algebra-simps)
  qed
  have rewr:  $4 * ((a_o - a_e) / 2) = 2 * (a_o - a_e)$  by simp
  from  $\langle a_o > a_e \rangle \langle$ ego-other.p  $t = 0 \rangle$  ego-other.p-all-zeroes[OF D2[symmetric],
of  $t$ ]
  have  $0 \leq D2$  and disj:  $(t = (- (v_o - v_e) + \text{sqrt } D2) / (a_o - a_e) \vee t = (- (v_o - v_e) - \text{sqrt } D2) / (a_o - a_e))$ 
  using hyps assms
  unfolding rewr by simp-all
  show  $0 \leq D2$  by fact
  from add-strict-mono[OF  $\langle t < - (v_e / a_e) \rangle \langle t < - (v_o / a_o) \rangle \langle 0 < t \rangle \langle a_o > a_e \rangle$ ]
  have  $0 < - (v_e / a_e) + - (v_o / a_o)$  by (simp add: divide-simps)
  then have  $0 > v_e * a_o + a_e * v_o$  using hyps
  by (simp add: field-split-simps split: if-splits)
  show  $v_o < v_e$ 
  using  $\langle a_e < a_o \rangle \langle$ movement.p  $(a_o - a_e) (v_o - v_e) (s_o - s_e) t = 0 \rangle$  in-front
  t(2)
  apply (auto simp: movement.p-def divide-less-0-iff algebra-simps power2-eq-square)
  by (smt divide-less-0-iff mult-le-cancel-right mult-mono mult-nonneg-nonneg
nonneg-vel-ego)
  from disj have  $x2 < ?min$ 
  proof rule
    assume  $t = (- (v_o - v_e) - \text{sqrt } D2) / (a_o - a_e)$ 
    then show ?thesis
    using  $\langle t < ?min \rangle$ 
    by (simp add: x2)
  next
  assume  $t = (- (v_o - v_e) + \text{sqrt } D2) / (a_o - a_e)$ 
  also have ...  $\geq x2$ 
  unfolding x2
  apply (rule divide-right-mono)
  apply (subst (2) diff-conv-add-uminus)

```

```

    apply (rule add-left-mono)
    using ⟨ao > ae⟩ ⟨D2 ≥ 0⟩
    by auto
    also (xtrans) note ⟨t < ?min⟩
    finally show ?thesis .
qed
then show sqrt D2 > ve - ae / ao * vo
    using hyps ⟨ao > ae⟩
    by (auto simp: x2 field-split-simps other.t-stop-def)
qed
finally show ?thesis .
qed

```

### 1.4.2 Formalising Safe Distance

First definition for Safe Distance based on *cond-1*.

**definition** *absolute-safe-distance* :: real **where**  
*absolute-safe-distance* = - v<sub>e</sub><sup>2</sup> / (2 \* a<sub>e</sub>)

**lemma** *absolute-safe-distance*:  
**assumes** s<sub>o</sub> - s<sub>e</sub> > *absolute-safe-distance*  
**shows** *no-collision* {0..}  
**proof** -  
**from** *assms hyps absolute-safe-distance-def* **have** *ego.s-stop* < s<sub>o</sub>  
**by** (auto simp *add:ego.s-def ego.p-def ego.t-stop-def power-def*)  
**thus** ?thesis **by** (rule *cond-1*)  
**qed**

First Fallback for Safe Distance.

**definition** *fst-safe-distance* :: real **where**  
*fst-safe-distance* = v<sub>o</sub><sup>2</sup> / (2 \* a<sub>o</sub>) - v<sub>e</sub><sup>2</sup> / (2 \* a<sub>e</sub>)

**definition** *distance-leq-d2* :: real **where**  
*distance-leq-d2* = (a<sub>e</sub> + a<sub>o</sub>) / (2 \* a<sub>o</sub><sup>2</sup>) \* v<sub>o</sub><sup>2</sup> - v<sub>o</sub> \* v<sub>e</sub> / a<sub>o</sub>

**lemma** *snd-leq-fst-exp*: *distance-leq-d2* ≤ *fst-safe-distance*

**proof** -  
**have** 0 ≤ (*other.t-stop* - *ego.t-stop*)<sup>2</sup> **by** *auto*  
**hence** - *ego.t-stop*<sup>2</sup> ≤ *other.t-stop*<sup>2</sup> - 2 \* *other.t-stop* \* *ego.t-stop* **by** (*simp add:power-def algebra-simps*)  
**with** *hyps*(3) **have** - *ego.t-stop*<sup>2</sup> \* (a<sub>e</sub> / 2) ≥ (*other.t-stop*<sup>2</sup> - 2 \* *other.t-stop* \* *ego.t-stop*) \* (a<sub>e</sub> / 2)  
**by** (*smt half-gt-zero-iff mult-le-cancel-right*)  
**with** *ego.t-stop-def other.t-stop-def hyps*  
**have** - v<sub>e</sub><sup>2</sup> / (2 \* a<sub>e</sub>) ≥ a<sub>e</sub> \* v<sub>o</sub><sup>2</sup> / (2 \* a<sub>o</sub><sup>2</sup>) - v<sub>o</sub> \* v<sub>e</sub> / a<sub>o</sub> **by** (*simp add:power-def algebra-simps*)  
**with** *fst-safe-distance-def distance-leq-d2-def*  
**have** 1: *fst-safe-distance* ≥ a<sub>e</sub> \* v<sub>o</sub><sup>2</sup> / (2 \* a<sub>o</sub><sup>2</sup>) - v<sub>o</sub> \* v<sub>e</sub> / a<sub>o</sub> + v<sub>o</sub><sup>2</sup> / (2 \* a<sub>o</sub>) **by** (*auto simp add:algebra-simps*)

**have**  $a_e * v_o^2 / (2 * a_o^2) - v_o * v_e / a_o + v_o^2 / (2 * a_o) = \text{distance-leq-d2}$  (is  
 $?LHS = -$ )  
**proof** –  
**have**  $?LHS = a_e * v_o^2 / (2 * a_o^2) - v_o * v_e / a_o + a_o * v_o^2 / (2 * a_o^2)$   
**by** (auto simp add: algebra-simps power-def)  
**also have** ... = distance-leq-d2  
**by** (auto simp add: power-def field-split-simps distance-leq-d2-def)  
**finally show** ?thesis by auto  
**qed**  
**with 1 show** ?thesis by auto  
**qed**

**lemma** *sqrt-D2-leq-stop-time-diff*:

**assumes**  $a_e < a_o$   
**assumes**  $0 \leq v_e - a_e / a_o * v_o$   
**assumes**  $s_o - s_e \geq \text{distance-leq-d2}$   
**shows**  $\text{sqrt } D2 \leq v_e - a_e / a_o * v_o$   
**proof** –  
**from** *assms* **have**  $-2 * (a_o - a_e) * (s_o - s_e) \leq -2 * (a_o - a_e) * \text{distance-leq-d2}$   
(is  $?L \leq ?R$ )  
**by** *simp*  
**hence**  $D2 \leq (v_o - v_e)^2 - 2 * (a_o - a_e) * \text{distance-leq-d2}$  **by** (*simp add:*  
*algebra-simps*)  
**also have** ... =  $(v_e - a_e / a_o * v_o)^2$   
**proof** –  
**from** *distance-leq-d2-def*  
**have** 1:  $(v_o - v_e)^2 - 2 * (a_o - a_e) * \text{distance-leq-d2} =$   
 $(v_o - v_e)^2 - (a_o - a_e) * (a_e + a_o) / a_o^2 * v_o^2 + 2 * (a_o - a_e) * v_o *$   
 $v_e / a_o$   
**by** (*auto simp add: field-split-simps*)  
**with** *hyps(4)* **have** ... =  $(v_e - a_e / a_o * v_o)^2$   
**by** (*auto simp add: power-def field-split-simps*)  
**with 1 show** ?thesis by auto  
**qed**  
**finally show** ?thesis **by** (*smt assms(2) real-le-lsqrt real-sqrt-le-0-iff*)  
**qed**

**lemma** *cond2-imp-pos-vo*:

**assumes**  $s_o \leq \text{ego.s-stop} \text{ ego.s-stop} < \text{other.s-stop}$   
**shows**  $v_o \neq 0$   
**proof** (*rule ccontr*)  
**assume**  $\neg v_o \neq 0$   
**with** *other.s-def other.t-stop-def* **have**  $\text{other.s-stop} = s_o$  **by** *auto*  
**with** *assms(2)* **have**  $\text{ego.s-stop} < s_o$  **by** *auto*  
**with** *assms(1)* **show** *False* **by** *auto*  
**qed**

**lemma** *cond2-imp-gt-fst-sd*:

**assumes**  $s_o \leq \text{ego.s-stop} \text{ ego.s-stop} < \text{other.s-stop}$

**shows**  $fst\text{-safe-distance} < s_o - s_e$   
**proof** (cases  $v_e \neq 0$ )  
    **case** *True*  
    **from**  $fst\text{-safe-distance-def}$  *assms*  $ego.s\text{-def}$   $ego.t\text{-stop-pos}[OF \langle v_e \neq 0 \rangle]$   $ego.p\text{-def}$   $ego.t\text{-stop-def}$   
         $other.s\text{-def}$   $other.t\text{-stop-pos}[OF \text{ cond2-imp-pos-vo}[OF \text{ assms}]]$   $other.p\text{-def}$   $other.t\text{-stop-def}$  *hyps*  
    **show** *?thesis* **by** (*simp add: power-def algebra-simps*)  
**next**  
    **case** *False*  
    **with**  $fst\text{-safe-distance-def}$  **have**  $fst\text{-safe-distance} = v_o^2 / (2 * a_o)$  **by** *auto*  
    **also have**  $\dots \leq 0$  **by** (*simp add: divide-nonneg-neg hyps*)  
    **also have**  $\dots < s_o - s_e$  **by** (*simp add: algebra-simps hyps*)  
    **finally show** *?thesis* **by** *auto*  
**qed**

Second Fallback for Safe Distance.

**definition**  $snd\text{-safe-distance} :: \text{real}$  **where**  
     $snd\text{-safe-distance} = (v_o - v_e)^2 / (2 * (a_o - a_e))$

**lemma**  $fst\text{-leq-snd-safe-distance}$ :

**assumes**  $a_e < a_o$   
**shows**  $fst\text{-safe-distance} \leq snd\text{-safe-distance}$   
**proof** –  
    **have**  $0 \leq (v_o / a_o - v_e / a_e)^2$  **by** *auto*  
    **hence**  $1: 0 \leq (v_o / a_o)^2 - 2 * v_o * v_e / (a_o * a_e) + (v_e / a_e)^2$  **by** (*auto simp add: power-def algebra-simps*)  
    **from** *hyps* **have**  $0 \leq a_o * a_e$  **by** (*simp add: mult-nonneg-nonneg*)  
    **from**  $mult\text{-right-mono}[OF 1 \text{ this}]$  *hyps*  
    **have**  $0 \leq v_o^2 * a_e / a_o - 2 * v_o * v_e + v_e^2 * a_o / a_e$  **by** (*auto simp add: power-def algebra-simps*)  
    **with** *hyps* **have**  $2: (v_o^2 / (2 * a_o) - v_e^2 / (2 * a_e)) * (2 * (a_o - a_e)) \leq (v_o - v_e)^2$   
    **by** (*auto simp add: power-def field-split-simps*)  
    **from** *assms* **have**  $0 \leq 2 * (a_o - a_e)$  **by** *auto*  
    **from**  $divide\text{-right-mono}[OF 2 \text{ this}]$  *assms*  $fst\text{-safe-distance-def}$   $snd\text{-safe-distance-def}$   
    **show** *?thesis* **by** *auto*  
**qed**

**lemma**  $snd\text{-safe-distance-iff-nonneg-D2}$ :

**assumes**  $a_e < a_o$   
**shows**  $s_o - s_e \leq snd\text{-safe-distance} \iff 0 \leq D2$   
**proof** –  
    **from**  $snd\text{-safe-distance-def}$  *assms*  $pos\text{-le-divide-eq}[of 2 * (a_o - a_e)]$   
    **have**  $s_o - s_e \leq snd\text{-safe-distance} \iff (s_o - s_e) * (2 * (a_o - a_e)) \leq (v_o - v_e)^2$   
**by** *auto*  
    **also have**  $\dots \iff 0 \leq D2$  **by** (*auto simp add: algebra-simps*)  
    **finally show** *?thesis* **by** *auto*  
**qed**

**lemma** *t-stop-diff-neg-means-leq-D2*:

**assumes**  $s_o \leq \text{ego.s-stop}$   $\text{ego.s-stop} < \text{other.s-stop}$   $a_e < a_o$   $0 \leq D2$

**shows**  $v_e - a_e / a_o * v_o < 0 \iff \text{sqrt } D2 > v_e - a_e / a_o * v_o$

**proof**

**assume** *only-if*:  $v_e - a_e / a_o * v_o < 0$

**from** *assms* **have**  $\dots \leq \text{sqrt } D2$  **by** *auto*

**with** *only-if* **show**  $v_e - a_e / a_o * v_o < \text{sqrt } D2$  **by** *linarith*

**next**

**assume** *if-part*:  $v_e - a_e / a_o * v_o < \text{sqrt } D2$

**from** *cond2-imp-gt-fst-sd*[*OF* *assms*(1) *assms*(2)] *snd-leq-fst-exp* **have** *distance-leq-d2*  $\leq s_o - s_e$  **by** *auto*

**from** *if-part* **and** *sqrt-D2-leq-stop-time-diff* [*OF*  $\langle a_e < a_o \rangle - \langle \text{distance-leq-d2} \leq s_o - s_e \rangle$ ]

**show**  $v_e - a_e / a_o * v_o < 0$  **by** *linarith*

**qed**

**theorem** *cond-3'*:

**assumes**  $s_o \leq \text{ego.s-stop}$   $\text{ego.s-stop} < \text{other.s-stop}$

**shows** *collision*  $\{0..\}$   $\iff (a_o > a_e \wedge v_o < v_e \wedge s_o - s_e \leq \text{snd-safe-distance} \wedge v_e - a_e / a_o * v_o < 0)$

**proof** (*cases*  $a_o \leq a_e \vee v_o \geq v_e$ )

**case** *True*

**with** *cond-3*[*OF* *assms*] **show** *?thesis* **by** *auto*

**next**

**case** *False*

**from**  $\langle \neg (a_o \leq a_e \vee v_e \leq v_o) \rangle$  **have**  $a_o > a_e$  **by** *auto*

**from**  $\langle \neg (a_o \leq a_e \vee v_e \leq v_o) \rangle$  **have**  $v_o < v_e$  **by** *auto*

**show** *?thesis*

**proof** -

**from** *snd-safe-distance-iff-nonneg-D2* [*OF*  $\langle a_o > a_e \rangle$ ]

**have** 1:  $(a_e < a_o \wedge v_o < v_e \wedge s_o - s_e \leq \text{snd-safe-distance} \wedge v_e - a_e / a_o * v_o < 0) \iff$

$(a_e < a_o \wedge v_o < v_e \wedge 0 \leq D2 \wedge v_e - a_e / a_o * v_o < 0)$  **by** *auto*

**from** *t-stop-diff-neg-means-leq-D2*[*OF* *assms*  $\langle a_e < a_o \rangle$ ]

**have**  $\dots = (a_e < a_o \wedge v_o < v_e \wedge 0 \leq D2 \wedge \text{sqrt } D2 > v_e - a_e / a_o * v_o)$  **by** *auto*

**with** 1 *cond-3*[*OF* *assms*] **show** *?thesis* **by** *blast*

**qed**

**qed**

**definition** *d* :: *real*  $\Rightarrow$  *real* **where**

*d* *t* = (

*if*  $t \leq 0$  *then*  $s_o - s_e$

*else if*  $t \leq \text{ego.t-stop} \wedge t \leq \text{other.t-stop}$  *then*  $\text{ego-other.p } t$

*else if*  $\text{ego.t-stop} \leq t \wedge t \leq \text{other.t-stop}$  *then*  $\text{other.p } t - \text{ego.s-stop}$

*else if*  $\text{other.t-stop} \leq t \wedge t \leq \text{ego.t-stop}$  *then*  $\text{other.s-stop} - \text{ego.p } t$



```

    else other.s-stop - ego.s-stop
  )

lemma d-diff: d t = other.s t - ego.s t
by (auto simp: d-def ego.s-eq-s-stop other.s-eq-s-stop ego.s-cond-def other.s-cond-def
      movement.p-def field-split-simps)

lemma collision-d: collision S  $\longleftrightarrow$  ( $\exists t \in S. d t = 0$ )
by (force simp: d-diff collision-def )

lemma collision-restrict: collision {0..}  $\longleftrightarrow$  collision {0..max ego.t-stop other.t-stop}

by (auto simp: max.coboundedI1 ego.t-stop-nonneg min-def
      ego.s-eq-s-stop other.s-eq-s-stop collision-def
      intro!: bezI[where x = min t (max (movement.t-stop a_e v_e) (movement.t-stop
a_o v_o)) for t])

lemma collision-union: collision (A  $\cup$  B)  $\longleftrightarrow$  collision A  $\vee$  collision B
by (auto simp: collision-def)

lemma symbolic-checker:
  collision {0..}  $\longleftrightarrow$ 
    (quadroot-in 0 (min ego.t-stop other.t-stop) (1/2 * (a_o - a_e)) (v_o - v_e) (s_o -
s_e))  $\vee$ 
    (quadroot-in ego.t-stop other.t-stop (1/2 * a_o) v_o (s_o - ego.s-stop))  $\vee$ 
    (quadroot-in other.t-stop ego.t-stop (1/2 * a_e) v_e (s_e - other.s-stop))
  (is -  $\longleftrightarrow$  ?q1  $\vee$  ?q2  $\vee$  ?q3)
proof -
  have *: {0..max ego.t-stop other.t-stop} =
    {0 .. min ego.t-stop other.t-stop}  $\cup$  {ego.t-stop .. other.t-stop}  $\cup$  {other.t-stop
.. ego.t-stop}
    using ego.t-stop-nonneg other.t-stop-nonneg
    by auto
  have collision {0..min (movement.t-stop a_e v_e) (movement.t-stop a_o v_o)} = ?q1
    by (force simp: collision-def quadroot-in-def root-in-def d-def
      power2-eq-square field-split-simps movement.p-def movement.s-cond-def)
  moreover
  have collision {ego.t-stop .. other.t-stop} = ?q2
    using ego.t-stop-nonneg
    by (force simp: collision-def quadroot-in-def root-in-def d-def
      ego.s-eq-s-stop movement.s-cond-def movement.p-def)
  moreover
  have collision {other.t-stop .. ego.t-stop} = ?q3
    using other.t-stop-nonneg
    by (force simp: collision-def quadroot-in-def root-in-def d-def
      other.s-eq-s-stop movement.s-cond-def movement.p-def)
  ultimately
  show ?thesis
    unfolding collision-restrict * collision-union

```

by *auto*  
qed

end

## 1.5 Checker Design

**definition** *rel-dist-to-stop* :: *real*  $\Rightarrow$  *real*  $\Rightarrow$  *real* **where**  
*rel-dist-to-stop* *v a*  $\equiv -v^2 / (2 * a)$

**context includes** *floatarith-notation* **begin**

**definition** *rel-dist-to-stop-expr* :: *nat*  $\Rightarrow$  *nat*  $\Rightarrow$  *floatarith* **where**  
*rel-dist-to-stop-expr* *v a* = *Mult* (*Minus* (*Power* (*Var* *v*) 2)) (*Inverse* (*Mult* (*Num* 2) (*Var* *a*)))

**definition** *rel-dist-to-stop'* :: *nat*  $\Rightarrow$  *float interval option*  $\Rightarrow$  *float interval option*  
 $\Rightarrow$  *float interval option* **where**  
*rel-dist-to-stop'* *p v a* = *approx p* (*rel-dist-to-stop-expr* 0 1) [*v, a*]

**lemma** *rel-dist-to-stop'*: *interpret-floatarith* (*rel-dist-to-stop-expr* 0 1) [*v, a*] = *rel-dist-to-stop* *v a*  
**by** (*simp add: rel-dist-to-stop-def rel-dist-to-stop-expr-def inverse-eq-divide*)

**definition** *first-safe-dist* :: *real*  $\Rightarrow$  *real*  $\Rightarrow$  *real* **where**  
*first-safe-dist* *v<sub>e</sub> a<sub>e</sub>*  $\equiv$  *rel-dist-to-stop* *v<sub>e</sub> a<sub>e</sub>*

**definition** *second-safe-dist* :: *real*  $\Rightarrow$  *real*  $\Rightarrow$  *real*  $\Rightarrow$  *real*  $\Rightarrow$  *real* **where**  
*second-safe-dist* *v<sub>e</sub> a<sub>e</sub> v<sub>o</sub> a<sub>o</sub>*  $\equiv$  *rel-dist-to-stop* *v<sub>e</sub> a<sub>e</sub>* - *rel-dist-to-stop* *v<sub>o</sub> a<sub>o</sub>*

**definition** *second-safe-dist-expr* :: *nat*  $\Rightarrow$  *nat*  $\Rightarrow$  *nat*  $\Rightarrow$  *nat*  $\Rightarrow$  *floatarith* **where**  
*second-safe-dist-expr* *ve ae vo ao* = *Add* (*rel-dist-to-stop-expr* *ve ae*) (*Minus* (*rel-dist-to-stop-expr* *vo ao*))

**definition** *second-safe-dist'* :: *nat*  $\Rightarrow$  *float interval option*  $\Rightarrow$  *float interval option*  
 $\Rightarrow$  *float interval option*  $\Rightarrow$  *float interval option*  $\Rightarrow$  *float interval option* **where**  
*second-safe-dist'* *p v<sub>e</sub> a<sub>e</sub> v<sub>o</sub> a<sub>o</sub>* = *approx p* (*second-safe-dist-expr* 0 1 2 3) [*v<sub>e</sub>, a<sub>e</sub>, v<sub>o</sub>, a<sub>o</sub>*]

**lemma** *second-safe-dist'*:  
*interpret-floatarith* (*second-safe-dist-expr* 0 1 2 3) [*v, a, v', a'*] = *second-safe-dist* *v a v' a'*  
**by** (*simp add: second-safe-dist-def second-safe-dist-expr-def rel-dist-to-stop-def rel-dist-to-stop-expr-def inverse-eq-divide*)

**definition** *t-stop* :: *real*  $\Rightarrow$  *real*  $\Rightarrow$  *real* **where**  
*t-stop* *v a*  $\equiv -v / a$

**definition** *t-stop-expr* :: *nat*  $\Rightarrow$  *nat*  $\Rightarrow$  *floatarith* **where**  
*t-stop-expr* *v a* = *Minus* (*Mult* (*Var* *v*) (*Inverse* (*Var* *a*)))

**end**

**definition**  $s\text{-stop} :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real}$  **where**

$s\text{-stop } s \ v \ a \equiv s + \text{rel-dist-to-stop } v \ a$

**definition**  $\text{discriminant} :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real}$  **where**

$\text{discriminant } s_e \ v_e \ a_e \ s_o \ v_o \ a_o \equiv (v_o - v_e)^2 - 2 * (a_o - a_e) * (s_o - s_e)$

**definition**  $\text{suff-cond-safe-dist2} :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{bool}$  **where**

$\text{suff-cond-safe-dist2 } s_e \ v_e \ a_e \ s_o \ v_o \ a_o \equiv$   
 $\text{let } D2 = \text{discriminant } s_e \ v_e \ a_e \ s_o \ v_o \ a_o$   
 $\text{in } \neg (a_e < a_o \wedge v_o < v_e \wedge 0 \leq D2 \wedge v_e - a_e / a_o * v_o < \text{sqrt } D2$   
 $)$

**lemma**  $\text{less-sqrt-iff}: y \geq 0 \implies x < \text{sqrt } y \longleftrightarrow (x \geq 0 \longrightarrow x^2 < y)$

**by** ( $\text{smt real-le-lsqrt real-less-rsqrt real-sqrt-ge-zero}$ )

**lemma**  $\text{suff-cond-safe-dist2-code}[code]:$

$\text{suff-cond-safe-dist2 } s_e \ v_e \ a_e \ s_o \ v_o \ a_o =$   
 $(\text{let } D2 = \text{discriminant } s_e \ v_e \ a_e \ s_o \ v_o \ a_o \ \text{in}$   
 $(a_e < a_o \longrightarrow v_o < v_e \longrightarrow 0 \leq D2 \longrightarrow (v_e - a_e / a_o * v_o \geq 0 \wedge (v_e - a_e /$   
 $a_o * v_o)^2 \geq D2)))$

**using**  $\text{real-sqrt-ge-zero real-less-rsqrt less-sqrt-iff}$

**by** ( $\text{auto simp: suff-cond-safe-dist2-def Let-def}$ )

There are two expressions for safe distance. The first safe distance  $\text{first-safe-dist}$  is always valid. Whenever the distance is bigger than  $\text{first-safe-dist}$ , it is guaranteed to be collision free. The second one is  $\text{second-safe-dist}$ . If the sufficient condition  $\text{suff-cond-safe-dist2}$  is satisfied and the distance is bigger than  $\text{second-safe-dist}$ , it is guaranteed to be collision free.

**definition**  $\text{check-precond} :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{bool}$  **where**

$\text{check-precond } s_e \ v_e \ a_e \ s_o \ v_o \ a_o \longleftrightarrow s_o > s_e \wedge 0 \leq v_e \wedge 0 \leq v_o \wedge a_e < 0 \wedge a_o < 0$

**lemma**  $\text{check-precond-safe-distance}:$

$\text{check-precond } s_e \ v_e \ a_e \ s_o \ v_o \ a_o = \text{safe-distance } a_e \ v_e \ s_e \ a_o \ v_o \ s_o$

**proof**

**assume**  $\text{safe-distance } a_e \ v_e \ s_e \ a_o \ v_o \ s_o$

**then interpret**  $\text{safe-distance } a_e \ v_e \ s_e \ a_o \ v_o \ s_o$  .

**show**  $\text{check-precond } s_e \ v_e \ a_e \ s_o \ v_o \ a_o$

**by** ( $\text{auto simp: check-precond-def in-front nonneg-vel-ego other.nonneg-vel ego.decel other.decel}$ )

**qed** ( $\text{unfold-locales; auto simp: check-precond-def}$ )

### 1.5.1 Prescriptive Checker

**definition**  $\text{checker} :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{bool}$  **where**

*checker*  $s_e v_e a_e s_o v_o a_o \equiv$   
*let* *distance* =  $s_o - s_e$ ;  
*precond* = *check-precond*  $s_e v_e a_e s_o v_o a_o$ ;  
*safe-dist1* = *first-safe-dist*  $v_e a_e$ ;  
*safe-dist2* = *second-safe-dist*  $v_e a_e v_o a_o$ ;  
*cond2* = *suff-cond-safe-dist2*  $s_e v_e a_e s_o v_o a_o$   
*in* *precond*  $\wedge$  (*safe-dist1*  $<$  *distance*  $\vee$  (*safe-dist2*  $<$  *distance*  $\wedge$  *distance*  $\leq$   
*safe-dist1*  $\wedge$  *cond2*))

**lemma** *aux-logic*:

**assumes**  $a \implies b$   
**assumes**  $b \implies a \longleftrightarrow c$   
**shows**  $a \longleftrightarrow b \wedge c$   
**using** *assms* **by** *blast*

**theorem** *soundness-correctness*:

*checker*  $s_e v_e a_e s_o v_o a_o \longleftrightarrow$  *check-precond*  $s_e v_e a_e s_o v_o a_o \wedge$  *safe-distance.no-collision*  
 $a_e v_e s_e a_o v_o s_o \{0..\}$

**proof** (*rule aux-logic, simp add: checker-def Let-def*)

**assume** *cp*: *check-precond*  $s_e v_e a_e s_o v_o a_o$

**then have** *in-front'*:  $s_o > s_e$

**and** *nonneg-vel-ego*:  $0 \leq v_e$

**and** *nonneg-vel-other*:  $0 \leq v_o$

**and** *decelerate-ego*:  $a_e < 0$

**and** *decelerate-other*:  $a_o < 0$

**by** (*auto simp: check-precond-def*)

**from** *in-front'* **have** *in-front*:  $0 < s_o - s_e$  **by** *arith*

**interpret** *safe-distance*  $a_e v_e s_e a_o v_o s_o$  **by** (*unfold-locales; fact*)

**interpret** *ego*: *braking-movement*  $a_e v_e s_e$  **by** (*unfold-locales; fact*)

**interpret** *other*: *braking-movement*  $a_o v_o s_o$  **by** (*unfold-locales; fact*)

**have** *ego.p-max*  $<$   $s_o \vee$  *other.p-max*  $\leq$  *ego.p-max*  $\vee$   $s_o \leq$  *ego.p-max*  $\wedge$  *ego.p-max*  
 $<$  *other.p-max*

**by** *arith*

**then show** *checker*  $s_e v_e a_e s_o v_o a_o =$  *safe-distance.no-collision*  $a_e v_e s_e a_o v_o s_o \{0..\}$

**proof** (*elim disjE*)

**assume** *ego.p-max*  $<$   $s_o$

**then have** *checker*  $s_e v_e a_e s_o v_o a_o$

**using**  $\langle a_e < 0 \rangle$  *cp*

**by** (*simp add: checker-def Let-def first-safe-dist-def rel-dist-to-stop-def ego.p-max-def*  
*ego.p-def ego.t-stop-def algebra-simps power2-eq-square*)

**moreover**

**have** *no-collision*  $\{0..\}$

**using**  $\langle ego.p-max < s_o \rangle$

**by** (*intro cond-1*) (*auto simp: ego.s-t-stop*)

**ultimately show** *?thesis* **by** *auto*

```

next
  assume other.p-max ≤ ego.p-max
  then have ¬ checker se ve ae so vo ao
    using ⟨ae < 0⟩ ⟨ao < 0⟩ other.nonneg-vel
    by (auto simp add: checker-def Let-def first-safe-dist-def second-safe-dist-def
      rel-dist-to-stop-def movement.p-max-def
      movement.p-def movement.t-stop-def algebra-simps power2-eq-square)
      (smt divide-nonneg-neg mult-nonneg-nonneg)
    moreover have collision {0..}
      using ⟨other.p-max ≤ ego.p-max⟩
      by (intro cond-2) (auto simp: other.s-t-stop ego.s-t-stop)
    ultimately show ?thesis by auto
next
  assume H: so ≤ ego.p-max ∧ ego.p-max < other.p-max
  then have checker se ve ae so vo ao = (¬ (ae < ao ∧ vo < ve ∧ 0 ≤ D2 ∧ ve
- ae / ao * vo < sqrt D2))
    using ⟨ae < 0⟩ ⟨ao < 0⟩ cp
    by (simp add: checker-def Let-def first-safe-dist-def rel-dist-to-stop-def ego.p-max-def
      ego.p-def ego.t-stop-def algebra-simps power2-eq-square second-safe-dist-def
      suff-cond-safe-dist2-def discriminant-def not-less other.p-max-def other.p-def
      other.t-stop-def)
    also have ... = no-collision {0..}
      using H
      unfolding Not-eq-iff
      by (intro cond-3[symmetric]) (auto simp: ego.s-t-stop other.s-t-stop)
    finally show ?thesis by auto
qed
qed

```

```

definition checker2 :: real ⇒ real ⇒ real ⇒ real ⇒ real ⇒ real ⇒ bool where
  checker2 se ve ae so vo ao ≡
    let distance = so - se;
        precond = check-precond se ve ae so vo ao;
        safe-dist1 = first-safe-dist ve ae;
        safe-dist2 = second-safe-dist ve ae vo ao;
        safe-dist3 = - rel-dist-to-stop (vo - ve) (ao - ae)
    in
      if ¬ precond then False
      else if distance > safe-dist1 then True
      else if ao > ae ∧ vo < ve ∧ ve - ae / ao * vo < 0 then distance > safe-dist3
      else distance > safe-dist2

```

```

theorem checker-eq-checker2: checker se ve ae so vo ao ↔ checker2 se ve ae so
vo ao
proof (cases check-precond se ve ae so vo ao)
  case False
    with checker-def checker2-def
    show ?thesis by auto
next

```

**case** *True*  
**with** *check-precond-def safe-distance-def*  
**have** *safe-distance*  $a_e v_e s_e a_o v_o s_o$  **by** (*simp add: check-precond-safe-distance*)  
  
**from** *this* **interpret** *safe-distance*  $a_e v_e s_e a_o v_o s_o$  **by** *auto*  
**interpret** *ego: braking-movement*  $a_e v_e s_e$  **by** (*unfold-locales; fact*)  
**interpret** *other: braking-movement*  $a_o v_o s_o$  **by** (*unfold-locales; fact*)  
  
**from**  $\langle$ *check-precond*  $s_e v_e a_e s_o v_o a_o$  $\rangle$  *cond-3 cond-3* [*symmetric*] *fst-leq-snd-safe-distance*  
*ego.s-t-stop ego.p-max-def ego.p-def ego.t-stop-def hyps other.s-t-stop other.p-max-def*  
*other.p-def*  
*other.t-stop-def checker2-def checker-def suff-cond-safe-dist2-def fst-safe-distance-def*  
  
*first-safe-dist-def snd-safe-distance-def second-safe-dist-def rel-dist-to-stop-def dis-*  
*criminant-def*  
**show** *?thesis*  
**by** (*auto simp add:power-def Let-def split: if-splits*)  
**qed**

**definition** *checker3* :: *real*  $\Rightarrow$  *real*  $\Rightarrow$  *real*  $\Rightarrow$  *real*  $\Rightarrow$  *real*  $\Rightarrow$  *real*  $\Rightarrow$  *bool* **where**  
*checker3*  $s_e v_e a_e s_o v_o a_o \equiv$   
*let* *distance* =  $s_o - s_e$ ;  
*precond* = *check-precond*  $s_e v_e a_e s_o v_o a_o$ ;  
*s-stop-e* =  $s_e + \text{rel-dist-to-stop } v_e a_e$ ;  
*s-stop-o* =  $s_o + \text{rel-dist-to-stop } v_o a_o$   
*in* *precond*  $\wedge$  (*s-stop-e*  $<$   $s_o$   
 $\vee$  ( $s_o \leq s\text{-stop-e} \wedge s\text{-stop-e} < s\text{-stop-o} \wedge$   
 $(\neg(a_o > a_e \wedge v_o < v_e \wedge v_e - a_e / a_o * v_o < 0 \wedge \text{distance} * (a_o -$   
 $a_e) \leq (v_o - v_e)^2 / 2))))$ )

**theorem** *checker2-eq-checker3*:  
*checker2*  $s_e v_e a_e s_o v_o a_o \longleftrightarrow$  *checker3*  $s_e v_e a_e s_o v_o a_o$   
**apply** (*auto simp: checker2-def checker3-def Let-def first-safe-dist-def not-less*  
*suff-cond-safe-dist2-def second-safe-dist-def rel-dist-to-stop-def check-precond-def*)  
**proof** *goal-cases*  
**case** *1*  
**then interpret** *safe-distance*  
**by** *unfold-locales auto*  
**from** *fst-leq-snd-safe-distance 1*  
**show** *?case*  
**by** (*auto simp: fst-safe-distance-def snd-safe-distance-def*)  
**next**  
**case** *2*  
**then interpret** *safe-distance*  
**by** *unfold-locales auto*  
**from** *fst-leq-snd-safe-distance 2*  
**show** *?case*  
**by** (*auto simp: fst-safe-distance-def snd-safe-distance-def field-split-simps*)  
**next**

```

case 3
then interpret safe-distance
  by unfold-locales auto
from fst-leq-snd-safe-distance 3
show ?case
  by (auto simp: fst-safe-distance-def snd-safe-distance-def field-split-simps)
qed

```

### 1.5.2 Approximate Checker

```

lemma checker2-def': checker2 a b c d e f = (
  let distance = d - a;
      precondition = check-precond a b c d e f;
      safe-dist1 = first-safe-dist b c;
      safe-dist2 = second-safe-dist b c e f;
      C = c < f ∧ e < b ∧ b * f > c * e;
      P1 = (e - b)2 < 2 * distance * (f - c);
      P2 = - b2 / c + e2 / f < 2 * distance
  in precondition ∧ (safe-dist1 < distance ∨
    safe-dist1 ≥ distance ∧ (C ∧ P1 ∨ ¬C ∧ P2)))
unfolding checker2-def
by (auto simp: Let-def field-split-simps check-precond-def second-safe-dist-def
  rel-dist-to-stop-def)

```

```

lemma power2-less-sqrt-iff: (x::real)2 < y ↔ (y ≥ 0 ∧ abs x < sqrt y)
apply (auto simp: real-less-rsqrt abs-real-def less-sqrt-iff)
apply (meson le-less le-less-trans not-less power2-less-0)+
done

```

```

schematic-goal checker-form: interpret-form ?x ?y ⇒ checker se ve ae so vo ao
unfolding checker-eq-checker2 checker2-eq-checker3 checker3-def check-precond-def
first-safe-dist-def second-safe-dist-def
  suff-cond-safe-dist2-def Let-def t-stop-def s-stop-def
  rel-dist-to-stop-def
  discriminant-def
  not-le not-less
  de-Morgan-conj
  de-Morgan-disj
  power2-less-sqrt-iff
apply (tactic ⟨(Reification.tac @ {context} @ {thms interpret-form.simps interpret-floatarith.simps interpret-floatarith-divide interpret-floatarith-diff}) NONE 1⟩)
apply assumption
done

```

**context includes floatarith-notation begin**

```

definition checker' p se ve ae so vo ao = approx-form p
  (Conj (Conj (Less (Var (Suc (Suc 0))) (Var (Suc (Suc (Suc 0)))))
    (Conj (LessEqual (Var (Suc (Suc (Suc (Suc (Suc (Suc (Suc (Suc 0)))))
      (Var (Suc (Suc (Suc (Suc (Suc 0)))))

```

$$\begin{aligned}
& (Conj (LessEqual (Var (Suc (Suc (Suc (Suc (Suc (Suc (Suc 0))))))) (Var (Suc (Suc (Suc (Suc (Suc 0))))))) \\
& (Conj (Less (Var 0) (Var (Suc (Suc (Suc (Suc (Suc (Suc (Suc 0))))))) \\
& (Less (Var (Suc 0)) (Var (Suc (Suc (Suc (Suc (Suc (Suc (Suc 0))))))) \\
& (Disj (Less (Add (Var (Suc (Suc 0))) \\
& (Mult (Minus (Power (Var (Suc (Suc (Suc (Suc (Suc 0)))))) \\
2)) (Inverse (Mult (Var (Suc (Suc (Suc (Suc 0)))) (Var 0)))) \\
& (Var (Suc (Suc (Suc 0)))) \\
& (Conj (LessEqual (Var (Suc (Suc (Suc 0))) \\
& (Add (Var (Suc (Suc 0))) \\
& (Mult (Minus (Power (Var (Suc (Suc (Suc (Suc (Suc 0)))))) \\
2)) (Inverse (Mult (Var (Suc (Suc (Suc (Suc 0)))) (Var 0)))) \\
& (Conj (Less (Add (Var (Suc (Suc 0))) \\
& (Mult (Minus (Power (Var (Suc (Suc (Suc (Suc (Suc (Suc 0)))))) \\
0)))))) 2)) (Inverse (Mult (Var (Suc (Suc (Suc (Suc 0)))) (Var 0)))) \\
& (Add (Var (Suc (Suc (Suc 0)))) \\
& (Mult (Minus (Power (Var (Suc (Suc (Suc (Suc (Suc (Suc (Suc 0)))))) \\
0)))))) 2)) \\
& (Inverse (Mult (Var (Suc (Suc (Suc (Suc 0)))) (Var (Suc 0)))) \\
& (Disj (LessEqual (Var (Suc 0)) (Var 0)) \\
& (Disj (LessEqual (Var (Suc (Suc (Suc (Suc (Suc 0)))) (Var (Suc (Suc (Suc (Suc (Suc (Suc 0))))))) \\
& (Disj (LessEqual (Var (Suc (Suc (Suc (Suc (Suc (Suc (Suc 0))))))) \\
& (Add (Var (Suc (Suc (Suc (Suc (Suc 0)))))) \\
& (Minus (Mult (Mult (Var 0) (Inverse (Var (Suc 0)))) \\
(Var (Suc (Suc (Suc (Suc (Suc (Suc 0))))))) \\
& (Less (Mult (Power (Add (Var (Suc (Suc (Suc (Suc (Suc (Suc (Suc 0)))))) \\
0)))))) (Minus (Var (Suc (Suc (Suc (Suc 0)))))) 2) \\
& (Inverse (Var (Suc (Suc (Suc (Suc 0)))))) \\
& (Mult (Add (Var (Suc (Suc (Suc 0)))) (Minus (Var (Suc (Suc 0)))) (Add (Var (Suc 0)) (Minus (Var 0)))))) \\
& ([ $a_e$ ,  $a_o$ ,  $s_e$ ,  $s_o$ , Interval' (Float 2 0) (Float 2 0),  $v_e$ ,  $v_o$ , Interval' (Float 0 1) (Float 0 1)]) (replicate 8 0)
\end{aligned}$$

**lemma** *less-Suc-iff-disj*:  $i < Suc\ x \longleftrightarrow i = x \vee i < x$   
**by** *auto*

**lemma** *checker'-soundness-correctness*:

**assumes**  $a \in \{\text{real-of-float } al \dots \text{real-of-float } au\}$   
**assumes**  $b \in \{\text{real-of-float } bl \dots \text{real-of-float } bu\}$   
**assumes**  $c \in \{\text{real-of-float } cl \dots \text{real-of-float } cu\}$   
**assumes**  $d \in \{\text{real-of-float } dl \dots \text{real-of-float } du\}$   
**assumes**  $e \in \{\text{real-of-float } el \dots \text{real-of-float } eu\}$   
**assumes**  $f \in \{\text{real-of-float } fl \dots \text{real-of-float } fu\}$   
**assumes** *chk*: *checker'*  $p$  (Interval'  $al\ au$ ) (Interval'  $bl\ bu$ ) (Interval'  $cl\ cu$ )



```

(Interval' dl du) (Interval' el eu) (Interval' fl fu)
shows checker a b c d e f
apply (rule checker-form)
apply (rule approx-form-aux)
apply (rule chk[unfolded checker'-def])
using assms(1-6)
unfolding bounded-by-def
proof (auto split: option.splits)
fix i x2
assume *: [Interval' cl cu, Interval' fl fu, Interval' al au, Interval' dl du,
Interval' (Float 2 0) (Float 2 0), Interval' bl bu, Interval' el eu, Interval'
0 0] ! i = Some x2
assume i < Suc (Suc (Suc (Suc (Suc (Suc (Suc (Suc 0)))))))
then consider i = 0 | i = 1 | i = 2 | i = 3 | i = 4 | i = 5 | i = 6 | i = 7
by linarith
thus [c, f, a, d, 2, b, e, 0] ! i ∈r x2
apply cases using assms(1-6) *
by (auto intro!: in-real-intervalI dest!: Interval'-eq-Some)
qed

```

**lemma** *approximate-soundness-correctness*:

```

assumes a ∈ {real-of-float al .. real-of-float au}
assumes b ∈ {real-of-float bl .. real-of-float bu}
assumes c ∈ {real-of-float cl .. real-of-float cu}
assumes d ∈ {real-of-float dl .. real-of-float du}
assumes e ∈ {real-of-float el .. real-of-float eu}
assumes f ∈ {real-of-float fl .. real-of-float fu}
assumes chk: checker' p (Interval' al au) (Interval' bl bu) (Interval' cl cu)
(Interval' dl du) (Interval' el eu) (Interval' fl fu)
shows checker'-precond: check-precond a b c d e f
and checker'-no-collision: safe-distance.no-collision c b a f e d {0..}
unfolding atomize-conj
apply (subst soundness-correctness[symmetric])
using checker'-soundness-correctness[OF assms]
by (auto simp: checker-def Let-def)

```

### 1.5.3 Symbolic Checker

**definition** *symbolic-checker* ::  $real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow bool$   
**where**

```

symbolic-checker se ve ae so vo ao ≡
let e-stop = - ve / ae;
o-stop = - vo / ao
in check-precond se ve ae so vo ao ∧
(¬quadroot-in 0 (min e-stop o-stop) (1/2 * (ao - ae)) (vo - ve) (so - se) ∧
¬quadroot-in e-stop o-stop (1/2 * ao) vo (so - movement.p ae ve se e-stop)
∧
¬quadroot-in o-stop e-stop (1/2 * ae) ve (se - movement.p ao vo so o-stop))

```

**theorem** *symbolic-soundness-correctness*:  
 $symbolic-checker\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o \longleftrightarrow check-precond\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o \wedge$   
 $safe-distance.no-collision\ a_e\ v_e\ s_e\ a_o\ v_o\ s_o\ \{0..\}$

**proof** –

```

{
  assume c: check-precond s_e v_e a_e s_o v_o a_o
  then interpret safe-distance a_e v_e s_e a_o v_o s_o
    by (simp add: check-precond-safe-distance)
  have symbolic-checker s_e v_e a_e s_o v_o a_o = no-collision {0..}
    using c
    unfolding symbolic-checker symbolic-checker-def ego.s-t-stop other.s-t-stop
    ego.p-max-def other.p-max-def
    by (auto simp: Let-def movement.t-stop-def)
}
then show ?thesis
  by (auto simp: symbolic-checker-def Let-def)
qed
end

end

```

## 2 Safe Distance with Reaction Time

**theory** *Safe-Distance-Reaction*  
**imports**  
*Safe-Distance*  
**begin**

### 2.1 Normal Safe Distance

**locale** *safe-distance-normal* = *safe-distance* +  
**fixes**  $\delta :: real$   
**assumes** *pos-react*:  $0 < \delta$   
**begin**

**sublocale** *ego2*: *braking-movement*  $a_e\ v_e\ (ego.q\ \delta) ..$

**lemma** *ego2-s-init*:  $ego2.s\ 0 = ego.q\ \delta$  **unfolding** *ego2.s-def* **by** *auto*

**definition**  $\tau :: real \Rightarrow real$  **where**  
 $\tau\ t = t - \delta$

**definition**  $\tau' :: real \Rightarrow real$  **where**  
 $\tau'\ t = 1$

**lemma**  $\tau$ -*continuous*[*continuous-intros*]: *continuous-on*  $T\ \tau$   
**unfolding**  $\tau$ -*def* **by** (*auto intro: continuous-intros*)

**lemma** *isCont*- $\tau$ [*continuous-intros*]: *isCont*  $\tau\ x$

**using**  $\tau$ -continuous[of UNIV] **by** (auto simp: continuous-on-eq-continuous-at)

**lemma** del-has-vector-derivative[derivative-intros]: ( $\tau$  has-vector-derivative  $\tau'$   $t$ )  
(at  $t$  within  $u$ )  
**by** (auto simp:  $\tau$ -def[abs-def]  $\tau'$ -def has-vector-derivative-def algebra-simps  
intro!: derivative-eq-intros)

**lemma** del-has-real-derivative[derivative-intros]: ( $\tau$  has-real-derivative  $\tau'$   $t$ ) (at  $t$   
within  $u$ )  
**using** del-has-vector-derivative  
**by** (simp add: has-field-derivative-iff-has-vector-derivative)

**lemma** delay-image:  $\tau \text{ ' } \{\delta..\} = \{0..\}$   
**proof** (rule subset-antisym, unfold image-def, unfold  $\tau$ -def)  
**show**  $\{y. \exists x \in \{\delta..\}. y = x - \delta\} \subseteq \{0..\}$  **by** auto  
**next**  
**show**  $\{0..\} \subseteq \{y. \exists x \in \{\delta..\}. y = x - \delta\}$   
**proof** (rule subsetI)  
**fix**  $a$   
**assume** ( $a::\text{real}$ )  $\in \{0..\}$   
**hence**  $0 \leq a$  **by** simp  
**hence**  $\exists x \in \{\delta..\}. a = x - \delta$  **using** bexI[where  $x = a + \delta$ ] **by** auto  
**thus**  $a \in \{y. \exists x \in \{\delta..\}. y = x - \delta\}$  **by** auto  
**qed**  
**qed**

**lemma** s-delayed-has-real-derivative[derivative-intros]:  
**assumes**  $\delta \leq t$   
**shows** ((ego2.s  $\circ$   $\tau$ ) has-field-derivative ego2.s' ( $t - \delta$ ) \*  $\tau'$   $t$ ) (at  $t$  within  $\{\delta..\}$ )  
**proof** (rule DERIV-image-chain)  
**from** assms **have**  $0: 0 \leq t - \delta$  **by** simp  
**from** ego2.t-stop-nonneg **have**  $1: v_e / a_e \leq 0$  **unfolding** ego2.t-stop-def **by**  
simp  
**from** ego2.decel **have**  $2: a_e \neq 0$  **by** simp  
**show** (ego2.s has-real-derivative ego2.s' ( $t - \delta$ )) (at ( $\tau$   $t$ ) within  $\tau \text{ ' } \{\delta..\}$ )  
**using** ego2.s-has-real-derivative[OF 0 1 2] sym[OF delay-image]  
**unfolding**  $\tau$ -def **by** simp  
**next**  
**from** del-has-real-derivative **show** ( $\tau$  has-real-derivative  $\tau'$   $t$ ) (at  $t$  within  $\{\delta..\}$ )  
**by** auto  
**qed**

**lemma** s-delayed-has-real-derivative' [derivative-intros]:  
**assumes**  $\delta \leq t$   
**shows** ((ego2.s  $\circ$   $\tau$ ) has-field-derivative (ego2.s'  $\circ$   $\tau$ )  $t$ ) (at  $t$  within  $\{\delta..\}$ )  
**proof** –  
**from** s-delayed-has-real-derivative[OF assms] **have**  
((ego2.s  $\circ$   $\tau$ ) has-field-derivative ego2.s' ( $t - \delta$ ) \*  $\tau'$   $t$ ) (at  $t$  within  $\{\delta..\}$ )  
**by** auto

**hence**  $((\text{ego2}.s \circ \tau) \text{ has-field-derivative } \text{ego2}.s' (t - \delta) * 1)$  (at  $t$  within  $\{\delta..\}$ )  
**using**  $\tau'$ -def[of  $t$ ] **by** *metis*  
**hence**  $((\text{ego2}.s \circ \tau) \text{ has-field-derivative } \text{ego2}.s' (t - \delta))$  (at  $t$  within  $\{\delta..\}$ )  
**by** (*simp add: algebra-simps*)  
**thus** *?thesis unfolding comp-def  $\tau$ -def* **by** *auto*  
**qed**

**lemma** *s-delayed-has-vector-derivative'* [*derivative-intros*]:  
**assumes**  $\delta \leq t$   
**shows**  $((\text{ego2}.s \circ \tau) \text{ has-vector-derivative } (\text{ego2}.s' \circ \tau) t)$  (at  $t$  within  $\{\delta..\}$ )  
**using** *s-delayed-has-real-derivative'*[*OF assms*]  
**by** (*simp add: has-field-derivative-iff-has-vector-derivative*)

**definition**  $u :: \text{real} \Rightarrow \text{real}$  **where**

$u \ t = ($   
     if  $t \leq 0$  then  $s_e$   
     else if  $t \leq \delta$  then  $\text{ego}.q \ t$   
     else  $(\text{ego2}.s \circ \tau) \ t$

**lemma** *init-u*:  $t \leq 0 \implies u \ t = s_e$  **unfolding** *u-def* **by** *auto*

**lemma** *u-delta*:  $u \ \delta = \text{ego2}.s \ 0$

**proof** –

**have**  $u \ \delta = \text{ego}.q \ \delta$  **using** *pos-react* **unfolding** *u-def* **by** *auto*

**also have**  $\dots = \text{ego2}.s \ 0$  **unfolding** *ego2.s-def* **by** *auto*

**finally show**  $u \ \delta = \text{ego2}.s \ 0$  .

**qed**

**lemma** *q-delta*:  $\text{ego}.q \ \delta = \text{ego2}.s \ 0$  **using** *u-delta* *pos-react* **unfolding** *u-def* **by** *auto*

**definition**  $u' :: \text{real} \Rightarrow \text{real}$  **where**

$u' \ t = (\text{if } t \leq \delta \text{ then } \text{ego}.q' \ t \text{ else } \text{ego2}.s' (t - \delta))$

**lemma** *u'-delta*:  $u' \ \delta = \text{ego2}.s' \ 0$

**proof** –

**have**  $u' \ \delta = \text{ego}.q' \ \delta$  **unfolding** *u'-def* **by** *auto*

**also have**  $\dots = v_e$  **unfolding** *ego2.q'-def* **by** *simp*

**also have**  $\dots = \text{ego2}.p' \ 0$  **unfolding** *ego2.p'-def* **by** *simp*

**also have**  $\dots = \text{ego2}.s' \ 0$  **using** *ego2.t-stop-nonneg* **unfolding** *ego2.s'-def* **by** *auto*

**finally show**  $u' \ \delta = \text{ego}.s' \ 0$  .

**qed**

**lemma** *q'-delta*:  $\text{ego}.q' \ \delta = \text{ego2}.s' \ 0$  **using** *u'-delta* **unfolding** *u'-def* **by** *auto*

**lemma** *u-has-real-derivative*[*derivative-intros*]:

**assumes** *nonneg-t*:  $t \geq 0$

**shows**  $(u \ \text{has-real-derivative } u' \ t)$  (at  $t$  within  $\{0..\}$ )

**proof** –

**from** *pos-react* **have**  $0 \leq \delta$  **by** *simp*

**have** *temp*:  $((\lambda t. \text{if } t \in \{0 \dots \delta\} \text{ then } \text{ego}.q \ t \ \text{else } (\text{ego2}.s \circ \tau) \ t) \ \text{has-real-derivative} \ (\text{if } t \in \{0 \dots \delta\} \text{ then } \text{ego}.q' \ t \ \text{else } (\text{ego2}.s' \circ \tau) \ t)) \ (\text{at } t \ \text{within } \{0 \dots\}) \ (\text{is } (?f1 \ \text{has-real-derivative } ?f2) \ (?net))$

**unfolding** *u-def[abs-def]* *u'-def*  
*has-field-derivative-iff-has-vector-derivative*

**apply**  $(\text{rule } \text{has-vector-derivative-If-within-closures}[\text{where } T = \{\delta \dots\}])$

**using**  $\langle 0 \leq \delta \rangle$  *q-delta* *q'-delta* *ego.s-has-vector-derivative[OF assms]* *ego.decel*  
*ego.t-stop-nonneg*  
*s-delayed-has-vector-derivative'[of t]*  *$\tau$ -def*

**unfolding** *comp-def*

**by**  $(\text{auto } \text{simp} : \text{assms } \text{max-def } \text{insert-absorb} \ \text{intro!} : \text{ego}.q\text{-has-vector-derivative})$

**show** *?thesis*

**unfolding** *has-vector-derivative-def* *has-field-derivative-iff-has-vector-derivative*  
*u'-def* *u-def[abs-def]*

**proof**  $(\text{rule } \text{has-derivative-transform}[\text{where } f = (\lambda t. \text{if } t \in \{0 \dots \delta\} \text{ then } \text{ego}.q \ t \ \text{else } (\text{ego2}.s \circ \tau) \ t)])$

**from** *nonneg-t* **show**  $t \in \{0 \dots\}$  **by** *auto*

**next**

**fix** *x*

**assume**  $(x :: \text{real}) \in \{0 \dots\}$

**hence**  $x \leq \delta \iff x \in \{0 \dots \delta\}$  **by** *simp*

**thus**  $(\text{if } x \leq 0 \text{ then } s_e \ \text{else if } x \leq \delta \text{ then } \text{ego}.q \ x \ \text{else } (\text{ego2}.s \circ \tau) \ x) =$   
 $(\text{if } x \in \{0 \dots \delta\} \text{ then } \text{ego}.q \ x \ \text{else } (\text{ego2}.s \circ \tau) \ x)$  **using** *pos-react* **unfolding**  
*ego.q-def* **by** *auto*

**next**

**from** *temp* **have**  $(?f1 \ \text{has-vector-derivative } ?f2) \ ?net$

**using** *has-field-derivative-iff-has-vector-derivative* **by** *auto*

**moreover with** *assms* **have**  $t \in \{0 \dots \delta\} \iff t \leq \delta$  **by** *auto*

**ultimately show**  $((\lambda t. \text{if } t \in \{0 \dots \delta\} \text{ then } \text{ego}.q \ t \ \text{else } (\text{ego2}.s \circ \tau) \ t) \ \text{has-derivative} \ (\lambda x. x *_R (\text{if } t \leq \delta \text{ then } \text{ego2}.q' \ t \ \text{else } \text{ego2}.s' \ (t - \delta)))) \ (\text{at } t \ \text{within } \{0 \dots\})$

**unfolding** *comp-def*  *$\tau$ -def* *has-vector-derivative-def* **by** *auto*

**qed**

**qed**

**definition** *t-stop* :: *real* **where**  
 $t\text{-stop} = \text{ego2}.t\text{-stop} + \delta$

**lemma** *t-stop-nonneg*:  $0 \leq t\text{-stop}$

**unfolding** *t-stop-def*

**using** *ego2.t-stop-nonneg* *pos-react*

**by** *auto*

**lemma** *t-stop-pos*:  $0 < t\text{-stop}$

**unfolding** *t-stop-def*

using *ego2.t-stop-nonneg pos-react*  
by *auto*

**lemma** *t-stop-zero*:

assumes  $t\text{-stop} \leq x$

assumes  $x \leq \delta$

shows  $v_e = 0$

using *assms unfolding t-stop-def using ego2.t-stop-nonneg pos-react ego2.t-stop-zero*  
by *auto*

**lemma** *u'-stop-zero*:  $u' t\text{-stop} = 0$

unfolding *u'-def t-stop-def ego2.q'-def ego2.s'-def*

using *ego2.t-stop-nonneg ego2.p'-stop-zero decelerate-ego ego2.t-stop-zero* by  
*auto*

**definition** *u-max* :: *real* where

$u\text{-max} = u (ego2.t\text{-stop} + \delta)$

**lemma** *u-max-eq*:  $u\text{-max} = ego.q \delta - v_e^2 / a_e / 2$

**proof** (*cases ego2.t-stop = 0*)

assume  $ego2.t\text{-stop} = 0$

hence  $v_e = 0$  using *ego2.t-stop-zero* by *simp*

with  $\langle ego2.t\text{-stop} = 0 \rangle$  show  $u\text{-max} = ego.q \delta - v_e^2 / a_e / 2$  unfolding  
*u-max-def u-def* using *pos-react* by *auto*

**next**

assume  $ego2.t\text{-stop} \neq 0$

hence  $u\text{-max} = (ego2.s \circ \tau) (ego2.t\text{-stop} + \delta)$

unfolding *u-max-def u-def* using *ego2.t-stop-nonneg pos-react* by *auto*

moreover have  $\dots = ego2.s \ ego2.t\text{-stop}$  unfolding *comp-def  $\tau$ -def* by *auto*

moreover have  $\dots = ego2.p\text{-max}$

unfolding *ego2.s-def ego2.p-max-def* using  $\langle ego2.t\text{-stop} \neq 0 \rangle$  *ego2.t-stop-nonneg*  
by *auto*

moreover have  $\dots = ego.q \delta - v_e^2 / a_e / 2$  using *ego2.p-max-eq* .

ultimately show *?thesis* by *auto*

**qed**

**lemma** *u-mono*:

assumes  $x \leq y$   $y \leq t\text{-stop}$

shows  $u x \leq u y$

**proof** –

have  $y \leq 0 \vee (0 < y \wedge y \leq \delta) \vee \delta < y$  by *auto*

moreover

{ assume  $y \leq 0$

with *assms* have  $x \leq 0$  by *auto*

with  $\langle y \leq 0 \rangle$  have  $u x \leq u y$  unfolding *u-def* by *auto* }

moreover

{ assume  $0 < y \wedge y \leq \delta$

**with** *assms* **have**  $x \leq \delta$  **by** *auto*  
**hence**  $u x \leq u y$   
**proof** (*cases*  $x \leq 0$ )  
    **assume**  $x \leq 0$   
    **with**  $\langle x \leq \delta \rangle$  **and**  $\langle 0 < y \wedge y \leq \delta \rangle$  **show**  $u x \leq u y$  **unfolding** *u-def* **using**  
*ego.q-min* **by** *auto*  
**next**  
    **assume**  $\neg x \leq 0$   
    **with**  $\langle 0 < y \wedge y \leq \delta \rangle$  **and** *assms* **show**  $u x \leq u y$   
    **unfolding** *u-def* **using** *ego.q-mono* **by** *auto*  
**qed** }

**moreover**  
{ **assume**  $\delta < y$   
**have**  $u x \leq u y$   
**proof** (*cases*  $\delta < x$ )  
    **assume**  $\delta < x$   
    **with** *pos-react* **have**  $\neg x \leq 0$  **by** *auto*  
    **moreover from**  $\langle \delta < y \rangle$  **and** *pos-react* **have**  $\neg y \leq 0$  **by** *auto*  
    **ultimately show**  $u x \leq u y$  **unfolding** *u-def comp-def*  
    **using** *assms* *ego2.s-mono*[*of*  $x - \delta$   $y - \delta$ ]  $\langle \delta < y \rangle$   $\langle \delta < x \rangle$  **by** (*auto*  
*simp*: $\tau$ -*def*)  
**next**  
    **assume**  $\neg \delta < x$   
    **hence**  $x \leq \delta$  **by** *simp*  
    **hence**  $u x \leq ego.q \delta$  **unfolding** *u-def* **using** *pos-react nonneg-vel-ego*  
    **by** (*auto simp add:ego.q-def mult-left-mono*)  
    **also have**  $\dots = ego2.s (\tau \delta)$  **unfolding** *ego2.s-def* **unfolding**  $\tau$ -*def* **by** *auto*  
    **also have**  $\dots \leq ego2.s (\tau y)$  **unfolding**  $\tau$ -*def* **using**  $\langle \delta < y \rangle$  **by** (*auto simp*  
*add:ego2.s-mono*)  
    **also have**  $\dots = u y$  **unfolding** *u-def* **using**  $\langle \delta < y \rangle$  *pos-react* **by** *auto*  
    **ultimately show**  $u x \leq u y$  **by** *auto*  
**qed** }

**ultimately show**  $u x \leq u y$  **by** *auto*  
**qed**

**lemma** *u-antimono*:  $x \leq y \implies t\text{-stop} \leq x \implies u y \leq u x$

**proof** –

**assume** *1*:  $x \leq y$   
**assume** *2*:  $t\text{-stop} \leq x$   
**hence**  $\delta \leq x$  **unfolding**  $\tau$ -*def* *t-stop-def* **using** *pos-react ego2.t-stop-nonneg* **by**  
*auto*  
**with** *1* **have**  $\delta \leq y$  **by** *auto*  
**from** *1* **and** *2* **have** *3*:  $t\text{-stop} \leq y$  **by** *auto*  
**show**  $u y \leq u x$   
**proof** (*cases*  $x \neq \delta \wedge y \neq \delta$ )  
    **assume**  $x \neq \delta \wedge y \neq \delta$   
    **hence**  $x \neq \delta$  **and**  $y \neq \delta$  **by** *auto*

**have**  $u y \leq (ego2.s \circ \tau) y$  **unfolding**  $u\text{-def}$  **using**  $\langle \delta \leq y \rangle \langle y \neq \delta \rangle$   $pos\text{-react}$   
**by**  $auto$   
**also have**  $\dots \leq (ego2.s \circ \tau) x$  **unfolding**  $comp\text{-def}$   
**proof** ( $intro\ ego2.s\text{-antimono}$ )  
**show**  $\tau x \leq \tau y$  **unfolding**  $\tau\text{-def}$  **using**  $\langle x \leq y \rangle$  **by**  $auto$   
**next**  
**show**  $ego2.t\text{-stop} \leq \tau x$  **unfolding**  $\tau\text{-def}$  **using**  $\langle t\text{-stop} \leq x \rangle$  **by** ( $auto\ simp:$   
 $t\text{-stop}\text{-def}$ )  
**qed**  
**also have**  $\dots \leq u x$  **unfolding**  $u\text{-def}$  **using**  $\langle \delta \leq x \rangle \langle x \neq \delta \rangle$   $pos\text{-react}$  **by**  $auto$   
**ultimately show**  $u y \leq u x$  **by**  $auto$   
**next**  
**assume**  $\neg (x \neq \delta \wedge y \neq \delta)$   
**have**  $x \neq \delta \rightarrow y \neq \delta$   
**proof** ( $rule\ impI; erule\ contrapos\text{-pp}[\text{where } Q = \neg x = \delta]$ )  
**assume**  $\neg y \neq \delta$   
**hence**  $y = \delta$  **by**  $simp$   
**with**  $\langle t\text{-stop} \leq y \rangle$  **have**  $ego2.t\text{-stop} = 0$  **unfolding**  $t\text{-stop}\text{-def}$   
**using**  $ego2.t\text{-stop}\text{-nonneg}$  **by**  $auto$   
**with**  $\langle t\text{-stop} \leq x \rangle$  **have**  $x = \delta$  **unfolding**  $t\text{-stop}\text{-def}$  **using**  $\langle x \leq y \rangle \langle y = \delta \rangle$   
**by**  $auto$   
**thus**  $\neg x \neq \delta$  **by**  $auto$   
**qed**  
**with**  $\langle \neg (x \neq \delta \wedge y \neq \delta) \rangle$  **have**  $(x = \delta \wedge y = \delta) \vee (x = \delta)$  **by**  $auto$   
  
**moreover**  
**{** **assume**  $x = \delta \wedge y = \delta$   
**hence**  $x = \delta$  **and**  $y = \delta$  **by**  $auto$   
**hence**  $u y \leq ego.q\ \delta$  **unfolding**  $u\text{-def}$  **using**  $pos\text{-react}$  **by**  $auto$   
**also have**  $\dots \leq u x$  **unfolding**  $u\text{-def}$  **using**  $\langle x = \delta \rangle$   $pos\text{-react}$  **by**  $auto$   
**ultimately have**  $u y \leq u x$  **by**  $auto$  **}**  
  
**moreover**  
**{** **assume**  $x = \delta$   
**hence**  $ego2.t\text{-stop} = 0$  **using**  $\langle t\text{-stop} \leq x \rangle$   $ego2.t\text{-stop}\text{-nonneg}$  **by** ( $auto$   
 $simp:t\text{-stop}\text{-def}$ )  
**hence**  $v_e = 0$  **by** ( $rule\ ego2.t\text{-stop}\text{-zero}$ )  
**hence**  $u y \leq ego.q\ \delta$   
**using**  $pos\text{-react}$   $\langle x = \delta \rangle \langle x \leq y \rangle \langle v_e = 0 \rangle$   
**unfolding**  $u\text{-def}$   $comp\text{-def}$   $\tau\text{-def}$   $ego2.s\text{-def}$   $ego2.p\text{-def}$   $ego2.p\text{-max}\text{-def}$   
 $ego2.t\text{-stop}\text{-def}$   
**by**  $auto$   
**also have**  $\dots \leq u x$  **using**  $\langle x = \delta \rangle$   $pos\text{-react}$  **unfolding**  $u\text{-def}$  **by**  $auto$   
**ultimately have**  $u y \leq u x$  **by**  $auto$  **}**  
  
**ultimately show**  $?thesis$  **by**  $auto$   
**qed**  
**qed**



**lemma** *u-max*:  $u\ x \leq u\text{-max}$   
**unfolding** *u-max-def* **using** *t-stop-def*  
**by** (*cases*  $x \leq t\text{-stop}$ ) (*auto intro: u-mono u-antimono*)

**lemma** *u-eq-u-stop*: *NO-MATCH*  $t\text{-stop}\ x \implies x \geq t\text{-stop} \implies u\ x = u\text{-max}$   
**proof** –  
**assume**  $t\text{-stop} \leq x$   
**with** *t-stop-pos* **have**  $0 < x$  **by** *auto*  
**from**  $\langle t\text{-stop} \leq x \rangle$  **have**  $\delta \leq x$  **unfolding** *t-stop-def* **using** *ego2.t-stop-nonneg*  
**by** *auto*  
**show**  $u\ x = u\text{-max}$   
**proof** (*cases*  $x \leq \delta$ )  
**assume**  $x \leq \delta$   
**with**  $\langle t\text{-stop} \leq x \rangle$  **have**  $v_e = 0$  **by** (*rule t-stop-zero*)  
**also** **have**  $x = \delta$  **using**  $\langle x \leq \delta \rangle$  **and**  $\langle \delta \leq x \rangle$  **by** *auto*  
**ultimately** **have**  $u\ x = \text{ego}.q\ \delta$  **unfolding** *u-def* **using** *pos-react* **by** *auto*  
**also** **have**  $\dots = u\text{-max}$  **unfolding** *u-max-eq* **using**  $\langle v_e = 0 \rangle$  **by** *auto*  
**ultimately** **show**  $u\ x = u\text{-max}$  **by** *simp*  
**next**  
**assume**  $\neg x \leq \delta$   
**hence**  $\delta < x$  **by** *auto*  
**hence**  $u\ x = (\text{ego2}.s \circ \tau)\ x$  **unfolding** *u-def* **using** *pos-react* **by** *auto*  
**also** **have**  $\dots = \text{ego2}.s\ \text{ego2}.t\text{-stop}$   
**proof** (*unfold comp-def; unfold  $\tau$ -def; intro order.antisym*)  
**have**  $x - \delta \geq \text{ego2}.t\text{-stop}$  **using**  $\langle t\text{-stop} \leq x \rangle$  **unfolding** *t-stop-def* **by** *auto*  
**thus**  $\text{ego2}.s\ (x - \delta) \leq \text{ego2}.s\ \text{ego2}.t\text{-stop}$  **by** (*rule ego2.s-antimono*) *simp*  
**next**  
**have**  $x - \delta \geq \text{ego2}.t\text{-stop}$  **using**  $\langle t\text{-stop} \leq x \rangle$  **unfolding** *t-stop-def* **by** *auto*  
**thus**  $\text{ego2}.s\ \text{ego2}.t\text{-stop} \leq \text{ego2}.s\ (x - \delta)$  **using** *ego2.t-stop-nonneg* **by** (*rule ego2.s-mono*)  
**qed**  
**also** **have**  $\dots = u\text{-max}$  **unfolding** *u-max-eq* *ego2.s-t-stop* *ego2.p-max-eq* **by** *auto*  
**ultimately** **show**  $u\ x = u\text{-max}$  **by** *auto*  
**qed**  
**qed**

**lemma** *at-least-delta*:  
**assumes**  $x \leq \delta$   
**assumes**  $t\text{-stop} \leq x$   
**shows**  $\text{ego}.q\ x = \text{ego2}.s\ (x - \delta)$   
**using** *assms ego2.t-stop-nonneg*  
**unfolding** *t-stop-def* *ego2.s-def* *less-eq-real-def* **by** *auto*

**lemma** *continuous-on-u[continuous-intros]*: *continuous-on*  $T\ u$   
**unfolding** *u-def[abs-def]*  
**using** *t-stop-nonneg* *pos-react* *at-least-delta*  
**proof** (*intro continuous-on-subset[where  $t=T$  and  $s = \{..0\} \cup (\{0..\delta\}) \cup (\{\delta\} \cup \{t\text{-stop}\} \cup \{t\text{-stop}..\})$ ]]* *continuous-on-If* *continuous-intros*)

```

fix  $x$ 
assume  $\neg x \leq \delta$ 
assume  $x \in \{0.. \delta\}$ 
hence  $0 \leq x$  and  $x \leq \delta$  by auto
thus  $ego.q\ x = (ego2.s \circ \tau)\ x$ 
  unfolding comp-def  $\tau$ -def ego2.s-def
  using  $\langle \neg x \leq \delta \rangle$  by auto
next
fix  $x$ 
assume  $x \in \{\delta..t-stop\} \cup \{t-stop..\}$ 
hence  $\delta \leq x$  unfolding t-stop-def using pos-react ego.t-stop-nonneg by auto
also assume  $x \leq \delta$ 
ultimately have  $x = \delta$  by auto
thus  $ego.q\ x = (ego2.s \circ \tau)\ x$  unfolding comp-def  $\tau$ -def ego2.s-def by auto
next
fix  $t::real$ 
assume  $t \in \{.. 0\}$ 
hence  $t \leq 0$  by auto
also assume  $\neg t \leq 0$ 
ultimately have  $t = 0$  by auto
hence  $s_e = ego.q\ t$  unfolding ego.q-def by auto
with pos-react  $\langle t = 0 \rangle$  show  $s_e = (if\ t \leq \delta\ then\ ego.q\ t\ else\ (ego2.s \circ \tau)\ t)$  by
auto
next
fix  $t::real$ 
assume  $t \in \{0.. \delta\} \cup (\{\delta..t-stop\} \cup \{t-stop..\})$ 
hence  $0 \leq t$  using pos-react ego2.t-stop-nonneg by (auto simp: t-stop-def)
also assume  $t \leq 0$ 
ultimately have  $t = 0$  by auto
hence  $s_e = (if\ t \leq \delta\ then\ ego.q\ t\ else\ (ego2.s \circ \tau)\ t)$  using pos-react ego.init-q
by auto
thus  $s_e = (if\ t \leq \delta\ then\ ego.q\ t\ else\ (ego2.s \circ \tau)\ t)$  by auto
next
show  $T \subseteq \{..0\} \cup (\{0.. \delta\} \cup (\{\delta..t-stop\} \cup \{t-stop..\}))$  by auto
qed

```

**lemma** *isCont-u[continuous-intros]*: *isCont*  $u\ x$   
**using** *continuous-on-u[of UNIV]*  
**by** (*auto simp:continuous-on-eq-continuous-at*)

**definition** *collision-react*  $:: real\ set \Rightarrow bool$  **where**  
*collision-react*  $time-set \equiv (\exists t \in time-set. u\ t = other.s\ t)$

**abbreviation** *no-collision-react*  $:: real\ set \Rightarrow bool$  **where**  
*no-collision-react*  $time-set \equiv \neg\ collision-react\ time-set$

**lemma** *no-collision-reactI*:  
**assumes**  $\bigwedge t. t \in S \Longrightarrow u\ t \neq other.s\ t$   
**shows** *no-collision-react*  $S$

**using** *assms*  
**unfolding** *collision-react-def*  
**by** *blast*

**lemma** *no-collision-union*:  
**assumes** *no-collision-react S*  
**assumes** *no-collision-react T*  
**shows** *no-collision-react (S ∪ T)*  
**using** *assms*  
**unfolding** *collision-react-def*  
**by** *auto*

**lemma** *collision-trim-subset*:  
**assumes** *collision-react S*  
**assumes** *no-collision-react T*  
**assumes**  $T \subseteq S$   
**shows** *collision-react (S - T)*  
**using** *assms*  
**unfolding** *collision-react-def* **by** *auto*

**theorem** *cond-1r* :  $u\text{-max} < s_o \implies \text{no-collision-react } \{0..\}$

**proof** (*rule no-collision-reactI, simp*)

**fix**  $t :: \text{real}$   
**assume**  $0 \leq t$   
**have**  $u\ t \leq u\text{-max}$  **by** (*rule u-max*)  
**also assume**  $\dots < s_o$   
**also have**  $\dots = \text{other.s } 0$   
**by** (*simp add: other.init-s*)  
**also have**  $\dots \leq \text{other.s } t$   
**using**  $\langle 0 \leq t \rangle$  *hyps*  
**by** (*intro other.s-mono*) *auto*  
**finally show**  $u\ t \neq \text{other.s } t$   
**by** *simp*

qed

**definition** *safe-distance-1r* :: *real* **where**

$$\text{safe-distance-1r} = v_e * \delta - v_e^2 / a_e / 2$$

**lemma** *sd-1r-eq*:  $(s_o - s_e > \text{safe-distance-1r}) = (u\text{-max} < s_o)$

**proof** –

**have**  $(s_o - s_e > \text{safe-distance-1r}) = (s_o - s_e > v_e * \delta - v_e^2 / a_e / 2)$  **unfolding**  
*safe-distance-1r-def* **by** *auto*

**moreover have**  $\dots = (s_e + v_e * \delta - v_e^2 / a_e / 2 < s_o)$  **by** *auto*

**ultimately show** *?thesis* **using** *u-max-eq ego.q-def* **by** *auto*

qed

**lemma** *sd-1r-correct*:

**assumes**  $s_o - s_e > \text{safe-distance-1r}$

**shows** *no-collision-react {0..}*

**proof** –  
**from** *assms* **have**  $u\text{-max} < s_o$  **using** *sd-1r-eq* **by** *auto*  
**thus** *?thesis* **by** (*rule cond-1r*)  
**qed**

**lemma** *u-other-strict-ivt*:  
**assumes**  $u\ t > \text{other.s}\ t$   
**shows** *collision-react*  $\{0..<t\}$   
**proof** *cases*  
**assume**  $0 \leq t$   
**with** *assms in-front*  
**have**  $\exists x \geq 0. x \leq t \wedge \text{other.s}\ x - u\ x = 0$   
**by** (*intro IVT2*) (*auto simp: init-u other.init-s continuous-diff isCont-u other.isCont-s*)  
**then show** *?thesis*  
**using** *assms*  
**by** (*auto simp add: algebra-simps collision-react-def Bex-def order.order-iff-strict*)  
**qed**(*insert assms hyps, auto simp: collision-react-def init-u other.init-s*)

**lemma** *collision-react-subset*: *collision-react*  $s \implies s \subseteq t \implies \text{collision-react}\ t$   
**by** (*auto simp: collision-react-def*)

**lemma** *u-other-ivt*:  
**assumes**  $u\ t \geq \text{other.s}\ t$   
**shows** *collision-react*  $\{0 .. t\}$   
**proof** *cases*  
**assume**  $u\ t > \text{other.s}\ t$   
**from** *u-other-strict-ivt*[*OF this*]  
**show** *?thesis*  
**by** (*rule collision-react-subset*) *auto*  
**qed** (*insert hyps assms; cases t ≥ 0; force simp: collision-react-def init-u other.init-s*)

**theorem** *cond-2r*:  
**assumes**  $u\text{-max} \geq \text{other.s-stop}$   
**shows** *collision-react*  $\{0 ..\}$   
**using** *assms*  
**apply**(*intro collision-react-subset*[**where**  $t = \{0.. \}$  **and**  $s = \{0 .. \max\ t\text{-stop}\ \text{other.t-stop}\}$ ])  
**apply**(*intro u-other-ivt*[**where**  $t = \max\ t\text{-stop}\ \text{other.t-stop}$ ])  
**apply**(*auto simp: u-eq-u-stop other.s-eq-s-stop*)  
**done**

**definition** *ego-other2* :: *real*  $\Rightarrow$  *real* **where**  
*ego-other2*  $t = \text{other.s}\ t - u\ t$

**lemma** *continuous-on-ego-other2*[*continuous-intros*]: *continuous-on*  $T$  *ego-other2*  
**unfolding** *ego-other2-def*[*abs-def*]  
**by** (*intro continuous-intros*)

**lemma** *isCont-ego-other2*[*continuous-intros*]: *isCont* *ego-other2*  $x$   
**using** *continuous-on-ego-other2*[*of UNIV*]

by (auto simp: continuous-on-eq-continuous-at)

**definition** *ego-other2'* :: real  $\Rightarrow$  real **where**

*ego-other2' t = other.s' t - u' t*

**lemma** *ego-other2-has-real-derivative*[*derivative-intros*]:

**assumes**  $0 \leq t$

**shows** (*ego-other2 has-real-derivative ego-other2' t*) (at  $t$  within  $\{0..\}$ )

**using** *assms other.t-stop-nonneg decelerate-other*

**unfolding** *other.t-stop-def*

**by** (auto simp: *ego-other2-def*[*abs-def*] *ego-other2'-def algebra-simps*  
*intro!*: *derivative-eq-intros*)

**theorem** *cond-3r-1*:

**assumes**  $u \delta \geq other.s \delta$

**shows** *collision-react*  $\{0 .. \delta\}$

**proof** (*unfold collision-react-def*)

**have**  $1: \exists t \geq 0. t \leq \delta \wedge ego-other2 t = 0$

**proof** (*intro IVT2*)

**show** *ego-other2*  $\delta \leq 0$  **unfolding** *ego-other2-def* **using** *assms* **by** *auto*

**next**

**show**  $0 \leq ego-other2 0$  **unfolding** *ego-other2-def*

**using** *other.init-s*[*of 0*] *init-u*[*of 0*] *in-front* **by** *auto*

**next**

**show**  $0 \leq \delta$  **using** *pos-react* **by** *auto*

**next**

**show**  $\forall t. 0 \leq t \wedge t \leq \delta \longrightarrow isCont\ ego-other2\ t$

**using** *isCont-ego-other2* **by** *auto*

**qed**

**then obtain**  $t$  **where**  $0 \leq t \wedge t \leq \delta \wedge ego-other2 t = 0$  **by** *auto*

**hence**  $t \in \{0 .. \delta\}$  **and**  $u t = other.s t$  **unfolding** *ego-other2-def* **by** *auto*

**thus**  $\exists t \in \{0 .. \delta\}. u t = other.s t$  **by** (*intro bexI*)

**qed**

**definition** *distance0* :: real **where**

*distance0 = v<sub>e</sub> \*  $\delta$  - v<sub>o</sub> \*  $\delta$  - a<sub>o</sub> \*  $\delta^2 / 2$*

**definition** *distance0-2* :: real **where**

*distance0-2 = v<sub>e</sub> \*  $\delta$  + 1 / 2 \* v<sub>o</sub><sup>2</sup> / a<sub>o</sub>*

**theorem** *cond-3r-1'*:

**assumes**  $s_o - s_e \leq distance0$

**assumes**  $\delta \leq other.t-stop$

**shows** *collision-react*  $\{0 .. \delta\}$

**proof** –

**from** *assms* **have**  $u \delta \geq other.s \delta$  **unfolding** *distance0-def other.s-def*

*other.p-def u-def ego.q-def* **using** *pos-react* **by** *auto*

**thus** *?thesis* **using** *cond-3r-1* **by** *auto*

**qed**

**theorem** *distance0-2-eq*:

**assumes**  $\delta > \text{other.t-stop}$

**shows**  $(u \delta < \text{other.s } \delta) = (s_o - s_e > \text{distance0-2})$

**proof** –

**from** *assms* **have**  $(u \delta < \text{other.s } \delta) = (\text{ego.q } \delta < \text{other.p-max})$

**using** *u-def other.s-def pos-react* **by** *auto*

**also have**  $\dots = (s_e + v_e * \delta < s_o + v_o * (-v_o / a_o) + 1 / 2 * a_o * (-v_o / a_o)^2)$

**using** *ego.q-def other.p-max-def other.p-def other.t-stop-def* **by** *auto*

**also have**  $\dots = (v_e * \delta - v_o * (-v_o / a_o) - 1 / 2 * a_o * (-v_o / a_o)^2 < s_o - s_e)$  **by** *linarith*

**also have**  $\dots = (v_e * \delta + v_o^2 / a_o - 1 / 2 * v_o^2 / a_o < s_o - s_e)$

**using** *other.p-def other.p-max-def other.p-max-eq other.t-stop-def* **by** *auto*

**also have**  $\dots = (v_e * \delta + 1 / 2 * v_o^2 / a_o < s_o - s_e)$  **by** *linarith*

**thus** *?thesis* **using** *distance0-2-def* **by** (*simp add: calculation*)

**qed**

**theorem** *cond-3r-1'-2*:

**assumes**  $s_o - s_e \leq \text{distance0-2}$

**assumes**  $\delta > \text{other.t-stop}$

**shows** *collision-react*  $\{0 .. \delta\}$

**proof** –

**from** *assms distance0-2-eq* **have**  $u \delta \geq \text{other.s } \delta$  **unfolding** *distance0-def other.s-def other.p-def u-def ego.q-def* **using** *pos-react* **by** *auto*

**thus** *?thesis* **using** *cond-3r-1* **by** *auto*

**qed**

**definition** *safe-distance-3r* :: *real* **where**

$\text{safe-distance-3r} = v_e * \delta - v_e^2 / 2 / a_e - v_o * \delta - 1/2 * a_o * \delta^2$

**lemma** *distance0-at-most-sd3r*:

$\text{distance0} \leq \text{safe-distance-3r}$

**unfolding** *distance0-def safe-distance-3r-def* **using** *nonneg-vel-ego decelerate-ego* **by** (*auto simp add:field-simps*)

**definition** *safe-distance-4r* :: *real* **where**

$\text{safe-distance-4r} = (v_o + a_o * \delta - v_e)^2 / 2 / (a_o - a_e) - v_o * \delta - 1/2 * a_o * \delta^2 + v_e * \delta$

**lemma** *distance0-at-most-sd4r*:

**assumes**  $a_o > a_e$

**shows**  $\text{distance0} \leq \text{safe-distance-4r}$

**proof** –

**from** *assms* **have**  $a_o \geq a_e$  **by** *auto*

**have**  $0 \leq (v_o + a_o * \delta - v_e)^2 / (2 * a_o - 2 * a_e)$

**by** (*rule divide-nonneg-nonneg*) (*auto simp add:assms <a\_e ≤ a\_o>*)

**thus** *?thesis* **unfolding** *distance0-def safe-distance-4r-def*

**by** *auto*

qed

**definition** *safe-distance-2r* :: real **where**

$$\text{safe-distance-2r} = v_e * \delta - v_e^2 / 2 / a_e + v_o^2 / 2 / a_o$$

**lemma** *vo-start-geq-ve*:

**assumes**  $\delta \leq \text{other.t-stop}$

**assumes**  $\text{other.s}' \delta \geq v_e$

**shows**  $u \delta < \text{other.s} \delta$

**proof** –

**from** *assms* **have**  $v_e \leq v_o + a_o * \delta$  **unfolding** *other.s'-def other.p'-def* **by** *auto*

**with** *mult-right-mono[OF this, of  $\delta$ ]* **have**  $v_e * \delta \leq v_o * \delta + a_o * \delta^2$  (**is**  $?l0 \leq ?r0$ )

**using** *pos-react* **by** (*auto simp add:field-simps power-def*)

**hence**  $s_e + ?l0 \leq s_e + ?r0$  **by** *auto*

**also have**  $\dots < s_o + ?r0$  **using** *in-front* **by** *auto*

**also have**  $\dots < s_o + v_o * \delta + a_o * \delta^2 / 2$  **using** *decelerate-other pos-react* **by** *auto*

**finally show** *?thesis* **using** *pos-react assms(1)*

**unfolding** *u-def ego.q-def other.s-def other.t-stop-def other.p-def* **by** *auto*

qed

**theorem** *so-star-stop-leq-se-stop*:

**assumes**  $\delta \leq \text{other.t-stop}$

**assumes**  $\text{other.s}' \delta < v_e$

**assumes**  $\neg (a_o > a_e \wedge \text{other.s}' \delta < v_e \wedge v_e - a_e / a_o * \text{other.s}' \delta < 0)$

**shows**  $0 \leq -v_e^2 / a_e / 2 + (v_o + a_o * \delta)^2 / a_o / 2$

**proof** –

**consider**  $v_e - a_e / a_o * \text{other.s}' \delta \geq 0 \mid \neg (v_e - a_e / a_o * \text{other.s}' \delta \geq 0)$  **by** *auto*

**thus** *?thesis*

**proof** (*cases*)

**case** 1

**hence**  $v_e - a_e / a_o * (v_o + a_o * \delta) \geq 0$  **unfolding** *other.s'-def other.p'-def* **by** (*auto simp add:assms(1)*)

**hence**  $v_e - a_e / a_o * v_o - a_e * \delta \geq 0$  (**is**  $?l0 \geq 0$ ) **using** *decelerate-other* **by** (*auto simp add:field-simps*)

**hence**  $?l0 / a_e \leq 0$  **using** *divide-right-mono-neg[OF  $\langle ?l0 \geq 0 \rangle$ ]* *decelerate-ego* **by** *auto*

**hence**  $0 \geq v_e / a_e - v_o / a_o - \delta$  **using** *decelerate-ego* **by** (*auto simp add:field-simps*)

**hence**  $*$ :  $-v_e / a_e \geq -(v_o + a_o * \delta) / a_o$  **using** *decelerate-other* **by** (*auto simp add:field-simps*)

**from** *assms* **have**  $**$ :  $v_o + a_o * \delta \leq v_e$  **unfolding** *other.s'-def other.p'-def* **by** *auto*

**have** *vo-star-nneg*:  $v_o + a_o * \delta \geq 0$

**proof** –

**from** *assms(1)* **have**  $-v_o \leq a_o * \delta$  **unfolding** *other.t-stop-def* **using**

*decelerate-other*  
**by** (*auto simp add:field-simps*)  
**thus** ?thesis **by** *auto*  
**qed**  
**from** *mult-mono*[*OF* \* \*\* -  $\langle 0 \leq v_o + a_o * \delta \rangle$ ]  
**have**  $-(v_o + a_o * \delta) / a_o * (v_o + a_o * \delta) \leq -v_e / a_e * v_e$  **using** *nonneg-vel-ego*  
*decelerate-ego*  
**by** (*auto simp add:field-simps*)  
**hence**  $-(v_o + a_o * \delta)^2 / a_o \leq -v_e^2 / a_e$  **by** (*auto simp add: field-simps*  
*power-def*)  
**thus** ?thesis **by** (*auto simp add:field-simps*)  
**next**  
**case** 2  
**with** *assms* **have**  $a_o \leq a_e$  **by** *auto*  
**from** *assms*(2) **have**  $(v_o + a_o * \delta) \leq v_e$  **unfolding** *other.s'-def* **using** *assms*  
**unfolding** *other.p'-def*  
**by** *auto*  
**have** *vo-star-nneg*:  $v_o + a_o * \delta \geq 0$   
**proof** -  
**from** *assms*(1) **have**  $-v_o \leq a_o * \delta$  **unfolding** *other.t-stop-def* **using**  
*decelerate-other*  
**by** (*auto simp add:field-simps*)  
**thus** ?thesis **by** *auto*  
**qed**  
**with** *mult-mono*[*OF*  $\langle v_o + a_o * \delta \leq v_e \rangle$   $\langle v_o + a_o * \delta \leq v_e \rangle$ ] **have** \*:  $(v_o + a_o$   
 $* \delta)^2 \leq v_e^2$   
**using** *nonneg-vel-ego* **by** (*auto simp add:power-def*)  
**from**  $\langle a_o \leq a_e \rangle$  **have**  $-1 / a_o \leq -1 / a_e$  **using** *decelerate-ego* *decelerate-other*  
**by** (*auto simp add:field-simps*)  
**from** *mult-mono*[*OF* \* *this*] **have**  $(v_o + a_o * \delta)^2 * (-1 / a_o) \leq v_e^2 * (-1 /$   
 $a_e)$   
**using** *nonneg-vel-ego* *decelerate-other* **by** (*auto simp add:field-simps*)  
**then show** ?thesis **by** *auto*  
**qed**  
**qed**

**theorem** *distance0-at-most-distance2r*:  
**assumes**  $\delta \leq \text{other.t-stop}$   
**assumes** *other.s'*  $\delta < v_e$   
**assumes**  $\neg (a_o > a_e \wedge \text{other.s}' \delta < v_e \wedge v_e - a_e / a_o * \text{other.s}' \delta < 0)$   
**shows** *distance0*  $\leq$  *safe-distance-2r*  
**proof** -  
**from** *so-star-stop-leq-se-stop*[*OF* *assms*] **have**  $0 \leq -v_e^2 / a_e / 2 + (v_o + a_o$   
 $* \delta)^2 / a_o / 2$  (**is**  $0 \leq$  ?term)  
**by** *auto*  
**have** *safe-distance-2r*  $= v_e * \delta - v_e^2 / 2 / a_e + v_o^2 / 2 / a_o$  **unfolding**  
*safe-distance-2r-def* **by** *auto*  
**also have** ...  $= v_e * \delta - v_e^2 / 2 / a_e + (v_o + a_o * \delta)^2 / 2 / a_o - v_o * \delta - a_o$   
 $* \delta^2 / 2$



**using** *decelerate-other* **by** (*auto simp add:field-simps power-def*)  
**also have**  $\dots = v_e * \delta - v_o * \delta - a_o * \delta^2 / 2 + ?term$  (**is - = ?left + ?term**)  
**by** (*auto simp add:field-simps*)  
**finally have**  $safe\_distance\_2r = distance0 + ?term$  **unfolding** *distance0-def* **by**  
*auto*  
**with**  $\langle 0 \leq ?term \rangle$  **show**  $distance0 \leq safe\_distance\_2r$  **by** *auto*  
**qed**

**theorem** *dist0-sd2r-1*:

**assumes**  $\delta \leq other.t\text{-stop}$   
**assumes**  $\neg (a_o > a_e \wedge other.s' \delta < v_e \wedge v_e - a_e / a_o * other.s' \delta < 0)$   
**assumes**  $s_o - s_e > safe\_distance\_2r$   
**shows**  $s_o - s_e > distance0$   
**proof** (*cases other.s'  $\delta \geq v_e$* )  
**assume**  $v_e \leq other.s' \delta$   
**from** *vo-start-geq-ve*[*OF assms(1) this*] **have**  $u \delta < other.s \delta$  **by** *auto*  
**thus** *?thesis* **unfolding** *distance0-def u-def* **using** *pos-react assms(1) unfolding*  
*ego.q-def*  
*other.s-def other.p-def* **by** *auto*  
**next**  
**assume**  $\neg v_e \leq other.s' \delta$   
**hence**  $v_e > other.s' \delta$  **by** *auto*  
**from** *distance0-at-most-distance2r*[*OF assms(1) this assms(2)*] **have**  $distance0$   
 $\leq safe\_distance\_2r$   
**by** *auto*  
**with** *assms(3)* **show** *?thesis* **by** *auto*  
**qed**

**theorem** *sd2r-eq*:

**assumes**  $\delta > other.t\text{-stop}$   
**shows**  $(u\text{-max} < other.s \delta) = (s_o - s_e > safe\_distance\_2r)$   
**proof** –  
**from** *assms* **have**  $(u\text{-max} < other.s \delta) = (ego2.s (- v_e / a_e) < other.p\text{-max})$   
**using** *u-max-def ego2.t-stop-def u-def other.s-def  $\tau$ -def pos-react ego2.p-max-eq*  
*ego2.s-t-stop u-max-eq* **by** *auto*  
**also have**  $\dots = (s_e + v_e * \delta + v_e * (- v_e / a_e) + 1 / 2 * a_e * (- v_e / a_e)^2 <$   
 $s_o + v_o * (- v_o / a_o) + 1 / 2 * a_o * (- v_o / a_o)^2)$   
**using** *ego2.s-def ego2.p-def ego.q-def other.p-max-def other.p-def other.t-stop-def*  
*ego2.p-max-def ego2.s-t-stop ego2.t-stop-def* **by** *auto*  
**also have**  $\dots = (v_e * \delta + v_e * (- v_e / a_e) + 1 / 2 * a_e * (- v_e / a_e)^2 - v_o * (- v_o / a_o) - 1 / 2 * a_o * (- v_o / a_o)^2 < s_o - s_e)$  **by** *linarith*  
**also have**  $\dots = (v_e * \delta - v_e^2 / a_e + 1 / 2 * v_e^2 / a_e + v_o^2 / a_o - 1 / 2 * v_o^2 / a_o < s_o - s_e)$   
**using** *ego2.p-def ego2.p-max-def ego2.p-max-eq ego2.t-stop-def other.p-def other.p-max-def*  
*other.p-max-eq other.t-stop-def* **by** *auto*  
**also have**  $\dots = (v_e * \delta - v_e^2 / 2 / a_e + v_o^2 / 2 / a_o < s_o - s_e)$  **by** *linarith*  
**thus** *?thesis* **using** *distance0-2-def* **by** (*simp add: calculation safe-distance-2r-def*)  
**qed**

**theorem** *dist0-sd2r-2*:  
**assumes**  $\delta > -v_o / a_o$   
**assumes**  $s_o - s_e > \text{safe-distance-2r}$   
**shows**  $s_o - s_e > \text{distance0-2}$   
**proof** –  
**have**  $-v_e^2 / 2 / a_e \geq 0$  **using** *zero-le-power2 hyps(3) divide-nonneg-neg* **by**  
*(auto simp add:field-simps)*  
**hence**  $v_e * \delta - v_e^2 / 2 / a_e + v_o^2 / 2 / a_o \geq v_e * \delta + v_o^2 / 2 / a_o$  **by** *simp*  
**hence**  $\text{safe-distance-2r} \geq \text{distance0-2}$  **using** *safe-distance-2r-def distance0-2-def*  
**by** *auto*  
**thus** *?thesis* **using** *assms(2)* **by** *linarith*  
**qed**  
**end**

## 2.2 Safe Distance Delta

**locale** *safe-distance-no-collision-delta* = *safe-distance-normal* +  
**assumes** *no-collision-delta*:  $u \delta < \text{other.s} \delta$   
**begin**

**sublocale** *delayed-safe-distance*: *safe-distance*  $a_e v_e \text{ego.q} \delta a_o \text{other.s}' \delta \text{other.s} \delta$   
**proof** (*unfold-locales*)  
**from** *nonneg-vel-ego* **show**  $0 \leq v_e$  **by** *auto*  
**next**  
**from** *nonneg-vel-other* **show**  $0 \leq \text{other.s}' \delta$  **unfolding** *other.s'-def other.p'-def*  
*other.t-stop-def*  
**using** *decelerate-other* **by** (*auto simp add: field-split-simps*)  
**next**  
**from** *decelerate-ego* **show**  $a_e < 0$  **by** *auto*  
**next**  
**from** *decelerate-other* **show**  $a_o < 0$  **by** *auto*  
**next**  
**from** *no-collision-delta* **show**  $\text{ego.q} \delta < \text{other.s} \delta$  **unfolding** *u-def* **using**  
*pos-react* **by** *auto*  
**qed**

**lemma** *no-collision-react-initially-strict*:

**assumes**  $s_o \leq u\text{-max}$   
**assumes**  $u\text{-max} < \text{other.s-stop}$   
**shows** *no-collision-react*  $\{0 < .. < \delta\}$   
**proof** (*rule no-collision-reactI*)  
**fix**  $t::\text{real}$   
**assume**  $t \in \{0 < .. < \delta\}$   
**show**  $u t \neq \text{other.s} t$   
**proof** (*rule ccontr*)  
**assume**  $\neg u t \neq \text{other.s} t$   
**hence** *ego-other2*  $t = 0$  **unfolding** *ego-other2-def* **by** *auto*  
**from**  $\langle t \in \{0 < .. < \delta\} \rangle$  **have** *ego-other2*  $t = \text{other.s} t - \text{ego.q} t$   
**unfolding** *ego-other2-def u-def* **using** *ego.init-q* **by** *auto*

**have**  $\delta \leq \text{other.t-stop} \vee \text{other.t-stop} < \delta$  **by** *auto*

**moreover**

**{** **assume** *le-t-stop*:  $\delta \leq \text{other.t-stop}$

**with**  $\langle \text{ego-other2 } t = \text{other.s } t - \text{ego.q } t \rangle$  **have**  $\text{ego-other2 } t = \text{other.p } t - \text{ego.q } t$

**unfolding** *other.s-def* **using**  $\langle t \in \{0 <..< \delta\} \rangle$  **by** *auto*

**with**  $\langle \text{ego-other2 } t = 0 \rangle$  **have**  $\text{other.p } t - \text{ego.q } t = 0$  **by** *auto*

**hence**  $\text{eq}: (s_o - s_e) + (v_o - v_e) * t + (1/2 * a_o) * t^2 = 0$

**unfolding** *other.p-def* *ego.q-def* **by** (*auto simp: algebra-simps*)

**define** *p* **where**  $p \equiv \lambda x. (1/2 * a_o) * x^2 + (v_o - v_e) * x + (s_o - s_e)$

**have**  $0 < 1/2 * a_o$

**proof** (*intro p-convex* [**where**  $p=p$  **and**  $b=v_o - v_e$  **and**  $c=s_o - s_e$ ])

**show**  $0 < t$  **using**  $\langle t \in \{0 <..< \delta\} \rangle$  **by** *auto*

**next**

**show**  $t < \delta$  **using**  $\langle t \in \{0 <..< \delta\} \rangle$  **by** *auto*

**next**

**show**  $p \ t < p \ 0$  **unfolding** *p-def* **using** *eq in-front* **by** (*auto simp: algebra-simps*)

**next**

**from** *eq* **have**  $p \ t = 0$  **unfolding** *p-def* **by** *auto*

**also** **have**  $\dots < p \ \delta$  **using** *no-collision-delta pos-react le-t-stop*

**unfolding** *p-def* *u-def* *other.s-def* *ego.q-def* *other.p-def* **by** (*auto simp: algebra-simps*)

**finally** **have**  $p \ t < p \ \delta$  **by** *simp*

**thus**  $p \ t \leq p \ \delta$  **by** *auto*

**next**

**show**  $p = (\lambda x. 1 / 2 * a_o * x^2 + (v_o - v_e) * x + (s_o - s_e))$  **unfolding**

*p-def*

**by** (*rule refl*)

**qed**

**hence**  $0 < a_o$  **by** *auto*

**with** *decelerate-other* **have** *False* **by** *simp* **}**

**moreover**

**{** **assume** *gt-t-stop*:  $\delta > \text{other.t-stop}$

**have** *lt-t-stop*:  $t < \text{other.t-stop}$

**proof** (*rule ccontr*)

**assume**  $\neg t < \text{other.t-stop}$

**hence**  $\text{other.t-stop} \leq t$  **by** *simp*

**from**  $\langle \text{ego-other2 } t = 0 \rangle$  **have**  $\text{ego.q } t = \text{other.p-max}$

**unfolding** *ego-other2-def* *u-def* *other.s-def* *comp-def* *tau-def* *other.p-max-def*

**using**  $\langle t \in \{0 <..< \delta\} \rangle$   $\langle \text{other.t-stop} \leq t \rangle$  *gt-t-stop* **by** (*auto split:if-splits*)

**have**  $\text{ego.q } t = u \ t$  **unfolding** *u-def* **using**  $\langle t \in \{0 <..< \delta\} \rangle$  **by** *auto*

**also** **have**  $\dots \leq u\text{-max}$  **using** *u-max* **by** *auto*

**also** **have**  $\dots < \text{other.p-max}$  **using** *assms(2)* *other.s-t-stop* **by** *auto*

**finally** **have**  $\text{ego.q } t < \text{other.p-max}$  **by** *auto*

**with**  $\langle \text{ego.q } t = \text{other.p-max} \rangle$  **show** *False* **by** *auto*

**qed**

**with**  $\langle \text{ego-other2 } t = \text{other.s } t - \text{ego.q } t \rangle$  **have**  $\text{ego-other2 } t = \text{other.p } t - \text{ego.q } t$   
**unfolding**  $\text{other.s-def}$  **using**  $\langle t \in \{0 <..< \delta\} \rangle$  **by** *auto*  
**with**  $\langle \text{ego-other2 } t = 0 \rangle$  **have**  $\text{other.p } t - \text{ego.q } t = 0$  **by** *auto*  
**hence**  $\text{eq: } (s_o - s_e) + (v_o - v_e) * t + (1/2 * a_o) * t^2 = 0$   
**unfolding**  $\text{other.p-def}$   $\text{ego.q-def}$  **by** (*auto simp: algebra-simps*)  
**define**  $p$  **where**  $p \equiv \lambda x. (1/2 * a_o) * x^2 + (v_o - v_e) * x + (s_o - s_e)$   
**have**  $0 < 1/2 * a_o$   
**proof** (*intro p-convex[where p=p and b=v\_o - v\_e and c=s\_o - s\_e]*)  
**show**  $0 < t$  **using**  $\langle t \in \{0 <..< \delta\} \rangle$  **by** *auto*  
**next**  
**show**  $t < \text{other.t-stop}$  **using**  $t\text{-lt-t-stop}$  **by** *auto*  
**next**  
**show**  $p \ t < p \ 0$  **unfolding**  $p\text{-def}$  **using**  $\text{eq in-front}$  **by** (*auto simp: algebra-simps*)  
**next**  
**from**  $\text{eq}$  **have**  $\text{zero: } p \ t = 0$  **unfolding**  $p\text{-def}$  **by** *auto*  
**have**  $\text{eq: } p \ \text{other.t-stop} = \text{ego-other2 } \text{other.t-stop}$   
**unfolding**  $\text{ego-other2-def}$   $\text{other.s-t-stop}$   $u\text{-def}$   $\text{ego.q-def}$   
 $\text{other.s-def}$   $\text{other.p-def}$   $p\text{-def}$   
**using**  $\langle \delta > \text{other.t-stop} \rangle$   $\text{other.t-stop-nonneg}$   $\text{other.t-stop-def}$   
**by** (*auto simp: diff-divide-distrib left-diff-distrib'*)  
**have**  $u \ \text{other.t-stop} \leq u\text{-max}$  **using**  $u\text{-max}$  **by** *auto*  
**also** **have**  $\dots < \text{other.s-stop}$  **using**  $\text{assms}$  **by** *auto*  
**finally** **have**  $0 \leq \text{other.s-stop} - u \ \text{other.t-stop}$  **by** *auto*  
**hence**  $0 \leq \text{ego-other2 } \text{other.t-stop}$  **unfolding**  $\text{ego-other2-def}$  **by** *auto*  
**hence**  $0 \leq p \ \text{other.t-stop}$  **using**  $\text{eq}$  **by** *auto*  
**with**  $\text{zero}$  **show**  $p \ t \leq p \ \text{other.t-stop}$  **by** *auto*  
**next**  
**show**  $p = (\lambda x. 1 / 2 * a_o * x^2 + (v_o - v_e) * x + (s_o - s_e))$   
**unfolding**  $p\text{-def}$  **by** (*rule refl*)  
**qed**  
**hence** *False* **using**  $\text{decelerate-other}$  **by** *auto* }

**ultimately show** *False* **by** *auto*

**qed**

**qed**

**lemma** *no-collision-react-initially:*

**assumes**  $s_o \leq u\text{-max}$

**assumes**  $u\text{-max} < \text{other.s-stop}$

**shows**  $\text{no-collision-react } \{0 .. \delta\}$

**proof** –

**have**  $\text{no-collision-react } \{0 <..< \delta\}$  **by** (*rule no-collision-react-initially-strict[OF assms]*)

**have**  $u \ 0 \neq \text{other.s } 0$  **using**  $\text{init-u}$   $\text{other.init-s}$  *in-front* **by** *auto*

**hence**  $\text{no-collision-react } \{0\}$  **unfolding**  $\text{collision-react-def}$  **by** *auto*

**with**  $\langle \text{no-collision-react } \{0 <..< \delta\} \rangle$  **have**  $\text{no-collision-react } (\{0\} \cup \{0 <..< \delta\})$

using *no-collision-union*[of  $\{0\}$   $\{0 <..] by *auto*  
 moreover have  $\{0\} \cup \{0 <.. using *pos-react* by *auto*  
 ultimately have *no-collision-react*  $\{0 ..<delta\}$  by *auto*$$

have  $u \delta \neq other.s \delta$  using *no-collision-delta* by *auto*  
 hence *no-collision-react*  $\{\delta\}$  unfolding *collision-react-def* by *auto*  
 with  $\langle no-collision-react \{0 ..<delta\} \rangle$  have *no-collision-react*  $(\{\delta\} \cup \{0 ..<delta\})$   
 using *no-collision-union*[of  $\{\delta\}$   $\{0 ..<delta\}$ ] by *auto*  
 moreover have  $\{\delta\} \cup \{0 ..<delta\} = \{0 ..\delta\}$  using *pos-react* by *auto*  
 ultimately show *no-collision-react*  $\{0 ..\delta\}$  by *auto*  
 qed

lemma *collision-after-delta*:

assumes  $s_o \leq u-max$   
 assumes  $u-max < other.s-stop$   
 shows *collision-react*  $\{0 ..\}$   $\longleftrightarrow$  *collision-react*  $\{\delta ..\}$

proof

assume *collision-react*  $\{0 ..\}$   
 have *no-collision-react*  $\{0 ..\delta\}$  by (rule *no-collision-react-initially*[*OF assms*])  
 with  $\langle collision-react \{0 ..\} \rangle$  have *collision-react*  $(\{0 ..\} - \{0 ..\delta\})$   
 using *pos-react* by (auto intro: *collision-trim-subset*)

moreover have  $\{0 ..\} - \{0 ..\delta\} = \{\delta <..\}$  using *pos-react* by *auto*  
 ultimately have *collision-react*  $\{\delta <..\}$  by *auto*  
 thus *collision-react*  $\{\delta ..\}$  by (auto intro: *collision-react-subset*)

next

assume *collision-react*  $\{\delta ..\}$   
 moreover have  $\{\delta ..\} \subseteq \{0 ..\}$  using *pos-react* by *auto*  
 ultimately show *collision-react*  $\{0 ..\}$  by (rule *collision-react-subset*)

qed

lemma *collision-react-strict*:

assumes  $s_o \leq u-max$   
 assumes  $u-max < other.s-stop$   
 shows *collision-react*  $\{\delta ..\}$   $\longleftrightarrow$  *collision-react*  $\{\delta <..\}$

proof

assume *asm*: *collision-react*  $\{\delta ..\}$   
 have *no-collision-react*  $\{\delta\}$  using *no-collision-delta* unfolding *collision-react-def*  
 by *auto*  
 moreover have  $\{\delta <..\} \subseteq \{\delta ..\}$  by *auto*  
 ultimately have *collision-react*  $(\{\delta ..\} - \{\delta\})$  using *asm* *collision-trim-subset*  
 by *simp*  
 moreover have  $\{\delta <..\} = \{\delta ..\} - \{\delta\}$  by *auto*  
 ultimately show *collision-react*  $\{\delta <..\}$  by *auto*

next

assume *collision-react*  $\{\delta <..\}$   
 thus *collision-react*  $\{\delta ..\}$   
 using *collision-react-subset*[where  $t=\{\delta ..\}$  and  $s=\{\delta <..\}$ ] by *fastforce*

qed

**lemma** *delayed-other-s-stop-eq*:  $\text{delayed-safe-distance.other.s-stop} = \text{other.s-stop}$   
**proof** (*unfold other.s-t-stop*; *unfold delayed-safe-distance.other.s-t-stop*; *unfold movement.p-max-eq*)

**have**  $\delta \leq \text{other.t-stop} \vee \text{other.t-stop} < \delta$  **by** *auto*

**moreover**

{ **assume**  $\delta \leq \text{other.t-stop}$   
**hence**  $\text{other.s } \delta - (\text{other.s}' \delta)^2 / a_o / 2 = s_o - v_o^2 / a_o / 2$   
**unfolding** *other.s-def other.s'-def*  
**using** *pos-react decelerate-other*  
**by** (*auto simp add: other.p-def other.p'-def power2-eq-square field-split-simps*)  
}

**moreover**

{ **assume**  $\text{other.t-stop} < \delta$   
**hence**  $\text{other.s } \delta - (\text{other.s}' \delta)^2 / a_o / 2 = s_o - v_o^2 / a_o / 2$   
**unfolding** *other.s-def other.s'-def other.p-max-eq*  
**using** *pos-react decelerate-other*  
**by** (*auto*) }

**ultimately show**  $\text{other.s } \delta - (\text{other.s}' \delta)^2 / a_o / 2 = s_o - v_o^2 / a_o / 2$  **by**  
*auto*  
**qed**

**lemma** *delayed-cond3'*:

**assumes**  $\text{other.s } \delta \leq u\text{-max}$

**assumes**  $u\text{-max} < \text{other.s-stop}$

**shows**  $\text{delayed-safe-distance.collision } \{0 \dots\} \longleftrightarrow$

$(a_o > a_e \wedge \text{other.s}' \delta < v_e \wedge \text{other.s } \delta - \text{ego.q } \delta \leq \text{delayed-safe-distance.snd-safe-distance}$   
 $\wedge v_e - a_e / a_o * \text{other.s}' \delta < 0)$

**proof** (*rule delayed-safe-distance.cond-3'*)

**have**  $\text{other.s } \delta \leq u\text{-max}$  **using**  $\langle \text{other.s } \delta \leq u\text{-max} \rangle$  .

**also have**  $\dots = \text{ego2.s-stop}$  **unfolding** *u-max-eq ego2.s-t-stop ego2.p-max-eq*

**by** (*rule refl*)

**finally show**  $\text{other.s } \delta \leq \text{ego2.s-stop}$  **by** *auto*

**next**

**have**  $\text{ego2.s-stop} = u\text{-max}$  **unfolding** *ego2.s-t-stop ego2.p-max-eq u-max-eq* **by**  
(*rule refl*)

**also have**  $\dots < \text{other.s-stop}$  **using** *assms* **by** *auto*

**also have**  $\dots \leq \text{delayed-safe-distance.other.s-stop}$  **using** *delayed-other-s-stop-eq*

**by** *auto*

**finally show**  $\text{ego2.s-stop} < \text{delayed-safe-distance.other.s-stop}$  **by** *auto*

**qed**

**lemma** *delayed-other-t-stop-eq*:

**assumes**  $\delta \leq \text{other.t-stop}$

**shows**  $\text{delayed-safe-distance.other.t-stop} + \delta = \text{other.t-stop}$

**using** *assms decelerate-other*

**unfolding** *delayed-safe-distance.other.t-stop-def other.t-stop-def other.s'-def  
movement.t-stop-def other.p'-def*  
**by** (*auto simp add: field-split-simps*)

**lemma** *delayed-other-s-eq:*

**assumes**  $0 \leq t$

**shows** *delayed-safe-distance.other.s t = other.s (t +  $\delta$ )*

**proof** (*cases  $\delta \leq other.t-stop$* )

**assume** *1:  $\delta \leq other.t-stop$*

**have**  $t + \delta \leq other.t-stop \vee other.t-stop < t + \delta$  **by** *auto*

**moreover**

{ **assume**  $t + \delta \leq other.t-stop$

**hence** *delayed-safe-distance.other.s t = delayed-safe-distance.other.p t*

**using** *delayed-other-t-stop-eq [OF 1] assms*

**unfolding** *delayed-safe-distance.other.s-def* **by** *auto*

**also have**  $\dots = other.p (t + \delta)$

**unfolding** *movement.p-def other.s-def other.s'-def other.p'-def*

**using** *pos-react 1*

**by** (*auto simp add: power2-eq-square field-split-simps*)

**also have**  $\dots = other.s (t + \delta)$

**unfolding** *other.s-def*

**using** *assms pos-react  $\langle t + \delta \leq other.t-stop \rangle$*  **by** *auto*

**finally have** *delayed-safe-distance.other.s t = other.s (t +  $\delta$ )* **by** *auto* }

**moreover**

{ **assume**  $other.t-stop < t + \delta$

**hence** *delayed-safe-distance.other.s t = delayed-safe-distance.other.p-max*

**using** *delayed-other-t-stop-eq [OF 1] assms delayed-safe-distance.other.t-stop-nonneg*

**unfolding** *delayed-safe-distance.other.s-def* **by** *auto*

**also have**  $\dots = other.p-max$

**unfolding** *movement.p-max-eq other.s-def other.s'-def other.p-def other.p'-def*

**using** *pos-react 1 decelerate-other*

**by** (*auto simp add: power2-eq-square field-split-simps*)

**also have**  $\dots = other.s (t + \delta)$

**unfolding** *other.s-def*

**using** *assms pos-react  $\langle other.t-stop < t + \delta \rangle$*  **by** *auto*

**finally have** *delayed-safe-distance.other.s t = other.s (t +  $\delta$ )* **by** *auto* }

**ultimately show** *?thesis* **by** *auto*

**next**

**assume**  $\neg \delta \leq other.t-stop$

**hence**  $other.t-stop < \delta$  **by** *auto*

**hence**  $other.s' \delta = 0$  **and**  $other.s \delta = other.p-max$

**unfolding** *other.s'-def other.s-def* **using** *pos-react* **by** *auto*  
**hence** *delayed-safe-distance.other.s t = delayed-safe-distance.other.p-max*  
**unfolding** *delayed-safe-distance.other.s-def* **using** *assms decelerate-other*  
**by** (*auto simp add:movement.p-max-eq movement.p-def movement.t-stop-def*)  
**also have**  $\dots = \textit{other.p-max}$   
**unfolding** *movement.p-max-eq* **using**  $\langle \textit{other.s'} \delta = 0 \rangle \langle \textit{other.s} \delta = \textit{other.p-max} \rangle$   
**using** *other.p-max-eq* **by** *auto*  
**also have**  $\dots = \textit{other.s} (t + \delta)$   
**unfolding** *other.s-def* **using** *pos-react assms*  $\langle \textit{other.t-stop} < \delta \rangle$  **by** *auto*  
**finally show** *delayed-safe-distance.other.s t = other.s (t +  $\delta$ )* **by** *auto*  
**qed**

**lemma** *translate-collision-range:*

**assumes**  $s_o \leq u\text{-max}$

**assumes**  $u\text{-max} < \textit{other.s-stop}$

**shows** *delayed-safe-distance.collision*  $\{0 \dots\} \longleftrightarrow \textit{collision-react} \{\delta \dots\}$

**proof**

**assume** *delayed-safe-distance.collision*  $\{0 \dots\}$

**then obtain**  $t$  **where** *eq:*  $\textit{ego2.s} t = \textit{delayed-safe-distance.other.s} t$  **and**  $0 \leq t$

**unfolding** *delayed-safe-distance.collision-def* **by** *auto*

**have**  $\textit{ego2.s} t = (\textit{ego2.s} \circ \tau) (t + \delta)$  **unfolding** *comp-def  $\tau$ -def* **by** *auto*

**also have**  $\dots = u (t + \delta)$  **unfolding** *u-def* **using**  $\langle 0 \leq t \rangle$  *pos-react*

**by** (*auto simp:  $\tau$ -def ego2.init-s*)

**finally have** *left:ego2.s t = u (t +  $\delta$ )* **by** *auto*

**have** *right: delayed-safe-distance.other.s t = other.s (t +  $\delta$ )*

**using** *delayed-other-s-eq pos-react*  $\langle 0 \leq t \rangle$  **by** *auto*

**with** *eq* **and** *left* **have**  $u (t + \delta) = \textit{other.s} (t + \delta)$  **by** *auto*

**moreover have**  $\delta \leq t + \delta$  **using**  $\langle 0 \leq t \rangle$  **by** *auto*

**ultimately show** *collision-react*  $\{\delta \dots\}$  **unfolding** *collision-react-def* **by** *auto*

**next**

**assume** *collision-react*  $\{\delta \dots\}$

**hence** *collision-react*  $\{\delta < \dots\}$  **using** *collision-react-strict[OF assms]* **by** *simp*

**then obtain**  $t$  **where** *eq:*  $u t = \textit{other.s} t$  **and**  $\delta < t$

**unfolding** *collision-react-def* **by** *auto*

**moreover hence**  $u t = (\textit{ego2.s} \circ \tau) t$  **unfolding** *u-def* **using** *pos-react* **by** *auto*

**moreover have**  $\textit{other.s} t = \textit{delayed-safe-distance.other.s} (t - \delta)$

**using** *delayed-other-s-eq*  $\langle \delta < t \rangle$  **by** *auto*

**ultimately have**  $\textit{ego2.s} (t - \delta) = \textit{delayed-safe-distance.other.s} (t - \delta)$

**unfolding** *comp-def  $\tau$ -def* **by** *auto*

**with**  $\langle \delta < t \rangle$  **show** *delayed-safe-distance.collision*  $\{0 \dots\}$

**unfolding** *delayed-safe-distance.collision-def* **by** *auto*

**qed**

**theorem** *cond-3r-2:*

**assumes**  $s_o \leq u\text{-max}$

**assumes**  $u\text{-max} < \textit{other.s-stop}$



**assumes**  $other.s \delta \leq u-max$   
**shows**  $collision-react \{0 ..\} \longleftrightarrow$   
 $(a_o > a_e \wedge other.s' \delta < v_e \wedge other.s \delta - ego.q \delta \leq delayed-safe-distance.snd-safe-distance$   
 $\wedge v_e - a_e / a_o * other.s' \delta < 0)$   
**proof** –  
**have**  $collision-react \{0 ..\} \longleftrightarrow collision-react \{\delta ..\}$  **by** (rule *collision-after-delta*[*OF*  
*assms(1) assms(2)*])  
**also have** ...  $\longleftrightarrow delayed-safe-distance.collision \{0 ..\}$  **by** (*simp add: trans-*  
*late-collision-range*[*OF assms(1) assms(2)*])  
**also have** ...  $\longleftrightarrow (a_o > a_e \wedge other.s' \delta < v_e \wedge other.s \delta - ego.q \delta \leq de-$   
 $layed-safe-distance.snd-safe-distance \wedge v_e - a_e / a_o * other.s' \delta < 0)$   
**by** (rule *delayed-cond3'*[*OF assms(3) assms(2)*])  
**finally show**  $collision-react \{0 ..\} \longleftrightarrow (a_o > a_e \wedge other.s' \delta < v_e \wedge other.s \delta$   
 $- ego.q \delta \leq delayed-safe-distance.snd-safe-distance \wedge v_e - a_e / a_o * other.s' \delta <$   
 $0)$   
**by** *auto*  
**qed**

**lemma** *sd-2r-correct-for-3r-2*:

**assumes**  $s_o - s_e > safe-distance-2r$   
**assumes**  $other.s \delta \leq u-max$   
**assumes**  $\neg (a_o > a_e \wedge other.s' \delta < v_e \wedge v_e - a_e / a_o * other.s' \delta < 0)$   
**shows**  $no-collision-react \{0.. \}$   
**proof** –  
**from** *assms* **have**  $s_o - s_e > v_e * \delta - v_e^2 / 2 / a_e + v_o^2 / 2 / a_o$  **unfolding**  
*safe-distance-2r-def* **by** *auto*  
**hence**  $s_o - v_o^2 / 2 / a_o > s_e + v_e * \delta - v_e^2 / 2 / a_e$  **by** *auto*  
**hence**  $s_o - v_o^2 / a_o + v_o^2 / 2 / a_o > s_e + v_e * \delta - v_e^2 / 2 / a_e$  **by** *auto*  
**hence**  $s_o + v_o * (-v_o / a_o) + 1/2 * a_o * (-v_o / a_o)^2 > s_e + v_e * \delta - v_e^2 /$   
 $2 / a_e$   
**using** *other.p-def other.p-max-def other.p-max-eq other.t-stop-def* **by** *auto*  
**hence**  $other.s-stop > u-max$  **unfolding** *other.s-def* **using** *u-max-eq other.t-stop-def*  
**using** *ego.q-def other.p-def other.p-max-def other.s-def other.s-t-stop* **by** *auto*  
**thus** *?thesis*  
**using** *assms(2) assms(3) collision-after-delta cond-1r delayed-cond3' trans-*  
*late-collision-range* **by** *linarith*  
**qed**

**lemma** *sd2-at-most-sd4*:

**assumes**  $a_o > a_e$   
**shows**  $safe-distance-2r \leq safe-distance-4r$   
**proof** –  
**have**  $a_o \neq 0$  **and**  $a_e \neq 0$  **and**  $a_o - a_e \neq 0$  **and**  $0 < 2 * (a_o - a_e)$  **using** *hyps*  
*assms(1)* **by** *auto*  
**have**  $0 \leq (-v_e * a_o + v_o * a_e + a_o * a_e * \delta) * (-v_e * a_o + v_o * a_e + a_o * a_e * \delta)$   
**(is**  $0 \leq (?l1 + ?l2 + ?l3) * ?r$ ) **by** *auto*  
**also have** ...  $= v_e^2 * a_o^2 + v_o^2 * a_e^2 + a_o^2 * a_e^2 * \delta^2 - 2 * v_e * a_o * v_o * a_e$   
 $- 2 * a_o^2 * a_e * \delta * v_e + 2 * a_o * a_e^2 * \delta * v_o$

(is ?lhs = ?rhs)  
 by (auto simp add: algebra-simps power-def)  
 finally have  $0 \leq ?rhs$  by auto  
 hence  $(-v_e^2 * a_o / a_e - v_o^2 * a_e / a_o) * (a_o * a_e) \leq (a_o * a_e * \delta^2 - 2 * v_e * v_o - 2 * a_o * \delta * v_e + 2 * a_e * \delta * v_o) * (a_o * a_e)$   
 by (auto simp add: algebra-simps power-def)  
 hence  $2 * v_e * \delta * (a_o - a_e) - v_e^2 * a_o / a_e + v_e^2 + v_o^2 - v_o^2 * a_e / a_o \leq v_o^2 + a_o^2 * \delta^2 + v_e^2 + 2 * v_o * \delta * a_o - 2 * v_e * v_o - 2 * a_o * \delta * v_e - 2 * v_o * \delta * a_o + 2 * a_e * \delta * v_o - a_o^2 * \delta^2 + a_o * a_e * \delta^2 + 2 * v_e * \delta * (a_o - a_e)$   
 by (auto simp add: ego2.decel other.decel)  
 hence  $2 * v_e * \delta * (a_o - a_e) - v_e^2 * a_o / a_e + v_e^2 + v_o^2 - v_o^2 * a_e / a_o \leq (v_o + \delta * a_o - v_e)^2 - 2 * v_o * \delta * a_o + 2 * a_e * \delta * v_o - a_o^2 * \delta^2 + a_o * a_e * \delta^2 + 2 * v_e * \delta * (a_o - a_e)$   
 by (auto simp add: algebra-simps power-def)  
 hence  $v_e * \delta * 2 * (a_o - a_e) - v_e^2 / 2 / a_e * 2 * a_o + v_e^2 / 2 / a_e * 2 * a_e + v_o^2 / 2 / a_o * 2 * a_o - v_o^2 / 2 / a_o * 2 * a_e \leq (v_o + \delta * a_o - v_e)^2 - v_o * \delta * 2 * a_o - v_o * \delta * 2 * -a_e - a_o * \delta^2 / 2 * 2 * a_o - a_o * \delta^2 / 2 * 2 * -a_e + v_e * \delta * 2 * (a_o - a_e)$   
 (is ?lhs1  $\leq$  ?rhs1)  
 by (simp add:  $\langle a_o \neq 0 \rangle \langle a_e \neq 0 \rangle$  power2-eq-square algebra-simps)  
 hence  $v_e * \delta * 2 * (a_o - a_e) - v_e^2 / 2 / a_e * 2 * (a_o - a_e) + v_o^2 / 2 / a_o * 2 * (a_o - a_e) \leq (v_o + a_o * \delta - v_e)^2 / 2 / (a_o - a_e) * 2 * (a_o - a_e) - v_o * \delta * 2 * (a_o - a_e) - a_o * \delta^2 / 2 * 2 * (a_o - a_e) + v_e * \delta * 2 * (a_o - a_e)$   
 (is ?lhs2  $\leq$  ?rhs2)  
 proof -  
 assume ?lhs1  $\leq$  ?rhs1  
 have ?lhs1 = ?lhs2 by (auto simp add: field-simps)  
 moreover  
 have ?rhs1 = ?rhs2 using  $\langle a_o - a_e \neq 0 \rangle$  by (auto simp add: field-simps)  
 ultimately show ?thesis using  $\langle ?lhs1 \leq ?rhs1 \rangle$  by auto  
 qed  
 hence  $(v_e * \delta - v_e^2 / 2 / a_e + v_o^2 / 2 / a_o) * 2 * (a_o - a_e) \leq ((v_o + a_o * \delta - v_e)^2 / 2 / (a_o - a_e) - v_o * \delta - 1/2 * a_o * \delta^2 + v_e * \delta) * 2 * (a_o - a_e)$   
 by (simp add: algebra-simps)  
 hence  $v_e * \delta - v_e^2 / 2 / a_e + v_o^2 / 2 / a_o \leq (v_o + a_o * \delta - v_e)^2 / 2 / (a_o - a_e) - v_o * \delta - 1/2 * a_o * \delta^2 + v_e * \delta$   
 using  $\langle a_o > a_e \rangle$  mult-le-cancel-iff1[OF  $\langle 0 < 2 * (a_o - a_e) \rangle$ , of  $(v_e * \delta - v_e^2 / 2 / a_e + v_o^2 / 2 / a_o)$   
 $(v_o + a_o * \delta - v_e)^2 / 2 / (a_o - a_e) - v_o * \delta - 1/2 * a_o * \delta^2 + v_e * \delta]$   
 semiring-normalization-rules(18)  
 by (metis (no-types, lifting))  
 thus ?thesis using safe-distance-2r-def safe-distance-4r-def by auto  
 qed

lemma sd-4r-correct:

assumes  $s_o - s_e > \text{safe-distance-4r}$   
 assumes other.s  $\delta \leq u\text{-max}$   
 assumes  $\delta \leq \text{other.t-stop}$   
 assumes  $a_o > a_e$

**shows** *no-collision-react* {0..}  
**proof** –  
**from** *assms* **have**  $s_o - s_e > (v_o + a_o * \delta - v_e)^2 / 2 / (a_o - a_e) - v_o * \delta - 1/2 * a_o * \delta^2 + v_e * \delta$   
**unfolding** *safe-distance-4r-def* **by** *auto*  
**hence**  $s_o + v_o * \delta + 1/2 * a_o * \delta^2 - s_e - v_e * \delta > (v_o + a_o * \delta - v_e)^2 / 2 / (a_o - a_e)$  **by** *linarith*  
**hence** *other.s*  $\delta -$  *ego.q*  $\delta > (other.s' \delta - v_e)^2 / 2 / (a_o - a_e)$   
**using** *assms(3)* *ego.q-def* *other.p-def* *other.s-def* *other.p'-def* *other.s'-def* *pos-react*  
**by** *auto*  
**hence** *other.s*  $\delta -$  *ego.q*  $\delta >$  *delayed-safe-distance.snd-safe-distance*  
**by** (*simp add: delayed-safe-distance.snd-safe-distance-def*)  
**hence**  $c: \neg (other.s \delta - ego.q \delta \leq delayed-safe-distance.snd-safe-distance)$  **by** *linarith*  
**have**  $u-max < other.s-stop$   
**unfolding** *u-max-eq* *other.s-t-stop* *other.p-max-eq* *ego.q-def* **using** *assms(1)*  
*sd2-at-most-sd4* [*OF* *assms(4)*]  
**unfolding** *safe-distance-4r-def* *safe-distance-2r-def* **by** *auto*  
**consider**  $s_o \leq u-max \mid s_o > u-max$  **by** *linarith*  
**thus** *?thesis*  
**proof** (*cases*)  
**case** 1  
**from** *cond-3r-2* [*OF* *this*  $\langle u-max < other.s-stop \rangle$  *assms(2)*] **show** *?thesis*  
**using** *c* **by** *auto*  
**next**  
**case** 2  
**then show** *?thesis* **using** *cond-1r* **by** *auto*  
**qed**  
**qed**

Irrelevant, this Safe Distance is unreachable in the checker.

**definition** *safe-distance-5r* :: *real* **where**  
 $safe-distance-5r = v_e^2 / 2 / (a_o - a_e) + v_o^2 / 2 / a_o + v_e * \delta$

**lemma** *sd-5r-correct*:

**assumes**  $s_o - s_e > safe-distance-5r$   
**assumes**  $u-max < other.s-stop$   
**assumes**  $other.s \delta \leq u-max$   
**assumes**  $\delta > other.t-stop$   
**shows** *no-collision-react* {0..}  
**proof** –  
**from** *assms* **have**  $s_o - s_e > v_e^2 / 2 / (a_o - a_e) + v_o^2 / 2 / a_o + v_e * \delta$   
**unfolding** *safe-distance-5r-def* **by** *auto*  
**hence**  $s_o + v_o^2 / 2 / a_o - s_e - v_e * \delta > (0 - v_e)^2 / 2 / (a_o - a_e)$   
**using** *assms(2-4)* **unfolding** *other.s-def* *other.s-t-stop*  
**apply** (*auto simp: movement.p-t-stop split: if-splits*)  
**using** *cond-1r* *cond-2r* *other.s-t-stop* **by** *linarith* +  
**hence** *other.s*  $\delta -$  *ego.q*  $\delta > (other.s' \delta - v_e)^2 / 2 / (a_o - a_e)$   
**using** *assms(2)* *assms(3)* *assms(4)* *other.s-def* *other.s-t-stop* **by** *auto*

**hence**  $other.s \delta - ego.q \delta > delayed-safe-distance.snd-safe-distance$   
**by** (*simp add: delayed-safe-distance.snd-safe-distance-def*)  
**hence**  $\neg (other.s \delta - ego.q \delta \leq delayed-safe-distance.snd-safe-distance)$  **by**  
*linarith*  
**thus** *?thesis using assms(2) assms(3) cond-1r cond-3r-2 by linarith*  
**qed**

**lemma** *translate-no-collision-range:*

$delayed-safe-distance.no-collision \{0 \dots\} \longleftrightarrow no-collision-react \{\delta \dots\}$

**proof**

**assume** *left: delayed-safe-distance.no-collision {0 ..}*

**show** *no-collision-react {δ ..}*

**proof** (*unfold collision-react-def; simp; rule ballI*)

**fix** *t::real*

**assume**  $t \in \{\delta \dots\}$

**hence**  $\delta \leq t$  **by** *simp*

**with** *pos-react* **have**  $0 \leq t - \delta$  **by** *simp*

**with** *left* **have** *ineq: ego2.s (t - δ) ≠ delayed-safe-distance.other.s (t - δ)*

**unfolding** *delayed-safe-distance.collision-def* **by** *auto*

**have**  $ego2.s (t - \delta) = (ego2.s \circ \tau) t$  **unfolding** *comp-def τ-def* **by** *auto*

**also have**  $\dots = u t$  **unfolding** *u-def* **using**  $\langle \delta \leq t \rangle$  *pos-react*

**by** (*auto simp: τ-def ego2.init-s*)

**finally have**  $ego2.s (t - \delta) = u t$  **by** *auto*

**moreover have**  $delayed-safe-distance.other.s (t - \delta) = other.s t$

**using** *delayed-other-s-eq pos-react ⟨δ ≤ t⟩* **by** *auto*

**ultimately show**  $u t \neq other.s t$  **using** *ineq* **by** *auto*

**qed**

**next**

**assume** *right: no-collision-react {δ ..}*

**show** *delayed-safe-distance.no-collision {0 ..}*

**proof** (*unfold delayed-safe-distance.collision-def; simp; rule ballI*)

**fix** *t ::real*

**assume**  $t \in \{0 \dots\}$

**hence**  $0 \leq t$  **by** *auto*

**hence**  $\delta \leq t + \delta$  **by** *auto*

**with** *right* **have** *ineq: u (t + δ) ≠ other.s (t + δ)* **unfolding** *collision-react-def*

**by** *auto*

**have**  $u (t + \delta) = ego2.s t$  **unfolding** *u-def comp-def τ-def*

**using**  $\langle 0 \leq t \rangle$  *pos-react ⟨δ ≤ t+ δ⟩* **by** (*auto simp add: ego2.init-s*)

**moreover have**  $other.s (t + \delta) = delayed-safe-distance.other.s t$

**using** *delayed-other-s-eq[of t]* **using**  $\langle 0 \leq t \rangle$  **by** *auto*

**ultimately show**  $ego2.s t \neq delayed-safe-distance.other.s t$  **using** *ineq* **by** *auto*

**qed**

**qed**

**lemma** *delayed-cond1:*

**assumes**  $other.s \delta > u-max$   
**shows**  $delayed-safe-distance.no-collision \{0 ..\}$   
**proof** –  
**have**  $ego2.s-stop = u-max$  **unfolding**  $ego2.s-t-stop$   $ego2.p-max-eq$   $u-max-eq$  **by**  $auto$   
**also have**  $... < other.s \delta$  **using**  $assms$  **by**  $simp$   
**finally have**  $ego2.s-stop < other.s \delta$  **by**  $auto$   
**thus**  $delayed-safe-distance.no-collision \{0 ..\}$  **by** ( $simp$   $add: delayed-safe-distance.cond-1$ )  
**qed**

**theorem**  $cond-3r-3$ :

**assumes**  $s_o \leq u-max$   
**assumes**  $u-max < other.s-stop$   
**assumes**  $other.s \delta > u-max$   
**shows**  $no-collision-react \{0 ..\}$   
**proof** –  
**have**  $eq: \{0 ..\} = \{0 .. \delta\} \cup \{\delta ..\}$  **using**  $pos-react$  **by**  $auto$   
**show**  $?thesis$  **unfolding**  $eq$   
**proof** ( $intro$   $no-collision-union$ )  
**show**  $no-collision-react \{0 .. \delta\}$  **by** ( $rule$   $no-collision-react-initially[OF$   $assms(1)$   $assms(2)]$ )  
**next**  
**have**  $delayed-safe-distance.no-collision \{0 ..\}$  **by** ( $rule$   $delayed-cond1[OF$   $assms(3)]$ )  
**with**  $translate-no-collision-range$  **show**  $no-collision-react \{\delta ..\}$  **by**  $auto$   
**qed**  
**qed**

**lemma**  $sd-2r-correct-for-3r-3$ :

**assumes**  $s_o - s_e > safe-distance-2r$   
**assumes**  $other.s \delta > u-max$   
**shows**  $no-collision-react \{0.. \}$   
**proof** –  
**from**  $assms$  **have**  $s_o - s_e > v_e * \delta - v_e^2 / 2 / a_e + v_o^2 / 2 / a_o$  **unfolding**  $safe-distance-2r-def$  **by**  $auto$   
**hence**  $s_o - v_o^2 / 2 / a_o > s_e + v_e * \delta - v_e^2 / 2 / a_e$  **by**  $auto$   
**hence**  $s_o - v_o^2 / a_o + v_o^2 / 2 / a_o > s_e + v_e * \delta - v_e^2 / 2 / a_e$  **by**  $auto$   
**hence**  $s_o + v_o * (-v_o / a_o) + 1/2 * a_o * (-v_o / a_o)^2 > s_e + v_e * \delta - v_e^2 / 2 / a_e$   
**using**  $other.p-def$   $other.p-max-def$   $other.p-max-eq$   $other.t-stop-def$  **by**  $auto$   
**hence**  $other.s-stop > u-max$  **unfolding**  $other.s-def$  **using**  $u-max-eq$   $other.t-stop-def$   
**using**  $ego.q-def$   $other.p-def$   $other.p-max-def$   $other.s-def$   $other.s-t-stop$  **by**  $auto$   
**thus**  $?thesis$   
**using**  $assms(2)$   $cond-1r$   $cond-3r-3$  **by**  $linarith$   
**qed**

**lemma**  $sd-3r-correct$ :

**assumes**  $s_o - s_e > safe-distance-3r$   
**assumes**  $\delta \leq other.t-stop$   
**shows**  $no-collision-react \{0 ..\}$

**proof** –

**from** *assms* **have**  $s_o - s_e > v_e * \delta - v_e^2 / 2 / a_e - v_o * \delta - 1/2 * a_o * \delta^2$

**unfolding** *safe-distance-3r-def* **by** *auto*

**hence**  $s_o + v_o * \delta + 1/2 * a_o * \delta^2 > s_e + v_e * \delta - v_e^2 / 2 / a_e$  **by** *auto*

**hence** *other.s*  $\delta > u\text{-max}$  **using** *other.s-def* *u-max-eq* *assms(2)* *ego.q-def* *other.p-def*

*pos-react* **by** *auto*

**thus** *?thesis* **using** *cond-1r* *cond-3r-3* *delayed-other-s-stop-eq* *delayed-safe-distance.other.s0-le-s-stop*

**by** *linarith*

**qed**

**lemma** *sd-2-at-least-sd-3*:

**assumes**  $\delta \leq \text{other.t-stop}$

**shows** *safe-distance-3r*  $\geq$  *safe-distance-2r*

**proof** –

**from** *assms* **have**  $\delta = \text{other.t-stop} \vee \delta < \text{other.t-stop}$  **by** *auto*

**then** **have** *safe-distance-3r*  $=$  *safe-distance-2r*  $\vee$  *safe-distance-3r*  $>$  *safe-distance-2r*

**proof** (*rule* *Meson.disj-forward*)

**assume**  $\delta = \text{other.t-stop}$

**hence**  $\delta = -v_o / a_o$  **unfolding** *other.t-stop-def* **by** *auto*

**hence**  $-v_o * \delta - 1/2 * a_o * \delta^2 = -v_o * \text{other.t-stop} - 1/2 * a_o * \text{other.t-stop}^2$  **by** (*simp add: movement.t-stop-def*)

**thus** *safe-distance-3r*  $=$  *safe-distance-2r*

**using** *other.p-def* *other.p-max-def* *other.p-max-eq* *safe-distance-2r-def* *safe-distance-3r-def*

**by** *auto*

**next**

**assume**  $\delta < \text{other.t-stop}$

**hence**  $\delta < -v_o / a_o$  **unfolding** *other.t-stop-def* **by** *auto*

**hence**  $0 < v_o + a_o * \delta$

**using** *other.decel* *other.p'-def* *other.p'-pos-iff* **by** *auto*

**hence**  $0 < v_o + 1/2 * a_o * (\delta + \text{other.t-stop})$  **by** (*auto simp add: field-simps other.t-stop-def*)

**hence**  $0 > v_o * (\delta - \text{other.t-stop}) + 1/2 * a_o * (\delta + \text{other.t-stop}) * (\delta - \text{other.t-stop})$

**using**  $\langle \delta < \text{other.t-stop} \rangle$  *mult-less-cancel-left-neg* [**where**  $c = \delta - \text{other.t-stop}$

**and**  $a = v_o + 1/2 * a_o * (\delta + \text{other.t-stop})$  **and**  $b = 0$ ]

**by** (*auto simp add: field-simps*)

**hence**  $(\delta + \text{other.t-stop}) * (\delta - \text{other.t-stop}) = (\delta^2 - \text{other.t-stop}^2)$

**by** (*simp add: power2-eq-square square-diff-square-factored*)

**hence**  $0 > v_o * (\delta - \text{other.t-stop}) + 1/2 * a_o * (\delta^2 - \text{other.t-stop}^2)$

**by** (*metis (no-types, opaque-lifting) <v\_o \* (\delta - other.t-stop) + 1/2 \* a\_o \* (\delta + other.t-stop) \* (\delta - other.t-stop) < 0>* *divide-divide-eq-left* *divide-divide-eq-right* *times-divide-eq-left*)

**hence**  $0 > v_o * \delta - v_o * \text{other.t-stop} + 1/2 * a_o * \delta^2 - 1/2 * a_o * \text{other.t-stop}^2$

**by** (*simp add: right-diff-distrib*)

**hence**  $-v_o * \delta - 1/2 * a_o * \delta^2 > -v_o * (-v_o / a_o) - 1/2 * a_o * (-v_o / a_o)^2$

**unfolding** *movement.t-stop-def* **by** *argo*

**thus** *safe-distance-3r*  $>$  *safe-distance-2r*

```

using other.p-def other.p-max-def other.p-max-eq other.t-stop-def safe-distance-2r-def
safe-distance-3r-def by auto
qed
thus ?thesis by auto
qed
end

```

## 2.3 Checker Design

We define two checkers for different cases:

- one checker for the case that  $\delta \leq \text{other.t-stop}$  ( $\text{other.t-stop} = -v_o / a_o$ )
- a second checker for the case that  $\delta > \text{other.t-stop}$

**definition** *check-precond-r1* ::  $\text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{bool}$  **where**

*check-precond-r1*  $s_e v_e a_e s_o v_o a_o \delta \longleftrightarrow s_o > s_e \wedge 0 \leq v_e \wedge 0 \leq v_o \wedge a_e < 0 \wedge a_o < 0 \wedge 0 < \delta \wedge \delta \leq -v_o / a_o$

**definition** *safe-distance0* ::  $\text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real}$  **where**

*safe-distance0*  $v_e a_o v_o \delta = v_e * \delta - v_o * \delta - a_o * \delta^2 / 2$

**definition** *safe-distance-1r* ::  $\text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real}$  **where**

*safe-distance-1r*  $a_e v_e \delta = v_e * \delta - v_e^2 / a_e / 2$

**definition** *safe-distance-2r* ::  $\text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real}$  **where**

*safe-distance-2r*  $a_e v_e a_o v_o \delta = v_e * \delta - v_e^2 / 2 / a_e + v_o^2 / 2 / a_o$

**definition** *safe-distance-4r* ::  $\text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real}$  **where**

*safe-distance-4r*  $a_e v_e a_o v_o \delta = (v_o + a_o * \delta - v_e)^2 / 2 / (a_o - a_e) - v_o * \delta - 1 / 2 * a_o * \delta^2 + v_e * \delta$

**definition** *safe-distance-3r* ::  $\text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real}$  **where**

*safe-distance-3r*  $a_e v_e a_o v_o \delta = v_e * \delta - v_e^2 / 2 / a_e - v_o * \delta - 1 / 2 * a_o * \delta^2$

**definition** *checker-r1* ::  $\text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{bool}$  **where**

*checker-r1*  $s_e v_e a_e s_o v_o a_o \delta \equiv$   
*let* *distance* =  $s_o - s_e$ ;  
*precond* = *check-precond-r1*  $s_e v_e a_e s_o v_o a_o \delta$ ;  
*vo-star* =  $v_o + a_o * \delta$ ;  
*t-stop-o-star* =  $-vo-star / a_o$ ;  
*t-stop-e* =  $-v_e / a_e$ ;  
*safe-dist0* = *safe-distance-1r*  $a_e v_e \delta$ ;  
*safe-dist1* = *safe-distance-2r*  $a_e v_e a_o v_o \delta$ ;  
*safe-dist2* = *safe-distance-4r*  $a_e v_e a_o v_o \delta$ ;

$safe-dist3 = safe-distance-3r\ a_e\ v_e\ a_o\ v_o\ \delta$   
in  
if  $\neg\ precond$  then *False*  
else if  $distance > safe-dist0 \vee distance > safe-dist3$  then *True*  
else if  $(a_o > a_e \wedge vo-star < v_e \wedge t-stop-e < t-stop-o-star)$  then  $distance >$   
 $safe-dist2$   
else  $distance > safe-dist1$

**theorem** *checker-r1-correctness:*

$(checker-r1\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta \longleftrightarrow check-precond-r1\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta \wedge$   
 $safe-distance-normal.no-collision-react\ a_e\ v_e\ s_e\ a_o\ v_o\ s_o\ \delta\ \{0..\})$

**proof**

**assume** *asm*:  $checker-r1\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta$

**have** *pre*:  $check-precond-r1\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta$

**proof** (*rule ccontr*)

**assume**  $\neg\ check-precond-r1\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta$

**with** *asm* **show** *False* **unfolding** *checker-r1-def* *Let-def* **by** *auto*

**qed**

**from** *pre* **have** *sdn'*:  $safe-distance-normal\ a_e\ v_e\ s_e\ a_o\ v_o\ s_o\ \delta$

**by** (*unfold-locales*) (*auto simp add: check-precond-r1-def*)

**interpret** *sdn*:  $safe-distance-normal\ a_e\ v_e\ s_e\ a_o\ v_o\ s_o\ \delta$

**rewrites**  $sdn.distance0 = safe-distance0\ v_e\ a_o\ v_o\ \delta$  **and**

$sdn.safe-distance-1r = safe-distance-1r\ a_e\ v_e\ \delta$  **and**

$sdn.safe-distance-2r = safe-distance-2r\ a_e\ v_e\ a_o\ v_o\ \delta$  **and**

$sdn.safe-distance-4r = safe-distance-4r\ a_e\ v_e\ a_o\ v_o\ \delta$  **and**

$sdn.safe-distance-3r = safe-distance-3r\ a_e\ v_e\ a_o\ v_o\ \delta$

**proof** –

**from** *sdn'* **show**  $safe-distance-normal\ a_e\ v_e\ s_e\ a_o\ v_o\ s_o\ \delta$  **by** *auto*

**next**

**show**  $safe-distance-normal.distance0\ v_e\ a_o\ v_o\ \delta = safe-distance0\ v_e\ a_o\ v_o\ \delta$

**unfolding**  $safe-distance-normal.distance0-def[OF\ sdn]$   $safe-distance0-def$  **by**

*auto*

**next**

**show**  $safe-distance-normal.safe-distance-1r\ a_e\ v_e\ \delta = safe-distance-1r\ a_e\ v_e\ \delta$

**unfolding**  $safe-distance-normal.safe-distance-1r-def[OF\ sdn]$   $safe-distance-1r-def$

**by** *auto*

**next**

**show**  $safe-distance-normal.safe-distance-2r\ a_e\ v_e\ a_o\ v_o\ \delta = safe-distance-2r\ a_e$

$v_e\ a_o\ v_o\ \delta$

**unfolding**  $safe-distance-normal.safe-distance-2r-def[OF\ sdn]$   $safe-distance-2r-def$

**by** *auto*

**next**

**show**  $safe-distance-normal.safe-distance-4r\ a_e\ v_e\ a_o\ v_o\ \delta = safe-distance-4r\ a_e$

$v_e\ a_o\ v_o\ \delta$

**unfolding**  $safe-distance-normal.safe-distance-4r-def[OF\ sdn]$   $safe-distance-4r-def$

**by** *auto*

**next**

**show**  $safe-distance-normal.safe-distance-3r\ a_e\ v_e\ a_o\ v_o\ \delta = safe-distance-3r\ a_e$

$v_e\ a_o\ v_o\ \delta$



```

unfolding safe-distance-normal.safe-distance-3r-def[OF sdn^] safe-distance-3r-def
by auto
qed
have  $0 < \delta$  and  $\delta \leq -v_o / a_o$  using pre unfolding check-precond-r1-def by
auto
define so-delta where  $so-delta = s_o + v_o * \delta + a_o * \delta^2 / 2$ 
define q-e-delta where  $q-e-delta \equiv s_e + v_e * \delta$ 
define u-stop-e where  $u-stop-e \equiv q-e-delta - v_e^2 / (2 * a_e)$ 
define vo-star where  $vo-star = v_o + a_o * \delta$ 
define t-stop-o-star where  $t-stop-o-star \equiv -vo-star / a_o$ 
define t-stop-e where  $t-stop-e = -v_e / a_e$ 
define distance where  $distance \equiv s_o - s_e$ 
define distance0 where  $distance0 = safe-distance0\ v_e\ a_o\ v_o\ \delta$ 
define safe-dist0 where  $safe-dist0 = safe-distance-1r\ a_e\ v_e\ \delta$ 
define safe-dist2 where  $safe-dist2 \equiv safe-distance-4r\ a_e\ v_e\ a_o\ v_o\ \delta$ 
define safe-dist1 where  $safe-dist1 \equiv safe-distance-2r\ a_e\ v_e\ a_o\ v_o\ \delta$ 
define safe-dist3 where  $safe-dist3 = safe-distance-3r\ a_e\ v_e\ a_o\ v_o\ \delta$ 
note abb = so-delta-def q-e-delta-def u-stop-e-def vo-star-def t-stop-o-star-def
t-stop-e-def
distance-def safe-dist2-def safe-dist1-def safe-dist0-def safe-dist3-def
distance0-def
consider  $distance > safe-dist0 \mid distance > safe-dist3 \mid distance \leq safe-dist0 \wedge$ 
 $distance \leq safe-dist3$ 
by linarith
hence sdn.no-collision-react {0..}
proof (cases)
  case 1
    then show ?thesis using sdn.sd-1r-correct unfolding abb by auto
  next
    case 2
      hence pre2: distance > distance0 using sdn.distance0-at-most-sd3r unfolding
abb by auto
      hence  $sdn.u\ \delta < sdn.other.s\ \delta$  using pre unfolding sdn.u-def sdn.ego.q-def
sdn.other.s-def sdn.other.t-stop-def sdn.other.p-def abb check-precond-r1-def
sdn.distance0-def
      by auto
      from pre interpret sdr: safe-distance-no-collision-delta a_e v_e s_e a_o v_o s_o \delta
      by (unfold-locales) (auto simp add:check-precond-r1-def  $\langle sdn.u\ \delta < sdn.other.s$ 
 $\delta \rangle$ )
      show ?thesis using sdr.sd-3r-correct 2 pre unfolding check-precond-r1-def abb
sdn.other.t-stop-def
      by auto
    next
      case 3
        hence  $distance \leq safe-dist3$  by auto
        hence  $sdn.other.s\ \delta \leq sdn.u-max$  using pre unfolding check-precond-r1-def
sdn.other.s-def sdn.other.t-stop-def
         $sdn.other.p-def sdn.u-max-eq sdn.ego.q-def abb sdn.safe-distance-3r-def$  by
auto

```

```

have ( $a_o > a_e \wedge vo\text{-star} < v_e \wedge t\text{-stop}\text{-}e < t\text{-stop}\text{-}o\text{-star}$ )  $\vee \neg$  ( $a_o > a_e \wedge$ 
 $vo\text{-star} < v_e \wedge t\text{-stop}\text{-}e < t\text{-stop}\text{-}o\text{-star}$ )
  by auto
moreover
  { assume cond: ( $a_o > a_e \wedge vo\text{-star} < v_e \wedge t\text{-stop}\text{-}e < t\text{-stop}\text{-}o\text{-star}$ )
    with  $\exists$  pre have  $distance > safe\text{-}dist2$  using asm unfolding checker-r1-def
      Let-def abb by auto
    with sdn.distance0-at-most-sd4r have  $distance > distance0$  unfolding abb
using cond by auto
    hence sdn.u  $\delta < sdn.other.s \delta$  using pre unfolding sdn.u-def sdn.ego.q-def
      sdn.other.s-def sdn.other.t-stop-def sdn.other.p-def abb check-precond-r1-def
      sdn.distance0-def
    by auto
    from pre interpret sdr: safe-distance-no-collision-delta a_e v_e s_e a_o v_o s_o \delta
    by (unfold-locales) (auto simp add:check-precond-r1-def  $\langle sdn.u \delta < sdn.other.s$ 
 $\delta \rangle$ )
    from sdr.sd-4r-correct[OF -  $\langle sdn.other.s \delta \leq sdn.u\text{-max} \rangle$   $\langle distance > safe\text{-}dist2 \rangle$ 
    have ?thesis using pre cond unfolding check-precond-r1-def sdn.other.t-stop-def
abb by auto }
  moreover
  { assume not-cond:  $\neg$  ( $a_o > a_e \wedge vo\text{-star} < v_e \wedge t\text{-stop}\text{-}e < t\text{-stop}\text{-}o\text{-star}$ )
    with  $\exists$  pre have  $distance > safe\text{-}dist1$  using asm unfolding checker-r1-def
      Let-def abb by auto
    with sdn.dist0-sd2r-1 have  $distance > distance0$  using pre not-cond un-
folding check-precond-r1-def
      sdn.other.t-stop-def sdn.other.s'-def sdn.other.p'-def abb by (auto simp
add:field-simps)
    hence sdn.u  $\delta < sdn.other.s \delta$  using pre unfolding sdn.u-def sdn.ego.q-def
      sdn.other.s-def sdn.other.t-stop-def sdn.other.p-def abb check-precond-r1-def
      sdn.distance0-def
    by auto
    from pre interpret sdr: safe-distance-no-collision-delta a_e v_e s_e a_o v_o s_o \delta
    by (unfold-locales) (auto simp add:check-precond-r1-def  $\langle sdn.u \delta < sdn.other.s$ 
 $\delta \rangle$ )
    from sdr.sd-2r-correct-for-3r-2[OF -  $\langle sdn.other.s \delta \leq sdn.u\text{-max} \rangle$  not-cond
 $\langle distance > safe\text{-}dist1 \rangle$ 
    have ?thesis using pre unfolding abb sdn.other.s'-def check-precond-r1-def
sdn.other.t-stop-def sdn.other.p'-def
    by (auto simp add:field-simps) }
    ultimately show ?thesis by auto
  qed
  with pre show check-precond-r1 s_e v_e a_e s_o v_o a_o \delta \wedge sdn.no-collision-react
  {0..} by auto
next
  assume check-precond-r1 s_e v_e a_e s_o v_o a_o \delta \wedge safe-distance-normal.no-collision-react
a_e v_e s_e a_o v_o s_o \delta {0..}
  hence pre: check-precond-r1 s_e v_e a_e s_o v_o a_o \delta and as2: safe-distance-normal.no-collision-react
a_e v_e s_e a_o v_o s_o \delta {0..}
  by auto

```

```

show checker-r1 se ve ae so vo ao δ
proof (rule ccontr)
  assume as1: ¬ checker-r1 se ve ae so vo ao δ
  from pre have 0 < δ and δ ≤ - vo / ao unfolding check-precond-r1-def by
auto
  define so-delta where so-delta = so + vo * δ + ao * δ2 / 2
  define q-e-delta where q-e-delta ≡ se + ve * δ
  define u-stop-e where u-stop-e ≡ q-e-delta - ve2 / (2 * ae)
  define vo-star where vo-star ≡ vo + ao * δ
  define t-stop-o-star where t-stop-o-star ≡ - vo-star / ao
  define t-stop-e where t-stop-e ≡ - ve / ae
  define distance where distance ≡ so - se
  define distance0 where distance0 ≡ safe-distance0 ve ao vo δ
  define safe-dist0 where safe-dist0 ≡ safe-distance-1r ae ve δ
  define safe-dist2 where safe-dist2 ≡ safe-distance-4r ae ve ao vo δ
  define safe-dist1 where safe-dist1 ≡ safe-distance-2r ae ve ao vo δ
  define safe-dist3 where safe-dist3 ≡ safe-distance-3r ae ve ao vo δ
  note abb = so-delta-def q-e-delta-def u-stop-e-def vo-star-def t-stop-o-star-def
t-stop-e-def
distance-def safe-dist2-def safe-dist1-def safe-dist0-def safe-dist3-def
distance0-def
  from pre have sdn': safe-distance-normal ae ve se ao vo so δ
  by (unfold-locales) (auto simp add: check-precond-r1-def)
  interpret sdn: safe-distance-normal ae ve se ao vo so δ
  rewrites sdn.distance0 = safe-distance0 ve ao vo δ and
    sdn.safe-distance-1r = safe-distance-1r ae ve δ and
    sdn.safe-distance-2r = safe-distance-2r ae ve ao vo δ and
    sdn.safe-distance-4r = safe-distance-4r ae ve ao vo δ and
    sdn.safe-distance-3r = safe-distance-3r ae ve ao vo δ
  proof -
  from sdn' show safe-distance-normal ae ve se ao vo so δ by auto
  next
  show safe-distance-normal.distance0 ve ao vo δ = safe-distance0 ve ao vo δ
  unfolding safe-distance-normal.distance0-def[OF sdn'] safe-distance0-def
by auto
  next
  show safe-distance-normal.safe-distance-1r ae ve δ = safe-distance-1r ae ve δ
  unfolding safe-distance-normal.safe-distance-1r-def[OF sdn'] safe-distance-1r-def
by auto
  next
  show safe-distance-normal.safe-distance-2r ae ve ao vo δ = safe-distance-2r
ae ve ao vo δ
  unfolding safe-distance-normal.safe-distance-2r-def[OF sdn'] safe-distance-2r-def
by auto
  next
  show safe-distance-normal.safe-distance-4r ae ve ao vo δ = safe-distance-4r
ae ve ao vo δ
  unfolding safe-distance-normal.safe-distance-4r-def[OF sdn'] safe-distance-4r-def
by auto

```

```

next
  show safe-distance-normal.safe-distance-3r a_e v_e a_o v_o δ = safe-distance-3r
a_e v_e a_o v_o δ
  unfolding safe-distance-normal.safe-distance-3r-def[OF sdn] safe-distance-3r-def
by auto
qed
have ¬ distance > distance0 ∨ distance > distance0 by auto
moreover
{ assume ¬ distance > distance0
  hence distance ≤ distance0 by auto
  with sdn.cond-3r-1' have sdn.collision-react {0..δ} using pre unfolding
check-precond-r1-def abb
  sdn.other.t-stop-def by auto
  with sdn.collision-react-subset have sdn.collision-react {0..} by auto
  with as2 have False by auto }
moreover
{ assume if2: distance > distance0
  have ¬ (distance > safe-dist0 ∨ distance > safe-dist3)
  proof (rule ccontr)
    assume ¬ ¬ (safe-dist0 < distance ∨ safe-dist3 < distance)
    hence (safe-dist0 < distance ∨ safe-dist3 < distance) by auto
    with as1 show False using pre if2 unfolding checker-r1-def Let-def abb
    by auto
  qed
  hence if31: distance ≤ safe-dist0 and if32: distance ≤ safe-dist3 by auto
  have sdn.u δ < sdn.other.s δ using if2 pre unfolding sdn.u-def sdn.ego.q-def
  sdn.other.s-def sdn.other.t-stop-def sdn.other.p-def abb check-precond-r1-def
sdn.distance0-def
  by auto
  from pre interpret sdr: safe-distance-no-collision-delta a_e v_e s_e a_o v_o s_o δ
  by (unfold-locales) (auto simp add:check-precond-r1-def ‹sdn.u δ < sdn.other.s
δ›)
  have s_o ≤ sdn.u-max using if31 unfolding sdn.u-max-eq sdn.ego.q-def abb
  sdn.safe-distance-1r-def by auto
  have sdn.other.s δ ≤ sdn.u-max using if32 pre unfolding sdn.other.s-def
check-precond-r1-def
  sdn.other.t-stop-def sdn.other.p-def sdn.u-max-eq sdn.ego.q-def abb sdn.safe-distance-3r-def
  by auto
  consider (a_o > a_e ∧ vo-star < v_e ∧ t-stop-e < t-stop-o-star) |
  ¬ (a_o > a_e ∧ vo-star < v_e ∧ t-stop-e < t-stop-o-star) by auto
  hence False
  proof (cases)
    case 1
      hence rest-conjunct:(a_e < a_o ∧ sdn.other.s' δ < v_e ∧ v_e - a_e / a_o *
sdn.other.s' δ < 0)
      using pre unfolding check-precond-r1-def unfolding sdn.other.s'-def
sdn.other.t-stop-def
      sdn.other.p'-def abb by (auto simp add:field-simps)
      from 1 have distance ≤ safe-dist2 using as1 pre if2 if31 if32 unfolding

```

```

checker-r1-def
  Let-def abb by auto
  hence cond-f: sdn.other.s  $\delta$  - sdn.ego.q  $\delta \leq$  sdr.delayed-safe-distance.snd-safe-distance
  using pre unfolding check-precond-r1-def sdn.other.s-def sdn.other.t-stop-def
sdn.other.p-def
  sdn.ego.q-def sdr.delayed-safe-distance.snd-safe-distance-def using sdn.other.s'-def[of
 $\delta$ ]
  unfolding sdn.other.t-stop-def sdn.other.p'-def abb sdn.safe-distance-4r-def
  by auto
  have distance > safe-dist1  $\vee$  distance  $\leq$  safe-dist1 by auto
  moreover
  { assume distance > safe-dist1
  hence sdn.u-max < sdn.other.s-stop unfolding sdn.u-max-eq sdn.ego.q-def
sdn.other.s-t-stop
    sdn.other.p-max-eq abb sdn.safe-distance-2r-def by (auto simp
add:field-simps)
    from sdr.cond-3r-2[OF  $\langle s_o \leq$  sdn.u-max  $\rangle$  this  $\langle$  sdn.other.s  $\delta \leq$  sdn.u-max  $\rangle$ ]
    have sdn.collision-react {0..} using cond-f rest-conjunct by auto
    with as2 have False by auto }
  moreover
  { assume distance  $\leq$  safe-dist1
  hence sdn.u-max  $\geq$  sdn.other.s-stop unfolding sdn.u-max-eq sdn.ego.q-def
sdn.other.s-t-stop
    sdn.other.p-max-eq abb sdn.safe-distance-2r-def by (auto simp
add:field-simps)
    with sdn.cond-2r[OF this] have sdn.collision-react {0..} by auto
    with as2 have False by auto }
  ultimately show ?thesis by auto
next
case 2
hence distance  $\leq$  safe-dist1 using as1 pre if2 if31 if32 unfolding checker-r1-def
  Let-def abb by auto
  hence sdn.u-max  $\geq$  sdn.other.s-stop unfolding sdn.u-max-eq sdn.ego.q-def
sdn.other.s-t-stop
    sdn.other.p-max-eq abb sdn.safe-distance-2r-def by (auto simp add:field-simps)
  with sdn.cond-2r[OF this] have sdn.collision-react {0..} by auto
  with as2 show False by auto
qed }
ultimately show False by auto
qed
qed

```

**definition** check-precond-r2 :: real  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  bool **where**

check-precond-r2  $s_e v_e a_e s_o v_o a_o \delta \iff s_o > s_e \wedge 0 \leq v_e \wedge 0 \leq v_o \wedge a_e < 0$   
 $\wedge a_o < 0 \wedge 0 < \delta \wedge \delta > -v_o / a_o$

**definition** safe-distance0-2 :: real  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  real **where**

safe-distance0-2  $v_e a_o v_o \delta = v_e * \delta + 1 / 2 * v_o^2 / a_o$

**definition** *checker-r2* :: *real* ⇒ *real* ⇒ *real* ⇒ *real* ⇒ *real* ⇒ *real* ⇒ *real* ⇒ *bool*

**where**

*checker-r2* *s<sub>e</sub>* *v<sub>e</sub>* *a<sub>e</sub>* *s<sub>o</sub>* *v<sub>o</sub>* *a<sub>o</sub>* *δ* ≡  
 let *distance* = *s<sub>o</sub>* - *s<sub>e</sub>*;  
*precond* = *check-precond-r2* *s<sub>e</sub>* *v<sub>e</sub>* *a<sub>e</sub>* *s<sub>o</sub>* *v<sub>o</sub>* *a<sub>o</sub>* *δ*;  
*safe-dist0* = *safe-distance-1r* *a<sub>e</sub>* *v<sub>e</sub>* *δ*;  
*safe-dist1* = *safe-distance-2r* *a<sub>e</sub>* *v<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *δ*  
 in  
 if ¬ *precond* then *False*  
 else if *distance* > *safe-dist0* then *True*  
 else *distance* > *safe-dist1*

**theorem** *checker-r2-correctness*:

(*checker-r2* *s<sub>e</sub>* *v<sub>e</sub>* *a<sub>e</sub>* *s<sub>o</sub>* *v<sub>o</sub>* *a<sub>o</sub>* *δ* ↔ *check-precond-r2* *s<sub>e</sub>* *v<sub>e</sub>* *a<sub>e</sub>* *s<sub>o</sub>* *v<sub>o</sub>* *a<sub>o</sub>* *δ* ∧  
*safe-distance-normal.no-collision-react* *a<sub>e</sub>* *v<sub>e</sub>* *s<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *s<sub>o</sub>* *δ* {0..})

**proof**

**assume** *asm*: *checker-r2* *s<sub>e</sub>* *v<sub>e</sub>* *a<sub>e</sub>* *s<sub>o</sub>* *v<sub>o</sub>* *a<sub>o</sub>* *δ*

**have** *pre*: *check-precond-r2* *s<sub>e</sub>* *v<sub>e</sub>* *a<sub>e</sub>* *s<sub>o</sub>* *v<sub>o</sub>* *a<sub>o</sub>* *δ*

**proof** (rule *ccontr*)

**assume** ¬ *check-precond-r2* *s<sub>e</sub>* *v<sub>e</sub>* *a<sub>e</sub>* *s<sub>o</sub>* *v<sub>o</sub>* *a<sub>o</sub>* *δ*

**with** *asm* **show** *False* **unfolding** *checker-r2-def* *Let-def* **by** *auto*

**qed**

**from** *pre* **have** *sdn'*: *safe-distance-normal* *a<sub>e</sub>* *v<sub>e</sub>* *s<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *s<sub>o</sub>* *δ*

**by** (*unfold-locales*) (*auto simp add: check-precond-r2-def*)

**interpret** *sdn*: *safe-distance-normal* *a<sub>e</sub>* *v<sub>e</sub>* *s<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *s<sub>o</sub>* *δ*

**rewrites** *sdn.distance0-2* = *safe-distance0-2* *v<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *δ* **and**

*sdn.safe-distance-1r* = *safe-distance-1r* *a<sub>e</sub>* *v<sub>e</sub>* *δ* **and**

*sdn.safe-distance-2r* = *safe-distance-2r* *a<sub>e</sub>* *v<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *δ*

**proof** –

**from** *sdn'* **show** *safe-distance-normal* *a<sub>e</sub>* *v<sub>e</sub>* *s<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *s<sub>o</sub>* *δ* **by** *auto*

**next**

**show** *safe-distance-normal.distance0-2* *v<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *δ* = *safe-distance0-2* *v<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *δ*

**unfolding** *safe-distance-normal.distance0-2-def*[*OF sdn*] *safe-distance0-2-def*

**by** *auto*

**next**

**show** *safe-distance-normal.safe-distance-1r* *a<sub>e</sub>* *v<sub>e</sub>* *δ* = *safe-distance-1r* *a<sub>e</sub>* *v<sub>e</sub>* *δ*

**unfolding** *safe-distance-normal.safe-distance-1r-def*[*OF sdn*] *safe-distance-1r-def*

**by** *auto*

**next**

**show** *safe-distance-normal.safe-distance-2r* *a<sub>e</sub>* *v<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *δ* = *safe-distance-2r* *a<sub>e</sub>*  
*v<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *δ*

**unfolding** *safe-distance-normal.safe-distance-2r-def*[*OF sdn*] *safe-distance-2r-def*

**by** *auto*

**qed**

**have**  $0 < \delta$  **and**  $\delta > -v_o / a_o$  **using** *pre* **unfolding** *check-precond-r2-def* **by**  
*auto*

**define** *distance* **where** *distance* ≡ *s<sub>o</sub>* - *s<sub>e</sub>*

**define** *distance0-2* **where** *distance0-2* = *safe-distance0-2* *v<sub>e</sub>* *a<sub>o</sub>* *v<sub>o</sub>* *δ*

```

define safe-dist0 where safe-dist0 = safe-distance-1r ae ve  $\delta$ 
define safe-dist1 where safe-dist1  $\equiv$  safe-distance-2r ae ve ao vo  $\delta$ 
note abb = distance-def safe-dist1-def safe-dist0-def distance0-2-def
consider distance > safe-dist0 | distance  $\leq$  safe-dist0
  by linarith
hence sdn.no-collision-react {0..}
proof (cases)
  case 1
    then show ?thesis using sdn.sd-1r-correct unfolding abb by auto
  next
    case 2
      hence (so  $\leq$  sdn.u-max) using distance-def safe-dist0-def sdn.sd-1r-eq by
linarith
      with 2 pre have distance > safe-dist1 using asm unfolding checker-r2-def
Let-def abb by auto
      with sdn.dist0-sd2r-2 have distance > distance0-2 using abb  $\langle -v_o / a_o < \delta \rangle$ 
by auto
      hence sdn.u  $\delta <$  sdn.other.s  $\delta$  using abb sdn.distance0-2-eq  $\langle \delta > -v_o / a_o \rangle$ 
sdn.other.t-stop-def by auto
      have sdn.u-max < sdn.other.s  $\delta$  using abb sdn.sd2r-eq  $\langle \delta > -v_o / a_o \rangle$ 
sdn.other.t-stop-def  $\langle \textit{distance} > \textit{safe-dist1} \rangle$  by auto
      from pre interpret sdr: safe-distance-no-collision-delta ae ve se ao vo so  $\delta$ 
      by (unfold-locales) (auto simp add:check-precond-r2-def  $\langle \textit{sdn.u} \delta < \textit{sdn.other.s} \delta \rangle$ )
      from sdr.sd-2r-correct-for-3r-3[OF]  $\langle \textit{distance} > \textit{safe-dist1} \rangle$   $\langle \textit{sdn.u} \delta < \textit{sdn.other.s} \delta \rangle$ 
 $\langle \textit{sdn.u-max} < \textit{sdn.other.s} \delta \rangle$ 
      show ?thesis using pre unfolding abb sdn.other.s'-def check-precond-r2-def
sdn.other.t-stop-def sdn.other.p'-def
      by (auto simp add:field-simps)
    qed
    with pre show check-precond-r2 se ve ae so vo ao  $\delta \wedge$  sdn.no-collision-react
{0..} by auto
  next
    assume check-precond-r2 se ve ae so vo ao  $\delta \wedge$  safe-distance-normal.no-collision-react
ae ve se ao vo so  $\delta$  {0..}
    hence pre: check-precond-r2 se ve ae so vo ao  $\delta$  and as2: safe-distance-normal.no-collision-react
ae ve se ao vo so  $\delta$  {0..}
    by auto
    show checker-r2 se ve ae so vo ao  $\delta$ 
    proof (rule ccontr)
      assume as1:  $\neg$  checker-r2 se ve ae so vo ao  $\delta$ 
      from pre have  $0 < \delta$  and  $\delta > -v_o / a_o$  unfolding check-precond-r2-def by
auto
      define distance where distance  $\equiv$  so - se
      define distance0-2 where distance0-2 = safe-distance0-2 ve ao vo  $\delta$ 
      define safe-dist0 where safe-dist0 = safe-distance-1r ae ve  $\delta$ 
      define safe-dist1 where safe-dist1  $\equiv$  safe-distance-2r ae ve ao vo  $\delta$ 
      note abb = distance-def safe-dist1-def safe-dist0-def distance0-2-def
      from pre have sdn': safe-distance-normal ae ve se ao vo so  $\delta$ 

```

```

    by (unfold-locales) (auto simp add: check-precond-r2-def)
interpret sdn: safe-distance-normal  $a_e v_e s_e a_o v_o s_o \delta$ 
rewrites sdn.distance0-2 = safe-distance0-2  $v_e a_o v_o \delta$  and
           sdn.safe-distance-1r = safe-distance-1r  $a_e v_e \delta$  and
           sdn.safe-distance-2r = safe-distance-2r  $a_e v_e a_o v_o \delta$ 
proof –
  from sdn' show safe-distance-normal  $a_e v_e s_e a_o v_o s_o \delta$  by auto
next
  show safe-distance-normal.distance0-2  $v_e a_o v_o \delta =$  safe-distance0-2  $v_e a_o v_o$ 
 $\delta$ 
  unfolding safe-distance-normal.distance0-2-def[OF sdn'] safe-distance0-2-def
by auto
next
  show safe-distance-normal.safe-distance-1r  $a_e v_e \delta =$  safe-distance-1r  $a_e v_e \delta$ 
  unfolding safe-distance-normal.safe-distance-1r-def[OF sdn'] safe-distance-1r-def
by auto
next
  show safe-distance-normal.safe-distance-2r  $a_e v_e a_o v_o \delta =$  safe-distance-2r
 $a_e v_e a_o v_o \delta$ 
  unfolding safe-distance-normal.safe-distance-2r-def[OF sdn'] safe-distance-2r-def
by auto
qed
have  $\neg$  distance > distance0-2  $\vee$  distance > distance0-2 by auto
moreover
  { assume  $\neg$  distance > distance0-2
    hence distance  $\leq$  distance0-2 by auto
    with sdn.cond-3r-1'-2 have sdn.collision-react {0.. $\delta$ } using pre unfolding
check-precond-r2-def abb sdn.other.t-stop-def by auto
    with sdn.collision-react-subset have sdn.collision-react {0..} by auto
    with as2 have False by auto }
moreover
  { assume if2: distance > distance0-2
    have  $\neg$  (distance > safe-dist0)
    proof (rule ccontr)
      assume  $\neg$   $\neg$  (safe-dist0 < distance)
      hence (safe-dist0 < distance) by auto
      with as1 show False using pre if2 unfolding checker-r2-def Let-def abb
by auto
    qed
    hence if3: distance  $\leq$  safe-dist0 by auto
    with pre have distance  $\leq$  safe-dist1 using as1 unfolding checker-r2-def
Let-def abb by auto

    have sdn.u  $\delta <$  sdn.other.s  $\delta$  using abb if2 sdn.distance0-2-eq  $\langle \delta > - v_o /$ 
 $a_o \rangle$  sdn.other.t-stop-def by auto
    from pre interpret sdr: safe-distance-no-collision-delta  $a_e v_e s_e a_o v_o s_o \delta$ 
    by (unfold-locales) (auto simp add:check-precond-r2-def  $\langle$ sdn.u  $\delta <$ 
sdn.other.s  $\delta \rangle$ )
    have sdn.u-max  $\geq$  sdn.other.s  $\delta$  using abb sdn.sd2r-eq  $\langle \delta > - v_o / a_o \rangle$ 

```



```

sdn.other.t-stop-def ‹distance ≤ safe-dist1› by auto
  with ‹δ > - vo / ao› have sdn.u-max ≥ sdn.other.s-stop
  using sdn.other.s-mono sdn.other.t-stop-nonneg sdn.other.p-t-stop sdn.other.p-zero
sdn.other.t-stop-def
  apply (auto simp: sdn.other.s-def movement.t-stop-def split: if-splits)
  using sdn.other.p-zero by auto
  hence sdn.collision-react {0..} using sdn.cond-2r by auto
  with as2 have False by auto }
ultimately show False by auto
qed
qed

```

Combine the two checkers into one.

**definition** *check-precond-r* ::  $real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow bool$  **where**

$check\text{-}precond\text{-}r\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta \longleftrightarrow s_o > s_e \wedge 0 \leq v_e \wedge 0 \leq v_o \wedge a_e < 0 \wedge a_o < 0 \wedge 0 < \delta$

**definition** *checker-r* ::  $real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow real \Rightarrow bool$  **where**

$checker\text{-}r\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta \equiv$

$let\ distance = s_o - s_e;$

$precond = check\text{-}precond\text{-}r\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta;$

$vo\text{-}star = v_o + a_o * \delta;$

$t\text{-}stop\text{-}o\text{-}star = -vo\text{-}star / a_o;$

$t\text{-}stop\text{-}e = -v_e / a_e;$

$t\text{-}stop\text{-}o = -v_o / a_o;$

$safe\text{-}dist0 = safe\text{-}distance\text{-}1r\ a_e\ v_e\ \delta;$

$safe\text{-}dist1 = safe\text{-}distance\text{-}2r\ a_e\ v_e\ a_o\ v_o\ \delta;$

$safe\text{-}dist2 = safe\text{-}distance\text{-}4r\ a_e\ v_e\ a_o\ v_o\ \delta;$

$safe\text{-}dist3 = safe\text{-}distance\text{-}3r\ a_e\ v_e\ a_o\ v_o\ \delta$

*in*

*if*  $\neg precond$  *then* *False*

*else if*  $distance > safe\text{-}dist0$  *then* *True*

*else if*  $\delta \leq t\text{-}stop\text{-}o \wedge distance > safe\text{-}dist3$  *then* *True*

*else if*  $\delta \leq t\text{-}stop\text{-}o \wedge (a_o > a_e \wedge vo\text{-}star < v_e \wedge t\text{-}stop\text{-}e < t\text{-}stop\text{-}o\text{-}star)$

*then*  $distance > safe\text{-}dist2$

*else*  $distance > safe\text{-}dist1$

**theorem** *checker-eq-1*:

$checker\text{-}r\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta \wedge \delta \leq -v_o / a_o \longleftrightarrow checker\text{-}r1\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta$

**proof** –

**have**  $checker\text{-}r\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta \wedge \delta \leq -v_o / a_o \longleftrightarrow check\text{-}precond\text{-}r\ s_e\ v_e\ a_e\ s_o\ v_o\ a_o\ \delta$

$\wedge (s_o - s_e > safe\text{-}distance\text{-}1r\ a_e\ v_e\ \delta$

$\vee s_o - s_e > safe\text{-}distance\text{-}3r\ a_e\ v_e\ a_o\ v_o\ \delta$

$\vee (((a_o > a_e \wedge v_o + a_o * \delta < v_e \wedge -v_e / a_e < -(v_o + a_o * \delta) / a_o) \longrightarrow$

$s_o - s_e > safe\text{-}distance\text{-}4r\ a_e\ v_e\ a_o\ v_o\ \delta)$

$\wedge (\neg (a_o > a_e \wedge v_o + a_o * \delta < v_e \wedge -v_e / a_e < -(v_o + a_o * \delta) / a_o)$   
 $\longrightarrow s_o - s_e > \text{safe-distance-2r } a_e v_e a_o v_o \delta))$   
 $\wedge \delta \leq -v_o / a_o$  **using checker-r-def by metis**  
**also have ...**  $\longleftrightarrow \text{check-precond-r1 } s_e v_e a_e s_o v_o a_o \delta$   
 $\wedge (s_o - s_e > \text{safe-distance-1r } a_e v_e \delta$   
 $\vee s_o - s_e > \text{safe-distance-3r } a_e v_e a_o v_o \delta$   
 $\vee (((a_o > a_e \wedge v_o + a_o * \delta < v_e \wedge -v_e / a_e < -(v_o + a_o * \delta) / a_o) \longrightarrow$   
 $s_o - s_e > \text{safe-distance-4r } a_e v_e a_o v_o \delta))$   
 $\wedge (\neg (a_o > a_e \wedge v_o + a_o * \delta < v_e \wedge -v_e / a_e < -(v_o + a_o * \delta) / a_o)$   
 $\longrightarrow s_o - s_e > \text{safe-distance-2r } a_e v_e a_o v_o \delta))$   
**by (auto simp add:check-precond-r-def check-precond-r1-def)**  
**also have ...**  $\longleftrightarrow \text{checker-r1 } s_e v_e a_e s_o v_o a_o \delta$  **by (metis checker-r1-def)**  
**finally show ?thesis by auto**  
**qed**

**theorem checker-eq-2:**

$\text{checker-r } s_e v_e a_e s_o v_o a_o \delta \wedge \delta > -v_o / a_o \longleftrightarrow \text{checker-r2 } s_e v_e a_e s_o v_o a_o \delta$   
**proof -**  
**have**  $\text{checker-r } s_e v_e a_e s_o v_o a_o \delta \wedge \delta > -v_o / a_o \longleftrightarrow \text{check-precond-r } s_e v_e$   
 $a_e s_o v_o a_o \delta \wedge (\neg \text{check-precond-r } s_e v_e a_e s_o v_o a_o \delta \vee$   
 $s_o - s_e > \text{safe-distance-1r } a_e v_e \delta \vee$   
 $(\delta \leq -v_o / a_o \wedge s_o - s_e > \text{safe-distance-3r } a_e v_e a_o v_o \delta) \vee$   
 $(\delta \leq -v_o / a_o \wedge a_o > a_e \wedge v_o + a_o * \delta < v_e \wedge -v_e / a_e < -(v_o + a_o * \delta)$   
 $/ a_o \wedge s_o - s_e > \text{safe-distance-4r } a_e v_e a_o v_o \delta) \vee$   
 $s_o - s_e > \text{safe-distance-2r } a_e v_e a_o v_o \delta) \wedge \delta > -v_o / a_o$  **unfolding checker-r-def**  
**Let-def if-splits by auto**  
**also have**  
 $\dots \longleftrightarrow \text{check-precond-r } s_e v_e a_e s_o v_o a_o \delta$   
 $\wedge (s_o - s_e > \text{safe-distance-1r } a_e v_e \delta \vee s_o - s_e > \text{safe-distance-2r } a_e v_e a_o v_o$   
 $\delta)$   
 $\wedge \delta > -v_o / a_o$  **by (auto simp add:HOL.disjE)**  
**also have**  
 $\dots \longleftrightarrow \text{check-precond-r2 } s_e v_e a_e s_o v_o a_o \delta$   
 $\wedge (s_o - s_e > \text{safe-distance-1r } a_e v_e \delta \vee s_o - s_e > \text{safe-distance-2r } a_e v_e a_o v_o$   
 $\delta)$   
**by (auto simp add:check-precond-r-def check-precond-r2-def)**  
**also have ...**  $\longleftrightarrow \text{checker-r2 } s_e v_e a_e s_o v_o a_o \delta$  **by (auto simp add:checker-r2-def**  
**Let-def if-splits)**  
**thus ?thesis using calculation by auto**  
**qed**

**theorem checker-r-correctness:**

$(\text{checker-r } s_e v_e a_e s_o v_o a_o \delta \longleftrightarrow \text{check-precond-r } s_e v_e a_e s_o v_o a_o \delta \wedge$   
 $\text{safe-distance-normal.no-collision-react } a_e v_e s_e a_o v_o s_o \delta \{0..\})$   
**proof -**  
**have**  $\text{checker-r } s_e v_e a_e s_o v_o a_o \delta \longleftrightarrow (\text{checker-r } s_e v_e a_e s_o v_o a_o \delta \wedge \delta \leq -$   
 $v_o / a_o) \vee (\text{checker-r } s_e v_e a_e s_o v_o a_o \delta \wedge \delta > -v_o / a_o)$  **by auto**  
**also have ...**  $\longleftrightarrow \text{checker-r1 } s_e v_e a_e s_o v_o a_o \delta \vee \text{checker-r2 } s_e v_e a_e s_o v_o a_o$   
 $\delta$  **using checker-eq-1 checker-eq-2 by auto**

```

also have ...  $\longleftrightarrow$  (check-precond-r1  $s_e v_e a_e s_o v_o a_o \delta \wedge$  safe-distance-normal.no-collision-react
 $a_e v_e s_e a_o v_o s_o \delta \{0..\}$ )
   $\vee$  (check-precond-r2  $s_e v_e a_e s_o v_o a_o \delta \wedge$  safe-distance-normal.no-collision-react
 $a_e v_e s_e a_o v_o s_o \delta \{0..\}$ )
using checker-r1-correctness checker-r2-correctness by auto
also have ...  $\longleftrightarrow$  ( $\delta \leq -v_o / a_o \wedge$  check-precond-r  $s_e v_e a_e s_o v_o a_o \delta \wedge$ 
safe-distance-normal.no-collision-react  $a_e v_e s_e a_o v_o s_o \delta \{0..\}$ )
   $\vee$  ( $\delta > -v_o / a_o \wedge$  check-precond-r  $s_e v_e a_e s_o v_o a_o \delta \wedge$  safe-distance-normal.no-collision-react
 $a_e v_e s_e a_o v_o s_o \delta \{0..\}$ )
by (auto simp add:check-precond-r-def check-precond-r1-def check-precond-r2-def)
also have ...  $\longleftrightarrow$  check-precond-r  $s_e v_e a_e s_o v_o a_o \delta \wedge$  safe-distance-normal.no-collision-react
 $a_e v_e s_e a_o v_o s_o \delta \{0..\}$ 
by auto
finally show ?thesis by auto
qed

end

```

### 3 Evaluation

```

theory Evaluation
imports
  Safe-Distance
  HOL-Library.Float
begin

```

#### 3.1 Code Generation Setup for Numeric Values

```

definition real-div-down ::  $\text{nat} \Rightarrow \text{int} \Rightarrow \text{int} \Rightarrow \text{real}$  where
  real-div-down  $p\ i\ j = \text{truncate-down} (\text{Suc } p) (i / j)$ 

```

```

definition real-div-up ::  $\text{nat} \Rightarrow \text{int} \Rightarrow \text{int} \Rightarrow \text{real}$  where
  real-div-up  $p\ i\ j = \text{truncate-up} (\text{Suc } p) (i / j)$ 

```

```

context includes float.lifting begin

```

```

lift-definition float-div-down ::  $\text{nat} \Rightarrow \text{int} \Rightarrow \text{int} \Rightarrow \text{float}$  is real-div-down
by (simp add: real-div-down-def)

```

```

lift-definition float-div-up ::  $\text{nat} \Rightarrow \text{int} \Rightarrow \text{int} \Rightarrow \text{float}$  is real-div-up
by (simp add: real-div-up-def)

```

```

end

```

```

lemma compute-float-div-up[code]:  $\text{float-div-up } p\ i\ j = - \text{float-div-down } p\ (-i)\ j$ 
including float.lifting
by transfer (simp add: real-div-up-def real-div-down-def truncate-up-eq-truncate-down)

```

```

lemma compute-float-div-down[code]:
   $\text{float-div-down } \text{prec } m1\ m2 = \text{lapprox-rat} (\text{Suc } \text{prec})\ m1\ m2$ 
including float.lifting by transfer (simp add: real-div-down-def)

```

```

definition real2-of-real :: nat  $\Rightarrow$  real  $\Rightarrow$  (real * real) where
  real2-of-real p x = (truncate-down (Suc p) x, truncate-up (Suc p) x)

context includes float.lifting begin
lift-definition float2-of-real :: nat  $\Rightarrow$  real  $\Rightarrow$  float  $\times$  float is real2-of-real
  by (auto simp: real2-of-real-def)
end

definition float2-opt-of-real :: nat  $\Rightarrow$  real  $\Rightarrow$  float interval option where
  float2-opt-of-real prec x = Interval' (fst (float2-of-real prec x)) (snd (float2-of-real
  prec x))

hide-const (open) Fraction-Field.Fract
lemma real-of-rat-Fract[simp]: real-of-rat (Fract a b) = a / b
  by (simp add: Fract-of-int-quotient of-rat-divide)

lemma [code]: float2-of-real p (Ratreal r) =
  (let (a, b) = quotient-of r in
  (float-div-down p a b, float-div-up p a b))
  including float.lifting
  apply transfer
  apply (auto split: prod.split simp: real2-of-real-def real-div-down-def real-div-up-def)
  apply (metis of-rat-divide of-rat-of-int-eq quotient-of-div)+
  done

fun real-of-dec :: integer  $\times$  integer  $\Rightarrow$  real where
  real-of-dec (m, e) =
    real-of-int (int-of-integer m) *
    (if e  $\geq$  0 then 10  $^$ (nat-of-integer e) else inverse (10  $^$ (nat  $-$ (int-of-integer
  e))))

lemma real-of-dec (m, e) = int-of-integer m * 10 powr (int-of-integer e)
proof -
  have 1: e  $\geq$  0  $\implies$  real (nat-of-integer e) = real-of-int (int-of-integer e)
    using less-eq-integer.rep-eq nat-of-integer.rep-eq by auto
  have 2: e  $\leq$  0  $\implies$  real-of-int (int-of-integer e) = - real (nat  $-$  int-of-integer
  e))
    using less-eq-integer.rep-eq by auto
  show ?thesis
    using 1
    apply (auto simp: powr-realpow[symmetric] divide-simps)
    apply (subst (asm) 2)
    apply (auto simp: powr-add[symmetric])
  done
qed

```

## 3.2 Data Evaluation

**definition** *trans6* where

```
trans6 c chk se ve ae so vo ao =
      chk (c se) (c ve) (c ae) (c so) (c vo) (c ao)
```

**definition** *checker-dec* where

```
checker-dec chk p u =
  trans6 (float2-opt-of-real (nat-of-integer u) o real-of-dec) (chk (nat-of-integer
p))
```

**definition** *checker-interval* = *checker-dec checker'*

**definition** *checker-symbolic* = *trans6 real-of-dec symbolic-checker*

**definition** *checker-rational* = *trans6 real-of-dec checker*

**lemmas**[*code*] = *movement.p-def*

**ML** <

```
exception InvalidArgument of string;
```

```
fun split-string s = String.fields (fn c => c = the (Char.fromString s))
```

```
fun dec-of-string s =
  case split-string . s
  of [r] => (the (IntInf.fromString r), 0)
  | [d1, d2] => (the (IntInf.fromString (d1 ^ d2)), ~ (String.size d2))
  | - => raise (InvalidArgument s)
```

```
fun check-string chk data =
  case split-string , data of
  [-, so, -, ve, ae, -, vo, ao] => chk data (0, 0)
  (dec-of-string ve) (dec-of-string ae) (dec-of-string so) (dec-of-string vo)
(dec-of-string ao)
  | - => raise (InvalidArgument data)
```

>

The precision of the input data is roughly 12 and yields similar performance as Sturm

**ML** <

```
val prec = 12
```

*local*

```
exception Result of int * int;
```

```
fun check-line chk n l (y, i) =
  let
    val l = String.substring (l, 0, String.size l - 1)
    val c = check-string chk l
  in if i < n then (if c then y + 1 else y, i + 1) else raise Result (y, i) end
```

*in*

```
fun check-file chk path n =  
  File.fold-lines (check-line chk n) path (0, 0)  
  handle Result res => res;
```

*end*

```
val check-file-symbolic      = check-file (fn - => code <checker-symbolic>)  
fun check-file-interval prec uncer = check-file (fn - => code <checker-interval> prec  
  uncer)  
val check-file-rational    = check-file (fn - => code <checker-rational>)  
>
```

Number of data points:

- data01: 1121215
- data02: 1341135
- data03: 1452656

**ML** <

```
val data01 = master-dir + path <data/data01.csv>  
val data02 = master-dir + path <data/data02.csv>  
val data03 = master-dir + path <data/data03.csv>  
>
```

**ML** <

```
val t-start1 = Timing.start ();  
val result1 = check-file-rational data01 100;  
val t-end1 = Timing.result t-start1;  
assert (result1 = (96, 100));  
>
```

**ML** <

```
val t-start2 = Timing.start ();  
val result2 = check-file-rational data02 100;  
val t-end2 = Timing.result t-start2;  
assert (result2 = (100, 100));  
>
```

**ML** <

```
val t-start3 = Timing.start ();  
val result3 = check-file-rational data03 100;  
val t-end3 = Timing.result t-start3;  
assert (result3 = (76, 100));  
>
```

Precision: 12, Uncertainty: 7 digits

```
ML <assert (check-file-interval 12 7 data01 100 = (95, 100))>  
ML <assert (check-file-rational data01 100 = (96, 100))>  
ML <assert (check-file-symbolic data01 100 = (96, 100))>
```

end

## References

- [1] A. Rizaldi, F. Immler, and M. Althoff. A formally verified checker of the safe distance traffic rules for autonomous vehicles. In S. Rayadurgam and O. Tkachuk, editors, *NASA Formal Methods*, pages 175–190, Cham, 2016. Springer International Publishing.