The Incompatibility of $SD$-Efficiency and $SD$-Strategy-Proofness

Manuel Eberl

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Abstract

This formalisation contains the proof that there is no anonymous and neutral Social Decision Scheme for at least four voters and alternatives that fulfils both $SD$-Efficiency and $SD$-Strategy-Proofness. The proof is a fully structured and quasi-human-redable one. It was derived from the (unstructured) SMT proof of the case for exactly four voters and alternatives by Brandl et al. [1].

Their proof relies on an unverified translation of the original problem to SMT, and the proof that lifts the argument for exactly four voters and alternatives to the general case is also not machine-checked. In this Isabelle proof, on the other hand, all of these steps are also fully proven and machine-checked. This is particularly important seeing as a previously published informal proof of a weaker statement contained a mistake in precisely this lifting step. [2]

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1 Incompatibility of SD-Efficiency and SD-Strategy-Proofness

theory SDS-Impossibility
imports
  Randomised-Social-Choice.SDS-Automation
  Randomised-Social-Choice.Randomised-Social-Choice
begin

1.1 Preliminary Definitions

locale sds-impossibility =
  anonymous-sds agents alts sds +
  neutral-sds agents alts sds +
  sd-efficient-sds agents alts sds +
  strategy-proof-sds agents alts sds
for agents :: 'agent set and alts :: 'alt set and sds
assumes agents-ge-4: card agents ≥ 4
  and alts-ge-4: card alts ≥ 4

locale sds-impossibility-4-4 = sds-impossibility agents alts sds
for agents :: 'agent set and alts :: 'alt set and sds
fixes A1 A2 A3 A4 :: 'agent and a b c d :: 'alt
assumes distinct-agents: distinct [A1, A2, A3, A4]
  and distinct-alts: distinct [a, b, c, d]
  and agents: agents = {A1, A2, A3, A4}
  and alts: alts = {a, b, c, d}
begin

lemma an-sds: an-sds agents alts sds (proof)
lemma ex-post-efficient-sds: ex-post-efficient-sds agents alts sds (proof)
lemma sd-efficient-sds: sd-efficient-sds agents alts sds (proof)
lemma strategy-proof-an-sds: strategy-proof-an-sds agents alts sds (proof)

lemma distinct-agents' [simp]:
  A1 ≠ A2 A1 ≠ A3 A1 ≠ A4 A2 ≠ A1 A2 ≠ A3 A2 ≠ A4
  A3 ≠ A1 A3 ≠ A2 A3 ≠ A4 A4 ≠ A1 A4 ≠ A2 A4 ≠ A3
  (proof)

lemma distinct-alts' [simp]:
  a ≠ b a ≠ c a ≠ d b ≠ a b ≠ c b ≠ d
  c ≠ a c ≠ b c ≠ d d ≠ a d ≠ b d ≠ c
  (proof)

lemma card-agents [simp]: card agents = 4 and card-alts [simp]: card alts = 4
  (proof)

lemma in-agents [simp]: A1 ∈ agents A2 ∈ agents A3 ∈ agents A4 ∈ agents
  (proof)
lemma in-alts [simp]: \( a \in \text{alts} \Rightarrow b \in \text{alts} \Rightarrow c \in \text{alts} \Rightarrow d \in \text{alts} \)

\( \langle \text{proof} \rangle \)

lemma agent-iff: \( x \in \text{agents} \iff x \in \{A1, A2, A3, A4\} \)

\( (\forall x \in \text{agents}.\ P x) \iff P A1 \land P A2 \land P A3 \land P A4 \)

\( (\exists x \in \text{agents}.\ P x) \iff P A1 \lor P A2 \lor P A3 \lor P A4 \)

\( \langle \text{proof} \rangle \)

lemma alt-iff: \( x \in \text{alts} \iff x \in \{a, b, c, d\} \)

\( (\forall x \in \text{alts}.\ P x) \iff P a \land P b \land P c \land P d \)

\( (\exists x \in \text{alts}.\ P x) \iff P a \lor P b \lor P c \lor P d \)

\( \langle \text{proof} \rangle \)

1.2 Definition of Preference Profiles and Fact Gathering

preference-profile

agents: \text{agents}

alts: \text{alts}

where \(R1 = A1: [c, d], [a, b] \quad A2: [b, d], a, c \quad A3: a, b, [c, d] \quad A4: [a, c, b, d]\)

\(R2 = A1: [a, c], [b, d] \quad A2: [c, d], a, b \quad A3: [b, d], a, c \quad A4: a, b, c, a\)

\(R3 = A1: [b, c], [a, d] \quad A2: [c, d], [a, b] \quad A3: d, [a, b], c \quad A4: c, a, b, d\)

\(R4 = A1: [a, b], [c, d] \quad A2: [a, d], [b, c] \quad A3: c, [a, b], d \quad A4: d, c, a, b\)

\(R5 = A1: [c, d], [a, b] \quad A2: [a, b], [c, d] \quad A3: [a, c], d, b \quad A4: d, a, c, b\)

\(R6 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: [a, c], [b, d] \quad A4: d, b, a, c\)

\(R7 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: a, c, d, b \quad A4: d, a, c, b\)

\(R8 = A1: [a, b], [c, d] \quad A2: [a, c], [b, d] \quad A3: d, [a, b], c \quad A4: d, c, a, b\)

\(R9 = A1: [a, b], [c, d] \quad A2: [a, d], [c, b] \quad A3: d, c, [a, b] \quad A4: [a, b, c, d]\)

\(R10 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: [a, c], d, b \quad A4: [b, a, c, d]\)

\(R11 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: d, [a, b], c \quad A4: c, a, b, d\)

\(R12 = A1: [c, d], [a, b] \quad A2: [a, b], [c, d] \quad A3: [a, c], d, b \quad A4: [a, b, c, d]\)

\(R13 = A1: [a, c], [b, d] \quad A2: [c, d], [a, b] \quad A3: [b, d], a, c \quad A4: a, b, c, d\)

\(R14 = A1: [a, b], [c, d] \quad A2: d, c, [a, b] \quad A3: [a, b, c], d \quad A4: a, c, d, b\)

\(R15 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: [b, d], a, c \quad A4: a, c, d, b\)
\begin{align*}
\text{and } R16 &= A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: a, c, d, b \quad A4: [a, b, d], c \\
\text{and } R17 &= A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: [a, c], [b, d] \quad A4: [a, d], c \\
\text{and } R18 &= A1: [a, b], [c, d] \quad A2: [a, d], [b, c] \quad A3: [a, b, c], d \quad A4: d, c, \\
\text{and } R19 &= A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: [b, d], a, c \quad A4: [a, b, d], c \\
\text{and } R20 &= A1: [b, d], a, c \quad A2: b, a, [c, d] \quad A3: a, c, [b, d] \quad A4: d, \\
\text{and } R21 &= A1: [a, d], c, b \quad A2: d, c, [a, b] \quad A3: [a, b, d], A4: a, \\
\text{and } R22 &= A1: [a, c], d, b \quad A2: d, c, [a, b] \quad A3: d, [a, b], c \quad A4: a, \\
\text{and } R23 &= A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: [a, c], [b, d] \quad A4: [a, b, d], c \\
\text{and } R24 &= A1: [c, d], [a, b] \quad A2: d, b, a, c \quad A3: c, a, [b, d] \quad A4: b, \\
\text{and } R25 &= A1: [c, d], [a, b] \quad A2: [b, d], a, c \quad A3: a, b, [c, d] \quad A4: a, \\
\text{and } R26 &= A1: [b, d], [a, c] \quad A2: [c, d], [a, b] \quad A3: a, b, [c, d] \quad A4: [a, b, d], c \\
\text{and } R27 &= A1: [a, b], [c, d] \quad A2: [b, d], a, c \quad A3: a, [a, b], [b, d] \quad A4: [a, b, c], d \\
\text{and } R28 &= A1: [c, d], a, b \quad A2: [b, d], a, c \quad A3: a, b, [c, d] \quad A4: a, \\
\text{and } R29 &= A1: [a, c], d, b \quad A2: [b, d], a, c \quad A3: a, b, [c, d] \quad A4: d, \\
\text{and } R30 &= A1: [a, c], d, b \quad A2: d, c, [a, b] \quad A3: c, [a, b], d \quad A4: [a, b, d], c \\
\text{and } R31 &= A1: [b, d], a, c \quad A2: [a, c], d, b \quad A3: c, d, [a, b] \quad A4: [a, b, d], c \\
\text{and } R32 &= A1: [a, c], d, b \quad A2: d, c, [a, b] \quad A3: d, [a, b], c \quad A4: [a, b, d], c \\
\text{and } R33 &= A1: [c, d], [a, b] \quad A2: [a, c], d, b \quad A3: a, b, [c, d] \quad A4: d, \\
\text{and } R34 &= A1: [a, b], [c, d] \quad A2: a, c, d, b \quad A3: b, [a, b], c \quad A4: a, \\
\text{and } R35 &= A1: [a, b], [c, d] \quad A2: a, b, [c, d] \quad A3: a, [a, b], [b, d] \quad A4: d, \\
\text{and } R36 &= A1: [c, d], [a, b] \quad A2: [a, c], d, b \quad A3: a, b, [c, d] \quad A4: d, \\
\text{and } R37 &= A1: [b, d], [a, c] \quad A2: [b, d], [a, c] \quad A3: a, b, [c, d] \quad A4: c, \\
\text{and } R38 &= A1: [c, d], a, b \quad A2: [b, d], a, c \quad A3: a, b, [c, d] \quad A4: [a, b, d], c \\
\text{and } R39 &= A1: [a, c], d, b \quad A2: [b, d], a, c \quad A3: a, b, [c, d] \quad A4: [c, b, d], a \\
\text{and } R40 &= A1: [a, d], c, b \quad A2: [a, b], c, d \quad A3: [a, b, c], d \quad A4: d, 
\end{align*}
c, [a, b] 
and R41 = A1: [a, d], c, b  A2: [a, b], d, c  A3: [a, b, c], d  A4: d,
c, [a, b] 
and R42 = A1: [c, d], [a, b]  A2: [a, b], [c, d]  A3: d, b, a, c  A4: c,a, [b, d] 
and R43 = A1: [a, b], [c, d]  A2: [c, d], [a, b]  A3: d, [a, b], c  A4: a,[c, d], b 
and R44 = A1: [c, d], [a, b]  A2: [a, c], d, b  A3: [a, b], d, c  A4: [a, b, d], c 
and R45 = A1: [a, c], d, b  A2: [b, d], a, c  A3: [a, b, c], d  A4: [c, d], b, a 
and R46 = A1: [b, d], a, c  A2: d, c, [a, b]  A3: [a, c], [b, d]  A4: b,a, [c, d] 
and R47 = A1: [a, b], [c, d]  A2: [a, d], c, b  A3: d, c, [a, b]  A4: c, [a, b], d 
(proof)

derive-orbit-equations (an-sds) 
R10 R26 R27 R28 R29 R43 R45 
(proof)

prove-inefficient-supports (ex-post-efficient-sds sd-efficient-sds) 
R3 [b] and R4 [b] and R5 [b] and R7 [b] and R8 [b] and R9 [b] and R11 [b] and R12 [b] and R14 [b] and R16 [b] and R17 [b] and R18 [b] and R21 [b] and R22 [b] and R23 [b] and R30 [b] and R32 [b] and R33 [b] and R35 [b] and R40 [b] and R41 [b] and R43 [b] and R44 [b] and R47 [b] and R10 [c, b] witness: [a: 1 / 2, b: 0, c: 0, d: 1 / 2] and R15 [c, b] witness: [a: 1 / 2, b: 0, c: 0, d: 1 / 2] and R19 [c, b] witness: [a: 1 / 2, b: 0, c: 0, d: 1 / 2] and R25 [b, c] witness: [c: 0, d: 1 / 2, a: 1 / 2, b: 0] and R26 [c, b] witness: [b: 0, d: 1 / 2, a: 1 / 2, c: 0] and R27 [c, b] witness: [a: 1 / 2, b: 0, c: 0, d: 1 / 2] and R28 [b, c] witness: [c: 0, d: 1 / 2, a: 1 / 2, b: 0] and R29 [b, c] witness: [a: 1 / 2, b: 0, c: 0, d: 1 / 2, b: 0] and R39 [b, c] witness: [a: 1 / 2, b: 0, c: 0, d: 1 / 2, b: 0] 
(proof)

derive-strategyproofness-conditions (strategyproof-an-sds) 
distance: 2 
(proof)

lemma lottery-conditions: 
assumes is-pref-profile R

5
shows \[ \text{pmf (sds R)} \ a \geq 0 \ \text{pmf (sds R)} \ b \geq 0 \ \text{pmf (sds R)} \ c \geq 0 \ \text{pmf (sds R)} \ d \geq 0 \]
\[ \text{pmf (sds R)} \ a + \text{pmf (sds R)} \ b + \text{pmf (sds R)} \ c + \text{pmf (sds R)} \ d = 1 \]

\[ \langle \text{proof} \rangle \]

1.3 Main Proof

lemma R45 [simp]: \[ \text{pmf (sds R45)} \ a = 1/4 \ \text{pmf (sds R45)} \ b = 1/4 \]
\[ \text{pmf (sds R45)} \ c = 1/4 \ \text{pmf (sds R45)} \ d = 1/4 \]
\[ \langle \text{proof} \rangle \]

lemma R10-bc [simp]: \[ \text{pmf (sds R10)} \ b = 0 \ \text{pmf (sds R10)} \ c = 0 \]
\[ \langle \text{proof} \rangle \]

lemma R10-ad [simp]: \[ \text{pmf (sds R10)} \ a = 1/2 \ \text{pmf (sds R10)} \ d = 1/2 \]
\[ \langle \text{proof} \rangle \]

lemma R26-bc [simp]: \[ \text{pmf (sds R26)} \ b = 0 \ \text{pmf (sds R26)} \ c = 0 \]
\[ \langle \text{proof} \rangle \]

lemma R26-d [simp]: \[ \text{pmf (sds R26)} \ d = 1 - \text{pmf (sds R26)} \ a \]
\[ \langle \text{proof} \rangle \]

lemma R27-bc [simp]: \[ \text{pmf (sds R27)} \ b = 0 \ \text{pmf (sds R27)} \ c = 0 \]
\[ \langle \text{proof} \rangle \]

lemma R27-d [simp]: \[ \text{pmf (sds R27)} \ d = 1 - \text{pmf (sds R27)} \ a \]
\[ \langle \text{proof} \rangle \]

lemma R28-bc [simp]: \[ \text{pmf (sds R28)} \ b = 0 \ \text{pmf (sds R28)} \ c = 0 \]
\[ \langle \text{proof} \rangle \]

lemma R28-d [simp]: \[ \text{pmf (sds R28)} \ d = 1 - \text{pmf (sds R28)} \ a \]
\[ \langle \text{proof} \rangle \]

lemma R29-bc [simp]: \[ \text{pmf (sds R29)} \ b = 0 \ \text{pmf (sds R29)} \ c = 0 \]
\[ \langle \text{proof} \rangle \]

lemma R29-ac [simp]: \[ \text{pmf (sds R29)} \ a = 1/2 \ \text{pmf (sds R29)} \ d = 1/2 \]
\[ \langle \text{proof} \rangle \]

lemmas R43-bc [simp] = R43.support

lemma R43-ad [simp]: \[ \text{pmf (sds R43)} \ a = 1/2 \ \text{pmf (sds R43)} \ d = 1/2 \]
lemma R39-b [simp]: \text{pmf (sds R39)} b = 0

lemma R36-a [simp]: \text{pmf (sds R36)} a = 1/2 \text{ and } R36-b [simp]: \text{pmf (sds R36)} b = 0

lemma R36-d [simp]: \text{pmf (sds R36)} d = 1/2 - \text{pmf (sds R36)} c

lemma R39-a [simp]: \text{pmf (sds R39)} a = 1/2

lemma R39-d [simp]: \text{pmf (sds R39)} d = 1/2 - \text{pmf (sds R39)} c

lemmas R12-b [simp] = R12.support

lemma R12-c [simp]: \text{pmf (sds R12)} c = 0

lemma R12-d [simp]: \text{pmf (sds R12)} d = 1 - \text{pmf (sds R12)} a

lemma R12-a-ge-one-half: \text{pmf (sds R12)} a \geq 1/2

lemma R44 [simp]:
\text{pmf (sds R44)} a = \text{pmf (sds R12)} a \text{ pmf (sds R44)} d = 1 - \text{pmf (sds R12)} a
\text{pmf (sds R44)} b = 0 \text{ pmf (sds R44)} c = 0

lemma R9-a [simp]: \text{pmf (sds R9)} a = \text{pmf (sds R35)} a

lemma R18-c [simp]: \text{pmf (sds R18)} c = \text{pmf (sds R9)} c

lemma R5-d-ge-one-half: \text{pmf (sds R5)} d \geq 1/2

lemma R7 [simp]: \text{pmf (sds R7)} a = 1/2 \text{ pmf (sds R7)} b = 0 \text{ pmf (sds R7)} c
\[
= 0 \text{ pmf } (sds \ R7) \ d = 1/2
\]
\langle \text{proof} \rangle

\textbf{lemma R5 [simp]: } pmf (sds \ R5) \ a = 1/2 \text{ pmf } (sds \ R5) \ b = 0 \text{ pmf } (sds \ R5) \ c = 0 \text{ pmf } (sds \ R5) \ d = 1/2
\langle \text{proof} \rangle

\textbf{lemma R15 [simp]: } pmf (sds \ R15) \ a = 1/2 \text{ pmf } (sds \ R15) \ b = 0 \text{ pmf } (sds \ R15) \ c = 0 \text{ pmf } (sds \ R15) \ d = 1/2
\langle \text{proof} \rangle

\textbf{lemma R13-aux: } pmf (sds \ R13) \ b = 0 \text{ pmf } (sds \ R13) \ c = 0 \text{ pmf } (sds \ R13) \ d = 1 - \text{ pmf } (sds \ R13) \ a
\textbf{and R27-R13 [simp]: } pmf (sds \ R27) \ a = \text{ pmf } (sds \ R13) \ a
\langle \text{proof} \rangle

\textbf{lemma R13 [simp]: } pmf (sds \ R13) \ a = 1/2 \text{ pmf } (sds \ R13) \ b = 0 \text{ pmf } (sds \ R13) \ c = 0 \text{ pmf } (sds \ R13) \ d = 1/2
\langle \text{proof} \rangle

\textbf{lemma R27 [simp]: } pmf (sds \ R27) \ a = 1/2 \text{ pmf } (sds \ R27) \ b = 0 \text{ pmf } (sds \ R27) \ c = 0 \text{ pmf } (sds \ R27) \ d = 1/2
\langle \text{proof} \rangle

\textbf{lemma R19 [simp]: } pmf (sds \ R19) \ a = 1/2 \text{ pmf } (sds \ R19) \ b = 0 \text{ pmf } (sds \ R19) \ c = 0 \text{ pmf } (sds \ R19) \ d = 1/2
\langle \text{proof} \rangle

\textbf{lemma R1 [simp]: } pmf (sds \ R1) \ a = 1/2 \text{ pmf } (sds \ R1) \ b = 0
\langle \text{proof} \rangle

\textbf{lemma R22 [simp]: } pmf (sds \ R22) \ a = 1/2 \text{ pmf } (sds \ R22) \ b = 0 \text{ pmf } (sds \ R22) \ c = 0 \text{ pmf } (sds \ R22) \ d = 1/2
\langle \text{proof} \rangle

\textbf{lemma R28 [simp]: } pmf (sds \ R28) \ a = 1/2 \text{ pmf } (sds \ R28) \ b = 0 \text{ pmf } (sds \ R28) \ c = 0 \text{ pmf } (sds \ R28) \ d = 1/2
\langle \text{proof} \rangle

\textbf{lemma R39 [simp]: } pmf (sds \ R39) \ a = 1/2 \text{ pmf } (sds \ R39) \ b = 0 \text{ pmf } (sds \ R39) \ c = 0 \text{ pmf } (sds \ R39) \ d = 1/2
\langle \text{proof} \rangle

\textbf{lemma R2 [simp]: } pmf (sds \ R2) \ a = 1/2 \text{ pmf } (sds \ R2) \ b = 0 \text{ pmf } (sds \ R2) \ c = 0 \text{ pmf } (sds \ R2) \ d = 1/2
\langle \text{proof} \rangle

\textbf{lemma R42 [simp]: } pmf (sds \ R42) \ a = 0 \text{ pmf } (sds \ R42) \ b = 0 \text{ pmf } (sds \ R42) \ c = 1/2 \text{ pmf } (sds \ R42) \ d = 1/2
lemma R37 [simp]: \( \text{pmf} \ (sds \ R37) \ a = \frac{1}{2} \ \text{pmf} \ (sds \ R37) \) \( b = 0 \ \text{pmf} \ (sds \ R37) \) \( c = \frac{1}{2} \ \text{pmf} \ (sds \ R37) \) \( d = 0 \)

lemma R24 [simp]: \( \text{pmf} \ (sds \ R24) \ a = 0 \ \text{pmf} \ (sds \ R24) \) \( b = 0 \ \text{pmf} \ (sds \ R24) \) \( d = 1 - \text{pmf} \ (sds \ R24) \) \( c \)

lemma R34 [simp]:
\[
\begin{align*}
\text{pmf} \ (sds \ R34) \ a &= 1 - \text{pmf} \ (sds \ R24) \ c \\
\text{pmf} \ (sds \ R34) \ b &= \text{pmf} \ (sds \ R24) \ c \\
\text{pmf} \ (sds \ R34) \ c &= 0 \ \text{pmf} \ (sds \ R34) \ d = 0 \\
\end{align*}
\]

lemma R14 [simp]: \( \text{pmf} \ (sds \ R14) \ b = 0 \ \text{pmf} \ (sds \ R14) \) \( d = 0 \ \text{pmf} \ (sds \ R14) \) \( c = 1 - \text{pmf} \ (sds \ R14) \) \( a \)

lemma R46 [simp]: \( \text{pmf} \ (sds \ R46) \ a = 0 \ \text{pmf} \ (sds \ R46) \) \( c = 0 \ \text{pmf} \ (sds \ R46) \) \( d = 1 - \text{pmf} \ (sds \ R46) \) \( b \)

lemma R20 [simp]: \( \text{pmf} \ (sds \ R20) \ a = 0 \ \text{pmf} \ (sds \ R20) \) \( c = 0 \ \text{pmf} \ (sds \ R20) \) \( d = 1 - \text{pmf} \ (sds \ R20) \) \( b \)

lemma R21 [simp]: \( \text{pmf} \ (sds \ R21) \ d = 1 - \text{pmf} \ (sds \ R21) \ a \) \( \text{pmf} \ (sds \ R21) \ b = 0 \ \text{pmf} \ (sds \ R21) \) \( c = 0 \)

lemma R16-R12: \( \text{pmf} \ (sds \ R16) \) \( c + \text{pmf} \ (sds \ R16) \ a \leq \text{pmf} \ (sds \ R12) \) \( a \)

lemma R16 [simp]: \( \text{pmf} \ (sds \ R16) \ b = 0 \ \text{pmf} \ (sds \ R16) \) \( c = 0 \ \text{pmf} \ (sds \ R16) \) \( d = 1 - \text{pmf} \ (sds \ R16) \) \( a \)

lemma R12-R14: \( \text{pmf} \ (sds \ R14) \) \( a \leq \text{pmf} \ (sds \ R12) \) \( a \)

lemma R12-a [simp]: \( \text{pmf} \ (sds \ R12) \) \( a = \text{pmf} \ (sds \ R9) \) \( a \)

lemma R9 [simp]: \( \text{pmf} \ (sds \ R9) \ b = 0 \ \text{pmf} \ (sds \ R9) \) \( d = 0 \ \text{pmf} \ (sds \ R14) \) \( a = \text{pmf} \ (sds \ R35) \ a \) \( \text{pmf} \ (sds \ R9) \ c = 1 - \text{pmf} \ (sds \ R35) \ a \)
lemma $R23 \ [\text{simp}]$: $\text{pmf}(\text{sds R23}) b = 0 \ \text{pmf}(\text{sds R23}) c = 0 \ \text{pmf}(\text{sds R23}) d = 1 - \text{pmf}(\text{sds R23}) a$

\langle\text{proof}\rangle

lemma $R35 \ [\text{simp}]$: $\text{pmf}(\text{sds R35}) a = \text{pmf}(\text{sds R21}) \ a \ \text{pmf}(\text{sds R35}) b = 0 \ \text{pmf}(\text{sds R35}) c = 0 \ \text{pmf}(\text{sds R35}) d = 1 - \text{pmf}(\text{sds R21}) a$

\langle\text{proof}\rangle

lemma $R18 \ [\text{simp}]$: $\text{pmf}(\text{sds R18}) a = \text{pmf}(\text{sds R14}) \ a \ \text{pmf}(\text{sds R18}) b = 0 \ \text{pmf}(\text{sds R18}) c = 0 \ \text{pmf}(\text{sds R18}) d = 1 - \text{pmf}(\text{sds R14}) a$

\langle\text{proof}\rangle

lemma $R4 \ [\text{simp}]$: $\text{pmf}(\text{sds R4}) a = \text{pmf}(\text{sds R21}) \ a \ \text{pmf}(\text{sds R4}) b = 0 \ \text{pmf}(\text{sds R4}) c = 1 - \text{pmf}(\text{sds R4}) a \ \text{pmf}(\text{sds R4}) d = 0$

\langle\text{proof}\rangle

lemma $R8-\ d \ [\text{simp}]$: $\text{pmf}(\text{sds R8}) d = 1 - \text{pmf}(\text{sds R8}) a$

and $R8-\ c \ [\text{simp}]$: $\text{pmf}(\text{sds R8}) c = 0$

and $R26-\ a \ [\text{simp}]$: $\text{pmf}(\text{sds R26}) a = 1 - \text{pmf}(\text{sds R8}) a$

\langle\text{proof}\rangle

lemma $R21-R47$: $\text{pmf}(\text{sds R21}) d \leq \text{pmf}(\text{sds R47}) c$

\langle\text{proof}\rangle

lemma $R30 \ [\text{simp}]$: $\text{pmf}(\text{sds R30}) a = \text{pmf}(\text{sds R47}) \ a \ \text{pmf}(\text{sds R30}) b = 0 \ \text{pmf}(\text{sds R30}) c = 0 \ \text{pmf}(\text{sds R30}) d = 1 - \text{pmf}(\text{sds R47}) a$

\langle\text{proof}\rangle

lemma $R31-\ c-\ge-\text{one-half}$: $\text{pmf}(\text{sds R31}) c \geq 1/2$

\langle\text{proof}\rangle

lemma $R31$: $\text{pmf}(\text{sds R31}) a = 0 \ \text{pmf}(\text{sds R31}) c = 1/2 \ \text{pmf}(\text{sds R31}) b + \text{pmf}(\text{sds R31}) d = 1/2$

\langle\text{proof}\rangle

\text{lemma} \ \text{absurd}: False

\langle\text{proof}\rangle

end

1.4 Lifting to more than 4 agents and alternatives

\text{lemma} \ \text{finite-list}':
\text{assumes} \ \text{finite} A

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obtains $xs$ where $A = \{x \in \text{set } xs \mid \text{distinct } xs\}$ length $xs = \text{card } A$

lemma finite-list-subset:
  assumes finite $A$ card $A \geq n$
  obtains $xs$ where set $xs \subseteq A$ distinct $xs$ length $xs = n$

lemma card-ge-4E:
  assumes finite $A$ card $A \geq 4$
  obtains $a \ b \ c \ d$ where distinct $\{a,b,c,d\}$ \{a,b,c,d\} $\subseteq A$

context sds-impossibility
begin

lemma absurd: False

end

end

References
