

# The Incompatibility of *SD*-Efficiency and *SD*-Strategy-Proofness

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## Abstract

This formalisation contains the proof that there is no anonymous and neutral Social Decision Scheme for at least four voters and alternatives that fulfils both *SD*-Efficiency and *SD*-Strategy-Proofness. The proof is a fully structured and quasi-human-readable one. It was derived from the (unstructured) SMT proof of the case for exactly four voters and alternatives by Brandl *et al.* [1].

Their proof relies on an unverified translation of the original problem to SMT, and the proof that lifts the argument for exactly four voters and alternatives to the general case is also not machine-checked.

In this Isabelle proof, on the other hand, all of these steps are also fully proven and machine-checked. This is particularly important seeing as a previously published informal proof of a weaker statement contained a mistake in precisely this lifting step. [2]

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# 1 Incompatibility of SD-Efficiency and SD-Strategy-Proofness

**theory** *SDS-Impossibility*

**imports**

*Randomised-Social-Choice.SDS-Automation*

*Randomised-Social-Choice.Randomised-Social-Choice*

**begin**

## 1.1 Preliminary Definitions

**locale** *sds-impossibility* =

*anonymous-sds agents alts sds* +

*neutral-sds agents alts sds* +

*sd-efficient-sds agents alts sds* +

*strategyproof-sds agents alts sds*

**for** *agents* :: 'agent set **and** *alts* :: 'alt set **and** *sds* +

**assumes** *agents-ge-4*: *card agents*  $\geq 4$

**and** *alts-ge-4*: *card alts*  $\geq 4$

**locale** *sds-impossibility-4-4* = *sds-impossibility agents alts sds*

**for** *agents* :: 'agent set **and** *alts* :: 'alt set **and** *sds* +

**fixes** *A1 A2 A3 A4* :: 'agent **and** *a b c d* :: 'alt

**assumes** *distinct-agents*: *distinct* [*A1, A2, A3, A4*]

**and** *distinct-alts*: *distinct* [*a, b, c, d*]

**and** *agents*: *agents* = {*A1, A2, A3, A4*}

**and** *alts*: *alts* = {*a, b, c, d*}

**begin**

**lemma** *an-sds*: *an-sds agents alts sds*  $\langle$ *proof* $\rangle$

**lemma** *ex-post-efficient-sds*: *ex-post-efficient-sds agents alts sds*  $\langle$ *proof* $\rangle$

**lemma** *sd-efficient-sds*: *sd-efficient-sds agents alts sds*  $\langle$ *proof* $\rangle$

**lemma** *strategyproof-an-sds*: *strategyproof-an-sds agents alts sds*  $\langle$ *proof* $\rangle$

**lemma** *distinct-agents'* [*simp*]:

*A1*  $\neq$  *A2* *A1*  $\neq$  *A3* *A1*  $\neq$  *A4* *A2*  $\neq$  *A1* *A2*  $\neq$  *A3* *A2*  $\neq$  *A4*

*A3*  $\neq$  *A1* *A3*  $\neq$  *A2* *A3*  $\neq$  *A4* *A4*  $\neq$  *A1* *A4*  $\neq$  *A2* *A4*  $\neq$  *A3*

$\langle$ *proof* $\rangle$

**lemma** *distinct-alts'* [*simp*]:

*a*  $\neq$  *b* *a*  $\neq$  *c* *a*  $\neq$  *d* *b*  $\neq$  *a* *b*  $\neq$  *c* *b*  $\neq$  *d*

*c*  $\neq$  *a* *c*  $\neq$  *b* *c*  $\neq$  *d* *d*  $\neq$  *a* *d*  $\neq$  *b* *d*  $\neq$  *c*

$\langle$ *proof* $\rangle$

**lemma** *card-agents* [*simp*]: *card agents* = 4 **and** *card-alts* [*simp*]: *card alts* = 4

$\langle$ *proof* $\rangle$

**lemma** *in-agents* [*simp*]: *A1*  $\in$  *agents* *A2*  $\in$  *agents* *A3*  $\in$  *agents* *A4*  $\in$  *agents*

$\langle$ *proof* $\rangle$

**lemma** *in-alt*s [simp]:  $a \in \text{alts } b \in \text{alts } c \in \text{alts } d \in \text{alts}$

*<proof>*

**lemma** *agent-iff*:  $x \in \text{agents} \longleftrightarrow x \in \{A1, A2, A3, A4\}$

$(\forall x \in \text{agents}. P x) \longleftrightarrow P A1 \wedge P A2 \wedge P A3 \wedge P A4$

$(\exists x \in \text{agents}. P x) \longleftrightarrow P A1 \vee P A2 \vee P A3 \vee P A4$

*<proof>*

**lemma** *alt-iff*:  $x \in \text{alts} \longleftrightarrow x \in \{a, b, c, d\}$

$(\forall x \in \text{alts}. P x) \longleftrightarrow P a \wedge P b \wedge P c \wedge P d$

$(\exists x \in \text{alts}. P x) \longleftrightarrow P a \vee P b \vee P c \vee P d$

*<proof>*

## 1.2 Definition of Preference Profiles and Fact Gathering

**preference-profile**

*agents: agents*

*alts: alts*

**where**  $R1 = A1: [c, d], [a, b] \quad A2: [b, d], a, c \quad A3: a, b, [c, d] \quad A4: [a, c], [b, d]$

**and**  $R2 = A1: [a, c], [b, d] \quad A2: [c, d], a, b \quad A3: [b, d], a, c \quad A4: a, b, [c, d]$

**and**  $R3 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: d, [a, b], c \quad A4: c, a, [b, d]$

**and**  $R4 = A1: [a, b], [c, d] \quad A2: [a, d], [b, c] \quad A3: c, [a, b], d \quad A4: d, c, [a, b]$

**and**  $R5 = A1: [c, d], [a, b] \quad A2: [a, b], [c, d] \quad A3: [a, c], d, b \quad A4: d, [a, b], c$

**and**  $R6 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: [a, c], [b, d] \quad A4: d, b, a, c$

**and**  $R7 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: a, c, d, b \quad A4: d, [a, b], c$

**and**  $R8 = A1: [a, b], [c, d] \quad A2: [a, c], [b, d] \quad A3: d, [a, b], c \quad A4: d, c, [a, b]$

**and**  $R9 = A1: [a, b], [c, d] \quad A2: [a, d], c, b \quad A3: d, c, [a, b] \quad A4: [a, b, c], d$

**and**  $R10 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: [a, c], d, b \quad A4: [b, d], a, c$

**and**  $R11 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: d, [a, b], c \quad A4: c, a, b, d$

**and**  $R12 = A1: [c, d], [a, b] \quad A2: [a, b], [c, d] \quad A3: [a, c], d, b \quad A4: [a, b, d], c$

**and**  $R13 = A1: [a, c], [b, d] \quad A2: [c, d], a, b \quad A3: [b, d], a, c \quad A4: a, b, d, c$

**and**  $R14 = A1: [a, b], [c, d] \quad A2: d, c, [a, b] \quad A3: [a, b, c], d \quad A4: a, d, c, b$

**and**  $R15 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: [b, d], a, c \quad A4: a, c, d, b$

**and**  $R16 = A1: [a, b], [c, d]$      $A2: [c, d], [a, b]$      $A3: a, c, d, b$      $A4: [a, b, d], c$   
**and**  $R17 = A1: [a, b], [c, d]$      $A2: [c, d], [a, b]$      $A3: [a, c], [b, d]$      $A4: d, [a, b], c$   
**and**  $R18 = A1: [a, b], [c, d]$      $A2: [a, d], [b, c]$      $A3: [a, b, c], d$      $A4: d, c, [a, b]$   
**and**  $R19 = A1: [a, b], [c, d]$      $A2: [c, d], [a, b]$      $A3: [b, d], a, c$      $A4: [a, c], [b, d]$   
**and**  $R20 = A1: [b, d], a, c$      $A2: b, a, [c, d]$      $A3: a, c, [b, d]$      $A4: d, c, [a, b]$   
**and**  $R21 = A1: [a, d], c, b$      $A2: d, c, [a, b]$      $A3: c, [a, b], d$      $A4: a, b, [c, d]$   
**and**  $R22 = A1: [a, c], d, b$      $A2: d, c, [a, b]$      $A3: d, [a, b], c$      $A4: a, b, [c, d]$   
**and**  $R23 = A1: [a, b], [c, d]$      $A2: [c, d], [a, b]$      $A3: [a, c], [b, d]$      $A4: [a, b, d], c$   
**and**  $R24 = A1: [c, d], [a, b]$      $A2: d, b, a, c$      $A3: c, a, [b, d]$      $A4: b, a, [c, d]$   
**and**  $R25 = A1: [c, d], [a, b]$      $A2: [b, d], a, c$      $A3: a, b, [c, d]$      $A4: a, c, [b, d]$   
**and**  $R26 = A1: [b, d], [a, c]$      $A2: [c, d], [a, b]$      $A3: a, b, [c, d]$      $A4: a, c, [b, d]$   
**and**  $R27 = A1: [a, b], [c, d]$      $A2: [b, d], a, c$      $A3: [a, c], [b, d]$      $A4: [c, d], a, b$   
**and**  $R28 = A1: [c, d], a, b$      $A2: [b, d], a, c$      $A3: a, b, [c, d]$      $A4: a, c, [b, d]$   
**and**  $R29 = A1: [a, c], d, b$      $A2: [b, d], a, c$      $A3: a, b, [c, d]$      $A4: d, c, [a, b]$   
**and**  $R30 = A1: [a, d], c, b$      $A2: d, c, [a, b]$      $A3: c, [a, b], d$      $A4: [a, b], d, c$   
**and**  $R31 = A1: [b, d], a, c$      $A2: [a, c], d, b$      $A3: c, d, [a, b]$      $A4: [a, b], c, d$   
**and**  $R32 = A1: [a, c], d, b$      $A2: d, c, [a, b]$      $A3: d, [a, b], c$      $A4: [a, b], d, c$   
**and**  $R33 = A1: [c, d], [a, b]$      $A2: [a, c], d, b$      $A3: a, b, [c, d]$      $A4: d, [a, b], c$   
**and**  $R34 = A1: [a, b], [c, d]$      $A2: a, c, d, b$      $A3: b, [a, d], c$      $A4: c, d, [a, b]$   
**and**  $R35 = A1: [a, d], c, b$      $A2: a, b, [c, d]$      $A3: [a, b, c], d$      $A4: d, c, [a, b]$   
**and**  $R36 = A1: [c, d], [a, b]$      $A2: [a, c], d, b$      $A3: [b, d], a, c$      $A4: a, b, [c, d]$   
**and**  $R37 = A1: [a, c], [b, d]$      $A2: [b, d], [a, c]$      $A3: a, b, [c, d]$      $A4: c, d, [a, b]$   
**and**  $R38 = A1: [c, d], a, b$      $A2: [b, d], a, c$      $A3: a, b, [c, d]$      $A4: [a, c], b, d$   
**and**  $R39 = A1: [a, c], d, b$      $A2: [b, d], a, c$      $A3: a, b, [c, d]$      $A4: [c, d], a, b$   
**and**  $R40 = A1: [a, d], c, b$      $A2: [a, b], c, d$      $A3: [a, b, c], d$      $A4: d, c, [a, b]$

$[a, b]$   
**and**  $R41 = A1: [a, d], c, b \quad A2: [a, b], d, c \quad A3: [a, b, c], d \quad A4: d, c,$   
 $[a, b]$   
**and**  $R42 = A1: [c, d], [a, b] \quad A2: [a, b], [c, d] \quad A3: d, b, a, c \quad A4: c, a,$   
 $[b, d]$   
**and**  $R43 = A1: [a, b], [c, d] \quad A2: [c, d], [a, b] \quad A3: d, [a, b], c \quad A4: a, [c,$   
 $d], b$   
**and**  $R44 = A1: [c, d], [a, b] \quad A2: [a, c], d, b \quad A3: [a, b], d, c \quad A4: [a, b,$   
 $d], c$   
**and**  $R45 = A1: [a, c], d, b \quad A2: [b, d], a, c \quad A3: [a, b], c, d \quad A4: [c,$   
 $d], b, a$   
**and**  $R46 = A1: [b, d], a, c \quad A2: d, c, [a, b] \quad A3: [a, c], [b, d] \quad A4: b, a,$   
 $[c, d]$   
**and**  $R47 = A1: [a, b], [c, d] \quad A2: [a, d], c, b \quad A3: d, c, [a, b] \quad A4: c,$   
 $[a, b], d$   
*<proof>*

**derive-orbit-equations** (*an-sds*)

$R10 \ R26 \ R27 \ R28 \ R29 \ R43 \ R45$   
*<proof>*

**prove-inefficient-supports** (*ex-post-efficient-sds sd-efficient-sds*)

$R3 [b]$  **and**  $R4 [b]$  **and**  $R5 [b]$  **and**  $R7 [b]$  **and**  $R8 [b]$  **and**  
 $R9 [b]$  **and**  $R11 [b]$  **and**  $R12 [b]$  **and**  $R14 [b]$  **and**  $R16 [b]$  **and**  
 $R17 [b]$  **and**  $R18 [b]$  **and**  $R21 [b]$  **and**  $R22 [b]$  **and**  $R23 [b]$  **and**  
 $R30 [b]$  **and**  $R32 [b]$  **and**  $R33 [b]$  **and**  $R35 [b]$  **and**  $R40 [b]$  **and**  
 $R41 [b]$  **and**  $R43 [b]$  **and**  $R44 [b]$  **and**  $R47 [b]$  **and**  
 $R10 [c, b]$  *witness:*  $[a: 1 / 2, b: 0, c: 0, d: 1 / 2]$  **and**  
 $R15 [c, b]$  *witness:*  $[a: 1 / 2, b: 0, c: 0, d: 1 / 2]$  **and**  
 $R19 [c, b]$  *witness:*  $[a: 1 / 2, b: 0, c: 0, d: 1 / 2]$  **and**  
 $R25 [b, c]$  *witness:*  $[c: 0, d: 1 / 2, a: 1 / 2, b: 0]$  **and**  
 $R26 [c, b]$  *witness:*  $[b: 0, d: 1 / 2, a: 1 / 2, c: 0]$  **and**  
 $R27 [c, b]$  *witness:*  $[a: 1 / 2, b: 0, c: 0, d: 1 / 2]$  **and**  
 $R28 [b, c]$  *witness:*  $[c: 0, d: 1 / 2, a: 1 / 2, b: 0]$  **and**  
 $R29 [b, c]$  *witness:*  $[a: 1 / 2, c: 0, d: 1 / 2, b: 0]$  **and**  
 $R39 [b, c]$  *witness:*  $[a: 1 / 2, c: 0, d: 1 / 2, b: 0]$   
*<proof>*

**derive-strategyproofness-conditions** (*strategyproof-an-sds*)

*distance: 2*  
 $R1 \ R2 \ R3 \ R4 \ R5 \ R6 \ R7 \ R8 \ R9 \ R10 \ R11 \ R12 \ R13 \ R14 \ R15 \ R16 \ R17 \ R18 \ R19$   
 $R20$   
 $R21 \ R22 \ R23 \ R24 \ R25 \ R26 \ R27 \ R28 \ R29 \ R30 \ R31 \ R32 \ R33 \ R34 \ R35 \ R36 \ R37$   
 $R38 \ R39 \ R40$   
 $R41 \ R42 \ R43 \ R44 \ R45 \ R46 \ R47$   
*<proof>*

**lemma** *lottery-conditions:*

**assumes** *is-pref-profile*  $R$

**shows**  $pmf (sds R) a \geq 0$   $pmf (sds R) b \geq 0$   $pmf (sds R) c \geq 0$   $pmf (sds R) d \geq 0$   
 $pmf (sds R) a + pmf (sds R) b + pmf (sds R) c + pmf (sds R) d = 1$   
 ⟨proof⟩

### 1.3 Main Proof

**lemma** *R45 [simp]*:  $pmf (sds R45) a = 1/4$   $pmf (sds R45) b = 1/4$   
 $pmf (sds R45) c = 1/4$   $pmf (sds R45) d = 1/4$   
 ⟨proof⟩

**lemma** *R10-bc [simp]*:  $pmf (sds R10) b = 0$   $pmf (sds R10) c = 0$   
 ⟨proof⟩

**lemma** *R10-ad [simp]*:  $pmf (sds R10) a = 1/2$   $pmf (sds R10) d = 1/2$   
 ⟨proof⟩

**lemma** *R26-bc [simp]*:  $pmf (sds R26) b = 0$   $pmf (sds R26) c = 0$   
 ⟨proof⟩

**lemma** *R26-d [simp]*:  $pmf (sds R26) d = 1 - pmf (sds R26) a$   
 ⟨proof⟩

**lemma** *R27-bc [simp]*:  $pmf (sds R27) b = 0$   $pmf (sds R27) c = 0$   
 ⟨proof⟩

**lemma** *R27-d [simp]*:  $pmf (sds R27) d = 1 - pmf (sds R27) a$   
 ⟨proof⟩

**lemma** *R28-bc [simp]*:  $pmf (sds R28) b = 0$   $pmf (sds R28) c = 0$   
 ⟨proof⟩

**lemma** *R28-d [simp]*:  $pmf (sds R28) d = 1 - pmf (sds R28) a$   
 ⟨proof⟩

**lemma** *R29-bc [simp]*:  $pmf (sds R29) b = 0$   $pmf (sds R29) c = 0$   
 ⟨proof⟩

**lemma** *R29-ac [simp]*:  $pmf (sds R29) a = 1/2$   $pmf (sds R29) d = 1/2$   
 ⟨proof⟩

**lemmas** *R43-bc [simp]* = *R43.support*

**lemma** *R43-ad [simp]*:  $pmf (sds R43) a = 1/2$   $pmf (sds R43) d = 1/2$

$\langle proof \rangle$

**lemma** *R39-b [simp]*:  $pmf (sds R39) b = 0$   
 $\langle proof \rangle$

**lemma** *R36-a [simp]*:  $pmf (sds R36) a = 1/2$  **and** *R36-b [simp]*:  $pmf (sds R36) b = 0$   
 $\langle proof \rangle$

**lemma** *R36-d [simp]*:  $pmf (sds R36) d = 1/2 - pmf (sds R36) c$   
 $\langle proof \rangle$

**lemma** *R39-a [simp]*:  $pmf (sds R39) a = 1/2$   
 $\langle proof \rangle$

**lemma** *R39-d [simp]*:  $pmf (sds R39) d = 1/2 - pmf (sds R39) c$   
 $\langle proof \rangle$

**lemmas** *R12-b [simp]* = *R12.support*

**lemma** *R12-c [simp]*:  $pmf (sds R12) c = 0$   
 $\langle proof \rangle$

**lemma** *R12-d [simp]*:  $pmf (sds R12) d = 1 - pmf (sds R12) a$   
 $\langle proof \rangle$

**lemma** *R12-a-ge-one-half*:  $pmf (sds R12) a \geq 1/2$   
 $\langle proof \rangle$

**lemma** *R44 [simp]*:  
 $pmf (sds R44) a = pmf (sds R12) a$   $pmf (sds R44) d = 1 - pmf (sds R12) a$   
 $pmf (sds R44) b = 0$   $pmf (sds R44) c = 0$   
 $\langle proof \rangle$

**lemma** *R9-a [simp]*:  $pmf (sds R9) a = pmf (sds R35) a$   
 $\langle proof \rangle$

**lemma** *R18-c [simp]*:  $pmf (sds R18) c = pmf (sds R9) c$   
 $\langle proof \rangle$

**lemma** *R5-d-ge-one-half*:  $pmf (sds R5) d \geq 1/2$   
 $\langle proof \rangle$

**lemma** *R7 [simp]*:  $pmf (sds R7) a = 1/2$   $pmf (sds R7) b = 0$   $pmf (sds R7) c =$

$0 \text{ pmf } (sds \ R7) \ d = 1/2$   
 $\langle \text{proof} \rangle$

**lemma** *R5* [*simp*]:  $\text{pmf } (sds \ R5) \ a = 1/2 \ \text{pmf } (sds \ R5) \ b = 0 \ \text{pmf } (sds \ R5) \ c = 0 \ \text{pmf } (sds \ R5) \ d = 1/2$   
 $\langle \text{proof} \rangle$

**lemma** *R15* [*simp*]:  $\text{pmf } (sds \ R15) \ a = 1/2 \ \text{pmf } (sds \ R15) \ b = 0 \ \text{pmf } (sds \ R15) \ c = 0 \ \text{pmf } (sds \ R15) \ d = 1/2$   
 $\langle \text{proof} \rangle$

**lemma** *R13-aux*:  $\text{pmf } (sds \ R13) \ b = 0 \ \text{pmf } (sds \ R13) \ c = 0 \ \text{pmf } (sds \ R13) \ d = 1 - \text{pmf } (sds \ R13) \ a$   
**and** *R27-R13* [*simp*]:  $\text{pmf } (sds \ R27) \ a = \text{pmf } (sds \ R13) \ a$   
 $\langle \text{proof} \rangle$

**lemma** *R13* [*simp*]:  $\text{pmf } (sds \ R13) \ a = 1/2 \ \text{pmf } (sds \ R13) \ b = 0 \ \text{pmf } (sds \ R13) \ c = 0 \ \text{pmf } (sds \ R13) \ d = 1/2$   
 $\langle \text{proof} \rangle$

**lemma** *R27* [*simp*]:  $\text{pmf } (sds \ R27) \ a = 1/2 \ \text{pmf } (sds \ R27) \ b = 0 \ \text{pmf } (sds \ R27) \ c = 0 \ \text{pmf } (sds \ R27) \ d = 1/2$   
 $\langle \text{proof} \rangle$

**lemma** *R19* [*simp*]:  $\text{pmf } (sds \ R19) \ a = 1/2 \ \text{pmf } (sds \ R19) \ b = 0 \ \text{pmf } (sds \ R19) \ c = 0 \ \text{pmf } (sds \ R19) \ d = 1/2$   
 $\langle \text{proof} \rangle$

**lemma** *R1* [*simp*]:  $\text{pmf } (sds \ R1) \ a = 1/2 \ \text{pmf } (sds \ R1) \ b = 0$   
 $\langle \text{proof} \rangle$

**lemma** *R22* [*simp*]:  $\text{pmf } (sds \ R22) \ a = 1/2 \ \text{pmf } (sds \ R22) \ b = 0 \ \text{pmf } (sds \ R22) \ c = 0 \ \text{pmf } (sds \ R22) \ d = 1/2$   
 $\langle \text{proof} \rangle$

**lemma** *R28* [*simp*]:  $\text{pmf } (sds \ R28) \ a = 1/2 \ \text{pmf } (sds \ R28) \ b = 0 \ \text{pmf } (sds \ R28) \ c = 0 \ \text{pmf } (sds \ R28) \ d = 1/2$   
 $\langle \text{proof} \rangle$

**lemma** *R39* [*simp*]:  $\text{pmf } (sds \ R39) \ a = 1/2 \ \text{pmf } (sds \ R39) \ b = 0 \ \text{pmf } (sds \ R39) \ c = 0 \ \text{pmf } (sds \ R39) \ d = 1/2$   
 $\langle \text{proof} \rangle$

**lemma** *R2* [*simp*]:  $\text{pmf } (sds \ R2) \ a = 1/2 \ \text{pmf } (sds \ R2) \ b = 0 \ \text{pmf } (sds \ R2) \ c = 0 \ \text{pmf } (sds \ R2) \ d = 1/2$   
 $\langle \text{proof} \rangle$

**lemma** *R42* [*simp*]:  $\text{pmf } (sds \ R42) \ a = 0 \ \text{pmf } (sds \ R42) \ b = 0 \ \text{pmf } (sds \ R42) \ c = 1/2 \ \text{pmf } (sds \ R42) \ d = 1/2$



$\langle proof \rangle$

**lemma R37** [simp]:  $pmf (sds R37) a = 1/2 pmf (sds R37) b = 0 pmf (sds R37) c = 1/2 pmf (sds R37) d = 0$   
 $\langle proof \rangle$

**lemma R24** [simp]:  $pmf (sds R24) a = 0 pmf (sds R24) b = 0 pmf (sds R24) d = 1 - pmf (sds R24) c$   
 $\langle proof \rangle$

**lemma R34** [simp]:  
 $pmf (sds R34) a = 1 - pmf (sds R24) c pmf (sds R34) b = pmf (sds R24) c$   
 $pmf (sds R34) c = 0 pmf (sds R34) d = 0$   
 $\langle proof \rangle$

**lemma R14** [simp]:  $pmf (sds R14) b = 0 pmf (sds R14) d = 0 pmf (sds R14) c = 1 - pmf (sds R14) a$   
 $\langle proof \rangle$

**lemma R46** [simp]:  $pmf (sds R46) a = 0 pmf (sds R46) c = 0 pmf (sds R46) d = 1 - pmf (sds R46) b$   
 $\langle proof \rangle$

**lemma R20** [simp]:  $pmf (sds R20) a = 0 pmf (sds R20) c = 0 pmf (sds R20) d = 1 - pmf (sds R20) b$   
 $\langle proof \rangle$

**lemma R21** [simp]:  $pmf (sds R21) d = 1 - pmf (sds R21) a pmf (sds R21) b = 0 pmf (sds R21) c = 0$   
 $\langle proof \rangle$

**lemma R16-R12**:  $pmf (sds R16) c + pmf (sds R16) a \leq pmf (sds R12) a$   
 $\langle proof \rangle$

**lemma R16** [simp]:  $pmf (sds R16) b = 0 pmf (sds R16) c = 0 pmf (sds R16) d = 1 - pmf (sds R16) a$   
 $\langle proof \rangle$

**lemma R12-R14**:  $pmf (sds R14) a \leq pmf (sds R12) a$   
 $\langle proof \rangle$

**lemma R12-a** [simp]:  $pmf (sds R12) a = pmf (sds R9) a$   
 $\langle proof \rangle$

**lemma R9** [simp]:  $pmf (sds R9) b = 0 pmf (sds R9) d = 0 pmf (sds R14) a = pmf (sds R35) a pmf (sds R9) c = 1 - pmf (sds R35) a$   
 $\langle proof \rangle$

**lemma** *R23* [*simp*]:  $\text{pmf } (sds \ R23) \ b = 0 \ \text{pmf } (sds \ R23) \ c = 0 \ \text{pmf } (sds \ R23) \ d = 1 - \text{pmf } (sds \ R23) \ a$   
 ⟨*proof*⟩

**lemma** *R35* [*simp*]:  $\text{pmf } (sds \ R35) \ a = \text{pmf } (sds \ R21) \ a \ \text{pmf } (sds \ R35) \ b = 0$   
 $\text{pmf } (sds \ R35) \ c = 0 \ \text{pmf } (sds \ R35) \ d = 1 - \text{pmf } (sds \ R21) \ a$   
 ⟨*proof*⟩

**lemma** *R18* [*simp*]:  $\text{pmf } (sds \ R18) \ a = \text{pmf } (sds \ R14) \ a \ \text{pmf } (sds \ R18) \ b = 0$   
 $\text{pmf } (sds \ R18) \ d = 0 \ \text{pmf } (sds \ R18) \ c = 1 - \text{pmf } (sds \ R14) \ a$   
 ⟨*proof*⟩

**lemma** *R4* [*simp*]:  $\text{pmf } (sds \ R4) \ a = \text{pmf } (sds \ R21) \ a \ \text{pmf } (sds \ R4) \ b = 0$   
 $\text{pmf } (sds \ R4) \ c = 1 - \text{pmf } (sds \ R4) \ a \ \text{pmf } (sds \ R4) \ d = 0$   
 ⟨*proof*⟩

**lemma** *R8-d* [*simp*]:  $\text{pmf } (sds \ R8) \ d = 1 - \text{pmf } (sds \ R8) \ a$   
**and** *R8-c* [*simp*]:  $\text{pmf } (sds \ R8) \ c = 0$   
**and** *R26-a* [*simp*]:  $\text{pmf } (sds \ R26) \ a = 1 - \text{pmf } (sds \ R8) \ a$   
 ⟨*proof*⟩

**lemma** *R21-R47*:  $\text{pmf } (sds \ R21) \ d \leq \text{pmf } (sds \ R47) \ c$   
 ⟨*proof*⟩

**lemma** *R30* [*simp*]:  $\text{pmf } (sds \ R30) \ a = \text{pmf } (sds \ R47) \ a \ \text{pmf } (sds \ R30) \ b = 0$   
 $\text{pmf } (sds \ R30) \ c = 0 \ \text{pmf } (sds \ R30) \ d = 1 - \text{pmf } (sds \ R47) \ a$   
 ⟨*proof*⟩

**lemma** *R31-c-ge-one-half*:  $\text{pmf } (sds \ R31) \ c \geq 1/2$   
 ⟨*proof*⟩

**lemma** *R31*:  $\text{pmf } (sds \ R31) \ a = 0 \ \text{pmf } (sds \ R31) \ c = 1/2 \ \text{pmf } (sds \ R31) \ b +$   
 $\text{pmf } (sds \ R31) \ d = 1/2$   
 ⟨*proof*⟩

**lemma** *absurd*: *False*  
 ⟨*proof*⟩

**end**

## 1.4 Lifting to more than 4 agents and alternatives

**lemma** *finite-list'*:  
 assumes *finite A*

**obtains**  $xs$  **where**  $A = \text{set } xs \text{ distinct } xs \text{ length } xs = \text{card } A$   
*<proof>*

**lemma** *finite-list-subset*:

**assumes** *finite*  $A$   $\text{card } A \geq n$

**obtains**  $xs$  **where**  $\text{set } xs \subseteq A$  *distinct*  $xs$   $\text{length } xs = n$   
*<proof>*

**lemma** *card-ge-4E*:

**assumes** *finite*  $A$   $\text{card } A \geq 4$

**obtains**  $a \ b \ c \ d$  **where** *distinct*  $[a,b,c,d]$   $\{a,b,c,d\} \subseteq A$   
*<proof>*

**context** *sds-impossibility*

**begin**

**lemma** *absurd*: *False*

*<proof>*

**end**

**end**

## References

- [1] F. Brandl, F. Brandt, and C. Geist. Proving the incompatibility of Efficiency and Strategyproofness via SMT solving. *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI)*, 2016. Forthcoming.
- [2] F. Brandl, F. Brandt, and W. Suksompong. The impossibility of extending Random Dictatorship to weak preferences. *Economics Letters*, 141:pp. 44 – 47, 2016.