The Incompatibility of *SD*-Efficiency and *SD*-Strategy-Proofness

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Abstract

This formalisation contains the proof that there is no anonymous and neutral Social Decision Scheme for at least four voters and alternatives that fulfils both *SD*-Efficiency and *SD*-Strategy-Proofness. The proof is a fully structured and quasi-human-redable one. It was derived from the (unstructured) SMT proof of the case for exactly four voters and alternatives by Brandl *et al.* [1].

Their proof relies on an unverified translation of the original problem to SMT, and the proof that lifts the argument for exactly four voters and alternatives to the general case is also not machine-checked.

In this Isabelle proof, on the other hand, all of these steps are also fully proven and machine-checked. This is particularly important seeing as a previously published informal proof of a weaker statement contained a mistake in precisely this lifting step. [2]

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1 Incompatibility of SD-Efficiency and SD-Strategy-Proofness

theory *SDS-Impossibility*
imports
Randomised-Social-Choice.SDS-Automation
Randomised-Social-Choice.Randomised-Social-Choice
begin

1.1 Preliminary Definitions

locale *sds-impossibility* =
anonymous-sds agents alts sds +
neutral-sds agents alts sds +
sd-efficient-sds agents alts sds +
strategyproof-sds agents alts sds
for *agents* :: *'agent set* **and** *alts* :: *'alt set* **and** *sds* +
assumes *agents-ge-4*: *card agents* $\geq$ *4*
and *alts-ge-4*: *card alts* $\geq$ *4*

locale *sds-impossibility-4-4* = *sds-impossibility agents alts sds*
for *agents* :: *'agent set* **and** *alts* :: *'alt set* **and** *sds* +
fixes *A1 A2 A3 A4* :: *'agent* **and** *a b c d* :: *'alt*
assumes *distinct-agents*: *distinct* [*A1*, *A2*, *A3*, *A4*]
and *distinct-alts*: *distinct* [*a*, *b*, *c*, *d*]
and *agents*: *agents* = {*A1*, *A2*, *A3*, *A4*}
and *alts*: *alts* = {*a*, *b*, *c*, *d*}
begin

lemma *an-sds*: *an-sds agents alts sds* ⟨*proof*⟩
lemma *ex-post-efficient-sds*: *ex-post-efficient-sds agents alts sds* ⟨*proof*⟩
lemma *sd-efficient-sds*: *sd-efficient-sds agents alts sds* ⟨*proof*⟩
lemma *strategyproof-an-sds*: *strategyproof-an-sds agents alts sds* ⟨*proof*⟩

lemma *distinct-agents'* [*simp*]:
$A1 \neq A2$ $A1 \neq A3$ $A1 \neq A4$ $A2 \neq A1$ $A2 \neq A3$ $A2 \neq A4$
$A3 \neq A1$ $A3 \neq A2$ $A3 \neq A4$ $A4 \neq A1$ $A4 \neq A2$ $A4 \neq A3$
⟨*proof*⟩

lemma *distinct-alts'* [*simp*]:
$a \neq b$ $a \neq c$ $a \neq d$ $b \neq a$ $b \neq c$ $b \neq d$
$c \neq a$ $c \neq b$ $c \neq d$ $d \neq a$ $d \neq b$ $d \neq c$
⟨*proof*⟩

lemma *card-agents* [*simp*]: *card agents* = *4* **and** *card-alts* [*simp*]: *card alts* = *4*
⟨*proof*⟩

lemma *in-agents* [*simp*]: $A1 \in$ *agents* $A2 \in$ *agents* $A3 \in$ *agents* $A4 \in$ *agents*
⟨*proof*⟩

lemma *in-alts* [*simp*]: $a \in alts$ $b \in alts$ $c \in alts$ $d \in alts$
⟨*proof*⟩

lemma *agent-iff*: $x \in agents \longleftrightarrow x \in \{A1, A2, A3, A4\}$
$(\forall x \in agents.\ P\ x) \longleftrightarrow P\ A1 \wedge P\ A2 \wedge P\ A3 \wedge P\ A4$
$(\exists x \in agents.\ P\ x) \longleftrightarrow P\ A1 \vee P\ A2 \vee P\ A3 \vee P\ A4$
⟨*proof*⟩

lemma *alt-iff*: $x \in alts \longleftrightarrow x \in \{a,b,c,d\}$
$(\forall x \in alts.\ P\ x) \longleftrightarrow P\ a \wedge P\ b \wedge P\ c \wedge P\ d$
$(\exists x \in alts.\ P\ x) \longleftrightarrow P\ a \vee P\ b \vee P\ c \vee P\ d$
⟨*proof*⟩

1.2 Definition of Preference Profiles and Fact Gathering

preference-profile
agents: *agents*
alts: *alts*
where *R1* = *A1*: [*c*, *d*], [*a*, *b*] *A2*: [*b*, *d*], *a*, *c* *A3*: *a*, *b*, [*c*, *d*] *A4*: [*a*, *c*], [*b*, *d*]
and *R2* = *A1*: [*a*, *c*], [*b*, *d*] *A2*: [*c*, *d*], *a*, *b* *A3*: [*b*, *d*], *a*, *c* *A4*: *a*, *b*, [*c*, *d*]
and *R3* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: [*c*, *d*], [*a*, *b*] *A3*: *d*, [*a*, *b*], *c* *A4*: *c*, *a*, [*b*, *d*]
and *R4* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: [*a*, *d*], [*b*, *c*] *A3*: *c*, [*a*, *b*], *d* *A4*: *d*, *c*, [*a*, *b*]
and *R5* = *A1*: [*c*, *d*], [*a*, *b*] *A2*: [*a*, *b*], [*c*, *d*] *A3*: [*a*, *c*], *d*, *b* *A4*: *d*, [*a*, *b*], *c*
and *R6* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: [*c*, *d*], [*a*, *b*] *A3*: [*a*, *c*], [*b*, *d*] *A4*: *d*, *b*, *a*, *c*
and *R7* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: [*c*, *d*], [*a*, *b*] *A3*: *a*, *c*, *d*, *b* *A4*: *d*, [*a*, *b*], *c*
and *R8* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: [*a*, *c*], [*b*, *d*] *A3*: *d*, [*a*, *b*], *c* *A4*: *d*, *c*, [*a*, *b*]
and *R9* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: [*a*, *d*], *c*, *b* *A3*: *d*, *c*, [*a*, *b*] *A4*: [*a*, *b*, *c*], *d*
and *R10* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: [*c*, *d*], [*a*, *b*] *A3*: [*a*, *c*], *d*, *b* *A4*: [*b*, *d*], *a*, *c*
and *R11* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: [*c*, *d*], [*a*, *b*] *A3*: *d*, [*a*, *b*], *c* *A4*: *c*, *a*, *b*, *d*
and *R12* = *A1*: [*c*, *d*], [*a*, *b*] *A2*: [*a*, *b*], [*c*, *d*] *A3*: [*a*, *c*], *d*, *b* *A4*: [*a*, *b*, *d*], *c*
and *R13* = *A1*: [*a*, *c*], [*b*, *d*] *A2*: [*c*, *d*], *a*, *b* *A3*: [*b*, *d*], *a*, *c* *A4*: *a*, *b*, *d*, *c*
and *R14* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: *d*, *c*, [*a*, *b*] *A3*: [*a*, *b*, *c*], *d* *A4*: *a*, *d*, *c*, *b*
and *R15* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: [*c*, *d*], [*a*, *b*] *A3*: [*b*, *d*], *a*, *c* *A4*: *a*, *c*, *d*, *b*

and $R16$ = $A1$: $[a, b], [c, d]$ $A2$: $[c, d], [a, b]$ $A3$: a, c, d, b $A4$: $[a, b, d], c$
and $R17$ = $A1$: $[a, b], [c, d]$ $A2$: $[c, d], [a, b]$ $A3$: $[a, c], [b, d]$ $A4$: $d, [a, b], c$
and $R18$ = $A1$: $[a, b], [c, d]$ $A2$: $[a, d], [b, c]$ $A3$: $[a, b, c], d$ $A4$: $d, c, [a, b]$
and $R19$ = $A1$: $[a, b], [c, d]$ $A2$: $[c, d], [a, b]$ $A3$: $[b, d], a, c$ $A4$: $[a, c], [b, d]$
and $R20$ = $A1$: $[b, d], a, c$ $A2$: $b, a, [c, d]$ $A3$: $a, c, [b, d]$ $A4$: $d, c, [a, b]$
and $R21$ = $A1$: $[a, d], c, b$ $A2$: $d, c, [a, b]$ $A3$: $c, [a, b], d$ $A4$: $a, b, [c, d]$
and $R22$ = $A1$: $[a, c], d, b$ $A2$: $d, c, [a, b]$ $A3$: $d, [a, b], c$ $A4$: $a, b, [c, d]$
and $R23$ = $A1$: $[a, b], [c, d]$ $A2$: $[c, d], [a, b]$ $A3$: $[a, c], [b, d]$ $A4$: $[a, b, d], c$
and $R24$ = $A1$: $[c, d], [a, b]$ $A2$: d, b, a, c $A3$: $c, a, [b, d]$ $A4$: $b, a, [c, d]$
and $R25$ = $A1$: $[c, d], [a, b]$ $A2$: $[b, d], a, c$ $A3$: $a, b, [c, d]$ $A4$: $a, c, [b, d]$
and $R26$ = $A1$: $[b, d], [a, c]$ $A2$: $[c, d], [a, b]$ $A3$: $a, b, [c, d]$ $A4$: $a, c, [b, d]$
and $R27$ = $A1$: $[a, b], [c, d]$ $A2$: $[b, d], a, c$ $A3$: $[a, c], [b, d]$ $A4$: $[c, d], a, b$
and $R28$ = $A1$: $[c, d], a, b$ $A2$: $[b, d], a, c$ $A3$: $a, b, [c, d]$ $A4$: $a, c, [b, d]$
and $R29$ = $A1$: $[a, c], d, b$ $A2$: $[b, d], a, c$ $A3$: $a, b, [c, d]$ $A4$: $d, c, [a, b]$
and $R30$ = $A1$: $[a, d], c, b$ $A2$: $d, c, [a, b]$ $A3$: $c, [a, b], d$ $A4$: $[a, b], d, c$
and $R31$ = $A1$: $[b, d], a, c$ $A2$: $[a, c], d, b$ $A3$: $c, d, [a, b]$ $A4$: $[a, b], c, d$
and $R32$ = $A1$: $[a, c], d, b$ $A2$: $d, c, [a, b]$ $A3$: $d, [a, b], c$ $A4$: $[a, b], d, c$
and $R33$ = $A1$: $[c, d], [a, b]$ $A2$: $[a, c], d, b$ $A3$: $a, b, [c, d]$ $A4$: $d, [a, b], c$
and $R34$ = $A1$: $[a, b], [c, d]$ $A2$: a, c, d, b $A3$: $b, [a, d], c$ $A4$: $c, d, [a, b]$
and $R35$ = $A1$: $[a, d], c, b$ $A2$: $a, b, [c, d]$ $A3$: $[a, b, c], d$ $A4$: $d, c, [a, b]$
and $R36$ = $A1$: $[c, d], [a, b]$ $A2$: $[a, c], d, b$ $A3$: $[b, d], a, c$ $A4$: $a, b, [c, d]$
and $R37$ = $A1$: $[a, c], [b, d]$ $A2$: $[b, d], [a, c]$ $A3$: $a, b, [c, d]$ $A4$: $c, d, [a, b]$
and $R38$ = $A1$: $[c, d], a, b$ $A2$: $[b, d], a, c$ $A3$: $a, b, [c, d]$ $A4$: $[a, c], b, d$
and $R39$ = $A1$: $[a, c], d, b$ $A2$: $[b, d], a, c$ $A3$: $a, b, [c, d]$ $A4$: $[c, d], a, b$
and $R40$ = $A1$: $[a, d], c, b$ $A2$: $[a, b], c, d$ $A3$: $[a, b, c], d$ $A4$: $d,$

c, [*a*, *b*]
and *R41* = *A1*: [*a*, *d*], *c*, *b* *A2*: [*a*, *b*], *d*, *c* *A3*: [*a*, *b*, *c*], *d* *A4*: *d*, *c*, [*a*, *b*]
and *R42* = *A1*: [*c*, *d*], [*a*, *b*] *A2*: [*a*, *b*], [*c*, *d*] *A3*: *d*, *b*, *a*, *c* *A4*: *c*, *a*, [*b*, *d*]
and *R43* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: [*c*, *d*], [*a*, *b*] *A3*: *d*, [*a*, *b*], *c* *A4*: *a*, [*c*, *d*], *b*
and *R44* = *A1*: [*c*, *d*], [*a*, *b*] *A2*: [*a*, *c*], *d*, *b* *A3*: [*a*, *b*], *d*, *c* *A4*: [*a*, *b*, *d*], *c*
and *R45* = *A1*: [*a*, *c*], *d*, *b* *A2*: [*b*, *d*], *a*, *c* *A3*: [*a*, *b*], *c*, *d* *A4*: [*c*, *d*], *b*, *a*
and *R46* = *A1*: [*b*, *d*], *a*, *c* *A2*: *d*, *c*, [*a*, *b*] *A3*: [*a*, *c*], [*b*, *d*] *A4*: *b*, *a*, [*c*, *d*]
and *R47* = *A1*: [*a*, *b*], [*c*, *d*] *A2*: [*a*, *d*], *c*, *b* *A3*: *d*, *c*, [*a*, *b*] *A4*: *c*, [*a*, *b*], *d*
⟨*proof*⟩

derive-orbit-equations (*an-sds*)
R10 R26 R27 R28 R29 R43 R45
⟨*proof*⟩

prove-inefficient-supports (*ex-post-efficient-sds sd-efficient-sds*)
R3 [*b*] **and** *R4* [*b*] **and** *R5* [*b*] **and** *R7* [*b*] **and** *R8* [*b*] **and**
R9 [*b*] **and** *R11* [*b*] **and** *R12* [*b*] **and** *R14* [*b*] **and** *R16* [*b*] **and**
R17 [*b*] **and** *R18* [*b*] **and** *R21* [*b*] **and** *R22* [*b*] **and** *R23* [*b*] **and**
R30 [*b*] **and** *R32* [*b*] **and** *R33* [*b*] **and** *R35* [*b*] **and** *R40* [*b*] **and**
R41 [*b*] **and** *R43* [*b*] **and** *R44* [*b*] **and** *R47* [*b*] **and**
R10 [*c*, *b*] *witness*: [*a*: *1* / *2*, *b*: *0*, *c*: *0*, *d*: *1* / *2*] **and**
R15 [*c*, *b*] *witness*: [*a*: *1* / *2*, *b*: *0*, *c*: *0*, *d*: *1* / *2*] **and**
R19 [*c*, *b*] *witness*: [*a*: *1* / *2*, *b*: *0*, *c*: *0*, *d*: *1* / *2*] **and**
R25 [*b*, *c*] *witness*: [*c*: *0*, *d*: *1* / *2*, *a*: *1* / *2*, *b*: *0*] **and**
R26 [*c*, *b*] *witness*: [*b*: *0*, *d*: *1* / *2*, *a*: *1* / *2*, *c*: *0*] **and**
R27 [*c*, *b*] *witness*: [*a*: *1* / *2*, *b*: *0*, *c*: *0*, *d*: *1* / *2*] **and**
R28 [*b*, *c*] *witness*: [*c*: *0*, *d*: *1* / *2*, *a*: *1* / *2*, *b*: *0*] **and**
R29 [*b*, *c*] *witness*: [*a*: *1* / *2*, *c*: *0*, *d*: *1* / *2*, *b*: *0*] **and**
R39 [*b*, *c*] *witness*: [*a*: *1* / *2*, *c*: *0*, *d*: *1* / *2*, *b*: *0*]
⟨*proof*⟩

derive-strategyproofness-conditions (*strategyproof-an-sds*)
distance: *2*
R1 R2 R3 R4 R5 R6 R7 R8 R9 R10 R11 R12 R13 R14 R15 R16 R17 R18 R19 R20
R21 R22 R23 R24 R25 R26 R27 R28 R29 R30 R31 R32 R33 R34 R35 R36 R37 R38 R39 R40
R41 R42 R43 R44 R45 R46 R47
⟨*proof*⟩

lemma *lottery-conditions*:
assumes *is-pref-profile R*

shows $pmf\ (sds\ R)\ a \geq 0$ $pmf\ (sds\ R)\ b \geq 0$ $pmf\ (sds\ R)\ c \geq 0$ $pmf\ (sds\ R)\ d \geq 0$
$pmf\ (sds\ R)\ a + pmf\ (sds\ R)\ b + pmf\ (sds\ R)\ c + pmf\ (sds\ R)\ d = 1$
⟨*proof*⟩

1.3 Main Proof

lemma *R45* [*simp*]: $pmf\ (sds\ R45)\ a = 1/4$ $pmf\ (sds\ R45)\ b = 1/4$
$pmf\ (sds\ R45)\ c = 1/4$ $pmf\ (sds\ R45)\ d = 1/4$
⟨*proof*⟩

lemma *R10-bc* [*simp*]: $pmf\ (sds\ R10)\ b = 0$ $pmf\ (sds\ R10)\ c = 0$
⟨*proof*⟩

lemma *R10-ad* [*simp*]: $pmf\ (sds\ R10)\ a = 1/2$ $pmf\ (sds\ R10)\ d = 1/2$
⟨*proof*⟩

lemma *R26-bc* [*simp*]: $pmf\ (sds\ R26)\ b = 0$ $pmf\ (sds\ R26)\ c = 0$
⟨*proof*⟩

lemma *R26-d* [*simp*]: $pmf\ (sds\ R26)\ d = 1 - pmf\ (sds\ R26)\ a$
⟨*proof*⟩

lemma *R27-bc* [*simp*]: $pmf\ (sds\ R27)\ b = 0$ $pmf\ (sds\ R27)\ c = 0$
⟨*proof*⟩

lemma *R27-d* [*simp*]: $pmf\ (sds\ R27)\ d = 1 - pmf\ (sds\ R27)\ a$
⟨*proof*⟩

lemma *R28-bc* [*simp*]: $pmf\ (sds\ R28)\ b = 0$ $pmf\ (sds\ R28)\ c = 0$
⟨*proof*⟩

lemma *R28-d* [*simp*]: $pmf\ (sds\ R28)\ d = 1 - pmf\ (sds\ R28)\ a$
⟨*proof*⟩

lemma *R29-bc* [*simp*]: $pmf\ (sds\ R29)\ b = 0$ $pmf\ (sds\ R29)\ c = 0$
⟨*proof*⟩

lemma *R29-ac* [*simp*]: $pmf\ (sds\ R29)\ a = 1/2$ $pmf\ (sds\ R29)\ d = 1/2$
⟨*proof*⟩

lemmas *R43-bc* [*simp*] = *R43.support*

lemma *R43-ad* [*simp*]: $pmf\ (sds\ R43)\ a = 1/2$ $pmf\ (sds\ R43)\ d = 1/2$

⟨*proof*⟩

lemma *R39-b* [*simp*]: *pmf* (*sds R39*) *b* = *0*
⟨*proof*⟩

lemma *R36-a* [*simp*]: *pmf* (*sds R36*) *a* = *1/2* **and** *R36-b* [*simp*]: *pmf* (*sds R36*) *b* = *0*
⟨*proof*⟩

lemma *R36-d* [*simp*]: *pmf* (*sds R36*) *d* = *1/2* − *pmf* (*sds R36*) *c*
⟨*proof*⟩

lemma *R39-a* [*simp*]: *pmf* (*sds R39*) *a* = *1/2*
⟨*proof*⟩

lemma *R39-d* [*simp*]: *pmf* (*sds R39*) *d* = *1/2* − *pmf* (*sds R39*) *c*
⟨*proof*⟩

lemmas *R12-b* [*simp*] = *R12.support*

lemma *R12-c* [*simp*]: *pmf* (*sds R12*) *c* = *0*
⟨*proof*⟩

lemma *R12-d* [*simp*]: *pmf* (*sds R12*) *d* = *1* − *pmf* (*sds R12*) *a*
⟨*proof*⟩

lemma *R12-a-ge-one-half*: *pmf* (*sds R12*) *a* ≥ *1/2*
⟨*proof*⟩

lemma *R44* [*simp*]:
pmf (*sds R44*) *a* = *pmf* (*sds R12*) *a* *pmf* (*sds R44*) *d* = *1* − *pmf* (*sds R12*) *a*
pmf (*sds R44*) *b* = *0* *pmf* (*sds R44*) *c* = *0*
⟨*proof*⟩

lemma *R9-a* [*simp*]: *pmf* (*sds R9*) *a* = *pmf* (*sds R35*) *a*
⟨*proof*⟩

lemma *R18-c* [*simp*]: *pmf* (*sds R18*) *c* = *pmf* (*sds R9*) *c*
⟨*proof*⟩

lemma *R5-d-ge-one-half*: *pmf* (*sds R5*) *d* ≥ *1/2*
⟨*proof*⟩

lemma *R7* [*simp*]: *pmf* (*sds R7*) *a* = *1/2* *pmf* (*sds R7*) *b* = *0* *pmf* (*sds R7*) *c*

= *0* *pmf* (*sds* *R7*) *d* = *1/2*
⟨*proof*⟩

lemma *R5* [*simp*]: *pmf* (*sds* *R5*) *a* = *1/2* *pmf* (*sds* *R5*) *b* = *0* *pmf* (*sds* *R5*) *c* = *0* *pmf* (*sds* *R5*) *d* = *1/2*
⟨*proof*⟩

lemma *R15* [*simp*]: *pmf* (*sds* *R15*) *a* = *1/2* *pmf* (*sds* *R15*) *b* = *0* *pmf* (*sds* *R15*) *c* = *0* *pmf* (*sds* *R15*) *d* = *1/2*
⟨*proof*⟩

lemma *R13-aux*: *pmf* (*sds* *R13*) *b* = *0* *pmf* (*sds* *R13*) *c* = *0* *pmf* (*sds* *R13*) *d* = *1* − *pmf* (*sds* *R13*) *a*
and *R27-R13* [*simp*]: *pmf* (*sds* *R27*) *a* = *pmf* (*sds* *R13*) *a*
⟨*proof*⟩

lemma *R13* [*simp*]: *pmf* (*sds* *R13*) *a* = *1/2* *pmf* (*sds* *R13*) *b* = *0* *pmf* (*sds* *R13*) *c* = *0* *pmf* (*sds* *R13*) *d* = *1/2*
⟨*proof*⟩

lemma *R27* [*simp*]: *pmf* (*sds* *R27*) *a* = *1/2* *pmf* (*sds* *R27*) *b* = *0* *pmf* (*sds* *R27*) *c* = *0* *pmf* (*sds* *R27*) *d* = *1/2*
⟨*proof*⟩

lemma *R19* [*simp*]: *pmf* (*sds* *R19*) *a* = *1/2* *pmf* (*sds* *R19*) *b* = *0* *pmf* (*sds* *R19*) *c* = *0* *pmf* (*sds* *R19*) *d* = *1/2*
⟨*proof*⟩

lemma *R1* [*simp*]: *pmf* (*sds* *R1*) *a* = *1/2* *pmf* (*sds* *R1*) *b* = *0*
⟨*proof*⟩

lemma *R22* [*simp*]: *pmf* (*sds* *R22*) *a* = *1/2* *pmf* (*sds* *R22*) *b* = *0* *pmf* (*sds* *R22*) *c* = *0* *pmf* (*sds* *R22*) *d* = *1/2*
⟨*proof*⟩

lemma *R28* [*simp*]: *pmf* (*sds* *R28*) *a* = *1/2* *pmf* (*sds* *R28*) *b* = *0* *pmf* (*sds* *R28*) *c* = *0* *pmf* (*sds* *R28*) *d* = *1/2*
⟨*proof*⟩

lemma *R39* [*simp*]: *pmf* (*sds* *R39*) *a* = *1/2* *pmf* (*sds* *R39*) *b* = *0* *pmf* (*sds* *R39*) *c* = *0* *pmf* (*sds* *R39*) *d* = *1/2*
⟨*proof*⟩

lemma *R2* [*simp*]: *pmf* (*sds* *R2*) *a* = *1/2* *pmf* (*sds* *R2*) *b* = *0* *pmf* (*sds* *R2*) *c* = *0* *pmf* (*sds* *R2*) *d* = *1/2*
⟨*proof*⟩

lemma *R42* [*simp*]: *pmf* (*sds* *R42*) *a* = *0* *pmf* (*sds* *R42*) *b* = *0* *pmf* (*sds* *R42*) *c* = *1/2* *pmf* (*sds* *R42*) *d* = *1/2*

⟨*proof*⟩

lemma *R37* [*simp*]: *pmf* (*sds R37*) *a* = *1/2 pmf* (*sds R37*) *b* = *0 pmf* (*sds R37*) *c* = *1/2 pmf* (*sds R37*) *d* = *0*
⟨*proof*⟩

lemma *R24* [*simp*]: *pmf* (*sds R24*) *a* = *0 pmf* (*sds R24*) *b* = *0 pmf* (*sds R24*) *d* = *1* − *pmf* (*sds R24*) *c*
⟨*proof*⟩

lemma *R34* [*simp*]:
pmf (*sds R34*) *a* = *1* − *pmf* (*sds R24*) *c pmf* (*sds R34*) *b* = *pmf* (*sds R24*) *c*
pmf (*sds R34*) *c* = *0 pmf* (*sds R34*) *d* = *0*
⟨*proof*⟩

lemma *R14* [*simp*]: *pmf* (*sds R14*) *b* = *0 pmf* (*sds R14*) *d* = *0 pmf* (*sds R14*) *c* = *1* − *pmf* (*sds R14*) *a*
⟨*proof*⟩

lemma *R46* [*simp*]: *pmf* (*sds R46*) *a* = *0 pmf* (*sds R46*) *c* = *0 pmf* (*sds R46*) *d* = *1* − *pmf* (*sds R46*) *b*
⟨*proof*⟩

lemma *R20* [*simp*]: *pmf* (*sds R20*) *a* = *0 pmf* (*sds R20*) *c* = *0 pmf* (*sds R20*) *d* = *1* − *pmf* (*sds R20*) *b*
⟨*proof*⟩

lemma *R21* [*simp*]: *pmf* (*sds R21*) *d* = *1* − *pmf* (*sds R21*) *a pmf* (*sds R21*) *b* = *0 pmf* (*sds R21*) *c* = *0*
⟨*proof*⟩

lemma *R16-R12*: *pmf* (*sds R16*) *c* + *pmf* (*sds R16*) *a* ≤ *pmf* (*sds R12*) *a*
⟨*proof*⟩

lemma *R16* [*simp*]: *pmf* (*sds R16*) *b* = *0 pmf* (*sds R16*) *c* = *0 pmf* (*sds R16*) *d* = *1* − *pmf* (*sds R16*) *a*
⟨*proof*⟩

lemma *R12-R14*: *pmf* (*sds R14*) *a* ≤ *pmf* (*sds R12*) *a*
⟨*proof*⟩

lemma *R12-a* [*simp*]: *pmf* (*sds R12*) *a* = *pmf* (*sds R9*) *a*
⟨*proof*⟩

lemma *R9* [*simp*]: *pmf* (*sds R9*) *b* = *0 pmf* (*sds R9*) *d* = *0 pmf* (*sds R14*) *a* = *pmf* (*sds R35*) *a pmf* (*sds R9*) *c* = *1* − *pmf* (*sds R35*) *a*
⟨*proof*⟩

lemma *R23* [*simp*]: *pmf* (*sds R23*) *b* = *0* *pmf* (*sds R23*) *c* = *0* *pmf* (*sds R23*) *d* = *1* − *pmf* (*sds R23*) *a*
⟨*proof*⟩

lemma *R35* [*simp*]: *pmf* (*sds R35*) *a* = *pmf* (*sds R21*) *a* *pmf* (*sds R35*) *b* = *0* *pmf* (*sds R35*) *c* = *0* *pmf* (*sds R35*) *d* = *1* − *pmf* (*sds R21*) *a*
⟨*proof*⟩

lemma *R18* [*simp*]: *pmf* (*sds R18*) *a* = *pmf* (*sds R14*) *a* *pmf* (*sds R18*) *b* = *0*
pmf (*sds R18*) *d* = *0* *pmf* (*sds R18*) *c* = *1* − *pmf* (*sds R14*) *a*
⟨*proof*⟩

lemma *R4* [*simp*]: *pmf* (*sds R4*) *a* = *pmf* (*sds R21*) *a* *pmf* (*sds R4*) *b* = *0*
pmf (*sds R4*) *c* = *1* − *pmf* (*sds R4*) *a* *pmf* (*sds R4*) *d* = *0*
⟨*proof*⟩

lemma *R8-d* [*simp*]: *pmf* (*sds R8*) *d* = *1* − *pmf* (*sds R8*) *a*
and *R8-c* [*simp*]: *pmf* (*sds R8*) *c* = *0*
and *R26-a* [*simp*]: *pmf* (*sds R26*) *a* = *1* − *pmf* (*sds R8*) *a*
⟨*proof*⟩

lemma *R21-R47*: *pmf* (*sds R21*) *d* ≤ *pmf* (*sds R47*) *c*
⟨*proof*⟩

lemma *R30* [*simp*]: *pmf* (*sds R30*) *a* = *pmf* (*sds R47*) *a* *pmf* (*sds R30*) *b* = *0*
pmf (*sds R30*) *c* = *0* *pmf* (*sds R30*) *d* = *1* − *pmf* (*sds R47*) *a*
⟨*proof*⟩

lemma *R31-c-ge-one-half*: *pmf* (*sds R31*) *c* ≥ *1/2*
⟨*proof*⟩

lemma *R31*: *pmf* (*sds R31*) *a* = *0* *pmf* (*sds R31*) *c* = *1/2* *pmf* (*sds R31*) *b* + *pmf* (*sds R31*) *d* = *1/2*
⟨*proof*⟩

lemma *absurd*: *False*
⟨*proof*⟩

end

1.4 Lifting to more than 4 agents and alternatives

lemma *finite-list′*:
assumes *finite A*

obtains *xs* **where** $A = set\ xs$ *distinct xs* $length\ xs = card\ A$
⟨*proof*⟩

lemma *finite-list-subset*:
assumes *finite A* $card\ A \geq n$
obtains *xs* **where** $set\ xs \subseteq A$ *distinct xs* $length\ xs = n$
⟨*proof*⟩

lemma *card-ge-4E*:
assumes *finite A* $card\ A \geq 4$
obtains *a b c d* **where** *distinct* $[a,b,c,d]$ $\{a,b,c,d\} \subseteq A$
⟨*proof*⟩

context *sds-impossibility*
begin

lemma *absurd*: *False*
⟨*proof*⟩

end

end

References

[1] F. Brandl, F. Brandt, and C. Geist. Proving the incompatibility of Efficiency and Strategyproofness via SMT solving. *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI)*, 2016. Forthcoming.

[2] F. Brandl, F. Brandt, and W. Suksompong. The impossibility of extending Random Dictatorship to weak preferences. *Economics Letters*, 141:pp. 44 – 47, 2016.