

Correctness of a Set-based Algorithm for Computing Strongly Connected Components of a Graph

Stephan Merz and Vincent Trélat

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Abstract

We prove the correctness of a sequential algorithm for computing maximal strongly connected components (SCCs) of a graph due to Vincent Bloemen.

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1 Overview

Computing the maximal strongly connected components (SCCs) of a finite directed graph is a celebrated problem in the theory of graph algorithms. Although Tarjan’s algorithm [5] is perhaps the best-known solution, there are many others. In his PhD thesis, Bloemen [1] presents an algorithm that is itself based on earlier algorithms by Munro [4] and Dijkstra [2]. Just like these algorithms, Bloemen’s solution is based on enumerating SCCs in a depth-first traversal of the graph. Gabow’s algorithm that has already been formalized in Isabelle [3] also falls into this category of solutions. Nevertheless, Bloemen goes on to present a parallel variant of the algorithm suitable for execution on multi-core processors, based on clever data structures that minimize locking.

In the following, we encode the sequential version of the algorithm in the proof assistant Isabelle/HOL, and prove its correctness. Bloemen’s thesis briefly and informally explains why the algorithm is correct. Our proof expands on these arguments, making them completely formal. The encoding is based on a direct representation of the algorithm as a pair of mutually recursive functions; we are not aiming at extracting executable code.

```
theory SCC-Bloemen-Sequential  
imports Main  
begin
```

The record below represents the variables of the algorithm. Most variables correspond to those used in Bloemen’s presentation. Thus, the variable \mathcal{S} associates to every node the set of nodes that have already been determined to be part of the same SCC. A core invariant of the algorithm will be that this mapping represents equivalence classes of nodes: for all nodes v and w , we maintain the relationship

$$v \in \mathcal{S} w \iff \mathcal{S} v = \mathcal{S} w.$$

In an actual implementation of this algorithm, this variable could conveniently be represented by a union-find structure. Variable *stack* holds the list of roots of these (not yet maximal) SCCs, in depth-first order, *visited* and *explored* represent the nodes that have already been seen, respectively that have been completely explored, by the algorithm, and *sccs* is the set of maximal SCCs that the algorithm has found so far.

Additionally, the record holds some auxiliary variables that are used in the proof of correctness. In particular, *root* denotes the node on which the algorithm was called, *cstack* represents the call stack of the recursion of function *dfs*, and *vsuccs* stores the successors of each node that have already been visited by the function *dfss* that loops over all successors of a given node.

```
record 'v env =
```

```

root :: 'v
S :: 'v ⇒ 'v set
explored :: 'v set
visited :: 'v set
vsuccs :: 'v ⇒ 'v set
sccs :: 'v set set
stack :: 'v list
cstack :: 'v list

```

The algorithm is initially called with an environment that initializes the root node and trivializes all other components.

definition *init-env* **where**

```

init-env v = (
  root = v,
  S = (λu. {u}),
  explored = {},
  visited = {},
  vsuccs = (λu. {}),
  sccs = {},
  stack = [],
  cstack = []
)

```

— Make the simplifier expand let-constructions automatically.

declare *Let-def*[simp]

2 Auxiliary lemmas about lists

We use the precedence order on the elements that appear in a list. In particular, stacks are represented as lists, and a node x precedes another node y on the stack if x was pushed on the stack later than y .

definition *precedes* ($- \preceq -$ in $- [100,100,100] 39$) **where**

```

x ≼ y in xs ≡ ∃ l r. xs = l @ (x # r) ∧ y ∈ set (x # r)

```

lemma *precedes-mem*:

```

assumes x ≼ y in xs
shows x ∈ set xs y ∈ set xs
using assms unfolding precedes-def by auto

```

lemma *head-precedes*:

```

assumes y ∈ set (x # xs)
shows x ≼ y in (x # xs)
using assms unfolding precedes-def by force

```

lemma *precedes-in-tail*:

```

assumes x ≠ z
shows x ≼ y in (z # zs) ⟷ x ≼ y in zs

```

using *assms* **unfolding** *precedes-def* **by** (*auto simp: Cons-eq-append-conv*)

lemma *tail-not-precedes*:

assumes $y \preceq x$ *in* $(x \# xs)$ $x \notin \text{set } xs$

shows $x = y$

using *assms* **unfolding** *precedes-def*

by (*metis Cons-eq-append-conv Un-iff list.inject set-append*)

lemma *split-list-precedes*:

assumes $y \in \text{set } (ys @ [x])$

shows $y \preceq x$ *in* $(ys @ x \# xs)$

using *assms* **unfolding** *precedes-def*

by (*metis append-Cons append-assoc in-set-conv-decomp rotate1.simps(2) set-ConsD set-rotate1*)

lemma *precedes-refl* [*simp*]: $(x \preceq x \text{ in } xs) = (x \in \text{set } xs)$

proof

assume $x \preceq x$ *in* xs **thus** $x \in \text{set } xs$

by (*simp add: precedes-mem*)

next

assume $x \in \text{set } xs$

from *this* [*THEN split-list*] **show** $x \preceq x$ *in* xs

unfolding *precedes-def* **by** *auto*

qed

lemma *precedes-append-left*:

assumes $x \preceq y$ *in* xs

shows $x \preceq y$ *in* $(ys @ xs)$

using *assms* **unfolding** *precedes-def* **by** (*metis append.assoc*)

lemma *precedes-append-left-iff*:

assumes $x \notin \text{set } ys$

shows $x \preceq y$ *in* $(ys @ xs) \iff x \preceq y$ *in* xs (**is** *?lhs = ?rhs*)

proof

assume *?lhs*

then obtain $l r$ **where** $lr: ys @ xs = l @ (x \# r)$ $y \in \text{set } (x \# r)$

unfolding *precedes-def* **by** *blast*

then obtain us **where**

$(ys = l @ us \wedge us @ xs = x \# r) \vee (ys @ us = l \wedge xs = us @ (x \# r))$

by (*auto simp: append-eq-append-conv2*)

thus *?rhs*

proof

assume $us: ys = l @ us \wedge us @ xs = x \# r$

with *assms* **have** $us = []$

by (*metis Cons-eq-append-conv in-set-conv-decomp*)

with $us lr$ **show** *?rhs*

unfolding *precedes-def* **by** *auto*

next

assume $us: ys @ us = l \wedge xs = us @ (x \# r)$

```

    with  $\langle y \in \text{set } (x \# r) \rangle$  show ?rhs
      unfolding precedes-def by blast
qed
next
  assume ?rhs thus ?lhs by (rule precedes-append-left)
qed

lemma precedes-append-right:
  assumes  $x \preceq y$  in xs
  shows  $x \preceq y$  in (xs @ ys)
  using assms unfolding precedes-def by force

lemma precedes-append-right-iff:
  assumes  $y \notin \text{set } ys$ 
  shows  $x \preceq y$  in (xs @ ys)  $\longleftrightarrow$   $x \preceq y$  in xs (is ?lhs = ?rhs)
proof
  assume ?lhs
  then obtain l r where lr:  $xs @ ys = l @ (x \# r)$   $y \in \text{set } (x \# r)$ 
    unfolding precedes-def by blast
  then obtain us where
    ( $xs = l @ us \wedge us @ ys = x \# r$ )  $\vee$  ( $xs @ us = l \wedge ys = us @ (x \# r)$ )
    by (auto simp: append-eq-append-conv2)
  thus ?rhs
proof
  assume us:  $xs = l @ us \wedge us @ ys = x \# r$ 
  with  $\langle y \in \text{set } (x \# r) \rangle$  assms show ?rhs
    unfolding precedes-def by (metis Cons-eq-append-conv Un-iff set-append)
next
  assume us:  $xs @ us = l \wedge ys = us @ (x \# r)$ 
  with  $\langle y \in \text{set } (x \# r) \rangle$  assms
  show ?rhs by auto — contradiction
qed
next
  assume ?rhs thus ?lhs by (rule precedes-append-right)
qed

```

Precedence determines an order on the elements of a list, provided elements have unique occurrences. However, consider a list such as $[2,3,1,2]$: then 1 precedes 2 and 2 precedes 3, but 1 does not precede 3.

```

lemma precedes-trans:
  assumes  $x \preceq y$  in xs and  $y \preceq z$  in xs and distinct xs
  shows  $x \preceq z$  in xs
  using assms unfolding precedes-def
  by (smt Un-iff append.assoc append-Cons-eq-iff distinct-append
    not-distinct-conv-prefix set-append split-list-last)

```

```

lemma precedes-antisym:
  assumes  $x \preceq y$  in xs and  $y \preceq x$  in xs and distinct xs
  shows  $x = y$ 

```

```

proof –
  from  $\langle x \preceq y \text{ in } xs \rangle \langle \text{distinct } xs \rangle$  obtain  $as\ bs$  where
    1:  $xs = as @ (x \# bs)$   $y \in \text{set } (x \# bs)$   $y \notin \text{set } as$ 
    unfolding precedes-def by force
  from  $\langle y \preceq x \text{ in } xs \rangle \langle \text{distinct } xs \rangle$  obtain  $cs\ ds$  where
    2:  $xs = cs @ (y \# ds)$   $x \in \text{set } (y \# ds)$   $x \notin \text{set } cs$ 
    unfolding precedes-def by force
  from 1 2 have  $as @ (x \# bs) = cs @ (y \# ds)$ 
    by simp
  then obtain  $zs$  where
     $(as = cs @ zs \wedge zs @ (x \# bs) = y \# ds)$ 
     $\vee (as @ zs = cs \wedge x \# bs = zs @ (y \# ds))$  (is  $?P \vee ?Q$ )
    by (auto simp: append-eq-append-conv2)
  then show  $?thesis$ 
  proof
    assume  $?P$  with  $\langle y \notin \text{set } as \rangle$  show  $?thesis$ 
    by (cases zs auto)
  next
    assume  $?Q$  with  $\langle x \notin \text{set } cs \rangle$  show  $?thesis$ 
    by (cases zs auto)
  qed
qed

```

3 Finite directed graphs

We represent a graph as an Isabelle locale that identifies a finite set of vertices (of some base type $'v$) and associates to each vertex its set of successor vertices.

```

locale graph =
  fixes vertices ::  $'v \text{ set}$ 
    and successors ::  $'v \Rightarrow 'v \text{ set}$ 
  assumes vfin: finite vertices
    and sclosed:  $\forall x \in \text{vertices. successors } x \subseteq \text{vertices}$ 

```

```

context graph
begin

```

```

abbreviation edge where
  edge  $x\ y \equiv y \in \text{successors } x$ 

```

We inductively define reachability of nodes in the graph.

```

inductive reachable where
  reachable-refl[iff]: reachable  $x\ x$ 
| reachable-succ[elim]:  $[[\text{edge } x\ y; \text{reachable } y\ z]] \Longrightarrow \text{reachable } x\ z$ 

```

```

lemma reachable-edge: edge  $x\ y \Longrightarrow \text{reachable } x\ y$ 
by auto

```

lemma *succ-reachable*:
assumes *reachable x y* **and** *edge y z*
shows *reachable x z*
using *assms* **by** *induct auto*

lemma *reachable-trans*:
assumes *y: reachable x y* **and** *z: reachable y z*
shows *reachable x z*
using *assms* **by** *induct auto*

In order to derive a “reverse” induction rule for *reachable*, we define an alternative reachability predicate and prove that the two coincide.

inductive *reachable-end* **where**
re-refl[*iff*]: *reachable-end x x*
| *re-succ*: $\llbracket \text{reachable-end } x \ y; \text{ edge } y \ z \rrbracket \implies \text{reachable-end } x \ z$

lemma *succ-re*:
assumes *y: edge x y* **and** *z: reachable-end y z*
shows *reachable-end x z*
using *z y* **by** (*induction*) (*auto intro: re-succ*)

lemma *reachable-re*:
assumes *reachable x y*
shows *reachable-end x y*
using *assms* **by** (*induction*) (*auto intro: succ-re*)

lemma *re-reachable*:
assumes *reachable-end x y*
shows *reachable x y*
using *assms* **by** (*induction*) (*auto intro: succ-reachable*)

lemma *reachable-end-induct*:
assumes *r: reachable x y*
and *base*: $\bigwedge x. P \ x \ x$
and *step*: $\bigwedge x \ y \ z. \llbracket P \ x \ y; \text{ edge } y \ z \rrbracket \implies P \ x \ z$
shows $P \ x \ y$
using *r*[*THEN reachable-re*] **proof** (*induction*)
case (*re-refl x*)
from *base* **show** *?case* .
next
case (*re-succ x y z*)
with *step* **show** *?case* **by** *blast*
qed

We also need the following variant of reachability avoiding certain edges. More precisely, *y* is reachable from *x* avoiding a set *E* of edges if there exists a path such that no edge from *E* appears along the path.

inductive *reachable-avoiding* **where**
ra-refl[*iff*]: *reachable-avoiding x x E*

| *ra-succ[elim]*: $\llbracket \text{reachable-avoiding } x \ y \ E; \text{ edge } y \ z; (y,z) \notin E \rrbracket \implies \text{reachable-avoiding } x \ z \ E$

lemma *edge-ra*:

assumes *edge* $x \ y$ **and** $(x,y) \notin E$
shows *reachable-avoiding* $x \ y \ E$
using *assms* **by** (*meson reachable-avoiding.simps*)

lemma *ra-trans*:

assumes *1*: *reachable-avoiding* $x \ y \ E$ **and** *2*: *reachable-avoiding* $y \ z \ E$
shows *reachable-avoiding* $x \ z \ E$
using *2 1* **by** *induction auto*

lemma *ra-cases*:

assumes *reachable-avoiding* $x \ y \ E$
shows $x=y \vee (\exists z. z \in \text{successors } x \wedge (x,z) \notin E \wedge \text{reachable-avoiding } z \ y \ E)$
using *assms* **proof** (*induction*)
case (*ra-refl* $x \ S$)
then show *?case* **by** *simp*
next
case (*ra-succ* $x \ y \ S \ z$)
then show *?case*
by (*metis ra-refl reachable-avoiding.ra-succ*)
qed

lemma *ra-mono*:

assumes *reachable-avoiding* $x \ y \ E$ **and** $E' \subseteq E$
shows *reachable-avoiding* $x \ y \ E'$
using *assms* **by** *induction auto*

lemma *ra-add-edge*:

assumes *reachable-avoiding* $x \ y \ E$
shows *reachable-avoiding* $x \ y \ (E \cup \{(v,w)\})$
 $\vee (\text{reachable-avoiding } x \ v \ (E \cup \{(v,w)\}) \wedge \text{reachable-avoiding } w \ y \ (E \cup \{(v,w)\}))$
using *assms* **proof** (*induction*)
case (*ra-refl* $x \ E$)
then show *?case* **by** *simp*
next
case (*ra-succ* $x \ y \ E \ z$)
then show *?case*
using *reachable-avoiding.ra-succ* **by** *auto*
qed

Reachability avoiding some edges obviously implies reachability. Conversely, reachability implies reachability avoiding the empty set.

lemma *ra-reachable*:

reachable-avoiding $x \ y \ E \implies \text{reachable } x \ y$
by (*induction rule: reachable-avoiding.induct*) (*auto intro: succ-reachable*)

lemma *ra-empty*:
reachable-avoiding $x\ y\ \{\}$ = *reachable* $x\ y$
proof
assume *reachable-avoiding* $x\ y\ \{\}$
thus *reachable* $x\ y$
 by (*rule ra-reachable*)
next
assume *reachable* $x\ y$
hence *reachable-end* $x\ y$
 by (*rule reachable-re*)
thus *reachable-avoiding* $x\ y\ \{\}$
 by *induction auto*
qed

4 Strongly connected components

A strongly connected component is a set S of nodes such that any two nodes in S are reachable from each other. This concept is represented by the predicate *is-subsc* below. We are ultimately interested in non-empty, maximal strongly connected components, represented by the predicate *is-scc*.

definition *is-subsc* **where**
is-subsc $S \equiv \forall x \in S. \forall y \in S. \text{reachable } x\ y$

definition *is-scc* **where**
is-scc $S \equiv S \neq \{\} \wedge \text{is-subsc } S \wedge (\forall S'. S \subseteq S' \wedge \text{is-subsc } S' \longrightarrow S' = S)$

lemma *subsc-add*:
assumes *is-subsc* S **and** $x \in S$
 and *reachable* $x\ y$ **and** *reachable* $y\ x$
shows *is-subsc* (*insert* $y\ S$)
using *assms* **unfolding** *is-subsc-def* **by** (*metis insert-iff reachable-trans*)

lemma *sccE*:
— Two nodes that are reachable from each other are in the same SCC.
assumes *is-scc* S **and** $x \in S$
 and *reachable* $x\ y$ **and** *reachable* $y\ x$
shows $y \in S$
using *assms* **unfolding** *is-scc-def*
by (*metis insertI1 subsc-add subset-insertI*)

lemma *scc-partition*:
— Two SCCs that contain a common element are identical.
assumes *is-scc* S **and** *is-scc* S' **and** $x \in S \cap S'$
shows $S = S'$
using *assms* **unfolding** *is-scc-def is-subsc-def*
by (*metis IntE assms(2) sccE subsetI*)

5 Algorithm for computing strongly connected components

We now introduce our representation of Bloemen's algorithm in Isabelle/HOL. The auxiliary function *unite* corresponds to the inner *while* loop in Bloemen's pseudo-code [1, p.32]. It is applied to two nodes v and w (and the environment e holding the current values of the program variables) when a loop is found, i.e. when w is a successor of v in the graph that has already been visited in the depth-first search. In that case, the root of the SCC of node w determined so far must appear below the root of v 's SCC in the *stack* maintained by the algorithm. The effect of the function is to merge the SCCs of all nodes on the top of the stack above (and including) w . Node w 's root will be the root of the merged SCC.

definition *unite* :: ' $v \Rightarrow 'v \Rightarrow 'v \text{ env} \Rightarrow 'v \text{ env}$ **where**
unite $v w e \equiv$
 let $pfx = \text{takeWhile } (\lambda x. w \notin \mathcal{S} e x) (\text{stack } e);$
 $sfx = \text{dropWhile } (\lambda x. w \notin \mathcal{S} e x) (\text{stack } e);$
 $cc = \bigcup \{ \mathcal{S} e x \mid x. x \in \text{set } pfx \cup \{hd \text{ } sfx\} \}$
 in $e(\mathcal{S} := \lambda x. \text{if } x \in cc \text{ then } cc \text{ else } \mathcal{S} e x,$
 $\text{stack} := sfx)$

We now represent the algorithm as two mutually recursive functions *dfs* and *dfss* in Isabelle/HOL. The function *dfs* corresponds to Bloemen's function *SetBased*, whereas *dfss* corresponds to the *forall* loop over the successors of the node on which *dfs* was called. Instead of using global program variables in imperative style, our functions explicitly pass environments that hold the current values of these variables.

A technical complication in the development of the algorithm in Isabelle is the fact that the functions need not terminate when their pre-conditions (introduced below) are violated, for example when *dfs* is called for a node that was already visited previously. We therefore cannot prove termination at this point, but will later show that the explicitly given pre-conditions ensure termination.

function (*domintros*) *dfs* :: ' $v \Rightarrow 'v \text{ env} \Rightarrow 'v \text{ env}$
and *dfss* :: ' $v \Rightarrow 'v \text{ env} \Rightarrow 'v \text{ env}$ **where**
dfs $v e =$
 (let $e1 = e(\text{visited} := \text{visited } e \cup \{v\},$
 $\text{stack} := (v \# \text{stack } e),$
 $\text{cstack} := (v \# \text{cstack } e));$
 $e' = \text{dfss } v e1$
 in if $v = hd(\text{stack } e')$
 then $e'(\text{sccs} := \text{sccs } e' \cup \{\mathcal{S} e' v\},$
 $\text{explored} := \text{explored } e' \cup (\mathcal{S} e' v),$
 $\text{stack} := tl(\text{stack } e'),$
 $\text{cstack} := tl(\text{cstack } e'))$

```

    else e'(|cstack := tl(cstack e'|))
| dfss v e =
  (let vs = successors v - vsucce e v
   in if vs = {} then e
      else let w = SOME x. x ∈ vs;
            e' = (if w ∈ explored e then e
                  else if w ∉ visited e
                    then dfs w e
                    else unite v w e);
            e'' = (e'(|vsucce :=
                    (λx. if x=v then vsucce e' v ∪ {w}
                     else vsucce e' x)|))
               in dfss v e''))
by pat-completeness (force+)

```

6 Definition of the predicates used in the correctness proof

Environments are partially ordered according to the following definition.

definition *sub-env* where

```

sub-env e e' ≡
  root e' = root e
  ∧ visited e ⊆ visited e'
  ∧ explored e ⊆ explored e'
  ∧ (∀ v. vsucce e v ⊆ vsucce e' v)
  ∧ (∀ v. S e v ⊆ S e' v)
  ∧ (∪ {S e v | v . v ∈ set (stack e)})
    ⊆ (∪ {S e' v | v . v ∈ set (stack e')})

```

lemma *sub-env-trans*:

assumes *sub-env e e'* **and** *sub-env e' e''*

shows *sub-env e e''*

using *assms unfolding sub-env-def*

by (*metis (no-types, lifting) subset-trans*)

The set *unvisited e u* contains all edges (a,b) such that node a is in the same SCC as node u and the edge has not yet been followed, in the sense represented by variable *vsucce*.

definition *unvisited* where

```

unvisited e u ≡
  {(a,b) | a b. a ∈ S e u ∧ b ∈ successors a - vsucce e a}

```

6.1 Main invariant

The following definition characterizes well-formed environments. This predicate will be shown to hold throughout the execution of the algorithm. In

words, it asserts the following facts:

- Only nodes reachable from the root (for which the algorithm was originally called) are visited.
- The two stacks *stack* and *cstack* do not contain duplicate nodes, and *stack* contains a subset of the nodes on *cstack*, in the same order.
- Any node higher on the *stack* (i.e., that was pushed later) is reachable from nodes lower in the *stack*. This property also holds for nodes on the call stack, but this is not needed for the correctness proof.
- Every explored node, and every node on the call stack, has been visited.
- Nodes reachable from fully explored nodes have themselves been fully explored.
- The set $vsuccs\ e\ n$, for any node n , is a subset of n 's successors, and all these nodes are in *visited*. The set is empty if $n \notin visited$, and it contains all successors if n has been fully explored or if n has been visited, but is no longer on the call stack.
- The sets $\mathcal{S}\ e\ n$ represent an equivalence relation. The equivalence classes of nodes that have not yet been visited are singletons. Also, equivalence classes for two distinct nodes on the *stack* are disjoint because the stack only stores roots of SCCs, and the union of the equivalence classes for these root nodes corresponds to the set of live nodes, i.e. those nodes that have already been visited but not yet fully explored.
- More precisely, an equivalence class is represented on the stack by the oldest node in the sense of the call order: any node in the class that is still on the call stack precedes the representative on the call stack and was therefore pushed later.
- Equivalence classes represent the maximal available information about strong connectedness: nodes represented by some node n on the *stack* can reach some node m that is lower in the stack only by taking an edge from some node in n 's equivalence class that has not yet been followed. (Remember that m can reach n by one of the previous conjuncts.)
- Equivalence classes represent partial SCCs in the sense of the predicate *is-subsc*. Variable *sccs* holds maximal SCCs in the sense of the predicate *is-scc*, and their union corresponds to the set of explored nodes.

definition *wf-env* where

$$\begin{aligned}
& \text{wf-env } e \equiv \\
& (\forall n \in \text{visited } e. \text{reachable } (\text{root } e) \ n) \\
& \wedge \text{distinct } (\text{stack } e) \\
& \wedge \text{distinct } (\text{cstack } e) \\
& \wedge (\forall n \ m. n \preceq m \text{ in stack } e \longrightarrow n \preceq m \text{ in cstack } e) \\
& \wedge (\forall n \ m. n \preceq m \text{ in stack } e \longrightarrow \text{reachable } m \ n) \\
& \wedge \text{explored } e \subseteq \text{visited } e \\
& \wedge \text{set } (\text{cstack } e) \subseteq \text{visited } e \\
& \wedge (\forall n \in \text{explored } e. \forall m. \text{reachable } n \ m \longrightarrow m \in \text{explored } e) \\
& \wedge (\forall n. \text{vsuccs } e \ n \subseteq \text{successors } n \cap \text{visited } e) \\
& \wedge (\forall n. n \notin \text{visited } e \longrightarrow \text{vsuccs } e \ n = \{\}) \\
& \wedge (\forall n \in \text{explored } e. \text{vsuccs } e \ n = \text{successors } n) \\
& \wedge (\forall n \in \text{visited } e - \text{set } (\text{cstack } e). \text{vsuccs } e \ n = \text{successors } n) \\
& \wedge (\forall n \ m. m \in \mathcal{S} \ e \ n \longleftrightarrow (\mathcal{S} \ e \ n = \mathcal{S} \ e \ m)) \\
& \wedge (\forall n. n \notin \text{visited } e \longrightarrow \mathcal{S} \ e \ n = \{n\}) \\
& \wedge (\forall n \in \text{set } (\text{stack } e). \forall m \in \text{set } (\text{stack } e). n \neq m \longrightarrow \mathcal{S} \ e \ n \cap \mathcal{S} \ e \ m = \{\}) \\
& \wedge \bigcup \{\mathcal{S} \ e \ n \mid n. n \in \text{set } (\text{stack } e)\} = \text{visited } e - \text{explored } e \\
& \wedge (\forall n \in \text{set } (\text{stack } e). \forall m \in \mathcal{S} \ e \ n. m \in \text{set } (\text{cstack } e) \longrightarrow m \preceq n \text{ in cstack } e) \\
& \wedge (\forall n \ m. n \preceq m \text{ in stack } e \wedge n \neq m \longrightarrow \\
& \quad (\forall u \in \mathcal{S} \ e \ n. \neg \text{reachable-avoiding } u \ m \ (\text{unvisited } e \ n))) \\
& \wedge (\forall n. \text{is-subsc} (\mathcal{S} \ e \ n)) \\
& \wedge (\forall S \in \text{sccs } e. \text{is-scc } S) \\
& \wedge \bigcup (\text{sccs } e) = \text{explored } e
\end{aligned}$$

6.2 Consequences of the invariant

Since every node on the call stack is an element of *visited* and every node on the *stack* also appears on *cstack*, all these nodes are also in *visited*.

lemma *stack-visited*:

assumes *wf-env* $e \ n \in \text{set } (\text{stack } e)$
shows $n \in \text{visited } e$
using *assms* **unfolding** *wf-env-def*
by (*meson precedes-refl subset-iff*)

Classes represented on the stack consist of visited nodes that have not yet been fully explored.

lemma *stack-class*:

assumes *wf-env* $e \ n \in \text{set } (\text{stack } e) \ m \in \mathcal{S} \ e \ n$
shows $m \in \text{visited } e - \text{explored } e$
using *assms* **unfolding** *wf-env-def* **by** *blast*

Conversely, every such node belongs to some class represented on the stack.

lemma *visited-unexplored*:

assumes *wf-env* $e \ m \in \text{visited } e \ m \notin \text{explored } e$
obtains n **where** $n \in \text{set } (\text{stack } e) \ m \in \mathcal{S} \ e \ n$
using *assms* **unfolding** *wf-env-def*
by (*smt (verit, ccfv-threshold) Diff-iff Union-iff mem-Collect-eq*)

Every node belongs to its own equivalence class.

lemma *S-reflexive*:
assumes *wf-env e*
shows $n \in S \ e \ n$
using *assms* **by** (*auto simp: wf-env-def*)

No node on the stack has been fully explored.

lemma *stack-unexplored*:
assumes *1: wf-env e*
and *2: n ∈ set (stack e)*
and *3: n ∈ explored e*
shows *P*
using *stack-class[OF 1 2] S-reflexive[OF 1] 3*
by *blast*

If *w* is reachable from visited node *v*, but no unvisited successor of a node reachable from *v* can reach *w*, then *w* must be visited.

lemma *reachable-visited*:
assumes *e: wf-env e*
and *v: v ∈ visited e*
and *w: reachable v w*
and *s: ∀ n ∈ visited e. ∀ m ∈ successors n - vsuccs e n. reachable v n ⟶ ¬ reachable m w*
shows $w \in \text{visited } e$
using *w v s* **proof** (*induction*)
case (*reachable-refl x*)
then show *?case* **by** *simp*
next
case (*reachable-succ x y z*)
then have $y \in \text{vsuccs } e \ x$ **by** *blast*
with *e* **have** $y \in \text{visited } e$
unfolding *wf-env-def* **by** (*meson le-infE subset-eq*)
with *reachable-succ reachable.reachable-succ* **show** *?case*
by *blast*
qed

Edges towards explored nodes do not contribute to reachability of unexplored nodes avoiding some set of edges.

lemma *avoiding-explored*:
assumes *e: wf-env e*
and *xy: reachable-avoiding x y E*
and *y: y ∉ explored e*
and *w: w ∈ explored e*
shows *reachable-avoiding x y (E ∪ {(v,w)})*
using *xy y* **proof** (*induction*)
case (*ra-refl x E*)
then show *?case* **by** *simp*
next

```

case (ra-succ x y E z)
from e  $\langle z \in \text{successors } y \rangle \langle z \notin \text{explored } e \rangle$ 
have y  $\notin \text{explored } e$ 
  unfolding wf-env-def by (meson reachable-edge)
with ra-succ.IH have reachable-avoiding x y ( $E \cup \{(v,w)\}$ ) .
moreover
from w  $\langle (y,z) \notin E \rangle \langle z \notin \text{explored } e \rangle$  have  $(y,z) \notin E \cup \{(v,w)\}$ 
  by auto
ultimately show ?case
  using  $\langle z \in \text{successors } y \rangle$  by auto
qed

```

6.3 Pre- and post-conditions of function *dfs*

Function *dfs* should be called for a well-formed environment and a node *v* that has not yet been visited and that is reachable from the root node, as well as from all nodes in the stack. No outgoing edges from node *v* have yet been followed.

definition *pre-dfs* **where**

```

pre-dfs v e  $\equiv$ 
  wf-env e
   $\wedge v \notin \text{visited } e$ 
   $\wedge \text{reachable}(\text{root } e) v$ 
   $\wedge \text{vsuccs } e v = \{\}$ 
   $\wedge (\forall n \in \text{set}(\text{stack } e). \text{reachable } n v)$ 

```

Function *dfs* maintains the invariant *wf-env* and returns an environment *e'* that extends the input environment *e*. Node *v* has been visited and all its outgoing edges have been followed. Because the algorithm works in depth-first fashion, no new outgoing edges of nodes that had already been visited in the input environment have been followed, and the stack of *e'* is a suffix of the one of *e* such that *v* is still reachable from all nodes on the stack. The stack may have been shortened because SCCs represented at the top of the stack may have been merged. The call stack is reestablished as it was in *e*. There are two possible outcomes of the algorithm:

- Either *v* has been fully explored, in which case the stacks of *e* and *e'* are the same, and the equivalence classes of all nodes represented on the stack are unchanged. This corresponds to the case where *v* is the root node of its (maximal) SCC.
- Alternatively, the stack of *e'* must be non-empty and *v* must be represented by the node at the top of the stack. The SCCs of the nodes lower on the stack are unchanged. This corresponds to the case where *v* is not the root node of its SCC, but some SCCs at the top of the stack may have been merged.

definition *post-dfs* where

post-dfs $v\ e\ e' \equiv$
 $wf\text{-env}\ e'$
 $\wedge v \in visited\ e'$
 $\wedge sub\text{-env}\ e\ e'$
 $\wedge vsuccs\ e'\ v = successors\ v$
 $\wedge (\forall w \in visited\ e. vsuccs\ e'\ w = vsuccs\ e\ w)$
 $\wedge (\forall n \in set\ (stack\ e').\ reachable\ n\ v)$
 $\wedge (\exists ns.\ stack\ e = ns\ @\ (stack\ e'))$
 $\wedge (\ (v \in explored\ e' \wedge stack\ e' = stack\ e$
 $\quad \wedge (\forall n \in set\ (stack\ e').\ \mathcal{S}\ e'\ n = \mathcal{S}\ e\ n))$
 $\vee (stack\ e' \neq [] \wedge v \in \mathcal{S}\ e'\ (hd\ (stack\ e'))$
 $\quad \wedge (\forall n \in set\ (tl\ (stack\ e')).\ \mathcal{S}\ e'\ n = \mathcal{S}\ e\ n)))$
 $\wedge cstack\ e' = cstack\ e$

The initial environment is easily seen to satisfy *dfs*'s pre-condition.

lemma *init-env-pre-dfs*: *pre-dfs* $v\ (init\text{-env}\ v)$

by (*auto simp*: *pre-dfs-def wf-env-def init-env-def is-subsc-def*
dest: *precedes-mem*)

Any node represented by the top stack element of the input environment is still represented by the top element of the output stack.

lemma *dfs-S-hd-stack*:

assumes *wf*: *wf-env* e
and *post*: *post-dfs* $v\ e\ e'$
and *n*: $stack\ e \neq [] \wedge n \in \mathcal{S}\ e\ (hd\ (stack\ e))$
shows $stack\ e' \neq [] \wedge n \in \mathcal{S}\ e'\ (hd\ (stack\ e'))$

proof –

have 1: $stack\ e' \neq [] \wedge n \in \mathcal{S}\ e'\ (hd\ (stack\ e'))$

proof (*cases* $stack\ e' = stack\ e \wedge (\forall n \in set\ (stack\ e').\ \mathcal{S}\ e'\ n = \mathcal{S}\ e\ n)$)

case *True*

with *n* show *?thesis*

by *auto*

next

case 2: *False*

with *post* have $stack\ e' \neq []$

by (*simp add*: *post-dfs-def*)

from *n* have $hd\ (stack\ e) \in set\ (stack\ e)$

by *simp*

with 2 *n* *post* obtain *u* where

$u: u \in set\ (stack\ e') \wedge n \in \mathcal{S}\ e'\ u$

unfolding *post-dfs-def sub-env-def* by *blast*

show *?thesis*

proof (*cases* $u = hd\ (stack\ e')$)

case *True*

with $u \in set\ (stack\ e' \neq [])$ show *?thesis*

by *simp*

next

case *False*


```

with  $u$  have  $u \in \text{set } (\text{tl } (\text{stack } e'))$ 
  by (metis empty-set equals0D list.collapse set-ConsD)
with  $u \neq \text{post}$  have  $u \in \text{set } (\text{tl } (\text{stack } e)) \wedge n \in \mathcal{S} e u$ 
  unfolding post-dfs-def
  by (metis Un-iff append-self-conv2 set-append tl-append2)
with  $n \text{ wf } \langle \text{hd } (\text{stack } e) \in \text{set } (\text{stack } e) \rangle$  show ?thesis
  unfolding wf-env-def
  by (metis (no-types, opaque-lifting) disjoint-iff-not-equal distinct.simps(2))
list.collapse list.set-sel(2)
qed
qed
from 1 show  $\text{stack } e' \neq []$  by simp
from 1 show  $n \in \mathcal{S} e' (\text{hd } (\text{stack } e'))$  by simp
qed

```

Function *dfs* leaves the SCCs represented by elements in the (new) tail of the *stack* unchanged.

lemma *dfs-S-tl-stack*:

```

assumes post: post-dfs v e e'
  and nempty:  $\text{stack } e \neq []$ 
shows  $\text{stack } e' \neq [] \wedge \forall n \in \text{set } (\text{tl } (\text{stack } e')). \mathcal{S} e' n = \mathcal{S} e n$ 
proof –
  have 1:  $\text{stack } e' \neq [] \wedge (\forall n \in \text{set } (\text{tl } (\text{stack } e')). \mathcal{S} e' n = \mathcal{S} e n)$ 
  proof (cases stack e' = stack e  $\wedge (\forall n \in \text{set } (\text{stack } e'). \mathcal{S} e' n = \mathcal{S} e n)$ )
    case True
    with nempty show ?thesis
    by (simp add: list.set-sel(2))
  next
    case False
    with post show ?thesis
    by (auto simp: post-dfs-def)
  qed
from 1 show  $\text{stack } e' \neq []$ 
  by simp
from 1 show  $\forall n \in \text{set } (\text{tl } (\text{stack } e')). \mathcal{S} e' n = \mathcal{S} e n$ 
  by simp
qed

```

6.4 Pre- and post-conditions of function *dfss*

The pre- and post-conditions of function *dfss* correspond to the invariant of the loop over all outgoing edges from node v . The environment must be well-formed, node v must be visited and represented by the top element of the (non-empty) stack. Node v must be reachable from all nodes on the stack, and it must be the top node on the call stack. All outgoing edges of node v that have already been followed must either lead to completely explored nodes (that are no longer represented on the stack) or to nodes that are part of the same SCC as v .

definition *pre-dfss* where

$$\begin{aligned}
\text{pre-dfss } v \ e \equiv & \\
& \text{wf-env } e \\
& \wedge v \in \text{visited } e \\
& \wedge (\text{stack } e \neq []) \\
& \wedge (v \in \mathcal{S} \ e \ (\text{hd} \ (\text{stack } e))) \\
& \wedge (\forall w \in \text{vsuccs } e \ v. w \in \text{explored } e \cup \mathcal{S} \ e \ (\text{hd} \ (\text{stack } e))) \\
& \wedge (\forall n \in \text{set} \ (\text{stack } e). \text{reachable } n \ v) \\
& \wedge (\exists ns. \text{cstack } e = v \ \# \ ns)
\end{aligned}$$

The post-condition establishes that all outgoing edges of node v have been followed. As for function *dfs*, no new outgoing edges of previously visited nodes have been followed. Also as before, the new stack is a suffix of the old one, and the call stack is restored. In case node v is still on the stack (and therefore is the root node of its SCC), no node that is lower on the stack can be reachable from v . This condition guarantees the maximality of the computed SCCs.

definition *post-dfss* where

$$\begin{aligned}
\text{post-dfss } v \ e \ e' \equiv & \\
& \text{wf-env } e' \\
& \wedge \text{vsuccs } e' \ v = \text{successors } v \\
& \wedge (\forall w \in \text{visited } e - \{v\}. \text{vsuccs } e' \ w = \text{vsuccs } e \ w) \\
& \wedge \text{sub-env } e \ e' \\
& \wedge (\forall w \in \text{successors } v. w \in \text{explored } e' \cup \mathcal{S} \ e' \ (\text{hd} \ (\text{stack } e'))) \\
& \wedge (\forall n \in \text{set} \ (\text{stack } e'). \text{reachable } n \ v) \\
& \wedge (\text{stack } e' \neq []) \\
& \wedge (\exists ns. \text{stack } e = ns \ @ \ (\text{stack } e')) \\
& \wedge v \in \mathcal{S} \ e' \ (\text{hd} \ (\text{stack } e')) \\
& \wedge (\forall n \in \text{set} \ (\text{tl} \ (\text{stack } e')). \mathcal{S} \ e' \ n = \mathcal{S} \ e \ n) \\
& \wedge (\text{hd} \ (\text{stack } e') = v \longrightarrow (\forall n \in \text{set} \ (\text{tl} \ (\text{stack } e')). \neg \text{reachable } v \ n)) \\
& \wedge \text{cstack } e' = \text{cstack } e
\end{aligned}$$

7 Proof of partial correctness

7.1 Lemmas about function *unite*

We start by establishing a few lemmas about function *unite* in the context where it is called.

lemma *unite-stack*:

$$\begin{aligned}
& \text{fixes } e \ v \ w \\
& \text{defines } e' \equiv \text{unite } v \ w \ e \\
& \text{assumes } \text{wf}: \text{wf-env } e \\
& \quad \text{and } w: w \in \text{successors } v \ w \notin \text{vsuccs } e \ v \ w \in \text{visited } e \ w \notin \text{explored } e \\
& \text{obtains } \text{pfx} \ \text{where } \text{stack } e = \text{pfx} \ @ \ (\text{stack } e') \\
& \quad \text{stack } e' \neq [] \\
& \quad \text{let } cc = \bigcup \{ \mathcal{S} \ e \ n \mid n. n \in \text{set } \text{pfx} \cup \{ \text{hd} \ (\text{stack } e') \} \} \\
& \quad \text{in } \mathcal{S} \ e' = (\lambda x. \text{if } x \in cc \ \text{then } cc \ \text{else } \mathcal{S} \ e \ x)
\end{aligned}$$

$$w \in \mathcal{S} e' (\text{hd } (\text{stack } e'))$$

proof –

define *pfx* **where** *pfx* = *takeWhile* ($\lambda x. w \notin \mathcal{S} e x$) (*stack e*)
define *sfx* **where** *sfx* = *dropWhile* ($\lambda x. w \notin \mathcal{S} e x$) (*stack e*)
define *cc* **where** *cc* = $\bigcup \{ \mathcal{S} e x \mid x. x \in \text{set } pfx \cup \{\text{hd } sfx\} \}$

have *stack e* = *pfx* @ *sfx*
by (*simp add: pfx-def sfx-def*)

moreover

have *stack e'* = *sfx*
by (*simp add: e'-def unite-def sfx-def*)

moreover

from *wf w* **have** $w \in \bigcup \{ \mathcal{S} e n \mid n. n \in \text{set } (\text{stack } e) \}$
by (*simp add: wf-env-def*)

then obtain *n* **where** $n \in \text{set } (\text{stack } e)$ $w \in \mathcal{S} e n$
by *auto*

hence *sfx*: *sfx* $\neq []$ $\wedge w \in \mathcal{S} e (\text{hd } sfx)$

unfolding *sfx-def*

by (*metis dropWhile-eq-Nil-conv hd-dropWhile*)

moreover

have $\mathcal{S} e' = (\lambda x. \text{if } x \in \text{cc} \text{ then } \text{cc} \text{ else } \mathcal{S} e x)$

by (*rule*,

auto simp add: e'-def unite-def pfx-def sfx-def cc-def)

moreover

from *sfx* **have** $w \in \text{cc}$

by (*auto simp: cc-def*)

from *S-reflexive[OF wf, of hd sfx]*

have $\text{hd } sfx \in \text{cc}$

by (*auto simp: cc-def*)

with $\langle w \in \text{cc} \rangle \langle \mathcal{S} e' = (\lambda x. \text{if } x \in \text{cc} \text{ then } \text{cc} \text{ else } \mathcal{S} e x) \rangle$

have $w \in \mathcal{S} e' (\text{hd } sfx)$

by *simp*

ultimately show *?thesis*

using *that e'-def unite-def pfx-def sfx-def cc-def*

by *meson*

qed

Function *unite* leaves intact the equivalence classes represented by the tail of the new stack.

lemma *unite-S-tl*:

fixes *e v w*

defines $e' \equiv \text{unite } v w e$

assumes *wf*: *wf-env e*

and *w*: $w \in \text{successors } v w \notin \text{vsuccs } e v w \in \text{visited } e w \notin \text{explored } e$

and *n*: $n \in \text{set } (\text{tl } (\text{stack } e'))$

shows $\mathcal{S} e' n = \mathcal{S} e n$

proof –

from *assms* **obtain** *pfx* **where**

pfx: *stack e* = *pfx* @ (*stack e'*) *stack e'* $\neq []$

```

    let cc =  $\bigcup \{\mathcal{S} e n \mid n. n \in \text{set pfx} \cup \{\text{hd}(\text{stack } e')\}\}$ 
    in  $\mathcal{S} e' = (\lambda x. \text{if } x \in \text{cc} \text{ then } \text{cc} \text{ else } \mathcal{S} e x)$ 
  by (blast dest: unite-stack)
define cc where cc  $\equiv \bigcup \{\mathcal{S} e n \mid n. n \in \text{set pfx} \cup \{\text{hd}(\text{stack } e')\}\}$ 

have n  $\notin$  cc
proof
  assume n  $\in$  cc
  then obtain m where
    m  $\in \text{set pfx} \cup \{\text{hd}(\text{stack } e')\}$  n  $\in \mathcal{S} e m$ 
  by (auto simp: cc-def)
  with S-reflexive[OF wf, of n] n wf  $\langle \text{stack } e = \text{pfx} @ \text{stack } e' \rangle \langle \text{stack } e' \neq [] \rangle$ 
  show False
  unfolding wf-env-def
  by (smt (z3) Diff-triv Un-iff Un-insert-right append.right-neutral disjoint-insert(1)

    distinct.simps(2) distinct-append empty-set insertE insert-Diff
  list.exhaust-sel
    list.simps(15) set-append)
qed
with pfx show  $\mathcal{S} e' n = \mathcal{S} e n$ 
  by (auto simp add: cc-def)
qed

```

The stack of the result of *unite* represents the same vertices as the input stack, potentially in fewer equivalence classes.

lemma *unite-S-equal*:

```

fixes e v w
defines e'  $\equiv \text{unite } v w e$ 
assumes wf: wf-env e
  and w: w  $\in \text{successors } v w \notin \text{vsuccs } e v w \in \text{visited } e w \notin \text{explored } e$ 
shows  $(\bigcup \{\mathcal{S} e' n \mid n. n \in \text{set}(\text{stack } e')\}) = (\bigcup \{\mathcal{S} e n \mid n. n \in \text{set}(\text{stack } e)\})$ 
proof –
from assms obtain pfx where
  pfx:  $\text{stack } e = \text{pfx} @ (\text{stack } e') \text{ stack } e' \neq []$ 
  let cc =  $\bigcup \{\mathcal{S} e n \mid n. n \in \text{set pfx} \cup \{\text{hd}(\text{stack } e')\}\}$ 
  in  $\mathcal{S} e' = (\lambda x. \text{if } x \in \text{cc} \text{ then } \text{cc} \text{ else } \mathcal{S} e x)$ 
  by (blast dest: unite-stack)
define cc where cc  $\equiv \bigcup \{\mathcal{S} e n \mid n. n \in \text{set pfx} \cup \{\text{hd}(\text{stack } e')\}\}$ 

from pfx have Se':  $\forall x. \mathcal{S} e' x = (\text{if } x \in \text{cc} \text{ then } \text{cc} \text{ else } \mathcal{S} e x)$ 
  by (auto simp: cc-def)
from S-reflexive[OF wf, of hd (stack e')]
have S-hd:  $\mathcal{S} e' (\text{hd}(\text{stack } e')) = \text{cc}$ 
  by (auto simp: Se' cc-def)
from  $\langle \text{stack } e' \neq [] \rangle$ 
have ste':  $\text{set}(\text{stack } e') = \{\text{hd}(\text{stack } e')\} \cup \text{set}(\text{tl}(\text{stack } e'))$ 
  by (metis insert-is-Un list.exhaust-sel list.simps(15))

```

```

from ⟨stack e = pfx @ stack e'⟩ ⟨stack e' ≠ []⟩
have stack e = pfx @ (hd (stack e') # tl (stack e'))
  by auto
hence  $\bigcup \{\mathcal{S} e n \mid n. n \in \text{set } (\text{stack } e)\}$ 
  =  $cc \cup (\bigcup \{\mathcal{S} e n \mid n. n \in \text{set } (\text{tl } (\text{stack } e'))\})$ 
  by (auto simp add: cc-def)
also from S-hd unite-S-tl[OF wf w]
have ... =  $\mathcal{S} e' (\text{hd } (\text{stack } e')) \cup (\bigcup \{\mathcal{S} e' n \mid n. n \in \text{set } (\text{tl } (\text{stack } e'))\})$ 
  by (auto simp: e'-def)
also from ste'
have ... =  $\bigcup \{\mathcal{S} e' n \mid n. n \in \text{set } (\text{stack } e')\}$ 
  by auto
finally show ?thesis
  by simp
qed

```

The head of the stack represents a (not necessarily maximal) SCC.

lemma *unite-subsc*:

```

fixes e v w
defines  $e' \equiv \text{unite } v w e$ 
assumes pre: pre-dfss v e
  and w: w ∈ successors v w ∉ vsuccs e v w ∈ visited e w ∉ explored e
  shows is-subsc (S e' (hd (stack e')))

```

proof –

```

from pre have wf: wf-env e
  by (simp add: pre-dfss-def)
from assms obtain pfx where
  pfx: stack e = pfx @ (stack e') stack e' ≠ []
  let cc =  $\bigcup \{\mathcal{S} e n \mid n. n \in \text{set } pfx \cup \{\text{hd } (\text{stack } e')\}\}$ 
  in S e' = ( $\lambda x. \text{if } x \in cc \text{ then } cc \text{ else } \mathcal{S} e x$ )
  by (blast dest: unite-stack[OF wf])

```

```

define cc where  $cc \equiv \bigcup \{\mathcal{S} e n \mid n. n \in \text{set } pfx \cup \{\text{hd } (\text{stack } e')\}\}$ 

```

```

from wf w have  $w \in \bigcup \{\mathcal{S} e n \mid n. n \in \text{set } (\text{stack } e)\}$ 
  by (simp add: wf-env-def)
hence  $w \in \mathcal{S} e (\text{hd } (\text{stack } e'))$ 
  apply (simp add: e'-def unite-def)
  by (metis dropWhile-eq-Nil-conv hd-dropWhile)

```

have *is-subsc cc*

proof (*clarsimp simp: is-subsc-def*)

fix *x y*

assume $x \in cc \ y \in cc$

then obtain *nx ny* **where**

nx: nx ∈ set pfx ∪ {hd (stack e')} x ∈ S e nx **and**

ny: ny ∈ set pfx ∪ {hd (stack e')} y ∈ S e ny

by (*auto simp: cc-def*)

with *wf* **have** *reachable x nx reachable ny y*

```

    by (auto simp: wf-env-def is-subsc-def)
  from w pre have reachable v w
    by (auto simp: pre-dfss-def)
  from pre have reachable (hd (stack e)) v
    by (auto simp: pre-dfss-def wf-env-def is-subsc-def)
  from pre have stack e = hd (stack e) # tl (stack e)
    by (auto simp: pre-dfss-def)
  with nx ⟨stack e = pfx @ (stack e')⟩ ⟨stack e' ≠ []⟩
  have hd (stack e) ≤ nx in stack e
    by (metis Un-iff Un-insert-right head-precedes list.exhaust-sel list.simps(15)
      set-append sup-bot.right-neutral)
  with wf have reachable nx (hd (stack e))
    by (auto simp: wf-env-def)
  from ⟨stack e = pfx @ (stack e')⟩ ⟨stack e' ≠ []⟩ ny
  have ny ≤ hd (stack e') in stack e
    by (metis List.set-insert empty-set insert-Nil list.exhaust-sel set-append split-list-precedes)
  with wf have reachable (hd (stack e')) ny
    by (auto simp: wf-env-def is-subsc-def)
  from wf ⟨stack e' ≠ []⟩ ⟨w ∈ S e (hd (stack e'))⟩
  have reachable w (hd (stack e'))
    by (auto simp: wf-env-def is-subsc-def)

  from ⟨reachable x nx⟩ ⟨reachable nx (hd (stack e))⟩
    ⟨reachable (hd (stack e)) v⟩ ⟨reachable v w⟩
    ⟨reachable w (hd (stack e'))⟩
    ⟨reachable (hd (stack e')) ny⟩ ⟨reachable ny y⟩
  show reachable x y
    using reachable-trans by meson
qed
with S-reflexive[OF wf, of hd (stack e')] pfx
show ?thesis
  by (auto simp: cc-def)
qed

```

The environment returned by function *unite* extends the input environment.

lemma *unite-sub-env*:

```

  fixes e v w
  defines e' ≡ unite v w e
  assumes pre: pre-dfss v e
    and w: w ∈ successors v w ∉ vsucce e v w ∈ visited e w ∉ explored e
  shows sub-env e e'
proof –
  from pre have wf: wf-env e
    by (simp add: pre-dfss-def)
  from assms obtain pfx where
    pfx: stack e = pfx @ (stack e') stack e' ≠ []
    let cc = ∪ {S e n | n. n ∈ set pfx ∪ {hd (stack e')}}
    in S e' = (λx. if x ∈ cc then cc else S e x)
  by (blast dest: unite-stack[OF wf])

```

```

define cc where  $cc \equiv \bigcup \{\mathcal{S} \ e \ n \mid n. n \in \text{set } pfx \cup \{\text{hd } (\text{stack } e')\}\}$ 
have  $\forall n. \mathcal{S} \ e \ n \subseteq \mathcal{S} \ e' \ n$ 
proof (clarify)
  fix n u
  assume u:  $u \in \mathcal{S} \ e \ n$ 
  show  $u \in \mathcal{S} \ e' \ n$ 
  proof (cases  $n \in cc$ )
    case True
      then obtain m where
        m:  $m \in \text{set } pfx \cup \{\text{hd } (\text{stack } e')\} \ n \in \mathcal{S} \ e \ m$ 
        by (auto simp: cc-def)
        with wf S-reflexive[OF wf, of n] u have  $u \in \mathcal{S} \ e \ m$ 
        by (auto simp: wf-env-def)
        with m pfx show ?thesis
        by (auto simp: cc-def)
      next
        case False
          with pfx u show ?thesis
          by (auto simp: cc-def)
    qed
  moreover
    have  $\text{root } e' = \text{root } e \wedge \text{visited } e' = \text{visited } e$ 
       $\wedge \text{explored } e' = \text{explored } e \wedge \text{vsuccs } e' = \text{vsuccs } e$ 
      by (simp add: e'-def unite-def)
    ultimately show ?thesis
      using unite-S-equal[OF wf w]
      by (simp add: e'-def sub-env-def)
  qed

```

The environment returned by function *unite* is well-formed.

lemma *unite-wf-env*:

```

fixes e v w
defines  $e' \equiv \text{unite } v \ w \ e$ 
assumes pre: pre-dfss v e
  and w:  $w \in \text{successors } v \ w \notin \text{vsuccs } e \ v \ w \in \text{visited } e \ w \notin \text{explored } e$ 
shows wf-env e'
proof –
  from pre have wf: wf-env e
  by (simp add: pre-dfss-def)
  from assms obtain pfx where
    pfx:  $\text{stack } e = pfx \ @ \ (\text{stack } e') \ \text{stack } e' \neq []$ 
     $\text{let } cc = \bigcup \{\mathcal{S} \ e \ n \mid n. n \in \text{set } pfx \cup \{\text{hd } (\text{stack } e')\}\}$ 
     $\text{in } \mathcal{S} \ e' = (\lambda x. \text{if } x \in cc \text{ then } cc \text{ else } \mathcal{S} \ e \ x)$ 
  by (blast dest: unite-stack[OF wf])
  define cc where  $cc \equiv \bigcup \{\mathcal{S} \ e \ n \mid n. n \in \text{set } pfx \cup \{\text{hd } (\text{stack } e')\}\}$ 
  from pfx have Se':  $\forall x. \mathcal{S} \ e' \ x = (\text{if } x \in cc \text{ then } cc \text{ else } \mathcal{S} \ e \ x)$ 
  by (auto simp add: cc-def)

```

```

have cc-Un:  $cc = \bigcup \{\mathcal{S} \ e \ x \mid x. x \in cc\}$ 
proof
  from S-reflexive[OF wf]
  show  $cc \subseteq \bigcup \{\mathcal{S} \ e \ x \mid x. x \in cc\}$ 
    by (auto simp: cc-def)
next
  {
    fix n x
    assume  $x \in cc \ n \in \mathcal{S} \ e \ x$ 
    with wf have  $n \in cc$ 
      unfolding wf-env-def cc-def
      by (smt (verit) Union-iff mem-Collect-eq)
    }
  thus  $(\bigcup \{\mathcal{S} \ e \ x \mid x. x \in cc\}) \subseteq cc$ 
    by blast
qed

from S-reflexive[OF wf, of hd (stack e')]
have hd-cc:  $\mathcal{S} \ e' \ (hd \ (stack \ e')) = cc$ 
  by (auto simp: cc-def Se')

{
  fix n m
  assume  $n: n \in set \ (tl \ (stack \ e'))$ 
    and  $m: m \in \mathcal{S} \ e \ n \cap cc$ 
  from m obtain l where
     $l \in set \ pfx \cup \{hd \ (stack \ e')\} \ m \in \mathcal{S} \ e \ l$ 
    by (auto simp: cc-def)
  with n m wf  $\langle stack \ e = pfx \ @ \ stack \ e' \rangle \langle stack \ e' \neq [] \rangle$ 
  have False
    unfolding wf-env-def
  by (metis (no-types, lifting) Int-iff UnCI UnE disjoint-insert(1) distinct.simps(2)

      distinct-append emptyE hd-Cons-tl insert-iff list.set-sel(1) list.set-sel(2)

      mk-disjoint-insert set-append)
}
hence tl-cc:  $\forall n \in set \ (tl \ (stack \ e')). \ \mathcal{S} \ e \ n \cap cc = \{\}$ 
  by blast

from wf
have  $\forall n \in visited \ e'. \ reachable \ (root \ e') \ n$ 
  distinct (cstack e')
  explored e' ⊆ visited e'
  set (cstack e') ⊆ visited e'
   $\forall n \in explored \ e'. \ \forall m. \ reachable \ n \ m \longrightarrow m \in explored \ e'$ 
   $\forall n. \ vsuccs \ e' \ n \subseteq successors \ n \cap visited \ e'$ 
   $\forall n. \ n \notin visited \ e' \longrightarrow vsuccs \ e' \ n = \{\}$ 

```


$\forall n \in \text{explored } e'. \text{vsuccs } e' n = \text{successors } n$
 $\forall n \in \text{visited } e' - \text{set } (\text{cstack } e'). \text{vsuccs } e' n = \text{successors } n$
 $\forall S \in \text{sccs } e'. \text{is-scc } S$
 $\bigcup (\text{sccs } e') = \text{explored } e'$
by (*auto simp: wf-env-def e'-def unite-def*)

moreover
from $\langle \text{stack } e = \text{pfx } @ \text{stack } e' \rangle$
have $\text{distinct } (\text{stack } e')$
by (*auto simp: wf-env-def*)

moreover
have $\forall n m. n \preceq m \text{ in stack } e' \longrightarrow n \preceq m \text{ in cstack } e'$
proof (*clarify*)
fix $n m$
assume $n \preceq m \text{ in stack } e'$
with $\langle \text{stack } e = \text{pfx } @ \text{stack } e' \rangle \text{ wf}$
have $n \preceq m \text{ in cstack } e$
unfolding *wf-env-def*
by (*metis precedes-append-left*)
thus $n \preceq m \text{ in cstack } e'$
by (*simp add: e'-def unite-def*)
qed

moreover
from $\langle \text{stack } e = \text{pfx } @ \text{stack } e' \rangle$
have $\forall n m. n \preceq m \text{ in stack } e' \longrightarrow \text{reachable } m n$
unfolding *wf-env-def* **by** (*metis precedes-append-left*)

moreover
have $\forall n m. m \in \mathcal{S} e' n \longleftrightarrow (\mathcal{S} e' n = \mathcal{S} e' m)$
proof (*clarify*)
fix $n m$
show $m \in \mathcal{S} e' n \longleftrightarrow (\mathcal{S} e' n = \mathcal{S} e' m)$
proof
assume $l: m \in \mathcal{S} e' n$
show $\mathcal{S} e' n = \mathcal{S} e' m$
proof (*cases n ∈ cc*)
case *True*
with l **show** *?thesis*
by (*simp add: Se'*)
next
case *False*
with $l \text{ wf}$ **have** $\mathcal{S} e' n = \mathcal{S} e' m$
by (*simp add: wf-env-def Se'*)
with *False cc-Un wf* **have** $m \notin \text{cc}$
unfolding *wf-env-def e'-def*
by (*smt (verit, best) Union-iff mem-Collect-eq*)
with $\langle \mathcal{S} e' n = \mathcal{S} e' m \rangle \text{ False}$ **show** *?thesis*

```

    by (simp add: Se')
  qed
next
assume r:  $\mathcal{S} e' n = \mathcal{S} e' m$ 
show  $m \in \mathcal{S} e' n$ 
proof (cases  $n \in cc$ )
  case True
  with r pfx have  $\mathcal{S} e' m = cc$ 
  by (auto simp: cc-def)
  have  $m \in cc$ 
  proof (rule ccontr)
    assume  $m \notin cc$ 
    with pfx have  $\mathcal{S} e' m = \mathcal{S} e m$ 
    by (auto simp: cc-def)
    with S-reflexive[OF wf, of m]  $\langle \mathcal{S} e' m = cc \rangle \langle m \notin cc \rangle$ 
    show False
    by simp
  qed
  with pfx True show  $m \in \mathcal{S} e' n$ 
  by (auto simp: cc-def)
next
case False
hence  $\mathcal{S} e' n = \mathcal{S} e n$ 
  by (simp add: Se')
have  $m \notin cc$ 
proof
  assume m:  $m \in cc$ 
  with  $\langle \mathcal{S} e' n = \mathcal{S} e n \rangle r$  have  $\mathcal{S} e n = cc$ 
  by (simp add: Se')
  with S-reflexive[OF wf, of n] have  $n \in cc$ 
  by simp
  with  $\langle n \notin cc \rangle$  show False ..
qed
with r  $\langle \mathcal{S} e' n = \mathcal{S} e n \rangle$  have  $\mathcal{S} e m = \mathcal{S} e n$ 
  by (simp add: Se')
with S-reflexive[OF wf, of m] have  $m \in \mathcal{S} e n$ 
  by simp
with  $\langle \mathcal{S} e' n = \mathcal{S} e n \rangle$  show ?thesis
  by simp
qed
qed
qed

moreover
have  $\forall n. n \notin \text{visited } e' \longrightarrow \mathcal{S} e' n = \{n\}$ 
proof (clarify)
  fix n
  assume  $n \notin \text{visited } e'$ 
  hence  $n \notin \text{visited } e$ 

```

```

    by (simp add: e'-def unite-def)
  moreover have  $n \notin cc$ 
  proof
    assume  $n \in cc$ 
    then obtain  $m$  where  $m \in set\ pfx \cup \{hd\ (stack\ e')\}$   $n \in \mathcal{S}\ e\ m$ 
      by (auto simp: cc-def)
    with  $\langle stack\ e = pfx\ @\ stack\ e' \rangle \langle stack\ e' \neq [] \rangle$ 
    have  $m \in set\ (stack\ e)$ 
      by auto
    with stack-class[OF wf this  $\langle n \in \mathcal{S}\ e\ m \rangle$ ]  $\langle n \notin visited\ e \rangle$ 
    show False
      by simp
  qed
  ultimately show  $\mathcal{S}\ e'\ n = \{n\}$ 
    using wf by (auto simp: wf-env-def Se')
qed

moreover
have  $\forall n \in set\ (stack\ e'). \forall m \in set\ (stack\ e'). n \neq m \longrightarrow \mathcal{S}\ e'\ n \cap \mathcal{S}\ e'\ m = \{\}$ 
  proof (clarify)
    fix  $n\ m$ 
    assume  $n \in set\ (stack\ e')\ m \in set\ (stack\ e')\ n \neq m$ 
    show  $\mathcal{S}\ e'\ n \cap \mathcal{S}\ e'\ m = \{\}$ 
      proof (cases  $n = hd\ (stack\ e')$ )
        case True
          with  $\langle m \in set\ (stack\ e') \rangle \langle n \neq m \rangle \langle stack\ e' \neq [] \rangle$ 
          have  $m \in set\ (tl\ (stack\ e'))$ 
            by (metis hd-Cons-tl set-ConsD)
          with True hd-cc tl-cc unite-S-tl[OF wf w]
          show ?thesis
            by (auto simp: e'-def)
        case False
          next
          with  $\langle n \in set\ (stack\ e') \rangle \langle stack\ e' \neq [] \rangle$ 
          have  $n \in set\ (tl\ (stack\ e'))$ 
            by (metis hd-Cons-tl set-ConsD)
          show ?thesis
            proof (cases  $m = hd\ (stack\ e')$ )
              case True
                with  $\langle n \in set\ (tl\ (stack\ e')) \rangle\ hd-cc\ tl-cc\ unite-S-tl[OF\ wf\ w]$ 
                show ?thesis
                  by (auto simp: e'-def)
              case False
                next
                with  $\langle m \in set\ (stack\ e') \rangle \langle stack\ e' \neq [] \rangle$ 
                have  $m \in set\ (tl\ (stack\ e'))$ 
                  by (metis hd-Cons-tl set-ConsD)
                with  $\langle n \in set\ (tl\ (stack\ e')) \rangle$ 
                have  $\mathcal{S}\ e'\ m = \mathcal{S}\ e\ m \wedge \mathcal{S}\ e'\ n = \mathcal{S}\ e\ n$ 

```

by (auto simp: e'-def unite-S-tl[OF wf w])
 moreover
 from $\langle m \in \text{set } (\text{stack } e') \rangle \langle n \in \text{set } (\text{stack } e') \rangle \langle \text{stack } e = \text{pfx } @ \text{stack } e' \rangle$
 have $m \in \text{set } (\text{stack } e) \wedge n \in \text{set } (\text{stack } e)$
 by auto
 ultimately show ?thesis
 using wf $\langle n \neq m \rangle$ by (auto simp: wf-env-def)
 qed
 qed
 qed

moreover
 {
 from unite-S-equal[OF wf w]
 have $\bigcup \{ \mathcal{S} \ e' \ n \mid n. n \in \text{set } (\text{stack } e') \} = \bigcup \{ \mathcal{S} \ e \ n \mid n. n \in \text{set } (\text{stack } e) \}$
 by (simp add: e'-def)
 with wf
 have $\bigcup \{ \mathcal{S} \ e' \ n \mid n. n \in \text{set } (\text{stack } e') \} = \text{visited } e - \text{explored } e$
 by (simp add: wf-env-def)
 }
 hence $\bigcup \{ \mathcal{S} \ e' \ n \mid n. n \in \text{set } (\text{stack } e') \} = \text{visited } e' - \text{explored } e'$
 by (simp add: e'-def unite-def)

moreover
 have $\forall n \in \text{set } (\text{stack } e'). \forall m \in \mathcal{S} \ e' \ n. m \in \text{set } (\text{cstack } e') \longrightarrow m \preceq n \text{ in } \text{cstack } e'$
 proof (clarify)
 fix n m
 assume $n \in \text{set } (\text{stack } e') \ m \in \mathcal{S} \ e' \ n \ m \in \text{set } (\text{cstack } e')$
 from $\langle m \in \text{set } (\text{cstack } e') \rangle$ have $m \in \text{set } (\text{cstack } e)$
 by (simp add: e'-def unite-def)
 have $m \preceq n \text{ in } \text{cstack } e$
 proof (cases $n = \text{hd } (\text{stack } e')$)
 case True
 with $\langle m \in \mathcal{S} \ e' \ n \rangle$ have $m \in \text{cc}$
 by (simp add: hd-cc)
 then obtain l where
 $l \in \text{set } \text{pfx} \cup \{ \text{hd } (\text{stack } e') \} \ m \in \mathcal{S} \ e \ l$
 by (auto simp: cc-def)
 with $\langle \text{stack } e = \text{pfx } @ \text{stack } e' \rangle \langle \text{stack } e' \neq [] \rangle$
 have $l \in \text{set } (\text{stack } e)$
 by auto
 with $\langle m \in \mathcal{S} \ e \ l \rangle \langle m \in \text{set } (\text{cstack } e) \rangle$ wf
 have $m \preceq l \text{ in } \text{cstack } e$
 by (auto simp: wf-env-def)
 moreover
 from $\langle l \in \text{set } \text{pfx} \cup \{ \text{hd } (\text{stack } e') \} \rangle$ True
 $\langle \text{stack } e = \text{pfx } @ \text{stack } e' \rangle \langle \text{stack } e' \neq [] \rangle$
 have $l \preceq n \text{ in } \text{stack } e$

by (metis List.set-insert empty-set hd-Cons-tl insert-Nil set-append split-list-precedes)
 with wf have $l \preceq n$ in cstack e
 by (auto simp: wf-env-def)
 ultimately show ?thesis
 using wf unfolding wf-env-def
 by (meson precedes-trans)
 next
 case False
 with $\langle n \in \text{set } (\text{stack } e') \rangle \langle \text{stack } e' \neq [] \rangle$
 have $n \in \text{set } (\text{tl } (\text{stack } e'))$
 by (metis list.collapse set-ConsD)
 with unite-S-tl[OF wf w] $\langle m \in \mathcal{S} \ e' \ n \rangle$
 have $m \in \mathcal{S} \ e \ n$
 by (simp add: e'-def)
 with $\langle n \in \text{set } (\text{stack } e') \rangle \langle \text{stack } e = \text{pfx} @ \text{stack } e' \rangle$
 $\langle m \in \text{set } (\text{cstack } e) \rangle$ wf
 show ?thesis
 by (auto simp: wf-env-def)
 qed
 thus $m \preceq n$ in cstack e'
 by (simp add: e'-def unite-def)
 qed

 moreover
 have $\forall n \ m. n \preceq m$ in stack e' $\wedge n \neq m \longrightarrow$
 $(\forall u \in \mathcal{S} \ e' \ n. \neg \text{reachable-avoiding } u \ m \ (\text{unvisited } e' \ n))$
 proof (clarify)
 fix x y u
 assume xy: $x \preceq y$ in stack e' $x \neq y$
 and u: $u \in \mathcal{S} \ e' \ x$ reachable-avoiding u y (unvisited e' x)
 show False
 proof (cases $x = \text{hd } (\text{stack } e')$)
 case True
 hence $\mathcal{S} \ e' \ x = \text{cc}$
 by (simp add: hd-cc)
 with $\langle u \in \mathcal{S} \ e' \ x \rangle$ obtain x' where
 $x': x' \in \text{set } \text{pfx} \cup \{\text{hd } (\text{stack } e')\}$ $u \in \mathcal{S} \ e \ x'$
 by (auto simp: cc-def)
 from $\langle \text{stack } e = \text{pfx} @ \text{stack } e' \rangle \langle \text{stack } e' \neq [] \rangle$
 have $\text{stack } e = \text{pfx} @ (\text{hd } (\text{stack } e') \# \text{tl } (\text{stack } e'))$
 by auto
 with x' True have $x' \preceq x$ in stack e
 by (simp add: split-list-precedes)
 moreover
 from xy $\langle \text{stack } e = \text{pfx} @ \text{stack } e' \rangle$ have $x \preceq y$ in stack e
 by (simp add: precedes-append-left)
 ultimately have $x' \preceq y$ in stack e
 using wf by (auto simp: wf-env-def elim: precedes-trans)
 from $\langle x' \preceq x$ in stack e $\rangle \langle x \preceq y$ in stack e \rangle wf $\langle x \neq y \rangle$

```

have  $x' \neq y$ 
  by (auto simp: wf-env-def dest: precedes-antisym)
let ?unv =  $\bigcup \{unvisited\ e\ y \mid y. y \in set\ pfx \cup \{hd\ (stack\ e')\}\}$ 
from  $\langle \mathcal{S}\ e'\ x = cc \rangle$  have ?unv = unvisited  $e'\ x$ 
  by (auto simp: unvisited-def cc-def  $e'$ -def unite-def)
with  $\langle reachable\ avoiding\ u\ y\ (unvisited\ e'\ x) \rangle$ 
have reachable-avoiding  $u\ y\ ?unv$ 
  by simp
with  $x'$  have reachable-avoiding  $u\ y\ (unvisited\ e'\ x')$ 
  by (blast intro: ra-mono)
with  $\langle x' \preceq y\ in\ stack\ e \rangle \langle x' \neq y \rangle \langle u \in \mathcal{S}\ e\ x' \rangle$  wf
show ?thesis
  by (auto simp: wf-env-def)
next
case False
with  $\langle x \preceq y\ in\ stack\ e' \rangle \langle stack\ e' \neq [] \rangle$ 
have  $x \in set\ (tl\ (stack\ e'))$ 
  by (metis list.exhaust-sel precedes-mem(1) set-ConsD)
with  $\langle u \in \mathcal{S}\ e'\ x \rangle$  have  $u \in \mathcal{S}\ e\ x$ 
  by (auto simp add: unite-S-tl[OF wf w]  $e'$ -def)
moreover
from  $\langle x \preceq y\ in\ stack\ e' \rangle \langle stack\ e = pfx\ @\ stack\ e' \rangle$ 
have  $x \preceq y\ in\ stack\ e$ 
  by (simp add: precedes-append-left)
moreover
from unite-S-tl[OF wf w]  $\langle x \in set\ (tl\ (stack\ e')) \rangle$ 
have unvisited  $e'\ x = unvisited\ e\ x$ 
  by (auto simp: unvisited-def  $e'$ -def unite-def)
ultimately show ?thesis
  using  $\langle x \neq y \rangle \langle reachable\ avoiding\ u\ y\ (unvisited\ e'\ x) \rangle$  wf
  by (auto simp: wf-env-def)
qed
qed

moreover
have  $\forall n. is\ subsc\ (\mathcal{S}\ e'\ n)$ 
proof
fix  $n$ 
show is-subsc  $(\mathcal{S}\ e'\ n)$ 
proof (cases  $n \in cc$ )
case True
hence  $\mathcal{S}\ e'\ n = cc$ 
  by (simp add: Se')
with unite-subsc[OF pre w] hd-cc
show ?thesis
  by (auto simp:  $e'$ -def)
next
case False

```

with wf show *?thesis*
by (*simp add: Se' wf-env-def*)
qed
qed

ultimately show *?thesis*
unfolding wf-env-def by blast
qed

7.2 Lemmas establishing the pre-conditions

The precondition of function *dfs* ensures the precondition of *dfss* at the call of that function.

lemma *pre-dfs-pre-dfss*:
assumes *pre-dfs v e*
shows *pre-dfss v (e(|visited := visited e \cup {v},*
stack := v # stack e,
cstack := v # cstack e))
(is pre-dfss v ?e')

proof –

from *assms have wf: wf-env e*
by (*simp add: pre-dfs-def*)

from *assms have v: v \notin visited e*
by (*simp add: pre-dfs-def*)

from *assms stack-visited[OF wf]*

have $\forall n \in \text{visited } ?e'. \text{reachable } (\text{root } ?e') n$
 $\text{distinct } (\text{stack } ?e')$
 $\text{distinct } (\text{cstack } ?e')$
 $\text{explored } ?e' \subseteq \text{visited } ?e'$
 $\text{set } (\text{cstack } ?e') \subseteq \text{visited } ?e'$
 $\forall n \in \text{explored } ?e'. \forall m. \text{reachable } n m \longrightarrow m \in \text{explored } ?e'$
 $\forall n. \text{vsuccs } ?e' n \subseteq \text{successors } n$
 $\forall n \in \text{explored } ?e'. \text{vsuccs } ?e' n = \text{successors } n$
 $\forall n \in \text{visited } ?e' - \text{set}(\text{cstack } ?e'). \text{vsuccs } ?e' n = \text{successors } n$
 $\forall n. n \notin \text{visited } ?e' \longrightarrow \text{vsuccs } ?e' n = \{\}$
 $(\forall n m. m \in \mathcal{S } ?e' n \longleftrightarrow (\mathcal{S } ?e' n = \mathcal{S } ?e' m))$
 $(\forall n. n \notin \text{visited } ?e' \longrightarrow \mathcal{S } ?e' n = \{n\})$
 $\forall n. \text{is-subsc} (\mathcal{S } ?e' n)$
 $\forall S \in \text{sccs } ?e'. \text{is-scc } S$
 $\bigcup (\text{sccs } ?e') = \text{explored } ?e'$
by (*auto simp: pre-dfs-def wf-env-def*)

moreover

have $\forall n m. n \preceq m \text{ in stack } ?e' \longrightarrow \text{reachable } m n$

proof (*clarify*)

fix *x y*

assume $x \preceq y \text{ in stack } ?e'$

```

show reachable y x
proof (cases x=v)
  assume  $x=v$ 
  with  $\langle x \preceq y \text{ in stack } ?e' \rangle$  assms show ?thesis
    apply (simp add: pre-dfs-def)
    by (metis insert-iff list.simps(15) precedes-mem(2) reachable-refl)
next
  assume  $x \neq v$ 
  with  $\langle x \preceq y \text{ in stack } ?e' \rangle$  wf show ?thesis
    by (simp add: pre-dfs-def wf-env-def precedes-in-tail)
qed
qed

moreover
from wf v have  $\forall n. \text{vsuccs } ?e' n \subseteq \text{visited } ?e'$ 
  by (auto simp: wf-env-def)

moreover
from wf v
have  $(\forall n \in \text{set } (\text{stack } ?e'). \forall m \in \text{set } (\text{stack } ?e'). n \neq m \longrightarrow \mathcal{S} ?e' n \cap \mathcal{S} ?e'$ 
 $m = \{\})$ 
  apply (simp add: wf-env-def)
  by (metis singletonD)

moreover
have  $\bigcup \{\mathcal{S} ?e' v \mid v. v \in \text{set } (\text{stack } ?e')\} = \text{visited } ?e' - \text{explored } ?e'$ 
proof -
  have  $\bigcup \{\mathcal{S} ?e' v \mid v. v \in \text{set } (\text{stack } ?e')\} =$ 
 $(\bigcup \{\mathcal{S} e v \mid v. v \in \text{set } (\text{stack } e)\}) \cup \mathcal{S} e v$ 
  by auto
  also from wf v have  $\dots = \text{visited } ?e' - \text{explored } ?e'$ 
  by (auto simp: wf-env-def)
  finally show ?thesis .
qed

moreover
have  $\forall n m. n \preceq m \text{ in stack } ?e' \wedge n \neq m \longrightarrow$ 
 $(\forall u \in \mathcal{S} ?e' n. \neg \text{reachable-avoiding } u m (\text{unvisited } ?e' n))$ 
proof (clarify)
  fix  $x y u$ 
  assume asm:  $x \preceq y \text{ in stack } ?e' x \neq y u \in \mathcal{S} ?e' x$ 
 $\text{reachable-avoiding } u y (\text{unvisited } ?e' x)$ 
  show False
  proof (cases x = v)
  case True
  with wf v  $\langle u \in \mathcal{S} ?e' x \rangle$  have  $u = v \text{ vsuccs } ?e' v = \{\}$ 
  by (auto simp: wf-env-def)
  with  $\langle \text{reachable-avoiding } u y (\text{unvisited } ?e' x) \rangle$  [THEN ra-cases]
   $\text{True } \langle x \neq y \rangle$  wf

```



```

    show ?thesis
      by (auto simp: wf-env-def unvisited-def)
  next
    case False
    with asm wf show ?thesis
      by (auto simp: precedes-in-tail wf-env-def unvisited-def)
  qed
qed

moreover
have  $\forall n m. n \preceq m \text{ in } \text{stack } ?e' \longrightarrow n \preceq m \text{ in } \text{cstack } ?e'$ 
proof (clarsimp)
  fix n m
  assume  $n \preceq m \text{ in } (v \# \text{stack } e)$ 
  with assms show  $n \preceq m \text{ in } (v \# \text{cstack } e)$ 
    unfolding pre-dfs-def wf-env-def
    by (metis head-precedes insertI1 list.simps(15) precedes-in-tail precedes-mem(2)
    precedes-refl)
  qed

moreover
have  $\forall n \in \text{set } (\text{stack } ?e'). \forall m \in \mathcal{S} ?e'. n. m \in \text{set } (\text{cstack } ?e') \longrightarrow m \preceq n \text{ in } \text{cstack } ?e'$ 
proof (clarify)
  fix n m
  assume  $n \in \text{set } (\text{stack } ?e') m \in \mathcal{S} ?e' n m \in \text{set } (\text{cstack } ?e')$ 
  show  $m \preceq n \text{ in } \text{cstack } ?e'$ 
  proof (cases  $n = v$ )
    case True
    with wf v  $\langle m \in \mathcal{S} ?e' n \rangle$  show ?thesis
      by (auto simp: wf-env-def)
    next
    case False
    with  $\langle n \in \text{set } (\text{stack } ?e') \rangle \langle m \in \mathcal{S} ?e' n \rangle$ 
    have  $n \in \text{set } (\text{stack } e) m \in \mathcal{S} e n$ 
      by auto
    with wf v False  $\langle m \in \mathcal{S} e n \rangle \langle m \in \text{set } (\text{cstack } ?e') \rangle$ 
    show ?thesis
      apply (simp add: wf-env-def)
      by (metis (mono-tags, lifting) precedes-in-tail singletonD)
  qed
qed

ultimately have wf-env ?e'
  unfolding wf-env-def by (meson le-inf-iff)

moreover
from assms
have  $\forall w \in \text{vsuccs } ?e' v. w \in \text{explored } ?e' \cup \mathcal{S} ?e' (\text{hd } (\text{stack } ?e'))$ 

```

by (*simp add: pre-dfs-def*)

moreover

from $\langle \forall n m. n \preceq m \text{ in } \text{stack } ?e' \longrightarrow \text{reachable } m n \rangle$

have $\forall n \in \text{set } (\text{stack } ?e'). \text{reachable } n v$

by (*simp add: head-precedes*)

moreover

from $\text{wf } v \text{ have } \mathcal{S} ?e' (\text{hd } (\text{stack } ?e')) = \{v\}$

by (*simp add: pre-dfs-def wf-env-def*)

ultimately show *?thesis*

by (*auto simp: pre-dfss-def*)

qed

Similarly, we now show that the pre-conditions of the different function calls in the body of function *dfss* are satisfied. First, it is very easy to see that the pre-condition of *dfs* holds at the call of that function.

lemma *pre-dfss-pre-dfs*:

assumes *pre-dfss v e* **and** $w \notin \text{visited } e$ **and** $w \in \text{successors } v$

shows *pre-dfs w e*

using *assms* **unfolding** *pre-dfss-def pre-dfs-def wf-env-def*

by (*meson succ-reachable*)

The pre-condition of *dfss* holds when the successor considered in the current iteration has already been explored.

lemma *pre-dfss-explored-pre-dfss*:

fixes $e v w$

defines $e'' \equiv e(\text{vsuccs} := (\lambda x. \text{if } x=v \text{ then } \text{vsuccs } e v \cup \{w\} \text{ else } \text{vsuccs } e x))$

assumes $1: \text{pre-dfss } v e$ **and** $2: w \in \text{successors } v$ **and** $3: w \in \text{explored } e$

shows *pre-dfss v e''*

proof –

from 1 **have** $v: v \in \text{visited } e$

by (*simp add: pre-dfss-def*)

have *wf-env e''*

proof –

from 1 **have** *wf: wf-env e*

by (*simp add: pre-dfss-def*)

hence $\forall v \in \text{visited } e''. \text{reachable } (\text{root } e'') v$

distinct (stack e'')

distinct (cstack e'')

$\forall n m. n \preceq m \text{ in } \text{stack } e'' \longrightarrow n \preceq m \text{ in } \text{cstack } e''$

$\forall n m. n \preceq m \text{ in } \text{stack } e'' \longrightarrow \text{reachable } m n$

$\text{explored } e'' \subseteq \text{visited } e''$

$\text{set } (\text{cstack } e'') \subseteq \text{visited } e''$

$\forall n \in \text{explored } e''. \forall m. \text{reachable } n m \longrightarrow m \in \text{explored } e''$

$\forall n m. m \in \mathcal{S} e'' n \longleftrightarrow (\mathcal{S} e'' n = \mathcal{S} e'' m)$

$\forall n. n \notin \text{visited } e'' \longrightarrow \mathcal{S} e'' n = \{n\}$

$\forall n \in \text{set } (\text{stack } e''). \forall m \in \text{set } (\text{stack } e'').$

$n \neq m \longrightarrow \mathcal{S} e'' n \cap \mathcal{S} e'' m = \{\}$
 $\bigcup \{\mathcal{S} e'' n \mid n. n \in \text{set}(\text{stack } e'')\} = \text{visited } e'' - \text{explored } e''$
 $\forall n \in \text{set}(\text{stack } e''). \forall m \in \mathcal{S} e'' n.$
 $m \in \text{set}(\text{cstack } e'') \longrightarrow m \preceq n \text{ in } \text{cstack } e''$
 $\forall n. \text{is-subsc}(\mathcal{S} e'' n)$
 $\forall S \in \text{sccs } e''. \text{is-scc } S$
 $\bigcup (\text{sccs } e'') = \text{explored } e''$
by (*auto simp: wf-env-def e''-def*)
moreover
from *wf 2 3* **have** $\forall v. \text{vsuccs } e'' v \subseteq \text{successors } v \cap \text{visited } e''$
by (*auto simp: wf-env-def e''-def*)
moreover
from *wf v* **have** $\forall n. n \notin \text{visited } e'' \longrightarrow \text{vsuccs } e'' n = \{\}$
by (*auto simp: wf-env-def e''-def*)
moreover
from *wf 2*
have $\forall v. v \in \text{explored } e'' \longrightarrow \text{vsuccs } e'' v = \text{successors } v$
by (*auto simp: wf-env-def e''-def*)
moreover
have $\forall x y. x \preceq y \text{ in } \text{stack } e'' \wedge x \neq y \longrightarrow$
 $(\forall u \in \mathcal{S} e'' x. \neg \text{reachable-avoiding } u y (\text{unvisited } e'' x))$
proof (*clarify*)
fix $x y u$
assume $x \preceq y \text{ in } \text{stack } e'' \wedge x \neq y$
 $u \in \mathcal{S} e'' x$
 $\text{reachable-avoiding } u y (\text{unvisited } e'' x)$
hence *prec: $x \preceq y \text{ in } \text{stack } e'' \wedge u \in \mathcal{S} e'' x$*
by (*auto simp: e''-def*)
with *stack-unexplored[OF wf]* **have** $y \notin \text{explored } e''$
by (*blast dest: precedes-mem*)
have (*unvisited $e'' x = \text{unvisited } e'' x$*)
 $\vee (\text{unvisited } e'' x = \text{unvisited } e'' x \cup \{(v, w)\})$
by (*auto simp: e''-def unvisited-def split: if-splits*)
thus *False*
proof
assume *unvisited $e'' x = \text{unvisited } e'' x$*
with *prec $\langle x \neq y \rangle \langle \text{reachable-avoiding } u y (\text{unvisited } e'' x) \rangle$* *wf*
show *?thesis*
unfolding *wf-env-def* **by** *metis*
next
assume *unvisited $e'' x = \text{unvisited } e'' x \cup \{(v, w)\}$*
with *wf $\langle \text{reachable-avoiding } u y (\text{unvisited } e'' x) \rangle$*
 $\langle y \notin \text{explored } e'' \rangle \langle w \in \text{explored } e'' \rangle$ *prec $\langle x \neq y \rangle$*
show *?thesis*
using *avoiding-explored[OF wf]* **unfolding** *wf-env-def*
by (*metis (no-types, lifting)*)
qed
qed
moreover

```

from wf 2
have  $\forall n \in \text{visited } e'' - \text{set } (\text{cstack } e''). \text{vsuccs } e'' n = \text{successors } n$ 
  by (auto simp: e''-def wf-env-def)
ultimately show ?thesis
  unfolding wf-env-def by meson
qed
with 1 3 show ?thesis
  by (auto simp: pre-dfss-def e''-def)
qed

```

The call to *dfs* establishes the pre-condition for the recursive call to *dfss* in the body of *dfss*.

lemma *pre-dfss-post-dfs-pre-dfss*:

```

fixes e v w
defines e'  $\equiv \text{dfs } w e$ 
defines e''  $\equiv e' \setminus (\text{vsuccs } := (\lambda x. \text{if } x=v \text{ then vsuccs } e' v \cup \{w\} \text{ else vsuccs } e' x))$ 
assumes pre: pre-dfss v e
  and w:  $w \in \text{successors } v w \notin \text{visited } e$ 
  and post: post-dfs w e e'
shows pre-dfss v e''
proof –
from pre
have wf-env e v  $\in \text{visited } e \text{ stack } e \neq [] v \in \mathcal{S} e (\text{hd } (\text{stack } e))$ 
  by (auto simp: pre-dfss-def)
with post have stack e'  $\neq [] v \in \mathcal{S} e' (\text{hd } (\text{stack } e'))$ 
  by (auto dest: dfs-S-hd-stack)

from post have w  $\in \text{visited } e'$ 
  by (simp add: post-dfs-def)

```

have wf-env e''

proof –

```

from post have wf': wf-env e'
  by (simp add: post-dfs-def)
hence  $\forall n \in \text{visited } e''. \text{reachable } (\text{root } e'') n$ 
   $\text{distinct } (\text{stack } e'')$ 
   $\text{distinct } (\text{cstack } e'')$ 
   $\forall n m. n \preceq m \text{ in stack } e'' \longrightarrow n \preceq m \text{ in cstack } e''$ 
   $\forall n m. n \preceq m \text{ in stack } e'' \longrightarrow \text{reachable } m n$ 
   $\text{explored } e'' \subseteq \text{visited } e''$ 
   $\text{set } (\text{cstack } e'') \subseteq \text{visited } e''$ 
   $\forall n \in \text{explored } e''. \forall m. \text{reachable } n m \longrightarrow m \in \text{explored } e''$ 
   $\forall n m. m \in \mathcal{S} e'' n \longleftrightarrow (\mathcal{S} e'' n = \mathcal{S} e'' m)$ 
   $\forall n. n \notin \text{visited } e'' \longrightarrow \mathcal{S} e'' n = \{n\}$ 
   $\forall n \in \text{set } (\text{stack } e''). \forall m \in \text{set } (\text{stack } e'').$ 
     $n \neq m \longrightarrow \mathcal{S} e'' n \cap \mathcal{S} e'' m = \{\}$ 
   $\bigcup \{\mathcal{S} e'' n \mid n. n \in \text{set } (\text{stack } e'')\} = \text{visited } e'' - \text{explored } e''$ 
   $\forall n \in \text{set } (\text{stack } e''). \forall m \in \mathcal{S} e'' n. m \in \text{set } (\text{cstack } e'') \longrightarrow m \preceq n \text{ in}$ 
  cstack e''

```

$\forall n. \text{is-subsc} (\mathcal{S} e'' n)$
 $\forall S \in \text{sccs } e''. \text{is-scc } S$
 $\bigcup (\text{sccs } e'') = \text{explored } e''$
by (*auto simp: wf-env-def e''-def*)

moreover
from $wf' w$ **have** $\forall n. \text{vsuccs } e'' n \subseteq \text{successors } n$
by (*auto simp: wf-env-def e''-def*)

moreover
from $wf' \langle w \in \text{visited } e' \rangle$ **have** $\forall n. \text{vsuccs } e'' n \subseteq \text{visited } e''$
by (*auto simp: wf-env-def e''-def*)

moreover
from $\text{post} \langle v \in \text{visited } e \rangle$
have $\forall n. n \notin \text{visited } e'' \longrightarrow \text{vsuccs } e'' n = \{\}$
apply (*simp add: post-dfs-def wf-env-def sub-env-def e''-def*)
by (*meson subsetD*)

moreover
from $wf' w$
have $\forall n \in \text{explored } e''. \text{vsuccs } e'' n = \text{successors } n$
by (*auto simp: wf-env-def e''-def*)

moreover
have $\forall n m. n \preceq m \text{ in stack } e'' \wedge n \neq m \longrightarrow$
 $(\forall u \in \mathcal{S} e'' n. \neg \text{reachable-avoiding } u m (\text{unvisited } e'' n))$

proof (*clarify*)
fix $x y u$
assume $x \preceq y \text{ in stack } e'' \ x \neq y$
 $u \in \mathcal{S} e'' x$
 $\text{reachable-avoiding } u y (\text{unvisited } e'' x)$
hence $1: x \preceq y \text{ in stack } e' \ u \in \mathcal{S} e' x$
by (*auto simp: e''-def*)
with $\text{stack-unexplored}[OF wf']$ **have** $y \notin \text{explored } e'$
by (*auto dest: precedes-mem*)
have $(\text{unvisited } e' x = \text{unvisited } e'' x)$
 $\vee (\text{unvisited } e' x = \text{unvisited } e'' x \cup \{(v,w)\})$
by (*auto simp: e''-def unvisited-def split: if-splits*)
thus *False*

proof
assume $\text{unvisited } e' x = \text{unvisited } e'' x$
with $1 \langle x \neq y \rangle \langle \text{reachable-avoiding } u y (\text{unvisited } e'' x) \rangle wf'$
show *?thesis*
unfolding *wf-env-def* **by** *metis*

next
assume $\text{unv}: \text{unvisited } e' x = \text{unvisited } e'' x \cup \{(v,w)\}$
from *post*
have $w \in \text{explored } e'$
 $\vee (w \in \mathcal{S} e' (\text{hd } (\text{stack } e')) \wedge (\forall n \in \text{set } (\text{tl } (\text{stack } e')). \mathcal{S} e' n = \mathcal{S} e n))$
by (*auto simp: post-dfs-def*)
thus *?thesis*

proof
assume $w \in \text{explored } e'$

```

with  $wf' \text{ unv } \langle \text{reachable-avoiding } u \ y \ (\text{unvisited } e'' \ x) \rangle$ 
   $\langle y \notin \text{explored } e' \rangle \ 1 \ \langle x \neq y \rangle$ 
show ?thesis
  using avoiding-explored[OF wf'] unfolding wf-env-def
  by (metis (no-types, lifting))
next
assume  $w: w \in \mathcal{S} \ e' \ (\text{hd} \ (\text{stack } e'))$ 
   $\wedge (\forall n \in \text{set} \ (\text{tl} \ (\text{stack } e')). \ \mathcal{S} \ e' \ n = \mathcal{S} \ e \ n)$ 
from  $\langle \text{reachable-avoiding } u \ y \ (\text{unvisited } e'' \ x) \rangle$  [THEN ra-add-edge]
  unv
have  $\text{reachable-avoiding } u \ y \ (\text{unvisited } e' \ x)$ 
   $\vee \text{reachable-avoiding } w \ y \ (\text{unvisited } e' \ x)$ 
by auto
thus ?thesis
proof
  assume  $\text{reachable-avoiding } u \ y \ (\text{unvisited } e' \ x)$ 
  with  $\langle x \preceq y \ \text{in} \ \text{stack } e'' \rangle \ \langle x \neq y \rangle \ \langle u \in \mathcal{S} \ e'' \ x \rangle \ wf'$ 
  show ?thesis
  by (auto simp: e''-def wf-env-def)
next
assume  $\text{reachable-avoiding } w \ y \ (\text{unvisited } e' \ x)$ 
from unv have  $v \in \mathcal{S} \ e' \ x$ 
by (auto simp: unvisited-def)
from  $\langle x \preceq y \ \text{in} \ \text{stack } e'' \rangle$  have  $x \in \text{set} \ (\text{stack } e')$ 
by (simp add: e''-def precedes-mem)
have  $x = \text{hd} \ (\text{stack } e')$ 
proof (rule ccontr)
  assume  $x \neq \text{hd} \ (\text{stack } e')$ 
  with  $\langle x \in \text{set} \ (\text{stack } e') \rangle \ \langle \text{stack } e' \neq [] \rangle$ 
  have  $x \in \text{set} \ (\text{tl} \ (\text{stack } e'))$ 
  by (metis hd-Cons-tl set-ConsD)
  with  $w \ \langle v \in \mathcal{S} \ e' \ x \rangle$  have  $v \in \mathcal{S} \ e \ x$ 
  by auto
moreover
from post  $\langle \text{stack } e' \neq [] \rangle \ \langle x \in \text{set} \ (\text{stack } e') \rangle \ \langle x \in \text{set} \ (\text{tl} \ (\text{stack } e')) \rangle$ 
have  $x \in \text{set} \ (\text{tl} \ (\text{stack } e))$ 
unfolding post-dfs-def
by (metis Un-iff self-append-conv2 set-append tl-append2)
moreover
from pre have  $wf\text{-env } e \ \text{stack } e \neq [] \ v \in \mathcal{S} \ e \ (\text{hd} \ (\text{stack } e))$ 
by (auto simp: pre-dfs-def)
ultimately show False
unfolding wf-env-def
  by (metis (no-types, lifting) distinct.simps(2) hd-Cons-tl insert-disjoint(2))
  list.set-sel(1) list.set-sel(2) mk-disjoint-insert)
qed
with  $\langle \text{reachable-avoiding } w \ y \ (\text{unvisited } e' \ x) \rangle$ 
   $\langle x \preceq y \ \text{in} \ \text{stack } e'' \rangle \ \langle x \neq y \rangle \ w \ wf'$ 

```

```

    show ?thesis
      by (auto simp add: e''-def wf-env-def)
    qed
  qed
  qed
  qed

  moreover
  from wf' ⟨ $\forall n. vsuccs\ e''\ n \subseteq successors\ n$ ⟩
  have  $\forall n \in visited\ e'' - set\ (cstack\ e''). vsuccs\ e''\ n = successors\ n$ 
    by (auto simp: wf-env-def e''-def split: if-splits)
  ultimately show ?thesis
    unfolding wf-env-def by (meson le-inf-iff)
  qed

show pre-dfss v e''
proof -
  from pre post
  have  $v \in visited\ e''$ 
    by (auto simp: pre-dfss-def post-dfs-def sub-env-def e''-def)
  moreover
  {
    fix u
    assume u:  $u \in vsuccs\ e''\ v$ 
    have  $u \in explored\ e'' \cup \mathcal{S}\ e''\ (hd\ (stack\ e''))$ 
    proof (cases u = w)
      case True
      with post show ?thesis
        by (auto simp: post-dfs-def e''-def)
    next
      case False
      with u pre post
      have  $u \in explored\ e \cup \mathcal{S}\ e\ (hd\ (stack\ e))$ 
        by (auto simp: pre-dfss-def post-dfs-def e''-def)
      then show ?thesis
      proof
        assume  $u \in explored\ e$ 
        with post show ?thesis
          by (auto simp: post-dfs-def sub-env-def e''-def)
      next
        assume  $u \in \mathcal{S}\ e\ (hd\ (stack\ e))$ 
        with ⟨wf-env e⟩ post ⟨stack e ≠ []⟩
        show ?thesis
          by (auto simp: e''-def dest: dfs-S-hd-stack)
      qed
    qed
  }
  moreover
  from pre post

```

```

have  $\forall n \in \text{set } (\text{stack } e''). \text{reachable } n \ v$ 
  unfolding pre-dfss-def post-dfs-def
  using e''-def by force
moreover
from  $\langle \text{stack } e' \neq [] \rangle$  have  $\text{stack } e'' \neq []$ 
  by (simp add: e''-def)
moreover
from  $\langle v \in \mathcal{S} \ e' \ (\text{hd } (\text{stack } e')) \rangle$  have  $v \in \mathcal{S} \ e'' \ (\text{hd } (\text{stack } e''))$ 
  by (simp add: e''-def)
moreover
from pre post have  $\exists ns. \text{cstack } e'' = v \ \# \ ns$ 
  by (auto simp: pre-dfss-def post-dfs-def e''-def)
ultimately show ?thesis
  using  $\langle \text{wf-env } e'' \rangle$  unfolding pre-dfss-def by blast
qed
qed

```

Finally, the pre-condition for the recursive call to *dfss* at the end of the body of function *dfss* also holds if *unite* was applied.

lemma *pre-dfss-unite-pre-dfss*:

```

fixes e v w
defines  $e' \equiv \text{unite } v \ w \ e$ 
defines  $e'' \equiv e' \setminus \text{vsuccs} := (\lambda x. \text{if } x=v \text{ then } \text{vsuccs } e' \ v \cup \{w\} \text{ else } \text{vsuccs } e' \ x)$ 
assumes pre: pre-dfss v e
  and  $w: w \in \text{successors } v \ w \notin \text{vsuccs } e \ v \ w \in \text{visited } e \ w \notin \text{explored } e$ 
shows pre-dfss v e''

```

proof –

```

from pre have wf: wf-env e
  by (simp add: pre-dfss-def)
from pre have  $v \in \text{visited } e$ 
  by (simp add: pre-dfss-def)
from pre w have  $v \notin \text{explored } e$ 
  unfolding pre-dfss-def wf-env-def
  by (meson reachable-edge)

```

from *unite-stack[OF wf w]* **obtain** *px* **where**

```

  px: stack e = px @ stack e' stack e' ≠ []
  let cc =  $\bigcup \{ \mathcal{S} \ e \ n \mid n. n \in \text{set } px \cup \{ \text{hd } (\text{stack } e') \} \}$ 
  in  $\mathcal{S} \ e' = (\lambda x. \text{if } x \in cc \text{ then } cc \text{ else } \mathcal{S} \ e \ x)$ 
   $w \in \mathcal{S} \ e' \ (\text{hd } (\text{stack } e'))$ 

```

by (*auto simp: e'-def*)

define *cc* **where** $cc \equiv \bigcup \{ \mathcal{S} \ e \ n \mid n. n \in \text{set } px \cup \{ \text{hd } (\text{stack } e') \} \}$

from *unite-wf-env[OF pre w]* **have** *wf': wf-env e'*

by (*simp add: e'-def*)

from $\langle \text{stack } e = px \ @ \ \text{stack } e' \ \langle \text{stack } e' \neq [] \rangle$

have $\text{hd } (\text{stack } e) \in \text{set } px \cup \{ \text{hd } (\text{stack } e') \}$

by (*simp add: hd-append*)

with pre **have** $v \in cc$
by (*auto simp: pre-dfss-def cc-def*)
from S -*reflexive*[OF wf , *of* hd ($stack\ e'$)]
have hd ($stack\ e'$) $\in cc$
by (*auto simp: cc-def*)
with pfx $\langle v \in cc \rangle$ **have** $v \in \mathcal{S}\ e'$ (hd ($stack\ e'$))
by (*auto simp: cc-def*)

from *unite-sub-env*[OF $pre\ w$] **have** *sub-env* e'
by (*simp add: e'-def*)

have *wf-env* e''

proof –

from wf'

have $\forall n \in visited\ e''.$ *reachable* ($root\ e''$) n
distinct ($stack\ e''$)
distinct ($cstack\ e''$)
 $\forall n\ m. n \preceq m$ *in* $stack\ e'' \longrightarrow n \preceq m$ *in* $cstack\ e''$
 $\forall n\ m. n \preceq m$ *in* $stack\ e'' \longrightarrow reachable\ m\ n$
 $explored\ e'' \subseteq visited\ e''$
 $set\ (cstack\ e'') \subseteq visited\ e''$
 $\forall n \in explored\ e''.$ $\forall m.$ *reachable* $n\ m \longrightarrow m \in explored\ e''$
 $\forall n\ m. m \in \mathcal{S}\ e''\ n \longleftrightarrow (\mathcal{S}\ e''\ n = \mathcal{S}\ e''\ m)$
 $\forall n. n \notin visited\ e'' \longrightarrow \mathcal{S}\ e''\ n = \{n\}$
 $\forall n \in set\ (stack\ e''). \forall m \in set\ (stack\ e'').$
 $n \neq m \longrightarrow \mathcal{S}\ e''\ n \cap \mathcal{S}\ e''\ m = \{\}$
 $\bigcup \{\mathcal{S}\ e''\ n \mid n. n \in set\ (stack\ e'')\} = visited\ e'' - explored\ e''$
 $\forall n \in set\ (stack\ e''). \forall m \in \mathcal{S}\ e''\ n.$
 $m \in set\ (cstack\ e'') \longrightarrow m \preceq n$ *in* $cstack\ e''$
 $\forall n.$ *is-subsec* ($\mathcal{S}\ e''\ n$)
 $\forall S \in sccs\ e''.$ *is-scc* S
 $\bigcup (sccs\ e'') = explored\ e''$
by (*auto simp: wf-env-def e''-def*)

moreover

from wf' w $\langle sub\text{-env}\ e' \rangle$

have $\forall n. vsuccs\ e''\ n \subseteq successors\ n \cap visited\ e''$
by (*auto simp: wf-env-def sub-env-def e''-def*)

moreover

from wf' $\langle v \in visited\ e \rangle$ $\langle sub\text{-env}\ e' \rangle$

have $\forall n. n \notin visited\ e'' \longrightarrow vsuccs\ e''\ n = \{\}$
by (*auto simp: wf-env-def sub-env-def e''-def*)

moreover

from wf' $\langle v \notin explored\ e \rangle$

have $\forall n \in explored\ e''.$ $vsuccs\ e''\ n = successors\ n$
by (*auto simp: wf-env-def e''-def e'-def unite-def*)

moreover
from $wf' \langle w \in \text{successors } v \rangle$
have $\forall n \in \text{visited } e'' - \text{set } (cstack \ e''). \text{vsuccs } e'' \ n = \text{successors } n$
by $(\text{auto simp: wf-env-def } e''\text{-def } e'\text{-def unite-def})$

moreover
have $\forall x \ y. x \preceq y \text{ in stack } e'' \wedge x \neq y \longrightarrow$
 $(\forall u \in \mathcal{S} \ e'' \ x. \neg \text{reachable-avoiding } u \ y \ (\text{unvisited } e'' \ x))$

proof (clarify)
fix $x \ y \ u$
assume $xy: x \preceq y \text{ in stack } e'' \ x \neq y$
and $u: u \in \mathcal{S} \ e'' \ x \text{ reachable-avoiding } u \ y \ (\text{unvisited } e'' \ x)$
hence $\text{prec: } x \preceq y \text{ in stack } e' \ u \in \mathcal{S} \ e' \ x$
by $(\text{simp add: } e''\text{-def})+$
show False
proof $(\text{cases } x = \text{hd } (\text{stack } e'))$
case True
with $\langle v \in \mathcal{S} \ e' \ (\text{hd } (\text{stack } e')) \rangle$
have $\text{unvisited } e' \ x = \text{unvisited } e'' \ x$
 $\vee (\text{unvisited } e' \ x = \text{unvisited } e'' \ x \cup \{(v,w)\})$
by $(\text{auto simp: } e''\text{-def unvisited-def split: if-splits})$
thus False
proof
assume $\text{unvisited } e' \ x = \text{unvisited } e'' \ x$
with $\text{prec } \langle x \neq y \rangle \langle \text{reachable-avoiding } u \ y \ (\text{unvisited } e'' \ x) \rangle wf'$
show $?thesis$
unfolding $wf\text{-env-def}$ **by** metis
next
assume $\text{unvisited } e' \ x = \text{unvisited } e'' \ x \cup \{(v,w)\}$
with $\langle \text{reachable-avoiding } u \ y \ (\text{unvisited } e'' \ x) \rangle [THEN \ \text{ra-add-edge}]$
have $\text{reachable-avoiding } u \ y \ (\text{unvisited } e' \ x)$
 $\vee \text{reachable-avoiding } w \ y \ (\text{unvisited } e' \ x)$
by auto
thus $?thesis$
proof
assume $\text{reachable-avoiding } u \ y \ (\text{unvisited } e' \ x)$
with $\text{prec } \langle x \neq y \rangle wf'$ **show** $?thesis$
by $(\text{auto simp: wf-env-def})$
next
assume $\text{reachable-avoiding } w \ y \ (\text{unvisited } e' \ x)$
with $\langle x = \text{hd } (\text{stack } e') \rangle \langle w \in \mathcal{S} \ e' \ (\text{hd } (\text{stack } e')) \rangle$
 $\langle x \preceq y \text{ in stack } e' \rangle \langle x \neq y \rangle wf'$
show $?thesis$
by $(\text{auto simp: wf-env-def})$
qed
qed
next
case False
with $\langle x \preceq y \text{ in stack } e' \rangle \langle \text{stack } e' \neq [] \rangle$

```

have  $x \in \text{set } (\text{tl } (\text{stack } e'))$ 
  by (metis list.exhaust-sel precedes-mem(1) set-ConsD)
with  $\text{unite-}S\text{-tl}[OF \text{ wf } w] \langle u \in \mathcal{S} \ e' \ x \rangle$ 
have  $u \in \mathcal{S} \ e \ x$ 
  by (simp add: e'-def)
moreover
from  $\langle x \preceq y \text{ in } \text{stack } e' \rangle \langle \text{stack } e = \text{pfx } @ \ \text{stack } e' \rangle$ 
have  $x \preceq y \text{ in } \text{stack } e$ 
  by (simp add: precedes-append-left)
moreover
from  $\langle v \in \mathcal{S} \ e' \ (\text{hd } (\text{stack } e')) \rangle \langle x \in \text{set } (\text{tl } (\text{stack } e')) \rangle$ 
   $\langle \text{stack } e' \neq [] \rangle \text{ wf}'$ 
have  $v \notin \mathcal{S} \ e' \ x$ 
  unfolding wf-env-def
  by (metis (no-types, lifting) Diff-cancel Diff-triv distinct.simps(2) insert-not-empty
    list.exhaust-sel list.set-sel(1) list.set-sel(2) mk-disjoint-insert)
hence  $\text{unvisited } e'' \ x = \text{unvisited } e' \ x$ 
  by (auto simp: unvisited-def e''-def split: if-splits)
moreover
from  $\langle x \in \text{set } (\text{tl } (\text{stack } e')) \rangle \text{ unite-}S\text{-tl}[OF \ \text{wf } w]$ 
have  $\text{unvisited } e' \ x = \text{unvisited } e \ x$ 
  by (simp add: unvisited-def e'-def unite-def)
ultimately show ?thesis
  using  $\langle x \neq y \rangle \langle \text{reachable-avoiding } u \ y \ (\text{unvisited } e'' \ x) \rangle \text{ wf}$ 
  by (auto simp: wf-env-def)
qed
qed

ultimately show ?thesis
  unfolding wf-env-def by meson
qed

show pre-dfss v e''
proof –
  from pre have  $v \in \text{visited } e''$ 
  by (simp add: pre-dfss-def e''-def e'-def unite-def)

moreover
{
  fix  $u$ 
  assume  $u: u \in \text{vsuccs } e'' \ v$ 
  have  $u \in \text{explored } e'' \cup \mathcal{S} \ e'' \ (\text{hd } (\text{stack } e'))$ 
  proof (cases u = w)
  case True
  with  $\langle w \in \mathcal{S} \ e' \ (\text{hd } (\text{stack } e')) \rangle$  show ?thesis
  by (simp add: e''-def)
  next
  case False

```

```

with u have u ∈ vsuccs e v
  by (simp add: e''-def e'-def unite-def)
with pre have u ∈ explored e ∪ S e (hd (stack e))
  by (auto simp: pre-dfss-def)
then show ?thesis
proof
  assume u ∈ explored e
  thus ?thesis
    by (simp add: e''-def e'-def unite-def)
next
  assume u ∈ S e (hd (stack e))
  with ⟨hd (stack e) ∈ set pfx ∪ {hd (stack e')}⟩
  have u ∈ cc
    by (auto simp: cc-def)
  moreover
  from S-reflexive[OF wf, of hd (stack e')] pfx
  have S e' (hd (stack e')) = cc
    by (auto simp: cc-def)
  ultimately show ?thesis
    by (simp add: e''-def)
qed
qed
}
hence ∀ w ∈ vsuccs e'' v. w ∈ explored e'' ∪ S e'' (hd (stack e''))
  by blast

moreover
from pre ⟨stack e = pfx @ stack e'⟩
have ∀ n ∈ set (stack e''). reachable n v
  by (auto simp: pre-dfss-def e''-def)

moreover
from ⟨stack e' ≠ []⟩ have stack e'' ≠ []
  by (simp add: e''-def)

moreover
from ⟨v ∈ S e' (hd (stack e'))⟩ have v ∈ S e'' (hd (stack e''))
  by (simp add: e''-def)

moreover
from pre have ∃ ns. cstack e'' = v # ns
  by (auto simp: pre-dfss-def e''-def e'-def unite-def)

ultimately show ?thesis
  using ⟨wf-env e''⟩ unfolding pre-dfss-def by blast
qed
qed

```

7.3 Lemmas establishing the post-conditions

Assuming the pre-condition of function dfs and the post-condition of the call to $dfss$ in the body of that function, the post-condition of dfs is established.

```

lemma pre-dfs-implies-post-dfs:
  fixes  $v\ e$ 
  defines  $e1 \equiv e(\text{visited} := \text{visited } e \cup \{v\},$ 
            $\text{stack} := (v \# \text{stack } e),$ 
            $\text{cstack} := (v \# \text{cstack } e))$ 
  defines  $e' \equiv \text{dfss } v\ e1$ 
  defines  $e'' \equiv e'(\text{cstack} := \text{tl}(\text{cstack } e'))$ 
  assumes 1: pre-dfs  $v\ e$ 
           and 2: dfs-dfss-dom ( $\text{Inl}(v, e)$ )
           and 3: post-dfss  $v\ e1\ e'$ 
  shows post-dfs  $v\ e\ (dfs\ v\ e)$ 
proof –
  from 1 have  $wf: wf\text{-env } e$ 
    by (simp add: pre-dfs-def)
  from 1 have  $v: v \notin \text{visited } e$ 
    by (simp add: pre-dfs-def)
  from 3 have  $wf': wf\text{-env } e'$ 
    by (simp add: post-dfss-def)
  from 3 have  $cst': \text{cstack } e' = v \# \text{cstack } e$ 
    by (simp add: post-dfss-def e1-def)
  show ?thesis
proof (cases  $v = \text{hd}(\text{stack } e')$ )
  case True
  have notempty:  $\text{stack } e' = v \# \text{stack } e$ 
  proof –
  from 3 obtain  $ns$  where
     $ns: \text{stack } e1 = ns @ (\text{stack } e')\ \text{stack } e' \neq []$ 
    by (auto simp: post-dfss-def)
  have  $ns = []$ 
  proof (rule ccontr)
    assume  $ns \neq []$ 
    with  $ns$  have  $\text{hd } ns = v$ 
      apply (simp add: e1-def)
      by (metis hd-append2 list.sel(1))
    with  $\text{True } ns \langle ns \neq [] \rangle$  have  $\neg \text{distinct } (\text{stack } e1)$ 
      by (metis disjoint-iff-not-equal distinct-append hd-in-set)
    with  $wf\ v\ \text{stack-visited}[OF\ wf]$  show False
      by (auto simp: wf-env-def e1-def)
  qed
  with  $ns$  show ?thesis
    by (simp add: e1-def)
  qed
have  $e2: \text{dfs } v\ e = e'(\text{sccs} := \text{sccs } e' \cup \{\mathcal{S}\ e'\ v\},$ 
            $\text{explored} := \text{explored } e' \cup (\mathcal{S}\ e'\ v),$ 
            $\text{stack} := \text{tl}(\text{stack } e'),$ 

```

$cstack := tl (cstack e')$ (is - = ?e2)
using *True 2 dfs.psimps*[of v e] **unfolding** *e1-def e'-def*
by (*fastforce simp: e1-def e'-def*)

have *sub: sub-env e e1*
by (*auto simp: sub-env-def e1-def*)

from *notempty* **have** *stack2: stack ?e2 = stack e*
by (*simp add: e1-def*)

moreover from *3* **have** $v \in visited ?e2$
by (*auto simp: post-dfss-def sub-env-def e1-def*)

moreover
from *sub 3* **have** *sub-env e e'*
unfolding *post-dfss-def* **by** (*auto elim: sub-env-trans*)
with *stack2* **have** *subenv: sub-env e ?e2*
by (*fastforce simp: sub-env-def*)

moreover have *wf-env ?e2*

proof –

from *wf'*
have $\forall n \in visited ?e2. reachable (root ?e2) n$
 $distinct (stack ?e2)$
 $\forall n. vsuccs ?e2 n \subseteq successors n \cap visited ?e2$
 $\forall n. n \notin visited ?e2 \longrightarrow vsuccs ?e2 n = \{\}$
 $\forall n m. m \in \mathcal{S} ?e2 n \longleftrightarrow (\mathcal{S} ?e2 n = \mathcal{S} ?e2 m)$
 $\forall n. n \notin visited ?e2 \longrightarrow \mathcal{S} ?e2 n = \{n\}$
 $\forall n. is-subsc (\mathcal{S} ?e2 n)$
 $\bigcup (sccs ?e2) = explored ?e2$
by (*auto simp: wf-env-def distinct-tl*)

moreover
from *1 cst'* **have** $distinct (cstack ?e2)$
by (*auto simp: pre-dfs-def wf-env-def*)

moreover
from *1 stack2* **have** $\forall n m. n \preceq m \text{ in } stack ?e2 \longrightarrow reachable m n$
by (*auto simp: pre-dfs-def wf-env-def*)

moreover
from *1 stack2 cst'*
have $\forall n m. n \preceq m \text{ in } stack ?e2 \longrightarrow n \preceq m \text{ in } cstack ?e2$
by (*auto simp: pre-dfs-def wf-env-def*)

moreover
from *notempty wf'* **have** $explored ?e2 \subseteq visited ?e2$
apply (*simp add: wf-env-def*)
using *stack-class*[OF *wf'*]

```

by (smt (verit, del-Insts) Diff-iff insert-subset list.simps(15) subset-eq)

moreover
from  $\exists cst'$  have set (cstack ?e2)  $\subseteq$  visited ?e2
  by (simp add: post-dfss-def wf-env-def e1-def)

moreover
{
  fix u
  assume  $u \in explored\ ?e2$ 
  have vsuccs ?e2 u = successors u
  proof (cases  $u \in explored\ e'$ )
    case True
    with wf' show ?thesis
      by (auto simp: wf-env-def)
    next
    case False
    with  $\langle u \in explored\ ?e2 \rangle$  have  $u \in \mathcal{S}\ e'\ v$ 
      by simp
    show ?thesis
    proof (cases  $u = v$ )
      case True
      with  $\exists$  show ?thesis
        by (auto simp: post-dfss-def)
    next
    case False
    have  $u \in visited\ e' - set\ (cstack\ e')$ 
    proof
      from notempty  $\langle u \in \mathcal{S}\ e'\ v \rangle$  stack-class[OF wf'] False
      show  $u \in visited\ e'$ 
        by auto
    next
    show  $u \notin set\ (cstack\ e')$ 
    proof
      assume  $u: u \in set\ (cstack\ e')$ 
      with notempty  $\langle u \in \mathcal{S}\ e'\ v \rangle$   $\langle wf-env\ e' \rangle$  have  $u \preceq v$  in cstack e'
        by (auto simp: wf-env-def)
      with  $cst'\ u\ False\ wf'$  show False
        unfolding wf-env-def
        by (metis head-precedes precedes-antisym)
    qed
  qed
  with  $\exists$  show ?thesis
    by (auto simp: post-dfss-def wf-env-def)
  qed
  note explored-vsuccs = this
}

```

```

moreover have  $\forall n \in \text{explored } ?e2. \forall m. \text{reachable } n \ m \longrightarrow m \in \text{explored } ?e2$ 
proof (clarify)
  fix  $x \ y$ 
  assume  $asm: x \in \text{explored } ?e2 \ \text{reachable } x \ y$ 
  show  $y \in \text{explored } ?e2$ 
  proof (cases  $x \in \text{explored } e'$ )
    case True
      with  $\langle wf\text{-env } e' \rangle \langle \text{reachable } x \ y \rangle$  show  $?thesis$ 
      by (simp add: wf-env-def)
    next
      case False
      with  $asm$  have  $x \in \mathcal{S} \ e' \ v$ 
      by simp
      with  $\langle \text{explored } ?e2 \subseteq \text{visited } ?e2 \rangle$  have  $x \in \text{visited } e'$ 
      by auto
      from  $\langle x \in \mathcal{S} \ e' \ v \rangle \ wf'$  have  $\text{reachable } v \ x$ 
      by (auto simp: wf-env-def is-subsc-def)
      have  $y \in \text{visited } e'$ 
      proof (rule ccontr)
        assume  $y \notin \text{visited } e'$ 
        with  $\text{reachable-visited}[OF \ wf' \ \langle x \in \text{visited } e' \rangle \ \langle \text{reachable } x \ y \rangle]$ 
        obtain  $n \ m$  where
           $n \in \text{visited } e' \ m \in \text{successors } n - \text{vsuccs } e' \ n$ 
           $\text{reachable } x \ n \ \text{reachable } m \ y$ 
        by blast
        from  $wf' \ \langle m \in \text{successors } n - \text{vsuccs } e' \ n \rangle$ 
        have  $n \notin \text{explored } e'$ 
        by (auto simp: wf-env-def)
        obtain  $n'$  where
           $n' \in \text{set } (\text{stack } e') \ n \in \mathcal{S} \ e' \ n'$ 
        by (rule visited-unexplored[OF  $wf' \ \langle n \in \text{visited } e' \rangle \ \langle n \notin \text{explored } e' \rangle$ ])
        have  $n' = v$ 
        proof (rule ccontr)
          assume  $n' \neq v$ 
          with  $\langle n' \in \text{set } (\text{stack } e') \rangle \ \langle v = \text{hd } (\text{stack } e') \rangle$ 
          have  $n' \in \text{set } (\text{tl } (\text{stack } e'))$ 
          by (metis emptyE hd-Cons-tl set-ConsD set-empty)
        moreover
        from  $\langle n \in \mathcal{S} \ e' \ n' \rangle \ \langle wf\text{-env } e' \rangle$  have  $\text{reachable } n \ n'$ 
        by (auto simp: wf-env-def is-subsc-def)
        with  $\langle \text{reachable } v \ x \rangle \ \langle \text{reachable } x \ n \rangle \ \text{reachable-trans}$ 
        have  $\text{reachable } v \ n'$ 
        by blast
        ultimately show False
        using  $\exists \ \langle v = \text{hd } (\text{stack } e') \rangle$ 
        by (auto simp: post-dfss-def)
      qed
      with  $\langle n \in \mathcal{S} \ e' \ n' \rangle \ \langle m \in \text{successors } n - \text{vsuccs } e' \ n \rangle \ \text{explored-vsuccs}$ 
      show False

```



```

      by auto
    qed
  show ?thesis
  proof (cases y ∈ explored e')
    case True
    then show ?thesis
      by simp
  next
    case False
    obtain n where ndef: n ∈ set (stack e') (y ∈ S e' n)
      by (rule visited-unexplored[OF wf' ⟨y ∈ visited e'⟩ False])
    show ?thesis
    proof (cases n = v)
      case True
      with ndef show ?thesis by simp
    next
      case False
      with ndef notempty have n ∈ set (tl (stack e'))
        by simp
      moreover
      from wf' ndef have reachable y n
        by (auto simp: wf-env-def is-subsc-def)
      with ⟨reachable v x⟩ ⟨reachable x y⟩
      have reachable v n
        by (meson reachable-trans)
      ultimately show ?thesis
        using ⟨v = hd (stack e')⟩ 3
        by (simp add: post-dfss-def)
    qed
  qed
  qed
  qed
  qed

  moreover
  from 3 cst'
  have ∀ n ∈ visited ?e2 - set (cstack ?e2). vsuccs ?e2 n = successors n
    apply (simp add: post-dfss-def wf-env-def)
    by (metis (no-types, lifting) Diff-empty Diff-iff empty-set insertE
      list.exhaust-sel list.sel(1) list.simps(15))

  moreover
  from wf' notempty
  have ∀ n m. n ∈ set (stack ?e2) ∧ m ∈ set (stack ?e2) ∧ n ≠ m
    → (S ?e2 n ∩ S ?e2 m = {})
    by (simp add: wf-env-def)

  moreover
  have ⋃ {S ?e2 n | n . n ∈ set (stack ?e2)} = visited ?e2 - explored ?e2
  proof -

```

```

from wf' notempty
have  $(\bigcup \{\mathcal{S} \ ?e2 \ n \mid n . n \in \text{set} (\text{stack} \ ?e2)\}) \cap \mathcal{S} \ e' \ v = \{\}$ 
  by (auto simp: wf-env-def)
with notempty
have  $\bigcup \{\mathcal{S} \ ?e2 \ n \mid n . n \in \text{set} (\text{stack} \ ?e2)\} =$ 
   $(\bigcup \{\mathcal{S} \ e' \ n \mid n . n \in \text{set} (\text{stack} \ e')\}) - \mathcal{S} \ e' \ v$ 
  by auto
also from wf'
have  $\dots = (\text{visited } e' - \text{explored } e') - \mathcal{S} \ e' \ v$ 
  by (simp add: wf-env-def)
finally show ?thesis
  by auto
qed

moreover
have  $\forall n \in \text{set} (\text{stack} \ ?e2). \forall m \in \mathcal{S} \ ?e2 \ n. m \in \text{set} (\text{cstack} \ ?e2) \longrightarrow m \preceq n$ 
  in cstack ?e2
proof (clarsimp simp: cst')
  fix n m
  assume  $n \in \text{set} (\text{tl} (\text{stack} \ e'))$ 
   $m \in \mathcal{S} \ e' \ n \ m \in \text{set} (\text{cstack} \ e)$ 
  with  $\exists$  have  $m \in \mathcal{S} \ e \ n$ 
  by (auto simp: post-dfss-def e1-def)
  with wf notempty  $\langle n \in \text{set} (\text{tl} (\text{stack} \ e')) \rangle \langle m \in \text{set} (\text{cstack} \ e) \rangle$ 
  show  $m \preceq n$  in cstack e
  by (auto simp: wf-env-def)
qed

moreover
{
  fix x y u
  assume  $xy: x \preceq y$  in stack ?e2  $x \neq y$ 
  and  $u: u \in \mathcal{S} \ ?e2 \ x$  reachable-avoiding u y (unvisited ?e2 x)
  from xy notempty stack2
  have  $x \preceq y$  in stack e'
  by (metis head-precedes insert-iff list.simps(15) precedes-in-tail precedes-mem(2))
  with wf'  $\langle x \neq y \rangle u$  have False
  by (auto simp: wf-env-def unvisited-def)
}

moreover have  $\forall S \in \text{sccs} \ ?e2. \text{is-scc } S$ 
proof (clarify)
  fix S
  assume asm: S  $\in \text{sccs} \ ?e2$ 
  show is-scc S
  proof (cases S = S e' v)
  case True
  with S-reflexive[OF wf'] have  $S \neq \{\}$ 

```

```

    by blast
  from wf' True have subsc: is-subsc S
  by (simp add: wf-env-def)
  {
    assume  $\neg$  is-scc S
    with  $\langle S \neq \{\} \rangle$   $\langle$ is-subsc S $\rangle$  obtain S' where
      S'-def:  $S' \neq S \subseteq S'$  is-subsc S'
      unfolding is-scc-def by blast
    then obtain x where  $x \in S' \wedge x \notin S$ 
      by blast
    with True S'-def wf'
    have xv: reachable v x  $\wedge$  reachable x v
      unfolding wf-env-def is-subsc-def by (metis in-mono)
    from  $\langle \forall v w. w \in S \ ?e2\ v \longleftrightarrow (S \ ?e2\ v = S \ ?e2\ w) \rangle$ 
    have v  $\in$  explored ?e2
      by auto
    with  $\langle \forall x \in$  explored ?e2.  $\forall y.$  reachable x y  $\longrightarrow$  y  $\in$  explored ?e2 $\rangle$ 
      xv  $\langle S = S \ e' \ v \rangle$   $\langle x \in S' \wedge x \notin S \rangle$ 
    have x  $\in$  explored e'
      by auto
    with wf' xv have v  $\in$  explored e'
      by (auto simp: wf-env-def)
    with notempty have False
      by (auto intro: stack-unexplored[OF wf'])
  }
  then show ?thesis
    by blast
  next
  case False
  with asm wf' show ?thesis
    by (auto simp: wf-env-def)
  qed
  qed

  ultimately show ?thesis
    unfolding wf-env-def by meson
  qed

  moreover
  from  $\langle$ wf-env ?e2 $\rangle$  have v  $\in$  explored ?e2
    by (auto simp: wf-env-def)

  moreover
  from 3 have vsuccs ?e2 v = successors v
    by (simp add: post-dfss-def)

  moreover
  from 1 3 have  $\forall w \in$  visited e. vsuccs ?e2 w = vsuccs e w
    by (auto simp: pre-dfs-def post-dfss-def e1-def)

```

moreover
from *stack2 1*
have $\forall n \in \text{set } (\text{stack } ?e2). \text{reachable } n \ v$
by (*simp add: pre-dfs-def*)

moreover
from *stack2* **have** $\exists ns. \text{stack } e = ns \ @ \ (\text{stack } ?e2)$
by *auto*

moreover
from \exists **have** $\forall n \in \text{set } (\text{stack } ?e2). \mathcal{S} \ ?e2 \ n = \mathcal{S} \ e \ n$
by (*auto simp: post-dfs-def e1-def*)

moreover
from *cst'* **have** $\text{cstack } ?e2 = \text{cstack } e$
by *simp*

ultimately show *?thesis*
unfolding *post-dfs-def* **using** *e2* **by** *simp*

next
case *False*
with \mathcal{Q} **have** $e': \text{dfs } v \ e = e''$
by (*simp add: dfs.psimps e''-def e'-def e1-def*)

moreover have $\text{wf-env } e''$
proof –
from *wf'*
have $\forall n \in \text{visited } e''. \text{reachable } (\text{root } e'') \ n$
 $\text{distinct } (\text{stack } e'')$
 $\text{distinct } (\text{cstack } e'')$
 $\forall n \ m. n \preceq m \text{ in stack } e'' \longrightarrow \text{reachable } m \ n$
 $\text{explored } e'' \subseteq \text{visited } e''$
 $\forall n \in \text{explored } e''. \forall m. \text{reachable } n \ m \longrightarrow m \in \text{explored } e''$
 $\forall n. \text{vsuccs } e'' \ n \subseteq \text{successors } n \cap \text{visited } e''$
 $\forall n. n \notin \text{visited } e'' \longrightarrow \text{vsuccs } e'' \ n = \{\}$
 $\forall n \in \text{explored } e''. \text{vsuccs } e'' \ n = \text{successors } n$
 $\forall n \ m. m \in \mathcal{S} \ e'' \ n \longleftrightarrow (\mathcal{S} \ e'' \ n = \mathcal{S} \ e'' \ m)$
 $\forall n. n \notin \text{visited } e'' \longrightarrow \mathcal{S} \ e'' \ n = \{n\}$
 $\forall n \in \text{set } (\text{stack } e''). \forall m \in \text{set } (\text{stack } e'').$
 $n \neq m \longrightarrow \mathcal{S} \ e'' \ n \cap \mathcal{S} \ e'' \ m = \{\}$
 $\bigcup \{\mathcal{S} \ e'' \ n \mid n. n \in \text{set } (\text{stack } e'')\} = \text{visited } e'' - \text{explored } e''$
 $\forall n. \text{is-subsc} (\mathcal{S} \ e'' \ n)$
 $\forall S \in \text{sccs } e''. \text{is-scc } S$
 $\bigcup (\text{sccs } e'') = \text{explored } e''$
by (*auto simp: e''-def wf-env-def distinct-tl*)

moreover have $\forall n \ m. n \preceq m \text{ in stack } e'' \longrightarrow n \preceq m \text{ in cstack } e''$
proof (*clarsimp simp add: e''-def*)

```

fix  $n\ m$ 
assume  $nm: n \preceq m$  in stack  $e'$ 
with  $\exists$  have  $n \preceq m$  in cstack  $e'$ 
  unfolding  $post\text{-}dfss\text{-}def\ wf\text{-}env\text{-}def$ 
  by  $meson$ 
moreover
have  $n \neq v$ 
proof
  assume  $n = v$ 
  with  $nm$  have  $n \in set (stack\ e')$ 
    by  $(simp\ add: precedes\text{-}mem)$ 
  with  $\exists \langle n = v \rangle$  have  $v = hd (stack\ e')$ 
    unfolding  $post\text{-}dfss\text{-}def\ wf\text{-}env\text{-}def$ 
    by  $(metis (no\text{-}types, opaque\text{-}lifting) IntI\ equals0D\ list.set\text{-}sel(1))$ 
  with  $\langle v \neq hd (stack\ e') \rangle$  show  $False$ 
    by  $simp$ 
qed
ultimately show  $n \preceq m$  in tl (cstack  $e'$ )
  by  $(simp\ add: cst'\ precedes\text{-}in\text{-}tail)$ 
qed

moreover
from  $\exists$  have  $set (cstack\ e'') \subseteq visited\ e''$ 
  by  $(simp\ add: post\text{-}dfss\text{-}def\ wf\text{-}env\text{-}def\ e''\text{-}def\ e1\text{-}def\ subset\text{-}eq)$ 

moreover
from  $\exists$ 
have  $\forall n \in visited\ e'' - set (cstack\ e''). vsuccs\ e''\ n = successors\ n$ 
  apply  $(simp\ add: post\text{-}dfss\text{-}def\ wf\text{-}env\text{-}def\ e''\text{-}def\ e1\text{-}def)$ 
  by  $(metis (no\text{-}types, opaque\text{-}lifting) DiffE\ DiffI\ set\text{-}ConsD)$ 

moreover
have  $\forall n \in set (stack\ e''). \forall m \in \mathcal{S}\ e''\ n.$ 
   $m \in set (cstack\ e'') \longrightarrow m \preceq n$  in cstack  $e''$ 
proof  $(clarsimp\ simp: e''\text{-}def)$ 
  fix  $m\ n$ 
  assume  $asm: n \in set (stack\ e'')\ m \in \mathcal{S}\ e''\ n$ 
   $m \in set (tl (cstack\ e''))$ 
  with  $wf'\ cst'$  have  $m \neq v\ m \preceq n$  in cstack  $e'$ 
    by  $(auto\ simp: wf\text{-}env\text{-}def)$ 
  with  $cst'$  show  $m \preceq n$  in tl (cstack  $e'$ )
    by  $(simp\ add: precedes\text{-}in\text{-}tail)$ 
qed

moreover
from  $wf'$ 
have  $(\forall x\ y. x \preceq y$  in stack  $e'' \wedge x \neq y \longrightarrow$ 
   $(\forall u \in \mathcal{S}\ e''\ x. \neg reachable\text{-}avoiding\ u\ y (unvisited\ e''\ x)))$ 
  by  $(force\ simp: e''\text{-}def\ wf\text{-}env\text{-}def\ unvisited\text{-}def)$ 

```

ultimately show *?thesis*
 unfolding *wf-env-def* by *blast*
 qed

moreover
 from \exists have $v \in \text{visited } e''$
 by (*auto simp: post-dfss-def sub-env-def e''-def e1-def*)

moreover
 have *subenv*: *sub-env e e''*
 proof –
 have *sub-env e e1*
 by (*auto simp: sub-env-def e1-def*)
 with \exists have *sub-env e e'*
 by (*auto simp: post-dfss-def elim: sub-env-trans*)
 thus *?thesis*
 by (*auto simp add: sub-env-def e''-def*)
 qed

moreover
 from \exists have $\text{vsuccs } e'' v = \text{successors } v$
 by (*simp add: post-dfss-def e''-def*)

moreover
 from $1 \exists$ have $\forall w \in \text{visited } e. \text{vsuccs } e'' w = \text{vsuccs } e w$
 by (*auto simp: pre-dfs-def post-dfss-def e1-def e''-def*)

moreover
 from \exists have $\forall n \in \text{set } (\text{stack } e''). \text{reachable } n v$
 by (*auto simp: e''-def post-dfss-def*)

moreover
 from $\exists \langle v \neq \text{hd } (\text{stack } e') \rangle$
 have $\exists ns. \text{stack } e = ns @ (\text{stack } e'')$
 apply (*simp add: post-dfss-def e''-def e1-def*)
 by (*metis append-Nil list.sel(1) list.sel(3) tl-append2*)

moreover
 from \exists
 have $\text{stack } e'' \neq [] \vee v \in \mathcal{S} e'' (\text{hd } (\text{stack } e''))$
 $\forall n \in \text{set } (\text{tl } (\text{stack } e'')). \mathcal{S} e'' n = \mathcal{S} e n$
 by (*auto simp: post-dfss-def e1-def e''-def*)

moreover
 from *cst'* have $\text{cstack } e'' = \text{cstack } e$
 by (*simp add: e''-def*)

ultimately show *?thesis* unfolding *post-dfs-def*

by *blast*
qed
qed

The following lemma is central for proving partial correctness: assuming termination (represented by the predicate $dfs\text{-}dfss\text{-}dom$) and the pre-condition of the functions, both dfs and $dfss$ establish their post-conditions. The first part of the theorem follows directly from the preceding lemma and the computational induction rule generated by Isabelle, the second part is proved directly, distinguishing the different cases in the definition of function $dfss$.

lemma *pre-post*:

shows

$\llbracket dfs\text{-}dfss\text{-}dom\ (Inl(v,e));\ pre\text{-}dfs\ v\ e \rrbracket \implies post\text{-}dfs\ v\ e\ (dfs\ v\ e)$

$\llbracket dfs\text{-}dfss\text{-}dom\ (Inr(v,e));\ pre\text{-}dfss\ v\ e \rrbracket \implies post\text{-}dfss\ v\ e\ (dfss\ v\ e)$

proof (*induct rule: dfs-dfss.pinduct*)

fix $v\ e$

assume $dom: dfs\text{-}dfss\text{-}dom\ (Inl(v,e))$

and $predfs: pre\text{-}dfs\ v\ e$

and $prepostdfs: \bigwedge e1.\ \llbracket e1 = e\ (\!visited := visited\ e \cup \{v\},\ stack := v \# stack\ e,$

$cstack := v \# cstack\ e); pre\text{-}dfss\ v\ e1 \rrbracket$

$\implies post\text{-}dfss\ v\ e1\ (dfss\ v\ e1)$

then show $post\text{-}dfs\ v\ e\ (dfs\ v\ e)$

using *pre-dfs-implies-post-dfs pre-dfs-pre-dfss* **by** *auto*

next

fix $v\ e$

assume $dom: dfs\text{-}dfss\text{-}dom\ (Inr(v,e))$

and $predfss: pre\text{-}dfss\ v\ e$

and $prepostdfs:$

$\bigwedge vs\ w.$

$\llbracket vs = successors\ v - vsuccs\ e\ v;\ vs \neq \{\};\ w = (SOME\ x.\ x \in vs);$

$w \notin explored\ e;\ w \notin visited\ e;\ pre\text{-}dfs\ w\ e \rrbracket$

$\implies post\text{-}dfs\ w\ e\ (dfs\ w\ e)$

and $prepostdfss:$

$\bigwedge vs\ w\ e'\ e''.$

$\llbracket vs = successors\ v - vsuccs\ e\ v;\ vs \neq \{\};\ w = (SOME\ x.\ x \in vs);$

$e' = (if\ w \in explored\ e\ then\ e$

$else\ if\ w \notin visited\ e\ then\ dfs\ w\ e$

$else\ unite\ v\ w\ e);$

$e'' = e'(\!vsuccs := \lambda x.\ if\ x = v\ then\ vsuccs\ e'\ v \cup \{w\}$

$else\ vsuccs\ e'\ x);$

$pre\text{-}dfss\ v\ e'' \rrbracket$

$\implies post\text{-}dfss\ v\ e''\ (dfss\ v\ e'')$

show $post\text{-}dfss\ v\ e\ (dfss\ v\ e)$

proof –

let $?vs = successors\ v - vsuccs\ e\ v$

from $predfss$ **have** $wf: wf\text{-}env\ e$

by (*simp add: pre-dfss-def*)

from $predfss$ **have** $v \in visited\ e$

```

    by (simp add: pre-dfss-def)
  from predfss have  $v \notin \text{explored } e$ 
    by (meson DiffD2 list.set-sel(1) pre-dfss-def stack-class)

show ?thesis
proof (cases ?vs = {})
  case True
  with dom have  $\text{dfss } v \ e = e$ 
    by (simp add: dfss.psimps)
  moreover
  from True wf have  $\text{vsuccs } e \ v = \text{successors } v$ 
    unfolding wf-env-def
    by (meson Diff-eq-empty-iff le-infE subset-antisym)
  moreover
  have sub-env  $e \ e$ 
    by (simp add: sub-env-def)
  moreover
  from predfss  $\langle \text{vsuccs } e \ v = \text{successors } v \rangle$ 
  have  $\forall w \in \text{successors } v. w \in \text{explored } e \cup \mathcal{S} \ e \ (\text{hd } (\text{stack } e))$ 
     $\forall n \in \text{set } (\text{stack } e). \text{reachable } n \ v$ 
     $\text{stack } e \neq []$ 
     $v \in \mathcal{S} \ e \ (\text{hd } (\text{stack } e))$ 
    by (auto simp: pre-dfss-def)
  moreover have  $\exists ns. \text{stack } e = ns \ @ \ (\text{stack } e)$ 
    by simp
  moreover
  {
    fix n
    assume asm:  $\text{hd } (\text{stack } e) = v$ 
       $n \in \text{set } (\text{tl } (\text{stack } e))$ 
       $\text{reachable } v \ n$ 
    with  $\langle \text{stack } e \neq [] \rangle$  have  $v \preceq n$  in  $\text{stack } e$ 
      by (metis head-precedes hd-Cons-tl list.set-sel(2))
    moreover
    from wf  $\langle \text{stack } e \neq [] \rangle$  asm have  $v \neq n$ 
      unfolding wf-env-def
      by (metis distinct.simps(2) list.exhaust-sel)
    moreover
    from wf have  $v \in \mathcal{S} \ e \ v$ 
      by (rule S-reflexive)
    moreover
    {
      fix a b
      assume  $a \in \mathcal{S} \ e \ v \ b \in \text{successors } a - \text{vsuccs } e \ a$ 
      with  $\langle \text{vsuccs } e \ v = \text{successors } v \rangle$  have  $a \neq v$ 
        by auto
      from  $\langle \text{stack } e \neq [] \rangle \langle \text{hd } (\text{stack } e) = v \rangle$ 
      have  $v \in \text{set } (\text{stack } e)$ 
        by auto
    }
  }

```



```

with  $\langle a \neq v \rangle \langle a \in \mathcal{S} \ e \ v \rangle \text{ wf}$  have  $a \in \text{visited } e$ 
  unfolding wf-env-def by (metis singletonD)
have False
proof (cases a \in set (cstack e))
  case True
    with  $\langle v \in \text{set } (stack \ e) \rangle \langle a \in \mathcal{S} \ e \ v \rangle \langle \text{wf-env } e \rangle$ 
    have  $a \preceq v$  in cstack e
      by (auto simp: wf-env-def)
    moreover
    from predfss obtain ns where  $cstack \ e = v \ \# \ ns$ 
      by (auto simp: pre-dfss-def)
    moreover
    from wf have distinct (cstack e)
      by (simp add: wf-env-def)
    ultimately have  $a = v$ 
      using tail-not-precedes by force
    with  $\langle a \neq v \rangle$  show ?thesis ..
  next
    case False
    with  $\langle a \in \text{visited } e \rangle \text{ wf}$  have  $vsuccs \ e \ a = \text{successors } a$ 
      by (auto simp: wf-env-def)
    with  $\langle b \in \text{successors } a - vsuccs \ e \ a \rangle$  show ?thesis
      by simp
    qed
  }
hence  $\text{unvisited } e \ v = \{\}$ 
  by (auto simp: unvisited-def)

ultimately have  $\neg \text{reachable-avoiding } v \ n \ \{\}$ 
  using wf unfolding wf-env-def by metis
with  $\langle \text{reachable } v \ n \rangle$  have False
  by (simp add: ra-empty)
}
ultimately show ?thesis
using wf by (auto simp: post-dfss-def)
next
case vs-case: False
define w where  $w = (\text{SOME } x. x \in ?vs)$ 
define e' where  $e' = (\text{if } w \in \text{explored } e \text{ then } e$ 
   $\text{else if } w \notin \text{visited } e \text{ then } \text{dfs } w \ e$ 
   $\text{else } \text{unite } v \ w \ e)$ 
define e'' where  $e'' = (e' \setminus \{vsuccs \ := \ \lambda x. \text{if } x=v \text{ then } vsuccs \ e' \ v \cup \{w\} \text{ else } vsuccs \ e' \ x\})$ 

from dom vs-case have  $dfss: dfss \ v \ e = dfss \ v \ e''$ 
  apply (simp add: dfss.psimps e''-def)
  using e'-def w-def by auto

from vs-case have  $w \in ?vs$ 

```

```

  unfolding w-def by (metis some-in-eq)
show ?thesis
proof (cases w ∈ explored e)
  case True
  hence e': e' = e
    by (simp add: e'-def)
  with predfss wvs True
  have pre-dfss v e''
    by (auto simp: e''-def pre-dfss-explored-pre-dfss)
  with prepostdfss vs-case
  have post'': post-dfss v e'' (dfss v e'')
    by (auto simp: w-def e'-def e''-def)

  moreover
  from post''
  have ∀ u ∈ visited e - {v}. vsuccs (dfss v e'') u = vsuccs e u
    by (auto simp: post-dfss-def e' e''-def)

  moreover
  have sub-env e e''
    by (auto simp: sub-env-def e' e''-def)
  with post'' have sub-env e (dfss v e'')
    by (auto simp: post-dfss-def elim: sub-env-trans)

  moreover
  from e' have stack e'' = stack e S e'' = S e
    by (auto simp add: e''-def)

  moreover
  have cstack e'' = cstack e
    by (simp add: e''-def e')

  ultimately show ?thesis
    by (auto simp: dfss post-dfss-def)
next
case notexplored: False
then show ?thesis
proof (cases w ∉ visited e)
  case True
  with e'-def notexplored have e' = dfs w e
    by auto
  with True notexplored pre-dfss-pre-dfs predfss
    prepostdfs vs-case w-def
  have postdfsw: post-dfs w e e'
    by (metis DiffD1 some-in-eq)
  with predfss wvs True ⟨e' = dfs w e⟩
  have pre-dfss v e''
    by (auto simp: e''-def pre-dfss-post-dfs-pre-dfss)
  with prepostdfss vs-case

```

have $post''$: $post\text{-}dfss\ v\ e''\ (dfss\ v\ e'')$
by (*auto simp: w-def e'-def e''-def*)

moreover

have $\forall u \in visited\ e - \{v\}. vsuccs\ (dfss\ v\ e'')\ u = vsuccs\ e\ u$

proof

fix u

assume $u \in visited\ e - \{v\}$

with $postdfsw$

have $u: vsuccs\ e'\ u = vsuccs\ e\ u\ u \in visited\ e'' - \{v\}$

by (*auto simp: post-dfs-def sub-env-def e''-def*)

with $post''$ **have** $vsuccs\ (dfss\ v\ e'')\ u = vsuccs\ e''\ u$

by (*auto simp: post-dfss-def*)

with u **show** $vsuccs\ (dfss\ v\ e'')\ u = vsuccs\ e\ u$

by (*simp add: e''-def*)

qed

moreover

have $sub\text{-}env\ e\ (dfss\ v\ e'')$

proof –

from $postdfsw$ **have** $sub\text{-}env\ e\ e'$

by (*simp add: post-dfs-def*)

moreover

have $sub\text{-}env\ e'\ e''$

by (*auto simp: sub-env-def e''-def*)

moreover

from $post''$ **have** $sub\text{-}env\ e''\ (dfss\ v\ e'')$

by (*simp add: post-dfss-def*)

ultimately show *?thesis*

by (*metis sub-env-trans*)

qed

moreover

from $postdfsw\ post''$

have $\exists ns. stack\ e = ns\ @\ (stack\ (dfss\ v\ e''))$

by (*auto simp: post-dfs-def post-dfss-def e''-def*)

moreover

{

fix n

assume $n: n \in set\ (tl\ (stack\ (dfss\ v\ e'')))$

with $post''$ **have** $\mathcal{S}\ (dfss\ v\ e'')\ n = \mathcal{S}\ e'\ n$

by (*simp add: post-dfss-def e''-def*)

moreover

from $\langle pre\text{-}dfss\ v\ e'' \rangle\ n\ post''$

have $stack\ e' \neq [] \wedge n \in set\ (tl\ (stack\ e''))$

apply (*simp add: pre-dfss-def post-dfss-def e''-def*)

by (*metis (no-types, lifting) Un-iff list.set-sel(2) self-append-conv2*

set-append tl-append2)

with *postdfsw* **have** $\mathcal{S} e' n = \mathcal{S} e n$
apply (*simp add: post-dfs-def e''-def*)
by (*metis list.set-sel(2)*)
ultimately have $\mathcal{S} (dfss v e'') n = \mathcal{S} e n$
by *simp*
}

moreover
from *postdfsw* **have** $cstack e'' = cstack e$
by (*auto simp: post-dfs-def e''-def*)

ultimately show *?thesis*
by (*auto simp: dfss post-dfss-def*)

next

case *False*
hence $e': e' = unite v w e$ **using** *notexplored*
using *e'-def* **by** *simp*
from *False* **have** $w \in visited e$
by *simp*
from *wf wvs notexplored False* **obtain** *px* **where**
px: stack e = px @ (stack e') stack e' ≠ []
unfolding e' **by** (*blast dest: unite-stack*)

from *predfss wvs notexplored False <e' = unite v w e>*
have *pre-dfss v e''*
by (*auto simp: e''-def pre-dfss-unite-pre-dfss*)

with *prepostdfss vs-case <e' = unite v w e> <w ∉ explored e> <w ∈ visited*

e>

have *post''*: *post-dfss v e'' (dfss v e'')*
by (*auto simp: w-def e''-def*)

moreover
from *post''*
have $\forall u \in visited e - \{v\}. vsuccs (dfss v e'') u = vsuccs e u$
by (*auto simp: post-dfss-def e''-def e' unite-def*)

moreover
have *sub-env e (dfss v e'')*
proof –
from *predfss wvs <w ∈ visited e> notexplored*
have *sub-env e e'*
unfolding e' **by** (*blast dest: unite-sub-env*)
moreover
have *sub-env e' e''*
by (*auto simp: sub-env-def e''-def*)
moreover
from *post''* **have** *sub-env e'' (dfss v e'')*

```

      by (simp add: post-dfss-def)
    ultimately show ?thesis
      by (metis sub-env-trans)
  qed

  moreover
  from post'' ‹stack e = pfx @ stack e'›
  have ∃ ns. stack e = ns @ (stack (dfss v e''))
    by (auto simp: post-dfss-def e''-def)

  moreover
  {
    fix n
    assume n: n ∈ set (tl (stack (dfss v e'')))
    with post'' have S (dfss v e'') n = S e'' n
      by (simp add: post-dfss-def)
    moreover
    from n post'' ‹stack e' ≠ []›
    have n ∈ set (tl (stack e''))
      apply (simp add: post-dfss-def e''-def)
      by (metis (no-types, lifting) Un-iff list.set-sel(2) self-append-conv2
        set-append tl-append2)
    with wf wvs ‹w ∈ visited e› notexplored
    have S e'' n = S e n
      by (auto simp: e''-def e' dest: unite-S-tl)
    ultimately have S (dfss v e'') n = S e n
      by simp
  }

  moreover
  from post'' have cstack (dfss v e'') = cstack e
    by (simp add: post-dfss-def e''-def e' unite-def)

  ultimately show ?thesis
    by (simp add: dfss post-dfss-def)
  qed
qed
qed
qed
qed

```

We can now show partial correctness of the algorithm: applied to some node v and the empty environment, it computes the set of strongly connected components in the subgraph reachable from node v . In particular, if v is a root of the graph, the algorithm computes the set of SCCs of the graph.

theorem *partial-correctness*:

```

  fixes v
  defines e ≡ dfs v (init-env v)
  assumes dfs-dfss-dom (Inl (v, init-env v))

```

shows $sccs\ e = \{S . is\text{-}scc\ S \wedge (\forall n \in S. reachable\ v\ n)\}$
 (**is** $- = ?rhs$)

proof –

from $assms\ init\text{-}env\text{-}pre\text{-}dfs[of\ v]$
have $post: post\text{-}dfs\ v\ (init\text{-}env\ v)\ e$
by ($auto\ dest: pre\text{-}post$)
hence $wf: wf\text{-}env\ e$
by ($simp\ add: post\text{-}dfs\text{-}def$)
from $post$ **have** $cstack\ e = []$
by ($auto\ simp: post\text{-}dfs\text{-}def\ init\text{-}env\text{-}def$)
have $stack\ e = []$
proof ($rule\ ccontr$)
assume $stack\ e \neq []$
hence $hd\ (stack\ e) \preceq hd\ (stack\ e)\ in\ stack\ e$
by $simp$
with $wf\ \langle cstack\ e = [] \rangle$ **show** $False$
unfolding $wf\text{-}env\text{-}def$
by ($metis\ empty\text{-}iff\ empty\text{-}set\ precedes\ mem(2)$)

qed

with $post$ **have** $vexp: v \in explored\ e$
by ($simp\ add: post\text{-}dfs\text{-}def$)
from $wf\ \langle stack\ e = [] \rangle$ **have** $explored\ e = visited\ e$
by ($auto\ simp: wf\text{-}env\text{-}def$)
have $sccs\ e \subseteq ?rhs$

proof

fix S
assume $S: S \in sccs\ e$
with wf **have** $is\text{-}scc\ S$
by ($simp\ add: wf\text{-}env\text{-}def$)
moreover
from $S\ wf$ **have** $S \subseteq explored\ e$
unfolding $wf\text{-}env\text{-}def$
by $blast$
with $post\ \langle explored\ e = visited\ e \rangle$ **have** $\forall n \in S. reachable\ v\ n$
by ($auto\ simp: post\text{-}dfs\text{-}def\ wf\text{-}env\text{-}def\ sub\text{-}env\text{-}def\ init\text{-}env\text{-}def$)
ultimately show $S \in ?rhs$
by $auto$

qed

moreover

{
fix S
assume $is\text{-}scc\ S\ \forall n \in S. reachable\ v\ n$
from $\langle \forall n \in S. reachable\ v\ n \rangle\ vexp\ wf$
have $S \subseteq \bigcup (sccs\ e)$
unfolding $wf\text{-}env\text{-}def$ **by** ($metis\ subset\text{-}eq$)
with $\langle is\text{-}scc\ S \rangle$ **obtain** S' **where** $S': S' \in sccs\ e \wedge S \cap S' \neq \{\}$
unfolding $is\text{-}scc\text{-}def$
by ($metis\ Union\text{-}disjoint\ inf.\ absorb\text{-}iff2\ inf\text{-}commute$)
with wf **have** $is\text{-}scc\ S'$

```

    by (simp add: wf-env-def)
  with S' ‹is-scc S› have S ∈ sccs e
    by (auto dest: scc-partition)
}
ultimately show ?thesis by blast
qed

```

8 Proof of termination and total correctness

We define a binary relation on the arguments of functions dfs and $dfss$, and prove that this relation is well-founded and that all calls within the function bodies respect the relation, assuming that the pre-conditions of the initial function call are satisfied. By well-founded induction, we conclude that the pre-conditions of the functions are sufficient to ensure termination.

Following the internal representation of the two mutually recursive functions in Isabelle as a single function on the disjoint sum of the types of arguments, our relation is defined as a set of argument pairs injected into the sum type. The left injection Inl takes arguments of function dfs , the right injection Inr takes arguments of function $dfss$.¹ The conditions on the arguments in the definition of the relation overapproximate the arguments in the actual calls.

definition $dfs-dfss-term::('v \times 'v\ env + 'v \times 'v\ env) \times ('v \times 'v\ env + 'v \times 'v\ env)$ set **where**

$$\begin{aligned}
 dfs-dfss-term \equiv & \\
 & \{ (Inr(v, e1), Inl(v, e)) \mid v \in e1. \\
 & \quad v \in vertices - visited\ e \wedge visited\ e1 = visited\ e \cup \{v\} \} \\
 & \cup \{ (Inl(w, e), Inr(v, e)) \mid v \ w \ e. v \in vertices \} \\
 & \cup \{ (Inr(v, e''), Inr(v, e)) \mid v \in e''. \\
 & \quad v \in vertices \wedge sub-env\ e\ e'' \\
 & \quad \wedge (\exists w \in vertices. w \notin vsuccs\ e\ v \wedge w \in vsuccs\ e''\ v) \}
 \end{aligned}$$

Informally, termination is ensured because at each call, either a new vertex is visited (hence the complement of the set of visited nodes w.r.t. the finite set of vertices decreases) or a new successor is added to the set $vsuccs\ e\ v$ of some vertex v .

In order to make this argument formal, we inject the argument tuples that appear in our relation into tuples consisting of the sets mentioned in the informal argument. However, there is one added complication because the call of dfs from $dfss$ does not immediately add the vertex to the set of visited nodes (this happens only at the beginning of function dfs). We therefore add a third component of 0 or 1 to these tuples, reflecting the fact that there can only be one call of dfs from $dfss$ for a given vertex v .

fun $dfs-dfss-to-tuple$ **where**

¹Note that the types of the arguments of dfs and $dfss$ are actually identical. We nevertheless use the sum type in order to remember the function that was called.

$$\begin{aligned}
& \text{dfs-dfss-to-tuple } (\text{Inl}(v::'v, e::'v \text{ env})) = \\
& \quad (\text{vertices} - \text{visited } e, \text{vertices} \times \text{vertices} - \{(u, u') \mid u \ u'. \ u' \in \text{vsuccs } e \ u\}, 0) \\
| & \text{dfs-dfss-to-tuple } (\text{Inr}(v::'v, e::'v \text{ env})) = \\
& \quad (\text{vertices} - \text{visited } e, \text{vertices} \times \text{vertices} - \{(u, u') \mid u \ u'. \ u' \in \text{vsuccs } e \ u\}, 1::\text{nat})
\end{aligned}$$

The triples defined in this way can be ordered lexicographically (with the first two components ordered as finite subsets and the third one following the predecessor relation on natural numbers). We prove that the injection of the above relation into sets of triples respects the lexicographic ordering and conclude that our relation is well-founded.

lemma *wf-term: wf dfs-dfss-term*

proof –

let $?r = (\text{finite-psubset} :: ('v \text{ set} \times 'v \text{ set}) \text{ set})$
 $\langle *lex* \rangle (\text{finite-psubset} :: ((('v \times 'v) \text{ set}) \times ('v \times 'v) \text{ set}) \text{ set})$
 $\langle *lex* \rangle \text{pred-nat}$

have $wf (\text{finite-psubset} :: ('v \text{ set} \times 'v \text{ set}) \text{ set})$
by (*rule wf-finite-psubset*)

moreover

have $wf (\text{finite-psubset} :: ((('v \times 'v) \text{ set}) \times ('v \times 'v) \text{ set}) \text{ set})$
by (*rule wf-finite-psubset*)

ultimately have $wf ?r$

using *wf-pred-nat* **by** *blast*

moreover

have $\text{dfs-dfss-term} \subseteq \text{inv-image } ?r \ \text{dfs-dfss-to-tuple}$

proof (*clarify*)

fix $a \ b$

assume $(a, b) \in \text{dfs-dfss-term}$

hence $(\exists v \ w \ e \ e''. \ a = \text{Inr}(v, e'') \wedge b = \text{Inr}(v, e) \wedge v \in \text{vertices} \wedge \text{sub-env } e \ e''$

$\wedge w \in \text{vertices} \wedge w \notin \text{vsuccs } e \ v \wedge w \in \text{vsuccs } e'' \ v)$

$\vee (\exists v \ e \ e1. \ a = \text{Inl}(v, e1) \wedge b = \text{Inl}(v, e) \wedge v \in \text{vertices} - \text{visited } e$

$\wedge \text{visited } e1 = \text{visited } e \cup \{v\})$

$\vee (\exists v \ w \ e. \ a = \text{Inl}(w, e) \wedge b = \text{Inr}(v, e))$

(**is** $?c1 \ \vee \ ?c2 \ \vee \ ?c3$)

by (*auto simp: dfs-dfss-term-def*)

then show $(a, b) \in \text{inv-image } ?r \ \text{dfs-dfss-to-tuple}$

proof

assume $?c1$

then obtain $v \ w \ e \ e''$ **where**

$ab: a = \text{Inr}(v, e'') \ b = \text{Inr}(v, e)$ **and**

$wv: v \in \text{vertices} \ w \in \text{vertices} \ w \in \text{vsuccs } e'' \ v \ w \notin \text{vsuccs } e \ v$ **and**

$sub: \text{sub-env } e \ e''$

by *blast*

from sub **have** $\text{vertices} - \text{visited } e'' \subseteq \text{vertices} - \text{visited } e$

by (*auto simp: sub-env-def*)

moreover

from $sub \ wv$

have $(\text{vertices} \times \text{vertices} - \{(u, u') \mid u \ u'. \ u' \in \text{vsuccs } e'' \ u\})$

$\subseteq (\text{vertices} \times \text{vertices} - \{(u, u') \mid u \ u'. \ u' \in \text{vsuccs } e \ u\})$

by (*auto simp: sub-env-def*)


```

ultimately show ?thesis
  using vfin ab by auto
next
assume ?c2 ∨ ?c3
with vfin show ?thesis
  by (auto simp: pred-nat-def)
qed
qed
ultimately show ?thesis
  using wf-inv-image wf-subset by blast
qed

```

The following theorem establishes sufficient conditions that ensure termination of the two functions *dfs* and *dfss*. The proof proceeds by well-founded induction using the relation *dfs-dfss-term*. Isabelle represents the termination domains of the functions by the predicate *dfs-dfss-dom* and generates a theorem *dfs-dfss.domintros* for proving membership of arguments in the termination domains. The actual formulation is a little technical because the mutual induction must again be encoded in a single induction argument over the sum type representing the arguments of both functions.

theorem *dfs-dfss-termination*:

$\llbracket v \in \text{vertices} ; \text{pre-dfs } v \ e \rrbracket \implies \text{dfs-dfss-dom}(\text{Inl}(v, e))$

$\llbracket v \in \text{vertices} ; \text{pre-dfss } v \ e \rrbracket \implies \text{dfs-dfss-dom}(\text{Inr}(v, e))$

proof –

{ **fix** *args*

have (*case args*

 of *Inl(v,e)* \implies

$v \in \text{vertices} \wedge \text{pre-dfs } v \ e$

 | *Inr(v,e)* \implies

$v \in \text{vertices} \wedge \text{pre-dfss } v \ e$

$\longrightarrow \text{dfs-dfss-dom } \text{args} \text{ (is } ?P \ \text{args} \longrightarrow ?Q \ \text{args})$

proof (*rule wf-induct[OF wf-term]*)

fix *arg* :: (*'v* \times *'v env*) + (*'v* \times *'v env*)

assume *ih*: $\forall \ \text{arg}'. (\text{arg}', \text{arg}) \in \text{dfs-dfss-term} \longrightarrow (?P \ \text{arg}' \longrightarrow ?Q \ \text{arg}')$

show $?P \ \text{arg} \longrightarrow ?Q \ \text{arg}$

proof

assume *P*: $?P \ \text{arg}$

show $?Q \ \text{arg}$

proof (*cases arg*)

case (*Inl a*)

then obtain *v e* **where** *a*: $\text{arg} = \text{Inl}(v, e)$

using *dfs.cases* **by** *metis*

with *P* **have** *pre*: $v \in \text{vertices} \wedge \text{pre-dfs } v \ e$

by *simp*

let $?e1 = e(\text{visited} := \text{visited } e \cup \{v\}, \text{stack} := v \# \text{stack } e, \text{cstack} := v \# \text{cstack } e)$

let $?recarg = \text{Inr}(v, ?e1)$

```

from  $a$  pre
have  $(?recarg, arg) \in \text{dfs-dfss-term}$ 
  by (auto simp: pre-dfs-def dfs-dfss-term-def)
moreover
from pre have  $?P ?recarg$ 
  by (auto dest: pre-dfs-pre-dfss)
ultimately have  $?Q ?recarg$ 
  using ih a by auto
then have  $?Q (\text{Inl}(v, e))$ 
  by (auto intro: dfs-dfss.domintros)
then show  $?thesis$ 
  by (simp add: a)
next
case  $(\text{Inr } b)$ 
then obtain  $v e$  where  $b: arg = \text{Inr}(v, e)$ 
  using dfs.cases by metis
with  $P$  have  $pre: v \in \text{vertices} \wedge \text{pre-dfss } v e$ 
  by simp
let  $?sw = \text{SOME } w. w \in \text{successors } v \wedge w \notin \text{vsuccs } e v$ 
have  $?Q (\text{Inr}(v, e))$ 
proof (rule dfs-dfss.domintros)
  fix  $w$ 
  assume  $w \in \text{successors } v$ 
     $?sw \notin \text{explored } e$ 
     $?sw \notin \text{visited } e$ 
     $\neg \text{dfs-dfss-dom } (\text{Inl } (?sw, e))$ 
  show  $w \in \text{vsuccs } e v$ 
  proof (rule ccontr)
    assume  $w \notin \text{vsuccs } e v$ 
    with  $\langle w \in \text{successors } v \rangle$  have  $sw: ?sw \in \text{successors } v - \text{vsuccs } e v$ 
      by (metis (mono-tags, lifting) Diff-iff some-eq-imp)
    with  $pre \langle ?sw \notin \text{visited } e \rangle$  have  $\text{pre-dfs } ?sw e$ 
      by (blast intro: pre-dfss-pre-dfs)
    moreover
    from pre sw sclosed have  $?sw \in \text{vertices}$ 
      by blast
    moreover
    from pre have  $(\text{Inl}(?sw, e), \text{Inr}(v, e)) \in \text{dfs-dfss-term}$ 
      by (simp add: dfs-dfss-term-def)
    ultimately have  $\text{dfs-dfss-dom } (\text{Inl}(?sw, e))$ 
      using ih b by auto
    with  $\langle \neg \text{dfs-dfss-dom } (\text{Inl } (?sw, e)) \rangle$ 
    show False ..
  qed
next
let  $?e' = \text{dfs } ?sw e$ 
let  $?e'' = ?e'(\text{vsuccs} := \lambda x. \text{if } x = v \text{ then vsuccs } ?e' v \cup \{?sw\} \text{ else vsuccs } ?e' x)$ 
fix  $w$ 

```

```

assume asm:  $w \in \text{successors } v \ w \notin \text{vsuccs } e \ v$ 
            $?sw \notin \text{visited } e \ ?sw \notin \text{explored } e$ 
from  $\langle w \in \text{successors } v \ \langle w \notin \text{vsuccs } e \ v \rangle$ 
have sw:  $?sw \in \text{successors } v - \text{vsuccs } e \ v$ 
      by (metis (no-types, lifting) Diff-iff some-eq-imp)
with pre  $\langle ?sw \notin \text{visited } e \rangle$  have pre-dfs  $?sw \ e$ 
      by (blast intro: pre-dfss-pre-dfs)
moreover
from pre sw sclosed have  $?sw \in \text{vertices}$ 
      by blast
moreover
from pre have  $(\text{Inl}(?sw, e), \text{Inr}(v, e)) \in \text{dfs-dfss-term}$ 
      by (simp add: dfs-dfss-term-def)
ultimately have  $\text{dfs-dfss-dom } (\text{Inl}(?sw, e))$ 
      using ih b by auto
from this  $\langle \text{pre-dfs } ?sw \ e \rangle$  have post:  $\text{post-dfs } ?sw \ e \ ?e'$ 
      by (rule pre-post)
hence sub-env  $e \ ?e'$ 
      by (simp add: post-dfs-def)
moreover
have sub-env  $?e' \ ?e''$ 
      by (auto simp: sub-env-def)
ultimately have sub-env  $e \ ?e''$ 
      by (rule sub-env-trans)
with pre  $\langle ?sw \in \text{vertices} \rangle$  sw
have  $(\text{Inr}(v, ?e''), \text{Inr}(v, e)) \in \text{dfs-dfss-term}$ 
      by (auto simp: dfs-dfss-term-def)
moreover
from pre post sw  $\langle ?sw \notin \text{visited } e \rangle$  have pre-dfss  $v \ ?e''$ 
      by (blast intro: pre-dfss-post-dfs-pre-dfss)
ultimately show  $\text{dfs-dfss-dom}(\text{Inr}(v, ?e''))$ 
      using pre ih b by auto
next
let  $?e'' = e(\text{vsuccs} := \lambda x. \text{if } x = v \text{ then vsuccs } e \ v \cup \{?sw\} \text{ else vsuccs}$ 
e x)
      fix w
assume  $w \in \text{successors } v \ w \notin \text{vsuccs } e \ v$ 
            $?sw \notin \text{visited } e \ ?sw \in \text{explored } e$ 
with pre have False
      unfolding pre-dfss-def wf-env-def
      by (meson subsetD)
thus  $?Q (\text{Inr}(v, ?e''))$ 
      by simp
next
fix w
assume asm:  $w \in \text{successors } v \ w \notin \text{vsuccs } e \ v$ 
            $?sw \in \text{visited } e \ ?sw \in \text{explored } e$ 
let  $?e'' = e(\text{vsuccs} := \lambda x. \text{if } x = v \text{ then vsuccs } e \ v \cup \{?sw\} \text{ else vsuccs}$ 
e x)

```

```

let ?recarg = Inr(v, ?e'')

from ⟨w ∈ successors v⟩ ⟨w ∉ vsuccs e v⟩
have sw: ?sw ∈ successors v - vsuccs e v
  by (metis (no-types, lifting) Diff-iff some-eq-imp)

have (?recarg, arg) ∈ dfs-dfss-term
proof -
  have sub-env e ?e''
    by (auto simp: sub-env-def)
  moreover
  from sw pre sclosed
  have ∃ u ∈ vertices. u ∉ vsuccs e v ∧ u ∈ vsuccs ?e'' v
    by auto
  ultimately show ?thesis
    using pre b unfolding dfs-dfss-term-def by blast
qed

moreover
from pre sw ⟨?sw ∈ explored e⟩ have ?P ?recarg
  by (auto dest: pre-dfss-explored-pre-dfss)

ultimately show ?Q ?recarg
  using ih b by blast
next
fix w
assume asm: w ∈ successors v w ∉ vsuccs e v
  ?sw ∈ visited e ?sw ∉ explored e
let ?eu = unite v ?sw e
  let ?e'' = ?eu(|vsuccs := λx. if x = v then vsuccs ?eu v ∪ {?sw} else
vsuccs ?eu x)
let ?recarg = Inr(v, ?e'')

from ⟨w ∈ successors v⟩ ⟨w ∉ vsuccs e v⟩
have sw: ?sw ∈ successors v - vsuccs e v
  by (metis (no-types, lifting) Diff-iff some-eq-imp)

have (?recarg, arg) ∈ dfs-dfss-term
proof -
  from pre asm sw have sub-env e ?eu
    by (blast dest: unite-sub-env)
  hence sub-env e ?e''
    by (auto simp: sub-env-def)
  moreover
  from sw pre sclosed
  have ∃ u ∈ vertices. u ∉ vsuccs e v ∧ u ∈ vsuccs ?e'' v
    by auto
  ultimately show ?thesis
    using pre b unfolding dfs-dfss-term-def by blast

```

```

    qed

    moreover
    from pre sw ⟨?sw ∈ visited e⟩ ⟨?sw ∉ explored e⟩ have ?P ?recarg
    by (auto dest: pre-dfss-unite-pre-dfss)

    ultimately show ?Q ?recarg
    using ih b by auto
  qed
  then show ?thesis
  by (simp add: b)
  qed
  qed
  qed
}
note dom=this
from dom
show [ v ∈ vertices ; pre-dfs v e ] ⇒ dfs-dfss-dom(Inl(v, e))
  by auto
from dom
show [ v ∈ vertices ; pre-dfss v e ] ⇒ dfs-dfss-dom(Inr(v, e))
  by auto
qed

```

Putting everything together, we prove the total correctness of the algorithm when applied to some (root) vertex.

```

theorem correctness:
  assumes v ∈ vertices
  shows sccs (dfs v (init-env v)) = {S . is-scc S ∧ (∀ n ∈ S. reachable v n)}
  using assms init-env-pre-dfs[of v]
  by (simp add: dfs-dfss-termination partial-correctness)

```

```

end
end

```

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