The Rogers–Ramanujan Identities

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Abstract

This entry formalises the Rogers–Ramanujan Identities:

$$\sum_{k=-\infty}^{\infty} \frac{q^{k^2}}{\prod_{j=1}^{k}(1-q^j)} = \left(\prod_{n=0}^{\infty}(1-q^{1+5n})(1-q^{4+5n})\right)^{-1}$$
$$\sum_{k=-\infty}^{\infty} \frac{q^{k^2+k}}{\prod_{j=1}^{k}(1-q^j)} = \left(\prod_{n=0}^{\infty}(1-q^{2+5n})(1-q^{3+5n})\right)^{-1}$$

The formalisation follows the elegant proof given in Andrews and Eriksson *Integer Partitions*, using the Jacobi triple product.

1 The Rogers–Ramanujan identities

```
theory Rogers_Ramanujan
  imports "Theta_Functions.Jacobi_Triple_Product"
begin
```

Acknowledgement: I would like to thank George Andrews for giving me a crucial hint about a uniform convergence issue that I struggled with.

```
unbundle qpochhammer_inf_notation
```

First of all, we show two auxiliary results concerned with the (absolute) convergence of two infinite sums that will appear in our proof of the identities.

```
lemma summable_rogers_ramanujan_aux1:
  fixes q :: "'a :: {real_normed_field, banach}" and M :: int
  assumes q: "q ≠ 0" "norm q < 1"
  shows "(λj. norm q powi (j*(5*j+M) div 2)) summable_on UNIV"
⟨proof⟩

lemma summable_rogers_ramanujan_aux2:
  fixes q :: "'a :: {real_normed_field, banach}" and M :: int
  assumes q: "q ≠ 0" "norm q < 1"
  shows "summable (λj. norm (q ^ (j² + c * j) / qpochhammer (int j) q
q))"
⟨proof⟩
```

Next, we apply the Jacobi triple product to show that for $N \in \{0,1\}$ we have

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+2N+1)/2} = \frac{(q;q)_\infty}{\prod_{i\in I}(q^i;q^5)_\infty}$$

where $I = \{1,\ldots,4\} \setminus \{2-N, 3+N\}$.

```
lemma rogers_ramanujan_aux:
  fixes q :: complex and N :: nat
  assumes q: "norm q < 1" and N: "N < 2"
  shows "((λn. (-1) powi n * q powi (n*(5*n+2*N+1) div 2)) has_sum
          (q; q)∞ / ((q^(1+N); q^5)∞ * (q^(4-N); q^5)∞)) UNIV"
⟨proof⟩

theorem rogers_ramanujan_complex:
  fixes q :: complex
  assumes "norm q < 1"
  shows   "((λj. q ^ (j²) / qpochhammer j q q) has_sum (1 / ((q;q^5)∞
* (q^4;q^5)∞))) UNIV"
    and   "((λj. q ^ (j² + j) / qpochhammer j q q) has_sum (1 / ((q^2;q^5)∞
* (q^3;q^5)∞))) UNIV"
⟨proof⟩

lemma rogers_ramanujan_real:
```

```
  fixes q :: real
  assumes "|q| < 1"
  shows   "((λj. q ^ (j²) / qpochhammer j q q) has_sum (1 / ((q;q^5)∞
* (q^4;q^5)∞))) UNIV"
    and   "((λj. q ^ (j² + j) / qpochhammer j q q) has_sum (1 / ((q^2;q^5)∞
* (q^3;q^5)∞))) UNIV"
⟨proof⟩

unbundle no_qpochhammer_inf_notation

end
```

References

[1] G. Andrews and K. Eriksson. *Integer Partitions.* Cambridge University Press, 2004.