

The Z Property

Bertram Felgenhauer, Julian Nagele, Vincent van Oostrom, Christian Sternagel*

May 26, 2024

Abstract

We formalize the Z property introduced by Dehornoy and van Oostrom [1]. First we show that for any abstract rewrite system, Z implies confluence. Then we give two examples of proofs using Z: confluence of lambda-calculus with respect to beta-reduction and confluence of combinatory logic.

Contents

1	The Z property	1
2	Lambda Calculus has the Church-Rosser property	2
2.1	Ad-hoc methods for nominal-functions over lambda terms . . .	2
2.2	Substitutions	3
3	Combinatory Logic has the Church-Rosser property	7

1 The Z property

```
theory Z
imports Abstract-Rewriting.Abstract-Rewriting
begin

locale z-property =
  fixes bullet :: 'a ⇒ 'a (-• [1000] 1000)
  and R :: 'a rel
  assumes Z: (a, b) ∈ R ⇒ (b, a•) ∈ R* ∧ (a•, b•) ∈ R*
begin

lemma monotonicity:
  assumes (a, b) ∈ R*
  shows (a•, b•) ∈ R*
```

*This work was partially supported by FWF (Austrian Science Fund) projects P27502 and P27528.

<proof>

lemma *semi-confluence*:

shows $(R^{-1} \circ R^*) \subseteq R^\downarrow$

<proof>

lemma *CR*: $CR\ R$

<proof>

definition $R_d = \{(a, b). (a, b) \in R^* \wedge (b, a^\bullet) \in R^*\}$

end

locale *angle-property* =

fixes *bullet* :: $'a \Rightarrow 'a (-^\bullet [1000] 1000)$

and $R :: 'a\ rel$

and $R_d :: 'a\ rel$

assumes *intermediate*: $R \subseteq R_d\ R_d \subseteq R^*$

and *angle*: $(a, b) \in R_d \implies (b, a^\bullet) \in R_d$

sublocale *angle-property* \subseteq *z-property*

<proof>

sublocale *z-property* \subseteq *angle-property* *bullet* $R\ z\text{-property}.R_d\ \text{bullet}\ R$

<proof>

end

2 Lambda Calculus has the Church-Rosser property

theory *Lambda-Z*

imports

Nominal2.Nominal2

HOL-Eisbach.Eisbach

Z

begin

atom-decl *name*

nominal-datatype *term* =

Var name

| *App term term*

| *Abs x::name t::term binds x in t*

2.1 Ad-hoc methods for nominal-functions over lambda terms

<ML>

method *without-alpha- λ st* **methods** $m =$
(match termI in H [simproc del: alpha- λ st]: - \Rightarrow $\langle m \rangle$)

method *Abs- λ st* =
(match premises in
atom ?x $\#$ c and P [thin]: [[atom -]] λ st. - = [[atom -]] λ st. - for c :: 'a::fs \Rightarrow
\langle rule Abs- λ st1-fcb2' [where c = c, OF P] \rangle
| P [thin]: [[atom -]] λ st. - = [[atom -]] λ st. - \Rightarrow \langle rule Abs- λ st1-fcb2' [where c = (),
OF P] \rangle)

method *pat-comp-aux* =
(match premises in
x = (- :: term) \Rightarrow - for x \Rightarrow \langle rule term.strong-exhaust [where y = x and c =
x] \rangle
| x = (Var -, -) \Rightarrow - for x :: - :: fs \Rightarrow
\langle rule term.strong-exhaust [where y = fst x and c = x] \rangle
| x = (-, Var -) \Rightarrow - for x :: - :: fs \Rightarrow
\langle rule term.strong-exhaust [where y = snd x and c = x] \rangle
| x = (-, -, Var -) \Rightarrow - for x :: - :: fs \Rightarrow
\langle rule term.strong-exhaust [where y = snd (snd x) and c = x] \rangle)

method *pat-comp* = (*pat-comp-aux*; *force simp: fresh-star-def fresh-Pair-elim*)

method *freshness uses fresh* =
(match conclusion in
- $\#$ - \Rightarrow \langle simp add: fresh-Unit fresh-Pair fresh \rangle
*| - $\#$ * - \Rightarrow \langle simp add: fresh-star-def fresh-Unit fresh-Pair fresh \rangle)*

method *solve-eqvt-at* =
(simp add: eqvt-at-def; simp add: perm-supp-eq fresh-star-Pair)+

method *nf uses fresh* = *without-alpha- λ st* \langle
eqvt-graph-aux, rule TrueI, pat-comp, auto, Abs- λ st,
auto simp: Abs-fresh-iff pure-fresh perm-supp-eq,
(freshness fresh: fresh)+,
solve-eqvt-at? \rangle

2.2 Substitutions

nominal-function *subst*

where

subst x s (Var y) = (if x = y then s else Var y)
| subst x s (App t u) = App (subst x s t) (subst x s u)
| atom y $\#$ (x, s) \Rightarrow subst x s (Abs y t) = Abs y (subst x s t)
 \langle proof \rangle

nominal-termination (*eqvt*) \langle *proof* \rangle

lemma *fresh-subst*:

$atom\ z\ \# \ s \implies z = y \vee atom\ z\ \# \ t \implies atom\ z\ \# \ subst\ y\ s\ t$
 ⟨proof⟩

lemma *fresh-subst-id* [*simp*]:
 $atom\ x\ \# \ t \implies subst\ x\ s\ t = t$
 ⟨proof⟩

The substitution lemma.

lemma *subst-subst*:
assumes $x \neq y$ **and** $atom\ x\ \# \ u$
shows $subst\ y\ u\ (subst\ x\ s\ t) = subst\ x\ (subst\ y\ u\ s)\ (subst\ y\ u\ t)$
 ⟨proof⟩

inductive-set *Beta* ($\{\rightarrow_\beta\}$)

where

$root: atom\ x\ \# \ t \implies (App\ (Abs\ x\ s)\ t, subst\ x\ t\ s) \in \{\rightarrow_\beta\}$
 $| App: (s, t) \in \{\rightarrow_\beta\} \implies (App\ s\ u, App\ t\ u) \in \{\rightarrow_\beta\}$
 $| Appr: (s, t) \in \{\rightarrow_\beta\} \implies (App\ u\ s, App\ u\ t) \in \{\rightarrow_\beta\}$
 $| Abs: (s, t) \in \{\rightarrow_\beta\} \implies (Abs\ x\ s, Abs\ x\ t) \in \{\rightarrow_\beta\}$

abbreviation *beta* ($(-/ \rightarrow_\beta -)$ [*56, 56*] *55*)

where

$s \rightarrow_\beta t \equiv (s, t) \in \{\rightarrow_\beta\}$

equivariance *Betap*

lemmas *Beta-eqvt* = *Betap.eqvt* [*to-set*]

nominal-inductive *Betap*

avoids *Abs*: x

$| root: x$

⟨proof⟩

lemmas *Beta-strong-induct* = *Betap.strong-induct* [*to-set*]

abbreviation *betas* (**infix** \rightarrow_β^* *50*)

where

$s \rightarrow_\beta^* t \equiv (s, t) \in \{\rightarrow_\beta\}^*$

nominal-function *app-beta* :: $term \Rightarrow term \Rightarrow term$

where

$atom\ x\ \# \ u \implies app\ beta\ (Abs\ x\ s')\ u = subst\ x\ u\ s'$
 $| app\ beta\ (Var\ x)\ u = App\ (Var\ x)\ u$
 $| app\ beta\ (App\ s\ t)\ u = App\ (App\ s\ t)\ u$
 ⟨proof⟩

nominal-termination (*eqvt*) ⟨proof⟩

nominal-function *bullet* :: $term \Rightarrow term$ ($- \bullet$ [*1000*] *1000*)

where

$(Var\ x) \bullet = Var\ x$

$| (Abs\ x\ t)^\bullet = Abs\ x\ t^\bullet$
 $| (App\ s\ t)^\bullet = app\text{-}beta\ s^\bullet\ t^\bullet$
 $\langle proof \rangle$
nominal-termination (*eqvt*) $\langle proof \rangle$

lemma *app-beta-exhaust* [*case-names Redex no-Redex*]:
fixes $c :: 'a :: fs$
assumes $\bigwedge x\ s'.\ atom\ x \# c \implies s = Abs\ x\ s' \implies thesis$
and $(\bigwedge t.\ app\text{-}beta\ s\ t = App\ s\ t) \implies thesis$
shows *thesis*
 $\langle proof \rangle$

lemma *App-Betas*:
assumes $s \rightarrow_{\beta^*} t$ **and** $u \rightarrow_{\beta^*} v$
shows $App\ s\ u \rightarrow_{\beta^*} App\ t\ v$
 $\langle proof \rangle$

lemma *Abs-Betas*:
assumes $s \rightarrow_{\beta^*} t$
shows $Abs\ x\ s \rightarrow_{\beta^*} Abs\ x\ t$
 $\langle proof \rangle$

lemma *self*:
 $t \rightarrow_{\beta^*} t^\bullet$
 $\langle proof \rangle$

lemma *fresh-atom-bullet*:
 $atom\ (x::name) \# t \implies atom\ x \# t^\bullet$
 $\langle proof \rangle$

lemma *subst-Beta*:
assumes $t \rightarrow_{\beta} t'$
shows $subst\ x\ s\ t \rightarrow_{\beta} subst\ x\ s\ t'$
 $\langle proof \rangle$

lemma *Beta-in-subst*:
assumes $s \rightarrow_{\beta} s'$
shows $subst\ x\ s\ t \rightarrow_{\beta^*} subst\ x\ s'\ t$
 $\langle proof \rangle$

lemma *subst-Betas*:
assumes $s \rightarrow_{\beta^*} s'$ **and** $t \rightarrow_{\beta^*} t'$
shows $subst\ x\ s\ t \rightarrow_{\beta^*} subst\ x\ s'\ t'$
 $\langle proof \rangle$

lemma *Beta-fresh*:
fixes $x :: name$
assumes $s \rightarrow_{\beta} t$ **and** $atom\ x \# s$
shows $atom\ x \# t$

<proof>

lemma *Abs-BetaD*:

assumes $Abs\ x\ s \rightarrow_{\beta}\ t$

shows $\exists u. t = Abs\ x\ u \wedge s \rightarrow_{\beta}\ u$

<proof>

lemma *Abs-BetaE*:

assumes $Abs\ x\ s \rightarrow_{\beta}\ t$

obtains u **where** $t = Abs\ x\ u$ **and** $s \rightarrow_{\beta}\ u$

<proof>

lemma *Abs-BetasE*:

assumes $Abs\ x\ s \rightarrow_{\beta^*}\ t$

obtains u **where** $t = Abs\ x\ u$ **and** $s \rightarrow_{\beta^*}\ u$

<proof>

lemma *bullet-App*:

$(App\ s^{\bullet}\ t^{\bullet}, (App\ s\ t)^{\bullet}) \in \{\rightarrow_{\beta}\}^=$

<proof>

lemma *rhs*:

$subst\ x\ s^{\bullet}\ t^{\bullet} \rightarrow_{\beta^*}\ (subst\ x\ s\ t)^{\bullet}$

<proof>

lemma *Betas-fresh*:

fixes $x :: name$

assumes $s \rightarrow_{\beta^*}\ t$ **and** $atom\ x \# s$

shows $atom\ x \# t$

<proof>

lemma *Var-BetaD*:

assumes $Var\ x \rightarrow_{\beta}\ t$

shows *False*

<proof>

lemma *Var-BetasD*:

assumes $Var\ x \rightarrow_{\beta^*}\ t$

shows $t = Var\ x$

<proof>

lemma *app-beta-Betas*:

assumes $s \rightarrow_{\beta^*}\ s'$ **and** $t \rightarrow_{\beta^*}\ t'$

shows $app\ beta\ s\ t \rightarrow_{\beta^*}\ app\ beta\ s'\ t'$

<proof>

lemma *lambda-Z*:

assumes $s \rightarrow_{\beta}\ t$

shows $t \rightarrow_{\beta^*}\ s^{\bullet} \wedge s^{\bullet} \rightarrow_{\beta^*}\ t^{\bullet}$

<proof>

interpretation *lambda-z: z-property* *bullet Beta*

<proof>

end

3 Combinatory Logic has the Church-Rosser property

theory *CL-Z imports Z*

begin

datatype *CL = S | K | I | App CL CL (' - - [999, 999] 999)*

inductive-set *red :: CL rel where*

L: (t, t') ∈ red ⇒ (' t u, ' t' u) ∈ red
| R: (u, u') ∈ red ⇒ (' t u, ' t u') ∈ red
| S: (' ' S x y z, ' ' x z ' y z) ∈ red
| K: (' ' K x y, x) ∈ red
| I: (' I x, x) ∈ red

lemma *App-mono:*

(t, t') ∈ red ⇒ (u, u') ∈ red* ⇒ (' t u, ' t' u') ∈ red**
<proof>

fun *bullet-app :: CL ⇒ CL ⇒ CL where*

bullet-app (' ' S x y) z = ' ' x z ' y z
| bullet-app (' K x) y = x
| bullet-app I x = x
| bullet-app t u = ' t u

lemma *bullet-app-red:*

(' t u, bullet-app t u) ∈ red=
<proof>

lemma *bullet-app-redsI:*

(s, ' t u) ∈ red ⇒ (s, bullet-app t u) ∈ red**
<proof>

lemma *bullet-app-redL:*

*(t, t') ∈ red ⇒ (bullet-app t u, bullet-app t' u) ∈ red**
<proof>

lemma *bullet-app-redR:*

*(u, u') ∈ red ⇒ (bullet-app t u, bullet-app t u') ∈ red**
<proof>

lemma *bullet-app-mono*:
 assumes $(t, t') \in \text{red}^*$ $(u, u') \in \text{red}^*$ **shows** $(\text{bullet-app } t \ u, \text{bullet-app } t' \ u') \in \text{red}^*$
 $\langle \text{proof} \rangle$

fun *bullet* :: $CL \Rightarrow CL$ **where**
 $\text{bullet } (' t \ u) = \text{bullet-app } (\text{bullet } t) (\text{bullet } u)$
 | $\text{bullet } t = t$

lemma *bullet-incremental*:
 $(t, \text{bullet } t) \in \text{red}^*$
 $\langle \text{proof} \rangle$

interpretation *CL:z-property bullet red*
 $\langle \text{proof} \rangle$

lemmas $CR\text{-red} = CL.CR$

end

References

- [1] P. Dehornoy and V. v. Oostrom. Z, proving confluence by monotonic single-step upperbound functions. In *Logical Models of Reasoning and Computation (LMRC'2008)*, 2008.