

The Z Property

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Abstract

We formalize the Z property introduced by Dehornoy and van Oostrom [1]. First we show that for any abstract rewrite system, Z implies confluence. Then we give two examples of proofs using Z: confluence of lambda-calculus with respect to beta-reduction and confluence of combinatory logic.

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1 The Z property

```
theory Z
imports Abstract-Rewriting.Abstract-Rewriting
begin

locale z-property =
  fixes bullet :: 'a ⇒ 'a (-• [1000] 1000)
  and R :: 'a rel
  assumes Z: (a, b) ∈ R ⇒ (b, a•) ∈ R* ∧ (a•, b•) ∈ R*
begin

lemma monotonicity:
  assumes (a, b) ∈ R*
  shows (a•, b•) ∈ R*
```

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using *assms*
by (*induct*) (*auto dest: Z*)

lemma *semi-confluence*:

shows $(R^{-1} \circ R^*) \subseteq R^\downarrow$

proof (*intro subrelI, elim relcompEpair, drule converseD*)

fix *d a c*

assume $(a, c) \in R^*$ **and** $(a, d) \in R$

then show $(d, c) \in R^\downarrow$

proof (*cases*)

case (*step b*)

then have $(a^\bullet, b^\bullet) \in R^*$ **by** (*auto simp: monotonicity*)

then have $(d, b^\bullet) \in R^*$ **using** $(a, d) \in R$ **by** (*auto dest: Z*)

then show *?thesis* **using** $(b, c) \in R$ **by** (*auto dest: Z*)

qed *auto*

qed

lemma *CR*: $CR \ R$

by (*intro semi-confluence-imp-CR semi-confluence*)

definition $R_d = \{(a, b). (a, b) \in R^* \wedge (b, a^\bullet) \in R^*\}$

end

locale *angle-property* =

fixes *bullet* :: $'a \Rightarrow 'a (-\bullet [1000] 1000)$

and $R :: 'a \text{ rel}$

and $R_d :: 'a \text{ rel}$

assumes *intermediate*: $R \subseteq R_d \ R_d \subseteq R^*$

and *angle*: $(a, b) \in R_d \implies (b, a^\bullet) \in R_d$

sublocale *angle-property* \subseteq *z-property*

proof

fix *a b*

assume $(a, b) \in R$

with *angle intermediate* **have** $(b, a^\bullet) \in R_d$ **and** $(a^\bullet, b^\bullet) \in R_d$ **by** *auto*

then show $(b, a^\bullet) \in R^* \wedge (a^\bullet, b^\bullet) \in R^*$ **using** *intermediate* **by** *auto*

qed

sublocale *z-property* \subseteq *angle-property bullet R z-property.R_d bullet R*

proof

show $R \subseteq R_d$ **and** $R_d \subseteq R^*$ **unfolding** *R_d-def* **using** *Z* **by** *auto*

fix *a b*

assume $(a, b) \in R_d$

then show $(b, a^\bullet) \in R_d$ **using** *monotonicity* **unfolding** *R_d-def* **by** *auto*

qed

end

2 Lambda Calculus has the Church-Rosser property

```

theory Lambda-Z
imports
  Nominal2.Nominal2
  HOL-Eisbach.Eisbach
  Z
begin

atom-decl name

nominal-datatype term =
  Var name
| App term term
| Abs x::name t::term binds x in t

```

2.1 Ad-hoc methods for nominal-functions over lambda terms

```

ML ⟨
  fun graph-aux-tac ctxt =
    SUBGOAL (fn (subgoal, i) =>
      (case subgoal of
        Const (@{const-name Trueprop}, -) $ (Const (@{const-name eqvt}, -) $ Free
          (f, -)) =>
          full-simp-tac (
            ctxt addsimps [@{thm eqvt-def}, Proof-Context.get-thm ctxt (f ^ -def)]) i
          | - => no-tac))
    )

```

```

method-setup eqvt-graph-aux =
  ⟨Scan.succeed (fn ctxt : Proof.context => SIMPLE-METHOD' (graph-aux-tac
    ctxt))⟩
  show equivariance of auxilliary graph construction for nominal functions

```

```

method without-alpha-lst methods m =
  (match termI in H [simproc del: alpha-lst]: - => ⟨m⟩)

```

```

method Abs-lst =
  (match premises in
    atom ?x # c and P [thin]: [[atom -]]lst. - = [[atom -]]lst. - for c :: 'a::fs =>
    ⟨rule Abs-lst1-fcb2' [where c = c, OF P]⟩
    | P [thin]: [[atom -]]lst. - = [[atom -]]lst. - => ⟨rule Abs-lst1-fcb2' [where c = (),
      OF P]⟩)

```

```

method pat-comp-aux =
  (match premises in
    x = (- :: term) ==> - for x => ⟨rule term.strong-exhaust [where y = x and c =
      x]⟩)

```

$| x = (\text{Var } -, -) \implies - \text{ for } x :: - :: fs \implies$
 $\langle \text{rule term.strong-exhaust [where } y = \text{fst } x \text{ and } c = x] \rangle$
 $| x = (-, \text{Var } -) \implies - \text{ for } x :: - :: fs \implies$
 $\langle \text{rule term.strong-exhaust [where } y = \text{snd } x \text{ and } c = x] \rangle$
 $| x = (-, -, \text{Var } -) \implies - \text{ for } x :: - :: fs \implies$
 $\langle \text{rule term.strong-exhaust [where } y = \text{snd } (\text{snd } x) \text{ and } c = x] \rangle$

method *pat-comp* = (*pat-comp-aux*; *force simp: fresh-star-def fresh-Pair-elim*)

method *freshness uses fresh* =
 (**match conclusion in**
 $- \# - \implies \langle \text{simp add: fresh-Unit fresh-Pair fresh} \rangle$
 $| - \#* - \implies \langle \text{simp add: fresh-star-def fresh-Unit fresh-Pair fresh} \rangle$)

method *solve-eqv-at* =
 (*simp add: eqv-at-def*; *simp add: perm-supp-eq fresh-star-Pair*)⁺

method *nf uses fresh* = *without-alpha-lst* (
eqv-graph-aux, *rule TrueI*, *pat-comp*, *auto*, *Abs-lst*,
auto simp: Abs-fresh-iff pure-fresh perm-supp-eq,
(freshness fresh: fresh)⁺,
solve-eqv-at?)

2.2 Substitutions

nominal-function *subst*

where

$\text{subst } x \ s \ (\text{Var } y) = (\text{if } x = y \text{ then } s \text{ else } \text{Var } y)$
 $| \text{subst } x \ s \ (\text{App } t \ u) = \text{App } (\text{subst } x \ s \ t) \ (\text{subst } x \ s \ u)$
 $| \text{atom } y \ \# \ (x, s) \implies \text{subst } x \ s \ (\text{Abs } y \ t) = \text{Abs } y \ (\text{subst } x \ s \ t)$

by *nf*

nominal-termination (*eqv*) **by** *lexicographic-order*

lemma *fresh-subst*:

$\text{atom } z \ \# \ s \implies z = y \vee \text{atom } z \ \# \ t \implies \text{atom } z \ \# \ \text{subst } y \ s \ t$
by (*nominal-induct t avoiding: z y s rule: term.strong-induct*) *auto*

lemma *fresh-subst-id* [*simp*]:

$\text{atom } x \ \# \ t \implies \text{subst } x \ s \ t = t$
by (*nominal-induct t avoiding: x s rule: term.strong-induct*) (*auto simp: fresh-at-base*)

The substitution lemma.

lemma *subst-subst*:

assumes $x \neq y$ **and** $\text{atom } x \ \# \ u$
shows $\text{subst } y \ u \ (\text{subst } x \ s \ t) = \text{subst } x \ (\text{subst } y \ u \ s) \ (\text{subst } y \ u \ t)$
using *assms* **by** (*nominal-induct t avoiding: x y u s rule: term.strong-induct*) (*auto simp: fresh-subst*)

inductive-set *Beta* ($\{\rightarrow_\beta\}$)

where

$root: atom\ x \# t \implies (App\ (Abs\ x\ s)\ t,\ subst\ x\ t\ s) \in \{\rightarrow_\beta\}$
| $Appl: (s,\ t) \in \{\rightarrow_\beta\} \implies (App\ s\ u,\ App\ t\ u) \in \{\rightarrow_\beta\}$
| $Appr: (s,\ t) \in \{\rightarrow_\beta\} \implies (App\ u\ s,\ App\ u\ t) \in \{\rightarrow_\beta\}$
| $Abs: (s,\ t) \in \{\rightarrow_\beta\} \implies (Abs\ x\ s,\ Abs\ x\ t) \in \{\rightarrow_\beta\}$

abbreviation $beta\ ((-/ \rightarrow_\beta -) [56,\ 56]\ 55)$

where

$s \rightarrow_\beta t \equiv (s,\ t) \in \{\rightarrow_\beta\}$

equivariance $Betap$

lemmas $Beta-eqvt = Betap.eqvt [to-set]$

nominal-inductive $Betap$

avoids $Abs: x$

| $root: x$

by ($simp-all\ add: fresh-star-def\ fresh-subst$)

lemmas $Beta-strong-induct = Betap.strong-induct [to-set]$

abbreviation $betas\ (infix\ \rightarrow_{\beta^*}\ 50)$

where

$s \rightarrow_{\beta^*} t \equiv (s,\ t) \in \{\rightarrow_\beta\}^*$

nominal-function $app-beta :: term \Rightarrow term \Rightarrow term$

where

$atom\ x \# u \implies app-beta\ (Abs\ x\ s')\ u = subst\ x\ u\ s'$

| $app-beta\ (Var\ x)\ u = App\ (Var\ x)\ u$

| $app-beta\ (App\ s\ t)\ u = App\ (App\ s\ t)\ u$

by ($nf\ fresh: fresh-subst$)

nominal-termination ($eqvt$) **by** $lexicographic-order$

nominal-function $bullet :: term \Rightarrow term\ (-\bullet [1000]\ 1000)$

where

$(Var\ x)^\bullet = Var\ x$

| $(Abs\ x\ t)^\bullet = Abs\ x\ t^\bullet$

| $(App\ s\ t)^\bullet = app-beta\ s^\bullet\ t^\bullet$

by nf

nominal-termination ($eqvt$) **by** $lexicographic-order$

lemma $app-beta-exhaust [case-names\ Redex\ no-Redex]:$

fixes $c :: 'a :: fs$

assumes $\bigwedge x\ s'. atom\ x \# c \implies s = Abs\ x\ s' \implies thesis$

and $(\bigwedge t. app-beta\ s\ t = App\ s\ t) \implies thesis$

shows $thesis$

by ($cases\ s\ rule: term.strong-exhaust [of\ -\ c]$) ($auto\ simp: fresh-star-def\ fresh-Pair\ intro: assms$)

lemma $App-Betas:$

```

  assumes  $s \rightarrow_{\beta^*} t$  and  $u \rightarrow_{\beta^*} v$ 
  shows  $App\ s\ u \rightarrow_{\beta^*} App\ t\ v$ 
using assms(1)
proof (induct)
  case base
  show ?case using assms(2) by (induct) (auto intro: Beta.intros rtrancl-into-rtrancl)
qed (auto intro: Beta.intros rtrancl-into-rtrancl)

```

```

lemma Abs-Betas:
  assumes  $s \rightarrow_{\beta^*} t$ 
  shows  $Abs\ x\ s \rightarrow_{\beta^*} Abs\ x\ t$ 
using assms by (induct) (auto intro: Beta.intros rtrancl-into-rtrancl)

```

```

lemma self:
   $t \rightarrow_{\beta^*} t^\bullet$ 
proof (nominal-induct t rule: term.strong-induct)
  case (App t u)
  then show ?case
    by (cases  $t^\bullet$  rule: app-beta-exhaust[of u^\bullet])
      (auto intro: App-Betas Beta.intros rtrancl-into-rtrancl)
qed (auto intro: Abs-Betas)

```

```

lemma fresh-atom-bullet:
   $atom\ (x::name) \# t \implies atom\ x \# t^\bullet$ 
proof (nominal-induct t avoiding: x rule: term.strong-induct)
  case (App t u)
  then show ?case by (cases  $t^\bullet$  rule: app-beta-exhaust[of u^\bullet]) (auto intro: fresh-subst)
qed auto

```

```

lemma subst-Beta:
  assumes  $t \rightarrow_{\beta} t'$ 
  shows  $subst\ x\ s\ t \rightarrow_{\beta} subst\ x\ s\ t'$ 
using assms
proof (nominal-induct avoiding: x s rule: Beta-strong-induct)
  case (root y t u)
  then show ?case by (auto simp: subst-subst fresh-subst intro: Beta.root)
qed (auto intro: Beta.intros)

```

```

lemma Beta-in-subst:
  assumes  $s \rightarrow_{\beta} s'$ 
  shows  $subst\ x\ s\ t \rightarrow_{\beta^*} subst\ x\ s'\ t$ 
using assms
by (nominal-induct t avoiding: x s s' rule: term.strong-induct)
  (auto intro: App-Betas Abs-Betas)

```

```

lemma subst-Betas:
  assumes  $s \rightarrow_{\beta^*} s'$  and  $t \rightarrow_{\beta^*} t'$ 
  shows  $subst\ x\ s\ t \rightarrow_{\beta^*} subst\ x\ s'\ t'$ 
using assms(1)

```

proof (*induct*)
case *base*
from *assms*(2) **show** ?*case* **by** (*induct*) (*auto simp: subst-Beta intro: rtrancl-into-rtrancl*)
next
case (*step u v*)
from *Beta-in-subst* [*OF this*(2), *of x t*] **and** *this*(3) **show** ?*case* **by** *auto*
qed

lemma *Beta-fresh*:
fixes *x :: name*
assumes $s \rightarrow_{\beta} t$ **and** *atom x* $\#$ *s*
shows *atom x* $\#$ *t*
using *assms* **by** (*nominal-induct rule: Beta-strong-induct*) (*auto simp: fresh-subst*)

lemma *Abs-BetaD*:
assumes $Abs\ x\ s \rightarrow_{\beta} t$
shows $\exists u. t = Abs\ x\ u \wedge s \rightarrow_{\beta} u$
using *assms*
proof (*nominal-induct Abs x s t avoiding: s rule: Beta-strong-induct*)
case (*Abs u v y*)
then show ?*case*
by (*auto simp: Abs1-eq-iff Beta-fresh fresh-permute-left intro!: exI [of - (y \leftrightarrow x) \cdot v]*)
(*metis Abs1-eq-iff(3) Beta-eqvt flip-commute*)
qed

lemma *Abs-BetaE*:
assumes $Abs\ x\ s \rightarrow_{\beta} t$
obtains *u* **where** $t = Abs\ x\ u$ **and** $s \rightarrow_{\beta} u$
using *assms* **by** (*blast dest: Abs-BetaD*)

lemma *Abs-BetasE*:
assumes $Abs\ x\ s \rightarrow_{\beta^*} t$
obtains *u* **where** $t = Abs\ x\ u$ **and** $s \rightarrow_{\beta^*} u$
using *assms* **by** (*induct Abs x s t*) (*auto elim: Abs-BetaE intro: rtrancl-into-rtrancl*)

lemma *bullet-App*:
 $(App\ s^{\bullet}\ t^{\bullet}, (App\ s\ t)^{\bullet}) \in \{\rightarrow_{\beta}\}^=$
by (*cases s[•] rule: term.strong-exhaust[of - - t[•]]*)
(*auto simp: fresh-star-def intro: Beta.root*)

lemma *rhs*:
 $subst\ x\ s^{\bullet}\ t^{\bullet} \rightarrow_{\beta^*} (subst\ x\ s\ t)^{\bullet}$
proof (*nominal-induct t avoiding: x s rule: term.strong-induct*)
case (*App t₁ t₂*)
show ?*case*
proof (*cases t₁[•] rule: app-beta-exhaust[of (x, t₂, s)]*)
case (*Redex y u*)
then have $Abs\ y\ (subst\ x\ s^{\bullet}\ u) \rightarrow_{\beta^*} (subst\ x\ s\ t_1)^{\bullet}$

using *App(1)* [of $x\ s$] **by** (*simp add: fresh-star-def fresh-atom-bullet*)
with *Abs-BetasE* **obtain** v **where** $(subst\ x\ s\ t_1)^\bullet = Abs\ y\ v$ **and** $subst\ x\ s^\bullet$
 $u \rightarrow_{\beta^*} v$ **by** *blast*
then show *?thesis* **using** *subst-subst* **and** *subst-Betas* **and** *App(2)* **and** *Redex*
by (*simp add: fresh-atom-bullet fresh-subst*)
next
case (*no-Redex*)
then have $subst\ x\ s^\bullet ((App\ t_1\ t_2)^\bullet) \rightarrow_{\beta^*} App\ ((subst\ x\ s\ t_1)^\bullet) ((subst\ x\ s\ t_2)^\bullet)$
using *App* **by** (*auto intro: App-Betas*)
then show *?thesis* **using** *bullet-App* **by** (*force intro: rtrancl-into-rtrancl*)
qed
qed (*force dest: fresh-atom-bullet intro: Abs-Betas*)+

lemma *Betas-fresh*:
fixes $x :: name$
assumes $s \rightarrow_{\beta^*} t$ **and** $atom\ x \# s$
shows $atom\ x \# t$
using *assms* **by** (*induct*) (*auto dest: Beta-fresh*)

lemma *Var-BetaD*:
assumes $Var\ x \rightarrow_{\beta} t$
shows *False*
using *assms* **by** (*induct Var x t*)

lemma *Var-BetasD*:
assumes $Var\ x \rightarrow_{\beta^*} t$
shows $t = Var\ x$
using *assms* **by** (*induct*) (*auto dest: Var-BetaD*)

lemma *app-beta-Betas*:
assumes $s \rightarrow_{\beta^*} s'$ **and** $t \rightarrow_{\beta^*} t'$
shows $app\ beta\ s\ t \rightarrow_{\beta^*} app\ beta\ s'\ t'$
using *assms*
proof (*cases s rule: term.strong-exhaust [of - - t]*)
case (*App s₁ s₂*)
with *assms* **show** *?thesis*
by (*cases s' rule: app-beta-exhaust [of t']*) (*auto intro: root rtrancl-into-rtrancl App-Betas*)
qed (*auto simp: fresh-star-def Betas-fresh subst-Betas elim: Abs-BetasE intro: App-Betas dest!: Var-BetasD*)

lemma *lambda-Z*:
assumes $s \rightarrow_{\beta} t$
shows $t \rightarrow_{\beta^*} s^\bullet \wedge s^\bullet \rightarrow_{\beta^*} t^\bullet$
using *assms*
proof (*nominal-induct rule: Beta-strong-induct*)
case (*Appl s t u*)
then have $App\ t\ u \rightarrow_{\beta^*} App\ s^\bullet\ u^\bullet$ **using** *self* **by** (*auto intro: App-Betas*)
also have $App\ s^\bullet\ u^\bullet \rightarrow_{\beta^*} (App\ s\ u)^\bullet$ **using** *bullet-App* **by** *force*

finally show *?case using Appl by (auto intro: App-Betas app-beta-Betas)*
next
case (*Appr s t u*)
then have $App\ u\ t \rightarrow_{\beta^*} App\ u^\bullet\ s^\bullet$ **using** *self by (auto intro: App-Betas)*
also have $App\ u^\bullet\ s^\bullet \rightarrow_{\beta^*} (App\ u\ s)^\bullet$ **using** *bullet-App by force*
finally show *?case using Appr by (auto intro: App-Betas app-beta-Betas)*
qed (*auto intro: Abs-Betas subst-Betas rhs simp: self fresh-atom-bullet*)

interpretation *lambda-z: z-property bullet Beta*
by (*standard (fact lambda-Z)*)

end

3 Combinatory Logic has the Church-Rosser property

theory *CL-Z imports Z*
begin

datatype *CL = S | K | I | App CL CL (' - - [999, 999] 999)*

inductive-set *red :: CL rel where*

$L: (t, t') \in red \implies (' t u, ' t' u) \in red$
 $R: (u, u') \in red \implies (' t u, ' t u') \in red$
 $S: (' ' S x y z, ' ' x z ' y z) \in red$
 $K: (' ' K x y, x) \in red$
 $I: (' I x, x) \in red$

lemma *App-mono:*

$(t, t') \in red^* \implies (u, u') \in red^* \implies (' t u, ' t' u') \in red^*$

proof –

assume $(t, t') \in red^*$ **hence** $(' t u, ' t' u) \in red^*$

by (*induct t' rule: rtrancl-induct (auto intro: rtrancl-into-rtrancl red.intros)*)

moreover assume $(u, u') \in red^*$ **hence** $(' t' u, ' t' u') \in red^*$

by (*induct u' rule: rtrancl-induct (auto intro: rtrancl-into-rtrancl red.intros)*)

ultimately show *?thesis by auto*

qed

fun *bullet-app :: CL \Rightarrow CL \Rightarrow CL where*

$bullet_app\ (' ' S x y) z = ' ' x z ' y z$
 $bullet_app\ (' K x) y = x$
 $bullet_app\ I x = x$
 $bullet_app\ t u = ' t u$

lemma *bullet-app-red:*

$(' t u, bullet_app\ t u) \in red^=$

by (*induct t u rule: bullet-app.induct (auto intro: red.intros)*)

```

lemma bullet-app-redsI:
  (s, ' t u) ∈ red* ⇒ (s, bullet-app t u) ∈ red*
using bullet-app-red[of t u] by auto

lemma bullet-app-redL:
  (t, t') ∈ red ⇒ (bullet-app t u, bullet-app t' u) ∈ red*
by (induct t u rule: bullet-app.induct)
  (auto 0 6 intro: App-mono bullet-app-redsI elim: red.cases simp only: bullet-app.simps)

lemma bullet-app-redR:
  (u, u') ∈ red ⇒ (bullet-app t u, bullet-app t u') ∈ red*
by (induct t u rule: bullet-app.induct) (auto intro: App-mono)

lemma bullet-app-mono:
  assumes (t, t') ∈ red* (u, u') ∈ red* shows (bullet-app t u, bullet-app t' u') ∈
  red*
proof –
  have (bullet-app t u, bullet-app t' u) ∈ red* using assms(1)
  by (induct t' rule: rtrancl-induct) (auto intro: rtrancl-trans bullet-app-redL)
  moreover have (bullet-app t' u, bullet-app t' u') ∈ red* using assms(2)
  by (induct u' rule: rtrancl-induct) (auto intro: rtrancl-trans bullet-app-redR)
  ultimately show ?thesis by auto
qed

fun bullet :: CL ⇒ CL where
  bullet (' t u) = bullet-app (bullet t) (bullet u)
  | bullet t = t

lemma bullet-incremental:
  (t, bullet t) ∈ red*
by (induct t rule: bullet.induct) (auto intro: App-mono bullet-app-redsI)

interpretation CL:z-property bullet red
proof (unfold-locales, intro conjI)
  fix t u assume (t, u) ∈ red thus (u, bullet t) ∈ red*
  proof (induct t arbitrary: u rule: bullet.induct)
    case (1 t1 t2) show ?case using 1(3)
    by (cases) (auto intro: 1 App-mono bullet-app-redsI bullet-incremental)
  qed (auto elim: red.cases)
next
  fix t u assume (t, u) ∈ red thus (bullet t, bullet u) ∈ red*
  by (induct t u rule: red.induct) (auto intro: App-mono bullet-app-mono bullet-app-redsI)
qed

lemmas CR-red = CL.CR

end

```

References

- [1] P. Dehornoy and V. v. Oostrom. Z, proving confluence by monotonic single-step upperbound functions. In *Logical Models of Reasoning and Computation (LMRC'2008)*, 2008.