# Relational Minimum Spanning Tree Algorithms

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**Abstract**

We verify the correctness of Prim's, Kruskal's and Borůvka's minimum spanning tree algorithms based on algebras for aggregation and minimisation.

## Contents



## 1 Overview

The theories described in this document prove the correctness of Prim's, Kruskal's and Borůvka's minimum spanning tree algorithms. Specifications and algorithms work in Stone-Kleene relation algebras extended by operations for aggregation and minimisation. The algorithms are implemented in a simple imperative language and their proof uses Hoare logic. The correctness proofs are discussed in [3, 5, 6, 8].

## 1.1 Prim's and Kruskal's minimum spanning tree algorithms

A framework based on Stone relation algebras and Kleene algebras and extended by operations for aggregation and minimisation was presented by the first author in [3, 5] and used to formally verify the correctness of Prim's minimum spanning tree algorithm. It was extended in [6] and applied to prove the correctness of Kruskal's minimum spanning tree algorithm.

Two theories, one each for Prim's and Kruskal's algorithms, prove total correctness of these algorithms. As case studies for the algebraic framework, these two theories combined were originally part of another AFP entry [4].

## 1.2 Borůvka's minimum spanning tree algorithm

Otakar Borůvka formalised the minimum spanning tree problem and proposed a solution to it [1]. Borůvka's original paper is written in Czech; translations of varying completeness can be found in [2, 7].

The theory for Borůvka's minimum spanning tree algorithm proves partial correctness of this algorithm. This work is based on the same algebraic framework as the proof of Kruskal's algorithm; in particular it uses many theories from the hierarchy underlying [4].

The theory for Borůvka's algorithm formally verifies results from the second author's Master's thesis [8].

# 2 Kruskal's Minimum Spanning Tree Algorithm

In this theory we prove total correctness of Kruskal's minimum spanning tree algorithm. The proof uses the following steps [6]. We first establish that the algorithm terminates and constructs a spanning tree. This is a constructive proof of the existence of a spanning tree; any spanning tree algorithm could be used for this. We then conclude that a minimum spanning tree exists. This is necessary to establish the invariant for the actual correctness proof, which shows that Kruskal's algorithm produces a minimum spanning tree.

**theory** *Kruskal*

**imports** *HOL−Hoare.Hoare-Logic Aggregation-Algebras.Aggregation-Algebras*

**begin**

**context** *m-kleene-algebra*
**begin**

**definition** *spanning-forest f g* $\equiv$ *forest f* $\wedge$ $f \leq --g$ $\wedge$ *components g* $\leq$ *forest-components f* $\wedge$ *regular f*
**definition** *minimum-spanning-forest f g* $\equiv$ *spanning-forest f g* $\wedge$ ($\forall$ *u* . *spanning-forest u g* $\longrightarrow$ *sum* $(f \sqcap g) \leq$ *sum* $(u \sqcap g)$)

**definition** *kruskal-spanning-invariant f g h* $\equiv$ *symmetric g* $\wedge$ $h = h^T \wedge g \sqcap --h = h \wedge$ *spanning-forest f* $(-h \sqcap g)$
**definition** *kruskal-invariant f g h* $\equiv$ *kruskal-spanning-invariant f g h* $\wedge$ $(\exists\, w\,.$ *minimum-spanning-forest w g* $\wedge f \leq w \sqcup w^T)$

We first show two verification conditions which are used in both correctness proofs.

**lemma** *kruskal-vc-1*:
  **assumes** *symmetric g*
    **shows** *kruskal-spanning-invariant bot g g*
**proof** (*unfold kruskal-spanning-invariant-def*, *intro conjI*)
  **show** *symmetric g*
    **using** *assms* **by** *simp*
**next**
  **show** $g = g^T$
    **using** *assms* **by** *simp*
**next**
  **show** $g \sqcap --g = g$
    **using** *inf.sup-monoid.add-commute selection-closed-id* **by** *simp*
**next**
  **show** *spanning-forest bot* $(-g \sqcap g)$
    **using** *star.circ-transitive-equal spanning-forest-def* **by** *simp*
**qed**

**lemma** *kruskal-vc-2*:
  **assumes** *kruskal-spanning-invariant f g h*
    **and** $h \neq$ *bot*
    **shows** (*minarc h* $\leq -$*forest-components f* $\longrightarrow$ *kruskal-spanning-invariant* $((f \sqcap -(top * minarc\ h * f^{T\star})) \sqcup (f \sqcap top * minarc\ h * f^{T\star})^T \sqcup minarc\ h)\ g\ (h \sqcap -minarc\ h \sqcap -minarc\ h^T)$
$\wedge$ *card* $\{\ x\ .\ regular\ x \wedge x \leq --h \wedge x \leq -minarc\ h \wedge x \leq -minarc\ h^T\ \} <$ *card* $\{\ x\ .\ regular\ x \wedge x \leq --h\ \}) \wedge$
(¬ *minarc h* $\leq -$*forest-components f* $\longrightarrow$ *kruskal-spanning-invariant f g* $(h \sqcap -minarc\ h \sqcap -minarc\ h^T)$
$\wedge$ *card* $\{\ x\ .\ regular\ x \wedge x \leq --h \wedge x \leq -minarc\ h \wedge x \leq -minarc\ h^T\ \} <$ *card* $\{\ x\ .\ regular\ x \wedge x \leq --h\ \})$
**proof** –
  **let** $?e = minarc\ h$
  **let** $?f = (f \sqcap -(top * ?e * f^{T\star})) \sqcup (f \sqcap top * ?e * f^{T\star})^T \sqcup ?e$
  **let** $?h = h \sqcap -?e \sqcap -?e^T$
  **let** $?F =$ *forest-components f*
  **let** $?n1 = card\ \{\ x\ .\ regular\ x \wedge x \leq --h\ \}$
  **let** $?n2 = card\ \{\ x\ .\ regular\ x \wedge x \leq --h \wedge x \leq -?e \wedge x \leq -?e^T\ \}$
  **have** *1*: *regular f* $\wedge$ *regular ?e*
    **by** (*metis assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def minarc-regular*)
  **hence** *2*: *regular ?f* $\wedge$ *regular ?F* $\wedge$ *regular* $(?e^T)$
    **using** *regular-closed-star regular-conv-closed regular-mult-closed* **by** *simp*
  **have** *3*: ¬ $?e \leq -?e$

**using** *assms*(*2*) *inf.orderE minarc-bot-iff* **by** *fastforce*
**have** *4*: *?n2* < *?n1*
**apply** (*rule psubset-card-mono*)
**using** *finite-regular* **apply** *simp*
**using** *1 3 kruskal-spanning-invariant-def minarc-below* **by** *auto*
**show** (*?e* ≤ −*?F* ⟶ *kruskal-spanning-invariant ?f g ?h* ∧ *?n2* < *?n1*) ∧ (¬ *?e* ≤ −*?F* ⟶ *kruskal-spanning-invariant f g ?h* ∧ *?n2* < *?n1*)
**proof** (*rule conjI*)
**have** *5*: *injective ?f*
**apply** (*rule kruskal-injective-inv*)
**using** *assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def* **apply** *simp*
**apply** (*simp add*: *covector-mult-closed*)
**apply** (*simp add*: *comp-associative comp-isotone star.right-plus-below-circ*)
**apply** (*meson mult-left-isotone order-lesseq-imp star-outer-increasing top.extremum*)
**using** *assms*(*1*,*2*) *kruskal-spanning-invariant-def kruskal-injective-inv-2 minarc-arc spanning-forest-def* **apply** *simp*
**using** *assms*(*2*) *arc-injective minarc-arc* **apply** *blast*
**using** *assms*(*1*,*2*) *kruskal-spanning-invariant-def kruskal-injective-inv-3 minarc-arc spanning-forest-def* **by** *simp*
**show** *?e* ≤ −*?F* ⟶ *kruskal-spanning-invariant ?f g ?h* ∧ *?n2* < *?n1*
**proof**
**assume** *6*: *?e* ≤ −*?F*
**have** *7*: *equivalence ?F*
**using** *assms*(*1*) *kruskal-spanning-invariant-def forest-components-equivalence spanning-forest-def* **by** *simp*
**have** $?e^T * top * ?e^T = ?e^T$
**using** *assms*(*2*) **by** (*simp add*: *arc-top-arc minarc-arc*)
**hence** $?e^T * top * ?e^T \le -?F$
**using** *6 7 conv-complement conv-isotone* **by** *fastforce*
**hence** *8*: *?e* ∗ *?F* ∗ *?e* = *bot*
**using** *le-bot triple-schroeder-p* **by** *simp*
**show** *kruskal-spanning-invariant ?f g ?h* ∧ *?n2* < *?n1*
**proof** (*unfold kruskal-spanning-invariant-def*, *intro conjI*)
**show** *symmetric g*
**using** *assms*(*1*) *kruskal-spanning-invariant-def* **by** *simp*
**next**
**show** $?h = ?h^T$
**using** *assms*(*1*) **by** (*simp add*: *conv-complement conv-dist-inf inf-commute inf-left-commute kruskal-spanning-invariant-def*)
**next**
**show** *g* ⊓ −−*?h* = *?h*
**using** *1 2* **by** (*metis* (*opaque-lifting*) *assms*(*1*) *kruskal-spanning-invariant-def inf-assoc pp-dist-inf*)
**next**
**show** *spanning-forest ?f* (−*?h* ⊓ *g*)
**proof** (*unfold spanning-forest-def*, *intro conjI*)
**show** *injective ?f*

using *5* **by** *simp*
**next**
**show** *acyclic ?f*
**apply** (*rule kruskal-acyclic-inv*)
**using** *assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def* **apply** *simp*
**apply** (*simp add*: *covector-mult-closed*)
**using** *8 assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def kruskal-acyclic-inv-1* **apply** *simp*
**using** *8* **apply** (*metis comp-associative mult-left-sub-dist-sup-left star.circ-loop-fixpoint sup-commute le-bot*)
**using** *6* **by** (*simp add*: *p-antitone-iff*)
**next**
**show** $?f \leq --(-?h \sqcap g)$
**apply** (*rule kruskal-subgraph-inv*)
**using** *assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def* **apply** *simp*
**using** *assms*(*1*) **apply** (*metis kruskal-spanning-invariant-def minarc-below order.trans pp-isotone-inf*)
**using** *assms*(*1*) *kruskal-spanning-invariant-def* **apply** *simp*
**using** *assms*(*1*) *kruskal-spanning-invariant-def* **by** *simp*
**next**
**show** *components* $(-?h \sqcap g) \leq$ *forest-components ?f*
**apply** (*rule kruskal-spanning-inv*)
**using** *5* **apply** *simp*
**using** *1 regular-closed-star regular-conv-closed regular-mult-closed* **apply** *simp*
**using** *1* **apply** *simp*
**using** *assms*(*1*) *kruskal-spanning-invariant-def spanning-forest-def* **by** *simp*
**next**
**show** *regular ?f*
**using** *2* **by** *simp*
**qed**
**next**
**show** $?n2 < ?n1$
**using** *4* **by** *simp*
**qed**
**qed**
**next**
**show** $\neg\ ?e \leq -?F \longrightarrow$ *kruskal-spanning-invariant f g ?h* $\wedge\ ?n2 < ?n1$
**proof**
**assume** $\neg\ ?e \leq -?F$
**hence** *9*: $?e \leq ?F$
**using** *2 assms*(*2*) *arc-in-partition minarc-arc* **by** *fastforce*
**show** *kruskal-spanning-invariant f g ?h* $\wedge\ ?n2 < ?n1$
**proof** (*unfold kruskal-spanning-invariant-def*, *intro conjI*)
**show** *symmetric g*
**using** *assms*(*1*) *kruskal-spanning-invariant-def* **by** *simp*

```
    next
      show ?h = ?h^T
        using assms(1) by (simp add: conv-complement conv-dist-inf
inf-commute inf-left-commute kruskal-spanning-invariant-def)
    next
      show g ⊓ −−?h = ?h
        using 1 2 by (metis (opaque-lifting) assms(1)
kruskal-spanning-invariant-def inf-assoc pp-dist-inf)
    next
      show spanning-forest f (−?h ⊓ g)
      proof (unfold spanning-forest-def, intro conjI)
        show injective f
          using assms(1) kruskal-spanning-invariant-def spanning-forest-def by
simp
      next
        show acyclic f
          using assms(1) kruskal-spanning-invariant-def spanning-forest-def by
simp
      next
        have f ≤ −−(−h ⊓ g)
          using assms(1) kruskal-spanning-invariant-def spanning-forest-def by
simp
        also have ... ≤ −−(−?h ⊓ g)
          using comp-inf.mult-right-isotone inf.sup-monoid.add-commute
inf-left-commute p-antitone-inf pp-isotone by presburger
        finally show f ≤ −−(−?h ⊓ g)
          by simp
      next
        show components (−?h ⊓ g) ≤ ?F
          apply (rule kruskal-spanning-inv-1)
          using 9 apply simp
          using 1 apply simp
          using assms(1) kruskal-spanning-invariant-def spanning-forest-def
apply simp
          using assms(1) kruskal-spanning-invariant-def
forest-components-equivalence spanning-forest-def by simp
      next
        show regular f
          using 1 by simp
      qed
    next
      show ?n2 < ?n1
        using 4 by simp
    qed
  qed
 qed
qed
```

The following result shows that Kruskal's algorithm terminates and constructs a spanning tree. We cannot yet show that this is a minimum spanning tree.

**theorem** *kruskal-spanning*:
```
  VARS e f h
  [ symmetric g ]
  f := bot;
  h := g;
  WHILE h ≠ bot
    INV { kruskal-spanning-invariant f g h }
    VAR { card { x . regular x ∧ x ≤ −−h } }
     DO e := minarc h;
       IF e ≤ −forest-components f THEN
         f := (f ⊓ −(top ∗ e ∗ f^T⋆)) ⊔ (f ⊓ top ∗ e ∗ f^T⋆)^T ⊔ e
       ELSE
         SKIP
       FI;
       h := h ⊓ −e ⊓ −e^T
     OD
  [ spanning-forest f g ]
  apply vcg-tc-simp
  using kruskal-vc-1 apply simp
  using kruskal-vc-2 apply simp
  using kruskal-spanning-invariant-def by auto
```

Because we have shown total correctness, we conclude that a spanning tree exists.

**lemma** *kruskal-exists-spanning*:
```
  symmetric g ⟹ ∃f . spanning-forest f g
  using tc-extract-function kruskal-spanning by blast
```

This implies that a minimum spanning tree exists, which is used in the subsequent correctness proof.

**lemma** *kruskal-exists-minimal-spanning*:
```
  assumes symmetric g
    shows ∃f . minimum-spanning-forest f g
proof −
  let ?s = { f . spanning-forest f g }
  have ∃m∈?s . ∀z∈?s . sum (m ⊓ g) ≤ sum (z ⊓ g)
    apply (rule finite-set-minimal)
    using finite-regular spanning-forest-def apply simp
    using assms kruskal-exists-spanning apply simp
    using sum-linear by simp
  thus ?thesis
    using minimum-spanning-forest-def by simp
qed
```

Kruskal's minimum spanning tree algorithm terminates and is correct. This is the same algorithm that is used in the previous correctness proof, with the same precondition and variant, but with a different invariant and postcondition.

**theorem** *kruskal*:
*VARS e f h*
[ *symmetric g* ]
$f := bot$;
$h := g$;
*WHILE* $h \neq bot$
*INV* { *kruskal-invariant f g h* }
*VAR* { *card* { $x$ . *regular* $x \wedge x \leq --h$ } }
*DO* $e :=$ *minarc* $h$;
*IF* $e \leq -$*forest-components* $f$ *THEN*
$f := (f \sqcap -(top * e * f^{T\star})) \sqcup (f \sqcap top * e * f^{T\star})^T \sqcup e$
*ELSE*
*SKIP*
*FI*;
$h := h \sqcap -e \sqcap -e^T$
*OD*
[ *minimum-spanning-forest f g* ]
**proof** *vcg-tc-simp*
**assume** *symmetric g*
**thus** *kruskal-invariant bot g g*
**using** *kruskal-vc-1 kruskal-exists-minimal-spanning kruskal-invariant-def* **by** *simp*
**next**
**fix** *f h*
**let** *?e* = *minarc h*
**let** $?f = (f \sqcap -(top * ?e * f^{T\star})) \sqcup (f \sqcap top * ?e * f^{T\star})^T \sqcup ?e$
**let** $?h = h \sqcap -?e \sqcap -?e^T$
**let** *?F* = *forest-components f*
**let** *?n1* = *card* { $x$ . *regular* $x \wedge x \leq --h$ }
**let** *?n2* = *card* { $x$ . *regular* $x \wedge x \leq --h \wedge x \leq -?e \wedge x \leq -?e^T$ }
**assume** *1*: *kruskal-invariant f g h* $\wedge h \neq bot$
**from** *1* **obtain** *w* **where** *2*: *minimum-spanning-forest w g* $\wedge f \leq w \sqcup w^T$
**using** *kruskal-invariant-def* **by** *auto*
**hence** *3*: *regular f* $\wedge$ *regular w* $\wedge$ *regular ?e*
**using** *1* **by** (*metis kruskal-invariant-def kruskal-spanning-invariant-def minimum-spanning-forest-def spanning-forest-def minarc-regular*)
**show** ($?e \leq -?F \longrightarrow$ *kruskal-invariant ?f g ?h* $\wedge$ *?n2* $<$ *?n1*) $\wedge$ ($\neg$ $?e \leq -?F \longrightarrow$ *kruskal-invariant f g ?h* $\wedge$ *?n2* $<$ *?n1*)
**proof** (*rule conjI*)
**show** $?e \leq -?F \longrightarrow$ *kruskal-invariant ?f g ?h* $\wedge$ *?n2* $<$ *?n1*
**proof**
**assume** *4*: $?e \leq -?F$
**have** *5*: *equivalence ?F*
**using** *1 kruskal-invariant-def kruskal-spanning-invariant-def forest-components-equivalence spanning-forest-def* **by** *simp*
**have** $?e^T * top * ?e^T = ?e^T$
**using** *1* **by** (*simp add*: *arc-top-arc minarc-arc*)
**hence** $?e^T * top * ?e^T \leq -?F$
**using** *4 5 conv-complement conv-isotone* **by** *fastforce*

```
hence 6: ?e * ?F * ?e = bot
  using le-bot triple-schroeder-p by simp
show kruskal-invariant ?f g ?h ∧ ?n2 < ?n1
proof (unfold kruskal-invariant-def, intro conjI)
  show kruskal-spanning-invariant ?f g ?h
    using 1 4 kruskal-vc-2 kruskal-invariant-def by simp
next
  show ∃ w . minimum-spanning-forest w g ∧ ?f ≤ w ⊔ w^T
  proof
    let ?p = w ⊓ top * ?e * w^T⋆
    let ?v = (w ⊓ −(top * ?e * w^T⋆)) ⊔ ?p^T
    have 7: regular ?p
      using 3 regular-closed-star regular-conv-closed regular-mult-closed by
simp
    have 8: injective ?v
      apply (rule kruskal-exchange-injective-inv-1)
      using 2 minimum-spanning-forest-def spanning-forest-def apply simp
      apply (simp add: covector-mult-closed)
      apply (simp add: comp-associative comp-isotone
star.right-plus-below-circ)
      using 1 2 kruskal-injective-inv-3 minarc-arc
minimum-spanning-forest-def spanning-forest-def by simp
    have 9: components g ≤ forest-components ?v
      apply (rule kruskal-exchange-spanning-inv-1)
      using 8 apply simp
      using 7 apply simp
      using 2 minimum-spanning-forest-def spanning-forest-def by simp
    have 10: spanning-forest ?v g
    proof (unfold spanning-forest-def, intro conjI)
      show injective ?v
        using 8 by simp
    next
      show acyclic ?v
        apply (rule kruskal-exchange-acyclic-inv-1)
        using 2 minimum-spanning-forest-def spanning-forest-def apply simp
        by (simp add: covector-mult-closed)
    next
      show ?v ≤ −−g
        apply (rule sup-least)
        using 2 inf.coboundedI1 minimum-spanning-forest-def
spanning-forest-def apply simp
        using 1 2 by (metis kruskal-invariant-def
kruskal-spanning-invariant-def conv-complement conv-dist-inf order.trans
inf.absorb2 inf.cobounded1 minimum-spanning-forest-def spanning-forest-def)
    next
      show components g ≤ forest-components ?v
        using 9 by simp
    next
      show regular ?v
```

**using** *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *simp*
**qed**
**have** *11*: $sum\ (?v \sqcap g) = sum\ (w \sqcap g)$
**proof** −
**have** $sum\ (?v \sqcap g) = sum\ (w \sqcap -(top * ?e * w^{T\star}) \sqcap g) + sum\ (?p^T \sqcap g)$
**using** *2* **by** (*metis conv-complement conv-top epm-8 inf-import-p inf-top-right regular-closed-top vector-top-closed minimum-spanning-forest-def spanning-forest-def sum-disjoint*)
**also have** ... $= sum\ (w \sqcap -(top * ?e * w^{T\star}) \sqcap g) + sum\ (?p \sqcap g)$
**using** *1 kruskal-invariant-def kruskal-spanning-invariant-def sum-symmetric* **by** *simp*
**also have** ... $= sum\ (((w \sqcap -(top * ?e * w^{T\star})) \sqcup ?p) \sqcap g)$
**using** *inf-commute inf-left-commute sum-disjoint* **by** *simp*
**also have** ... $= sum\ (w \sqcap g)$
**using** *3 7 maddux-3-11-pp* **by** *simp*
**finally show** *?thesis*
**by** *simp*
**qed**
**have** *12*: $?v \sqcup ?v^T = w \sqcup w^T$
**proof** −
**have** $?v \sqcup ?v^T = (w \sqcap -?p) \sqcup ?p^T \sqcup (w^T \sqcap -?p^T) \sqcup ?p$
**using** *conv-complement conv-dist-inf conv-dist-sup inf-import-p sup-assoc* **by** *simp*
**also have** ... $= w \sqcup w^T$
**using** *3 7 conv-complement conv-dist-inf inf-import-p maddux-3-11-pp sup-monoid.add-assoc sup-monoid.add-commute* **by** *simp*
**finally show** *?thesis*
**by** *simp*
**qed**
**have** *13*: $?v * ?e^T = bot$
**apply** (*rule kruskal-reroot-edge*)
**using** *1* **apply** (*simp add*: *minarc-arc*)
**using** *2 minimum-spanning-forest-def spanning-forest-def* **by** *simp*
**have** $?v \sqcap ?e \leq ?v \sqcap top * ?e$
**using** *inf.sup-right-isotone top-left-mult-increasing* **by** *simp*
**also have** ... $\leq ?v * (top * ?e)^T$
**using** *covector-restrict-comp-conv covector-mult-closed vector-top-closed* **by** *simp*
**finally have** *14*: $?v \sqcap ?e = bot$
**using** *13* **by** (*metis conv-dist-comp mult-assoc le-bot mult-left-zero*)
**let** $?d = ?v \sqcap top * ?e^T * ?v^{T\star} \sqcap ?F * ?e^T * top \sqcap top * ?e * -?F$
**let** $?w = (?v \sqcap -?d) \sqcup ?e$
**have** *15*: *regular ?d*
**using** *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *simp*
**have** *16*: $?F \leq -?d$
**apply** (*rule kruskal-edge-between-components-1*)

**using** *5* **apply** *simp*
**using** *1 conv-dist-comp minarc-arc mult-assoc* **by** *simp*
**have** *17*: $f \sqcup f^T \leq (?v \sqcap -?d \sqcap -?d^T) \sqcup (?v^T \sqcap -?d \sqcap -?d^T)$
**apply** (*rule kruskal-edge-between-components-2*)
**using** *16* **apply** *simp*
**using** *1 kruskal-invariant-def kruskal-spanning-invariant-def spanning-forest-def* **apply** *simp*
**using** *2 12* **by** (*metis conv-dist-sup conv-involutive conv-isotone le-supI sup-commute*)
**show** *minimum-spanning-forest ?w g* $\wedge$ $?f \leq ?w \sqcup ?w^T$
**proof** (*intro conjI*)
**have** *18*: $?e^T \leq ?v^\star$
**apply** (*rule kruskal-edge-arc-1*[**where** *g*=*g* **and** *h*=*h*])
**using** *minarc-below* **apply** *simp*
**using** *1* **apply** (*metis kruskal-invariant-def kruskal-spanning-invariant-def inf-le1*)
**using** *1 kruskal-invariant-def kruskal-spanning-invariant-def* **apply** *simp*
**using** *9* **apply** *simp*
**using** *13* **by** *simp*
**have** *19*: *arc ?d*
**apply** (*rule kruskal-edge-arc*)
**using** *5* **apply** *simp*
**using** *10 spanning-forest-def* **apply** *blast*
**using** *1* **apply** (*simp add*: *minarc-arc*)
**using** *3* **apply** (*metis conv-complement pp-dist-star regular-mult-closed*)
**using** *2 8 12* **apply** (*simp add*: *kruskal-forest-components-inf*)
**using** *10 spanning-forest-def* **apply** *simp*
**using** *13* **apply** *simp*
**using** *6* **apply** *simp*
**using** *18* **by** *simp*
**show** *minimum-spanning-forest ?w g*
**proof** (*unfold minimum-spanning-forest-def*, *intro conjI*)
**have** $(?v \sqcap -?d) * ?e^T \leq ?v * ?e^T$
**using** *inf-le1 mult-left-isotone* **by** *simp*
**hence** $(?v \sqcap -?d) * ?e^T = bot$
**using** *13 le-bot* **by** *simp*
**hence** *20*: $?e * (?v \sqcap -?d)^T = bot$
**using** *conv-dist-comp conv-involutive conv-bot* **by** *force*
**have** *21*: *injective ?w*
**apply** (*rule injective-sup*)
**using** *8* **apply** (*simp add*: *injective-inf-closed*)
**using** *20* **apply** *simp*
**using** *1 arc-injective minarc-arc* **by** *blast*
**show** *spanning-forest ?w g*
**proof** (*unfold spanning-forest-def*, *intro conjI*)
**show** *injective ?w*
**using** *21* **by** *simp*

**next**
**show** *acyclic ?w*
**apply** (*rule kruskal-exchange-acyclic-inv-2*)
**using** *10 spanning-forest-def* **apply** *blast*
**using** *8* **apply** *simp*
**using** *inf.coboundedI1* **apply** *simp*
**using** *19* **apply** *simp*
**using** *1* **apply** (*simp add*: *minarc-arc*)
**using** *inf.cobounded2 inf.coboundedI1* **apply** *simp*
**using** *13* **by** *simp*
**next**
**have** $?w \leq ?v \sqcup ?e$
**using** *inf-le1 sup-left-isotone* **by** *simp*
**also have** ... $\leq --g \sqcup ?e$
**using** *10 sup-left-isotone spanning-forest-def* **by** *blast*
**also have** ... $\leq --g \sqcup --h$
**by** (*simp add*: *le-supI2 minarc-below*)
**also have** ... $= --g$
**using** *1* **by** (*metis kruskal-invariant-def kruskal-spanning-invariant-def pp-isotone-inf sup.orderE*)
**finally show** $?w \leq --g$
**by** *simp*
**next**
**have** *22*: $?d \leq (?v \sqcap -?d)^{T\star} * ?e^T * top$
**apply** (*rule kruskal-exchange-spanning-inv-2*)
**using** *8* **apply** *simp*
**using** *13* **apply** (*metis semiring.mult-not-zero star-absorb star-simulation-right-equal*)
**using** *17* **apply** *simp*
**by** (*simp add*: *inf.coboundedI1*)
**have** *components g* $\leq$ *forest-components ?v*
**using** *10 spanning-forest-def* **by** *auto*
**also have** ... $\leq$ *forest-components ?w*
**apply** (*rule kruskal-exchange-forest-components-inv*)
**using** *21* **apply** *simp*
**using** *15* **apply** *simp*
**using** *1* **apply** (*simp add*: *arc-top-arc minarc-arc*)
**apply** (*simp add*: *inf.coboundedI1*)
**using** *13* **apply** *simp*
**using** *8* **apply** *simp*
**apply** (*simp add*: *le-infI1*)
**using** *22* **by** *simp*
**finally show** *components g* $\leq$ *forest-components ?w*
**by** *simp*
**next**
**show** *regular ?w*
**using** *3 7 regular-conv-closed* **by** *simp*
**qed**
**next**

**have** *23*: $?e \sqcap g \neq bot$
**using** *1* **by** (*metis kruskal-invariant-def kruskal-spanning-invariant-def comp-inf.semiring.mult-zero-right inf.sup-monoid.add-assoc inf.sup-monoid.add-commute minarc-bot-iff minarc-meet-bot*)
**have** $g \sqcap -h \leq (g \sqcap -h)^\star$
**using** *star.circ-increasing* **by** *simp*
**also have** ... $\leq (--(g \sqcap -h))^\star$
**using** *pp-increasing star-isotone* **by** *blast*
**also have** ... $\leq ?F$
**using** *1 kruskal-invariant-def kruskal-spanning-invariant-def inf.sup-monoid.add-commute spanning-forest-def* **by** *simp*
**finally have** *24*: $g \sqcap -h \leq ?F$
**by** *simp*
**have** $?d \leq --g$
**using** *10 inf.coboundedI1 spanning-forest-def* **by** *blast*
**hence** $?d \leq --g \sqcap -?F$
**using** *16 inf.boundedI p-antitone-iff* **by** *simp*
**also have** ... $= --(g \sqcap -?F)$
**by** *simp*
**also have** ... $\leq --h$
**using** *24 p-shunting-swap pp-isotone* **by** *fastforce*
**finally have** *25*: $?d \leq --h$
**by** *simp*
**have** $?d = bot \longrightarrow top = bot$
**using** *19* **by** (*metis mult-left-zero mult-right-zero*)
**hence** $?d \neq bot$
**using** *1 le-bot* **by** *auto*
**hence** *26*: $?d \sqcap h \neq bot$
**using** *25* **by** (*metis inf.absorb-iff2 inf-commute pseudo-complement*)
**have** $sum\ (?e \sqcap g) = sum\ (?e \sqcap --h \sqcap g)$
**by** (*simp add*: *inf.absorb1 minarc-below*)
**also have** ... $= sum\ (?e \sqcap h)$
**using** *1* **by** (*metis kruskal-invariant-def kruskal-spanning-invariant-def inf.left-commute inf.sup-monoid.add-commute*)
**also have** ... $\leq sum\ (?d \sqcap h)$
**using** *19 26 minarc-min* **by** *simp*
**also have** ... $= sum\ (?d \sqcap (--h \sqcap g))$
**using** *1 kruskal-invariant-def kruskal-spanning-invariant-def inf-commute* **by** *simp*
**also have** ... $= sum\ (?d \sqcap g)$
**using** *25* **by** (*simp add*: *inf.absorb2 inf-assoc inf-commute*)
**finally have** *27*: $sum\ (?e \sqcap g) \leq sum\ (?d \sqcap g)$
**by** *simp*
**have** $?v \sqcap ?e \sqcap -?d = bot$
**using** *14* **by** *simp*
**hence** $sum\ (?w \sqcap g) = sum\ (?v \sqcap -?d \sqcap g) + sum\ (?e \sqcap g)$
**using** *sum-disjoint inf-commute inf-assoc* **by** *simp*
**also have** ... $\leq sum\ (?v \sqcap -?d \sqcap g) + sum\ (?d \sqcap g)$

**using** *23 27 sum-plus-right-isotone* **by** *simp*
**also have** ... $= sum\ (((?v \sqcap -?d) \sqcup ?d) \sqcap g)$
**using** *sum-disjoint inf-le2 pseudo-complement* **by** *simp*
**also have** ... $= sum\ ((?v \sqcup ?d) \sqcap (-?d \sqcup ?d) \sqcap g)$
**by** (*simp add*: *sup-inf-distrib2*)
**also have** ... $= sum\ ((?v \sqcup ?d) \sqcap g)$
**using** *15* **by** (*metis inf-top-right stone*)
**also have** ... $= sum\ (?v \sqcap g)$
**by** (*simp add*: *inf.sup-monoid.add-assoc*)
**finally have** $sum\ (?w \sqcap g) \leq sum\ (?v \sqcap g)$
**by** *simp*
**thus** $\forall\, u$ . *spanning-forest* $u\ g \longrightarrow sum\ (?w \sqcap g) \leq sum\ (u \sqcap g)$
**using** *2 11 minimum-spanning-forest-def* **by** *auto*
**qed**
**next**
**have** $?f \leq f \sqcup f^T \sqcup ?e$
**using** *conv-dist-inf inf-le1 sup-left-isotone sup-mono* **by** *presburger*
**also have** ... $\leq (?v \sqcap -?d \sqcap -?d^T) \sqcup (?v^T \sqcap -?d \sqcap -?d^T) \sqcup ?e$
**using** *17 sup-left-isotone* **by** *simp*
**also have** ... $\leq (?v \sqcap -?d) \sqcup (?v^T \sqcap -?d \sqcap -?d^T) \sqcup ?e$
**using** *inf.cobounded1 sup-inf-distrib2* **by** *presburger*
**also have** ... $= ?w \sqcup (?v^T \sqcap -?d \sqcap -?d^T)$
**by** (*simp add*: *sup-assoc sup-commute*)
**also have** ... $\leq ?w \sqcup (?v^T \sqcap -?d^T)$
**using** *inf.sup-right-isotone inf-assoc sup-right-isotone* **by** *simp*
**also have** ... $\leq ?w \sqcup ?w^T$
**using** *conv-complement conv-dist-inf conv-dist-sup sup-right-isotone*
**by** *simp*
**finally show** $?f \leq ?w \sqcup ?w^T$
**by** *simp*
**qed**
**qed**
**next**
**show** $?n2 < ?n1$
**using** *1 kruskal-vc-2 kruskal-invariant-def* **by** *auto*
**qed**
**qed**
**next**
**show** $\neg\ ?e \leq -?F \longrightarrow$ *kruskal-invariant* $f\ g\ ?h \wedge ?n2 < ?n1$
**using** *1 kruskal-vc-2 kruskal-invariant-def* **by** *auto*
**qed**
**next**
**fix** $f$
**assume** *28*: *kruskal-invariant f g bot*
**hence** *29*: *spanning-forest f g*
**using** *kruskal-invariant-def kruskal-spanning-invariant-def* **by** *auto*
**from** *28* **obtain** $w$ **where** *30*: *minimum-spanning-forest* $w\ g \wedge f \leq w \sqcup w^T$
**using** *kruskal-invariant-def* **by** *auto*
**hence** $w = w \sqcap --g$

**by** (*simp add*: *inf.absorb1 minimum-spanning-forest-def spanning-forest-def*)
**also have** ... $\leq w \sqcap$ *components g*
**by** (*metis inf.sup-right-isotone star.circ-increasing*)
**also have** ... $\leq w \sqcap f^{T\star} * f^\star$
**using** *29 spanning-forest-def inf.sup-right-isotone* **by** *simp*
**also have** ... $\leq f \sqcup f^T$
**apply** (*rule cancel-separate-6*[**where** $z=w$ **and** $y=w^T$])
**using** *30 minimum-spanning-forest-def spanning-forest-def* **apply** *simp*
**using** *30* **apply** (*metis conv-dist-inf conv-dist-sup conv-involutive inf.cobounded2 inf.orderE*)
**using** *30* **apply** (*simp add*: *sup-commute*)
**using** *30 minimum-spanning-forest-def spanning-forest-def* **apply** *simp*
**using** *30* **by** (*metis acyclic-star-below-complement comp-inf.mult-right-isotone inf-p le-bot minimum-spanning-forest-def spanning-forest-def*)
**finally have** *31*: $w \leq f \sqcup f^T$
**by** *simp*
**have** *sum* $(f \sqcap g) =$ *sum* $((w \sqcup w^T) \sqcap (f \sqcap g))$
**using** *30* **by** (*metis inf-absorb2 inf.assoc*)
**also have** ... $=$ *sum* $(w \sqcap (f \sqcap g)) +$ *sum* $(w^T \sqcap (f \sqcap g))$
**using** *30 inf.commute acyclic-asymmetric sum-disjoint minimum-spanning-forest-def spanning-forest-def* **by** *simp*
**also have** ... $=$ *sum* $(w \sqcap (f \sqcap g)) +$ *sum* $(w \sqcap (f^T \sqcap g^T))$
**by** (*metis conv-dist-inf conv-involutive sum-conv*)
**also have** ... $=$ *sum* $(f \sqcap (w \sqcap g)) +$ *sum* $(f^T \sqcap (w \sqcap g))$
**using** *28 inf.commute inf.assoc kruskal-invariant-def kruskal-spanning-invariant-def* **by** *simp*
**also have** ... $=$ *sum* $((f \sqcup f^T) \sqcap (w \sqcap g))$
**using** *29 acyclic-asymmetric inf.sup-monoid.add-commute sum-disjoint spanning-forest-def* **by** *simp*
**also have** ... $=$ *sum* $(w \sqcap g)$
**using** *31* **by** (*metis inf-absorb2 inf.assoc*)
**finally show** *minimum-spanning-forest f g*
**using** *29 30 minimum-spanning-forest-def* **by** *simp*
**qed**

**end**

**end**

# 3 Prim's Minimum Spanning Tree Algorithm

In this theory we prove total correctness of Prim's minimum spanning tree algorithm. The proof has the same overall structure as the total-correctness proof of Kruskal's algorithm [6]. The partial-correctness proof of Prim's algorithm is discussed in [3, 5].

**theory** *Prim*

**imports** *HOL−Hoare.Hoare-Logic Aggregation-Algebras.Aggregation-Algebras*

**begin**

**context** *m-kleene-algebra*
**begin**

**abbreviation** $component\ g\ r \equiv r^T * (--g)^\star$
**definition** $spanning\text{-}tree\ t\ g\ r \equiv forest\ t \wedge t \leq (component\ g\ r)^T * (component\ g\ r) \sqcap --g \wedge component\ g\ r \leq r^T * t^\star \wedge regular\ t$
**definition** $minimum\text{-}spanning\text{-}tree\ t\ g\ r \equiv spanning\text{-}tree\ t\ g\ r \wedge (\forall u .\ spanning\text{-}tree\ u\ g\ r \longrightarrow sum\ (t \sqcap g) \leq sum\ (u \sqcap g))$
**definition** $prim\text{-}precondition\ g\ r \equiv g = g^T \wedge injective\ r \wedge vector\ r \wedge regular\ r$
**definition** $prim\text{-}spanning\text{-}invariant\ t\ v\ g\ r \equiv prim\text{-}precondition\ g\ r \wedge v^T = r^T * t^\star \wedge spanning\text{-}tree\ t\ (v * v^T \sqcap g)\ r$
**definition** $prim\text{-}invariant\ t\ v\ g\ r \equiv prim\text{-}spanning\text{-}invariant\ t\ v\ g\ r \wedge (\exists w .\ minimum\text{-}spanning\text{-}tree\ w\ g\ r \wedge t \leq w)$

**lemma** *span-tree-split*:
  **assumes** $vector\ r$
    **shows** $t \leq (component\ g\ r)^T * (component\ g\ r) \sqcap --g \longleftrightarrow (t \leq (component\ g\ r)^T \wedge t \leq component\ g\ r \wedge t \leq --g)$
**proof** –
  **have** $(component\ g\ r)^T * (component\ g\ r) = (component\ g\ r)^T \sqcap component\ g\ r$
    **by** (*metis assms conv-involutive covector-mult-closed vector-conv-covector vector-covector*)
  **thus** *?thesis*
    **by** *simp*
**qed**

**lemma** *span-tree-component*:
  **assumes** $spanning\text{-}tree\ t\ g\ r$
    **shows** $component\ g\ r = component\ t\ r$
  **using** *assms* **by** (*simp add*: *order.antisym mult-right-isotone star-isotone spanning-tree-def*)

We first show three verification conditions which are used in both correctness proofs.

**lemma** *prim-vc-1*:
  **assumes** $prim\text{-}precondition\ g\ r$
    **shows** $prim\text{-}spanning\text{-}invariant\ bot\ r\ g\ r$
**proof** (*unfold prim-spanning-invariant-def*, *intro conjI*)
  **show** $prim\text{-}precondition\ g\ r$
    **using** *assms* **by** *simp*
**next**
  **show** $r^T = r^T * bot^\star$
    **by** (*simp add*: *star-absorb*)
**next**
  **let** $?ss = r * r^T \sqcap g$
  **show** $spanning\text{-}tree\ bot\ ?ss\ r$

**proof** (*unfold spanning-tree-def*, *intro conjI*)
**show** *injective bot*
**by** *simp*
**next**
**show** *acyclic bot*
**by** *simp*
**next**
**show** $bot \leq (component\ ?ss\ r)^T * (component\ ?ss\ r) \sqcap --?ss$
**by** *simp*
**next**
**have** $component\ ?ss\ r \leq component\ (r * r^T)\ r$
**by** (*simp add*: *mult-right-isotone star-isotone*)
**also have** ... $\leq r^T * 1^\star$
**using** *assms* **by** (*metis order.eq-iff p-antitone regular-one-closed star-sub-one prim-precondition-def*)
**also have** ... $= r^T * bot^\star$
**by** (*simp add*: *star.circ-zero star-one*)
**finally show** $component\ ?ss\ r \leq r^T * bot^\star$
.
**next**
**show** *regular bot*
**by** *simp*
**qed**
**qed**

**lemma** *prim-vc-2*:
**assumes** *prim-spanning-invariant t v g r*
**and** $v * -v^T \sqcap g \neq bot$
**shows** $prim\text{-}spanning\text{-}invariant\ (t \sqcup minarc\ (v * -v^T \sqcap g))\ (v \sqcup minarc\ (v * -v^T \sqcap g)^T * top)\ g\ r \wedge card\ \{\ x\ .\ regular\ x \wedge x \leq component\ g\ r \wedge x \leq -(v \sqcup minarc\ (v * -v^T \sqcap g)^T * top)^T\ \} < card\ \{\ x\ .\ regular\ x \wedge x \leq component\ g\ r \wedge x \leq -v^T\ \}$
**proof** −
**let** $?vcv = v * -v^T \sqcap g$
**let** $?e = minarc\ ?vcv$
**let** $?t = t \sqcup ?e$
**let** $?v = v \sqcup ?e^T * top$
**let** $?c = component\ g\ r$
**let** $?g = --g$
**let** $?n1 = card\ \{\ x\ .\ regular\ x \wedge x \leq ?c \wedge x \leq -v^T\ \}$
**let** $?n2 = card\ \{\ x\ .\ regular\ x \wedge x \leq ?c \wedge x \leq -?v^T\ \}$
**have** *1*: $regular\ v \wedge regular\ (v * v^T) \wedge regular\ (?v * ?v^T) \wedge regular\ (top * ?e)$
**using** *assms*(*1*) **by** (*metis prim-spanning-invariant-def spanning-tree-def prim-precondition-def regular-conv-closed regular-closed-star regular-mult-closed conv-involutive regular-closed-top regular-closed-sup minarc-regular*)
**hence** *2*: $t \leq v * v^T \sqcap ?g$
**using** *assms*(*1*) **by** (*metis prim-spanning-invariant-def spanning-tree-def inf-pp-commute inf.boundedE*)
**hence** *3*: $t \leq v * v^T$

**by** *simp*
**have** *4*: $t \leq ?g$
**using** *2* **by** *simp*
**have** *5*: $?e \leq v * -v^T \sqcap ?g$
**using** *1* **by** (*metis minarc-below pp-dist-inf regular-mult-closed regular-closed-p*)
**hence** *6*: $?e \leq v * -v^T$
**by** *simp*
**have** *7*: *vector v*
**using** *assms*(*1*) *prim-spanning-invariant-def prim-precondition-def* **by** (*simp add*: *covector-mult-closed vector-conv-covector*)
**hence** *8*: $?e \leq v$
**using** *6* **by** (*metis conv-complement inf.boundedE vector-complement-closed vector-covector*)
**have** *9*: $?e * t = bot$
**using** *3 6 7 et*(*1*) **by** *blast*
**have** *10*: $?e * t^T = bot$
**using** *3 6 7 et*(*2*) **by** *simp*
**have** *11*: *arc ?e*
**using** *assms*(*2*) *minarc-arc* **by** *simp*
**have** $r^T \leq r^T * t^\star$
**by** (*metis mult-right-isotone order-refl semiring.mult-not-zero star.circ-separate-mult-1 star-absorb*)
**hence** *12*: $r^T \leq v^T$
**using** *assms*(*1*) **by** (*simp add*: *prim-spanning-invariant-def*)
**have** *13*: *vector r* $\wedge$ *injective r* $\wedge$ $v^T = r^T * t^\star$
**using** *assms*(*1*) *prim-spanning-invariant-def prim-precondition-def minimum-spanning-tree-def spanning-tree-def reachable-restrict* **by** *simp*
**have** $g = g^T$
**using** *assms*(*1*) *prim-invariant-def prim-spanning-invariant-def prim-precondition-def* **by** *simp*
**hence** *14*: $?g^T = ?g$
**using** *conv-complement* **by** *simp*
**show** *prim-spanning-invariant ?t ?v g r* $\wedge$ $?n2 < ?n1$
**proof** (*rule conjI*)
**show** *prim-spanning-invariant ?t ?v g r*
**proof** (*unfold prim-spanning-invariant-def*, *intro conjI*)
**show** *prim-precondition g r*
**using** *assms*(*1*) *prim-spanning-invariant-def* **by** *simp*
**next**
**show** $?v^T = r^T * ?t^\star$
**using** *assms*(*1*) *6 7 9* **by** (*simp add*: *reachable-inv prim-spanning-invariant-def prim-precondition-def spanning-tree-def*)
**next**
**let** $?G = ?v * ?v^T \sqcap g$
**show** *spanning-tree ?t ?G r*
**proof** (*unfold spanning-tree-def*, *intro conjI*)
**show** *injective ?t*
**using** *assms*(*1*) *10 11* **by** (*simp add*: *injective-inv*

*prim-spanning-invariant-def spanning-tree-def*)
**next**
**show** *acyclic ?t*
**using** *assms*(*1*) *3 6 7 acyclic-inv prim-spanning-invariant-def spanning-tree-def* **by** *simp*
**next**
**show** $?t \leq (component\ ?G\ r)^T * (component\ ?G\ r) \sqcap --?G$
**using** *1 2 5 7 13 prim-subgraph-inv inf-pp-commute mst-subgraph-inv-2* **by** *auto*
**next**
**show** $component\ (?v * ?v^T \sqcap g)\ r \leq r^T * ?t^\star$
**proof** −
**have** *15*: $r^T * (v * v^T \sqcap ?g)^\star \leq r^T * t^\star$
**using** *assms*(*1*) *1* **by** (*metis prim-spanning-invariant-def spanning-tree-def inf-pp-commute*)
**have** $component\ (?v * ?v^T \sqcap g)\ r = r^T * (?v * ?v^T \sqcap ?g)^\star$
**using** *1* **by** *simp*
**also have** $... \leq r^T * ?t^\star$
**using** *2 6 7 11 12 13 14 15* **by** (*metis span-inv*)
**finally show** *?thesis*
**.**
**qed**
**next**
**show** *regular ?t*
**using** *assms*(*1*) **by** (*metis prim-spanning-invariant-def spanning-tree-def regular-closed-sup minarc-regular*)
**qed**
**qed**
**next**
**have** *16*: $top * ?e \leq ?c$
**proof** −
**have** $top * ?e = top * ?e^T * ?e$
**using** *11* **by** (*metis arc-top-edge mult-assoc*)
**also have** $... \leq v^T * ?e$
**using** *7 8* **by** (*metis conv-dist-comp conv-isotone mult-left-isotone symmetric-top-closed*)
**also have** $... \leq v^T * ?g$
**using** *5 mult-right-isotone* **by** *auto*
**also have** $... = r^T * t^\star * ?g$
**using** *13* **by** *simp*
**also have** $... \leq r^T * ?g^\star * ?g$
**using** *4* **by** (*simp add*: *mult-left-isotone mult-right-isotone star-isotone*)
**also have** $... \leq ?c$
**by** (*simp add*: *comp-associative mult-right-isotone star.right-plus-below-circ*)
**finally show** *?thesis*
**by** *simp*
**qed**
**have** *17*: $top * ?e \leq -v^T$
**using** *6 7* **by** (*simp add*: *schroeder-4-p vTeT*)

**have** *18*: $\neg$ *top* $*$ *?e* $\leq -($*top* $*$ *?e*$)$
**by** (*metis assms*(*2*) *inf.orderE minarc-bot-iff conv-complement-sub-inf inf-p inf-top.left-neutral p-bot symmetric-top-closed vector-top-closed*)
**have** *19*: $-?v^T = -v^T \sqcap -(top * ?e)$
**by** (*simp add*: *conv-dist-comp conv-dist-sup*)
**hence** *20*: $\neg$ *top* $*$ *?e* $\leq -?v^T$
**using** *18* **by** *simp*
**show** *?n2* $<$ *?n1*
**apply** (*rule psubset-card-mono*)
**using** *finite-regular* **apply** *simp*
**using** *1 16 17 19 20* **by** *auto*
**qed**
**qed**

**lemma** *prim-vc-3*:
**assumes** *prim-spanning-invariant t v g r*
**and** $v * -v^T \sqcap g = bot$
**shows** *spanning-tree t g r*
**proof** $-$
**let** $?g = --g$
**have** *1*: *regular v* $\wedge$ *regular* $(v * v^T)$
**using** *assms*(*1*) **by** (*metis prim-spanning-invariant-def spanning-tree-def prim-precondition-def regular-conv-closed regular-closed-star regular-mult-closed conv-involutive*)
**have** *2*: $v * -v^T \sqcap ?g = bot$
**using** *assms*(*2*) *pp-inf-bot-iff pp-pp-inf-bot-iff* **by** *simp*
**have** *3*: $v^T = r^T * t^\star \wedge$ *vector v*
**using** *assms*(*1*) **by** (*simp add*: *covector-mult-closed prim-invariant-def prim-spanning-invariant-def vector-conv-covector prim-precondition-def*)
**have** *4*: $t \leq v * v^T \sqcap ?g$
**using** *assms*(*1*) *1* **by** (*metis prim-spanning-invariant-def inf-pp-commute spanning-tree-def inf.boundedE*)
**have** $r^T * (v * v^T \sqcap ?g)^\star \leq r^T * t^\star$
**using** *assms*(*1*) *1* **by** (*metis prim-spanning-invariant-def inf-pp-commute spanning-tree-def*)
**hence** *5*: *component g r* $= v^T$
**using** *1 2 3 4* **by** (*metis span-post*)
**have** *regular* $(v * v^T)$
**using** *assms*(*1*) **by** (*metis prim-spanning-invariant-def spanning-tree-def prim-precondition-def regular-conv-closed regular-closed-star regular-mult-closed conv-involutive*)
**hence** *6*: $t \leq v * v^T \sqcap ?g$
**by** (*metis assms*(*1*) *prim-spanning-invariant-def spanning-tree-def inf-pp-commute inf.boundedE*)
**show** *spanning-tree t g r*
**apply** (*unfold spanning-tree-def*, *intro conjI*)
**using** *assms*(*1*) *prim-spanning-invariant-def spanning-tree-def* **apply** *simp*
**using** *assms*(*1*) *prim-spanning-invariant-def spanning-tree-def* **apply** *simp*
**using** *5 6* **apply** *simp*

**using** *assms*(*1*) *5* *prim-spanning-invariant-def* **apply** *simp*
**using** *assms*(*1*) *prim-spanning-invariant-def* *spanning-tree-def* **by** *simp*
**qed**

The following result shows that Prim's algorithm terminates and constructs a spanning tree. We cannot yet show that this is a minimum spanning tree.

**theorem** *prim-spanning*:
$VARS\ t\ v\ e$
$[\ prim\text{-}precondition\ g\ r\ ]$
$t := bot;$
$v := r;$
$WHILE\ v * -v^T \sqcap g \neq bot$
$INV\ \{\ prim\text{-}spanning\text{-}invariant\ t\ v\ g\ r\ \}$
$VAR\ \{\ card\ \{\ x\ .\ regular\ x \wedge x \leq component\ g\ r \sqcap -v^T\ \}\ \}$
$DO\ e := minarc\ (v * -v^T \sqcap g);$
$t := t \sqcup e;$
$v := v \sqcup e^T * top$
$OD$
$[\ spanning\text{-}tree\ t\ g\ r\ ]$
**apply** *vcg-tc-simp*
**apply** (*simp add*: *prim-vc-1*)
**using** *prim-vc-2* **apply** *blast*
**using** *prim-vc-3* **by** *auto*

Because we have shown total correctness, we conclude that a spanning tree exists.

**lemma** *prim-exists-spanning*:
$prim\text{-}precondition\ g\ r \Longrightarrow \exists t\ .\ spanning\text{-}tree\ t\ g\ r$
**using** *tc-extract-function* *prim-spanning* **by** *blast*

This implies that a minimum spanning tree exists, which is used in the subsequent correctness proof.

**lemma** *prim-exists-minimal-spanning*:
**assumes** *prim-precondition g r*
**shows** $\exists t\ .\ minimum\text{-}spanning\text{-}tree\ t\ g\ r$
**proof** −
**let** $?s = \{\ t\ .\ spanning\text{-}tree\ t\ g\ r\ \}$
**have** $\exists m \in ?s\ .\ \forall z \in ?s\ .\ sum\ (m \sqcap g) \leq sum\ (z \sqcap g)$
**apply** (*rule finite-set-minimal*)
**using** *finite-regular* *spanning-tree-def* **apply** *simp*
**using** *assms* *prim-exists-spanning* **apply** *simp*
**using** *sum-linear* **by** *simp*
**thus** *?thesis*
**using** *minimum-spanning-tree-def* **by** *simp*
**qed**

Prim's minimum spanning tree algorithm terminates and is correct. This is the same algorithm that is used in the previous correctness proof, with the same precondition and variant, but with a different invariant and postcondition.

**theorem** *prim*:
*VARS t v e*
[ *prim-precondition g r* $\wedge$ ($\exists$ *w* . *minimum-spanning-tree w g r*) ]
$t := bot$;
$v := r$;
*WHILE* $v * -v^T \sqcap g \neq bot$
*INV* { *prim-invariant t v g r* }
*VAR* { *card* { $x$ . *regular* $x \wedge x \leq$ *component g r* $\sqcap -v^T$ } }
*DO* $e :=$ *minarc* $(v * -v^T \sqcap g)$;
$t := t \sqcup e$;
$v := v \sqcup e^T * top$
*OD*
[ *minimum-spanning-tree t g r* ]
**proof** *vcg-tc-simp*
**assume** *prim-precondition g r* $\wedge$ ($\exists$ *w* . *minimum-spanning-tree w g r*)
**thus** *prim-invariant bot r g r*
**using** *prim-invariant-def prim-vc-1* **by** *simp*
**next**
**fix** *t v*
**let** $?vcv = v * -v^T \sqcap g$
**let** $?vv = v * v^T \sqcap g$
**let** $?e =$ *minarc* $?vcv$
**let** $?t = t \sqcup ?e$
**let** $?v = v \sqcup ?e^T * top$
**let** $?c =$ *component g r*
**let** $?g = --g$
**let** $?n1 =$ *card* { $x$ . *regular* $x \wedge x \leq ?c \wedge x \leq -v^T$ }
**let** $?n2 =$ *card* { $x$ . *regular* $x \wedge x \leq ?c \wedge x \leq -?v^T$ }
**assume** *1*: *prim-invariant t v g r* $\wedge$ $?vcv \neq bot$
**hence** *2*: *regular* $v \wedge$ *regular* $(v * v^T)$
**by** (*metis* (*no-types*, *opaque-lifting*) *prim-invariant-def*
*prim-spanning-invariant-def spanning-tree-def prim-precondition-def*
*regular-conv-closed regular-closed-star regular-mult-closed conv-involutive*)
**have** *3*: $t \leq v * v^T \sqcap ?g$
**using** *1 2* **by** (*metis* (*no-types*, *opaque-lifting*) *prim-invariant-def*
*prim-spanning-invariant-def spanning-tree-def inf-pp-commute inf.boundedE*)
**hence** *4*: $t \leq v * v^T$
**by** *simp*
**have** *5*: $t \leq ?g$
**using** *3* **by** *simp*
**have** *6*: $?e \leq v * -v^T \sqcap ?g$
**using** *2* **by** (*metis minarc-below pp-dist-inf regular-mult-closed*
*regular-closed-p*)
**hence** *7*: $?e \leq v * -v^T$
**by** *simp*
**have** *8*: *vector v*
**using** *1 prim-invariant-def prim-spanning-invariant-def prim-precondition-def*

**by** (*simp add*: *covector-mult-closed vector-conv-covector*)
  **have** *9*: *arc ?e*
    **using** *1 minarc-arc* **by** *simp*
  **from** *1* **obtain** $w$ **where** *10*: *minimum-spanning-tree* $w\ g\ r \wedge t \leq w$
    **by** (*metis prim-invariant-def*)
  **hence** *11*: *vector* $r \wedge$ *injective* $r \wedge v^T = r^T * t^\star \wedge$ *forest* $w \wedge t \leq w \wedge w \leq ?c^T * ?c \sqcap ?g \wedge r^T * (?c^T * ?c \sqcap ?g)^\star \leq r^T * w^\star$
    **using** *1 2 prim-invariant-def prim-spanning-invariant-def prim-precondition-def minimum-spanning-tree-def spanning-tree-def reachable-restrict* **by** *simp*
  **hence** *12*: $w * v \leq v$
    **using** *predecessors-reachable reachable-restrict* **by** *auto*
  **have** *13*: $g = g^T$
    **using** *1 prim-invariant-def prim-spanning-invariant-def prim-precondition-def* **by** *simp*
  **hence** *14*: $?g^T = ?g$
    **using** *conv-complement* **by** *simp*
  **show** *prim-invariant ?t ?v g r* $\wedge$ *?n2* $<$ *?n1*
  **proof** (*unfold prim-invariant-def*, *intro conjI*)
    **show** *prim-spanning-invariant ?t ?v g r*
      **using** *1 prim-invariant-def prim-vc-2* **by** *blast*
  **next**
    **show** $\exists\, w$ . *minimum-spanning-tree* $w\ g\ r \wedge ?t \leq w$
    **proof**
      **let** $?f = w \sqcap v * -v^T \sqcap top * ?e * w^{T\star}$
      **let** $?p = w \sqcap -v * -v^T \sqcap top * ?e * w^{T\star}$
      **let** $?fp = w \sqcap -v^T \sqcap top * ?e * w^{T\star}$
      **let** $?w = (w \sqcap -?fp) \sqcup ?p^T \sqcup ?e$
      **have** *15*: *regular ?f* $\wedge$ *regular ?fp* $\wedge$ *regular ?w*
        **using** *2 10* **by** (*metis regular-conv-closed regular-closed-star regular-mult-closed regular-closed-top regular-closed-inf regular-closed-sup minarc-regular minimum-spanning-tree-def spanning-tree-def*)
      **show** *minimum-spanning-tree ?w g r* $\wedge$ $?t \leq ?w$
      **proof** (*intro conjI*)
        **show** *minimum-spanning-tree ?w g r*
        **proof** (*unfold minimum-spanning-tree-def*, *intro conjI*)
          **show** *spanning-tree ?w g r*
          **proof** (*unfold spanning-tree-def*, *intro conjI*)
            **show** *injective ?w*
              **using** *7 8 9 11 exchange-injective* **by** *blast*
          **next**
            **show** *acyclic ?w*
              **using** *7 8 11 12 exchange-acyclic* **by** *blast*
          **next**
            **show** $?w \leq ?c^T * ?c \sqcap --g$
            **proof** $-$
              **have** *16*: $w \sqcap -?fp \leq ?c^T * ?c \sqcap --g$
                **using** *10* **by** (*simp add*: *le-infI1 minimum-spanning-tree-def spanning-tree-def*)

**have** $?p^T \leq w^T$
**by** $($*simp add*: *conv-isotone inf.sup-monoid.add-assoc*$)$
**also have** ... $\leq (?c^T * ?c \sqcap --g)^T$
**using** *11 conv-order* **by** *simp*
**also have** ... $= ?c^T * ?c \sqcap --g$
**using** *2 14 conv-dist-comp conv-dist-inf* **by** *simp*
**finally have** *17*: $?p^T \leq ?c^T * ?c \sqcap --g$
**.**
**have** $?e \leq ?c^T * ?c \sqcap ?g$
**using** *5 6 11 mst-subgraph-inv* **by** *auto*
**thus** *?thesis*
**using** *16 17* **by** *simp*
**qed**
**next**
**show** $?c \leq r^T * ?w^\star$
**proof** $-$
**have** $?c \leq r^T * w^\star$
**using** *10 minimum-spanning-tree-def spanning-tree-def* **by** *simp*
**also have** ... $\leq r^T * ?w^\star$
**using** *4 7 8 10 11 12 15* **by** $($*metis mst-reachable-inv*$)$
**finally show** *?thesis*
**.**
**qed**
**next**
**show** *regular ?w*
**using** *15* **by** *simp*
**qed**
**next**
**have** *18*: $?f \sqcup ?p = ?fp$
**using** *2 8 epm-1* **by** *fastforce*
**have** *arc* $(w \sqcap --v * -v^T \sqcap top * ?e * w^{T\star})$
**using** *5 6 8 9 11 12 reachable-restrict arc-edge* **by** *auto*
**hence** *19*: *arc ?f*
**using** *2* **by** *simp*
**hence** $?f = bot \longrightarrow top = bot$
**by** $($*metis mult-left-zero mult-right-zero*$)$
**hence** $?f \neq bot$
**using** *1 le-bot* **by** *auto*
**hence** $?f \sqcap v * -v^T \sqcap ?g \neq bot$
**using** *2 11* **by** $($*simp add*: *inf.absorb1 le-infI1*$)$
**hence** $g \sqcap (?f \sqcap v * -v^T) \neq bot$
**using** *inf-commute pp-inf-bot-iff* **by** *simp*
**hence** *20*: $?f \sqcap ?vcv \neq bot$
**by** $($*simp add*: *inf-assoc inf-commute*$)$
**hence** *21*: $?f \sqcap g = ?f \sqcap ?vcv$
**using** *2* **by** $($*simp add*: *inf-assoc inf-commute inf-left-commute*$)$
**have** *22*: $?e \sqcap g = minarc\ ?vcv \sqcap ?vcv$
**using** *7* **by** $($*simp add*: *inf.absorb2 inf.assoc inf.commute*$)$
**hence** *23*: $sum\ (?e \sqcap g) \leq sum\ (?f \sqcap g)$

**using** *15 19 20 21* **by** (*simp add*: *minarc-min*)
**have** $?e \neq bot$
**using** *20 comp-inf.semiring.mult-not-zero semiring.mult-not-zero* **by** *blast*
**hence** *24*: $?e \sqcap g \neq bot$
**using** *22 minarc-meet-bot* **by** *auto*
**have** $sum\ (?w \sqcap g) = sum\ (w \sqcap -?fp \sqcap g) + sum\ (?p^T \sqcap g) + sum\ (?e \sqcap g)$
**using** *7 8 10* **by** (*metis sum-disjoint-3 epm-8 epm-9 epm-10 minimum-spanning-tree-def spanning-tree-def*)
**also have** ... $= sum\ (((w \sqcap -?fp) \sqcup ?p^T) \sqcap g) + sum\ (?e \sqcap g)$
**using** *11* **by** (*metis epm-8 sum-disjoint*)
**also have** ... $\leq sum\ (((w \sqcap -?fp) \sqcup ?p^T) \sqcap g) + sum\ (?f \sqcap g)$
**using** *23 24* **by** (*simp add*: *sum-plus-right-isotone*)
**also have** ... $= sum\ (w \sqcap -?fp \sqcap g) + sum\ (?p^T \sqcap g) + sum\ (?f \sqcap g)$
**using** *11* **by** (*metis epm-8 sum-disjoint*)
**also have** ... $= sum\ (w \sqcap -?fp \sqcap g) + sum\ (?p \sqcap g) + sum\ (?f \sqcap g)$
**using** *13 sum-symmetric* **by** *auto*
**also have** ... $= sum\ (((w \sqcap -?fp) \sqcup ?p \sqcup ?f) \sqcap g)$
**using** *2 8* **by** (*metis sum-disjoint-3 epm-11 epm-12 epm-13*)
**also have** ... $= sum\ (w \sqcap g)$
**using** *2 8 15 18 epm-2* **by** *force*
**finally have** $sum\ (?w \sqcap g) \leq sum\ (w \sqcap g)$
**.**
**thus** $\forall u\ .\ spanning\text{-}tree\ u\ g\ r \longrightarrow sum\ (?w \sqcap g) \leq sum\ (u \sqcap g)$
**using** *10 order-lesseq-imp minimum-spanning-tree-def* **by** *auto*
**qed**
**next**
**show** $?t \leq ?w$
**using** *4 8 10 mst-extends-new-tree* **by** *simp*
**qed**
**qed**
**next**
**show** $?n2 < ?n1$
**using** *1 prim-invariant-def prim-vc-2* **by** *auto*
**qed**
**next**
**fix** *t v*
**let** $?g = --g$
**assume** *25*: $prim\text{-}invariant\ t\ v\ g\ r \land v * -v^T \sqcap g = bot$
**hence** *26*: *regular v*
**by** (*metis prim-invariant-def prim-spanning-invariant-def spanning-tree-def prim-precondition-def regular-conv-closed regular-closed-star regular-mult-closed conv-involutive*)
**from** *25* **obtain** *w* **where** *27*: $minimum\text{-}spanning\text{-}tree\ w\ g\ r \land t \leq w$
**by** (*metis prim-invariant-def*)
**have** *spanning-tree t g r*
**using** *25 prim-invariant-def prim-vc-3* **by** *blast*
**hence** $component\ g\ r = v^T$

**by** (*metis 25 prim-invariant-def span-tree-component prim-spanning-invariant-def spanning-tree-def*)
**hence** *28*: $w \leq v * v^T$
**using** *26 27* **by** (*simp add*: *minimum-spanning-tree-def spanning-tree-def inf-pp-commute*)
**have** *vector* $r \wedge$ *injective* $r \wedge$ *forest* $w$
**using** *25 27* **by** (*simp add*: *prim-invariant-def prim-spanning-invariant-def prim-precondition-def minimum-spanning-tree-def spanning-tree-def*)
**hence** $w = t$
**using** *25 27 28 prim-invariant-def prim-spanning-invariant-def mst-post* **by** *blast*
**thus** *minimum-spanning-tree* $t\ g\ r$
**using** *27* **by** *simp*
**qed**

**end**

**end**

# 4 Borůvka's Minimum Spanning Tree Algorithm

In this theory we prove partial correctness of Borůvka's minimum spanning tree algorithm.

**theory** *Boruvka*

**imports**
*Aggregation-Algebras.M-Choose-Component*
*Relational-Disjoint-Set-Forests.Disjoint-Set-Forests*
*Kruskal*

**begin**

## 4.1 General results

The proof is carried out in $m$-$k$-Stone-Kleene relation algebras. In this section we give results that hold more generally.

**context** *stone-kleene-relation-algebra*
**begin**

**lemma** *He-eq-He-THe-star*:
**assumes** *arc e*
**and** *equivalence H*
**shows** $H * e = H * e * (top * H * e)^\star$
**proof** –
**let** $?x = H * e$
**have** *1*: $H * e \leq H * e * (top * H * e)^\star$
**using** *comp-isotone star.circ-reflexive* **by** *fastforce*
**have** $H * e * (top * H * e)^\star = H * e * (top * e)^\star$

**by** (*metis assms*(*2*) *preorder-idempotent surjective-var*)
**also have** ... $\leq H * e * (1 \sqcup top * (e * top)^\star * e)$
**by** (*metis eq-refl star.circ-mult-1*)
**also have** ... $\leq H * e * (1 \sqcup top * top * e)$
**by** (*simp add*: *star.circ-left-top*)
**also have** ... $= H * e \sqcup H * e * top * e$
**by** (*simp add*: *mult.semigroup-axioms semiring.distrib-left semigroup.assoc*)
**finally have** *2*: $H * e * (top * H * e)^\star \leq H * e$
**using** *assms arc-top-arc mult-assoc* **by** *auto*
**thus** *?thesis*
**using** *1 2* **by** *simp*
**qed**

**lemma** *path-through-components*:
**assumes** *equivalence H*
**and** *arc e*
**shows** $(H * (d \sqcup e))^\star = (H * d)^\star \sqcup (H * d)^\star * H * e * (H * d)^\star$
**proof** −
**have** $H * e * (H * d)^\star * H * e \leq H * e * top * H * e$
**by** (*simp add*: *comp-isotone*)
**also have** ... $= H * e * top * e$
**by** (*metis assms*(*1*) *preorder-idempotent surjective-var mult-assoc*)
**also have** ... $= H * e$
**using** *assms*(*2*) *arc-top-arc mult-assoc* **by** *auto*
**finally have** *1*: $H * e * (H * d)^\star * H * e \leq H * e$
**by** *simp*
**have** $(H * (d \sqcup e))^\star = (H * d \sqcup H * e)^\star$
**by** (*simp add*: *comp-left-dist-sup*)
**also have** ... $= (H * d)^\star \sqcup (H * d)^\star * H * e * (H * d)^\star$
**using** *1 star-separate-3* **by** (*simp add*: *mult-assoc*)
**finally show** *?thesis*
**by** *simp*
**qed**

**lemma** *simplify-f*:
**assumes** *regular p*
**and** *regular e*
**shows** $(f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap - e^T \sqcap p) \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup (f \sqcap - e^T \sqcap - p)^T \sqcup e^T \sqcup e = f \sqcup f^T \sqcup e \sqcup e^T$
**proof** −
**have** $(f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap - e^T \sqcap p) \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup (f \sqcap - e^T \sqcap - p)^T \sqcup e^T \sqcup e$
$= (f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap - e^T \sqcap p) \sqcup (f^T \sqcap - e \sqcap p^T) \sqcup (f^T \sqcap - e \sqcap - p^T) \sqcup e^T \sqcup e$
**by** (*simp add*: *conv-complement conv-dist-inf*)
**also have** ... =
$((f \sqcup (f \sqcap - e^T \sqcap p)) \sqcap (- e^T \sqcup (f \sqcap - e^T \sqcap p)) \sqcap (- p \sqcup (f \sqcap - e^T \sqcap p)))$
$\sqcup ((f^T \sqcup (f^T \sqcap - e \sqcap - p^T)) \sqcap (- e \sqcup (f^T \sqcap - e \sqcap - p^T)) \sqcap (p^T \sqcup (f^T \sqcap - e \sqcap - p^T)))$

$\sqcup\ e^T \sqcup e$
**by** (*metis sup-inf-distrib2 sup-assoc*)
**also have** ... =
$((f \sqcup f) \sqcap (f \sqcup -\ e^T) \sqcap (f \sqcup p) \sqcap (-\ e^T \sqcup f) \sqcap (-\ e^T \sqcup -\ e^T) \sqcap (-\ e^T \sqcup p) \sqcap (-\ p \sqcup f) \sqcap (-\ p \sqcup -\ e^T) \sqcap (-\ p \sqcup p))$
$\sqcup ((f^T \sqcup f^T) \sqcap (f^T \sqcup -\ e) \sqcap (f^T \sqcup -\ p^T) \sqcap (-\ e \sqcup f^T) \sqcap (-\ e \sqcup -\ e) \sqcap (-\ e \sqcup -\ p^T) \sqcap (p^T \sqcup f^T) \sqcap (p^T \sqcup -\ e) \sqcap (p^T \sqcup -\ p^T))$
$\sqcup\ e^T \sqcup e$
**using** *sup-inf-distrib1 sup-assoc inf-assoc sup-inf-distrib1* **by** *simp*
**also have** ... =
$((f \sqcup f) \sqcap (f \sqcup -\ e^T) \sqcap (f \sqcup p) \sqcap (f \sqcup -\ p) \sqcap (-\ e^T \sqcup f) \sqcap (-\ e^T \sqcup -\ e^T) \sqcap (-\ e^T \sqcup p) \sqcap (-\ e^T \sqcup -\ p) \sqcap (-\ p \sqcup p))$
$\sqcup ((f^T \sqcup f^T) \sqcap (f^T \sqcup -\ e) \sqcap (f^T \sqcup -\ p^T) \sqcap (-\ e \sqcup f^T) \sqcap (f^T \sqcup p^T) \sqcap (-\ e \sqcup -\ e) \sqcap (-\ e \sqcup -\ p^T) \sqcap (-\ e \sqcup p^T) \sqcap (p^T \sqcup -\ p^T))$
$\sqcup\ e^T \sqcup e$
**by** (*smt abel-semigroup.commute inf.abel-semigroup-axioms inf.left-commute sup.abel-semigroup-axioms*)
**also have** ... = $(f \sqcap -\ e^T \sqcap (-\ p \sqcup p)) \sqcup (f^T \sqcap -\ e \sqcap (p^T \sqcup -\ p^T)) \sqcup e^T \sqcup e$
**by** (*smt inf.sup-monoid.add-assoc inf.sup-monoid.add-commute inf-sup-absorb sup.idem*)
**also have** ... = $(f \sqcap -\ e^T) \sqcup (f^T \sqcap -\ e) \sqcup e^T \sqcup e$
**by** (*metis assms*(*1*) *conv-complement inf-top-right stone*)
**also have** ... = $(f \sqcup e^T) \sqcap (-\ e^T \sqcup e^T) \sqcup (f^T \sqcup e) \sqcap (-\ e \sqcup e)$
**by** (*metis sup.left-commute sup-assoc sup-inf-distrib2*)
**finally show** *?thesis*
**by** (*metis abel-semigroup.commute assms*(*2*) *conv-complement inf-top-right stone sup.abel-semigroup-axioms sup-assoc*)
**qed**

**lemma** *simplify-forest-components-f*:
**assumes** *regular p*
**and** *regular e*
**and** *injective* $(f \sqcap -\ e^T \sqcap -\ p \sqcup (f \sqcap -\ e^T \sqcap p)^T \sqcup e)$
**and** *injective f*
**shows** *forest-components* $((f \sqcap -\ e^T \sqcap -\ p) \sqcup (f \sqcap -e^T \sqcap p)^T \sqcup e) = (f \sqcup f^T \sqcup e \sqcup e^T)^\star$
**proof** −
**have** *forest-components* $((f \sqcap -\ e^T \sqcap -\ p) \sqcup (f \sqcap -e^T \sqcap p)^T \sqcup e) = wcc\ ((f \sqcap -\ e^T \sqcap -\ p) \sqcup (f \sqcap -\ e^T \sqcap p)^T \sqcup e)$
**by** (*simp add*: *assms*(*3*) *forest-components-wcc*)
**also have** ... = $((f \sqcap -\ e^T \sqcap -\ p) \sqcup (f \sqcap -\ e^T \sqcap p)^T \sqcup e \sqcup (f \sqcap -\ e^T \sqcap -\ p)^T \sqcup (f \sqcap -\ e^T \sqcap p) \sqcup e^T)^\star$
**using** *conv-dist-sup sup-assoc* **by** *auto*
**also have** ... = $((f \sqcap -\ e^T \sqcap -\ p) \sqcup (f \sqcap -\ e^T \sqcap p) \sqcup (f \sqcap -\ e^T \sqcap p)^T \sqcup (f \sqcap -\ e^T \sqcap -\ p)^T \sqcup e^T \sqcup e)^\star$
**using** *sup-assoc sup-commute* **by** *auto*
**also have** ... = $(f \sqcup f^T \sqcup e \sqcup e^T)^\star$
**using** *assms*(*1*, *2*, *3*, *4*) *simplify-f* **by** *auto*
**finally show** *?thesis*

**by** *simp*
**qed**

**lemma** *components-disj-increasing*:
**assumes** *regular p*
**and** *regular e*
**and** *injective* $(f \sqcap - e^T \sqcap - p \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup e)$
**and** *injective f*
**shows** *forest-components f* $\leq$ *forest-components* $(f \sqcap - e^T \sqcap - p \sqcup (f \sqcap - e^T \sqcap p)^T \sqcup e)$
**proof** $-$
**have** *1*: *forest-components* $((f \sqcap - e^T \sqcap - p) \sqcup (f \sqcap -e^T \sqcap p)^T \sqcup e) = (f \sqcup f^T \sqcup e \sqcup e^T)^\star$
**using** *simplify-forest-components-f assms*(*1*, *2*, *3*, *4*) **by** *blast*
**have** *forest-components f* $=$ *wcc f*
**by** (*simp add*: *assms*(*4*) *forest-components-wcc*)
**also have** ... $\leq (f \sqcup f^T \sqcup e^T \sqcup e)^\star$
**by** (*simp add*: *le-supI2 star-isotone sup-commute*)
**finally show** *?thesis*
**using** *1 sup.left-commute sup-commute* **by** *simp*
**qed**

**lemma** *fch-equivalence*:
**assumes** *forest h*
**shows** *equivalence* (*forest-components h*)
**by** (*simp add*: *assms forest-components-equivalence*)

**lemma** *forest-modulo-equivalence-path-split-1*:
**assumes** *arc a*
**and** *equivalence H*
**shows** $(H * d)^\star * H * a * top = (H * (d \sqcap - a))^\star * H * a * top$
**proof** $-$
**let** $?H = H$
**let** $?x = ?H * (d \sqcap -a)$
**let** $?y = ?H * a$
**let** $?a = ?H * a * top$
**let** $?d = ?H * d$
**have** *1*: $?d^\star * ?a \leq ?x^\star * ?a$
**proof** $-$
**have** $?x^\star * ?y * ?x^\star * ?a \leq ?x^\star * ?a * ?a$
**by** (*smt mult-left-isotone star.circ-right-top top-right-mult-increasing mult-assoc*)
**also have** ... $= ?x^\star * ?a * a * top$
**by** (*metis ex231e mult-assoc*)
**also have** ... $= ?x^\star * ?a$
**by** (*simp add*: *assms*(*1*) *mult-assoc*)
**finally have** *11*: $?x^\star * ?y * ?x^\star * ?a \leq ?x^\star * ?a$
**by** *simp*
**have** $?d^\star * ?a = (?H * (d \sqcap a) \sqcup ?H * (d \sqcap -a))^\star * ?a$

**proof** −
**have** *12*: *regular a*
**using** *assms*(*1*) *arc-regular* **by** *simp*
**have** $?H * ((d \sqcap a) \sqcup (d \sqcap - a)) = ?H * (d \sqcap top)$
**using** *12* **by** (*metis inf-top-right maddux-3-11-pp*)
**thus** *?thesis*
**using** *mult-left-dist-sup* **by** *auto*
**qed**
**also have** ... $\leq (?y \sqcup ?x)^\star * ?a$
**by** (*metis comp-inf.coreflexive-idempotent comp-isotone inf.cobounded1 inf.sup-monoid.add-commute semiring.add-mono star-isotone top.extremum*)
**also have** ... $= (?x \sqcup ?y)^\star * ?a$
**by** (*simp add*: *sup-commute mult-assoc*)
**also have** ... $= ?x^\star * ?a \sqcup (?x^\star * ?y * (?x^\star * ?y)^\star * ?x^\star) * ?a$
**by** (*smt mult-right-dist-sup star.circ-sup-9 star.circ-unfold-sum mult-assoc*)
**also have** ... $\leq ?x^\star * ?a \sqcup (?x^\star * ?y * (top * ?y)^\star * ?x^\star) * ?a$
**proof** −
**have** $(?x^\star * ?y)^\star \leq (top * ?y)^\star$
**by** (*simp add*: *mult-left-isotone star-isotone*)
**thus** *?thesis*
**by** (*metis comp-inf.coreflexive-idempotent comp-inf.transitive-star eq-refl mult-left-dist-sup top.extremum mult-assoc*)
**qed**
**also have** ... $= ?x^\star * ?a \sqcup (?x^\star * ?y * ?x^\star) * ?a$
**using** *assms*(*1*, *2*) *He-eq-He-THe-star arc-regular mult-assoc* **by** *auto*
**finally have** *13*: $(?H * d)^\star * ?a \leq ?x^\star * ?a \sqcup ?x^\star * ?y * ?x^\star * ?a$
**by** (*simp add*: *mult-assoc*)
**have** *14*: $?x^\star * ?y * ?x^\star * ?a \leq ?x^\star * ?a$
**using** *11 mult-assoc* **by** *auto*
**thus** *?thesis*
**using** *13 14 sup.absorb1* **by** *auto*
**qed**
**have** *2*: $?d^\star * ?a \geq ?x^\star * ?a$
**by** (*simp add*: *comp-isotone star-isotone*)
**thus** *?thesis*
**using** *1 2 order.antisym mult-assoc* **by** *simp*
**qed**

**lemma** *dTransHd-le-1*:
**assumes** *equivalence H*
**and** *univalent* $(H * d)$
**shows** $d^T * H * d \leq 1$
**proof** −
**have** $d^T * H^T * H * d \leq 1$
**using** *assms*(*2*) *conv-dist-comp mult-assoc* **by** *auto*
**thus** *?thesis*
**using** *assms*(*1*) *mult-assoc* **by** (*simp add*: *preorder-idempotent*)
**qed**

**lemma** *HcompaT-le-compHaT*:
**assumes** *equivalence* $H$
**and** *injective* $(a * top)$
**shows** $-H * a * top \leq - (H * a * top)$
**proof** −
**have** $a * top * a^T \leq 1$
**by** (*metis assms*(*2*) *conv-dist-comp symmetric-top-closed vector-top-closed mult-assoc*)
**hence** $a * top * a^T * H \leq H$
**using** *assms*(*1*) *comp-isotone order-trans* **by** *blast*
**hence** $a * top * top * a^T * H \leq H$
**by** (*simp add*: *vector-mult-closed*)
**hence** $a * top * (H * a * top)^T \leq H$
**by** (*metis assms*(*1*) *conv-dist-comp symmetric-top-closed vector-top-closed mult-assoc*)
**thus** *?thesis*
**using** *assms*(*2*) *comp-injective-below-complement mult-assoc* **by** *auto*
**qed**

## 4.2 Forests modulo an equivalence

In the graphical interpretation, the arcs of d are directed towards the root(s) of the *forest-modulo-equivalence*.

**definition** *forest-modulo-equivalence* $x$ $d$ $\equiv$ *equivalence* $x \wedge$ *univalent* $(x * d) \wedge x \sqcap d * d^T \leq 1 \wedge (x * d)^+ \sqcap x \leq bot$

**definition** *forest-modulo-equivalence-path* $a$ $b$ $H$ $d$ $\equiv$ *arc* $a \wedge$ *arc* $b \wedge a^T * top \leq (H * d)^\star * H * b * top$

**lemma** *d-separates-forest-modulo-equivalence-1*:
**assumes** *forest-modulo-equivalence* $x$ $d$
**shows** $x * d \leq -x$
**proof** −
**have** $x * d \leq (x * d)^+$
**using** *star.circ-mult-increasing* **by** *simp*
**also have** ... $\leq -x$
**using** *assms*(*1*) *forest-modulo-equivalence-def le-bot pseudo-complement* **by** *blast*
**finally show** *?thesis*
**by** *simp*
**qed**

**lemma** *d-separates-forest-modulo-equivalence-2*:
**shows** *forest-modulo-equivalence* $x$ $d \Longrightarrow d * x \leq -x$
**using** *forest-modulo-equivalence-def schroeder-6-p d-separates-forest-modulo-equivalence-1* **by** *metis*

**lemma** *d-separates-forest-modulo-equivalence-3*:
**assumes** *forest-modulo-equivalence* $x$ $d$

**shows** $d \leq -x$
**proof** $-$
**have** $1 \leq x$
**using** *assms*(*1*) *forest-modulo-equivalence-def* **by** *auto*
**then have** $d \leq x * d$
**using** *mult-left-isotone* **by** *fastforce*
**also have** ... $\leq (x * d)^+$
**using** *star.circ-mult-increasing* **by** *simp*
**also have** ... $\leq -x$
**using** *assms*(*1*) *forest-modulo-equivalence-def le-bot pseudo-complement* **by** *blast*
**finally show** *?thesis*
**by** *simp*
**qed**

**lemma** *d-separates-forest-modulo-equivalence-4*:
**shows** *forest-modulo-equivalence* $x\ d \Longrightarrow d^T \leq -x$
**using** *d-separates-forest-modulo-equivalence-3 forest-modulo-equivalence-def conv-isotone symmetric-complement-closed* **by** *metis*

**lemma** *d-separates-forest-modulo-equivalence-5*:
**shows** *forest-modulo-equivalence* $x\ d \Longrightarrow d \sqcup d^T \leq -x$
**using** *d-separates-forest-modulo-equivalence-3 d-separates-forest-modulo-equivalence-4 sup-least* **by** *blast*

**lemma** *d-separates-forest-modulo-equivalence-6*:
**shows** *forest-modulo-equivalence* $x\ d \Longrightarrow d * x \sqcup x * d \leq -x$
**using** *d-separates-forest-modulo-equivalence-1 d-separates-forest-modulo-equivalence-2 sup-least* **by** *blast*

**lemma** *d-separates-forest-modulo-equivalence-7*:
**shows** *forest-modulo-equivalence* $x\ d \Longrightarrow x * (d \sqcup d^T) * x \leq -x$
**using** *d-separates-forest-modulo-equivalence-5 forest-modulo-equivalence-def inf.sup-monoid.add-commute preorder-idempotent pseudo-complement triple-schroeder-p* **by** *metis*

**lemma** *d-separates-forest-modulo-equivalence-8*:
**shows** *forest-modulo-equivalence* $x\ d \Longrightarrow (d * x)^T \leq -x$
**using** *d-separates-forest-modulo-equivalence-2 forest-modulo-equivalence-def conv-isotone symmetric-complement-closed* **by** *metis*

**lemma** *d-separates-forest-modulo-equivalence-9*:
**shows** *forest-modulo-equivalence* $x\ d \Longrightarrow (x * d)^T \leq -x$
**by** (*metis d-separates-forest-modulo-equivalence-1 forest-modulo-equivalence-def conv-isotone symmetric-complement-closed*)

**lemma** *d-separates-forest-modulo-equivalence-10*:
**shows** *forest-modulo-equivalence* $x\ d \Longrightarrow (d * x)^+ \leq -x$
**using** *forest-modulo-equivalence-def le-bot pseudo-complement schroeder-5-p*

*star-slide mult-assoc* **by** *metis*

**lemma** *d-separates-forest-modulo-equivalence-11*:
**shows** *forest-modulo-equivalence* $x\ d \Longrightarrow (x * d)^{+} \leq -x$
**using** *forest-modulo-equivalence-def le-bot pseudo-complement* **by** *blast*

**lemma** *d-separates-forest-modulo-equivalence-12*:
**shows** *forest-modulo-equivalence* $x\ d \Longrightarrow (d * x)^{T+} \leq -x$
**using** *d-separates-forest-modulo-equivalence-10 forest-modulo-equivalence-def conv-isotone conv-plus-commute symmetric-complement-closed* **by** *metis*

**lemma** *d-separates-x-in-forest-13*:
**shows** *forest-modulo-equivalence* $x\ d \Longrightarrow (x * d)^{T+} \leq -x$
**using** *d-separates-forest-modulo-equivalence-11 forest-modulo-equivalence-def conv-isotone conv-plus-commute symmetric-complement-closed* **by** *metis*

**lemma** *irreflexive-d-in-forest-modulo-equivalence*:
**shows** *forest-modulo-equivalence* $x\ d \Longrightarrow$ *irreflexive* $d$
**by** (*metis d-separates-forest-modulo-equivalence-3 forest-modulo-equivalence-def inf.cobounded2 inf.left-commute inf.orderE pseudo-complement*)

**lemma** *univalent-d-in-forest-modulo-equivalence*:
**assumes** *forest-modulo-equivalence* $x\ d$
**shows** *univalent* $d$
**proof** −
**have** $d^T * d \leq d^T * x^T * x * d$
**using** *assms*(*1*) *forest-modulo-equivalence-def comp-isotone comp-right-one mult-sub-right-one* **by** *metis*
**also have** ... $\leq 1$
**using** *assms*(*1*) *forest-modulo-equivalence-def comp-associative conv-dist-comp* **by** *auto*
**finally show** *?thesis*
**by** *simp*
**qed**

**lemma** *acyclic-d-in-forest-modulo-equivalence*:
**assumes** *forest-modulo-equivalence* $x\ d$
**shows** *acyclic* $d$
**proof** −
**have** $d^\star \leq (x * d)^\star$
**using** *assms*(*1*) *forest-modulo-equivalence-def mult-left-isotone star.circ-circ-mult star.circ-circ-mult-1 star.circ-extra-circ star.left-plus-circ star-involutive star-isotone star-one star-slide mult-assoc* **by** *metis*
**then have** $d * d^\star \leq d * (x * d)^\star$
**using** *mult-right-isotone* **by** *blast*
**also have** ... $\leq x * d * (x * d)^\star$
**using** *assms*(*1*) *forest-modulo-equivalence-def eq-refl inf.order-trans mult-isotone star.circ-circ-mult-1 star-involutive star-one star-outer-increasing mult-assoc* **by** *metis*

**also have** ... $\leq -x$
**using** *assms d-separates-forest-modulo-equivalence-11* **by** *blast*
**also have** ... $\leq -1$
**using** *assms(1) forest-modulo-equivalence-def p-antitone* **by** *blast*
**finally show** *?thesis*
**by** *simp*
**qed**

**lemma** *acyclic-dt-d-in-forest-modulo-equivalence*:
**shows** *forest-modulo-equivalence x d* $\Longrightarrow$ *acyclic* $(d^T)$
**using** *acyclic-d-in-forest-modulo-equivalence conv-plus-commute irreflexive-conv-closed* **by** *fastforce*

**lemma** *dt-forest-modulo-equivalence-forest*:
**shows** *forest-modulo-equivalence x d* $\Longrightarrow$ *forest* $(d^T)$
**using** *acyclic-dt-d-in-forest-modulo-equivalence univalent-d-in-forest-modulo-equivalence* **by** *simp*

**lemma** *var-forest-modulo-equivalence-axiom*:
**shows** *forest-modulo-equivalence x d* $\Longrightarrow d^T * x * d \leq 1$
**using** *forest-modulo-equivalence-def comp-associative conv-dist-comp preorder-idempotent* **by** *metis*

**lemma** *forest-modulo-equivalence-wcc*:
**assumes** *forest-modulo-equivalence x d*
**shows** $(x * d)^\star * (x * d)^{T\star} = ((x * d) \sqcup (x * d)^T)^\star$
**using** *assms(1) forest-modulo-equivalence-def fc-wcc* **by** *force*

**lemma** *forest-modulo-equivalence-direction-1*:
**assumes** *forest-modulo-equivalence x d*
**shows** $(x * d)^\star \sqcap (x * d)^T = bot$
**using** *assms(1) d-separates-forest-modulo-equivalence-11 forest-modulo-equivalence-def acyclic-star-below-complement-1 order-lesseq-imp p-antitone-iff* **by** *meson*

**lemma** *forest-modulo-equivalence-direction-2*:
**assumes** *forest-modulo-equivalence x d*
**shows** $(x * d)^{T\star} \sqcap (x * d) \leq bot$
**using** *assms(1) forest-modulo-equivalence-direction-1 comp-inf.idempotent-bot-closed conv-inf-bot-iff conv-star-commute inf.sup-left-divisibility* **by** *metis*

**lemma** *forest-modulo-equivalence-separate*:
**assumes** *forest-modulo-equivalence x d*
**shows** $(x * d)^\star * (x * d)^{T\star} \sqcap (x * d)^T * (x * d) \leq 1$
**proof** –
**have** $(x * d)^\star \sqcap (x * d)^T * (x * d) = (1 \sqcup (x * d)^+) \sqcap (x * d)^T * (x * d)$
**using** *star-left-unfold-equal* **by** *simp*
**also have** ... $= (1 \sqcap (x * d)^T * (x * d)) \sqcup ((x * d)^+ \sqcap (x * d)^T * (x * d))$

**using** *comp-inf.semiring.distrib-right* **by** *simp*
**also have** ... $\leq$ *1* $\sqcup$ $((x * d)^+ \sqcap (x * d)^T * (x * d))$
**using** *inf.cobounded1 semiring.add-right-mono* **by** *blast*
**also have** ... $=$ *1* $\sqcup$ $((x * d)^\star \sqcap (x * d)^T) * (x * d)$
**using** *assms(1) forest-modulo-equivalence-def forest-modulo-equivalence-direction-1 comp-inf.semiring.mult-zero-right inf.sup-left-divisibility le-infI2 semiring.mult-not-zero sup.orderE* **by** *metis*
**also have** ... $\leq$ *1* $\sqcup$ *bot*
**using** *assms(1) forest-modulo-equivalence-direction-1* **by** *simp*
**finally have** *2*: $(x * d)^\star \sqcap (x * d)^T * (x * d) \leq$ *1*
**by** *simp*
**then have** *3*: $(x * d)^{T\star} \sqcap (x * d)^T * (x * d) \leq$ *1*
**using** *assms(1) forest-modulo-equivalence-def conv-dist-comp conv-dist-inf conv-involutive conv-star-commute coreflexive-symmetric* **by** *metis*
**have** $((x * d)^\star \sqcup (x * d)^{T\star}) \sqcap ((x * d)^T * (x * d)) \leq$ *1*
**using** *2 3 inf-sup-distrib2* **by** *simp*
**thus** *?thesis*
**using** *assms(1) le-infI2 forest-modulo-equivalence-def* **by** *blast*
**qed**

**lemma** *forest-modulo-equivalence-path-trans-closure*:
**assumes** *forest-modulo-equivalence x d*
**shows** $(x * d^T)^+ * x * d * x * d^T \leq (x * d^T)^+$
**proof** −
**have** $(x * d^T)^+ * x * d * x * d^T = (x * d^T)^\star * x * d^T * x * d * x * d^T$
**using** *comp-associative star.circ-plus-same* **by** *metis*
**also have** ... $\leq (x * d^T)^\star * x *$ *1* $* x * d^T$
**using** *assms(1) forest-modulo-equivalence-def var-forest-modulo-equivalence-axiom comp-associative mult-left-isotone mult-right-isotone* **by** *metis*
**also have** ... $\leq (x * d^T)^\star * x * d^T$
**using** *assms(1) forest-modulo-equivalence-def* **by** (*simp add*: *preorder-idempotent mult-assoc*)
**finally show** *?thesis*
**using** *star.circ-plus-same mult-assoc* **by** *simp*
**qed**

The *forest-modulo-equivalence* structure is preserved if d is decreased.

**lemma** *forest-modulo-equivalence-decrease-d*:
**assumes** *forest-modulo-equivalence x d*
**shows** *forest-modulo-equivalence x* $(d \sqcap c)$
**proof** (*unfold forest-modulo-equivalence-def*, *intro conjI*)
**show** *reflexive x*
**using** *assms(1) forest-modulo-equivalence-def* **by** *blast*
**next**
**show** *transitive x*
**using** *assms(1) forest-modulo-equivalence-def* **by** *blast*
**next**
**show** *symmetric x*

**using** *assms*(*1*) *forest-modulo-equivalence-def* **by** *blast*
**next**
**show** *univalent* $(x * (d \sqcap c))$
**proof** −
**have** $(x * (d \sqcap c))^T * x * (d \sqcap c) \leq (x * d)^T * x * d$
**using** *conv-order mult-isotone* **by** *auto*
**also have** ... $\leq 1$
**using** *assms*(*1*) *forest-modulo-equivalence-def mult-assoc* **by** *auto*
**finally show** *?thesis*
**using** *mult-assoc* **by** *auto*
**qed**
**next**
**show** *coreflexive* $(x \sqcap ((d \sqcap c) * (d \sqcap c)^T))$
**proof** −
**have** $x \sqcap (d \sqcap c) * (d \sqcap c)^T \leq x \sqcap d * d^T$
**using** *conv-dist-inf inf.sup-right-isotone mult-isotone* **by** *auto*
**thus** *?thesis*
**using** *assms*(*1*) *forest-modulo-equivalence-def order-lesseq-imp* **by** *blast*
**qed**
**next**
**show** $(x * (d \sqcap c))^+ \sqcap x \leq bot$
**proof** −
**have** $(x * (d \sqcap c))^+ \leq (x * d)^+$
**using** *comp-isotone star-isotone* **by** *simp*
**thus** *?thesis*
**using** *assms d-separates-forest-modulo-equivalence-11 dual-order.eq-iff dual-order.trans pseudo-complement* **by** *blast*
**qed**
**qed**

**lemma** *expand-forest-modulo-equivalence*:
**assumes** *forest-modulo-equivalence H d*
**shows** $(d^T * H)^\star * (H * d)^\star = (d^T * H)^\star \sqcup (H * d)^\star$
**proof** −
**have** $(H * d)^T * H * d \leq 1$
**using** *assms forest-modulo-equivalence-def mult-assoc* **by** *auto*
**hence** $d^T * H * H * d \leq 1$
**using** *assms forest-modulo-equivalence-def conv-dist-comp* **by** *auto*
**thus** *?thesis*
**by** (*simp add*: *cancel-separate-eq comp-associative*)
**qed**

**lemma** *forest-modulo-equivalence-path-bot*:
**assumes** *arc a*
**and** $a \leq d$
**and** *forest-modulo-equivalence H d*
**shows** $(d \sqcap - a)^T * (H * a * top) \leq bot$
**proof** −
**have** *1*: $d^T * H * d \leq 1$

**using** *assms*(*3*) *forest-modulo-equivalence-def dTransHd-le-1* **by** *blast*
**hence** $d * - 1 * d^T \leq - H$
**using** *triple-schroeder-p* **by** *force*
**hence** $d * - 1 * d^T \leq 1 \sqcup - H$
**by** (*simp add*: *le-supI2*)
**hence** $d * d^T \sqcup d * - 1 * d^T \leq 1 \sqcup - H$
**by** (*metis assms*(*3*) *forest-modulo-equivalence-def inf-commute regular-one-closed shunting-p le-supI*)
**hence** $d * 1 * d^T \sqcup d * - 1 * d^T \leq 1 \sqcup - H$
**by** *simp*
**hence** $d * (1 \sqcup - 1) * d^T \leq 1 \sqcup - H$
**using** *comp-associative mult-right-dist-sup* **by** (*simp add*: *mult-left-dist-sup*)
**hence** $d * top * d^T \leq 1 \sqcup - H$
**using** *regular-complement-top* **by** *auto*
**hence** $d * top * a^T \leq 1 \sqcup - H$
**using** *assms*(*2*) *conv-isotone dual-order.trans mult-right-isotone* **by** *blast*
**hence** $d * (a * top)^T \leq 1 \sqcup - H$
**by** (*simp add*: *comp-associative conv-dist-comp*)
**hence** $d \leq (1 \sqcup - H) * (a * top)$
**by** (*simp add*: *assms*(*1*) *shunt-bijective*)
**hence** $d \leq a * top \sqcup - H * a * top$
**by** (*simp add*: *comp-associative mult-right-dist-sup*)
**also have** ... $\leq a * top \sqcup - (H * a * top)$
**using** *assms*(*1*, *3*) *HcompaT-le-compHaT forest-modulo-equivalence-def sup-right-isotone* **by** *auto*
**finally have** $d \leq a * top \sqcup - (H * a * top)$
**by** *simp*
**hence** $d \sqcap --( H * a * top) \leq a * top$
**using** *shunting-var-p* **by** *auto*
**hence** *2*:$d \sqcap H * a * top \leq a * top$
**using** *inf.sup-right-isotone order.trans pp-increasing* **by** *blast*
**have** *3*:$d \sqcap H * a * top \leq top * a$
**proof** −
**have** $d \sqcap H * a * top \leq (H * a \sqcap d * top^T) * (top \sqcap (H * a)^T * d)$
**by** (*metis dedekind inf-commute*)
**also have** ... $= d * top \sqcap H * a * a^T * H^T * d$
**by** (*simp add*: *conv-dist-comp inf-vector-comp mult-assoc*)
**also have** ... $\leq d * top \sqcap H * a * d^T * H^T * d$
**using** *assms*(*2*) *mult-right-isotone mult-left-isotone conv-isotone inf.sup-right-isotone* **by** *auto*
**also have** ... $= d * top \sqcap H * a * d^T * H * d$
**using** *assms*(*3*) *forest-modulo-equivalence-def* **by** *auto*
**also have** ... $\leq d * top \sqcap H * a * 1$
**using** *1* **by** (*metis inf.sup-right-isotone mult-right-isotone mult-assoc*)
**also have** ... $\leq H * a$
**by** *simp*
**also have** ... $\leq top * a$
**by** (*simp add*: *mult-left-isotone*)
**finally have** $d \sqcap H * a * top \leq top * a$

**by** *simp*
**thus** *?thesis*
**by** *simp*
**qed**
**have** $d \sqcap H * a * top \le a * top \sqcap top * a$
**using** *2 3* **by** *simp*
**also have** ... $= a * top * top * a$
**by** (*metis comp-associative comp-inf.star.circ-decompose-9 comp-inf.star-star-absorb comp-inf-covector vector-inf-comp vector-top-closed*)
**also have** ... $= a * top * a$
**by** (*simp add*: *vector-mult-closed*)
**finally have** *4*:$d \sqcap H * a * top \le a$
**by** (*simp add*: *assms*(*1*) *arc-top-arc*)
**hence** $d \sqcap - a \le -(H * a * top)$
**using** *assms*(*1*) *arc-regular p-shunting-swap* **by** *fastforce*
**hence** $(d \sqcap - a) * top \le -(H * a * top)$
**using** *mult.semigroup-axioms p-antitone-iff schroeder-4-p semigroup.assoc* **by** *fastforce*
**thus** *?thesis*
**by** (*simp add*: *schroeder-3-p*)
**qed**

**lemma** *forest-modulo-equivalence-path-split-2*:
**assumes** *arc a*
**and** $a \le d$
**and** *forest-modulo-equivalence H d*
**shows** $(H * (d \sqcap - a))^\star * H * a * top = (H * ((d \sqcap - a) \sqcup (d \sqcap - a)^T))^\star * H * a * top$
**proof** −
**let** *?lhs* $= (H * (d \sqcap - a))^\star * H * a * top$
**have** *1*: $d^T * H * d \le 1$
**using** *assms*(*3*) *forest-modulo-equivalence-def dTransHd-le-1* **by** *blast*
**have** *2*: $H * a * top \le$ *?lhs*
**by** (*metis le-iff-sup star.circ-loop-fixpoint star.circ-transitive-equal star-involutive sup-commute mult-assoc*)
**have** $(d \sqcap - a)^T * (H * (d \sqcap - a))^\star * (H * a * top) = (d \sqcap - a)^T * H * (d \sqcap - a) * (H * (d \sqcap - a))^\star * (H * a * top)$
**proof** −
**have** $(d \sqcap - a)^T * (H * (d \sqcap - a))^\star * (H * a * top) = (d \sqcap - a)^T * (1 \sqcup H * (d \sqcap - a) * (H * (d \sqcap - a))^\star) * (H * a * top)$
**by** (*simp add*: *star-left-unfold-equal*)
**also have** ... $= (d \sqcap - a)^T * H * a * top \sqcup (d \sqcap - a)^T * H * (d \sqcap - a) * (H * (d \sqcap - a))^\star * (H * a * top)$
**by** (*smt mult-left-dist-sup star.circ-loop-fixpoint star.circ-mult-1 star-slide sup-commute mult-assoc*)
**also have** ... $= bot \sqcup (d \sqcap - a)^T * H * (d \sqcap - a) * (H * (d \sqcap - a))^\star * (H * a * top)$
**by** (*metis assms*(*1*, *2*, *3*) *forest-modulo-equivalence-path-bot mult-assoc le-bot*)

**thus** *?thesis*
**by** (*simp add*: *calculation*)
**qed**
**also have** ... $\leq d^T * H * d * (H * (d \sqcap - a))^\star * (H * a * top)$
**using** *conv-isotone inf.coboundedl mult-isotone* **by** *auto*
**also have** ... $\leq 1 * (H * (d \sqcap - a))^\star * (H * a * top)$
**using** *1* **by** (*metis le-iff-sup mult-right-dist-sup*)
**finally have** *3*: $(d \sqcap - a)^T * (H * (d \sqcap - a))^\star * (H * a * top) \leq$ *?lhs*
**using** *mult-assoc* **by** *auto*
**hence** *4*: $H * (d \sqcap - a)^T * (H * (d \sqcap - a))^\star * (H * a * top) \leq$ *?lhs*
**proof** −
**have** $H * (d \sqcap - a)^T * (H * (d \sqcap - a))^\star * (H * a * top) \leq H * (H * (d \sqcap - a))^\star * H * a * top$
**using** *3 mult-right-isotone mult-assoc* **by** *auto*
**also have** ... $= H * H * ((d \sqcap - a) * H)^\star * H * a * top$
**by** (*metis assms*(*3*) *forest-modulo-equivalence-def star-slide mult-assoc preorder-idempotent*)
**also have** ... $= H * ((d \sqcap - a) * H)^\star * H * a * top$
**using** *assms*(*3*) *forest-modulo-equivalence-def preorder-idempotent* **by** *fastforce*
**finally show** *?thesis*
**by** (*metis assms*(*3*) *forest-modulo-equivalence-def preorder-idempotent star-slide mult-assoc*)
**qed**
**have** *5*: $(H * (d \sqcap - a) \sqcup H * (d \sqcap - a)^T) * (H * (d \sqcap - a))^\star * H * a * top \leq$ *?lhs*
**proof** −
**have** *51*: $H * (d \sqcap - a) * (H * (d \sqcap - a))^\star * H * a * top \leq (H * (d \sqcap - a))^\star * H * a * top$
**using** *star.left-plus-below-circ mult-left-isotone* **by** *simp*
**have** *52*: $(H * (d \sqcap - a) \sqcup H * (d \sqcap - a)^T) * (H * (d \sqcap - a))^\star * H * a * top = H * (d \sqcap - a) * (H * (d \sqcap - a))^\star * H * a * top \sqcup H * (d \sqcap - a)^T * (H * (d \sqcap - a))^\star * H * a * top$
**using** *mult-right-dist-sup* **by** *auto*
**hence** ... $\leq (H * (d \sqcap - a))^\star * H * a * top \sqcup H * (d \sqcap - a)^T * (H * (d \sqcap - a))^\star * H * a * top$
**using** *star.left-plus-below-circ mult-left-isotone sup-left-isotone* **by** *auto*
**thus** *?thesis*
**using** *4 51 52 mult-assoc* **by** *auto*
**qed**
**hence** $(H * (d \sqcap - a) \sqcup H * (d \sqcap - a)^T)^\star * H * a * top \leq$ *?lhs*
**proof** −
**have** $(H * (d \sqcap - a) \sqcup H * (d \sqcap - a)^T)^\star * (H * (d \sqcap - a))^\star * H * a * top \leq$ *?lhs*
**using** *5 star-left-induct-mult-iff mult-assoc* **by** *auto*
**thus** *?thesis*
**using** *star.circ-decompose-11 star-decompose-1* **by** *auto*
**qed**
**hence** *6*: $(H * ((d \sqcap - a) \sqcup (d \sqcap - a)^T))^\star * H * a * top \leq$ *?lhs*

**using** *mult-left-dist-sup* **by** *auto*
**have** *7*: $(H * (d \sqcap - a))^\star * H * a * top \leq (H * ((d \sqcap - a) \sqcup (d \sqcap - a)^T))^\star * H * a * top$
**by** (*simp add*: *mult-left-isotone semiring.distrib-left star-isotone*)
**thus** *?thesis*
**using** *6 7* **by** (*simp add*: *mult-assoc*)
**qed**

**end**

## 4.3 An operation to select components

This section has been moved to theories *Stone-Relation-Algebras.Choose-Component* and *Aggregation-Algebras.M-Choose-Component*.

## 4.4 m-k-Stone-Kleene relation algebras

*m-k*-Stone-Kleene relation algebras are an extension of *m*-Kleene algebras where the *choose-component* operation has been added.

**context** *m-kleene-algebra-choose-component*
**begin**

A *selected-edge* is a minimum-weight edge whose source is in a component, with respect to $h$, $j$ and $g$, and whose target is not in that component.

**abbreviation** *selected-edge h j g* $\equiv$ *minarc* (*choose-component* (*forest-components h*) $j * -$ *choose-component* (*forest-components h*) $j^T \sqcap g$)

A *path* is any sequence of edges in the forest, $f$, of the graph, $g$, backwards from the target of the *selected-edge* to a root in $f$.

**abbreviation** *path f h j g* $\equiv$ *top* $*$ *selected-edge h j g* $* (f \sqcap -$ *selected-edge h j* $g^T)^{T\star}$

**definition** *boruvka-outer-invariant f g* $\equiv$
*symmetric g*
$\wedge$ *forest f*
$\wedge$ $f \leq --g$
$\wedge$ *regular f*
$\wedge$ ($\exists w$ . *minimum-spanning-forest w g* $\wedge f \leq w \sqcup w^T$)

**definition** *boruvka-inner-invariant j f h g d* $\equiv$
*boruvka-outer-invariant f g*
$\wedge$ $g \neq bot$
$\wedge$ *regular d*
$\wedge$ *regular j* $\wedge$ *vector j*
$\wedge$ *regular h* $\wedge$ *forest h*
$\wedge$ *forest-components h* $* j = j$
$\wedge$ *forest-modulo-equivalence* (*forest-components h*) *d*
$\wedge$ $d * top \leq - j$

$\wedge$ $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
$\wedge$ ($\forall$ *a b* . *forest-modulo-equivalence-path a b* (*forest-components h*) *d* $\wedge$ *a* $\leq$ $-$(*forest-components h*) $\sqcap$ $--$ *g* $\wedge$ *b* $\leq$ *d* $\longrightarrow$ *sum*(*b* $\sqcap$ *g*) $\leq$ *sum*(*a* $\sqcap$ *g*))

**lemma** *F-is-H-and-d*:
**assumes** $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
**and** *injective f*
**and** *injective h*
**shows** *forest-components f* = (*forest-components h* $*$ $(d \sqcup d^T)$)$^\star$ $*$ *forest-components h*
**proof** $-$
**have** *forest-components f* = $(f \sqcup f^T)^\star$
**using** *assms*(*2*) *cancel-separate-1* **by** *simp*
**also have** ... = $(h \sqcup h^T \sqcup d \sqcup d^T)^\star$
**using** *assms*(*1*) **by** *auto*
**also have** ... = $((h \sqcup h^T)^\star * (d \sqcup d^T))^\star * (h \sqcup h^T)^\star$
**using** *star.circ-sup-9 sup-assoc* **by** *metis*
**also have** ... = (*forest-components h* $*$ $(d \sqcup d^T)$)$^\star$ $*$ *forest-components h*
**using** *assms*(*3*) *forest-components-wcc* **by** *simp*
**finally show** *?thesis*
**by** *simp*
**qed**

**lemma** *H-below-F*:
**assumes** $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
**and** *injective f*
**and** *injective h*
**shows** *forest-components h* $\leq$ *forest-components f*
**using** *assms*(*1*, *2*, *3*) *cancel-separate-1 dual-order.trans star.circ-sub-dist* **by** *metis*

**lemma** *H-below-regular-g*:
**assumes** $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
**and** $f \leq --g$
**and** *symmetric g*
**shows** $h \leq --g$
**proof** $-$
**have** $h \leq f \sqcup f^T$
**using** *assms*(*1*) *sup-assoc* **by** *simp*
**also have** ... $\leq --g$
**using** *assms*(*2*, *3*) *conv-complement conv-isotone* **by** *fastforce*
**finally show** *?thesis*
**using** *order-trans* **by** *blast*
**qed**

**lemma** *expression-equivalent-without-e-complement*:
**assumes** *selected-edge h j g* $\leq$ $-$ *forest-components f*
**shows** $f \sqcap -$ (*selected-edge h j g*)$^T$ $\sqcap$ $-$ (*path f h j g*) $\sqcup$ ($f \sqcap -$ (*selected-edge h j g*)$^T$ $\sqcap$ (*path f h j g*))$^T$ $\sqcup$ (*selected-edge h j g*)

$= f \sqcap -\ (path\ f\ h\ j\ g) \sqcup (f \sqcap (path\ f\ h\ j\ g))^T \sqcup (selected\text{-}edge\ h\ j\ g)$
**proof** −
  **let** *?p* = *path f h j g*
  **let** *?e* = *selected-edge h j g*
  **let** *?F* = *forest-components f*
  **have** *1*: $?e \le -\ ?F$
    **by** (*simp add*: *assms*)
  **have** $f^T \le\ ?F$
    **by** (*metis conv-dist-comp conv-involutive conv-order conv-star-commute forest-components-increasing*)
  **hence** $-\ ?F \le -\ f^T$
    **using** *p-antitone* **by** *auto*
  **hence** $?e \le -\ f^T$
    **using** *1 dual-order.trans* **by** *blast*
  **hence** $f^T \le -\ ?e$
    **by** (*simp add*: *p-antitone-iff*)
  **hence** $f^{TT} \le -\ ?e^T$
    **by** (*metis conv-complement conv-dist-inf inf.orderE inf.orderI*)
  **hence** $f \le -\ ?e^T$
    **by** *auto*
  **hence** $f = f \sqcap -\ ?e^T$
    **using** *inf.orderE* **by** *blast*
  **hence** $f \sqcap -\ ?e^T \sqcap -\ ?p \sqcup (f \sqcap -\ ?e^T \sqcap\ ?p)^T \sqcup\ ?e = f \sqcap -\ ?p \sqcup (f \sqcap\ ?p)^T \sqcup\ ?e$
    **by** *auto*
  **thus** *?thesis* **by** *auto*
**qed**

The source of the *selected-edge* is contained in *j*, the vector describing those vertices still to be processed in the inner loop of Borůvka's algorithm.

**lemma** *et-below-j*:
  **assumes** *vector j*
    **and** *regular j*
    **and** $j \ne bot$
  **shows** $selected\text{-}edge\ h\ j\ g * top \le j$
**proof** −
  **let** *?e* = *selected-edge h j g*
  **let** *?c* = *choose-component* (*forest-components h*) *j*
  **have** $?e * top \le --(?c * -?c^T \sqcap g) * top$
    **using** *comp-isotone minarc-below* **by** *blast*
  **also have** ... $= (--(?c * -?c^T) \sqcap --g) * top$
    **by** *simp*
  **also have** ... $= (?c * -?c^T \sqcap --g) * top$
    **using** *component-is-regular regular-mult-closed* **by** *auto*
  **also have** ... $= (?c \sqcap -?c^T \sqcap --g) * top$
    **by** (*metis component-is-vector conv-complement vector-complement-closed vector-covector*)
  **also have** ... $\le\ ?c * top$
    **using** *inf.coboundedl mult-left-isotone order-trans* **by** *blast*

**also have** ... $\leq j * top$
**by** (*metis assms*(*2*) *comp-inf.star.circ-sup-2 comp-isotone component-in-v*)
**also have** ... $= j$
**by** (*simp add*: *assms*(*1*))
**finally show** *?thesis*
**by** *simp*
**qed**

### 4.4.1 Components of forests and forests modulo an equivalence

We prove a number of properties about *forest-modulo-equivalence* and *forest-components*.

**lemma** *fc-j-eq-j-inv*:
**assumes** *forest h*
**and** *forest-components* $h * j = j$
**shows** *forest-components* $h * (j \sqcap -$ *choose-component* (*forest-components* $h$) $j$) $= j \sqcap -$ *choose-component* (*forest-components* $h$) $j$
**proof** $-$
**let** $?c =$ *choose-component* (*forest-components* $h$) $j$
**let** $?H =$ *forest-components* $h$
**have** *1*:*equivalence ?H*
**by** (*simp add*: *assms*(*1*) *forest-components-equivalence*)
**have** $?H * (j \sqcap - ?c) = ?H * j \sqcap ?H * - ?c$
**using** *1* **by** (*metis assms*(*2*) *equivalence-comp-dist-inf inf.sup-monoid.add-commute*)
**hence** *2*: $?H * (j \sqcap - ?c) = j \sqcap ?H * - ?c$
**by** (*simp add*: *assms*(*2*))
**have** *3*: $j \sqcap - ?c \leq ?H * - ?c$
**using** *1* **by** (*metis assms*(*2*) *dedekind-1 dual-order.trans equivalence-comp-dist-inf inf.cobounded2*)
**have** $?H * ?c \leq ?c$
**using** *component-single* **by** *auto*
**hence** $?H^T * ?c \leq ?c$
**using** *1* **by** *simp*
**hence** $?H * - ?c \leq - ?c$
**using** *component-is-regular schroeder-3-p* **by** *force*
**hence** $j \sqcap ?H * - ?c \leq j \sqcap - ?c$
**using** *inf.sup-right-isotone* **by** *auto*
**thus** *?thesis*
**using** *2 3 order.antisym* **by** *simp*
**qed**

There is a path in the *forest-modulo-equivalence* between edges $a$ and $b$ if and only if there is either a path in the *forest-modulo-equivalence* from $a$ to $b$ or one from $a$ to $c$ and one from $c$ to $b$.

**lemma** *forest-modulo-equivalence-path-split-disj*:
**assumes** *equivalence H*
**and** *arc c*
**and** *regular* $a \wedge$ *regular* $b \wedge$ *regular* $c \wedge$ *regular* $d \wedge$ *regular* $H$

**shows** *forest-modulo-equivalence-path a b H* $(d \sqcup c) \longleftrightarrow$ *forest-modulo-equivalence-path a b H d* $\vee$ (*forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*)
**proof** −
**have** *1*: *forest-modulo-equivalence-path a b H* $(d \sqcup c) \longrightarrow$ *forest-modulo-equivalence-path a b H d* $\vee$ (*forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*)
**proof** (*rule impI*)
**assume** *11*: *forest-modulo-equivalence-path a b H* $(d \sqcup c)$
**hence** $a^T * top \leq (H * (d \sqcup c))^\star * H * b * top$
**by** (*simp add*: *forest-modulo-equivalence-path-def*)
**also have** ... $= ((H * d)^\star \sqcup (H * d)^\star * H * c * (H * d)^\star) * H * b * top$
**using** *assms*(*1*, *2*) *path-through-components* **by** *simp*
**also have** ... $= (H * d)^\star * H * b * top \sqcup (H * d)^\star * H * c * (H * d)^\star * H * b * top$
**by** (*simp add*: *mult-right-dist-sup*)
**finally have** *12*:$a^T * top \leq (H * d)^\star * H * b * top \sqcup (H * d)^\star * H * c * (H * d)^\star * H * b * top$
**by** *simp*
**have** *13*: $a^T * top \leq (H * d)^\star * H * b * top \vee a^T * top \leq (H * d)^\star * H * c * (H * d)^\star * H * b * top$
**proof** (*rule point-in-vector-sup*)
**show** *point* $(a^T * top)$
**using** *11 forest-modulo-equivalence-path-def mult-assoc* **by** *auto*
**next**
**show** *vector* $((H * d)^\star * H * b * top)$
**using** *vector-mult-closed* **by** *simp*
**next**
**show** *regular* $((H * d)^\star * H * b * top)$
**using** *assms*(*3*) *pp-dist-comp pp-dist-star* **by** *auto*
**next**
**show** $a^T * top \leq (H * d)^\star * H * b * top \sqcup (H * d)^\star * H * c * (H * d)^\star * H * b * top$
**using** *12* **by** *simp*
**qed**
**thus** *forest-modulo-equivalence-path a b H d* $\vee$ (*forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*)
**proof** (*cases* $a^T * top \leq (H * d)^\star * H * b * top$)
**case** *True*
**assume** $a^T * top \leq (H * d)^\star * H * b * top$
**hence** *forest-modulo-equivalence-path a b H d*
**using** *11 forest-modulo-equivalence-path-def* **by** *auto*
**thus** *?thesis*
**by** *simp*
**next**
**case** *False*
**have** *14*: $a^T * top \leq (H * d)^\star * H * c * (H * d)^\star * H * b * top$
**using** *13* **by** (*simp add*: *False*)
**hence** *15*: $a^T * top \leq (H * d)^\star * H * c * top$

**by** (*metis mult-right-isotone order-lesseq-imp top-greatest mult-assoc*)
**have** $c^T * top \leq (H * d)^\star * H * b * top$
**proof** (*rule ccontr*)
**assume** $\neg\ c^T * top \leq (H * d)^\star * H * b * top$
**hence** $c^T * top \leq -((H * d)^\star * H * b * top)$
**by** (*meson assms*(*2*, *3*) *point-in-vector-or-complement regular-closed-star regular-closed-top regular-mult-closed vector-mult-closed vector-top-closed*)
**hence** $c * (H * d)^\star * H * b * top \leq bot$
**using** *schroeder-3-p mult-assoc* **by** *auto*
**thus** *False*
**using** *13 False sup.absorb-iff1 mult-assoc* **by** *auto*
**qed**
**hence** *forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*
**using** *11 15 assms*(*2*) *forest-modulo-equivalence-path-def* **by** *auto*
**thus** *?thesis*
**by** *simp*
**qed**
**qed**
**have** *2*: *forest-modulo-equivalence-path a b H d* $\vee$ (*forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*) $\longrightarrow$ *forest-modulo-equivalence-path a b H* $(d \sqcup c)$
**proof** −
**have** *21*: *forest-modulo-equivalence-path a b H d* $\longrightarrow$ *forest-modulo-equivalence-path a b H* $(d \sqcup c)$
**proof** (*rule impI*)
**assume** *22*:*forest-modulo-equivalence-path a b H d*
**hence** $a^T * top \leq (H * d)^\star * H * b * top$
**using** *forest-modulo-equivalence-path-def* **by** *blast*
**hence** $a^T * top \leq (H * (d \sqcup c))^\star * H * b * top$
**by** (*simp add*: *mult-left-isotone mult-right-dist-sup mult-right-isotone order.trans star-isotone star-slide*)
**thus** *forest-modulo-equivalence-path a b H* $(d \sqcup c)$
**using** *22 forest-modulo-equivalence-path-def* **by** *blast*
**qed**
**have** *forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d* $\longrightarrow$ *forest-modulo-equivalence-path a b H* $(d \sqcup c)$
**proof** (*rule impI*)
**assume** *23*: *forest-modulo-equivalence-path a c H d* $\wedge$ *forest-modulo-equivalence-path c b H d*
**hence** $a^T * top \leq (H * d)^\star * H * c * top$
**using** *forest-modulo-equivalence-path-def* **by** *blast*
**also have** ... $\leq (H * d)^\star * H * c * c^T * c * top$
**by** (*metis ex231c comp-inf.star.circ-sup-2 mult-isotone mult-right-isotone mult-assoc*)
**also have** ... $\leq (H * d)^\star * H * c * c^T * top$
**by** (*simp add*: *mult-right-isotone mult-assoc*)
**also have** ... $\leq (H * d)^\star * H * c * (H * d)^\star * H * b * top$
**using** *23 mult-right-isotone mult-assoc* **by** (*simp add*:

*forest-modulo-equivalence-path-def*)
  **also have** ... $\leq (H * d)^\star * H * b * top \sqcup (H * d)^\star * H * c * (H * d)^\star * H * b * top$
  **by** (*simp add*: *forest-modulo-equivalence-path-def*)
  **finally have** $a^T * top \leq (H * (d \sqcup c))^\star * H * b * top$
  **using** *assms*(*1*, *2*) *path-through-components mult-right-dist-sup* **by** *simp*
  **thus** *forest-modulo-equivalence-path* $a$ $b$ $H$ $(d \sqcup c)$
  **using** *23 forest-modulo-equivalence-path-def* **by** *blast*
 **qed**
 **thus** *?thesis*
  **using** *21* **by** *auto*
 **qed**
 **thus** *?thesis*
  **using** *1 2* **by** *blast*
**qed**

**lemma** *dT-He-eq-bot*:
 **assumes** *vector j*
  **and** *regular j*
  **and** $d * top \leq - j$
  **and** *forest-components* $h * j = j$
  **and** $j \neq bot$
 **shows** $d^T *$ *forest-components* $h *$ *selected-edge* $h$ $j$ $g \leq bot$
**proof** −
 **let** *?e* = *selected-edge h j g*
 **let** *?H* = *forest-components h*
 **have** *1*: $?e * top \leq j$
  **using** *assms*(*1*, *2*, *5*) *et-below-j* **by** *auto*
 **have** $d^T * ?H * ?e \leq (d * top)^T * ?H * (?e * top)$
  **by** (*simp add*: *comp-isotone conv-isotone top-right-mult-increasing*)
 **also have** ... $\leq (d * top)^T * ?H * j$
  **using** *1 mult-right-isotone* **by** *auto*
 **also have** ... $\leq (- j)^T * ?H * j$
  **by** (*simp add*: *assms*(*3*) *conv-isotone mult-left-isotone*)
 **also have** ... $= (- j)^T * j$
  **using** *assms*(*4*) *comp-associative* **by** *auto*
 **also have** ... $= bot$
  **by** (*simp add*: *assms*(*1*) *conv-complement covector-vector-comp*)
 **finally show** *?thesis*
  **using** *coreflexive-bot-closed le-bot* **by** *blast*
**qed**

**lemma** *forest-modulo-equivalence-d-U-e*:
 **assumes** *forest f*
  **and** *vector j*
  **and** *regular j*
  **and** *forest h*
  **and** *forest-modulo-equivalence* (*forest-components h*) *d*
  **and** $d * top \leq - j$

**and** *forest-components h* $*$ *j* = *j*
**and** $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
**and** *selected-edge h j g* $\leq -$ *forest-components f*
**and** $j \neq$ *bot*
**shows** *forest-modulo-equivalence* (*forest-components h*) ($d \sqcup$ *selected-edge h j g*)
**proof** (*cases selected-edge h j g* = *bot*)
**let** *?e* = *selected-edge h j g*
**case** *True*
**assume** *?e* = *bot*
**thus** *?thesis*
**by** (*simp add*: *True assms*(*5*))
**next**
**let** *?H* = *forest-components h*
**let** *?F* = *forest-components f*
**let** *?e* = *selected-edge h j g*
**let** $?d' = d \sqcup ?e$
**case** *False*
**assume** *e-not-bot*: $?e \neq$ *bot*
**have** *forest-modulo-equivalence* (*forest-components h*) ($d \sqcup$ *selected-edge h j g*)
**proof** (*unfold forest-modulo-equivalence-def*, *intro conjI*)
**show** *01*: *reflexive ?H*
**by** (*simp add*: *assms*(*4*) *forest-components-equivalence*)
**show** *02*: *transitive ?H*
**by** (*simp add*: *assms*(*4*) *forest-components-equivalence*)
**show** *03*: *symmetric ?H*
**by** (*simp add*: *assms*(*4*) *forest-components-equivalence*)
**have** *04*: *equivalence ?H*
**by** (*simp add*: *01 02 03*)
**show** *univalent* ($?H * ?d'$)
**proof** $-$
**have** $(?H * ?d')^T * (?H * ?d') = ?d'^T * ?H^T * ?H * ?d'$
**using** *conv-dist-comp mult-assoc* **by** *auto*
**also have** ... $= ?d'^T * ?H * ?H * ?d'$
**by** (*simp add*: *conv-dist-comp conv-star-commute*)
**also have** ... $= ?d'^T * ?H * ?d'$
**using** *01 02* **by** (*metis preorder-idempotent mult-assoc*)
**finally have** *21*: *univalent* $(?H * ?d') \longleftrightarrow ?e^T * ?H * d \sqcup d^T * ?H * ?e \sqcup ?e^T * ?H * ?e \sqcup d^T * ?H * d \leq 1$
**using** *conv-dist-sup semiring.distrib-left semiring.distrib-right* **by** *auto*
**have** *22*: $?e^T * ?H * ?e \leq 1$
**proof** $-$
**have** *221*: $?e^T * ?H * ?e \leq ?e^T *$ *top* $* ?e$
**by** (*simp add*: *mult-left-isotone mult-right-isotone*)
**have** *arc ?e*
**using** *e-not-bot minarc-arc minarc-bot-iff* **by** *blast*
**hence** $?e^T *$ *top* $* ?e \leq 1$
**using** *arc-expanded* **by** *blast*
**thus** *?thesis*
**using** *221 dual-order.trans* **by** *blast*

**qed**
**have** *24*: $d^T * ?H * ?e \leq 1$
**by** (*metis assms*(*2*, *3*, *6*, *7*, *10*) *dT-He-eq-bot coreflexive-bot-closed le-bot*)
**hence** $(d^T * ?H * ?e)^T \leq 1^T$
**using** *conv-isotone* **by** *blast*
**hence** $?e^T * ?H^T * d^{TT} \leq 1$
**by** (*simp add*: *conv-dist-comp mult-assoc*)
**hence** *25*: $?e^T * ?H * d \leq 1$
**using** *assms*(*4*) *fch-equivalence* **by** *auto*
**have** *8*: $d^T * ?H * d \leq 1$
**using** *04 assms*(*5*) *dTransHd-le-1 forest-modulo-equivalence-def* **by** *blast*
**thus** *?thesis*
**using** *21 22 24 25* **by** *simp*
**qed**
**show** *coreflexive* $(?H \sqcap ?d' * ?d'^T)$
**proof** −
**have** *coreflexive* $(?H \sqcap ?d' * ?d'^T) \longleftrightarrow ?H \sqcap (d \sqcup ?e) * (d^T \sqcup ?e^T) \leq 1$
**by** (*simp add*: *conv-dist-sup*)
**also have** ... $\longleftrightarrow ?H \sqcap (d * d^T \sqcup d * ?e^T \sqcup ?e * d^T \sqcup ?e * ?e^T) \leq 1$
**by** (*metis mult-left-dist-sup mult-right-dist-sup sup.left-commute sup-commute*)
**finally have** *1*: *coreflexive* $(?H \sqcap ?d' * ?d'^T) \longleftrightarrow ?H \sqcap d * d^T \sqcup ?H \sqcap d * ?e^T \sqcup ?H \sqcap ?e * d^T \sqcup ?H \sqcap ?e * ?e^T \leq 1$
**by** (*simp add*: *inf-sup-distrib1*)
**have** *31*: $?H \sqcap d * d^T \leq 1$
**using** *assms*(*5*) *forest-modulo-equivalence-def* **by** *blast*
**have** *32*: $?H \sqcap ?e * d^T \leq 1$
**proof** −
**have** $?e * d^T \leq ?e * top * (d * top)^T$
**by** (*simp add*: *conv-isotone mult-isotone top-right-mult-increasing*)
**also have** ... $\leq ?e * top * - j^T$
**by** (*metis assms*(*6*) *conv-complement conv-isotone mult-right-isotone*)
**also have** ... $\leq j * - j^T$
**using** *assms*(*2*, *3*, *10*) *et-below-j mult-left-isotone* **by** *auto*
**also have** ... $\leq - ?H$
**using** *03* **by** (*metis assms*(*2*, *3*, *7*) *conv-complement conv-dist-comp equivalence-top-closed mult-left-isotone schroeder-3-p vector-top-closed*)
**finally have** $?e * d^T \leq - ?H$
**by** *simp*
**thus** *?thesis*
**by** (*metis inf.coboundedI1 p-antitone-iff p-shunting-swap regular-one-closed*)
**qed**
**have** *33*: $?H \sqcap d * ?e^T \leq 1$
**proof** −
**have** *331*: *injective h*
**by** (*simp add*: *assms*(*4*))
**have** $(?H \sqcap ?e * d^T)^T \leq 1$
**using** *32 coreflexive-conv-closed* **by** *auto*

**hence** $?H \sqcap (?e * d^T)^T \leq 1$
**using** *331 conv-dist-inf forest-components-equivalence* **by** *auto*
**thus** *?thesis*
**using** *conv-dist-comp* **by** *auto*
**qed**
**have** *34*: $?H \sqcap ?e * ?e^T \leq 1$
**proof** $-$
**have** *341*:$arc\ ?e \wedge arc\ (?e^T)$
**using** *e-not-bot minarc-arc minarc-bot-iff* **by** *auto*
**have** $?H \sqcap ?e * ?e^T \leq ?e * ?e^T$
**by** *auto*
**thus** *?thesis*
**using** *341 arc-injective le-infI2* **by** *blast*
**qed**
**thus** *?thesis*
**using** *1 31 32 33 34* **by** *simp*
**qed**
**show** *4*:$(?H * (d \sqcup ?e))^+ \sqcap ?H \leq bot$
**proof** $-$
**have** *40*: $(?H * d)^+ \leq -?H$
**using** *assms*(*5*) *forest-modulo-equivalence-def bot-unique pseudo-complement* **by** *blast*
**have** $?e \leq -\ ?F$
**by** (*simp add*: *assms*(*9*))
**hence** $?F \leq -\ ?e$
**by** (*simp add*: *p-antitone-iff*)
**hence** $?F^T * ?F \leq -\ ?e$
**using** *assms*(*1*) *fch-equivalence* **by** *fastforce*
**hence** $?F^T * ?F * ?F^T \leq -\ ?e$
**by** (*metis assms*(*1*) *fch-equivalence forest-components-star star.circ-decompose-9*)
**hence** *41*: $?F * ?e * ?F \leq -\ ?F$
**using** *triple-schroeder-p* **by** *blast*
**hence** *42*:$(?F * ?F)^\star * ?F * ?e * (?F * ?F)^\star \leq -\ ?F$
**proof** $-$
**have** *43*: $?F * ?F = ?F$
**using** *assms*(*1*) *forest-components-equivalence preorder-idempotent* **by** *auto*
**hence** $?F * ?e * ?F = ?F * ?F * ?e * ?F$
**by** *simp*
**also have** ... $= (?F)^\star * ?F * ?e * (?F)^\star$
**by** (*simp add*: *assms*(*1*) *forest-components-star*)
**also have** ... $= (?F * ?F)^\star * ?F * ?e * (?F * ?F)^\star$
**using** *43* **by** *simp*
**finally show** *?thesis*
**using** *41* **by** *simp*
**qed**
**hence** *44*: $(?H * d)^\star * ?H * ?e * (?H * d)^\star \leq -\ ?H$
**proof** $-$

**have** *45*: $?H \le ?F$
**using** *assms*(*1*, *4*, *8*) *H-below-F* **by** *blast*
**hence** *46*:$?H * ?e \le ?F * ?e$
**by** (*simp add*: *mult-left-isotone*)
**have** $d \le f \sqcup f^T$
**using** *assms*(*8*) *sup.left-commute sup-commute* **by** *auto*
**also have** ... $\le ?F$
**by** (*metis forest-components-increasing le-supI2 star.circ-back-loop-fixpoint star.circ-increasing sup.bounded-iff*)
**finally have** $d \le ?F$
**by** *simp*
**hence** $?H * d \le ?F * ?F$
**using** *45 mult-isotone* **by** *auto*
**hence** *47*: $(?H * d)^\star \le (?F * ?F)^\star$
**by** (*simp add*: *star-isotone*)
**hence** $(?H * d)^\star * ?H * ?e * (?H * d)^\star \le (?H * d)^\star * ?F * ?e * (?H * d)^\star$
**using** *46* **by** (*metis mult-left-isotone mult-right-isotone mult-assoc*)
**also have** ... $\le (?F * ?F)^\star * ?F * ?e * (?F * ?F)^\star$
**using** *47 mult-left-isotone mult-right-isotone* **by** (*simp add*: *comp-isotone*)
**also have** ... $\le - ?F$
**using** *42* **by** *simp*
**also have** ... $\le - ?H$
**using** *45* **by** (*simp add*: *p-antitone*)
**finally show** *?thesis*
**by** *simp*
**qed**
**have** $(?H * (d \sqcup ?e))^+ = (?H * (d \sqcup ?e))^\star * (?H * (d \sqcup ?e))$
**using** *star.circ-plus-same* **by** *auto*
**also have** ... $= ((?H * d)^\star \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star) * (?H * (d \sqcup ?e))$
**using** *assms*(*4*) *e-not-bot forest-components-equivalence minarc-arc minarc-bot-iff path-through-components* **by** *auto*
**also have** ... $= (?H * d)^\star * (?H * (d \sqcup ?e)) \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * (?H * (d \sqcup ?e))$
**using** *mult-right-dist-sup* **by** *auto*
**also have** ... $= (?H * d)^\star * (?H * d \sqcup ?H * ?e) \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * (?H * d \sqcup ?H * ?e)$
**by** (*simp add*: *mult-left-dist-sup*)
**also have** ... $= (?H * d)^\star * ?H * d \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * (?H * d \sqcup ?H * ?e)$
**using** *mult-left-dist-sup mult-assoc* **by** *auto*
**also have** ... $= (?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * (?H * d \sqcup ?H * ?e)$
**by** (*simp add*: *star.circ-plus-same mult-assoc*)
**also have** ... $= (?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * ?H * d \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * ?H * ?e$
**by** (*simp add*: *mult.semigroup-axioms semiring.distrib-left sup.semigroup-axioms semigroup.assoc*)

**also have** ... $\leq$ $(?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * ?H * d \sqcup (?H * d)^\star * ?H * ?e$
**proof** −
**have** $?e * (?H * d)^\star * ?H * ?e \leq ?e * top * ?e$
**by** (*metis comp-associative comp-inf.coreflexive-idempotent comp-inf.coreflexive-transitive comp-isotone top.extremum*)
**also have** ... $\leq$ $?e$
**using** *e-not-bot arc-top-arc minarc-arc minarc-bot-iff* **by** *auto*
**finally have** $?e * (?H * d)^\star * ?H * ?e \leq ?e$
**by** *simp*
**hence** $(?H * d)^\star * ?H * ?e * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
**by** (*simp add*: *comp-associative comp-isotone*)
**thus** *?thesis*
**using** *sup-right-isotone* **by** *blast*
**qed**
**also have** ... = $(?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star * ?H * d$
**by** (*simp add*: *order.eq-iff ac-simps*)
**also have** ... = $(?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^+$
**using** *star.circ-plus-same mult-assoc* **by** *auto*
**also have** ... = $(?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e * (1 \sqcup (?H * d)^+)$
**by** (*simp add*: *mult-left-dist-sup sup-assoc*)
**also have** ... = $(?H * d)^+ \sqcup (?H * d)^\star * ?H * ?e * (?H * d)^\star$
**by** (*simp add*: *star-left-unfold-equal*)
**also have** ... $\leq$ $- ?H$
**using** *40 44 assms*(*5*) *sup.boundedI* **by** *blast*
**finally show** *?thesis*
**using** *pseudo-complement* **by** *force*
**qed**
**qed**
**thus** *?thesis*
**by** *blast*
**qed**

### 4.4.2 Identifying arcs

The expression $d \sqcap \top e^\top H \sqcap (Hd^\top)^* Ha^\top \top$ identifies the edge incoming to the component that the *selected-edge*, $e$, is outgoing from and which is on the path from edge $a$ to $e$. Here, we prove this expression is an *arc*.

**lemma** *shows-arc-x*:
**assumes** *forest-modulo-equivalence H d*
**and** *forest-modulo-equivalence-path a e H d*
**and** $H * d * (H * d)^\star \leq - H$
**and** $\neg a^T * top \leq H * e * top$
**and** *regular a*
**and** *regular e*
**and** *regular H*
**and** *regular d*

**shows** *arc* $(d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$
**proof** −
**let** $?x = d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top$
**have** *1*:*regular ?x*
**using** *assms*(*5*, *6*, *7*, *8*) *regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
**have** *2*: $a^T * top * a \leq 1$
**using** *arc-expanded assms*(*2*) *forest-modulo-equivalence-path-def* **by** *auto*
**have** *3*: $e * top * e^T \leq 1$
**using** *assms*(*2*) *forest-modulo-equivalence-path-def arc-expanded* **by** *blast*
**have** *4*: $top * ?x * top = top$
**proof** −
**have** $a^T * top \leq (H * d)^\star * H * e * top$
**using** *assms*(*2*) *forest-modulo-equivalence-path-def* **by** *blast*
**also have** ... $= H * e * top \sqcup (H * d)^\star * H * d * H * e * top$
**by** (*metis star.circ-loop-fixpoint star.circ-plus-same sup-commute mult-assoc*)
**finally have** $a^T * top \leq H * e * top \sqcup (H * d)^\star * H * d * H * e * top$
**by** *simp*
**hence** $a^T * top \leq H * e * top \vee a^T * top \leq (H * d)^\star * H * d * H * e * top$
**using** *assms*(*2*, *6*, *7*) *point-in-vector-sup forest-modulo-equivalence-path-def regular-mult-closed vector-mult-closed* **by** *auto*
**hence** $a^T * top \leq (H * d)^\star * H * d * H * e * top$
**using** *assms*(*4*) **by** *blast*
**also have** ... $= (H * d)^\star * H * d * (H * e * top \sqcap H * e * top)$
**by** (*simp add*: *mult-assoc*)
**also have** ... $= (H * d)^\star * H * (d \sqcap (H * e * top)^T) * H * e * top$
**by** (*metis comp-associative covector-inf-comp-3 star.circ-left-top star.circ-top*)
**also have** ... $= (H * d)^\star * H * (d \sqcap top^T * e^T * H^T) * H * e * top$
**using** *conv-dist-comp mult-assoc* **by** *auto*
**also have** ... $= (H * d)^\star * H * (d \sqcap top * e^T * H) * H * e * top$
**using** *assms*(*1*) **by** (*simp add*: *forest-modulo-equivalence-def*)
**finally have** *2*: $a^T * top \leq (H * d)^\star * H * (d \sqcap top * e^T * H) * H * e * top$
**by** *simp*
**hence** $e * top \leq ((H * d)^\star * H * (d \sqcap top * e^T * H) * H)^T * a^T * top$
**proof** −
**have** *bijective* $(e * top) \wedge$ *bijective* $(a^T * top)$
**using** *assms*(*2*) *forest-modulo-equivalence-path-def* **by** *auto*
**thus** *?thesis*
**using** *2* **by** (*metis bijective-reverse mult-assoc*)
**qed**
**also have** ... $= H^T * (d \sqcap top * e^T * H)^T * H^T * (H * d)^{\star T} * a^T * top$
**by** (*simp add*: *conv-dist-comp mult-assoc*)
**also have** ... $= H * (d \sqcap top * e^T * H)^T * H * (H * d)^{\star T} * a^T * top$
**using** *assms*(*1*) *forest-modulo-equivalence-def* **by** *auto*
**also have** ... $= H * (d \sqcap top * e^T * H)^T * H * (d^T * H)^\star * a^T * top$
**using** *assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp conv-star-commute* **by** *auto*
**also have** ... $= H * (d^T \sqcap H * e * top) * H * (d^T * H)^\star * a^T * top$

**using** *assms*(*1*) *conv-dist-comp forest-modulo-equivalence-def comp-associative conv-dist-inf* **by** *auto*

**also have** ... = $H * (d^T \sqcap H * e * top) * (H * d^T)^\star * H * a^T * top$
**by** (*simp add*: *comp-associative star-slide*)

**also have** ... = $H * (d^T \sqcap H * e * top) * ((H * d^T)^\star * H * a^T * top \sqcap (H * d^T)^\star * H * a^T * top)$
**using** *mult-assoc* **by** *auto*

**also have** ... = $H * (d^T \sqcap H * e * top \sqcap ((H * d^T)^\star * H * a^T * top)^T) * (H * d^T)^\star * H * a^T * top$
**by** (*smt comp-inf-vector covector-comp-inf vector-conv-covector vector-top-closed mult-assoc*)

**also have** ... = $H * (d^T \sqcap (top * e^T * H)^T \sqcap ((H * d^T)^\star * H * a^T * top)^T) * (H * d^T)^\star * H * a^T * top$
**using** *assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp mult-assoc* **by** *auto*

**also have** ... = $H * (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)^T * (H * d^T)^\star * H * a^T * top$
**by** (*simp add*: *conv-dist-inf*)

**finally have** *3*: $e * top \le H * ?x^T * (H * d^T)^\star * H * a^T * top$
**by** *auto*

**have** $?x \ne bot$
**proof** (*rule ccontr*)
**assume** $\neg\ ?x \ne bot$
**hence** $e * top = bot$
**using** *3 le-bot* **by** *auto*
**thus** *False*
**using** *assms*(*2*, *4*) *forest-modulo-equivalence-path-def mult-assoc semiring.mult-zero-right* **by** *auto*
**qed**
**thus** *?thesis*
**using** *1* **using** *tarski* **by** *blast*
**qed**

**have** *5*: $?x * top * ?x^T \le 1$
**proof** −
**have** *51*: $H * (d * H)^\star \sqcap d * H * d^T \le 1$
**proof** −
**have** *511*: $d * (H * d)^\star \le -H$
**using** *assms*(*1*, *3*) *forest-modulo-equivalence-def preorder-idempotent schroeder-4-p triple-schroeder-p* **by** *fastforce*
**hence** $(d * H)^\star * d \le -H$
**using** *star-slide* **by** *auto*
**hence** $H * (d^T * H)^\star \le -d$
**by** (*smt assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp conv-star-commute schroeder-4-p star-slide*)
**hence** $H * (d * H)^\star \le -d^T$
**using** *511* **by** (*metis assms*(*1*) *forest-modulo-equivalence-def schroeder-5-p star-slide*)
**hence** $H * (d * H)^\star \le -(H * d^T)$
**by** (*metis assms*(*3*) *p-antitone-iff schroeder-4-p star-slide mult-assoc*)

**hence** $H * (d * H)^\star \sqcap H * d^T \le bot$
**by** (*simp add*: *bot-unique pseudo-complement*)
**hence** $H * d * (H * (d * H)^\star \sqcap H * d^T) \le 1$
**by** (*simp add*: *bot-unique*)
**hence** *512*: $H * d * H * (d * H)^\star \sqcap H * d * H * d^T \le 1$
**using** *univalent-comp-left-dist-inf assms*(*1*) *forest-modulo-equivalence-def mult-assoc* **by** *fastforce*
**hence** *513*: $H * d * H * (d * H)^\star \sqcap d * H * d^T \le 1$
**proof** −
**have** $d * H * d^T \le H * d * H * d^T$
**by** (*metis assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp conv-involutive mult-1-right mult-left-isotone*)
**thus** *?thesis*
**using** *512* **by** (*smt dual-order.trans p-antitone p-shunting-swap regular-one-closed*)
**qed**
**have** $d^T * H * d \le 1 \sqcup - H$
**using** *assms*(*1*) *forest-modulo-equivalence-def dTransHd-le-1 le-supI1* **by** *blast*
**hence** $(- 1 \sqcap H) * d^T * H \le - d^T$
**by** (*metis assms*(*1*) *forest-modulo-equivalence-def dTransHd-le-1 inf.sup-monoid.add-commute le-infI2 p-antitone-iff regular-one-closed schroeder-4-p mult-assoc*)
**hence** $d * (- 1 \sqcap H) * d^T \le - H$
**by** (*metis assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp schroeder-3-p triple-schroeder-p*)
**hence** $H \sqcap d * (- 1 \sqcap H) * d^T \le 1$
**by** (*metis inf.coboundedI1 p-antitone-iff p-shunting-swap regular-one-closed*)
**hence** $H \sqcap d * d^T \sqcup H \sqcap d * (- 1 \sqcap H) * d^T \le 1$
**using** *assms*(*1*) *forest-modulo-equivalence-def le-supI* **by** *blast*
**hence** $H \sqcap (d * 1 * d^T \sqcup d * (- 1 \sqcap H) * d^T) \le 1$
**using** *comp-inf.semiring.distrib-left* **by** *auto*
**hence** $H \sqcap (d * (1 \sqcup (- 1 \sqcap H)) * d^T) \le 1$
**by** (*simp add*: *mult-left-dist-sup mult-right-dist-sup*)
**hence** *514*: $H \sqcap d * H * d^T \le 1$
**by** (*metis assms*(*1*) *forest-modulo-equivalence-def comp-inf.semiring.distrib-left inf.le-iff-sup inf.sup-monoid.add-commute inf-top-right regular-one-closed stone*)
**thus** *?thesis*
**proof** −
**have** $H \sqcap d * H * d^T \sqcup H * d * H * (d * H)^\star \sqcap d * H * d^T \le 1$
**using** *513 514* **by** *simp*
**hence** $d * H * d^T \sqcap (H \sqcup H * d * H * (d * H)^\star) \le 1$
**by** (*simp add*: *comp-inf.semiring.distrib-left inf.sup-monoid.add-commute*)
**hence** $d * H * d^T \sqcap H * (1 \sqcup d * H * (d * H)^\star) \le 1$
**by** (*simp add*: *mult-left-dist-sup mult-assoc*)
**thus** *?thesis*
**by** (*simp add*: *inf.sup-monoid.add-commute star-left-unfold-equal*)

**qed**
**qed**
**have** $?x * top * ?x^T = (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top) * top * (d^T \sqcap H^T * e^{TT} * top^T \sqcap top^T * a^{TT} * H^T * (d^{TT} * H^T)^\star)$
**by** (*simp add*: *conv-dist-comp conv-dist-inf conv-star-commute mult-assoc*)
**also have** ... $= (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top) * top * (d^T \sqcap H * e * top \sqcap top * a * H * (d * H)^\star)$
**using** *assms*(*1*) *forest-modulo-equivalence-def* **by** *auto*
**also have** ... $= (H * d^T)^\star * H * a^T * top \sqcap (d \sqcap top * e^T * H) * top * (d^T \sqcap H * e * top \sqcap top * a * H * (d * H)^\star)$
**by** (*metis inf-vector-comp vector-export-comp*)
**also have** ... $= (H * d^T)^\star * H * a^T * top \sqcap (d \sqcap top * e^T * H) * top * top * (d^T \sqcap H * e * top \sqcap top * a * H * (d * H)^\star)$
**by** (*simp add*: *vector-mult-closed*)
**also have** ... $= (H * d^T)^\star * H * a^T * top \sqcap d * ((top * e^T * H)^T \sqcap top) * top * (d^T \sqcap H * e * top \sqcap top * a * H * (d * H)^\star)$
**by** (*simp add*: *covector-comp-inf-1 covector-mult-closed*)
**also have** ... $= (H * d^T)^\star * H * a^T * top \sqcap d * ((top * e^T * H)^T \sqcap (H * e * top)^T) * d^T \sqcap top * a * H * (d * H)^\star$
**by** (*smt comp-associative comp-inf.star-star-absorb comp-inf-vector conv-star-commute covector-comp-inf covector-conv-vector fc-top star.circ-top total-conv-surjective vector-conv-covector vector-inf-comp*)
**also have** ... $= (H * d^T)^\star * H * a^T * top \sqcap top * a * H * (d * H)^\star \sqcap d * ((top * e^T * H)^T \sqcap (H * e * top)^T) * d^T$
**using** *inf.sup-monoid.add-assoc inf.sup-monoid.add-commute* **by** *auto*
**also have** ... $= (H * d^T)^\star * H * a^T * top * top * a * H * (d * H)^\star \sqcap d * ((top * e^T * H)^T \sqcap (H * e * top)^T) * d^T$
**by** (*smt comp-inf.star.circ-decompose-9 comp-inf.star-star-absorb comp-inf-covector fc-top star.circ-decompose-11 star.circ-top vector-export-comp*)
**also have** ... $= (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star \sqcap d * (H * e * top \sqcap top * e^T * H) * d^T$
**using** *assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp mult-assoc* **by** *auto*
**also have** ... $= (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star \sqcap d * H * e * top * e^T * H * d^T$
**by** (*metis comp-inf-covector inf-top.left-neutral mult-assoc*)
**also have** ... $\leq (H * d^T)^\star * (H * d)^\star * H \sqcap d * H * e * top * e^T * H * d^T$
**proof** −
**have** $(H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star \leq (H * d^T)^\star * H * 1 * H * (d * H)^\star$
**using** *2* **by** (*metis comp-associative comp-isotone mult-left-isotone mult-semi-associative star.circ-transitive-equal*)
**also have** ... $= (H * d^T)^\star * H * (d * H)^\star$
**using** *assms*(*1*) *forest-modulo-equivalence-def mult.semigroup-axioms preorder-idempotent semigroup.assoc* **by** *fastforce*
**also have** ... $= (H * d^T)^\star * (H * d)^\star * H$
**by** (*metis star-slide mult-assoc*)
**finally show** *?thesis*
**using** *inf.sup-left-isotone* **by** *auto*

**qed**
**also have** ... $\le (H * d^T)^\star * (H * d)^\star * H \sqcap d * H * d^T$
**proof** −
**have** $d * H * e * top * e^T * H * d^T \le d * H * 1 * H * d^T$
**using** *3* **by** (*metis comp-isotone idempotent-one-closed mult-left-isotone mult-sub-right-one mult-assoc*)
**also have** ... $\le d * H * d^T$
**by** (*metis assms*(*1*) *forest-modulo-equivalence-def mult-left-isotone mult-one-associative mult-semi-associative preorder-idempotent*)
**finally show** *?thesis*
**using** *inf.sup-right-isotone* **by** *auto*
**qed**
**also have** ... $= H * (d^T * H)^\star * (H * d)^\star * H \sqcap d * H * d^T$
**by** (*metis assms*(*1*) *forest-modulo-equivalence-def comp-associative preorder-idempotent star-slide*)
**also have** ... $= H * ((d^T * H)^\star \sqcup (H * d)^\star) * H \sqcap d * H * d^T$
**by** (*simp add*: *assms*(*1*) *expand-forest-modulo-equivalence mult.semigroup-axioms semigroup.assoc*)
**also have** ... $= (H * (d^T * H)^\star * H \sqcup H * (H * d)^\star * H) \sqcap d * H * d^T$
**by** (*simp add*: *mult-left-dist-sup mult-right-dist-sup*)
**also have** ... $= (H * d^T)^\star * H \sqcap d * H * d^T \sqcup H * (d * H)^\star \sqcap d * H * d^T$
**by** (*smt assms*(*1*) *forest-modulo-equivalence-def inf-sup-distrib2 mult.semigroup-axioms preorder-idempotent star-slide semigroup.assoc*)
**also have** ... $\le (H * d^T)^\star * H \sqcap d * H * d^T \sqcup 1$
**using** *51 comp-inf.semiring.add-left-mono* **by** *blast*
**finally have** $?x * top * ?x^T \le 1$
**using** *51* **by** (*smt assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp conv-dist-inf conv-dist-sup conv-involutive conv-star-commute equivalence-one-closed mult.semigroup-axioms sup.absorb2 semigroup.assoc conv-isotone conv-order*)
**thus** *?thesis*
**by** *simp*
**qed**
**have** *6*: $?x^T * top * ?x \le 1$
**proof** −
**have** $?x^T * top * ?x = (d^T \sqcap H^T * e^{TT} * top^T \sqcap top^T * a^{TT} * H^T * (d^{TT} * H^T)^\star) * top * (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$
**by** (*simp add*: *conv-dist-comp conv-dist-inf conv-star-commute mult-assoc*)
**also have** ... $= (d^T \sqcap H * e * top \sqcap top * a * H * (d * H)^\star) * top * (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$
**using** *assms*(*1*) *forest-modulo-equivalence-def* **by** *auto*
**also have** ... $= H * e * top \sqcap (d^T \sqcap top * a * H * (d * H)^\star) * top * (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$
**by** (*smt comp-associative inf.sup-monoid.add-assoc inf.sup-monoid.add-commute star.circ-left-top star.circ-top vector-inf-comp*)
**also have** ... $= H * e * top \sqcap d^T * ((top * a * H * (d * H)^\star)^T \sqcap top) * (d \sqcap top * e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$
**by** (*simp add*: *covector-comp-inf-1 covector-mult-closed*)
**also have** ... $= H * e * top \sqcap d^T * (d * H)^{\star T} * H * a^T * top * (d \sqcap top *$

$e^T * H \sqcap (H * d^T)^\star * H * a^T * top)$
    **using** *assms*(*1*) *forest-modulo-equivalence-def comp-associative conv-dist-comp* **by** *auto*
  **also have** ... $= H * e * top \sqcap d^T * (d * H)^{\star T} * H * a^T * top * (d \sqcap (H * d^T)^\star * H * a^T * top) \sqcap top * e^T * H$
    **by** (*smt comp-associative comp-inf-covector inf.sup-monoid.add-assoc inf.sup-monoid.add-commute*)
  **also have** ... $= H * e * top \sqcap d^T * (d * H)^{\star T} * H * a^T * (top \sqcap ((H * d^T)^\star * H * a^T * top)^T) * d \sqcap top * e^T * H$
    **by** (*metis comp-associative comp-inf-vector vector-conv-covector vector-top-closed*)
  **also have** ... $= H * e * top \sqcap (H * e * top)^T \sqcap d^T * (d * H)^{\star T} * H * a^T * ((H * d^T)^\star * H * a^T * top)^T * d$
    **by** (*smt assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp inf.left-commute inf.sup-monoid.add-commute symmetric-top-closed mult-assoc inf-top.left-neutral*)
  **also have** ... $= H * e * top * (H * e * top)^T \sqcap d^T * (d * H)^{\star T} * H * a^T * ((H * d^T)^\star * H * a^T * top)^T * d$
    **using** *vector-covector vector-mult-closed* **by** *auto*
  **also have** ... $= H * e * top * top^T * e^T * H^T \sqcap d^T * (d * H)^{\star T} * H * a^T * top^T * a^{TT} * H^T * (H * d^T)^{\star T} * d$
    **by** (*smt conv-dist-comp mult.semigroup-axioms symmetric-top-closed semigroup.assoc*)
  **also have** ... $= H * e * top * top * e^T * H \sqcap d^T * (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star * d$
    **using** *assms*(*1*) *forest-modulo-equivalence-def conv-dist-comp conv-star-commute* **by** *auto*
  **also have** ... $= H * e * top * e^T * H \sqcap d^T * (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star * d$
    **using** *vector-top-closed mult-assoc* **by** *auto*
  **also have** ... $\leq H \sqcap d^T * (H * d^T)^\star * H * (d * H)^\star * d$
  **proof** −
    **have** $H * e * top * e^T * H \leq H * 1 * H$
      **using** *3* **by** (*metis comp-associative mult-left-isotone mult-right-isotone*)
    **also have** ... $= H$
      **using** *assms*(*1*) *forest-modulo-equivalence-def preorder-idempotent* **by** *auto*
    **finally have** *611*: $H * e * top * e^T * H \leq H$
      **by** *simp*
    **have** $d^T * (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star * d \leq d^T * (H * d^T)^\star * H * 1 * H * (d * H)^\star * d$
      **using** *2* **by** (*metis comp-associative mult-left-isotone mult-right-isotone*)
    **also have** ... $= d^T * (H * d^T)^\star * H * (d * H)^\star * d$
      **using** *assms*(*1*) *forest-modulo-equivalence-def mult.semigroup-axioms preorder-idempotent semigroup.assoc* **by** *fastforce*
    **finally have** $d^T * (H * d^T)^\star * H * a^T * top * a * H * (d * H)^\star * d \leq d^T * (H * d^T)^\star * H * (d * H)^\star * d$
      **by** *simp*
    **thus** *?thesis*
      **using** *611 comp-inf.comp-isotone* **by** *blast*

```
    qed
    also have ... = H ⊓ (d^T * H)⋆ * d^T * H * d * (H * d)⋆
      using star-slide mult-assoc by auto
    also have ... ≤ H ⊓ (d^T * H)⋆ * (H * d)⋆
    proof –
      have (d^T * H)⋆ * d^T * H * d * (H * d)⋆ ≤ (d^T * H)⋆ * 1 * (H * d)⋆
        by (smt assms(1) forest-modulo-equivalence-def conv-dist-comp
mult-left-isotone mult-right-isotone preorder-idempotent mult-assoc)
      also have ... = (d^T * H)⋆ * (H * d)⋆
        by simp
      finally show ?thesis
        using inf.sup-right-isotone by blast
    qed
    also have ... = H ⊓ ((d^T * H)⋆ ⊔ (H * d)⋆)
      by (simp add: assms(1) expand-forest-modulo-equivalence)
    also have ... = H ⊓ (d^T * H)⋆ ⊔ H ⊓ (H * d)⋆
      by (simp add: comp-inf.semiring.distrib-left)
    also have ... = 1 ⊔ H ⊓ (d^T * H)⁺ ⊔ H ⊓ (H * d)⁺
    proof –
      have 612: H ⊓ (H * d)⋆ = 1 ⊔ H ⊓ (H * d)⁺
        using assms(1) forest-modulo-equivalence-def reflexive-inf-star by blast
      have H ⊓ (d^T * H)⋆ = 1 ⊔ H ⊓ (d^T * H)⁺
        using assms(1) forest-modulo-equivalence-def reflexive-inf-star by auto
      thus ?thesis
        using 612 sup-assoc sup-commute by auto
    qed
    also have ... ≤ 1
    proof –
      have 613: H ⊓ (H * d)⁺ ≤ 1
        by (metis assms(3) inf.coboundedI1 p-antitone-iff p-shunting-swap
regular-one-closed)
      hence H ⊓ (d^T * H)⁺ ≤ 1
        by (metis assms(1) forest-modulo-equivalence-def conv-dist-comp
conv-dist-inf conv-plus-commute coreflexive-symmetric)
      thus ?thesis
        by (simp add: 613)
    qed
    finally show ?thesis
      by simp
  qed
  have 7:bijective (?x * top)
    using 4 5 6 arc-expanded by blast
  have bijective (?x^T * top)
    using 4 5 6 arc-expanded by blast
  thus ?thesis
    using 7 by simp
qed
```

To maintain that $f$ can be extended to a minimum spanning forest we identify an edge, $i = v \sqcap \overline{F}e\top \sqcap \top e^\top F$, that may be exchanged with the *selected-edge*, $e$. Here, we show that $i$ is an *arc*.

**lemma** *boruvka-edge-arc*:
  **assumes** *equivalence* $F$
    **and** *forest* $v$
    **and** *arc* $e$
    **and** *regular* $F$
    **and** $F \leq$ *forest-components* $(F \sqcap v)$
    **and** *regular* $v$
    **and** $v * e^T = bot$
    **and** $e * F * e = bot$
    **and** $e^T \leq v^\star$
    **and** $e \neq bot$
  **shows** *arc* $(v \sqcap -F * e * top \sqcap top * e^T * F)$
**proof** –
  **let** $?i = v \sqcap -F * e * top \sqcap top * e^T * F$
  **have** $1$: $?i^T * top * ?i \leq 1$
  **proof** –
    **have** $?i^T * top * ?i = (v^T \sqcap top * e^T * -F \sqcap F * e * top) * top * (v \sqcap -F * e * top \sqcap top * e^T * F)$
      **using** *assms*(*1*) *conv-complement conv-dist-comp conv-dist-inf mult.semigroup-axioms semigroup.assoc* **by** *fastforce*
    **also have** ... $= F * e * top \sqcap (v^T \sqcap top * e^T * -F) * top * (v \sqcap -F * e * top) \sqcap top * e^T * F$
      **by** (*smt covector-comp-inf covector-mult-closed inf-vector-comp vector-export-comp vector-top-closed*)
    **also have** ... $= F * e * top \sqcap (v^T \sqcap top * e^T * -F) * top * top * (v \sqcap -F * e * top) \sqcap top * e^T * F$
      **by** (*simp add*: *comp-associative*)
    **also have** ... $= F * e * top \sqcap v^T * (top \sqcap (top * e^T * -F)^T) * top * (v \sqcap -F * e * top) \sqcap top * e^T * F$
      **using** *comp-associative comp-inf-vector-1* **by** *auto*
    **also have** ... $= F * e * top \sqcap v^T * (top \sqcap (top * e^T * -F)^T) * (top \sqcap (-F * e * top)^T) * v \sqcap top * e^T * F$
      **by** (*smt comp-inf-vector conv-dist-comp mult.semigroup-axioms symmetric-top-closed semigroup.assoc*)
    **also have** ... $= F * e * top \sqcap v^T * (top * e^T * -F)^T * (-F * e * top)^T * v \sqcap top * e^T * F$
      **by** *simp*
    **also have** ... $= F * e * top \sqcap v^T * -F^T * e^{TT} * top^T * top^T * e^T * -F^T * v \sqcap top * e^T * F$
      **by** (*metis comp-associative conv-complement conv-dist-comp*)
    **also have** ... $= F * e * top \sqcap v^T * -F * e * top * top * e^T * -F * v \sqcap top * e^T * F$
      **by** (*simp add*: *assms*(*1*))
    **also have** ... $= F * e * top \sqcap v^T * -F * e * top \sqcap top * e^T * -F * v \sqcap top * e^T * F$
      **by** (*metis comp-associative comp-inf-covector inf.sup-monoid.add-assoc inf-top.left-neutral vector-top-closed*)
    **also have** ... $= (F \sqcap v^T * -F) * e * top \sqcap top * e^T * -F * v \sqcap top * e^T * F$

**using** *assms*(*3*) *injective-comp-right-dist-inf mult-assoc* **by** *auto*
**also have** ... $= (F \sqcap v^T * -F) * e * top \sqcap top * e^T * (F \sqcap -F * v)$
**using** *assms*(*3*) *conv-dist-comp inf.sup-monoid.add-assoc inf.sup-monoid.add-commute mult.semigroup-axioms univalent-comp-left-dist-inf semigroup.assoc* **by** *fastforce*
**also have** ... $= (F \sqcap v^T * -F) * e * top * top * e^T * (F \sqcap -F * v)$
**by** (*metis comp-associative comp-inf-covector inf-top.left-neutral vector-top-closed*)
**also have** ... $= (F \sqcap v^T * -F) * e * top * e^T * (F \sqcap -F * v)$
**by** (*simp add*: *comp-associative*)
**also have** ... $\leq (F \sqcap v^T * -F) * (F \sqcap -F * v)$
**by** (*smt assms*(*3*) *conv-dist-comp mult-left-isotone shunt-bijective symmetric-top-closed top-right-mult-increasing mult-assoc*)
**also have** ... $\leq (F \sqcap v^T * -F) * (F \sqcap -F * v) \sqcap F$
**by** (*metis assms*(*1*) *inf.absorb1 inf.cobounded1 mult-isotone preorder-idempotent*)
**also have** ... $\leq (F \sqcap v^T * -F) * (F \sqcap -F * v) \sqcap (F \sqcap v)^{T\star} * (F \sqcap v)^\star$
**using** *assms*(*5*) *comp-inf.mult-right-isotone* **by** *auto*
**also have** ... $\leq (-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^{T\star} * (F \sqcap v)^\star$
**proof** −
**have** $F \sqcap v^T * -F \leq (v^T \sqcap F * -F^T) * -F$
**by** (*metis conv-complement dedekind-2 inf-commute*)
**also have** ... $= (v^T \sqcap -F^T) * -F$
**using** *assms*(*1*) *equivalence-comp-left-complement* **by** *simp*
**finally have** $F \sqcap v^T * -F \leq F \sqcap (v^T \sqcap -F) * -F$
**using** *assms*(*1*) **by** *auto*
**hence** *11*: $F \sqcap v^T * -F = F \sqcap (-F \sqcap v^T) * -F$
**by** (*metis inf.antisym-conv inf.sup-monoid.add-commute comp-left-subdist-inf inf.boundedE inf.sup-right-isotone*)
**hence** $F^T \sqcap -F^T * v^{TT} = F^T \sqcap -F^T * (-F^T \sqcap v^{TT})$
**by** (*metis* (*full-types*) *assms*(*1*) *conv-complement conv-dist-comp conv-dist-inf*)
**hence** *12*: $F \sqcap -F * v = F \sqcap -F * (-F \sqcap v)$
**using** *assms*(*1*) **by** (*simp add*: *abel-semigroup.commute inf.abel-semigroup-axioms*)
**have** $(F \sqcap v^T * -F) * (F \sqcap -F * v) = (F \sqcap (-F \sqcap v^T) * -F) * (F \sqcap -F * (-F \sqcap v))$
**using** *11 12* **by** *auto*
**also have** ... $\leq (-F \sqcap v^T) * -F * -F * (-F \sqcap v)$
**by** (*metis comp-associative comp-isotone inf.cobounded2*)
**finally show** *?thesis*
**using** *comp-inf.mult-left-isotone* **by** *blast*
**qed**
**also have** ... $= ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star) \sqcup ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^\star)$
**by** (*metis comp-associative inf-sup-distrib1 star.circ-loop-fixpoint*)
**also have** ... $= ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v^T) * (F \sqcap v)^{T\star} * (F \sqcap v)^\star) \sqcup ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^\star)$
**using** *assms*(*1*) *conv-dist-inf* **by** *auto*

**also have** ... = $bot \sqcup ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^\star)$
**proof** −
**have** $(-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v^T) * (F \sqcap v)^{T\star} * (F \sqcap v)^\star \leq bot$
**using** *assms*(*1*, *2*) *forests-bot-2* **by** (*simp add*: *comp-associative*)
**thus** *?thesis*
**using** *le-bot* **by** *blast*
**qed**
**also have** ... = $(-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (1 \sqcup (F \sqcap v)^\star * (F \sqcap v))$
**by** (*simp add*: *star.circ-plus-same star-left-unfold-equal*)
**also have** ... = $((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap 1) \sqcup ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^\star * (F \sqcap v))$
**by** (*simp add*: *comp-inf.semiring.distrib-left*)
**also have** ... $\leq 1 \sqcup ((-F \sqcap v^T) * -F * -F * (-F \sqcap v) \sqcap (F \sqcap v)^\star * (F \sqcap v))$
**using** *sup-left-isotone* **by** *auto*
**also have** ... $\leq 1 \sqcup bot$
**using** *assms*(*1*, *2*) *forests-bot-3 comp-inf.semiring.add-left-mono* **by** *simp*
**finally show** *?thesis*
**by** *simp*
**qed**
**have** *2*: $?i * top * ?i^T \leq 1$
**proof** −
**have** $?i * top * ?i^T = (v \sqcap -F * e * top \sqcap top * e^T * F) * top * (v^T \sqcap (-F * e * top)^T \sqcap (top * e^T * F)^T)$
**by** (*simp add*: *conv-dist-inf*)
**also have** ... = $(v \sqcap -F * e * top \sqcap top * e^T * F) * top * (v^T \sqcap top^T * e^T * -F^T \sqcap F^T * e^{TT} * top^T)$
**by** (*simp add*: *conv-complement conv-dist-comp mult-assoc*)
**also have** ... = $(v \sqcap -F * e * top \sqcap top * e^T * F) * top * (v^T \sqcap top * e^T * -F \sqcap F * e * top)$
**by** (*simp add*: *assms*(*1*))
**also have** ... = $-F * e * top \sqcap (v \sqcap top * e^T * F) * top * (v^T \sqcap top * e^T * -F \sqcap F * e * top)$
**by** (*smt inf.left-commute inf.sup-monoid.add-assoc vector-export-comp*)
**also have** ... = $-F * e * top \sqcap (v \sqcap top * e^T * F) * top * (v^T \sqcap F * e * top) \sqcap top * e^T * -F$
**by** (*smt comp-inf-covector inf.sup-monoid.add-assoc inf.sup-monoid.add-commute mult-assoc*)
**also have** ... = $-F * e * top \sqcap (v \sqcap top * e^T * F) * top * top * (v^T \sqcap F * e * top) \sqcap top * e^T * -F$
**by** (*simp add*: *mult-assoc*)
**also have** ... = $-F * e * top \sqcap v * ((top * e^T * F)^T \sqcap top) * top * (v^T \sqcap F * e * top) \sqcap top * e^T * -F$
**by** (*simp add*: *comp-inf-vector-1 mult.semigroup-axioms semigroup.assoc*)
**also have** ... = $-F * e * top \sqcap v * ((top * e^T * F)^T \sqcap top) * (top \sqcap (F * e * top)^T) * v^T \sqcap top * e^T * -F$
**by** (*smt comp-inf-vector covector-comp-inf vector-conv-covector*

*vector-mult-closed vector-top-closed*)
**also have** ... $= -F * e * top \sqcap v * (top * e^T * F)^T * (F * e * top)^T * v^T \sqcap top * e^T * -F$
**by** *simp*
**also have** ... $= -F * e * top \sqcap v * F^T * e^{TT} * top^T * top^T * e^T * F^T * v^T \sqcap top * e^T * -F$
**by** (*metis comp-associative conv-dist-comp*)
**also have** ... $= -F * e * top \sqcap v * F * e * top * top * e^T * F * v^T \sqcap top * e^T * -F$
**using** *assms*(*1*) **by** *auto*
**also have** ... $= -F * e * top \sqcap v * F * e * top \sqcap top * e^T * F * v^T \sqcap top * e^T * -F$
**by** (*smt comp-associative comp-inf-covector inf.sup-monoid.add-assoc inf-top.left-neutral vector-top-closed*)
**also have** ... $= (-F \sqcap v * F) * e * top \sqcap top * e^T * F * v^T \sqcap top * e^T * -F$
**using** *injective-comp-right-dist-inf assms*(*3*) *mult.semigroup-axioms semigroup.assoc* **by** *fastforce*
**also have** ... $= (-F \sqcap v * F) * e * top \sqcap top * e^T * (F * v^T \sqcap -F)$
**using** *injective-comp-right-dist-inf assms*(*3*) *conv-dist-comp inf.sup-monoid.add-assoc mult.semigroup-axioms univalent-comp-left-dist-inf semigroup.assoc* **by** *fastforce*
**also have** ... $= (-F \sqcap v * F) * e * top * top * e^T * (F * v^T \sqcap -F)$
**by** (*metis inf-top-right vector-export-comp vector-top-closed*)
**also have** ... $= (-F \sqcap v * F) * e * top * e^T * (F * v^T \sqcap -F)$
**by** (*simp add*: *comp-associative*)
**also have** ... $\leq (-F \sqcap v * F) * (F * v^T \sqcap -F)$
**by** (*smt assms*(*3*) *conv-dist-comp mult.semigroup-axioms mult-left-isotone shunt-bijective symmetric-top-closed top-right-mult-increasing semigroup.assoc*)
**also have** ... $= (-F \sqcap v * F) * ((v * F)^T \sqcap -F)$
**by** (*simp add*: *assms*(*1*) *conv-dist-comp*)
**also have** ... $= (-F \sqcap v * F) * (-F \sqcap v * F)^T$
**using** *assms*(*1*) *conv-complement conv-dist-inf* **by** (*simp add*: *inf.sup-monoid.add-commute*)
**also have** ... $\leq (-F \sqcap v) * (F \sqcap v)^\star * (F \sqcap v)^{T\star} * (-F \sqcap v)^T$
**proof** −
**let** *?Fv* $= F \sqcap v$
**have** $-F \sqcap v * F \leq -F \sqcap v * (F \sqcap v)^{T\star} * (F \sqcap v)^\star$
**using** *assms*(*5*) *inf.sup-right-isotone mult-right-isotone comp-associative* **by** *auto*
**also have** ... $\leq -F \sqcap v * (F \sqcap v)^\star$
**proof** −
**have** $v * v^T \leq 1$
**by** (*simp add*: *assms*(*2*))
**hence** $v * v^T * F \leq F$
**using** *assms*(*1*) *dual-order.trans mult-left-isotone* **by** *blast*
**hence** $v * v^T * F^{T\star} * F^\star \leq F$
**by** (*metis assms*(*1*) *mult-1-right preorder-idempotent star.circ-sup-one-right-unfold star.circ-transitive-equal star-one star-simulation-right-equal mult-assoc*)

**hence** $v * (F \sqcap v)^T * F^{T\star} * F^\star \leq F$
**by** (*meson conv-isotone dual-order.trans inf.cobounded2 inf.sup-monoid.add-commute mult-left-isotone mult-right-isotone*)
**hence** $v * (F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star \leq F$
**by** (*meson conv-isotone dual-order.trans inf.cobounded2 inf.sup-monoid.add-commute mult-left-isotone mult-right-isotone comp-isotone conv-dist-inf inf.cobounded1 star-isotone*)
**hence** $-F \sqcap v * (F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star \leq bot$
**using** *order.eq-iff p-antitone pseudo-complement* **by** *auto*
**hence** $(-F \sqcap v * (F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star) \sqcup v * (v \sqcap F)^\star \leq v * (v \sqcap F)^\star$
**using** *bot-least le-bot* **by** *fastforce*
**hence** $(-F \sqcup v * (v \sqcap F)^\star) \sqcap (v * (F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star \sqcup v * (v \sqcap F)^\star) \leq v * (v \sqcap F)^\star$
**by** (*simp add*: *sup-inf-distrib2*)
**hence** $(-F \sqcup v * (v \sqcap F)^\star) \sqcap v * ((F \sqcap v)^T * (F \sqcap v)^{T\star} \sqcup 1) * (v \sqcap F)^\star \leq v * (v \sqcap F)^\star$
**by** (*simp add*: *inf.sup-monoid.add-commute mult.semigroup-axioms mult-left-dist-sup mult-right-dist-sup semigroup.assoc*)
**hence** $(-F \sqcup v * (v \sqcap F)^\star) \sqcap v * (F \sqcap v)^{T\star} * (v \sqcap F)^\star \leq v * (v \sqcap F)^\star$
**by** (*simp add*: *star-left-unfold-equal sup-commute*)
**hence** $-F \sqcap v * (F \sqcap v)^{T\star} * (v \sqcap F)^\star \leq v * (v \sqcap F)^\star$
**using** *comp-inf.mult-right-sub-dist-sup-left inf.order-lesseq-imp* **by** *blast*
**thus** *?thesis*
**by** (*simp add*: *inf.sup-monoid.add-commute*)
**qed**
**also have** ... $\leq (v \sqcap -F * (F \sqcap v)^{T\star}) * (F \sqcap v)^\star$
**by** (*metis dedekind-2 conv-star-commute inf.sup-monoid.add-commute*)
**also have** ... $\leq (v \sqcap -F * F^{T\star}) * (F \sqcap v)^\star$
**using** *conv-isotone inf.sup-right-isotone mult-left-isotone mult-right-isotone star-isotone* **by** *auto*
**also have** ... $= (v \sqcap -F * F) * (F \sqcap v)^\star$
**by** (*metis assms*(*1*) *equivalence-comp-right-complement mult-left-one star-one star-simulation-right-equal*)
**also have** ... $= (-F \sqcap v) * (F \sqcap v)^\star$
**using** *assms*(*1*) *equivalence-comp-right-complement inf.sup-monoid.add-commute* **by** *auto*
**finally have** $-F \sqcap v * F \leq (-F \sqcap v) * (F \sqcap v)^\star$
**by** *simp*
**hence** $(-F \sqcap v * F) * (-F \sqcap v * F)^T \leq (-F \sqcap v) * (F \sqcap v)^\star * ((-F \sqcap v) * (F \sqcap v)^\star)^T$
**by** (*simp add*: *comp-isotone conv-isotone*)
**also have** ... $= (-F \sqcap v) * (F \sqcap v)^\star * (F \sqcap v)^{T\star} * (-F \sqcap v)^T$
**by** (*simp add*: *comp-associative conv-dist-comp conv-star-commute*)
**finally show** *?thesis*
**by** *simp*
**qed**
**also have** ... $\leq (-F \sqcap v) * ((F \sqcap v^\star) \sqcup (F \sqcap v^{T\star})) * (-F \sqcap v)^T$
**proof** –

**have** $(F \sqcap v)^\star * (F \sqcap v)^{T\star} \leq F^\star * F^{T\star}$
**using** *fc-isotone* **by** *auto*
**also have** ... $\leq F * F$
**by** (*metis assms*(*1*) *preorder-idempotent star.circ-sup-one-left-unfold star.circ-transitive-equal star-right-induct-mult*)
**finally have** *21*: $(F \sqcap v)^\star * (F \sqcap v)^{T\star} \leq F$
**using** *assms*(*1*) *dual-order.trans* **by** *blast*
**have** $(F \sqcap v)^\star * (F \sqcap v)^{T\star} \leq v^\star * v^{T\star}$
**by** (*simp add*: *fc-isotone*)
**hence** $(F \sqcap v)^\star * (F \sqcap v)^{T\star} \leq F \sqcap v^\star * v^{T\star}$
**using** *21* **by** *simp*
**also have** ... $= F \sqcap (v^\star \sqcup v^{T\star})$
**by** (*simp add*: *assms*(*2*) *cancel-separate-eq*)
**finally show** *?thesis*
**by** (*metis assms*(*4*, *6*) *comp-associative comp-inf.semiring.distrib-left comp-isotone inf-pp-semi-commute mult-left-isotone regular-closed-inf*)
**qed**
**also have** ... $\leq (-F \sqcap v) * (F \sqcap v^{T\star}) * (-F \sqcap v)^T \sqcup (-F \sqcap v) * (F \sqcap v^\star) * (-F \sqcap v)^T$
**by** (*simp add*: *mult-left-dist-sup mult-right-dist-sup*)
**also have** ... $\leq (-F \sqcap v) * (-F \sqcap v)^T \sqcup (-F \sqcap v) * (-F \sqcap v)^T$
**proof** −
**have** $(-F \sqcap v) * (F \sqcap v^{T\star}) \leq (-F \sqcap v) * ((F \sqcap v)^{T\star} * (F \sqcap v)^\star \sqcap v^{T\star})$
**by** (*simp add*: *assms*(*5*) *inf.coboundedI1 mult-right-isotone*)
**also have** ... $= (-F \sqcap v) * ((F \sqcap v)^T * (F \sqcap v)^{T\star} * (F \sqcap v)^\star \sqcap v^{T\star}) \sqcup (-F \sqcap v) * ((F \sqcap v)^\star \sqcap v^{T\star})$
**by** (*metis comp-associative comp-inf.mult-right-dist-sup mult-left-dist-sup star.circ-loop-fixpoint*)
**also have** ... $\leq (-F \sqcap v) * (F \sqcap v)^T * top \sqcup (-F \sqcap v) * ((F \sqcap v)^\star \sqcap v^{T\star})$
**by** (*simp add*: *comp-associative comp-isotone inf.coboundedI2 inf.sup-monoid.add-commute le-supI1*)
**also have** ... $\leq (-F \sqcap v) * (F \sqcap v)^T * top \sqcup (-F \sqcap v) * (v^\star \sqcap v^{T\star})$
**by** (*smt comp-inf.mult-right-isotone comp-inf.semiring.add-mono order.eq-iff inf.cobounded2 inf.sup-monoid.add-commute mult-right-isotone star-isotone*)
**also have** ... $\leq bot \sqcup (-F \sqcap v) * (v^\star \sqcap v^{T\star})$
**by** (*metis assms*(*1*, *2*) *forests-bot-1 comp-associative comp-inf.semiring.add-right-mono mult-semi-associative vector-bot-closed*)
**also have** ... $\leq -F \sqcap v$
**by** (*simp add*: *assms*(*2*) *acyclic-star-inf-conv*)
**finally have** *22*: $(-F \sqcap v) * (F \sqcap v^{T\star}) \leq -F \sqcap v$
**by** *simp*
**have** $((-F \sqcap v) * (F \sqcap v^{T\star}))^T = (F \sqcap v^\star) * (-F \sqcap v)^T$
**by** (*simp add*: *assms*(*1*) *conv-dist-inf conv-star-commute conv-dist-comp*)
**hence** $(F \sqcap v^\star) * (-F \sqcap v)^T \leq (-F \sqcap v)^T$
**using** *22 conv-isotone* **by** *fastforce*
**thus** *?thesis*
**using** *22* **by** (*metis assms*(*4*, *6*) *comp-associative comp-inf.pp-comp-semi-commute comp-inf.semiring.add-mono comp-isotone*

*inf-pp-commute mult-left-isotone*)
**qed**
**also have** ... $= (-F \sqcap v) * (-F \sqcap v)^T$
**by** *simp*
**also have** ... $\leq v * v^T$
**by** (*simp add*: *comp-isotone conv-isotone*)
**also have** ... $\leq 1$
**by** (*simp add*: *assms*(*2*))
**thus** *?thesis*
**using** *calculation dual-order.trans* **by** *blast*
**qed**
**have** *3*: $top * ?i * top = top$
**proof** $-$
**have** *31*: *regular* $(e^T * -F * v * F * e)$
**using** *assms*(*3*, *4*, *6*) *arc-regular regular-mult-closed* **by** *auto*
**have** *32*: *bijective* $((top * e^T)^T)$
**using** *assms*(*3*) **by** (*simp add*: *conv-dist-comp*)
**have** $top * ?i * top = top * (v \sqcap -F * e * top) * ((top * e^T * F)^T \sqcap top)$
**by** (*simp add*: *comp-associative comp-inf-vector-1*)
**also have** ... $= (top \sqcap (-F * e * top)^T) * v * ((top * e^T * F)^T \sqcap top)$
**using** *comp-inf-vector conv-dist-comp* **by** *auto*
**also have** ... $= (-F * e * top)^T * v * (top * e^T * F)^T$
**by** *simp*
**also have** ... $= top^T * e^T * -F^T * v * F^T * e^{TT} * top^T$
**by** (*simp add*: *comp-associative conv-complement conv-dist-comp*)
**finally have** *33*: $top * ?i * top = top * e^T * -F * v * F * e * top$
**by** (*simp add*: *assms*(*1*))
**have** $top * ?i * top \neq bot$
**proof** (*rule ccontr*)
**assume** $\neg\ top * (v \sqcap -\ F * e * top \sqcap top * e^T * F) * top \neq bot$
**hence** $top * e^T * -F * v * F * e * top = bot$
**using** *33* **by** *auto*
**hence** $e^T * -F * v * F * e = bot$
**using** *31 tarski comp-associative le-bot* **by** *fastforce*
**hence** $top * (-F * v * F * e)^T \leq -(e^T)$
**by** (*metis comp-associative conv-complement-sub-leq conv-involutive p-bot schroeder-5-p*)
**hence** $top * e^T * F^T * v^T * -F^T \leq -(e^T)$
**by** (*simp add*: *comp-associative conv-complement conv-dist-comp*)
**hence** $v * F * e * top * e^T \leq F$
**by** (*metis assms*(*1*, *4*) *comp-associative conv-dist-comp schroeder-3-p symmetric-top-closed*)
**hence** $v * F * e * top * top * e^T \leq F$
**by** (*simp add*: *comp-associative*)
**hence** $v * F * e * top \leq F * (top * e^T)^T$
**using** *32* **by** (*metis shunt-bijective comp-associative conv-involutive*)
**hence** $v * F * e * top \leq F * e * top$
**using** *comp-associative conv-dist-comp* **by** *auto*
**hence** $v^\star * F * e * top \leq F * e * top$

**using** *comp-associative star-left-induct-mult-iff* **by** *auto*
**hence** $e^T * F * e * top \le F * e * top$
**by** (*meson assms*(*9*) *mult-left-isotone order-trans*)
**hence** $e^T * F * e * top * (e * top)^T \le F$
**using** *32 shunt-bijective assms*(*3*) *mult-assoc* **by** *auto*
**hence** *34*: $e^T * F * e * top * top * e^T \le F$
**by** (*metis conv-dist-comp mult.semigroup-axioms symmetric-top-closed semigroup.assoc*)
**hence** $e^T \le F$
**proof** −
**have** $e^T \le e^T * e * e^T$
**by** (*metis conv-involutive ex231c*)
**also have** ... $\le e^T * F * e * e^T$
**using** *assms*(*1*) *comp-associative mult-left-isotone mult-right-isotone* **by** *fastforce*
**also have** ... $\le e^T * F * e * top * top * e^T$
**by** (*simp add*: *mult-left-isotone top-right-mult-increasing vector-mult-closed*)
**finally show** *?thesis*
**using** *34* **by** *simp*
**qed**
**hence** *35*: $e \le F$
**using** *assms*(*1*) *conv-order* **by** *fastforce*
**have** $top * (F * e)^T \le - e$
**using** *assms*(*8*) *comp-associative schroeder-4-p* **by** *auto*
**hence** $top * e^T * F \le - e$
**by** (*simp add*: *assms*(*1*) *comp-associative conv-dist-comp*)
**hence** $(top * e^T)^T * e \le - F$
**using** *schroeder-3-p* **by** *auto*
**hence** $e * top * e \le - F$
**by** (*simp add*: *conv-dist-comp*)
**hence** $e \le - F$
**by** (*simp add*: *assms*(*3*) *arc-top-arc*)
**hence** $e \le F \sqcap - F$
**using** *35 inf.boundedI* **by** *blast*
**hence** $e = bot$
**using** *bot-unique* **by** *auto*
**thus** *False*
**using** *assms*(*10*) **by** *auto*
**qed**
**thus** *?thesis*
**by** (*metis assms*(*3*, *4*, *6*) *arc-regular regular-closed-inf regular-closed-top regular-conv-closed regular-mult-closed semiring.mult-not-zero tarski*)
**qed**
**have** *bijective* $(?i * top) \wedge$ *bijective* $(?i^T * top)$
**using** *1 2 3 arc-expanded* **by** *blast*
**thus** *?thesis*
**by** *blast*
**qed**

### 4.4.3 Comparison of edge weights

In this section we compare the weight of the *selected-edge* with other edges of interest. For example, Theorem *e-leq-c-c-complement-transpose-general* is used to show that the *selected-edge* has its source inside and its target outside the component it is chosen for.

**lemma** *e-leq-c-c-complement-transpose-general*:
  **assumes** $e = minarc\ (v * -(v)^T \sqcap g)$
    **and** *regular v*
  **shows** $e \leq v * -(v)^T$
**proof** –
  **have** $e \leq --\ (v * -\ v^T \sqcap g)$
    **using** *assms(1) minarc-below order-trans* **by** *blast*
  **also have** ... $\leq --\ (v * -\ v^T)$
    **using** *order-lesseq-imp pp-isotone-inf* **by** *blast*
  **also have** ... $= v * -\ v^T$
    **using** *assms(2) regular-mult-closed* **by** *auto*
  **finally show** *?thesis*
    **by** *simp*
**qed**

**lemma** *x-leq-c-transpose-general*:
  **assumes** *vector-classes x v*
    **and** $a^T * top \leq x * e * top$
    **and** $e \leq v * -v^T$
  **shows** $a \leq v^T$
**proof** –
  **have** *1*: *equivalence x*
    **using** *assms(1)* **using** *vector-classes-def* **by** *blast*
  **have** $a \leq top * a$
    **using** *top-left-mult-increasing* **by** *blast*
  **also have** ... $\leq (x * e * top)^T$
    **using** *assms(2) conv-dist-comp conv-isotone* **by** *fastforce*
  **also have** ... $= top * e^T * x$
    **using** *1* **by** (*simp add*: *conv-dist-comp mult-assoc*)
  **also have** ... $\leq top * (v * -\ v^T)^T * x$
    **by** (*metis assms(3) conv-dist-comp conv-isotone mult-left-isotone symmetric-top-closed*)
  **also have** ... $= top * (-\ v * v^T) * x$
    **by** (*simp add*: *conv-complement conv-dist-comp*)
  **also have** ... $\leq top * v^T * x$
    **by** (*metis mult-left-isotone top.extremum mult-assoc*)
  **also have** ... $= v^T * x$
    **using** *assms(1) vector-classes-def vector-conv-covector* **by** *auto*
  **also have** ... $= v^T$
    **by** (*metis assms(1) order.antisym conv-dist-comp conv-order dual-order.trans mult-right-isotone mult-sub-right-one vector-classes-def*)
  **finally show** *?thesis*
    **by** *simp*

**qed**

**lemma** *x-leq-c-complement-general*:
  **assumes** *vector* $v$
    **and** $v * v^T \leq x$
    **and** $a \leq v^T$
    **and** $a \leq -x$
  **shows** $a \leq -v$
**proof** –
  **have** $a \leq -x \sqcap v^T$
    **using** *assms*(*3*, *4*) **by** *auto*
  **also have** ... $\leq - v$
  **proof** –
    **have** $v \sqcap v^T \leq x$
      **using** *assms*(*1*, *2*) *vector-covector* **by** *auto*
    **hence** $-x \sqcap v \sqcap v^T \leq bot$
      **using** *inf.sup-monoid.add-assoc p-antitone pseudo-complement* **by** *fastforce*
    **thus** *?thesis*
      **using** *le-bot p-shunting-swap pseudo-complement* **by** *blast*
  **qed**
  **finally show** *?thesis*
    **by** *simp*
**qed**

**lemma** *sum-e-below-sum-a-when-outgoing-same-component-general*:
  **assumes** $e = minarc\ (v * -(v)^T \sqcap g)$
    **and** *symmetric* $g$
    **and** *arc* $a$
    **and** $a \leq -x \sqcap -- g$
    **and** $a^T * top \leq x * e * top$
    **and** *unique-vector-class* $x$ $v$
  **shows** $sum\ (e \sqcap g) \leq sum\ (a \sqcap g)$
**proof** –
  **have** *1*:$e \leq v * - v^T$
    **using** *assms*(*1*, *6*) *e-leq-c-c-complement-transpose-general*
*unique-vector-class-def vector-classes-def* **by** *auto*
  **have** *2*: $a \leq v^T$
    **using** *1 assms*(*5*) *assms*(*6*) *x-leq-c-transpose-general unique-vector-class-def*
**by** *blast*
  **hence** $a \leq -v$
    **using** *assms*(*4*, *6*) *inf.boundedE unique-vector-class-def vector-classes-def*
*x-leq-c-complement-general* **by** *meson*
  **hence** $a \leq - v \sqcap v^T$
    **using** *2* **by** *simp*
  **hence** $a \leq - v * v^T$
    **using** *assms*(*6*) *vector-complement-closed vector-covector*
*unique-vector-class-def vector-classes-def* **by** *metis*
  **hence** $a^T \leq v^{TT} * - v^T$
    **using** *conv-complement conv-dist-comp conv-isotone* **by** *metis*

**hence** *3*: $a^T \leq v * - v^T$
  **by** *simp*
**hence** $a \leq - - g$
  **using** *assms*(*4*) **by** *auto*
**hence** $a^T \leq - - g$
  **using** *assms*(*2*) *conv-complement conv-isotone* **by** *fastforce*
**hence** $a^T \sqcap v * - v^T \sqcap - - g \neq bot$
  **using** *3 assms*(*3*, *6*) *inf.orderE semiring.mult-not-zero unique-vector-class-def vector-classes-def* **by** *metis*
**hence** $a^T \sqcap v * - v^T \sqcap g \neq bot$
  **using** *inf.sup-monoid.add-commute pp-inf-bot-iff* **by** *auto*
**hence** *sum* (*minarc* $(v * - v^T \sqcap g) \sqcap (v * - v^T \sqcap g)) \leq$ *sum* $(a^T \sqcap v * - v^T \sqcap g)$
  **using** *assms*(*3*) *minarc-min inf.sup-monoid.add-assoc* **by** *simp*
**hence** *sum* $(e \sqcap v * - v^T \sqcap g) \leq$ *sum* $(a^T \sqcap v * - v^T \sqcap g)$
  **using** *assms*(*1*, *6*) *inf.sup-monoid.add-assoc* **by** *simp*
**hence** *sum* $(e \sqcap g) \leq$ *sum* $(a^T \sqcap g)$
  **using** *1 3* **by** (*metis inf.orderE*)
**hence** *sum* $(e \sqcap g) \leq$ *sum* $(a \sqcap g)$
  **by** (*simp add*: *assms*(*2*) *sum-symmetric*)
**thus** *?thesis*
  **by** *simp*
**qed**

**lemma** *sum-e-below-sum-x-when-outgoing-same-component*:
  **assumes** *symmetric g*
    **and** *vector j*
    **and** *forest h*
    **and** *regular h*
    **and** $x \leq -$ *forest-components* $h \sqcap - - g$
    **and** $x^T * top \leq$ *forest-components* $h *$ *selected-edge* $h\ j\ g * top$
    **and** $j \neq bot$
    **and** *arc x*
  **shows** *sum* (*selected-edge* $h\ j\ g \sqcap g) \leq$ *sum* $(x \sqcap g)$
**proof** −
  **let** *?e* = *selected-edge h j g*
  **let** *?c* = *choose-component* (*forest-components h*) *j*
  **let** *?H* = *forest-components h*
  **show** *?thesis*
  **proof** (*rule sum-e-below-sum-a-when-outgoing-same-component-general*)
  **next**
    **show** *?e* = *minarc* $(?c * - ?c^T \sqcap g)$
      **by** *simp*
  **next**
    **show** *symmetric g*
      **by** (*simp add*: *assms*(*1*))
  **next**
    **show** *arc x*
      **by** (*simp add*: *assms*(*8*))

```
  next
    show x ≤ −?H ⊓ −− g
      using assms(5) by auto
  next
    show x^T ∗ top ≤ ?H ∗ ?e ∗ top
      by (simp add: assms(6))
  next
    show unique-vector-class ?H ?c
    proof (unfold unique-vector-class-def, unfold vector-classes-def, intro conjI)
    next
      show regular ?H
        by (metis assms(4) conv-complement pp-dist-star regular-mult-closed)
    next
      show regular ?c
        using component-is-regular by auto
    next
      show reflexive ?H
        using assms(3) forest-components-equivalence by blast
    next
      show transitive ?H
        using assms(3) fch-equivalence by blast
    next
      show symmetric ?H
        by (simp add: assms(3) fch-equivalence)
    next
      show vector ?c
        by (simp add: assms(2, 6) component-is-vector)
    next
      show ?H ∗ ?c ≤ ?c
        using component-single by auto
    next
      show ?c ≠ bot
        using assms(2, 6 , 7, 8) inf-bot-left le-bot minarc-bot mult-left-zero
mult-right-zero by fastforce
    next
      show ?c ∗ ?c^T ≤ ?H
        by (simp add: component-is-connected)
    qed
  qed
qed
```

If there is a path in the *forest-modulo-equivalence* from an edge between components, $a$, to the *selected-edge*, $e$, then the weight of $e$ is no greater than the weight of $a$. This is because either,

* the edges $a$ and $e$ are adjacent the same component so that we can use *sum-e-below-sum-x-when-outgoing-same-component*, or

* there is at least one edge between $a$ and $e$, namely $x$, the edge incoming to the component that $e$ is outgoing from. The path from $a$ to $e$ is split

on $x$ using *forest-modulo-equivalence-path-split-disj*. We show that the weight of $e$ is no greater than the weight of $x$ by making use of lemma *sum-e-below-sum-x-when-outgoing-same-component*. We define $x$ in a way that we can show that the weight of $x$ is no greater than the weight of $a$ using the invariant. Then, it follows that the weight of $e$ is no greater than the weight of $a$ owing to transitivity.

**lemma** *a-to-e-in-forest-modulo-equivalence*:
  **assumes** *symmetric* $g$
    **and** $f \leq --g$
    **and** *vector* $j$
    **and** *forest* $h$
    **and** *forest-modulo-equivalence* (*forest-components* $h$) $d$
    **and** $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
    **and** ($\forall$ $a$ $b$ . *forest-modulo-equivalence-path* $a$ $b$ (*forest-components* $h$) $d \wedge a \leq$ $-$(*forest-components* $h$) $\sqcap --$ $g \wedge b \leq d \longrightarrow$ *sum*($b \sqcap g$) $\leq$ *sum*($a \sqcap g$))
    **and** *regular* $d$
    **and** $j \neq$ *bot*
    **and** $b =$ *selected-edge* $h$ $j$ $g$
    **and** *arc* $a$
    **and** *forest-modulo-equivalence-path* $a$ $b$ (*forest-components* $h$) ($d \sqcup$ *selected-edge* $h$ $j$ $g$)
    **and** $a \leq -$ *forest-components* $h \sqcap --$ $g$
    **and** *regular* $h$
  **shows** *sum* ($b \sqcap g$) $\leq$ *sum* ($a \sqcap g$)
**proof** $-$
  **let** *?p* = *path* $f$ $h$ $j$ $g$
  **let** *?e* = *selected-edge* $h$ $j$ $g$
  **let** *?F* = *forest-components* $f$
  **let** *?H* = *forest-components* $h$
  **have** *sum* ($b \sqcap g$) $\leq$ *sum* ($a \sqcap g$)
  **proof** (*cases* $a^T * top \leq$ *?H* $*$ *?e* $* top$)
    **case** *True*
    **show** $a^T * top \leq$ *?H* $*$ *?e* $* top \Longrightarrow$ *sum* ($b \sqcap g$) $\leq$ *sum* ($a \sqcap g$)
    **proof**$-$
      **have** *sum* (*?e* $\sqcap g$) $\leq$ *sum* ($a \sqcap g$)
      **proof** (*rule sum-e-below-sum-x-when-outgoing-same-component*)
        **show** *symmetric* $g$
          **using** *assms*(*1*) **by** *auto*
      **next**
        **show** *vector* $j$
          **using** *assms*(*3*) **by** *blast*
      **next**
        **show** *forest* $h$
          **by** (*simp add*: *assms*(*4*))
      **next**
        **show** $a \leq -$ *?H* $\sqcap --$ $g$
          **using** *assms*(*13*) **by** *auto*
      **next**

**show** $a^T * top \le ?H * ?e * top$
**using** *True* **by** *auto*
**next**
**show** $j \neq bot$
**by** (*simp add*: *assms*(*9*))
**next**
**show** *arc a*
**by** (*simp add*: *assms*(*11*))
**next**
**show** *regular h*
**using** *assms*(*14*) **by** *auto*
**qed**
**thus** *?thesis*
**using** *assms*(*10*) **by** *auto*
**qed**
**next**
**case** *False*
**show** $\neg\ a^T * top \le ?H * ?e * top \Longrightarrow sum\ (b \sqcap g) \le sum\ (a \sqcap g)$
**proof** −
**let** $?d' = d \sqcup ?e$
**let** $?x = d \sqcap top * ?e^T * ?H \sqcap (?H * d^T)^\star * ?H * a^T * top$
**have** *61*: *arc* (*?x*)
**proof** (*rule shows-arc-x*)
**show** *forest-modulo-equivalence ?H d*
**by** (*simp add*: *assms*(*5*))
**next**
**show** *forest-modulo-equivalence-path a ?e ?H d*
**proof** −
**have** *611*: *forest-modulo-equivalence-path a b ?H* $(d \sqcup b)$
**using** *assms*(*10*, *12*) **by** *auto*
**have** *616*: *regular h*
**using** *assms*(*14*) **by** *auto*
**have** *regular a*
**using** *611 forest-modulo-equivalence-path-def arc-regular* **by** *fastforce*
**thus** *?thesis*
**using** *616* **by** (*smt forest-modulo-equivalence-path-split-disj assms*(*4*, *8*, *10*, *12*) *forest-modulo-equivalence-path-def fch-equivalence minarc-regular regular-closed-star regular-conv-closed regular-mult-closed*)
**qed**
**next**
**show** $(?H * d)^+ \le -\ ?H$
**using** *assms*(*5*) *forest-modulo-equivalence-def le-bot pseudo-complement* **by** *blast*
**next**
**show** $\neg\ a^T * top \le ?H * ?e * top$
**by** (*simp add*: *False*)
**next**
**show** *regular a*
**using** *assms*(*12*) *forest-modulo-equivalence-path-def arc-regular* **by** *auto*

**next**
**show** *regular ?e*
**using** *minarc-regular* **by** *auto*
**next**
**show** *regular ?H*
**using** *assms*(*14*) *pp-dist-star regular-conv-closed regular-mult-closed* **by** *auto*
**next**
**show** *regular d*
**using** *assms*(*8*) **by** *auto*
**qed**
**have** *62*: *bijective* $(a^T * top)$
**by** (*simp add*: *assms*(*11*))
**have** *63*: *bijective* $(?x * top)$
**using** *61* **by** *simp*
**have** *64*: $?x \leq (?H * d^T)^\star * ?H * a^T * top$
**by** *simp*
**hence** $?x * top \leq (?H * d^T)^\star * ?H * a^T * top$
**using** *mult-left-isotone inf-vector-comp* **by** *auto*
**hence** $a^T * top \leq ((?H * d^T)^\star * ?H)^T * ?x * top$
**using** *62 63 64* **by** (*smt bijective-reverse mult-assoc*)
**also have** ... $= ?H * (d * ?H)^\star * ?x * top$
**using** *conv-dist-comp conv-star-commute* **by** *auto*
**also have** ... $= (?H * d)^\star * ?H * ?x * top$
**by** (*simp add*: *star-slide*)
**finally have** $a^T * top \leq (?H * d)^\star * ?H * ?x * top$
**by** *simp*
**hence** *65*: *forest-modulo-equivalence-path a ?x ?H d*
**using** *61 assms*(*12*) *forest-modulo-equivalence-path-def* **by** *blast*
**have** *66*: $?x \leq d$
**by** (*simp add*: *inf.sup-monoid.add-assoc*)
**hence** *x-below-a*: *sum* $(?x \sqcap g) \leq$ *sum* $(a \sqcap g)$
**using** *65 forest-modulo-equivalence-path-def assms*(*7*, *13*) **by** *blast*
**have** *sum* $(?e \sqcap g) \leq$ *sum* $(?x \sqcap g)$
**proof** (*rule sum-e-below-sum-x-when-outgoing-same-component*)
**show** *symmetric g*
**using** *assms*(*1*) **by** *auto*
**next**
**show** *vector j*
**using** *assms*(*3*) **by** *blast*
**next**
**show** *forest h*
**by** (*simp add*: *assms*(*4*))
**next**
**show** $?x \leq -\ ?H \sqcap --\ g$
**proof** –
**have** *67*: $?x \leq -\ ?H$
**proof** –
**have** $?x \leq d$

using *66* **by** *blast*
**also have** ... $\leq$ *?H* $*$ *d*
using *dual-order.trans star.circ-loop-fixpoint sup.cobounded2 mult-assoc* **by** *metis*
**also have** ... $\leq$ (*?H* $*$ *d*)$^+$
**using** *star.circ-mult-increasing* **by** *blast*
**also have** ... $\leq$ $-$ *?H*
**using** *assms*(*5*) *bot-unique pseudo-complement forest-modulo-equivalence-def* **by** *blast*
**thus** *?thesis*
**using** *calculation inf.order-trans* **by** *blast*
**qed**
**have** *?x* $\leq$ *d*
**by** (*simp add*: *conv-isotone inf.sup-monoid.add-assoc*)
**also have** ... $\leq$ *f* $\sqcup$ $f^T$
**proof** $-$
**have** $h \sqcup h^T \sqcup d \sqcup d^T = f \sqcup f^T$
**by** (*simp add*: *assms*(*6*))
**thus** *?thesis*
**by** (*metis* (*no-types*) *le-supE sup.absorb-iff2 sup.idem*)
**qed**
**also have** ... $\leq$ $--$ *g*
**using** *assms*(*1*, *2*) *conv-complement conv-isotone* **by** *fastforce*
**finally have** *?x* $\leq$ $--$ *g*
**by** *simp*
**thus** *?thesis*
**by** (*simp add*: *67*)
**qed**
**next**
**show** $?x^T * top \leq ?H * ?e * top$
**proof** $-$
**have** $?x \leq top * ?e^T * ?H$
**using** *inf.coboundedI1* **by** *auto*
**hence** $?x^T \leq ?H * ?e * top$
**using** *conv-dist-comp conv-dist-inf conv-star-commute inf.orderI inf.sup-monoid.add-assoc inf.sup-monoid.add-commute mult-assoc* **by** *auto*
**hence** $?x^T * top \leq ?H * ?e * top * top$
**by** (*simp add*: *mult-left-isotone*)
**thus** *?thesis*
**by** (*simp add*: *mult-assoc*)
**qed**
**next**
**show** $j \neq bot$
**by** (*simp add*: *assms*(*9*))
**next**
**show** *arc* (*?x*)
**using** *61* **by** *blast*
**next**
**show** *regular h*

**using** *assms*(*14*) **by** *auto*
**qed**
**hence** *sum* (*?e* $\sqcap$ *g*) $\leq$ *sum* (*a* $\sqcap$ *g*)
**using** *x-below-a order.trans* **by** *blast*
**thus** *?thesis*
**by** (*simp add*: *assms*(*10*))
**qed**
**qed**
**thus** *?thesis*
**by** *simp*
**qed**

### 4.4.4 Maintenance of algorithm invariants

In this section, most of the work is done to maintain the invariants of the inner and outer loops of the algorithm. In particular, we use *exists-a-w* to maintain that $f$ can be extended to a minimum spanning forest.

**lemma** *boruvka-exchange-spanning-inv*:
**assumes** *forest* $v$
**and** $v^\star * e^T = e^T$
**and** $i \leq v \sqcap top * e^T * w^{T\star}$
**and** *arc* $i$
**and** *arc* $e$
**and** $v \leq --g$
**and** $w \leq --g$
**and** $e \leq --g$
**and** *components* $g \leq$ *forest-components* $v$
**shows** $i \leq (v \sqcap -i)^{T\star} * e^T * top$
**proof** –
**have** *1*: $(v \sqcap -i \sqcap -i^T) * (v^T \sqcap -i \sqcap -i^T) \leq 1$
**using** *assms*(*1*) *comp-isotone order.trans inf.cobounded1* **by** *blast*
**have** *2*: *bijective* $(i * top) \wedge$ *bijective* $(e^T * top)$
**using** *assms*(*4*, *5*) *mult-assoc* **by** *auto*
**have** $i \leq v * (top * e^T * w^{T\star})^T$
**using** *assms*(*3*) *covector-mult-closed covector-restrict-comp-conv order-lesseq-imp vector-top-closed* **by** *blast*
**also have** ... $\leq v * w^{T\star T} * e^{TT} * top^T$
**by** (*simp add*: *comp-associative conv-dist-comp*)
**also have** ... $\leq v * w^\star * e * top$
**by** (*simp add*: *conv-star-commute*)
**also have** ... $= v * w^\star * e * e^T * e * top$
**using** *assms*(*5*) *arc-eq-1* **by** (*simp add*: *comp-associative*)
**also have** ... $\leq v * w^\star * e * e^T * top$
**by** (*simp add*: *comp-associative mult-right-isotone*)
**also have** ... $\leq (--g) * (--g)^\star * (--g) * e^T * top$
**using** *assms*(*6*, *7*, *8*) **by** (*simp add*: *comp-isotone star-isotone*)
**also have** ... $\leq (--g)^\star * e^T * top$
**by** (*metis comp-isotone mult-left-isotone star.circ-increasing star.circ-transitive-equal*)

**also have** ... $\leq v^{T\star} * v^\star * e^T * top$
  **by** (*simp add*: *assms*(*9*) *mult-left-isotone*)
**also have** ... $\leq v^{T\star} * e^T * top$
  **by** (*simp add*: *assms*(*2*) *comp-associative*)
**finally have** $i \leq v^{T\star} * e^T * top$
  **by** *simp*
**hence** $i * top \leq v^{T\star} * e^T * top$
  **by** (*metis comp-associative mult-left-isotone vector-top-closed*)
**hence** $e^T * top \leq v^{T\star T} * i * top$
  **using** *2* **by** (*metis bijective-reverse mult-assoc*)
**also have** ... $= v^\star * i * top$
  **by** (*simp add*: *conv-star-commute*)
**also have** ... $\leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
**proof** $-$
  **have** *3*: $i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
    **using** *star.circ-loop-fixpoint sup-right-divisibility mult-assoc* **by** *auto*
  **have** $(v \sqcap i) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq i * top * i * top$
    **by** (*metis comp-isotone inf.cobounded1 inf.sup-monoid.add-commute mult-left-isotone top.extremum*)
  **also have** ... $\leq i * top$
    **by** *simp*
  **finally have** *4*: $(v \sqcap i) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
    **using** *3 dual-order.trans* **by** *blast*
  **have** *5*: $(v \sqcap -i \sqcap -i^T) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
    **by** (*metis mult-left-isotone star.circ-increasing star.left-plus-circ*)
  **have** $v^+ \leq -1$
    **by** (*simp add*: *assms*(*1*))
  **hence** $v * v \leq -1$
    **by** (*metis mult-left-isotone order-trans star.circ-increasing star.circ-plus-same*)
  **hence** $v * 1 \leq -v^T$
    **by** (*simp add*: *schroeder-5-p*)
  **hence** $v \leq -v^T$
    **by** *simp*
  **hence** $v \sqcap v^T \leq bot$
    **by** (*simp add*: *bot-unique pseudo-complement*)
  **hence** *7*: $v \sqcap i^T \leq bot$
    **by** (*metis assms*(*3*) *comp-inf.mult-right-isotone conv-dist-inf inf.boundedE inf.le-iff-sup le-bot*)
  **hence** $(v \sqcap i^T) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq bot$
    **using** *le-bot semiring.mult-zero-left* **by** *fastforce*
  **hence** *6*: $(v \sqcap i^T) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
    **using** *bot-least le-bot* **by** *blast*
  **have** *8*: $v = (v \sqcap i) \sqcup (v \sqcap i^T) \sqcup (v \sqcap -i \sqcap -i^T)$
  **proof** $-$
    **have** *81*: *regular i*
      **by** (*simp add*: *assms*(*4*) *arc-regular*)

**have** $(v \sqcap i^T) \sqcup (v \sqcap -i \sqcap -i^T) = (v \sqcap -i)$
**using** *7* **by** (*metis comp-inf.coreflexive-comp-inf-complement inf-import-p inf-p le-bot maddux-3-11-pp top.extremum*)
**hence** $(v \sqcap i) \sqcup (v \sqcap i^T) \sqcup (v \sqcap -i \sqcap -i^T) = (v \sqcap i) \sqcup (v \sqcap -i)$
**by** (*simp add*: *sup.semigroup-axioms semigroup.assoc*)
**also have** ... $= v$
**using** *81* **by** (*metis maddux-3-11-pp*)
**finally show** *?thesis*
**by** *simp*
**qed**
**have** $(v \sqcap i) * (v \sqcap -i \sqcap -i^T)^\star * i * top \sqcup (v \sqcap i^T) * (v \sqcap -i \sqcap -i^T)^\star * i * top \sqcup (v \sqcap -i \sqcap -i^T) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
**using** *4 5 6* **by** *simp*
**hence** $((v \sqcap i) \sqcup (v \sqcap i^T) \sqcup (v \sqcap -i \sqcap -i^T)) * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
**by** (*simp add*: *mult-right-dist-sup*)
**hence** $v * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
**using** *8* **by** *auto*
**hence** $i * top \sqcup v * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
**using** *3* **by** *auto*
**hence** *9*:$v^\star * (v \sqcap -i \sqcap -i^T)^\star * i * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
**by** (*simp add*: *star-left-induct-mult mult-assoc*)
**have** $v^\star * i * top \leq v^\star * (v \sqcap -i \sqcap -i^T)^\star * i * top$
**using** *3 mult-right-isotone mult-assoc* **by** *auto*
**thus** *?thesis*
**using** *9 order.trans* **by** *blast*
**qed**
**finally have** $e^T * top \leq (v \sqcap -i \sqcap -i^T)^\star * i * top$
**by** *simp*
**hence** $i * top \leq (v \sqcap -i \sqcap -i^T)^{\star T} * e^T * top$
**using** *2* **by** (*metis bijective-reverse mult-assoc*)
**also have** ... $= (v^T \sqcap -i \sqcap -i^T)^\star * e^T * top$
**using** *comp-inf.inf-vector-comp conv-complement conv-dist-inf conv-star-commute inf.sup-monoid.add-commute* **by** *auto*
**also have** ... $\leq ((v \sqcap -i \sqcap -i^T) \sqcup (v^T \sqcap -i \sqcap -i^T))^\star * e^T * top$
**by** (*simp add*: *mult-left-isotone star-isotone*)
**finally have** $i \leq ((v^T \sqcap -i \sqcap -i^T) \sqcup (v \sqcap -i \sqcap -i^T))^\star * e^T * top$
**using** *dual-order.trans top-right-mult-increasing sup-commute* **by** *auto*
**also have** ... $= (v^T \sqcap -i \sqcap -i^T)^\star * (v \sqcap -i \sqcap -i^T)^\star * e^T * top$
**using** *1 cancel-separate-1* **by** (*simp add*: *sup-commute*)
**also have** ... $\leq (v^T \sqcap -i \sqcap -i^T)^\star * v^\star * e^T * top$
**by** (*simp add*: *inf-assoc mult-left-isotone mult-right-isotone star-isotone*)
**also have** ... $= (v^T \sqcap -i \sqcap -i^T)^\star * e^T * top$
**using** *assms*(*2*) *mult-assoc* **by** *simp*
**also have** ... $\leq (v^T \sqcap -i^T)^\star * e^T * top$
**by** (*metis mult-left-isotone star-isotone inf.cobounded2 inf.left-commute inf.sup-monoid.add-commute*)
**also have** ... $= (v \sqcap -i)^{T\star} * e^T * top$
**using** *conv-complement conv-dist-inf* **by** *auto*

**finally show** *?thesis*
**by** *simp*
**qed**

**lemma** *exists-a-w*:
**assumes** *symmetric g*
**and** *forest f*
**and** $f \leq --g$
**and** *regular f*
**and** $(\exists\, w\ .\ \textit{minimum-spanning-forest}\ w\ g \wedge f \leq w \sqcup w^T)$
**and** *vector j*
**and** *regular j*
**and** *forest h*
**and** *forest-modulo-equivalence* (*forest-components h*) *d*
**and** $d * top \leq -\ j$
**and** $\textit{forest-components}\ h * j = j$
**and** $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
**and** $(\forall\ a\ b\ .\ \textit{forest-modulo-equivalence-path}\ a\ b\ (\textit{forest-components}\ h)\ d \wedge a \leq -(\textit{forest-components}\ h) \sqcap --\ g \wedge b \leq d \longrightarrow sum(b \sqcap g) \leq sum(a \sqcap g))$
**and** *regular d*
**and** $\textit{selected-edge}\ h\ j\ g \leq -\ \textit{forest-components}\ f$
**and** $\textit{selected-edge}\ h\ j\ g \neq bot$
**and** $j \neq bot$
**and** *regular h*
**and** $h \leq --g$
**shows** $\exists\, w.\ \textit{minimum-spanning-forest}\ w\ g\ \wedge$
$f \sqcap -\ (\textit{selected-edge}\ h\ j\ g)^T \sqcap -\ (\textit{path}\ f\ h\ j\ g) \sqcup (f \sqcap -\ (\textit{selected-edge}\ h\ j\ g)^T \sqcap (\textit{path}\ f\ h\ j\ g))^T \sqcup (\textit{selected-edge}\ h\ j\ g) \leq w \sqcup w^T$
**proof** −
**let** $?p = \textit{path}\ f\ h\ j\ g$
**let** $?e = \textit{selected-edge}\ h\ j\ g$
**let** $?f = (f \sqcap -?e^T \sqcap -?p) \sqcup (f \sqcap -?e^T \sqcap ?p)^T \sqcup ?e$
**let** $?F = \textit{forest-components}\ f$
**let** $?H = \textit{forest-components}\ h$
**let** $?ec = \textit{choose-component}\ (\textit{forest-components}\ h)\ j * -\ \textit{choose-component}\ (\textit{forest-components}\ h)\ j^T \sqcap g$
**from** *assms*(*4*) **obtain** *w* **where** *2*: $\textit{minimum-spanning-forest}\ w\ g \wedge f \leq w \sqcup w^T$
**using** *assms*(*5*) **by** *blast*
**hence** *3*: $\textit{regular}\ w \wedge \textit{regular}\ f \wedge \textit{regular}\ ?e$
**by** (*metis assms*(*4*) *minarc-regular minimum-spanning-forest-def spanning-forest-def*)
**have** *5*: *equivalence ?F*
**using** *assms*(*2*) *forest-components-equivalence* **by** *auto*
**have** $?e^T * top * ?e^T = ?e^T$
**by** (*metis arc-conv-closed arc-top-arc coreflexive-bot-closed coreflexive-symmetric minarc-arc minarc-bot-iff semiring.mult-not-zero*)
**hence** $?e^T * top * ?e^T \leq -?F$
**using** *5 assms*(*15*) *conv-complement conv-isotone* **by** *fastforce*

**hence** *6*: $?e * ?F * ?e = bot$
**using** *assms*(*2*) *le-bot triple-schroeder-p* **by** *simp*
**let** $?q = w \sqcap top * ?e * w^{T\star}$
**let** $?v = (w \sqcap -(top * ?e * w^{T\star})) \sqcup ?q^T$
**have** *7*: *regular ?q*
**using** *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
**have** *8*: *injective ?v*
**proof** (*rule kruskal-exchange-injective-inv-1*)
**show** *injective w*
**using** *2 minimum-spanning-forest-def spanning-forest-def* **by** *blast*
**next**
**show** *covector* $(top * ?e * w^{T\star})$
**by** (*simp add*: *covector-mult-closed*)
**next**
**show** $top * ?e * w^{T\star} * w^T \leq top * ?e * w^{T\star}$
**by** (*simp add*: *mult-right-isotone star.right-plus-below-circ mult-assoc*)
**next**
**show** *coreflexive* $((top * ?e * w^{T\star})^T * (top * ?e * w^{T\star}) \sqcap w^T * w)$
**using** *2* **by** (*metis comp-inf.semiring.mult-not-zero forest-bot kruskal-injective-inv-3 minarc-arc minarc-bot-iff minimum-spanning-forest-def semiring.mult-not-zero spanning-forest-def*)
**qed**
**have** *9*: *components g* $\leq$ *forest-components ?v*
**proof** (*rule kruskal-exchange-spanning-inv-1*)
**show** *injective* $(w \sqcap - (top * ?e * w^{T\star}) \sqcup (w \sqcap top * ?e * w^{T\star})^T)$
**using** *8* **by** *simp*
**next**
**show** *regular* $(w \sqcap top * ?e * w^{T\star})$
**using** *7* **by** *simp*
**next**
**show** *components g* $\leq$ *forest-components w*
**using** *2 minimum-spanning-forest-def spanning-forest-def* **by** *blast*
**qed**
**have** *10*: *spanning-forest ?v g*
**proof** (*unfold spanning-forest-def*, *intro conjI*)
**show** *injective ?v*
**using** *8* **by** *auto*
**next**
**show** *acyclic ?v*
**proof** (*rule kruskal-exchange-acyclic-inv-1*)
**show** *pd-kleene-allegory-class.acyclic w*
**using** *2 minimum-spanning-forest-def spanning-forest-def* **by** *blast*
**next**
**show** *covector* $(top * ?e * w^{T\star})$
**by** (*simp add*: *covector-mult-closed*)
**qed**
**next**
**show** $?v \leq --g$
**proof** (*rule sup-least*)

**show** $w \sqcap - (top * ?e * w^{T\star}) \leq - - g$
**using** *7 inf.coboundedI1 minimum-spanning-forest-def spanning-forest-def 2* **by** *blast*
**next**
**show** $(w \sqcap top * ?e * w^{T\star})^T \leq - - g$
**using** *2* **by** (*metis assms*(*1*) *conv-complement conv-isotone inf.coboundedI1 minimum-spanning-forest-def spanning-forest-def*)
**qed**
**next**
**show** *components* $g \leq$ *forest-components* $?v$
**using** *9* **by** *simp*
**next**
**show** *regular* $?v$
**using** *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
**qed**
**have** *11*: *sum* $(?v \sqcap g) =$ *sum* $(w \sqcap g)$
**proof** −
**have** *sum* $(?v \sqcap g) =$ *sum* $(w \sqcap -(top * ?e * w^{T\star}) \sqcap g) +$ *sum* $(?q^T \sqcap g)$
**using** *2* **by** (*smt conv-complement conv-top epm-8 inf-import-p inf-top-right regular-closed-top vector-top-closed minimum-spanning-forest-def spanning-forest-def sum-disjoint*)
**also have** ... $=$ *sum* $(w \sqcap -(top * ?e * w^{T\star}) \sqcap g) +$ *sum* $(?q \sqcap g)$
**by** (*simp add*: *assms*(*1*) *sum-symmetric*)
**also have** ... $=$ *sum* $(((w \sqcap -(top * ?e * w^{T\star})) \sqcup ?q) \sqcap g)$
**using** *inf-commute inf-left-commute sum-disjoint* **by** *simp*
**also have** ... $=$ *sum* $(w \sqcap g)$
**using** *3 7 8 maddux-3-11-pp* **by** *auto*
**finally show** *?thesis*
**by** *simp*
**qed**
**have** *12*: $?v \sqcup ?v^T = w \sqcup w^T$
**proof** −
**have** $?v \sqcup ?v^T = (w \sqcap -?q) \sqcup ?q^T \sqcup (w^T \sqcap -?q^T) \sqcup ?q$
**using** *conv-complement conv-dist-inf conv-dist-sup inf-import-p sup-assoc* **by** *simp*
**also have** ... $= w \sqcup w^T$
**using** *3 7 conv-complement conv-dist-inf inf-import-p maddux-3-11-pp sup-monoid.add-assoc sup-monoid.add-commute* **by** *auto*
**finally show** *?thesis*
**by** *simp*
**qed**
**have** *13*: $?v * ?e^T = bot$
**proof** (*rule kruskal-reroot-edge*)
**show** *injective* $(?e^T * top)$
**using** *assms*(*16*) *minarc-arc minarc-bot-iff* **by** *blast*
**next**
**show** *pd-kleene-allegory-class.acyclic* $w$
**using** *2 minimum-spanning-forest-def spanning-forest-def* **by** *simp*
**qed**

**have** $?v \sqcap ?e \leq ?v \sqcap top * ?e$
  **using** *inf.sup-right-isotone top-left-mult-increasing* **by** *simp*
**also have** ... $\leq ?v * (top * ?e)^T$
  **using** *covector-restrict-comp-conv covector-mult-closed vector-top-closed* **by** *simp*
**finally have** *14*: $?v \sqcap ?e = bot$
  **using** *13* **by** (*metis conv-dist-comp mult-assoc le-bot mult-left-zero*)
**let** $?i = ?v \sqcap (- ?F) * ?e * top \sqcap top * ?e^T * ?F$
**let** $?w = (?v \sqcap -?i) \sqcup ?e$
**have** *15*: *regular* $?i$
  **using** *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *simp*
**have** *16*: $?F \leq -?i$
**proof** −
  **have** *161*: *bijective* $(?e * top)$
    **using** *assms*(*16*) *minarc-arc minarc-bot-iff* **by** *auto*
  **have** $?i \leq - ?F * ?e * top$
    **using** *inf.cobounded2 inf.coboundedI1* **by** *blast*
  **also have** ... $= - (?F * ?e * top)$
    **using** *161 comp-bijective-complement* **by** (*simp add*: *mult-assoc*)
  **finally have** $?i \leq - (?F * ?e * top)$
    **by** *blast*
  **hence** *162*: $?i \sqcap ?F \leq - (?F * ?e * top)$
    **using** *inf.coboundedI1* **by** *blast*
  **have** $?i \sqcap ?F \leq ?F \sqcap (top * ?e^T * ?F)$
    **by** (*meson inf-le1 inf-le2 le-infI order-trans*)
  **also have** ... $\leq ?F * (top * ?e^T * ?F)^T$
    **by** (*simp add*: *covector-mult-closed covector-restrict-comp-conv*)
  **also have** ... $= ?F * ?F^T * ?e^{TT} * top^T$
    **by** (*simp add*: *conv-dist-comp mult-assoc*)
  **also have** ... $= ?F * ?F * ?e * top$
    **by** (*simp add*: *conv-dist-comp conv-star-commute*)
  **also have** ... $= ?F * ?e * top$
    **by** (*simp add*: *5 preorder-idempotent*)
  **finally have** $?i \sqcap ?F \leq ?F * ?e * top$
    **by** *simp*
  **hence** $?i \sqcap ?F \leq ?F * ?e * top \sqcap - (?F * ?e * top)$
    **using** *162 inf.bounded-iff* **by** *blast*
  **also have** ... $= bot$
    **by** *simp*
  **finally show** *?thesis*
    **using** *le-bot p-antitone-iff pseudo-complement* **by** *blast*
**qed**
**have** *17*: $?i \leq top * ?e^T * (?F \sqcap ?v \sqcap -?i)^{T\star}$
**proof** −
  **have** $?i \leq ?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F \sqcap ?v)^{T\star} * (?F \sqcap ?v)^\star$
    **using** *2 8 12* **by** (*smt inf.sup-right-isotone kruskal-forest-components-inf mult-right-isotone mult-assoc*)
  **also have** ... $= ?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F \sqcap ?v)^{T\star} * (1 \sqcup (?F \sqcap ?v)^\star * (?F \sqcap ?v))$

**using** *star-left-unfold-equal star.circ-right-unfold-1* **by** *auto*
**also have** ... $= ?v \sqcap - ?F * ?e * top \sqcap (top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup top * ?e^T * (?F \sqcap ?v)^{T\star} * (?F \sqcap ?v)^\star * (?F \sqcap ?v))$
**by** (*simp add*: *mult-left-dist-sup mult-assoc*)
**also have** ... $= (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F \sqcap ?v)^{T\star}) \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F \sqcap ?v)^{T\star} * (?F \sqcap ?v)^\star * (?F \sqcap ?v))$
**using** *comp-inf.semiring.distrib-left* **by** *blast*
**also have** ... $\le top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F \sqcap ?v)^{T\star} * (?F \sqcap ?v)^\star * (?F \sqcap ?v))$
**using** *comp-inf.semiring.add-right-mono inf-le2* **by** *blast*
**also have** ... $\le top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F^T \sqcap ?v^T)^\star * (?F \sqcap ?v)^\star * (?F \sqcap ?v))$
**by** (*simp add*: *conv-dist-inf*)
**also have** ... $\le top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * ?F^{T\star} * ?F^\star * (?F \sqcap ?v))$
**proof** $-$
**have** $top * ?e^T * (?F^T \sqcap ?v^T)^\star * (?F \sqcap ?v)^\star * (?F \sqcap ?v) \le top * ?e^T * ?F^{T\star} * ?F^\star * (?F \sqcap ?v)$
**using** *star-isotone* **by** (*simp add*: *comp-isotone*)
**hence** $?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * (?F^T \sqcap ?v^T)^\star * (?F \sqcap ?v)^\star * (?F \sqcap ?v) \le ?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * ?F^{T\star} * ?F^\star * (?F \sqcap ?v)$
**using** *inf.sup-right-isotone* **by** *blast*
**thus** *?thesis*
**using** *sup-right-isotone* **by** *blast*
**qed**
**also have** ... $= top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * ?F^\star * ?F^\star * (?F \sqcap ?v))$
**using** *5* **by** *auto*
**also have** ... $= top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * ?F * ?F * (?F \sqcap ?v))$
**by** (*simp add*: *assms*(*2*) *forest-components-star*)
**also have** ... $= top * ?e^T * (?F \sqcap ?v)^{T\star} \sqcup (?v \sqcap - ?F * ?e * top \sqcap top * ?e^T * ?F * (?F \sqcap ?v))$
**using** *5 mult.semigroup-axioms preorder-idempotent semigroup.assoc* **by** *fastforce*
**also have** ... $= top * ?e^T * (?F \sqcap ?v)^{T\star}$
**proof** $-$
**have** $?e * top * ?e^T \le 1$
**using** *assms*(*16*) *arc-expanded minarc-arc minarc-bot-iff* **by** *auto*
**hence** $?F * ?e * top * ?e^T \le ?F * 1$
**by** (*metis comp-associative comp-isotone mult-semi-associative star.circ-transitive-equal*)
**hence** $?v * ?v^T * ?F * ?e * top * ?e^T \le 1 * ?F * 1$
**using** *8* **by** (*smt comp-isotone mult-assoc*)
**hence** *171*: $?v * ?v^T * ?F * ?e * top * ?e^T \le ?F$
**by** *simp*
**hence** $?v * (?F \sqcap ?v)^T * ?F * ?e * top * ?e^T \le ?F$
**proof** $-$
**have** $?v * (?F \sqcap ?v)^T * ?F * ?e * top * ?e^T \le ?v * ?v^T * ?F * ?e * top *$

$?e^T$
by (*simp add*: *conv-dist-inf mult-left-isotone mult-right-isotone*)
**thus** *?thesis*
**using** *171 order-trans* **by** *blast*
**qed**
**hence** *172*: $-?F * ((?F \sqcap ?v)^T * ?F * ?e * top * ?e^T)^T \leq -?v$
**by** (*smt schroeder-4-p comp-associative order-lesseq-imp pp-increasing*)
**have** $-?F * ((?F \sqcap ?v)^T * ?F * ?e * top * ?e^T)^T = -?F * ?e^{TT} * top^T * ?e^T * ?F^T * (?F \sqcap ?v)^{TT}$
**by** (*simp add*: *comp-associative conv-dist-comp*)
**also have** ... $= -?F * ?e * top * ?e^T * ?F * (?F \sqcap ?v)$
**using** *5* **by** *auto*
**also have** ... $= -?F * ?e * top * top * ?e^T * ?F * (?F \sqcap ?v)$
**using** *comp-associative* **by** *auto*
**also have** ... $= -?F * ?e * top \sqcap top * ?e^T * ?F * (?F \sqcap ?v)$
**by** (*smt comp-associative comp-inf.star.circ-decompose-9 comp-inf.star-star-absorb comp-inf-covector inf-vector-comp vector-top-closed*)
**finally have** $-?F * ((?F \sqcap ?v)^T * ?F * ?e * top * ?e^T)^T = -?F * ?e * top \sqcap top * ?e^T * ?F * (?F \sqcap ?v)$
**by** *simp*
**hence** $-?F * ?e * top \sqcap top * ?e^T * ?F * (?F \sqcap ?v) \leq -?v$
**using** *172* **by** *auto*
**hence** $?v \sqcap -?F * ?e * top \sqcap top * ?e^T * ?F * (?F \sqcap ?v) \leq bot$
**by** (*smt bot-unique inf.sup-monoid.add-commute p-shunting-swap pseudo-complement*)
**thus** *?thesis*
**using** *le-bot sup-monoid.add-0-right* **by** *blast*
**qed**
**also have** ... $= top * ?e^T * (?F \sqcap ?v \sqcap -?i)^{T\star}$
**using** *16* **by** (*smt comp-inf.coreflexive-comp-inf-complement inf-top-right p-bot pseudo-complement top.extremum*)
**finally show** *?thesis*
**by** *blast*
**qed**
**have** *18*: $?i \leq top * ?e^T * ?w^{T\star}$
**proof** −
**have** $?i \leq top * ?e^T * (?F \sqcap ?v \sqcap -?i)^{T\star}$
**using** *17* **by** *simp*
**also have** ... $\leq top * ?e^T * (?v \sqcap -?i)^{T\star}$
**using** *mult-right-isotone conv-isotone star-isotone inf.cobounded2 inf.sup-monoid.add-assoc* **by** (*simp add*: *inf.sup-monoid.add-assoc order.eq-iff inf.sup-monoid.add-commute*)
**also have** ... $\leq top * ?e^T * ((?v \sqcap -?i) \sqcup ?e)^{T\star}$
**using** *mult-right-isotone conv-isotone star-isotone sup-ge1* **by** *simp*
**finally show** *?thesis*
**by** *blast*
**qed**
**have** *19*: $?i \leq top * ?e^T * ?v^{T\star}$
**proof** −

have $?i \leq top * ?e^T * (?F \sqcap ?v \sqcap -?i)^{T\star}$
using *17* by *simp*
also have ... $\leq top * ?e^T * (?v \sqcap -?i)^{T\star}$
using *mult-right-isotone conv-isotone star-isotone inf.cobounded2 inf.sup-monoid.add-assoc* by (*simp add*: *inf.sup-monoid.add-assoc order.eq-iff inf.sup-monoid.add-commute*)
also have ... $\leq top * ?e^T * (?v)^{T\star}$
using *mult-right-isotone conv-isotone star-isotone* by *auto*
finally show *?thesis*
by *blast*
qed
have *20*: $f \sqcup f^T \leq (?v \sqcap -?i \sqcap -?i^T) \sqcup (?v^T \sqcap -?i \sqcap -?i^T)$
proof (*rule kruskal-edge-between-components-2*)
show $?F \leq -\ ?i$
using *16* by *simp*
next
show *injective f*
by (*simp add*: *assms*(*2*))
next
show $f \sqcup f^T \leq w \sqcap -\ (top * ?e * w^{T\star}) \sqcup (w \sqcap top * ?e * w^{T\star})^T \sqcup (w \sqcap -\ (top * ?e * w^{T\star}) \sqcup (w \sqcap top * ?e * w^{T\star})^T)^T$
using *2 12* by (*metis conv-dist-sup conv-involutive conv-isotone le-supI sup-commute*)
qed
have *minimum-spanning-forest ?w g* $\wedge$ $?f \leq ?w \sqcup ?w^T$
proof (*intro conjI*)
have *211*: $?e^T \leq ?v^\star$
proof (*rule kruskal-edge-arc-1*[where $g$=$g$ and $h$=$?ec$])
show $?e \leq --\ ?ec$
using *minarc-below* by *blast*
next
show $?ec \leq g$
using *assms*(*4*) *inf.cobounded2* by (*simp add*: *boruvka-inner-invariant-def boruvka-outer-invariant-def conv-dist-inf*)
next
show *symmetric g*
by (*meson assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*)
next
show *components g* $\leq$ *forest-components* $(w \sqcap -\ (top * ?e * w^{T\star}) \sqcup (w \sqcap top * ?e * w^{T\star})^T)$
using *9* by *simp*
next
show $(w \sqcap -\ (top * ?e * w^{T\star}) \sqcup (w \sqcap top * ?e * w^{T\star})^T) * ?e^T = bot$
using *13* by *blast*
qed
have *212*: *arc ?i*
proof (*rule boruvka-edge-arc*)
show *equivalence ?F*

**by** (*simp add*: *5*)
**next**
**show** *forest ?v*
**using** *10 spanning-forest-def* **by** *blast*
**next**
**show** *arc ?e*
**using** *assms*(*16*) *minarc-arc minarc-bot-iff* **by** *blast*
**next**
**show** *regular ?F*
**using** *3 regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
**next**
**show** $?F \le$ *forest-components* $(?F \sqcap ?v)$
**by** (*simp add*: *12 2 8 kruskal-forest-components-inf*)
**next**
**show** *regular ?v*
**using** *10 spanning-forest-def* **by** *blast*
**next**
**show** $?v * ?e^T = bot$
**using** *13* **by** *auto*
**next**
**show** $?e * ?F * ?e = bot$
**by** (*simp add*: *6*)
**next**
**show** $?e^T \le ?v^\star$
**using** *211* **by** *auto*

**next**
**show** $?e \ne bot$
**by** (*simp add*: *assms*(*16*))
**qed**
**show** *minimum-spanning-forest ?w g*
**proof** (*unfold minimum-spanning-forest-def*, *intro conjI*)
**have** $(?v \sqcap -?i) * ?e^T \le ?v * ?e^T$
**using** *inf-le1 mult-left-isotone* **by** *simp*
**hence** $(?v \sqcap -?i) * ?e^T = bot$
**using** *13 le-bot* **by** *simp*
**hence** *221*: $?e * (?v \sqcap -?i)^T = bot$
**using** *conv-dist-comp conv-involutive conv-bot* **by** *force*
**have** *222*: *injective ?w*
**proof** (*rule injective-sup*)
**show** *injective* $(?v \sqcap -?i)$
**using** *8* **by** (*simp add*: *injective-inf-closed*)
**next**
**show** *coreflexive* $(?e * (?v \sqcap -?i)^T)$
**using** *221* **by** *simp*
**next**
**show** *injective ?e*
**by** (*metis arc-injective minarc-arc coreflexive-bot-closed*
*coreflexive-injective minarc-bot-iff*)

**qed**
**show** *spanning-forest ?w g*
**proof** (*unfold spanning-forest-def*, *intro conjI*)
**show** *injective ?w*
**using** *222* **by** *simp*
**next**
**show** *acyclic ?w*
**proof** (*rule kruskal-exchange-acyclic-inv-2*)
**show** *acyclic ?v*
**using** *10 spanning-forest-def* **by** *blast*
**next**
**show** *injective ?v*
**using** *8* **by** *simp*
**next**
**show** $?i \leq ?v$
**using** *inf.coboundedI1* **by** *simp*
**next**
**show** *bijective* $(?i^T * top)$
**using** *212* **by** *simp*
**next**
**show** *bijective* $(?e * top)$
**using** *14 212* **by** (*smt assms*(*4*) *comp-inf.idempotent-bot-closed conv-complement minarc-arc minarc-bot-iff p-bot regular-closed-bot semiring.mult-not-zero symmetric-top-closed*)
**next**
**show** $?i \leq top * ?e^T * ?v^{T\star}$
**using** *19* **by** *simp*
**next**
**show** $?v * ?e^T * top = bot$
**using** *13* **by** *simp*
**qed**
**next**
**have** $?w \leq ?v \sqcup ?e$
**using** *inf-le1 sup-left-isotone* **by** *simp*
**also have** ... $\leq --g \sqcup ?e$
**using** *10 sup-left-isotone spanning-forest-def* **by** *blast*
**also have** ... $\leq --g \sqcup --h$
**proof** −
**have** *1*: $--g \leq --g \sqcup --h$
**by** *simp*
**have** *2*: $?e \leq --g \sqcup --h$
**by** (*metis inf.coboundedI1 inf.sup-monoid.add-commute minarc-below order.trans p-dist-inf p-dist-sup sup.cobounded1*)
**thus** *?thesis*
**using** *1 2* **by** *simp*
**qed**
**also have** ... $\leq --g$
**using** *assms*(*18*, *19*) **by** *auto*
**finally show** $?w \leq --g$

```
        by simp
      next
        have 223: ?i ≤ (?v ⊓ −?i)^T⋆ ∗ ?e^T ∗ top
        proof (rule boruvka-exchange-spanning-inv)
          show forest ?v
            using 10 spanning-forest-def by blast
        next
          show ?v⋆ ∗ ?e^T = ?e^T
            using 13 by (smt conv-complement conv-dist-comp conv-involutive
conv-star-commute dense-pp fc-top regular-closed-top star-absorb)
        next
          show ?i ≤ ?v ⊓ top ∗ ?e^T ∗ ?w^T⋆
            using 18 inf.sup-monoid.add-assoc by auto
        next
          show arc ?i
            using 212 by blast
        next
          show arc ?e
            using assms(16) minarc-arc minarc-bot-iff by auto
        next
          show ?v ≤ −−g
            using 10 spanning-forest-def by blast
        next
          show ?w ≤ −−g
          proof −
            have 2231: ?e ≤ −−g
              by (metis inf.boundedE minarc-below pp-dist-inf)
            have ?w ≤ ?v ⊔ ?e
              using inf-le1 sup-left-isotone by simp
            also have ... ≤ −−g
              using 2231 10 spanning-forest-def sup-least by blast
            finally show ?thesis
              by blast
          qed
        next
          show ?e ≤ −−g
            by (metis inf.boundedE minarc-below pp-dist-inf)
        next
          show components g ≤ forest-components ?v
            by (simp add: 9)
        qed
        have components g ≤ forest-components ?v
          using 10 spanning-forest-def by auto
        also have ... ≤ forest-components ?w
        proof (rule kruskal-exchange-forest-components-inv)
        next
          show injective ((?v ⊓ −?i) ⊔ ?e)
            using 222 by simp
        next
```

**show** *regular ?i*
**using** *15* **by** *simp*
**next**
**show** $?e * top * ?e = ?e$
**by** (*metis arc-top-arc minarc-arc minarc-bot-iff semiring.mult-not-zero*)
**next**
**show** $?i \le top * ?e^T * ?v^{T\star}$
**using** *19* **by** *blast*
**next**
**show** $?v * ?e^T * top = bot$
**using** *13* **by** *simp*
**next**
**show** *injective ?v*
**using** *8* **by** *simp*
**next**
**show** $?i \le ?v$
**by** (*simp add*: *le-infI1*)
**next**
**show** $?i \le (?v \sqcap -?i)^{T\star} * ?e^T * top$
**using** *223* **by** *blast*
**qed**
**finally show** *components g* $\le$ *forest-components ?w*
**by** *simp*
**next**
**show** *regular ?w*
**using** *3 7 regular-conv-closed* **by** *simp*
**qed**
**next**
**have** *224*: $?e \sqcap g \ne bot$
**using** *assms*(*16*) *inf.left-commute inf-bot-right minarc-meet-bot* **by** *fastforce*
**have** *225*: *sum* $(?e \sqcap g) \le$ *sum* $(?i \sqcap g)$
**proof** (*rule a-to-e-in-forest-modulo-equivalence*)
**show** *symmetric g*
**using** *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*
**by** *auto*
**next**
**show** $j \ne bot$
**by** (*simp add*: *assms*(*17*))
**next**
**show** $f \le --g$
**by** (*simp add*: *assms*(*3*))
**next**
**show** *vector j*
**using** *assms*(*6*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *forest h*
**by** (*simp add*: *assms*(*8*))
**next**
**show** *forest-modulo-equivalence* (*forest-components h*) *d*

by ($simp$ $add$: $assms(9)$)
next
show $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
by ($simp$ $add$: $assms(12)$)
next
show $\forall a\ b.$ *forest-modulo-equivalence-path* $a\ b\ (?H)\ d \wedge a \leq -\ ?H \sqcap --g \wedge b \leq d \longrightarrow sum\ (b \sqcap g) \leq sum\ (a \sqcap g)$
by ($simp$ $add$: $assms(13)$)
next
show *regular* $d$
using $assms(14)$ by *auto*
next
show $?e = ?e$
by *simp*
next
show *arc* $?i$
using *212* by *blast*
next
show *forest-modulo-equivalence-path* $?i\ ?e\ ?H\ (d \sqcup ?e)$
proof $-$
have $d^T * ?H * ?e = bot$
using $assms(6,\ 7,\ 10\ ,11,\ 17)$ *dT-He-eq-bot le-bot* by *blast*
hence *251*: $d^T * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by *simp*
hence $d^T * ?H * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by (*metis* $assms(8)$ *forest-components-star star.circ-decompose-9 mult-assoc*)
hence $d^T * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
proof $-$
have $d^T * ?H * d \leq 1$
using $assms(9)$ *forest-modulo-equivalence-def dTransHd-le-1* by *blast*
hence $d^T * ?H * d * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by (*metis mult-left-isotone star.circ-circ-mult star-involutive star-one*)
hence $d^T * ?H * ?e \sqcup d^T * ?H * d * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
using *251* by *simp*
hence $d^T * (1 \sqcup ?H * d * (?H * d)^\star) * ?H * ?e \leq (?H * d)^\star * ?H * ?e$
by (*simp add*: *comp-associative comp-left-dist-sup semiring.distrib-right*)
thus *?thesis*
by (*simp add*: *star-left-unfold-equal*)
qed
hence $?H * d^T * (?H * d)^\star * ?H * ?e \leq ?H * (?H * d)^\star * ?H * ?e$
by (*simp add*: *mult-right-isotone mult-assoc*)
hence $?H * d^T * (?H * d)^\star * ?H * ?e \leq ?H * ?H * (d * ?H)^\star * ?e$
by (*smt star-slide mult-assoc*)
hence $?H * d^T * (?H * d)^\star * ?H * ?e \leq ?H * (d * ?H)^\star * ?e$
by (*metis* $assms(8)$ *forest-components-star star.circ-decompose-9*)
hence $?H * d^T * (?H * d)^\star * ?H * ?e \leq (?H * d)^\star * ?H * ?e$

**using** *star-slide* **by** *auto*
**hence** $?H * d * (?H * d)^{\star} * ?H * ?e \sqcup ?H * d^T * (?H * d)^{\star} * ?H * ?e \leq (?H * d)^{\star} * ?H * ?e$
**by** (*smt le-supI star.circ-loop-fixpoint sup.cobounded2 sup-commute mult-assoc*)
**hence** $(?H * (d \sqcup d^T)) * (?H * d)^{\star} * ?H * ?e \leq (?H * d)^{\star} * ?H * ?e$
**by** (*simp add*: *semiring.distrib-left semiring.distrib-right*)
**hence** $(?H * (d \sqcup d^T))^{\star} * (?H * d)^{\star} * ?H * ?e \leq (?H * d)^{\star} * ?H * ?e$
**by** (*simp add*: *star-left-induct-mult mult-assoc*)
**hence** *252*: $(?H * (d \sqcup d^T))^{\star} * ?H * ?e \leq (?H * d)^{\star} * ?H * ?e$
**by** (*smt mult-left-dist-sup star.circ-transitive-equal star-slide star-sup-1 mult-assoc*)
**have** $?i \leq top * ?e^T * ?F$
**by** *auto*
**hence** $?i^T \leq ?F^T * ?e^{TT} * top^T$
**by** (*simp add*: *conv-dist-comp conv-dist-inf mult-assoc*)
**hence** $?i^T * top \leq ?F^T * ?e^{TT} * top^T * top$
**using** *comp-isotone* **by** *blast*
**also have** ... $= ?F^T * ?e^{TT} * top^T$
**by** (*simp add*: *vector-mult-closed*)
**also have** ... $= ?F * ?e^{TT} * top^T$
**by** (*simp add*: *conv-dist-comp conv-star-commute*)
**also have** ... $= ?F * ?e * top$
**by** *simp*
**also have** ... $= (?H * (d \sqcup d^T))^{\star} * ?H * ?e * top$
**using** *assms*(*2*, *8*, *12*) *F-is-H-and-d* **by** *simp*
**also have** ... $\leq (?H * d)^{\star} * ?H * ?e * top$
**by** (*simp add*: *252 comp-isotone*)
**also have** ... $\leq (?H * (d \sqcup ?e))^{\star} * ?H * ?e * top$
**by** (*simp add*: *comp-isotone star-isotone*)
**finally have** $?i^T * top \leq (?H * (d \sqcup ?e))^{\star} * ?H * ?e * top$
**by** *blast*
**thus** *?thesis*
**using** *212 assms*(*16*) *forest-modulo-equivalence-path-def minarc-arc minarc-bot-iff* **by** *blast*
**qed**
**next**
**show** $?i \leq - ?H \sqcap -- g$
**proof** −
**have** *forest-components h* $\leq$ *forest-components f*
**using** *assms*(*2*, *8*, *12*) *H-below-F* **by** *blast*
**then have** *241*: $?i \leq -?H$
**using** *16 assms*(*9*) *inf.order-lesseq-imp p-antitone-iff* **by** *blast*
**have** $?i \leq -- g$
**using** *10 inf.coboundedI1 spanning-forest-def* **by** *blast*
**thus** *?thesis*
**using** *241 inf-greatest* **by** *blast*
**qed**
**next**

**show** *regular h*
**using** *assms(18)* **by** *auto*
**qed**
**have** $?v \sqcap ?e \sqcap -?i = bot$
**using** *14* **by** *simp*
**hence** $sum\ (?w \sqcap g) = sum\ (?v \sqcap -?i \sqcap g) + sum\ (?e \sqcap g)$
**using** *sum-disjoint inf-commute inf-assoc* **by** *simp*
**also have** ... $\leq sum\ (?v \sqcap -?i \sqcap g) + sum\ (?i \sqcap g)$
**using** *224 225 sum-plus-right-isotone* **by** *simp*
**also have** ... $= sum\ (((?v \sqcap -?i) \sqcup ?i) \sqcap g)$
**using** *sum-disjoint inf-le2 pseudo-complement* **by** *simp*
**also have** ... $= sum\ ((?v \sqcup ?i) \sqcap (-?i \sqcup ?i) \sqcap g)$
**by** (*simp add*: *sup-inf-distrib2*)
**also have** ... $= sum\ ((?v \sqcup ?i) \sqcap g)$
**using** *15* **by** (*metis inf-top-right stone*)
**also have** ... $= sum\ (?v \sqcap g)$
**by** (*simp add*: *inf.sup-monoid.add-assoc*)
**finally have** $sum\ (?w \sqcap g) \leq sum\ (?v \sqcap g)$
**by** *simp*
**thus** $\forall u\ .\ spanning\text{-}forest\ u\ g \longrightarrow sum\ (?w \sqcap g) \leq sum\ (u \sqcap g)$
**using** *2 11 minimum-spanning-forest-def* **by** *auto*
**qed**
**next**
**have** $?f \leq f \sqcup f^T \sqcup ?e$
**by** (*smt conv-dist-inf inf-le1 sup-left-isotone sup-mono inf.order-lesseq-imp*)
**also have** ... $\leq (?v \sqcap -?i \sqcap -?i^T) \sqcup (?v^T \sqcap -?i \sqcap -?i^T) \sqcup ?e$
**using** *20 sup-left-isotone* **by** *simp*
**also have** ... $\leq (?v \sqcap -?i) \sqcup (?v^T \sqcap -?i \sqcap -?i^T) \sqcup ?e$
**by** (*metis inf.cobounded1 sup-inf-distrib2*)
**also have** ... $= ?w \sqcup (?v^T \sqcap -?i \sqcap -?i^T)$
**by** (*simp add*: *sup-assoc sup-commute*)
**also have** ... $\leq ?w \sqcup (?v^T \sqcap -?i^T)$
**using** *inf.sup-right-isotone inf-assoc sup-right-isotone* **by** *simp*
**also have** ... $\leq ?w \sqcup ?w^T$
**using** *conv-complement conv-dist-inf conv-dist-sup sup-right-isotone* **by** *simp*
**finally show** $?f \leq ?w \sqcup ?w^T$
**by** *simp*
**qed**
**thus** *?thesis* **by** *auto*
**qed**

**lemma** *boruvka-outer-invariant-when-e-not-bot*:
**assumes** *boruvka-inner-invariant j f h g d*
**and** $j \neq bot$
**and** $selected\text{-}edge\ h\ j\ g \leq -\ forest\text{-}components\ f$
**and** $selected\text{-}edge\ h\ j\ g \neq bot$
**shows** $boruvka\text{-}outer\text{-}invariant\ (f \sqcap -\ selected\text{-}edge\ h\ j\ g^T \sqcap -\ path\ f\ h\ j\ g \sqcup (f \sqcap -\ selected\text{-}edge\ h\ j\ g^T \sqcap path\ f\ h\ j\ g)^T \sqcup selected\text{-}edge\ h\ j\ g)\ g$
**proof** −

**let** *?c* = *choose-component* (*forest-components h*) *j*
**let** *?p* = *path f h j g*
**let** *?F* = *forest-components f*
**let** *?H* = *forest-components h*
**let** *?e* = *selected-edge h j g*
**let** $?f' = f \sqcap -?e^T \sqcap -?p \sqcup (f \sqcap -?e^T \sqcap ?p)^T \sqcup ?e$
**let** $?d' = d \sqcup ?e$
**let** $?j' = j \sqcap -?c$
**show** *boruvka-outer-invariant* $?f'$ *g*
**proof** (*unfold boruvka-outer-invariant-def*, *intro conjI*)
**show** *symmetric g*
**by** (*meson assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*)
**next**
**show** *injective* $?f'$
**proof** (*rule kruskal-injective-inv*)
**show** *injective* $(f \sqcap - ?e^T)$
**by** (*meson assms*(*1*) *boruvka-inner-invariant-def*
*boruvka-outer-invariant-def injective-inf-closed*)
**show** *covector* (*?p*)
**using** *covector-mult-closed* **by** *simp*
**show** $?p * (f \sqcap - ?e^T)^T \leq ?p$
**by** (*simp add*: *mult-right-isotone star.left-plus-below-circ star-plus*
*mult-assoc*)
**show** $?e \leq ?p$
**by** (*meson mult-left-isotone order.trans star-outer-increasing top.extremum*)
**show** $?p * (f \sqcap - ?e^T)^T \leq - ?e$
**proof** −
**have** $?p * (f \sqcap - ?e^T)^T \leq ?p * f^T$
**by** (*simp add*: *conv-dist-inf mult-right-isotone*)
**also have** ... $\leq top * ?e * (f)^{T\star} * f^T$
**using** *conv-dist-inf star-isotone comp-isotone* **by** *simp*
**also have** ... $\leq - ?e$
**using** *assms*(*1*, *4*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*
*kruskal-injective-inv-2 minarc-arc minarc-bot-iff* **by** *auto*
**finally show** *?thesis* **.**
**qed**
**show** *injective* (*?e*)
**by** (*metis arc-injective coreflexive-bot-closed minarc-arc minarc-bot-iff*
*semiring.mult-not-zero*)
**show** *coreflexive* $(?p^T * ?p \sqcap (f \sqcap - ?e^T)^T * (f \sqcap - ?e^T))$
**proof** −
**have** $(?p^T * ?p \sqcap (f \sqcap - ?e^T)^T * (f \sqcap - ?e^T)) \leq ?p^T * ?p \sqcap f^T * f$
**using** *conv-dist-inf inf.sup-right-isotone mult-isotone* **by** *simp*
**also have** ... $\leq (top * ?e * f^{T\star})^T * (top * ?e * f^{T\star}) \sqcap f^T * f$
**by** (*metis comp-associative comp-inf.coreflexive-transitive*
*comp-inf.mult-right-isotone comp-isotone conv-isotone inf.cobounded1 inf.idem*
*inf.sup-monoid.add-commute star-isotone top.extremum*)
**also have** ... $\leq 1$
**using** *assms*(*1*, *4*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*

*kruskal-injective-inv-3 minarc-arc minarc-bot-iff* **by** *auto*
**finally show** *?thesis*
**by** *simp*
**qed**
**qed**
**next**
**show** *acyclic ?f′*
**proof** (*rule kruskal-acyclic-inv*)
**show** *acyclic* $(f \sqcap -\ ?e^T)$
**proof** $-$
**have** *f-intersect-below*: $(f \sqcap -\ ?e^T) \le f$ **by** *simp*
**have** *acyclic f*
**by** (*meson assms*(*1*) *boruvka-inner-invariant-def*
*boruvka-outer-invariant-def*)
**thus** *?thesis*
**using** *comp-isotone dual-order.trans star-isotone f-intersect-below* **by** *blast*
**qed**
**next**
**show** *covector ?p*
**by** (*metis comp-associative vector-top-closed*)
**next**
**show** $(f \sqcap -\ ?e^T \sqcap\ ?p)^T * (f \sqcap -\ ?e^T)^\star *\ ?e = bot$
**proof** $-$
**have** $?e \le -\ (f^{T\star} * f^\star)$
**by** (*simp add*: *assms*(*3*))
**hence** $?e * top *\ ?e \le -\ (f^{T\star} * f^\star)$
**by** (*metis arc-top-arc minarc-arc minarc-bot-iff semiring.mult-not-zero*)
**hence** $?e^T * top *\ ?e^T \le -\ (f^{T\star} * f^\star)^T$
**by** (*metis comp-associative conv-complement conv-dist-comp conv-isotone*
*symmetric-top-closed*)
**hence** $?e^T * top *\ ?e^T \le -\ (f^{T\star} * f^\star)$
**by** (*simp add*: *conv-dist-comp conv-star-commute*)
**hence** $?e * (f^{T\star} * f^\star) *\ ?e \le bot$
**using** *triple-schroeder-p* **by** *auto*
**hence** *1*: $?e * f^{T\star} * f^\star *\ ?e \le bot$
**using** *mult-assoc* **by** *auto*
**have** *2*: $(f \sqcap -\ ?e^T)^{T\star} \le f^{T\star}$
**by** (*simp add*: *conv-dist-inf star-isotone*)
**have** $(f \sqcap -\ ?e^T \sqcap\ ?p)^T * (f \sqcap -\ ?e^T)^\star *\ ?e \le (f \sqcap\ ?p)^T * (f \sqcap -\ ?e^T)^\star$
$*\ ?e$
**by** (*simp add*: *comp-isotone conv-dist-inf inf.orderI*
*inf.sup-monoid.add-assoc*)
**also have** ... $\le (f \sqcap\ ?p)^T * f^\star *\ ?e$
**by** (*simp add*: *comp-isotone star-isotone*)
**also have** ... $\le (f \sqcap top *\ ?e * (f)^{T\star})^T * f^\star *\ ?e$
**using** *2* **by** (*metis comp-inf.comp-isotone comp-inf.coreflexive-transitive*
*comp-isotone conv-isotone inf.idem top.extremum*)
**also have** ... $= (f^T \sqcap (top *\ ?e * f^{T\star})^T) * f^\star *\ ?e$
**by** (*simp add*: *conv-dist-inf*)

also have ... $\leq$ *top* $*$ ($f^T \sqcap$ (*top* $*$ *?e* $*$ $f^{T\star})^T$) $*$ $f^\star$ $*$ *?e*
using *top-left-mult-increasing mult-assoc* **by** *auto*
also have ... = (*top* $\sqcap$ *top* $*$ *?e* $*$ $f^{T\star}$) $*$ $f^T$ $*$ $f^\star$ $*$ *?e*
by (*smt covector-comp-inf-1 covector-mult-closed order.eq-iff inf.sup-monoid.add-commute vector-top-closed*)
also have ... = *top* $*$ *?e* $*$ $f^{T\star}$ $*$ $f^T$ $*$ $f^\star$ $*$ *?e*
by *simp*
also have ... $\leq$ *top* $*$ *?e* $*$ $f^{T\star}$ $*$ $f^\star$ $*$ *?e*
by (*smt conv-dist-comp conv-isotone conv-star-commute mult-left-isotone mult-right-isotone star.left-plus-below-circ mult-assoc*)
also have ... $\leq$ *bot*
using *1 covector-bot-closed le-bot mult-assoc* **by** *fastforce*
finally show *?thesis*
using *le-bot* **by** *auto*
qed
next
show *?e* $*$ ($f \sqcap -$ *?e*$^T$)$^\star$ $*$ *?e* = *bot*
proof −
have *1*: *?e* $\leq -$ *?F*
by (*simp add*: *assms*(*3*))
have *2*: *injective f*
by (*meson assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def*)
have *3*: *equivalence ?F*
using *2 forest-components-equivalence* **by** *simp*
hence *4*: *?e*$^T$ = *?e*$^T$ $*$ *top* $*$ *?e*$^T$
by (*smt arc-conv-closed arc-top-arc covector-complement-closed covector-conv-vector ex231e minarc-arc minarc-bot-iff pp-surjective regular-closed-top vector-mult-closed vector-top-closed*)
also have ... $\leq -$ *?F* **using** *1 3 conv-isotone conv-complement calculation* **by** *fastforce*
finally have *5*: *?e* $*$ *?F* $*$ *?e* = *bot*
using *4* **by** (*smt triple-schroeder-p le-bot pp-total regular-closed-top vector-top-closed*)
have ($f \sqcap -$ *?e*$^T$)$^\star$ $\leq f^\star$
by (*simp add*: *star-isotone*)
hence *?e* $*$ ($f \sqcap -$ *?e*$^T$)$^\star$ $*$ *?e* $\leq$ *?e* $*$ $f^\star$ $*$ *?e*
using *mult-left-isotone mult-right-isotone* **by** *blast*
also have ... $\leq$ *?e* $*$ *?F* $*$ *?e*
by (*metis conv-star-commute forest-components-increasing mult-left-isotone mult-right-isotone star-involutive*)
also have *6*: ... = *bot*
using *5* **by** *simp*
finally show *?thesis* **using** *6 le-bot* **by** *blast*
qed
next
show *forest-components* ($f \sqcap -$*?e*$^T$) $\leq -$ *?e*
proof −
have *1*: *?e* $\leq -$ *?F*

```
        by (simp add: assms(3))
      have f ⊓ − ?e^T ≤ f
        by simp
      hence forest-components (f ⊓ − ?e^T) ≤ ?F
        using forest-components-isotone by blast
      thus ?thesis
        using 1 order-lesseq-imp p-antitone-iff by blast
    qed
  qed
next
  show ?f' ≤ −−g
  proof −
    have 1: (f ⊓ − ?e^T ⊓ − ?p) ≤ −−g
      by (meson assms(1) boruvka-inner-invariant-def
boruvka-outer-invariant-def inf.coboundedI1)
    have 2: (f ⊓ − ?e^T ⊓ ?p)^T ≤ −−g
    proof −
      have (f ⊓ − ?e^T ⊓ ?p)^T ≤ f^T
        by (simp add: conv-isotone inf.sup-monoid.add-assoc)
      also have ... ≤ −−g
        by (metis assms(1) boruvka-inner-invariant-def
boruvka-outer-invariant-def conv-complement conv-isotone)
      finally show ?thesis
        by simp
    qed
    have 3: ?e ≤ −−g
      by (metis inf.boundedE minarc-below pp-dist-inf)
    show ?thesis using 1 2 3
      by simp
  qed
next
  show regular ?f'
    using assms(1) boruvka-inner-invariant-def boruvka-outer-invariant-def
minarc-regular regular-closed-star regular-conv-closed regular-mult-closed by auto
next
  show ∃ w. minimum-spanning-forest w g ∧ ?f' ≤ w ⊔ w^T
  proof (rule exists-a-w)
    show symmetric g
      using assms(1) boruvka-inner-invariant-def boruvka-outer-invariant-def by
auto
  next
    show forest f
      using assms(1) boruvka-inner-invariant-def boruvka-outer-invariant-def by
auto
  next
    show f ≤ −−g
      using assms(1) boruvka-inner-invariant-def boruvka-outer-invariant-def by
auto
  next
```

**show** *regular f*
**using** *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
**next**
**show** $(\exists\, w\ .\ \textit{minimum-spanning-forest}\ w\ g \wedge f \le w \sqcup w^T)$
**using** *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
**next**
**show** *vector j*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *regular j*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *forest h*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *forest-modulo-equivalence* (*forest-components h*) *d*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $d * \textit{top} \le -\ j$
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $\textit{forest-components}\ h * j = j$
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $(\forall\ a\ b\ .\ \textit{forest-modulo-equivalence-path}\ a\ b\ (\textit{forest-components}\ h)\ d \wedge a \le -(\textit{forest-components}\ h) \sqcap --\ g \wedge b \le d \longrightarrow \textit{sum}(b \sqcap g) \le \textit{sum}(a \sqcap g))$
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *regular d*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $\textit{selected-edge}\ h\ j\ g \le -\ \textit{forest-components}\ f$
**by** (*simp add*: *assms*(*3*))
**next**
**show** $\textit{selected-edge}\ h\ j\ g \ne \textit{bot}$
**by** (*simp add*: *assms*(*4*))
**next**
**show** $j \ne \textit{bot}$
**by** (*simp add*: *assms*(*2*))
**next**
**show** *regular h*
**using** *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
**next**

**show** $h \leq --g$
**using** *H-below-regular-g assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
**qed**
**qed**
**qed**

**lemma** *second-inner-invariant-when-e-not-bot*:
**assumes** *boruvka-inner-invariant j f h g d*
**and** $j \neq bot$
**and** *selected-edge h j g* $\leq -$ *forest-components f*
**and** *selected-edge h j g* $\neq$ *bot*
**shows** *boruvka-inner-invariant*
($j \sqcap -$ *choose-component* (*forest-components h*) *j*)
($f \sqcap -$ *selected-edge h j* $g^T \sqcap -$ *path f h j g* $\sqcup$
($f \sqcap -$ *selected-edge h j* $g^T \sqcap$ *path f h j g*)$^T$ $\sqcup$
*selected-edge h j g*)
*h g* ($d \sqcup$ *selected-edge h j g*)
**proof** $-$
**let** *?c* = *choose-component* (*forest-components h*) *j*
**let** *?p* = *path f h j g*
**let** *?F* = *forest-components f*
**let** *?H* = *forest-components h*
**let** *?e* = *selected-edge h j g*
**let** $?f' = f \sqcap -?e^T \sqcap -?p \sqcup (f \sqcap -?e^T \sqcap ?p)^T \sqcup ?e$
**let** $?d' = d \sqcup ?e$
**let** $?j' = j \sqcap -?c$
**show** *boruvka-inner-invariant ?j' ?f' h g ?d'*
**proof** (*unfold boruvka-inner-invariant-def*, *intro conjI*)
**show** *1*: *boruvka-outer-invariant ?f' g*
**using** *assms*(*1*, *2*, *3*, *4*) *boruvka-outer-invariant-when-e-not-bot* **by** *blast*
**next**
**show** $g \neq bot$
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *force*
**next**
**show** *regular ?d'*
**using** *assms*(*1*) *boruvka-inner-invariant-def minarc-regular* **by** *auto*
**next**
**show** *regular ?j'*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *auto*
**next**
**show** *vector ?j'*
**using** *assms*(*1*, *2*) *boruvka-inner-invariant-def component-is-vector vector-complement-closed vector-inf-closed* **by** *simp*
**next**
**show** *regular h*
**by** (*meson assms*(*1*) *boruvka-inner-invariant-def*)
**next**
**show** *injective h*

**by** (*meson assms*(*1*) *boruvka-inner-invariant-def*)
**next**
**show** *pd-kleene-allegory-class.acyclic h*
**by** (*meson assms*(*1*) *boruvka-inner-invariant-def*)
**next**
**show** $?H * ?j' = ?j'$
**using** *fc-j-eq-j-inv assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *forest-modulo-equivalence* $?H$ $?d'$
**using** *assms*(*1*, *2*, *3*) *forest-modulo-equivalence-d-U-e boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *auto*
**next**
**show** $?d' * top \leq -?j'$
**proof** –
**have** *31*: $?d' * top = d * top \sqcup ?e * top$
**by** (*simp add*: *mult-right-dist-sup*)
**have** *32*: $d * top \leq -?j'$
**by** (*meson assms*(*1*) *boruvka-inner-invariant-def inf.coboundedI1 p-antitone-iff*)
**have** *regular* $(?c * - ?c^T)$
**using** *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def component-is-regular regular-conv-closed regular-mult-closed* **by** *presburger*
**then have** $minarc(?c * - ?c^T \sqcap g) = minarc(?c \sqcap - ?c^T \sqcap g)$
**by** (*metis component-is-vector covector-comp-inf inf-top.left-neutral vector-conv-compl*)
**also have** ... $\leq --(?c \sqcap - ?c^T \sqcap g)$
**using** *minarc-below* **by** *blast*
**also have** ... $\leq --?c$
**by** (*simp add*: *inf.sup-monoid.add-assoc*)
**also have** ... $= ?c$
**using** *component-is-regular* **by** *auto*
**finally have** $?e \leq ?c$
**by** *simp*
**then have** $?e * top \leq ?c$
**by** (*metis component-is-vector mult-left-isotone*)
**also have** ... $\leq -j \sqcup ?c$
**by** *simp*
**also have** ... $= -(j \sqcap - ?c)$
**using** *component-is-regular* **by** *auto*
**finally have** *33*: $?e * top \leq -(j \sqcap - ?c)$
**by** *simp*
**show** *?thesis*
**using** *31 32 33* **by** *auto*
**qed**
**next**
**show** $?f' \sqcup ?f'^T = h \sqcup h^T \sqcup ?d' \sqcup ?d'^T$
**proof** –
**have** $?f' \sqcup ?f'^T = f \sqcap - ?e^T \sqcap - ?p \sqcup (f \sqcap - ?e^T \sqcap ?p)^T \sqcup ?e \sqcup (f \sqcap - ?e^T \sqcap - ?p)^T \sqcup (f \sqcap - ?e^T \sqcap ?p) \sqcup ?e^T$

by (*simp add*: *conv-dist-sup sup-monoid.add-assoc*)
also have ... = $(f \sqcap - ?e^T \sqcap - ?p) \sqcup (f \sqcap - ?e^T \sqcap ?p) \sqcup (f \sqcap - ?e^T \sqcap ?p)^T \sqcup (f \sqcap - ?e^T \sqcap - ?p)^T \sqcup ?e^T \sqcup ?e$
by (*simp add*: *sup.left-commute sup-commute*)
also have ... = $f \sqcup f^T \sqcup ?e \sqcup ?e^T$
proof (*rule simplify-f*)
show *regular ?p*
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def minarc-regular regular-closed-star regular-conv-closed regular-mult-closed* by *auto*
next
show *regular ?e*
using *minarc-regular* by *blast*
qed
also have ... = $h \sqcup h^T \sqcup d \sqcup d^T \sqcup ?e \sqcup ?e^T$
using *assms*(*1*) *boruvka-inner-invariant-def* by *auto*
finally show *?thesis*
by (*smt conv-dist-sup sup.left-commute sup-commute*)
qed
next
show $\forall$ *a b* . *forest-modulo-equivalence-path a b ?H ?d'* $\wedge a \leq - ?H \sqcap -- g \wedge b \leq ?d' \longrightarrow sum\ (b \sqcap g) \leq sum\ (a \sqcap g)$
proof (*intro allI*, *rule impI*, *unfold forest-modulo-equivalence-path-def*)
fix *a b*
assume *1*: $(arc\ a \wedge arc\ b \wedge a^T * top \leq (?H * ?d')^\star * ?H * b * top) \wedge a \leq - ?H \sqcap -- g \wedge b \leq ?d'$
thus $sum\ (b \sqcap g) \leq sum\ (a \sqcap g)$
proof (*cases* $b = ?e$)
case *b-equals-e*: *True*
thus *?thesis*
proof (*cases* $a = ?e$)
case *True*
thus *?thesis*
using *b-equals-e* by *auto*
next
case *a-ne-e*: *False*
have $sum\ (b \sqcap g) \leq sum\ (a \sqcap g)$
proof (*rule a-to-e-in-forest-modulo-equivalence*)
show *symmetric g*
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* by *auto*
next
show $j \neq bot$
by (*simp add*: *assms*(*2*))
next
show $f \leq -- g$
using *assms*(*1*) *boruvka-inner-invariant-def boruvka-outer-invariant-def* by *auto*
next
show *vector j*

**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *forest h*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *forest-modulo-equivalence* (*forest-components h*) *d*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $\forall$ *a b*. *forest-modulo-equivalence-path a b* (*?H*) *d* $\wedge$ *a* $\leq -$ *?H* $\sqcap -$
$-$ *g* $\wedge$ *b* $\leq$ *d* $\longrightarrow$ *sum* (*b* $\sqcap$ *g*) $\leq$ *sum* (*a* $\sqcap$ *g*)
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *regular d*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *b* = *?e*
**using** *b-equals-e* **by** *simp*
**next**
**show** *arc a*
**using** *1* **by** *simp*
**next**
**show** *forest-modulo-equivalence-path a b ?H ?d'*
**using** *1 forest-modulo-equivalence-path-def* **by** *simp*
**next**
**show** *a* $\leq -$ *?H* $\sqcap --$ *g*
**using** *1* **by** *simp*
**next**
**show** *regular h*
**using** *assms*(*1*) *boruvka-inner-invariant-def*
*boruvka-outer-invariant-def* **by** *auto*
**qed**
**thus** *?thesis*
**by** *simp*
**qed**
**next**
**case** *b-not-equal-e*: *False*
**then have** *b-below-d*: *b* $\leq$ *d*
**using** *1 assms*(*4*) *different-arc-in-sup-arc minarc-arc minarc-bot-iff* **by**
*metis*
**thus** *?thesis*
**proof** (*cases ?e* $\leq$ *d*)
**case** *True*
**then have** *forest-modulo-equivalence-path a b ?H d* $\wedge$ *b* $\leq$ *d*
**using** *1 forest-modulo-equivalence-path-def sup.absorb1* **by** *auto*
**thus** *?thesis*
**using** *1 assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*

**next**
**case** *e-not-less-than-d*: *False*
**have** *71*:*equivalence ?H*
**using** *assms*(*1*) *fch-equivalence boruvka-inner-invariant-def* **by** *auto*
**then have** *72*: *forest-modulo-equivalence-path a b ?H ?d′* ⟷ *forest-modulo-equivalence-path a b ?H d* ∨ (*forest-modulo-equivalence-path a ?e ?H d* ∧ *forest-modulo-equivalence-path ?e b ?H d*)
**proof** (*rule forest-modulo-equivalence-path-split-disj*)
**show** *arc ?e*
**using** *assms*(*4*) *minarc-arc minarc-bot-iff* **by** *blast*
**next**
**show** *regular a* ∧ *regular b* ∧ *regular ?e* ∧ *regular d* ∧ *regular ?H*
**using** *assms*(*1*) *1 boruvka-inner-invariant-def boruvka-outer-invariant-def arc-regular minarc-regular regular-closed-star regular-conv-closed regular-mult-closed* **by** *auto*
**qed**
**thus** *?thesis*
**proof** (*cases forest-modulo-equivalence-path a b ?H d*)
**case** *True*
**have** *forest-modulo-equivalence-path a b ?H d* ∧ *b* ≤ *d*
**using** *1 True forest-modulo-equivalence-path-def sup.absorb1* **by** (*metis assms*(*4*) *b-not-equal-e minarc-arc minarc-bot-iff different-arc-in-sup-arc*)
**thus** *?thesis*
**using** *1 assms*(*1*) *b-below-d boruvka-inner-invariant-def* **by** *auto*
**next**
**case** *False*
**have** *73*:*forest-modulo-equivalence-path a ?e ?H d* ∧ *forest-modulo-equivalence-path ?e b ?H d*
**using** *1 72 False forest-modulo-equivalence-path-def* **by** *blast*
**have** *74*: *?e* ≤ −−*g*
**by** (*metis inf.boundedE minarc-below pp-dist-inf*)
**have** *?H* ≤ *?F*
**using** *assms*(*1*) *H-below-F boruvka-inner-invariant-def boruvka-outer-invariant-def* **by** *blast*
**then have** *?e* ≤ − *?H*
**using** *assms*(*3*) *order-trans p-antitone* **by** *blast*
**then have** *?e* ≤ − *?H* ⊓ −−*g*
**using** *74* **by** *simp*
**then have** *75*: *sum* (*b* ⊓ *g*) ≤ *sum* (*?e* ⊓ *g*)
**using** *assms*(*1*) *b-below-d 73 boruvka-inner-invariant-def* **by** *blast*
**have** *76*: *forest-modulo-equivalence-path a ?e ?H ?d′*
**by** (*meson 73 forest-modulo-equivalence-path-split-disj assms*(*1*) *forest-modulo-equivalence-path-def boruvka-inner-invariant-def boruvka-outer-invariant-def fch-equivalence arc-regular regular-closed-star regular-conv-closed regular-mult-closed*)
**have** *77*: *sum* (*?e* ⊓ *g*) ≤ *sum* (*a* ⊓ *g*)
**proof** (*rule a-to-e-in-forest-modulo-equivalence*)
**show** *symmetric g*
**using** *assms*(*1*) *boruvka-inner-invariant-def*

*boruvka-outer-invariant-def* **by** *auto*
**next**
**show** $j \neq bot$
**by** (*simp add*: *assms*(*2*))
**next**
**show** $f \leq -- g$
**using** *assms*(*1*) *boruvka-inner-invariant-def*
*boruvka-outer-invariant-def* **by** *auto*
**next**
**show** *vector j*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *forest h*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *forest-modulo-equivalence* (*forest-components h*) *d*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $\forall$ *a b*. *forest-modulo-equivalence-path a b* (*?H*) *d* $\wedge$ $a \leq -$ *?H* $\sqcap$ $- - g \wedge b \leq d \longrightarrow$ *sum* $(b \sqcap g) \leq$ *sum* $(a \sqcap g)$
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *regular d*
**using** *assms*(*1*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *?e* = *?e*
**by** *simp*
**next**
**show** *arc a*
**using** *1* **by** *simp*
**next**
**show** *forest-modulo-equivalence-path a ?e ?H ?d'*
**by** (*simp add*: *76*)
**next**
**show** $a \leq -$ *?H* $\sqcap --g$
**using** *1* **by** *simp*
**next**
**show** *regular h*
**using** *assms*(*1*) *boruvka-inner-invariant-def*
*boruvka-outer-invariant-def* **by** *auto*
**qed**
**thus** *?thesis*
**using** *75 order.trans* **by** *blast*
**qed**
**qed**
**qed**

**qed**
**qed**
**qed**

**lemma** *second-inner-invariant-when-e-bot*:
**assumes** *selected-edge h j g* = *bot*
**and** *selected-edge h j g* $\leq -$ *forest-components f*
**and** *boruvka-inner-invariant j f h g d*
**shows** *boruvka-inner-invariant*
$(j \sqcap -$ *choose-component* (*forest-components h*) $j)$
$(f \sqcap -$ *selected-edge h j g*$^T \sqcap -$ *path f h j g* $\sqcup$
$(f \sqcap -$ *selected-edge h j g*$^T \sqcap$ *path f h j g*$)^T \sqcup$
*selected-edge h j g*)
*h g* $(d \sqcup$ *selected-edge h j g*)
**proof** $-$
**let** *?c* = *choose-component* (*forest-components h*) *j*
**let** *?p* = *path f h j g*
**let** *?F* = *forest-components f*
**let** *?H* = *forest-components h*
**let** *?e* = *selected-edge h j g*
**let** *?f'* = $f \sqcap -?e^T \sqcap -?p \sqcup (f \sqcap -?e^T \sqcap ?p)^T \sqcup ?e$
**let** *?d'* = $d \sqcup ?e$
**let** *?j'* = $j \sqcap -?c$
**show** *boruvka-inner-invariant ?j' ?f' h g ?d'*
**proof** (*unfold boruvka-inner-invariant-def*, *intro conjI*)
**next**
**show** *boruvka-outer-invariant ?f' g*
**using** *assms*(*1*) *assms*(*3*) *boruvka-inner-invariant-def* **by** *auto*
**next**
**show** $g \neq$ *bot*
**using** *assms*(*3*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *regular ?d'*
**using** *assms*(*1*) *assms*(*3*) *boruvka-inner-invariant-def* **by** *auto*
**next**
**show** *regular ?j'*
**using** *assms*(*3*) *boruvka-inner-invariant-def* **by** *auto*
**next**
**show** *vector ?j'*
**by** (*metis assms*(*3*) *boruvka-inner-invariant-def component-is-vector*
*vector-complement-closed vector-inf-closed*)
**next**
**show** *regular h*
**using** *assms*(*3*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *injective h*
**using** *assms*(*3*) *boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *pd-kleene-allegory-class.acyclic h*

**using** *assms(3) boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $?H * ?j' = ?j'$
**using** *assms(3) fc-j-eq-j-inv boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *forest-modulo-equivalence* $?H$ $?d'$
**using** *assms(1) assms(3) boruvka-inner-invariant-def* **by** *auto*
**next**
**show** $?d' * top \le -?j'$
**using** *assms(1) assms(3) boruvka-inner-invariant-def* **by** (*metis order.trans p-antitone-inf sup-monoid.add-0-right*)
**next**
**show** $?f' \sqcup ?f'^T = h \sqcup h^T \sqcup ?d' \sqcup ?d'^T$
**using** *assms(1, 3) boruvka-inner-invariant-def* **by** *auto*
**next**
**show** $\forall a\ b.$ *forest-modulo-equivalence-path* $a\ b\ ?H\ ?d' \wedge a \le -?H \sqcap --g \wedge b \le ?d' \longrightarrow sum(b \sqcap g) \le sum(a \sqcap g)$
**using** *assms(1, 3) boruvka-inner-invariant-def* **by** *auto*
**qed**
**qed**

## 4.5 Formalization and correctness proof

The following result shows that Borůvka's algorithm constructs a minimum spanning forest. We have the same postcondition as the proof of Kruskal's minimum spanning tree algorithm. We show only partial correctness.

**theorem** *boruvka-mst*:
*VARS f j h c e d*
$\{$ *symmetric g* $\}$
$f := bot;$
*WHILE* $-($*forest-components* $f) \sqcap g \ne bot$
*INV* $\{$ *boruvka-outer-invariant f g* $\}$
*DO*
$j := top;$
$h := f;$
$d := bot;$
*WHILE* $j \ne bot$
*INV* $\{$ *boruvka-inner-invariant j f h g d* $\}$
*DO*
$c :=$ *choose-component* $($*forest-components* $h)\ j;$
$e := minarc(c * -c^T \sqcap g);$
*IF* $e \le -($*forest-components* $f)$ *THEN*
$f := f \sqcap -e^T;$
$f := (f \sqcap -(top * e * f^{T\star})) \sqcup (f \sqcap top * e * f^{T\star})^T \sqcup e;$
$d := d \sqcup e$
*ELSE*
*SKIP*
*FI*;
$j := j \sqcap -c$

        *OD*
    *OD*
  { *minimum-spanning-forest f g* }
**proof** *vcg-simp*
  **assume** *1*: *symmetric g*
  **show** *boruvka-outer-invariant bot g*
    **using** *1 boruvka-outer-invariant-def kruskal-exists-minimal-spanning* **by** *auto*
**next**
  **fix** *f*
  **let** *?F* = *forest-components f*
  **assume** *1*: *boruvka-outer-invariant f g* $\wedge$ $-$ *?F* $\sqcap$ *g* $\neq$ *bot*
  **have** *2*: *equivalence ?F*
    **using** *1 boruvka-outer-invariant-def forest-components-equivalence* **by** *auto*
  **show** *boruvka-inner-invariant top f f g bot*
  **proof** (*unfold boruvka-inner-invariant-def*, *intro conjI*)
    **show** *boruvka-outer-invariant f g*
      **by** (*simp add*: *1*)
  **next**
    **show** *g* $\neq$ *bot*
      **using** *1* **by** *auto*
  **next**
    **show** *surjective top*
      **by** *simp*
  **next**
    **show** *regular top*
      **by** *simp*
  **next**
    **show** *regular bot*
      **by** *auto*
  **next**
    **show** *regular f*
      **using** *1 boruvka-outer-invariant-def* **by** *blast*
  **next**
    **show** *injective f*
      **using** *1 boruvka-outer-invariant-def* **by** *blast*
  **next**
    **show** *pd-kleene-allegory-class.acyclic f*
      **using** *1 boruvka-outer-invariant-def* **by** *blast*
  **next**
    **show** *forest-modulo-equivalence ?F bot*
      **by** (*simp add*: *2 forest-modulo-equivalence-def*)
  **next**
    **show** *bot* $*$ *top* $\leq$ $-$ *top*
      **by** *simp*
  **next**
    **show** *times-top-class.total* (*?F*)
      **by** (*simp add*: *star.circ-right-top mult-assoc*)
  **next**
    **show** $f \sqcup f^T = f \sqcup f^T \sqcup bot \sqcup bot^T$

```
      by simp
    next
      show ∀ a b. forest-modulo-equivalence-path a b ?F bot ∧ a ≤ − ?F ⊓ −− g ∧
b ≤ bot ⟶ sum (b ⊓ g) ≤ sum (a ⊓ g)
        by (metis (full-types) forest-modulo-equivalence-path-def bot-unique
mult-left-zero mult-right-zero top.extremum)
    qed
  next
    fix f j h d
    let ?c = choose-component (forest-components h) j
    let ?p = path f h j g
    let ?F = forest-components f
    let ?H = forest-components h
    let ?e = selected-edge h j g
    let ?f' = f ⊓ −?e^T ⊓ −?p ⊔ (f ⊓ −?e^T ⊓ ?p)^T ⊔ ?e
    let ?d' = d ⊔ ?e
    let ?j' = j ⊓ −?c
    assume 1: boruvka-inner-invariant j f h g d ∧ j ≠ bot
    show (?e ≤ −?F ⟶ boruvka-inner-invariant ?j' ?f' h g ?d') ∧ (¬ ?e ≤ −?F
⟶ boruvka-inner-invariant ?j' f h g d)
    proof (intro conjI)
      show ?e ≤ −?F ⟶ boruvka-inner-invariant ?j' ?f' h g ?d'
      proof (cases ?e = bot)
        case True
        thus ?thesis
          using 1 second-inner-invariant-when-e-bot by simp
      next
        case False
        thus ?thesis
          using 1 second-inner-invariant-when-e-not-bot by simp
      qed
    next
      show ¬ ?e ≤ −?F ⟶ boruvka-inner-invariant ?j' f h g d
      proof (rule impI, unfold boruvka-inner-invariant-def, intro conjI)
        show boruvka-outer-invariant f g
          using 1 boruvka-inner-invariant-def by blast
      next
        show g ≠ bot
          using 1 boruvka-inner-invariant-def by blast
      next
        show vector ?j'
          using 1 boruvka-inner-invariant-def component-is-vector
vector-complement-closed vector-inf-closed by auto
      next
        show regular ?j'
          using 1 boruvka-inner-invariant-def by auto
      next
        show regular d
          using 1 boruvka-inner-invariant-def by blast
```

**next**
**show** *regular h*
**using** *1 boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *injective h*
**using** *1 boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *pd-kleene-allegory-class.acyclic h*
**using** *1 boruvka-inner-invariant-def* **by** *blast*
**next**
**show** *forest-modulo-equivalence ?H d*
**using** *1 boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $d * top \leq -?j'$
**using** *1* **by** (*meson boruvka-inner-invariant-def dual-order.trans p-antitone-inf*)
**next**
**show** $?H * ?j' = ?j'$
**using** *1 fc-j-eq-j-inv boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $f \sqcup f^T = h \sqcup h^T \sqcup d \sqcup d^T$
**using** *1 boruvka-inner-invariant-def* **by** *blast*
**next**
**show** $\neg\ ?e \leq -?F \Longrightarrow \forall\, a\ b.$ *forest-modulo-equivalence-path* $a\ b\ ?H\ d \wedge a \leq -?H \sqcap --g \wedge b \leq d \longrightarrow sum(b \sqcap g) \leq sum(a \sqcap g)$
**using** *1 boruvka-inner-invariant-def* **by** *blast*
**qed**
**qed**
**next**
**fix** *f h d*
**assume** *boruvka-inner-invariant bot f h g d*
**thus** *boruvka-outer-invariant f g*
**by** (*meson boruvka-inner-invariant-def*)
**next**
**fix** *f*
**assume** *1*: *boruvka-outer-invariant f g* $\wedge -$ *forest-components f* $\sqcap g = bot$
**hence** *2*:*spanning-forest f g*
**proof** (*unfold spanning-forest-def*, *intro conjI*)
**show** *injective f*
**using** *1 boruvka-outer-invariant-def* **by** *blast*
**next**
**show** *acyclic f*
**using** *1 boruvka-outer-invariant-def* **by** *blast*
**next**
**show** $f \leq --g$
**using** *1 boruvka-outer-invariant-def* **by** *blast*
**next**
**show** *components g* $\leq$ *forest-components f*
**proof** $-$

**let** $?F$ = *forest-components f*
**have** $-?F \sqcap g \leq bot$
**by** (*simp add*: *1*)
**hence** $--g \leq bot \sqcup --?F$
**using** *1 shunting-p p-antitone pseudo-complement* **by** *auto*
**hence** $--g \leq ?F$
**using** *1 boruvka-outer-invariant-def pp-dist-comp pp-dist-star regular-conv-closed* **by** *auto*
**hence** $(--g)^\star \leq ?F^\star$
**by** (*simp add*: *star-isotone*)
**thus** *?thesis*
**using** *1 boruvka-outer-invariant-def forest-components-star* **by** *auto*
**qed**
**next**
**show** *regular f*
**using** *1 boruvka-outer-invariant-def* **by** *auto*
**qed**
**from** *1* **obtain** *w* **where** *3*: *minimum-spanning-forest w g* $\wedge f \leq w \sqcup w^T$
**using** *boruvka-outer-invariant-def* **by** *blast*
**hence** $w = w \sqcap --g$
**by** (*simp add*: *inf.absorb1 minimum-spanning-forest-def spanning-forest-def*)
**also have** ... $\leq w \sqcap$ *components g*
**by** (*metis inf.sup-right-isotone star.circ-increasing*)
**also have** ... $\leq w \sqcap f^{T\star} * f^\star$
**using** *2 spanning-forest-def inf.sup-right-isotone* **by** *simp*
**also have** ... $\leq f \sqcup f^T$
**proof** (*rule cancel-separate-6*[**where** $z$=$w$ **and** $y$=$w^T$])
**show** *injective w*
**using** *3 minimum-spanning-forest-def spanning-forest-def* **by** *simp*
**next**
**show** $f^T \leq w^T \sqcup w$
**using** *3* **by** (*metis conv-dist-inf conv-dist-sup conv-involutive inf.cobounded2 inf.orderE*)
**next**
**show** $f \leq w^T \sqcup w$
**using** *3* **by** (*simp add*: *sup-commute*)
**next**
**show** *injective w*
**using** *3 minimum-spanning-forest-def spanning-forest-def* **by** *simp*
**next**
**show** $w \sqcap w^{T\star} = bot$
**using** *3* **by** (*metis acyclic-star-below-complement comp-inf.mult-right-isotone inf-p le-bot minimum-spanning-forest-def spanning-forest-def*)
**qed**
**finally have** *4*: $w \leq f \sqcup f^T$
**by** *simp*
**have** *sum* $(f \sqcap g) =$ *sum* $((w \sqcup w^T) \sqcap (f \sqcap g))$
**using** *3* **by** (*metis inf-absorb2 inf.assoc*)
**also have** ... = *sum* $(w \sqcap (f \sqcap g))$ + *sum* $(w^T \sqcap (f \sqcap g))$

**using** *3 inf.commute acyclic-asymmetric sum-disjoint minimum-spanning-forest-def spanning-forest-def* **by** *simp*
**also have** ... = *sum* $(w \sqcap (f \sqcap g))$ + *sum* $(w \sqcap (f^T \sqcap g^T))$
**by** (*metis conv-dist-inf conv-involutive sum-conv*)
**also have** ... = *sum* $(f \sqcap (w \sqcap g))$ + *sum* $(f^T \sqcap (w \sqcap g))$
**proof** −
**have** *51*:$f^T \sqcap (w \sqcap g) = f^T \sqcap (w \sqcap g^T)$
**using** *1 boruvka-outer-invariant-def* **by** *auto*
**have** *52*:$f \sqcap (w \sqcap g) = w \sqcap (f \sqcap g)$
**by** (*simp add*: *inf.left-commute*)
**thus** *?thesis*
**using** *51 52 abel-semigroup.left-commute inf.abel-semigroup-axioms* **by** *fastforce*
**qed**
**also have** ... = *sum* $((f \sqcup f^T) \sqcap (w \sqcap g))$
**using** *2 acyclic-asymmetric inf.sup-monoid.add-commute sum-disjoint spanning-forest-def* **by** *simp*
**also have** ... = *sum* $(w \sqcap g)$
**using** *4* **by** (*metis inf-absorb2 inf.assoc*)
**finally show** *minimum-spanning-forest f g*
**using** *2 3 minimum-spanning-forest-def* **by** *simp*
**qed**

**end**

**end**

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