Relational Divisibility

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Abstract

We formalise key concepts and axioms of the divisibility relation on natural numbers using relation algebras. They use standard relational constructions for extrema, bounds, suprema, the univalent part and symmetric quotients, which we also formalise. We moreover prove that mono-atomic elements correspond to join-irreducible elements under the divisibility axioms.

Contents

1 Relational Constructions

theory *Relational-Constructions*

imports *Stone-Relation-Algebras.Relation-Algebras*

begin

This theory defines relational constructions for extrema, bounds and suprema, the univalent part and symmetric quotients. All definitions and most properties are standard; for example, see [1, 3, 4, 5]. Some properties are new. We start with a few general properties of relations and orders.

context *bounded-distrib-allegory*
begin

lemma *transitive-mapping-idempotent*:
$\mathit{transitive}\ x \Longrightarrow \mathit{mapping}\ x \Longrightarrow \mathit{idempotent}\ x$
⟨*proof*⟩

end

context *pd-allegory*
begin

lemma *comp-univalent-complement*:
assumes *univalent x*
shows $x * -y = x * \mathit{top} \sqcap -(x * y)$
⟨*proof*⟩

lemma *comp-injective-complement*:
$\mathit{injective}\ x \Longrightarrow -y * x = \mathit{top} * x \sqcap -(y * x)$
⟨*proof*⟩

lemma *strict-order-irreflexive*:
$\mathit{irreflexive}\ (x \sqcap -1)$
⟨*proof*⟩

lemma *strict-order-transitive-1*:
$\mathit{antisymmetric}\ x \Longrightarrow \mathit{transitive}\ x \Longrightarrow x * (x \sqcap -1) \le x \sqcap -1$
⟨*proof*⟩

lemma *strict-order-transitive-2*:
$\mathit{antisymmetric}\ x \Longrightarrow \mathit{transitive}\ x \Longrightarrow (x \sqcap -1) * x \le x \sqcap -1$
⟨*proof*⟩

lemma *strict-order-transitive*:
$\mathit{antisymmetric}\ x \Longrightarrow \mathit{transitive}\ x \Longrightarrow (x \sqcap -1) * (x \sqcap -1) \le x \sqcap -1$
⟨*proof*⟩

lemma *strict-order-transitive-eq-1*:
$\mathit{order}\ x \Longrightarrow (x \sqcap -1) * x = x \sqcap -1$
⟨*proof*⟩

lemma *strict-order-transitive-eq-2*:

order $x \Longrightarrow x * (x \sqcap -1) = x \sqcap -1$
⟨*proof*⟩

lemma *strict-order-transitive-eq*:
order $x \Longrightarrow (x \sqcap -1) * x = x * (x \sqcap -1)$
⟨*proof*⟩

lemma *strict-order-asymmetric*:
antisymmetric $x \Longrightarrow$ *asymmetric* $(x \sqcap -1)$
⟨*proof*⟩

end

The following gives relational definitions for extrema, bounds, suprema, the univalent part and symmetric quotients.

context *relation-algebra-signature*
begin

definition *maximal* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
maximal $r\ s \equiv s \sqcap -((r \sqcap -1) * s)$

definition *minimal* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
minimal $r\ s \equiv s \sqcap -((r^T \sqcap -1) * s)$

definition *upperbound* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
upperbound $r\ s \equiv -(-r^T * s)$

definition *lowerbound* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
lowerbound $r\ s \equiv -(-r * s)$

definition *greatest* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
greatest $r\ s \equiv s \sqcap -(-r^T * s)$

definition *least* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
least $r\ s \equiv s \sqcap -(-r * s)$

definition *supremum* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
supremum $r\ s \equiv$ *least* r (*upperbound* $r\ s$)

definition *infimum* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
infimum $r\ s \equiv$ *greatest* r (*lowerbound* $r\ s$)

definition *univalent-part* :: $'a \Rightarrow 'a$ **where**
univalent-part $r \equiv r \sqcap -(r * -1)$

definition *symmetric-quotient* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
symmetric-quotient $r\ s \equiv -(r^T * -s) \sqcap -(-r^T * s)$

abbreviation *noyau* :: $'a \Rightarrow 'a$ **where**

noyau $r \equiv$ *symmetric-quotient* r r

end

context *relation-algebra*
begin

1.1 Extrema, bounds and suprema

lemma *maximal-comparable*:
$r \sqcap (\mathit{maximal}\ r\ s) * (\mathit{maximal}\ r\ s)^T \le r^T$
⟨*proof*⟩

lemma *maximal-comparable-same*:
assumes *antisymmetric* r
shows $r \sqcap (\mathit{maximal}\ r\ s) * (\mathit{maximal}\ r\ s)^T \le 1$
⟨*proof*⟩

lemma *transitive-lowerbound*:
$\mathit{transitive}\ r \Longrightarrow r * \mathit{lowerbound}\ r\ s \le \mathit{lowerbound}\ r\ s$
⟨*proof*⟩

lemma *transitive-least*:
$\mathit{transitive}\ r \Longrightarrow r * \mathit{least}\ r\ \mathit{top} \le \mathit{least}\ r\ \mathit{top}$
⟨*proof*⟩

lemma *transitive-minimal-not-least*:
assumes *transitive* r
shows $r^T * \mathit{minimal}\ r\ (-\mathit{least}\ r\ \mathit{top}) \le -\mathit{least}\ r\ \mathit{top}$
⟨*proof*⟩

lemma *least-injective*:
assumes *antisymmetric* r
shows *injective* (*least* r s)
⟨*proof*⟩

lemma *least-conv-greatest*:
$\mathit{least}\ r = \mathit{greatest}\ (r^T)$
⟨*proof*⟩

lemma *greatest-injective*:
$\mathit{antisymmetric}\ r \Longrightarrow \mathit{injective}\ (\mathit{greatest}\ r\ s)$
⟨*proof*⟩

lemma *supremum-upperbound*:
assumes *antisymmetric* r
and $s \le r$
shows $\mathit{supremum}\ r\ s = 1 \longleftrightarrow \mathit{upperbound}\ r\ s \le r^T$
⟨*proof*⟩

1.2 Univalent part

lemma *univalent-part-idempotent*:
univalent-part (*univalent-part* r) = *univalent-part* r
⟨*proof*⟩

lemma *univalent-part-univalent*:
univalent (*univalent-part* r)
⟨*proof*⟩

lemma *univalent-part-times-converse*:
r^T ∗ *univalent-part* r = (*univalent-part* r)^T ∗ *univalent-part* r
⟨*proof*⟩

lemma *univalent-part-times-converse-1*:
r^T ∗ *univalent-part* $r \leq 1$
⟨*proof*⟩

lemma *minimal-univalent-part*:
assumes *reflexive* r
and *vector* s
shows *minimal* r s = s ⊓ *univalent-part* (($r \sqcap s$)^T) ∗ *top*
⟨*proof*⟩

1.3 Symmetric quotients

lemma *univalent-part-syq*:
univalent-part r = *symmetric-quotient* (r^T) *1*
⟨*proof*⟩

lemma *minimal-syq*:
assumes *reflexive* r
and *vector* s
shows *minimal* r s = s ⊓ *symmetric-quotient* ($r \sqcap s$) *1* ∗ *top*
⟨*proof*⟩

lemma *syq-complement*:
symmetric-quotient ($-r$) ($-s$) = *symmetric-quotient* r s
⟨*proof*⟩

lemma *syq-converse*:
(*symmetric-quotient* r s)^T = *symmetric-quotient* s r
⟨*proof*⟩

lemma *syq-comp-transitive*:
symmetric-quotient r s ∗ *symmetric-quotient* s p ≤ *symmetric-quotient* r p
⟨*proof*⟩

lemma *syq-comp-syq-top*:
symmetric-quotient r s ∗ *symmetric-quotient* s p = *symmetric-quotient* r p ⊓

symmetric-quotient r s $*$ *top*
⟨*proof*⟩

lemma *syq-comp-top-syq*:
symmetric-quotient r s $*$ *symmetric-quotient s p* $=$ *symmetric-quotient r p* $\sqcap$ *top* $*$ *symmetric-quotient s p*
⟨*proof*⟩

lemma *comp-syq-below*:
r $*$ *symmetric-quotient r s* $\leq$ *s*
⟨*proof*⟩

lemma *comp-syq-top*:
r $*$ *symmetric-quotient r s* $=$ *s* $\sqcap$ *top* $*$ *symmetric-quotient r s*
⟨*proof*⟩

lemma *syq-comp-isotone*:
symmetric-quotient r s $\leq$ *symmetric-quotient* $(q * r)$ $(q * s)$
⟨*proof*⟩

lemma *syq-comp-isotone-eq*:
assumes *univalent q*
and *surjective q*
shows *symmetric-quotient r s* $=$ *symmetric-quotient* $(q * r)$ $(q * s)$
⟨*proof*⟩

lemma *univalent-comp-syq*:
assumes *univalent p*
shows *p* $*$ *symmetric-quotient r s* $=$ *p* $*$ *top* $\sqcap$ *symmetric-quotient* $(r * p^T)$ *s*
⟨*proof*⟩

lemma *coreflexive-comp-syq*:
coreflexive p $\Longrightarrow$ *p* $*$ *symmetric-quotient r s* $=$ *p* $*$ *symmetric-quotient* $(r * p)$ *s*
⟨*proof*⟩

lemma *injective-comp-syq*:
injective p $\Longrightarrow$ *symmetric-quotient r s* $*$ *p* $=$ *top* $*$ *p* $\sqcap$ *symmetric-quotient r* $(s * p)$
⟨*proof*⟩

lemma *syq-comp-coreflexive*:
coreflexive p $\Longrightarrow$ *symmetric-quotient r s* $*$ *p* $=$ *symmetric-quotient r* $(s * p) * p$
⟨*proof*⟩

lemma *coreflexive-comp-syq-comp-coreflexive*:
coreflexive p $\Longrightarrow$ *coreflexive q* $\Longrightarrow$ *p* $*$ *symmetric-quotient r s* $*$ *q* $=$ *p* $*$ *symmetric-quotient* $(r * p)$ $(s * q) * q$
⟨*proof*⟩

lemma *surjective-syq*:
surjective (*symmetric-quotient r s*) $\Longrightarrow$ *r* $*$ *symmetric-quotient r s* = *s*
⟨*proof*⟩

lemma *comp-syq-surjective*:
assumes *total* $(-(top * r))$
shows *surjective* (*symmetric-quotient r s*) $\longleftrightarrow$ *r* $*$ *symmetric-quotient r s* = *s*
⟨*proof*⟩

lemma *noyau-reflexive*:
reflexive (*noyau r*)
⟨*proof*⟩

lemma *noyau-equivalence*:
equivalence (*noyau r*)
⟨*proof*⟩

lemma *noyau-reflexive-comp*:
r $*$ *noyau r* = *r*
⟨*proof*⟩

lemma *syq-comp-reflexive*:
noyau r $*$ *symmetric-quotient r s* = *symmetric-quotient r s*
⟨*proof*⟩

lemma *reflexive-antisymmetric-noyau*:
assumes *reflexive r*
and *antisymmetric r*
shows *noyau r* = *1*
⟨*proof*⟩

end

end

2 Divisibility

theory *Relational-Divisibility*

imports *Relational-Constructions*

begin

This theory gives relational axioms and definitions for divisibility. We start with the definitions, which are based on standard relational constructions. Then follow the axioms, which are relational formulations of axioms expressed in predicate logic in [2].

context *bounded-distrib-allegory-signature*

begin

definition *antichain* :: $'a \Rightarrow 'a \Rightarrow bool$ **where**
$antichain\ r\ s \equiv vector\ s \wedge r \sqcap s \sqcap s^T \leq 1$

end

class *divisibility-op* =
fixes *divisibility* :: $'a$ (D)

class *divisibility-def* = *relation-algebra* + *divisibility-op*
begin

Dbot is the least element of the divisibility order, which represents the number 1.

definition *Dbot* :: $'a$ **where**
$Dbot \equiv least\ D\ top$

Datoms are the atoms of the divisibility order, which represent the prime numbers.

definition *Datoms* :: $'a$ **where**
$Datoms \equiv minimal\ D\ (-Dbot)$

Datoms are the mono-atomic elements of the divisibility order, which represent the prime powers.

definition *Dmono* :: $'a$ **where**
$Dmono \equiv univalent\text{-}part\ ((D \sqcap Datoms)^T) * top$

Dfactor relates p to x if and only if p is maximal prime power factor of x.

definition *Dfactor* :: $'a$ **where**
$Dfactor \equiv maximal\ D\ (D \sqcap Dmono)$

Dsupport relates x to y if and only if y is the product of all primes below x.

definition *Dsupport* :: $'a$ **where**
$Dsupport \equiv symmetric\text{-}quotient\ (Datoms \sqcap D)\ Dfactor$

Dsucc relates x to y if and only if y is the product of prime power x with its base prime.

definition *Dsucc* :: $'a$ **where**
$Dsucc \equiv greatest\ D\ (D \sqcap -1)$

Dinc relates x to y if and only if y is the product of x with all its base primes.

definition *Dinc* :: $'a$ **where**
$Dinc \equiv symmetric\text{-}quotient\ Dfactor\ (Dsucc * Dfactor)$

Datomsbot includes the number 1 with the prime numbers.

definition *Datomsbot* :: *'a* **where**
Datomsbot ≡ *Datoms* ⊔ *Dbot*

Dmonobot includes the number 1 with the prime powers.

definition *Dmonobot* :: *'a* **where**
Dmonobot ≡ *Dmono* ⊔ *Dbot*

Dfactorbot is like *Dfactor* except it also relates 1 to 1.

definition *Dfactorbot* :: *'a* **where**
Dfactorbot ≡ *maximal D* (*D* ⊓ *Dmonobot*)

We consider the following axioms for *D*. They correspond to axioms A1–A3, A6–A9, A11–A13 and A15–A16 of [2].

abbreviation *D1-reflexive* :: *'a* ⇒ *bool* **where** *D1-reflexive* - ≡ *reflexive D*
abbreviation *D2-antisymmetric* :: *'a* ⇒ *bool* **where** *D2-antisymmetric* - ≡ *antisymmetric D*
abbreviation *D3-transitive* :: *'a* ⇒ *bool* **where** *D3-transitive* - ≡ *transitive D*
abbreviation *D6-least-surjective* :: *'a* ⇒ *bool* **where** *D6-least-surjective* - ≡ *surjective Dbot*
abbreviation *D7-pre-f-decomposable* :: *'a* ⇒ *bool* **where** *D7-pre-f-decomposable* - ≡ *supremum D* (*D* ⊓ *Dmono*) = *1*
abbreviation *D8-fibered* :: *'a* ⇒ *bool* **where** *D8-fibered* - ≡ *Dmono* ⊓ D^T ∗ (*Datoms* ⊓ *D*) ⊓ $Dmono^T$ ≤ *D* ⊔ D^T
abbreviation *D9-f-decomposable* :: *'a* ⇒ *bool* **where** *D9-f-decomposable* - ≡ *Datoms* ⊓ *D* ≤ *D* ∗ *Dfactor*
abbreviation *D11-atomic* :: *'a* ⇒ *bool* **where** *D11-atomic* - ≡ D^T ∗ *Datoms* = −*Dbot*
abbreviation *D12-infinite-base* :: *'a* ⇒ *bool* **where** *D12-infinite-base* - ≡ −D^T ∗ *Datoms* = *top*
abbreviation *D13-supportable* :: *'a* ⇒ *bool* **where** *D13-supportable* - ≡ *total Dsupport*
abbreviation *D15a-discrete-fibers-succ* :: *'a* ⇒ *bool* **where** *D15a-discrete-fibers-succ* - ≡ *Dmono* ≤ *Dsucc* ∗ *top*
abbreviation *D15b-discrete-fibers-pred* :: *'a* ⇒ *bool* **where** *D15b-discrete-fibers-pred* - ≡ *Dmono* ≤ $Dsucc^T$ ∗ *top*
abbreviation *D16-incrementable* :: *'a* ⇒ *bool* **where** *D16-incrementable* - ≡ *total Dinc*

2.1 Partial order

lemma *div-antisymmetric-equal*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
shows *D* ⊓ D^T = *1*
⟨*proof*⟩

lemma *div-idempotent*:
  assumes *D1-reflexive* -
      and *D3-transitive* -
    shows *idempotent D*
  ⟨*proof*⟩

lemma *div-total*:
  assumes *D1-reflexive* -
    shows $D * top = top$
  ⟨*proof*⟩

lemma *div-surjective*:
  assumes *D1-reflexive* -
    shows $top * D = top$
  ⟨*proof*⟩

lemma *div-below-div-converse*:
  assumes *D2-antisymmetric* -
      and $x \leq D$
    shows $D \sqcap x^T \leq x$
  ⟨*proof*⟩

2.2 Bounds

The least element can be introduced equivalently by

* defining $Dbot = least\ D\ top$ and axiomatising either *surjective Dbot* or $Dbot \neq bot$, or
* axiomatising *point Dbot* and $Dbot \leq D$.

lemma *div-least-div*:
  $Dbot \leq D$
  ⟨*proof*⟩

lemma *div-least-vector*:
  vector Dbot
  ⟨*proof*⟩

lemma *div-least-injective*:
  assumes *D2-antisymmetric* -
    shows *injective Dbot*
  ⟨*proof*⟩

lemma *div-least-point*:
  assumes *D2-antisymmetric* -
      and *D6-least-surjective* -
    shows *point Dbot*
  ⟨*proof*⟩

lemma *div-point-least*:
assumes *D2-antisymmetric* -
and *point x*
and $x \le D$
shows $x = \mathit{least}\ D\ \mathit{top}$
⟨*proof*⟩

lemma *div-least-surjective-iff*:
assumes *D2-antisymmetric* -
shows *D6-least-surjective* - $\longleftrightarrow (\exists x\ .\ \mathit{point}\ x \wedge x \le D)$
⟨*proof*⟩

lemma *div-least-converse*:
assumes *D2-antisymmetric* -
shows $D \sqcap \mathit{Dbot}^T \le \mathit{Dbot}$
⟨*proof*⟩

lemma *bot-div-bot*:
assumes *D1-reflexive* -
and *D3-transitive* -
shows $D * \mathit{Dbot} = \mathit{Dbot}$
⟨*proof*⟩

lemma *all-div-bot*:
assumes *D2-antisymmetric* -
and *D6-least-surjective* -
shows $D^T * \mathit{Dbot} = \mathit{top}$
⟨*proof*⟩

lemma *div-strict-bot*:
assumes *D2-antisymmetric* -
shows $(D \sqcap -1) * \mathit{Dbot} = \mathit{bot}$
⟨*proof*⟩

2.3 Atoms

The atoms can be introduced equivalently by

* defining $\mathit{Datoms} = \mathit{minimal}\ D\ (-\mathit{Dbot})$ and axiomatising either $D^T * \mathit{Datoms} = -\mathit{Dbot}$ or $-\mathit{Dbot} \le D^T * \mathit{Datoms}$ or $-D \le D^T * \mathit{Datoms}$, or
* axiomatising *antichain D Datoms* and $D^T * \mathit{Datoms} = -\mathit{Dbot}$.

lemma *div-atoms-vector*:
vector Datoms
⟨*proof*⟩

lemma *div-atoms-bot-vector*:

vector Datomsbot
⟨*proof*⟩

lemma *div-least-not-atom*:
$Dbot \leq -Datoms$
⟨*proof*⟩

lemma *div-atoms-antichain*:
antichain D Datoms
⟨*proof*⟩

lemma *div-atomic-bot*:
assumes *D2-antisymmetric -*
and *D6-least-surjective -*
shows $D^T * Datomsbot = top$
⟨*proof*⟩

lemma *div-via-atom*:
assumes *D3-transitive -*
and *D11-atomic -*
shows $-Dbot \sqcap D \leq D^T * (D \sqcap Datoms)$
⟨*proof*⟩

lemma *div-via-atom-bot*:
assumes *D1-reflexive -*
and *D2-antisymmetric -*
and *D3-transitive -*
and *D6-least-surjective -*
shows $D \leq D^T * (D \sqcap Datomsbot)$
⟨*proof*⟩

lemma *div-converse-via-atom*:
assumes *D3-transitive -*
and *D11-atomic -*
shows $-Dbot^T \sqcap D^T \leq D^T * (D \sqcap Datoms)$
⟨*proof*⟩

lemma *div-converse-via-atom-bot*:
assumes *D1-reflexive -*
and *D2-antisymmetric -*
and *D3-transitive -*
and *D6-least-surjective -*
shows $D^T \leq D^T * (D \sqcap Datomsbot)$
⟨*proof*⟩

lemma *div-comparable-via-atom*:
assumes *D3-transitive -*
and *D11-atomic -*
shows $-Dbot \sqcap -Dbot^T \sqcap (D \sqcup D^T) \leq D^T * (D \sqcap Datoms)$

⟨proof⟩

lemma *div-comparable-via-atom-bot*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D6-least-surjective* -
shows $D \sqcup D^T \leq D^T * (D \sqcap Datomsbot)$
⟨proof⟩

lemma *div-atomic-iff-1*:
assumes *D3-transitive* -
shows *D11-atomic* - $\longleftrightarrow -Dbot \leq D^T * Datoms$
⟨proof⟩

lemma *div-atomic-iff-2*:
assumes *D3-transitive* -
shows *D11-atomic* - $\longleftrightarrow -D \leq D^T * Datoms$
⟨proof⟩

lemma *div-atoms-antichain-minimal*:
assumes *D2-antisymmetric* -
and *D3-transitive* -
and *antichain D x*
and $D^T * x = -Dbot$
shows $x = minimal\ D\ (-Dbot)$
⟨proof⟩

lemma *div-atomic-iff-3*:
assumes *D2-antisymmetric* -
and *D3-transitive* -
shows *D11-atomic* - $\longleftrightarrow (\exists x .\ antichain\ D\ x \wedge D^T * x = -Dbot)$
⟨proof⟩

The literal translation of axiom A12 is $-Dbot \leq -D^T * Datoms$. However, this allows a model without atoms, where $Dbot = top$ and $Datoms = Dmono = Dfactor = bot$. Nitpick finds a counterexample to *surjective Datoms*. With A2 and A12 the latter is equivalent to $-D^T * Datoms = top$, which we use as a replacement for axiom A12.

lemma *div-atom-surjective*:
assumes *D12-infinite-base* -
shows *surjective Datoms*
⟨proof⟩

lemma *div-infinite-base-upperbound*:
assumes *D12-infinite-base* -
shows $upperbound\ D\ Datoms = bot$
⟨proof⟩

lemma *div-atom-surjective-iff-infinite-base*:
assumes *D2-antisymmetric* -
and $-Dbot \le -D^T * Datoms$
shows *surjective Datoms* $\longleftrightarrow$ *D12-infinite-base* -
⟨*proof*⟩

2.4 Fibers

lemma *div-mono-vector*:
vector Dmono
⟨*proof*⟩

lemma *div-mono-bot-vector*:
vector Dmonobot
⟨*proof*⟩

lemma *div-atom-mono-atom*:
$Datoms \sqcap D * (D^T \sqcap Dmono) \sqcap Datoms^T \le 1$
⟨*proof*⟩

lemma *div-atoms-mono*:
assumes *D1-reflexive* -
shows $Datoms \le Dmono$
⟨*proof*⟩

lemma *div-mono-downclosed*:
assumes *D3-transitive* -
and *D11-atomic* -
shows $-Dbot \sqcap D * Dmono \le Dmono$
⟨*proof*⟩

lemma *div-mono-bot-downclosed*:
assumes *D1-reflexive* -
and *D3-transitive* -
and *D11-atomic* -
shows $D * Dmonobot \le Dmonobot$
⟨*proof*⟩

lemma *div-least-not-mono*:
assumes *D2-antisymmetric* -
shows $Dbot \le -Dmono$
⟨*proof*⟩

lemma *div-fibered-transitive-1*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D11-atomic* -
shows $Dmono \sqcap D^T * (Datoms \sqcap D) \sqcap Dmono^T = Dmono \sqcap (D \sqcup D^T) *$

$(Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T$
$\langle proof \rangle$

lemma *div-fibered-iff*:
assumes *D1-reflexive -*
and *D2-antisymmetric -*
and *D3-transitive -*
and *D11-atomic -*
shows *D8-fibered -* $\longleftrightarrow Dmono \sqcap (D \sqcup D^T) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T \leq D \sqcup D^T$
$\langle proof \rangle$

lemma *div-fibered-transitive*:
assumes *D1-reflexive -*
and *D2-antisymmetric -*
and *D3-transitive -*
and *D8-fibered -*
and *D11-atomic -*
shows $Dmono \sqcap (D \sqcup D^T) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T \leq D \sqcup D^T$
$\langle proof \rangle$

2.5 Fiber decomposition

lemma *div-factor-div-mono*:
$Dfactor \leq D \sqcap Dmono$
$\langle proof \rangle$

lemma *div-factor-div*:
$Dfactor \leq D$
$\langle proof \rangle$

lemma *div-factor-mono*:
$Dfactor \leq Dmono$
$\langle proof \rangle$

lemma *div-factor-one-mono*:
$Dfactor \sqcap 1 \leq Dmono$
$\langle proof \rangle$

lemma *div-pre-f-decomposable-1*:
assumes *D2-antisymmetric -*
and *D7-pre-f-decomposable -*
shows $upperbound\ D\ (D \sqcap Dmono) \leq D^T$
$\langle proof \rangle$

lemma *div-pre-f-decomposable-iff*:
assumes *D2-antisymmetric -*
shows *D7-pre-f-decomposable -* $\longleftrightarrow upperbound\ D\ (D \sqcap Dmono) \leq D^T$
$\langle proof \rangle$

lemma *div-pre-f-decomposable-char*:
  assumes *D2-antisymmetric* -
    and *D7-pre-f-decomposable* -
  shows $upperbound\ D\ (D \sqcap Dmono) \sqcap (upperbound\ D\ (D \sqcap Dmono))^T = 1$
⟨*proof*⟩

lemma *div-factor-bot*:
  assumes *D1-reflexive* -
    and *D2-antisymmetric* -
    and *D3-transitive* -
    and *D11-atomic* -
  shows $Dfactorbot = Dfactor \sqcup (Dbot \sqcap Dbot^T)$
⟨*proof*⟩

lemma *div-factor-surjective*:
  assumes *D1-reflexive* -
    and *D3-transitive* -
    and *D9-f-decomposable* -
    and *D11-atomic* -
  shows $surjective\ (Dbot^T \sqcup Dfactor)$
⟨*proof*⟩

lemma *div-factor-bot-surjective*:
  assumes *D1-reflexive* -
    and *D2-antisymmetric* -
    and *D3-transitive* -
    and *D9-f-decomposable* -
    and *D11-atomic* -
  shows $surjective\ Dfactorbot$
⟨*proof*⟩

lemma *div-factor-surjective-2*:
  assumes *D1-reflexive* -
    and *D3-transitive* -
    and *D9-f-decomposable* -
    and *D11-atomic* -
  shows $-D \leq Dfactor^T * top$
⟨*proof*⟩

lemma *div-conv-factor-div-factor*:
  assumes *D1-reflexive* -
  shows $Dmono \sqcap D^T * Dfactor \sqcap D \leq D * Dfactor$
⟨*proof*⟩

lemma *div-f-decomposable-mono*:
  assumes *D1-reflexive* -
    and *D2-antisymmetric* -
    and *D3-transitive* -

and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
shows *Dmono* $\sqcap$ *D* $\leq$ *D* $*$ *Dfactor*
⟨*proof*⟩

lemma *div-pre-f-decomposable-2*:
assumes *D2-antisymmetric* -
and *D7-pre-f-decomposable* -
shows $-D \leq (D \sqcap Dmono)^T * -D$
⟨*proof*⟩

lemma *div-f-decomposable-char-1*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D7-pre-f-decomposable* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
shows $Dfactor^T * -D = -D$
⟨*proof*⟩

lemma *div-f-decomposable-char-2*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D7-pre-f-decomposable* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
shows *noyau Dfactor* = *1*
⟨*proof*⟩

lemma *div-mono-one-div-factor*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
shows *Dmono* $\sqcap$ *1* $\leq$ *Dfactor*
⟨*proof*⟩

lemma *div-mono-one-div-factor-one*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
shows *Dmono* $\sqcap$ *1* = *Dfactor* $\sqcap$ *1*
⟨*proof*⟩

lemma *div-factor-div-mono-div-factor*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -

and *D3-transitive* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
shows $Dfactor * D = Dmono \sqcap D * Dfactor$
⟨*proof*⟩

lemma *div-mono-strict-div-factor*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
shows $Dmono \sqcap (D \sqcap -1) * Dfactor \leq Dfactor * (D \sqcap -1)$
⟨*proof*⟩

lemma *div-factor-div-strict*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
shows $Dfactor * D \sqcap -1 = Dfactor * (D \sqcap -1)$
⟨*proof*⟩

lemma *div-factor-strict*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
shows $Dfactor \sqcap -1 \leq Dfactor * (D \sqcap -1)$
⟨*proof*⟩

lemma *div-factor-div-mono-div*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
shows $Dfactor * D = Dmono \sqcap D$
⟨*proof*⟩

lemma *div-factor-div-div-factor*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
shows $Dfactor * D \leq D * Dfactor$

⟨*proof*⟩

lemma *div-f-decomposable-eq*:
assumes *D3-transitive* -
and *D9-f-decomposable* -
shows $Datoms \sqcap D = Datoms \sqcap D * Dfactor$
⟨*proof*⟩

lemma *div-f-decomposable-iff-1*:
assumes *D3-transitive* -
shows *D9-f-decomposable* - $\longleftrightarrow Datoms \sqcap D = Datoms \sqcap D * Dfactor$
⟨*proof*⟩

lemma *div-f-decomposable-iff-2*:
assumes *D3-transitive* -
shows $Dmono \sqcap D \leq D * Dfactor \longleftrightarrow Dmono \sqcap D = Dmono \sqcap D * Dfactor$
⟨*proof*⟩

lemma *div-factor-not-bot-conv*:
assumes *D2-antisymmetric* -
shows $Dfactor \leq -Dbot^T$
⟨*proof*⟩

lemma *div-total-top-factor*:
assumes *D2-antisymmetric* -
and *D6-least-surjective* -
shows *total* $(-(top * Dfactor))$
⟨*proof*⟩

lemma *dif-f-decomposable-iff-3*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D8-fibered* -
and *D11-atomic* -
shows *D9-f-decomposable* - $\longleftrightarrow Dmono \sqcap D \leq D * Dfactor$
⟨*proof*⟩

2.6 Support

lemma *div-support-div*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D7-pre-f-decomposable* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
shows $Dsupport \leq D^T$

⟨*proof*⟩

lemma *div-support-univalent*:
  assumes *D1-reflexive* -
    and *D2-antisymmetric* -
    and *D3-transitive* -
    and *D7-pre-f-decomposable* -
    and *D8-fibered* -
    and *D9-f-decomposable* -
    and *D11-atomic* -
   shows *univalent Dsupport*
  ⟨*proof*⟩

lemma *div-support-mapping*:
  assumes *D1-reflexive* -
    and *D2-antisymmetric* -
    and *D3-transitive* -
    and *D7-pre-f-decomposable* -
    and *D8-fibered* -
    and *D9-f-decomposable* -
    and *D11-atomic* -
    and *D13-supportable* -
   shows *mapping Dsupport*
  ⟨*proof*⟩

lemma *div-support-2*:
  assumes *D2-antisymmetric* -
    and *D3-transitive* -
    and *D9-f-decomposable* -
    and *D11-atomic* -
   shows $Dsupport = -((Datoms \sqcap D)^T * -Dfactor) \sqcap -(-D^T * (Datoms \sqcap D))$
⟨*proof*⟩

lemma *noyau-div-support*:
  assumes *D1-reflexive* -
    and *D2-antisymmetric* -
    and *D3-transitive* -
    and *D7-pre-f-decomposable* -
    and *D8-fibered* -
    and *D9-f-decomposable* -
    and *D11-atomic* -
    and *D13-supportable* -
   shows $noyau\ (Datoms \sqcap D) = Dsupport * Dsupport^T$
  ⟨*proof*⟩

lemma *div-support-transitive*:
  assumes *D1-reflexive* -
    and *D2-antisymmetric* -

and *D3-transitive* -
and *D7-pre-f-decomposable* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
and *D13-supportable* -
shows *idempotent Dsupport*
⟨*proof*⟩

lemma *div-support-below-noyau*:
assumes *D2-antisymmetric* -
and *D3-transitive* -
and *D9-f-decomposable* -
and *D11-atomic* -
shows $Dsupport \leq noyau\ (Datoms \sqcap D)$
⟨*proof*⟩

lemma *div-support-least-noyau*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D7-pre-f-decomposable* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
and *D13-supportable* -
shows $Dsupport = (least\ D\ (noyau\ (Datoms \sqcap D)))^T$
⟨*proof*⟩

lemma *div-factor-support*:
assumes *D13-supportable* -
shows $Datoms \sqcap D = Dfactor * Dsupport^T$
⟨*proof*⟩

lemma *div-supportable-iff*:
assumes *D2-antisymmetric* -
and *D6-least-surjective* -
shows *D13-supportable* - $\longleftrightarrow Datoms \sqcap D = Dfactor * Dsupport^T$
⟨*proof*⟩

2.7 Increments

lemma *least-div-atoms-succ*:
$Dbot \sqcap Datoms^T \leq Dsucc$
⟨*proof*⟩

lemma *least-div-succ*:
assumes *D12-infinite-base* -
shows $Dbot \leq Dsucc * top$

⟨*proof*⟩

lemma *noyau-div*:
assumes *D1-reflexive -*
and *D2-antisymmetric -*
shows *noyau D = 1*
⟨*proof*⟩

lemma *div-discrete-fibers-pred-geq*:
assumes *D1-reflexive -*
and *D2-antisymmetric -*
and *D3-transitive -*
and *D7-pre-f-decomposable -*
and *D8-fibered -*
and *D9-f-decomposable -*
and *D11-atomic -*
shows $Dsucc^T * top \leq Dmono$
⟨*proof*⟩

lemma *div-discrete-fibers-pred-eq*:
assumes *D1-reflexive -*
and *D2-antisymmetric -*
and *D3-transitive -*
and *D7-pre-f-decomposable -*
and *D8-fibered -*
and *D9-f-decomposable -*
and *D11-atomic -*
and *D15b-discrete-fibers-pred -*
shows $Dmono = Dsucc^T * top$
⟨*proof*⟩

lemma *div-discrete-fibers-pred-iff*:
assumes *D1-reflexive -*
and *D2-antisymmetric -*
and *D3-transitive -*
and *D7-pre-f-decomposable -*
and *D8-fibered -*
and *D9-f-decomposable -*
and *D11-atomic -*
shows *D15b-discrete-fibers-pred -* $\longleftrightarrow Dmono = Dsucc^T * top$
⟨*proof*⟩

lemma *div-succ-univalent*:
assumes *D1-reflexive -*
and *D2-antisymmetric -*
and *D3-transitive -*
and *D7-pre-f-decomposable -*
and *D8-fibered -*
and *D9-f-decomposable -*

and *D11-atomic* -
and *D15b-discrete-fibers-pred* -
shows $Dsucc^T * (-Dbot \sqcap Dsucc) \leq 1$
⟨*proof*⟩

lemma *div-succ-injective*:
assumes *D2-antisymmetric* -
shows *injective Dsucc*
⟨*proof*⟩

lemma *div-succ-below-div-irreflexive*:
$Dsucc \leq D \sqcap -1$
⟨*proof*⟩

lemma *div-succ-below-div*:
$Dsucc \leq D$
⟨*proof*⟩

lemma *div-succ-mono-bot*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D7-pre-f-decomposable* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
and *D12-infinite-base* -
and *D15a-discrete-fibers-succ* -
shows $Dsucc * top = Dmonobot$
⟨*proof*⟩

lemma *div-discrete-fibers-succ-iff*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D7-pre-f-decomposable* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
and *D12-infinite-base* -
shows *D15a-discrete-fibers-succ* - $\longleftrightarrow Dsucc * top = Dmonobot$
⟨*proof*⟩

lemma *div-succ-bot-atoms*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D6-least-surjective* -
shows $Dsucc^T * Dbot = Datoms$

⟨*proof*⟩

lemma *div-succ-inverse-poly*:
  assumes *D1-reflexive* -
    and *D2-antisymmetric* -
    and *D3-transitive* -
    and *D6-least-surjective* -
    and *D7-pre-f-decomposable* -
    and *D8-fibered* -
    and *D9-f-decomposable* -
    and *D11-atomic* -
    and *D15b-discrete-fibers-pred* -
  shows $Dsucc^T * Dsucc * (Dmono \sqcap -Datoms \sqcap 1) = Dmono \sqcap -Datoms \sqcap 1$
⟨*proof*⟩

lemma *div-inc-injective*:
  assumes *D1-reflexive* -
    and *D2-antisymmetric* -
    and *D3-transitive* -
    and *D7-pre-f-decomposable* -
    and *D8-fibered* -
    and *D9-f-decomposable* -
    and *D11-atomic* -
  shows *injective Dinc*
  ⟨*proof*⟩

lemma *div-factor-not-bot*:
  assumes *D2-antisymmetric* -
  shows $Dfactor \leq -Dbot$
  ⟨*proof*⟩

lemma *div-factor-conv-inc*:
  assumes *D1-reflexive* -
    and *D2-antisymmetric* -
    and *D3-transitive* -
    and *D6-least-surjective* -
  shows $Dfactor * Dinc^T \leq Dmono \sqcap -Datoms$
⟨*proof*⟩

lemma *div-inc-univalent*:
  assumes *D1-reflexive* -
    and *D2-antisymmetric* -
    and *D3-transitive* -
    and *D6-least-surjective* -
    and *D7-pre-f-decomposable* -
    and *D8-fibered* -
    and *D9-f-decomposable* -
    and *D11-atomic* -
    and *D15b-discrete-fibers-pred* -

shows *univalent Dinc*
⟨*proof*⟩

lemma *div-inc-mapping*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D6-least-surjective* -
and *D7-pre-f-decomposable* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
and *D15b-discrete-fibers-pred* -
and *D16-incrementable* -
shows *mapping Dinc*
⟨*proof*⟩

lemma *div-inc-mapping*:
assumes *D1-reflexive* -
and *D2-antisymmetric* -
and *D3-transitive* -
and *D6-least-surjective* -
and *D7-pre-f-decomposable* -
and *D8-fibered* -
and *D9-f-decomposable* -
and *D11-atomic* -
and *D13-supportable* -
and *D15a-discrete-fibers-succ* -
and *D15b-discrete-fibers-pred* -
and *D16-incrementable* -
shows *surjective Datoms*
nitpick[*expect=genuine,card=2*]
⟨*proof*⟩

end

end

3 Mono-Atomic Elements

theory *Mono-Atomic*

imports *Stone-Relation-Algebras.Relation-Algebras*

begin

This theory defines mono-atomic elements in a bounded semilattice and shows that they correspond to join-irreducible elements under the divisibility axioms A1–A17 of [2]. In the model of natural numbers both types of elements correspond to prime powers.

3.1 Mono-atomic

context *order-bot*
begin

Divisibility axioms A1 (reflexivity), A2 (antisymmetry), A3 (transitivity) and A6 (least element) are the axioms of class *order-bot*, so not mentioned explicitly.

An *atom* in a partial order is an element that is strictly above only the least element *bot*.

definition *atom a* $\equiv a \neq bot \wedge (\forall x . x \leq a \longrightarrow x = bot \vee x = a)$
abbreviation *atom-below a x* $\equiv$ *atom a* $\wedge a \leq x$

A mono-atomic element has exactly one atom below it.

definition *mono-atomic x* $\equiv (\exists! a$. *atom-below a x*$)$
definition *mono-atomic-with x a* $\equiv$ *atom-below a x* $\wedge (\forall b$. *atom-below b x* $\longrightarrow b = a)$
abbreviation *mono-atomic-below x y* $\equiv$ *mono-atomic x* $\wedge x \leq y$
abbreviation *mono-atomic-above x y* $\equiv$ *mono-atomic x* $\wedge y \leq x$
definition *mono-atomic-above-or-bot x y* $\equiv x = bot \vee$ *mono-atomic-above x y*

Divisibility axiom A11 (atomicity) states that every element except *bot* is above some atom.

abbreviation *A11-atomic* :: *'a* $\Rightarrow$ *bool* **where** *A11-atomic* - $\equiv$ $(\forall x . x \neq bot \longrightarrow (\exists a$. *atom-below a x*$))$

lemma *mono-atomic-above*:
mono-atomic x $\longleftrightarrow (\exists a$. *mono-atomic-with x a*$)$
⟨*proof*⟩

Among others, the following divisibility axioms are considered in [2]. In the model of natural numbers,

- A7 (pre-f-decomposability) expresses that every number x is the least upper bound of the prime powers below x;
- A8 (fibered) expresses that the prime powers can be partitioned into chains;
- A9 (f-decomposability) expresses that for every number x above an atom a there is a maximal prime power of a below x;
- A14 (truncability) express that the prime powers contained in a number y can be restricted to those whose atoms are not below a number x.

Their definitions are based on join-irreducible elements and given in class *bounded-semilattice-sup-bot* below. Here we introduce corresponding axioms B7, B8, B9 and B14 based on mono-atomic elements.

abbreviation *B7-pre-f-decomposable* :: $'a \Rightarrow bool$ **where** *B7-pre-f-decomposable* - $\equiv (\forall x\ y\ .\ (\forall z\ .\ \textit{mono-atomic-below}\ z\ x \longrightarrow z \leq y) \longrightarrow x \leq y)$
abbreviation *B8-fibered* :: $'a \Rightarrow bool$ **where** *B8-fibered* - $\equiv (\forall x\ y\ z\ .\ \textit{mono-atomic}\ x \wedge \textit{mono-atomic}\ y \wedge \textit{mono-atomic}\ z \wedge ((x \leq z \wedge y \leq z) \vee (z \leq x \wedge z \leq y)) \longrightarrow x \leq y \vee y \leq x)$
abbreviation *B9-f-decomposable* :: $'a \Rightarrow bool$ **where** *B9-f-decomposable* - $\equiv (\forall x\ a\ .\ \textit{atom}\ a \longrightarrow (\exists z\ .\ \textit{mono-atomic-above-or-bot}\ z\ a \wedge z \leq x \wedge (\forall w\ .\ \textit{mono-atomic-above-or-bot}\ w\ a \wedge w \leq x \longrightarrow w \leq z)))$

Function *mval* returns the value whose existence is asserted by axiom B9.

definition $\textit{mval}\ a\ x \equiv \textit{SOME}\ z\ .\ \textit{mono-atomic-above-or-bot}\ z\ a \wedge z \leq x \wedge (\forall w\ .\ \textit{mono-atomic-above-or-bot}\ w\ a \wedge w \leq x \longrightarrow w \leq z)$

lemma *mval-char*:
assumes *B9-f-decomposable* -
and *atom a*
shows $\textit{mono-atomic-above-or-bot}\ (\textit{mval}\ a\ x)\ a \wedge \textit{mval}\ a\ x \leq x \wedge (\forall w\ .\ \textit{mono-atomic-above-or-bot}\ w\ a \wedge w \leq x \longrightarrow w \leq \textit{mval}\ a\ x)$
⟨*proof*⟩

lemma *mval-unique*:
assumes *B9-f-decomposable* -
and *atom a*
and $\textit{mono-atomic-above-or-bot}\ z\ a \wedge z \leq x \wedge (\forall w\ .\ \textit{mono-atomic-above-or-bot}\ w\ a \wedge w \leq x \longrightarrow w \leq z)$
shows $z = \textit{mval}\ a\ x$
⟨*proof*⟩

lemma *atom-below-mval*:
assumes *B9-f-decomposable* -
and *atom a*
and $a \leq x$
shows $a \leq \textit{mval}\ a\ x$
⟨*proof*⟩

abbreviation *B14-truncability* :: $'a \Rightarrow bool$ **where** *B14-truncability* - $\equiv (\forall x\ y\ .\ \exists\ z\ .\ \forall a\ .\ \textit{atom}\ a \longrightarrow (\textit{if}\ a \leq x\ \textit{then}\ \textit{mval}\ a\ z = \textit{bot}\ \textit{else}\ \textit{mval}\ a\ z = \textit{mval}\ a\ y))$

Function *mtrunc* returns the value whose existence is asserted by axiom B14.

definition $\textit{mtrunc}\ x\ y \equiv \textit{SOME}\ z\ .\ \forall a\ .\ \textit{atom}\ a \longrightarrow (\textit{if}\ a \leq x\ \textit{then}\ \textit{mval}\ a\ z = \textit{bot}\ \textit{else}\ \textit{mval}\ a\ z = \textit{mval}\ a\ y)$

lemma *mtrunc-char*:
assumes *B14-truncability* -
shows $\forall\, a$. *atom a* $\longrightarrow$ (*if* $a \leq x$ *then mval a* (*mtrunc x y*) = *bot else mval a* (*mtrunc x y*) = *mval a y*)
⟨*proof*⟩

lemma *mtrunc-char-1*:
assumes *B14-truncability* -
and *atom a*
and $a \leq x$
shows *mval a* (*mtrunc x y*) = *bot*
⟨*proof*⟩

lemma *mtrunc-char-2*:
assumes *B14-truncability* -
and *atom a*
and $\neg\, a \leq x$
shows *mval a* (*mtrunc x y*) = *mval a y*
⟨*proof*⟩

lemma *mtrunc-unique*:
assumes *B14-truncability* -
and $\forall\, a$. *atom a* $\longrightarrow$ (*if* $a \leq x$ *then mval a z* = *bot else mval a z* = *mval a y*)
and *atom a*
shows *mval a z* = *mval a* (*mtrunc x y*)
⟨*proof*⟩

lemma *lesseq-iff-mval-below*:
assumes *B7-pre-f-decomposable* -
and *B9-f-decomposable* -
and *atom a*
shows $x \leq y \longleftrightarrow$ ($\forall\, a$. *atom a* $\longrightarrow$ *mval a x* $\leq y$)
⟨*proof*⟩

end

3.2 Join-irreducible

context *bounded-semilattice-sup-bot*
begin

Divisibility axioms A1 (reflexivity), A2 (antisymmetry), A3 (transitivity), A5 (least upper bound) and A6 (least element) are the axioms of class *bounded-semilattice-sup-bot*, so not mentioned explicitly.

A join-irreducible element cannot be expressed as the join of two incomparable elements.

definition *join-irreducible x* $\equiv x \neq$ *bot* $\wedge$ ($\forall\, y\ z$. $x = y \sqcup z \longrightarrow x = y \vee x = z$)
abbreviation *join-irreducible-below x y* $\equiv$ *join-irreducible x* $\wedge\ x \leq y$

abbreviation *join-irreducible-above x y* $\equiv$ *join-irreducible x* $\wedge$ $y \leq x$
definition *join-irreducible-above-or-bot x y* $\equiv$ $x = bot$ $\vee$ *join-irreducible-above x y*

Divisibility axioms A7, A8 and A9 based on join-irreducible elements are introduced here; axiom A14 is not needed for this development.

abbreviation *A7-pre-f-decomposable* :: *'a* $\Rightarrow$ *bool* **where** *A7-pre-f-decomposable* - $\equiv$ ($\forall$ *x y* . ($\forall$ *z* . *join-irreducible-below z x* $\longrightarrow$ $z \leq y$) $\longrightarrow$ $x \leq y$)
abbreviation *A8-fibered* :: *'a* $\Rightarrow$ *bool* **where** *A8-fibered* - $\equiv$ ($\forall$ *x y z* . *join-irreducible x* $\wedge$ *join-irreducible y* $\wedge$ *join-irreducible z* $\wedge$ (($x \leq z \wedge y \leq z$) $\vee$ ($z \leq x \wedge z \leq y$)) $\longrightarrow$ $x \leq y \vee y \leq x$)
abbreviation *A9-f-decomposable* :: *'a* $\Rightarrow$ *bool* **where** *A9-f-decomposable* - $\equiv$ ($\forall$ *x a* . *atom a* $\longrightarrow$ ($\exists$ *z* . *join-irreducible-above-or-bot z a* $\wedge$ $z \leq x$ $\wedge$ ($\forall$ *w* . *join-irreducible-above-or-bot w a* $\wedge$ $w \leq x \longrightarrow w \leq z$)))

lemma *atom-join-irreducible*:
  assumes *atom a*
    shows *join-irreducible a*
  ⟨*proof*⟩

lemma *mono-atomic-with-downclosed*:
  assumes *A11-atomic* -
      and *mono-atomic-with x a*
      and $y \neq bot$
      and $y \leq x$
    shows *mono-atomic-with y a*
  ⟨*proof*⟩

3.3 Equivalence

The following result shows that under divisibility axioms A1–A3, A5–A9 and A11, join-irreducible elements coincide with mono-atomic elements.

lemma *join-irreducible-iff-mono-atomic*:
  assumes *A7-pre-f-decomposable* -
      and *A8-fibered* -
      and *A9-f-decomposable* -
      and *A11-atomic* -
    shows *join-irreducible x* $\longleftrightarrow$ *mono-atomic x*
⟨*proof*⟩

The following result shows that under divisibility axioms A1–A3, A5–A6, B7–B9, A11 and B14, join-irreducible elements coincide with mono-atomic elements.

lemma *mono-atomic-iff-join-irreducible*:
  assumes *B7-pre-f-decomposable* -
      and *B8-fibered* -
      and *B9-f-decomposable* -
      and *A11-atomic* -
      and *B14-truncability* -

shows *mono-atomic x* $\longleftrightarrow$ *join-irreducible x*
⟨*proof*⟩

end

end

References

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